PACQ Chapter 1 Solutions

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May 29, 2024

Exercise 1.2

Let W be an $r \times s$ random matrix, and let A and C be $n \times r$ and $n \times s$ matrices of constants. Show that $\mathrm{E}(AW + C) = A\mathrm{E}(W) + C$. If B is $s \times t$ constant matrix, show that $\mathrm{E}(AWB) = A\mathrm{E}(W)B$. If $s = 1$, show that $\mathrm{Cov}(AW + C) = A\mathrm{Cov}(W)A'$.

For $\mathrm{E}(AW + C)$, by definition $(AW + C)_{ij} = \sum_{k=1}^{r} A_{ik}W_{kj} + C_{ij}$. Taking expectations, we have

$$\mathrm{E}((AW + C)_{ij}) = \sum_{k=1}^{r} A_{ik}\mathrm{E}(W_{kj}) + C_{ij}$$

and we see that this equals $A\mathrm{E}(W) + C$.

For $\mathrm{E}(AWB)$, $\mathrm{E}(AWB) = A\mathrm{E}(WB)$ by the above.

Exercise 1.3

Show that $\mathrm{Cov}(Y)$ is nonnegative definite for any random vector Y.

Let z be an arbitrary fixed vector with the same length as Y. Then

$$\begin{aligned} z'\mathrm{Cov}(Y)z &= z'\mathrm{E}[(Y - \mu)(Y - \mu)']z \\ &= \mathrm{E}[z'(Y - \mu)(Y - \mu)'z] \\ &= \mathrm{E}[||z'(Y - \mu)'||^2] \end{aligned}$$

and since $||z'(Y - \mu)'||^2 \geq 0$, its expectation is also ≥ 0.

Exercise 1.4

Let M be the ppo onto $C(X)$. Show that $(I - M)$ is the ppo onto $C(X)^{\perp}$.

$I - M$ is symmetric and $(I - M)(I - M) = I - 2M + M = I - M$ idempotent and thus is a ppo. It projects onto $C(X)^{\perp}$ because for $x \in C(X)$, $(I - M)x = x - x = 0$ and for $x \in C(X)^{\perp}$, $(I - M)x = x$. For the trace, $\mathrm{tr}(I - M + M) = n$ implies that $\mathrm{tr}(I - M) + \mathrm{tr}(M) = n$. Since $\mathrm{tr}(M) = r(X)$, we have that $\mathrm{tr}(I - M) = n - r(X)$.

Exercise 1.6

For a linear model $Y = X\beta + e$, $\mathrm{E}(e) = 0$, $\mathrm{Cov}(E) = \sigma^2 I$, the residuals are

$$\hat{e} = Y - X\hat{\beta} = (I - M)Y$$

where M is the perpendicular projection operator onto $C(X)$. Find the following.

Part a

$$\mathrm{E}(\hat{e}) = (I - M)\mathrm{E}(Y) = (I - M)X\beta = 0$$

because $X\beta \in C(X)$.

Part b

$$\mathrm{Cov}(\hat{e}) = (I - M)\mathrm{Cov}(Y)(I - M)' = \sigma^2(I - M)$$

because $I - M$ being a ppo implies $(I - M)(I - M)' = (I - M)$.

Part c

$$\mathrm{Cov}(\hat{e}, MY) = \mathrm{Cov}(\hat{e}, Y)M' = \mathrm{Cov}((I - M)Y, Y)M = \sigma^2(I - M)M = 0$$

Part d

$$\mathrm{E}(\hat{e}'\hat{e}) = \mathrm{E}[Y'(I - M)Y] = \mathrm{E}(e'(I - M)e)$$

where the second equality uses $(I - M)Y = (I - M)X\beta + e = (I - M)e$ and the symmetry of $I - M$. By the distribution of quadratic forms,

$$\mathrm{E}(e'(I - M)e) = \mathrm{tr}((I - M)(\sigma^2 I)) = \sigma^2(n - r)$$

Part e

To show that $\hat{e}'\hat{e} = Y'Y - Y'MY$, this immediately follows from $\hat{e}'\hat{e} = Y'(I - M)Y$ and distributing.

Part f

Using that $MY = X\hat{\beta}$, rewrite c. and e.

