

## Chapter 3 Functions of a Real Variable

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### Problem 1

Assume that  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfies  $|f(t) - f(x)| \leq |t - x|^2$  for all  $t, x$ . Prove that  $f$  is constant.

**Proof:** The assumption implies that for all  $t, x$ ,

$$0 \leq \left| \frac{f(t) - f(x)}{t - x} \right| = \frac{|f(t) - f(x)|}{|t - x|} \leq |t - x|$$

implies that  $f'(t) = \lim_{x \rightarrow t} \frac{f(t) - f(x)}{t - x} = 0$  at all  $t$ . The only functions with derivatives that are zero everywhere are constant functions.  $\square$

### Problem 2

A function  $f : (a, b) \rightarrow \mathbb{R}$  satisfies a Holder condition of order  $\alpha$  if  $\alpha > 0$ , and for some constant  $H$  and all  $u, x \in (a, b)$  we have

$$|f(u) - f(x)| \leq H|u - x|^\alpha$$

The function is said to be  $\alpha$ -Holder, with  $\alpha$ -Holder constant  $H$ .

#### Part a

Prove that the  $\alpha$ -Holder function defined on  $(a, b)$  is uniformly continuous and infer that it extends uniquely to a continuous function defined on  $[a, b]$ . Is the extended function  $\alpha$ -Holder?

**Proof:** Let  $\epsilon > 0$  and define  $\delta = (\frac{\epsilon}{H})^{1/\alpha}$ . Then for all  $u, x \in (a, b)$  such that  $|u - x| < \delta$ , we have

$$|f(u) - f(x)| \leq H|u - x|^\alpha < \epsilon$$

since  $\alpha > 0$ .  $\square$

By Problem 54 in Chapter 2, a uniformly continuous function defined on a metric space  $S$  has a unique continuous extension on  $\bar{S}$ . Since  $[a, b] = \overline{(a, b)}$ ,  $f : (a, b) \rightarrow \mathbb{R}$  being uniformly continuous implies that  $f$  extends uniquely to  $g : [a, b] \rightarrow \mathbb{R}$ , where  $g$  is continuous. In fact,  $g$  is uniformly continuous because it is continuous on a compact.

We claim that  $g$  is  $\alpha$ -Holder on  $[a, b]$ . Let  $x, y \in [a, b]$ . If  $x, y \in (a, b)$ , this just follows because  $g$  extends  $f$ .

Without loss of generality, let  $x = a$  and let  $y \in (a, b)$ . Let  $\epsilon > 0$  be fixed and arbitrary, and let  $\delta > 0$  be the corresponding continuity condition. Then

$$|g(c) - g(a)| \leq |g(c) - g(a + \delta)| + |g(a) - g(a + \delta)|$$

by the Triangle inequality. For the first term, because  $c$  and  $a + \delta$  are in the interval  $(a, b)$ , the Holder condition from  $f$  extends to  $g$ , so

$$|g(c) - g(a + \delta)| \leq H|c - a - \delta|^\alpha \leq H|c - a|^\alpha$$

because  $\alpha > 0$  and  $\delta > 0$ . For the second term, continuity of  $g$  means  $|g(a) - g(a + \delta)| < \epsilon$ . Thus

$$|g(c) - g(a)| \leq H|c - a|^\alpha + \epsilon$$

and  $\epsilon$  can be made arbitrarily small. The case where  $y = b$ , and the case where  $x = a$  and  $y = b$  simultaneously, are essentially the same.

## Part b

What does  $\alpha$ -Holder continuity mean when  $\alpha = 1$ ?

When  $\alpha = 1$ ,  $\alpha$ -Holder continuity simplifies to Lipschitz continuity.

## Part c

Prove that  $\alpha$ -Holder continuity when  $\alpha > 1$  implies that  $f$  is constant.

Let  $x$  in the domain of  $f$  be arbitrary. Dividing both sides by  $|u - x|$ ,

$$0 \leq \frac{|f(u) - f(x)|}{|u - x|} \leq H|u - x|^{\alpha-1}$$

Let  $u \rightarrow x$ . Since  $\alpha > 1$  the right side goes to 0, implying  $\frac{|f(u) - f(x)|}{|u - x|} \rightarrow 0$  and that  $f'(x) = 0$  for all  $x$  in  $f$ 's domain. The only functions with this property are constant functions.

