PACQ Chapter 2 Solutions

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Exercise 2.1

Show that for $\lambda'\beta$ estimable,

$$\frac{\lambda'\hat{\beta} - \lambda'\beta}{\sqrt{MSE\lambda'(X'X)^{-}\lambda}} \sim t(dfE) \tag{1}$$

Find the form of an α level test of $H_0 : \lambda'\beta = 0$ and the form for a $(1-\alpha)100\%$ confidence interval for $\lambda'\beta$.

I will make the usual assumptions for the linear model. Rewrite the expression as the following:

$$\frac{\frac{\lambda'\hat{\beta}-\lambda'\beta}{\sigma\sqrt{\lambda'(X'X)^{-}\lambda)}}}{\sqrt{\frac{MSE}{\sigma^2}}} \tag{2}$$

In the numerator,

$$\lambda'\hat{\beta} \sim N(\lambda\beta, \sigma^2\lambda'(X'X)^{-}\lambda)$$

implies

$$\frac{\lambda'\hat{\beta} - \lambda'\beta}{\sigma\sqrt{\lambda'(X'X)^{-}\lambda)}} \sim N(0, 1)$$

In the denominator,

$$(n-r)\frac{MSE}{\sigma^2} \sim X^2(n-r)$$

Since $\lambda'\beta$ is estimable, $\lambda'\hat{\beta} = \rho'MY$, and since $MSE = \frac{Y'(I-M)Y}{n-r}$, $\rho'MY$ and $Y'(I-M)Y$ are independent, given the usual linear model assumptions. Thus Equation 2 has a $t(dfE)$ distribution, where $dfE = n - r$.

For the α level test of $H_0 : \lambda\beta = 0$, the test rejects for large values of the test statistic in Equation 1. That is, we reject when

$$\frac{\lambda'\hat{\beta}}{\sqrt{MSE\lambda'(X'X)^{-}\lambda}} \geq t(1-\alpha/2, n-r)$$

or

$$\frac{\lambda'\hat{\beta}}{\sqrt{MSE\lambda'(X'X)^{-}\lambda}} \leq -t(1-\alpha/2, n-r)$$

For the $(1-\alpha)100\%$ confidence interval, we do not reject if

$$-t(1-\alpha/2, n-r) < \frac{\lambda'\hat{\beta} - \lambda'\beta}{\sqrt{MSE\lambda'(X'X)^{-}\lambda}} < t(1-\alpha/2, n-r)$$

which is equivalent to

$$\lambda'\hat{\beta} - t(1-\alpha/2, n-r)\sqrt{MSE\lambda'(X'X)^{-}\lambda} < \lambda'\beta < \lambda'\hat{\beta} + t(1-\alpha/2, n-r)\sqrt{MSE\lambda'(X'X)^{\lambda}}$$

Exercise 2.2

Let $y_{11}, y_{12} \ldots y_{1r}$ be $N(\mu_1, \sigma^2)$ and $y_{21}, y_{22} \ldots y_{2s}$ be $N(\mu_2, \sigma^2)$ with all y_{ij}s independent. Write this as a linear model. Using the results of Chapter 2, find estimates of $\mu_1, \mu_2, \mu_1 - \mu_2$, and σ^2. Form a $\alpha = 0.01$ test for $H_0 : \mu_1 = \mu_2$. Form 95% confidence intervals for $\mu_1 - \mu_2$ and μ_1. What is the test for $H_0 : \mu_1 = \mu_2 + \Delta$, where Δ is some known fixed quantity? How do these results compare with the usual analysis for two independent samples with a common variance?