Making the substitution, we get that $\mathrm{Cov}(\hat{\beta}, \hat{e}) = 0$ and $\hat{e}'\hat{e} = Y'Y - \hat{\beta}'Y$.

Exercise 1.7

Given that $Y \sim N(\mu, V)$ and V is nonsingular, show that the density of Y is

$$f(y) = (2\pi)^{-n/2}[\det(V)]^{-1/2}\exp(-(y-\mu)'V^{-1}(y-\mu)/2)$$

We can write V as $V = AA'$, where A is nonsingular. By the definition of normals, we can rewrite Y as $Y = G(Z) = \mu + AZ$, where Z, where Z is a standard normal. Thus $G^{-1}(Y) = A^{-1}(Y - \mu)$, $dG^{-1}(Y) = A^{-1}$, and

$$\det(dG^{-1}(Y)) = \det(A^{-1}) = \det(A)^{-1} = \det(V)^{-1/2}$$

since $AA' = V$ implies $\det(A) = \det(V)^{1/2}$. For the density of Z,

$$\begin{aligned} f_Z(z) &= f_Z(G^{-1}(y)) \\ &= (2\pi)^{-n/2}\exp(-(Y-\mu)'A'^{-1}A^{-1}(Y-\mu)/2) \\ &= (2\pi)^{-n/2}\exp(-(Y-\mu)'V^{-1}(Y-\mu)/2) \end{aligned}$$

since $AA' = V$ implies $A'^{-1}A^{-1} = V^{-1}$. Thus the final density of Y is

$$\begin{aligned} f_Y(y) &= f_Z(G^{-1}(y))|\det(dG^{-1}(Y))| \\ &= (2\pi)^{-n/2}\det(V)^{-1/2}\exp(-(Y-\mu)'V^{-1}(Y-\mu)/2) \end{aligned}$$

as desired.

Exercise 1.8

Show that if $Y \sim N(\mu, V)$ and B is a fixed $n \times r$ matrix, then $BY \sim N(B\mu, BVB')$.

Let Z be an r-dimensional standard normal. Define $Y = AZ + \mu$ where $AA' = V$. Then $BY = BAZ + B\mu$ is a normally distributed random variable with distribution $N(B\mu, BAA'B') = N(B\mu, BVB')$.

Exercise 1.11

Prove that if $Y \sim N(\mu, V)$ and $VAVBV = 0$, $VAVB\mu = 0$, $VBVA\mu = 0$, and the conditions from Theorem 1.3.6 hold for $Y'AY$ and $Y'BY$, then $Y'AY$ and $Y'BY$ are independent.

Let $V = QQ'$ and rewrite $Y = \mu + QZ$, where $Z \sim N(0, I)$. We now show that the following variables are independent:

$$\begin{bmatrix} Q'AQZ \\ \mu'AQZ \end{bmatrix} \perp\!\!\!\perp \begin{bmatrix} Q'BQZ \\ \mu'BQZ \end{bmatrix} \tag{1}$$

Since these variables are all normal, showing uncorrelatedness shows independence. We first show this for one of the terms.

$$\begin{aligned}\mathrm{Cov}(Q'AQZ, Q'BQZ) &= \mathrm{E}(Q'AQZZ'Q'B'Q) - \mathrm{E}(Q'AQZ)E(Z'Q'B'Q) \quad (2)\\ &= Q'AQQ'B'Q = Q'AVBQ \quad (3)\end{aligned}$$

because $Z \sim N(0, I)$ and $QQ' = V$. We have from the same argument as the proof of Theorem 1.3.6 in the book that $Q = Q_1Q_2$, where Q_1 has orthonormal columns and Q_2 is nonsingular. Thus by the same argument,

$$Q_2^{-1}Q_1'V = Q'$$

Applying this result to Equation 2, we get that

$$\mathrm{Cov}(Q'AQZ, Q'BQZ) = Q_2^{-1}Q_1'VAVBV'Q_1Q_2'^{-1} = 0$$

since by assumption $VAVABV = 0$. Similar results hold for the other cross terms.