## Problem 3

Assume that  $f : (a, b) \rightarrow \mathbb{R}$  is differentiable.

### Part a

If  $f'(x) > 0$  for all  $x$ , prove that  $f$  is strictly monotone increasing.

**Proof:** Let  $c, d \in (a, b)$ ,  $c < d$ . Then because  $f$  is differentiable on its domain, the Mean Value Theorem indicates that there is a point  $\theta \in (c, d)$  such that

$$f(c) - f(d) = f'(\theta)(d - c)$$

Since  $f'$  is always strictly positive and  $c < d$ , the right side is strictly positive.  
 $\square$

### Part b

If  $f'(x) \geq 0$  for all  $x$ , what can you prove?

We can prove that  $f$  is weakly monotone increasing. The proof is the same, except that  $f'(\theta)(d - c)$  can be zero.

## Problem 4

Prove that  $\sqrt{n+1} - \sqrt{n} \rightarrow 0$  as  $n \rightarrow \infty$ .

Consider the function  $f(x) = \sqrt{x}$ , and take a Taylor approximation of degree zero around  $x = n$ , where  $n$  is a positive natural number. Then  $P_0(x) = \sqrt{n}$ . Use the Taylor approximation to approximate  $x = n+1$ . The Taylor remainder term is

$$R(1) = \sqrt{n+1} - \sqrt{n}$$

$\sqrt{x}$  is smooth when  $x > 0$ , and  $n \geq 1$ . Therefore,  $f$  is smooth on  $(n, n+1)$ , and the Taylor approximation theorem states that there exists  $\theta \in (n, n+1)$  such that

$$R(1; n) = \sqrt{n+1} - \sqrt{n} = \frac{f'(\theta)}{1!}(1)^1 = \frac{1}{2}\theta^{-\frac{1}{2}}$$

As  $n \rightarrow \infty$ ,  $\theta > n$  implies  $\theta \rightarrow \infty$  implies  $R(1; n) \rightarrow 0$  implies  $\lim_{n \rightarrow \infty} \sqrt{n+1} - \sqrt{n} = 0$ .

## Problem 8

### Part b

Find a formula for a continuous function defined on  $[0, 1]$  that is differentiable on the interval  $(0, 1)$ , but not at the endpoints.

Consider the function

$$f(x) = \begin{cases} x \sin(\frac{1}{x}) & x \in (0, 1] \\ 0 & \text{else} \end{cases}$$

$f$  is the composition of continuous functions on  $(0, 1]$ , so it is continuous on that interval. At  $x = 0$ , we noting that for all  $x \in (0, 1]$ , we have

$$-x \leq x \sin(\frac{1}{x}) \leq x$$

implying that  $\lim_{x \rightarrow 0^+} f(x) = 0 = f(0)$  by the Squeeze theorem. This implies that  $f(x)$  is continuous at  $x = 0$ , and thus  $[0, 1]$ .  $\frac{1}{x}$  is differentiable on  $\mathbb{R} - 0$ , so  $f(x)$  is differentiable on  $(0, 1]$ .

Taking the definition of derivative to attempt to evaluate  $f'(0)$ ,

$$f'(0) = \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^+} \sin(\frac{1}{x})$$

which does not exist. Thus  $f(x)$  is differentiable on  $(0, 1]$ .

Consider the function

$$g(x) = f(x) + f(1 - x)$$

This consists of  $f$  and  $f$  reflected about the line  $x = \frac{1}{2}$  added together. From the above,  $g$  is continuous on  $[0, 1]$ , and differentiable on  $(0, 1)$ , but not 0 or 1.

## Part c

Does the Mean Value Theorem apply to such a function?

Yes, since the Mean Value Theorem only requires the function to be differentiable on the open interval. In this case, the Mean Value Theorem states there is a point  $\theta \in (0, 1)$  such that  $g'(\theta) = 0$ . We can probably prove that a point exists by using the Intermediate Value Theorem on  $g'(x)$  since it's continuous on  $(0, 1)$ , but I'm too lazy at the moment.

## Problem 10

Concoct a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  with a discontinuity of the second kind at  $x = 0$  such that  $f$  does not have the intermediate value property there. Infer that it is incorrect to assert that functions without jumps are Darboux continuous.

