PACQ Chapter 2 Solutions

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Exercise 2.1

Show that for $\lambda'\beta$ estimable,

$$\frac{\lambda'\hat{\beta} - \lambda'\beta}{\sqrt{MSE\lambda'(X'X)^{-}\lambda}} \sim t(dfE) \tag{1}$$

Find the form of an α level test of $H_0 : \lambda'\beta = 0$ and the form for a $(1-\alpha)100\%$ confidence interval for $\lambda'\beta$.

I will make the usual assumptions for the linear model. Rewrite the expression as the following:

$$\frac{\frac{\lambda'\hat{\beta}-\lambda'\beta}{\sigma\sqrt{\lambda'(X'X)^{-}\lambda)}}}{\sqrt{\frac{MSE}{\sigma^2}}} \tag{2}$$

In the numerator,

$$\lambda'\hat{\beta} \sim N(\lambda\beta, \sigma^2\lambda'(X'X)^{-}\lambda)$$

implies

$$\frac{\lambda'\hat{\beta} - \lambda'\beta}{\sigma\sqrt{\lambda'(X'X)^{-}\lambda)}} \sim N(0,1)$$

In the denominator,

$$(n-r)\frac{MSE}{\sigma^2} \sim X^2(n-r)$$

Since $\lambda'\beta$ is estimable, $\lambda'\hat{\beta} = \rho'MY$, and since $MSE = \frac{Y'(I-M)Y}{n-r}$, $\rho'MY$ and $Y'(I-M)Y$ are independent, given the usual linear model assumptions. Thus Equation 2 has a $t(dfE)$ distribution, where $dfE = n - r$.

For the α level test of $H_0 : \lambda'\beta = 0$, the test rejects for large values of the test statistic in Equation 1. That is, we reject when

$$\frac{\lambda'\hat{\beta}}{\sqrt{MSE\lambda'(X'X)^{-}\lambda}} \geq t(1-\alpha/2, n-r)$$

or

$$\frac{\lambda'\hat{\beta}}{\sqrt{MSE\lambda'(X'X)^{-}\lambda}} \leq -t(1-\alpha/2, n-r)$$

For the $(1-\alpha)100\%$ confidence interval, we do not reject if

$$-t(1-\alpha/2, n-r) < \frac{\lambda'\hat{\beta} - \lambda'\beta}{\sqrt{MSE\lambda'(X'X)^{-}\lambda}} < t(1-\alpha/2, n-r)$$

which is equivalent to

$$\lambda'\hat{\beta} - t(1-\alpha/2, n-r)\sqrt{MSE\lambda'(X'X)^{-}\lambda} < \lambda'\beta < \lambda'\hat{\beta} + t(1-\alpha/2, n-r)\sqrt{MSE\lambda'(X'X)^{-}\lambda}$$

and thus the $(1-\alpha)\%$ confidence interval for $\lambda'\beta$ is

$$\left(\lambda'\hat{\beta} - t(1-\alpha/2, n-r)\sqrt{MSE\lambda'(X'X)^{-}\lambda}, \lambda'\hat{\beta} + t(1-\alpha/2, n-r)\sqrt{MSE\lambda'(X'X)^{-}\lambda}\right)$$

Exercise 2.2

Let $y_{11}, y_{12} \ldots y_{1r}$ be $N(\mu_1, \sigma^2)$ and $y_{21}, y_{22} \ldots y_{2s}$ be $N(\mu_2, \sigma^2)$ with all y_{ij}s independent. Write this as a linear model. Using the results of Chapter 2, find estimates of $\mu_1, \mu_2, \mu_1 - \mu_2$, and σ^2. Form a $\alpha = 0.01$ test for $H_0 : \mu_1 = \mu_2$. Form 95% confidence intervals for $\mu_1 - \mu_2$ and μ_1. What is the test for $H_0 : \mu_1 = \mu_2 + \Delta$, where Δ is some known fixed quantity? How do these results compare with the usual analysis for two independent samples with a common variance?

I will write this as a linear model in $\mathbb{R}^{s+r}$, $Y = X\mu + \epsilon$ with data, design, and coefficient matrices

$$Y = \begin{pmatrix} y_{11} \\ \vdots \\ y_{1r} \\ y_{21} \\ \vdots \\ y_{2s} \end{pmatrix}, X = \begin{pmatrix} J_{r,1} & 0 \\ 0 & Js,1 \end{pmatrix}, \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

We will assume that ϵ is distributed $N(0, \sigma_2 I)$ as usual.

μ_1 is estimable. To show this, note that $\lambda_1' = (1, 0)$ implies that $\lambda_1'\mu = \mu_1$. For $\rho_1' = (1, 0 \ldots 0)$, $P_1'X\mu = \mu_1$. Thus $\hat{\mu}_1 = \rho_1'MY$ is the least squares estimate for μ_1. Similarly, the least squares estimate of μ_2 is estimated with $\rho_2' = (0 \ldots 0, 1)$, and $\mu_1 - \mu_2$ is estimable with $(\rho_1 - \rho_2)'$. σ^2 is estimated as usual with the MSE, $Y'(I-M)Y/(n-r(M))$.