We then have by the definition of $Y = \mu + QZ$ that

$$\begin{aligned}Y'AY &= (\mu' + Z'Q')A(QZ + \mu)\\ &= Z'Q'AQZ + Z'Q'A\mu + \mu'AQZ + \mu'A\mu\end{aligned}$$

The second, third, and fourth terms are trivially functions of variables on the left side of Equation 1. The first term is, too, because

$$\begin{aligned}Z'Q'AQZ &= Z'Q_2^{-1}Q_1'VAVQ_1Q_2'^{-1}Z\\ &= Z'Q_2^{-1}Q_1'VAVAVQ_1Q_2'^{-1}Z\\ &= Z'Q_2^{-1}Q_1'VAQQ'AVQ_1Q_2'^{-1}Z\\ &= Z'Q'VAQQ'AQZ\\ &= (Q'AQZ)'(Q'AQZ)\\ &= ||Q'AQZ||^2\end{aligned}$$

is a function of $Q'AQZ$, where the second line follows from the assumption that $VAVAV = VAV$ from Theorem 1.3.6 in the book. A similar argument holds for $Y'BY$. Thus $Y'AY$ is a function of variables on the left side of Equation 1, and $Y'BY$ a function of the right side. Since the two sides of Equation 1 are independent, $Y'AY$ and $Y'BY$ are independent.

Exercise 1.5.1

Let $Y = (y_1, y_2, y_3)'$ be a random vector. Suppose $E(Y) \in M$, where

$$M = \{(a, a - b, 2b)' | a, b \in \mathbb{R}\}$$

Part a

Show that M is a vector space.

Let $x = (a_1, a_1 - b_1, 2b_1)$ and $y = (a_2, a_2 - b_2, 2b_2)$ be in M. Then

$$cx + dy = \begin{pmatrix} ca_1 + da_2 \\ (ca_1 + da_2) - (cb_1 + db_2) \\ 2(cb_1 + db_2) \end{pmatrix}$$

is in M.

Part b

Find a basis for M.

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \right\}$$

is a basis for M.

Part c

Find a linear model for the problem.

A linear model is

$$X = \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$$

such that $Y = X\beta + e$, $\mathrm{E}(e) = 0$.

Part d

Find two vectors r and s such that $\mathrm{E}(r'Y) = r'X\beta = \beta_1$ and $\mathrm{E}(s'Y) = \beta_2$. Find another vector $t \neq r$ such that $\mathrm{E}(t'Y) = \beta_1$.

Let $r = (1, 0, 0)'$, $s = (0, 0, 1/2)'$, and $t = (0, 1, -1/2)$.

Exercise 1.5.2

Let

$$Y \sim N\left(\begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \\ 1 & 2 & 4 \end{pmatrix} \right)$$

Find

Part a

the marginal distribution of y_1.

$y_1 = AY$, where $A = (1, 0, 0)$. By Exercise 1.8, we have

$$AY \sim N(A\mu, AVA') = N(5, 2)$$

Part b

the joint distribution of y_1 and y_2.

Letting $B = (1, 1, 0)$, we have

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = BY \sim N\left(\begin{pmatrix} 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\right)$$

Part c

the conditional distribution of y_3 given $y_1 = u_1$ and $y_2 = u_2$.

By the formulas for conditional distributions of normals, $y_3|y_1, y_2$ is normal with mean and covariance

$$\mu_{3|1,2} = 7 + (1, 2)\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^{-1}\begin{pmatrix} u_1 - 5 \\ u_2 - 6 \end{pmatrix} = 7 + \frac{1}{2}(u_1 - 5) + \frac{2}{3}(u_2 - 6)$$

$$\Sigma_{3|1,2} = 4 - \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 4 - \frac{11}{6} = \frac{13}{6}$$

Part d

the conditional distribution of y_3 given $y_1 = u_1$.

From similar calculations to Part c we have that $y_3|y_1 = y_1$ is normal with mean and covariance

$$\mu_{3|1} = 7 + 1(2)^{-1}(u_1 - 5) = 7 + \frac{1}{2}(u_1 - 5)$$

$$\Sigma_{3|1} = 4 - 1(2)^{-1}1 = \frac{7}{2}$$

Part e

the conditional distribution of y_1 and y_2 given $y_3 = u_3$.