Consider the function

$$f(x) = \begin{cases} x & x \in \mathbb{R} - \mathbb{Q} \\ 1 & \text{else} \end{cases}$$

$f$  is continuous at  $x = 1$  and discontinuous everywhere else. These discontinuities are discontinuities of the second kind, since left and right limits don't exist when  $x$  is not 1.  $f(x)$  clearly does not have the intermediate value

property, as except for 1,  $f$  assumes no rational values. Since this is a function without jump discontinuities but does not possess the intermediate value property, functions without jumps are not necessarily Darboux continuous.

## Problem 11

Let  $f : (a, b) \rightarrow \mathbb{R}$  be given.

### Part a

If  $f''(x)$  exists, prove that

$$\lim_{h \rightarrow 0} \frac{f(x-h) - 2f(x) + f(x+h)}{h^2} = f''(x)$$

Denote  $F(x) = \lim_{h \rightarrow 0} \frac{f(x-h) - 2f(x) + f(x+h)}{h^2}$ . Since  $f$  is twice differentiable, we take a second-order Taylor expansion of  $f$  around  $x$ , getting

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + R(x)$$

where  $R(x)$  is second-order flat at  $h = 0$ , i.e.  $\lim_{h \rightarrow 0} R(x)/h^2 = 0$ . Similarly,

$$f(x-h) = f(x) - hf'(x) + \frac{1}{2}h^2 f''(x) + S(x)$$

where  $S(x)$  is second-order flat at  $h = 0$ . Substituting,

$$F(x) = \lim_{h \rightarrow 0} \frac{h^2 f''(x) + R(x) + S(x)}{h^2} = f''(x)$$

since the  $f(x)$  and  $hf'(x)$  terms cancel, and  $R(x)$  and  $S(x)$  are second-order flat.

### Part b

Find an example that this limit can exist even when  $f''(x)$  fails to exist.

Let  $f(x) = x|x|$ . Taking the first derivative, when  $x > 0$ ,  $f'(x) = x^2$ , so  $f'(x) = 2x$ . Similarly, when  $x < 0$ ,  $f'(x) = -2x$ . When  $x = 0$ ,

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h|h|}{h} = \lim_{h \rightarrow 0} |h| = 0$$

Thus

$$f'(x) = \begin{cases} 2x & x \geq 0 \\ -2x & x < 0 \end{cases}$$

As previously stated,  $f''(0)$  does not exist, since

$$f''(0) = \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \rightarrow 0} \frac{f'(h)}{h}$$

which does not exist, since the limit from the positive direction is 2 and the limit from the negative direction is  $-2$ .

Despite this, the partial difference approximation exists at  $x = 0$ . The partial difference approximation from the right is

$$\lim_{h \rightarrow 0^+} \frac{f(-h) + f(h)}{h^2} = \lim_{h \rightarrow 0^+} \frac{-h| - h| + h|h|}{h^2} = \lim_{h \rightarrow 0^+} \frac{0}{h^2} = \infty$$

Similarly,

$$\lim_{h \rightarrow 0^-} \frac{f(-h) + f(h)}{h^2} = \lim_{h \rightarrow 0^-} \frac{h|h| + -h| - h|}{h^2} = \lim_{h \rightarrow 0^-} \frac{0}{h^2} = \infty$$

Thus the difference approximation exists at  $x = 0$ , even though  $f''(0)$  does not exist.

## Problem 15

Define  $f(x) = x^2$  if  $x < 0$  and  $f(x) = x + x^2$  if  $x \geq 0$ . Differentiation gives  $f''(x) = 2$ . This is bogus. Why?

By the Fundamental Theorem of Calculus, if  $G$  is an antiderivative of  $g$ , then  $g$  equals the derivative of  $G$  where  $g$  is continuous. In this case, the standard power rule only applies when  $x \neq 0$ , since there is a discontinuity there.

Specifically, we have  $f''(0)$  does not exist, since  $f'(x) = 2x$  when  $x \geq 0$ , and  $f'(x) = 2x + 1$  when  $x < 0$ .  $f'(x)$  is discontinuous at  $x = 0$ , so its derivative does not exist there.

