PACQ Solutions

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1 Introduction

Exercise 1.2

Let W be an $r \times s$ random matrix, and let A and C be $n \times r$ and $n \times s$ matrices of constants. Show that $\boldsymbol{E}(AW + C) = A\boldsymbol{E}(W) + C$. If B is $s \times t$ constant matrix, show that $\boldsymbol{E}(AWB) = A\boldsymbol{E}(W)B$. If $s = 1$, show that $\boldsymbol{Cov}(AW + C) = A\boldsymbol{Cov}(W)A'$.

For $\boldsymbol{E}(AW + C)$, by definition $(AW + C)_{ij} = \sum_{k=1}^{r} A_{ik}W_{kj} + C_{ij}$. Taking expectations, we have

$$\boldsymbol{E}((AW + C)_{ij}) = \sum_{k=1}^{r} A_{ik}\boldsymbol{E}(W_{kj}) + C_{ij}$$

and we see that this equals $A\boldsymbol{E}(W) + C$.

For $\boldsymbol{E}(AWB)$, $\boldsymbol{E}(AWB) = A\boldsymbol{E}(WB)$ by the above.

Exercise 1.11

Prove that if $Y \sim N(\mu, V)$ and $VAVBV = 0$, $VAVB\mu = 0$, $VBVA\mu = 0$, and the conditions from Theorem 1.3.6 hold for $Y'AY$ and $Y'BY$, then $Y'AY$ and $Y'BY$ are independent.

Let $V = QQ'$ and rewrite $Y = \mu + QZ$, where $Z \sim N(0, I)$. We now show that the following variables are independent:

$$\begin{bmatrix} Q'AQZ \\ \mu'AQZ \end{bmatrix} \perp\!\!\!\perp \begin{bmatrix} Q'BQZ \\ \mu'BQZ \end{bmatrix} \tag{1}$$

Since these variables are all normal, showing uncorrelatedness shows independence. We first show this for one of the terms.

$$\begin{aligned} \boldsymbol{Cov}(Q'AQZ, Q'BQZ) &= \boldsymbol{E}(Q'AQZZ'Q'B'Q) - \boldsymbol{E}(Q'AQZ)E(Z'Q'B'Q) && (2) \\ &= Q'AQQ'B'Q = Q'AVBQ && (3) \end{aligned}$$

because $Z \sim N(0, I)$ and $QQ' = V$. We have from the same argument as the proof of Theorem 1.3.6 in the book that $Q = Q_1 Q_2$, where Q_1 has orthonormal columns and Q_2 is nonsingular. Thus by the same argument,

$$Q_2^{-1} Q_1' V = Q'$$

Applying this result to Equation **??**, we get that

$$\boldsymbol{Cov}(Q'AQZ, Q'BQZ) = Q_2^{-1} Q_1' V AV BV' Q_1 Q_2'^{-1} = 0$$

since by assumption $VAVABV = 0$. Similar results hold for the other cross terms.