$y_1, y_2|y_3$ is normal with mean and covariance

$$\mu_{1,2|3} = \begin{pmatrix} 5 \\ 6 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}(4)^{-1}(u_3 - 7) = \begin{pmatrix} 5 \\ 6 \end{pmatrix} + \begin{pmatrix} 1/4 \\ 1/2 \end{pmatrix}(u_3 - 7)$$

$$\Sigma_{1,2|3} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{pmatrix} 7/4 & -1/2 \\ -1/2 & 2 \end{pmatrix}$$

Part f

ρ_{12}, ρ_{23}, and ρ_{13}.

Reading off the numbers from the covariance matrix,

$$\rho_{12} = 0$$
$$\rho_{13} = \frac{1}{\sqrt{2}\sqrt{4}} = \frac{\sqrt{2}}{4}$$
$$\rho_{23} = \frac{2}{\sqrt{3}\sqrt{4}} = \frac{\sqrt{3}}{3}$$

Part g

The distribution of

$$Z = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} Y + \begin{pmatrix} -15 \\ -18 \end{pmatrix}$$

Multiplying out the matrices and using the formulas, we have that

$$\mu_Z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\Sigma_z = \begin{pmatrix} 11 & 11 \\ 11 & 15 \end{pmatrix}$$

Exercise 1.5.3

The density of $Y = (y_1, y_2, y_3)'$ is

$$(2\pi)^{-3/2}|V|^{-1/2}e^{-Q/2}$$

where

$$Q = 2y_1^2 + y_2^2 + 2y_1y_1 - 8y_1 - 4y_2 + 8$$

Find V^{-1} and μ.

We know that this is a normal in $\mathbb{R}^3$. We can see immediately from the fact that there is only one y_3 term, y_3^2, that $\mu_3 = 0 . \sigma_3^2 = 1$ and y_3 is uncorrelated, and thus independent, of y_1 and y_2.

From the y_1^2 and y_2^2 terms we see that $\sigma_1^2 = 2, \sigma_2^2 = 1$. The unknown quantities are $\mu_1, \mu_2, \sigma_{12}^2$. Putting these into the matrix multiplication, we have that the y_1 and y_2 parts of Q have the form

$$(y_1 - a, y_2 - b) \begin{pmatrix} 2 & c \\ c & 1 \end{pmatrix} (y_1 - a, y_2 - b)'$$
$$= 2(y_1 - a)^2 + 2c(y_1 - a)(y_2 - b) + (y_2 - b)^2$$

and we see with comparison with the $2y_1y_2$ term in Q that $c = 1$. Expanding out this term, we get

$$2y_1^2 - 2(2a+b)y_1 + 2y_1y_2 + y_2^2 - 2(a+b) + 2a^2 + 2ab + b^2$$

and comparing the terms with the y_1 and y_2 terms in Q gives that $a = 2, b = 0$. Thus

$$\mu = (2, 0, 0)'$$
$$V^{-1} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise 1.5.4

Let $Y \sim N(J\mu, \sigma^2 I)$ and let $O = [n^{-1/2}J, O_1]$ be an orthonormal matrix.

Part a

Find the distribution of $O'Y$.

$$\mu_{O'Y} = \begin{pmatrix} n^{-1/2}J' \\ O_1' \end{pmatrix} \mu J = \begin{pmatrix} n^{1/2}\mu \\ 0 \end{pmatrix}$$
$$\Sigma_{O'Y} = O'\sigma^2 IO = \sigma^2 I$$

Part b

Show that $\bar{y} = (1/n)J'Y$ and $s^2 = Y'O_1O_1'Y/(n-1)$.

$\bar{y}$ is immediate, because by definition $J'Y = \sum_{i=1}^n y_i$. For s^2, following the hint, we have

$$\begin{aligned} Y'Y = Y'OO'Y &= Y' \begin{pmatrix} n^{-1/2}J' \\ O_1' \end{pmatrix} \begin{pmatrix} n^{-1/2}J & O_1 \end{pmatrix} Y \\ &= Y'(1/n)JJ'Y + Y'O_1O_1'Y \end{aligned}$$

For the first term on the right, $Y'(1/n)JJ'Y = n\bar{y}^2$. Subtracting from both sides and expanding out $Y'Y$,

$$Y'Y - Y'(1/n)JJ'Y = \sum_{i=1}^n y_i^2 - \bar{y}^2 = \sum_{i=1}^n (y_i - \bar{y})^2$$

using the usual trick that the cross-terms sum to zero. Thus

$$\left(Y'Y - Y'(1/n)JJ'Y\right)/(n-1) = s^2 = Y'O_1O_1'Y/(n-1)$$

as desired.