Calculating M, we have that

$$X'X = \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}$$

implies

$$\begin{aligned}
(X'X)^{-1} &= \begin{pmatrix} 1/r & 0 \\ 0 & 1/s \end{pmatrix} \\
M = X(X'X)^{-1}X' &= \begin{pmatrix} \frac{1}{r}J_r & 0 \\ 0 & \frac{1}{s}J_s \end{pmatrix} \\
MY &= \begin{pmatrix} \left(\frac{1}{r}\sum_i^r y_{i1}\right) J_{r,1} \\ \left(\frac{1}{s}\sum_i^s y_{i2}\right) J_{s,1} \end{pmatrix} = \begin{pmatrix} \bar{Y_1} J_{r,1} \\ \bar{Y_2} J_{s,1} \end{pmatrix}
\end{aligned}$$

where $\bar{Y_1} = \frac{1}{r}\sum_{i=1}^r y_{1i}$, and $\bar{Y_2}$ similarly. Thus

$$\hat{\mu}_1 = \rho_1' MY = \frac{1}{r}\sum_i^r y_{i1} \tag{3}$$

$$\hat{\mu}_2 = \rho_2' MY = \frac{1}{s}\sum_i^s y_{i2} \tag{4}$$

$$\widehat{\mu_1 - \mu_2} = (\rho_1 - \rho_2)' MY = \hat{\mu}_1 - \hat{\mu}_2 \tag{5}$$

which match with the standard two-sample results. For $\hat{\sigma^2} = \frac{Y'(I-M)Y}{n-r(M)} = \frac{Y'(I-M)Y}{n-2}$, we have

$$\begin{aligned}
Y'Y &= \sum_{i=1}^r y_{1i}^2 + \sum_{i=1}^s y_{2i}^2 \\
Y'MY &= \sum_{i=1}^r y_{1i}\bar{Y_1} + \sum_{i=1}^s y_{2i}\bar{Y_2} \\
Y'(I-M)Y &= \sum_{i=1}^r y_{1i}(y_{1i} - \bar{Y_1}) + y_{2i}(y_{2i} - \bar{Y_2}) \\
&= \sum_{i=1}^r (y_{1i} - \bar{Y_1})^2 + \sum_{i=1}^s (y_{2i} - \bar{Y_2})^2
\end{aligned}$$

and so the estimate of σ^2, $\hat{\sigma^2} = MSE = \frac{Y'(I-M)Y}{n-2}$ matches the standard two-sample results.

For a $\alpha = .01$ level test for $H_0 : \mu_1 = \mu_2$, we have that $\lambda' = (1, -1)$ gives us a model of the form $H_0 : \lambda'\mu = 0$. Thus by Exercise 2.1,

$$\frac{\lambda'\hat{\mu}}{\sqrt{MSE\lambda'(X'X)^{-1}\lambda}} = \frac{\hat{\mu}_1 - \hat{\mu}_2}{\sqrt{MSE(\frac{1}{r}+\frac{1}{s})}} \geq t(.995, n-2)$$

forms a level $\alpha = .01$ test for H_0.

For the 95% confidence interval for $\mu_1 - \mu_2$, by Exercise 2.1,

$$\left(\hat{\mu}_1 - \hat{\mu}_2 - t(.975, n-2)\sqrt{MSE\left(\frac{1}{r}+\frac{1}{s}\right)}, \hat{\mu}_1 - \hat{\mu}_2 + t(.975, n-2)\sqrt{MSE\left(\frac{1}{r}+\frac{1}{s}\right)}\right)$$

forms a 95% confidence interval for $\mu_1 - \mu_2$. For a 95% confidence interval for μ_1, we have in this case that for $\lambda' = (1, 0)$, $\lambda\mu = \mu_1$, so $\lambda'(X'X)^{-1}\lambda = \frac{1}{r}$. Thus by Exercise 2.1,

$$\left(\hat{\mu}_1 - t(.975, n-2)\sqrt{\frac{MSE}{r}}, \hat{\mu}_1 + t(.975, n-2)\sqrt{\frac{MSE}{r}},\right)$$

forms a 95% confidence interval for μ_1.

To test the null hypothesis that $H_0 : \mu_1 = \mu_2 + \Delta$ where Δ is a fixed quantity, we know from Exercise 2.1 and Appendix E that

$$\frac{\hat{\mu}_1 - \hat{\mu}_2 - \Delta}{\sqrt{MSE\left(\frac{1}{r}+\frac{1}{s}\right)}} \sim t(n-2)$$

so a level $100(1-\alpha)\%$ test is to reject when

$$\left|\frac{\hat{\mu}_1 - \hat{\mu}_2 - \Delta}{\sqrt{MSE\left(\frac{1}{r}+\frac{1}{s}\right)}}\right| \geq t(.975, n-2)$$