Part c

Show that $\bar{y}$ and s^2 are independent. By Corollary 1.2.4, $J'Y$ and $O_1'Y$ are independent because $J'O_1 = 0$ by orthogonality. $\bar{y}$ and s^2 are functions of $J'Y$ and $O_1'Y$, respectively.

Exercise 1.5.6

Let $Y = (y_1, y_2, y_3)'$ have a $N(\mu, \sigma^2 I)$ distribution. Consider the quadratic forms given by the matrices below.

$$M_1 = \frac{1}{3}J_3^3, M_2 = \frac{1}{14}\begin{pmatrix} 9 & -3 & -6 \\ -3 & 1 & 2 \\ -6 & 2 & 4 \end{pmatrix}, M_3 = \frac{1}{42}\begin{pmatrix} 1 & -5 & 4 \\ -5 & 25 & -20 \\ 4 & -20 & 16 \end{pmatrix}$$

Part a

Find the distribution of each $Y'M_iY$. We first note that all of the M_i are symmetric idempotent matrices, so they are perpendicular projection matrices (ppo). They are all rank one. Since they are ppos, we can rewrite them as $M_i = O_iO_i'$. Quick reflection will show that

$$O_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, O_2 = \frac{1}{\sqrt{14}}\begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}, O_3 = \frac{1}{\sqrt{42}}\begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$$

By Theorem 1.3.3, we have that

$$Y'M_iY \sim X^2(1, \frac{\mu' M_i \mu}{2})$$

where

$$\mu' M_i \mu = \mu' O_i O_i' \mu = (\mu' O_i)^2$$

Part b

Show that the quadratic forms are pairwise independent.

We show that M_iY is independent of M_jY for $i \neq j$, which implies the pairwise independence of the quadratic forms. M_iY and M_jY are Gaussians, so showing their covariance is zero implies independence. We have that

$$\mathrm{Cov}(M_iY, M_jY) = M_i \mathrm{Cov}(Y, Y) M_j = \sigma^2 O_i O_i' O_j O_j'$$

A quick check will show that O_1, O_2, O_3 are orthonormal, so $O_i'O_j = 0$ for $i \neq j$. Thus M_iY and M_jY are pairwise independent for $i \neq j$.

Part c

Show that the quadratic forms are mutually independent.

This immediately follows from Part b because for Gaussians $1...k$, pairwise independence between all pairs implies mutual independence. The quadratic forms are functions of the Gaussians, so they too are mutually independent.

Exercise 1.5.7

Let A be symmetric, $Y \sim N(0, V)$, and $w_1 \dots w_s$ be independent $X^2(1)$ random variables. Show that for some values of s and numbers λ_i $Y'AY \sim \sum_{i=1}^{s} \lambda_i w_i$.

Using the hint, I will rewrite $Y \sim QZ$, where Z is a standard normal and $V = QQ'$. Thus $Y'AY \sim Z'Q'AQZ$.

Consider $Q'AQ$. Because A is symmetric, $Q'AQ$ is symmetric, so it is orthogonally diagonalizable. Thus $Q'AQ = PD(\lambda_i)P'$, where P is orthonormal and λ_i are the eigenvalues of $Q'AQ$.

Letting P_i be the ith row of P, we expand out $Z'Q'AQZ = Z'PD(\lambda_i)P'Z$ to see that

$$Z'PD(\lambda_i)P'Z = \begin{pmatrix} \lambda_1 Z_1 P_1 \\ \vdots \\ \lambda_n Z_n P_n \end{pmatrix} \begin{pmatrix} P_1' & \dots & P_n' \end{pmatrix} Z = D(\lambda_i Z_i)Z = \sum_{i=1}^{n} \lambda_i w_i$$

as desired, letting $s = n$.