## Problem 16

$\log x$  is defined to be  $\int_1^x 1/t dt$  for  $x > 0$ . Using only the mathematics explained in this chapter,

### Part a

Prove that  $\log$  is a smooth function.

By the Fundamental Theorem of Calculus, the indefinite integral of a Riemann integrable function is continuous with respect to  $x$ . Thus,  $\log x$  is continuous. Its derivative, again by the Fundamental Theorem of Calculus, is  $\frac{d}{dx} \int_1^x 1/t dx = 1/x$  when  $x > 0$ , which is continuous.  $1/x$  itself is smooth, so it has derivatives of all orders, which are continuous. Thus  $\log x$  is smooth.

## Part b

Prove that  $\log(xy) = \log x + \log y$  for all  $x, y > 0$ .

For any given  $y > 0$ , define  $f(x) = \log xy - \log x - \log y$ . By definition,

$$\begin{aligned} f(x) &= \int_1^{xy} 1/t dt - \int_1^x 1/t dt - \int_1^y 1/t dt \\ &= \int_x^{xy} 1/t dt - \int_1^y 1/t dt \end{aligned}$$

When  $x = 1$ ,  $f(x) = \int_1^y 1/t dt - \int_1^y 1/t dt = 0$ .

We now evaluate  $f'(x)$ . Splitting the integrals, for all  $x > 0$ , we can find a constant  $0 < c < x$ . Then

$$f(x) = \int_c^{xy} 1/t dt - \int_c^x 1/t dt - \int_1^y 1/t dt$$

By the Fundamental Theorem of Calculus,  $\frac{d}{dx} \int_c^x 1/t dt = 1/x$  since  $1/t$  is continuous on  $[c, \infty)$ . By the Chain Rule,  $\frac{d}{dx} \int_c^{xy} 1/t dt = y \frac{1}{xy} = 1/x$ .  $\int_1^y 1/t dt$  is constant with regards to  $x$ , and thus has derivative zero. Thus,  $f'(x) = 0$  for all  $x > 0$ . The only functions with derivatives equal to zero everywhere are constant functions, and since  $f(1) = 0$ , this implies that  $f(x) = 0$ . Thus  $\log xy = \log x + \log y$ .

## Part c

Prove that  $\log$  is strictly monotone increasing and its range is all of  $\mathbb{R}$ .

$\frac{d}{dx} \log x = 1/x$ , which is strictly positive for all  $x > 0$ . Thus  $\log x$  is strictly monotone increasing.

We know that  $\log(1) = 0$ . Going to the right, let  $a_k = \frac{1}{k}$ . Because  $\frac{1}{t}$  is decreasing, for all  $t \in [k, k+1]$ ,  $\frac{1}{t} \leq a_{k+1}$ . Thus because  $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{k}$  diverges to infinity, by the Integral Test,  $\int_1^{\infty} \frac{1}{t} dt$  diverges to infinity. This means that there is for large  $x$ ,  $\log(x) = \int_1^x \frac{1}{t} dt$  can be made arbitrarily large. This implies that when  $x \geq 0$ ,  $\log(x)$  takes on all values in  $[0, \infty)$ .

Going to the left, for  $x \in (0, 1]$ ,  $\log(x) = -\int_x^1 \frac{1}{t} dt$ . Let  $k \in \mathbb{N}$  and consider  $\log(\frac{1}{2^k}) = -\int_{\frac{1}{2^k}}^1 \frac{1}{t} dt$ .

To evaluate  $\int_{\frac{1}{2^k}}^1 \frac{1}{t} dt$ , consider the partition  $P$  such that  $x_i = \frac{1}{2^i}$  for  $i \in \mathbb{N}$ . Thus  $x_0 = 1$ ,  $x_1 = \frac{1}{2}$ ,  $x_2 = \frac{1}{4}$ , etc. Because  $\frac{1}{t}$  is strictly decreasing, the minimum of  $\frac{1}{t}$  occurs at the right endpoint of the interval. Thus the lower integral is greater than or equal to

$$\begin{aligned}
& 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) \dots \\
&= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \dots \\
&= \frac{k}{2}
\end{aligned}$$

because there are  $k$  intervals. Since  $\frac{1}{t}$  is Riemann integrable on  $(0, 1]$ ,  $\frac{k}{2}$  is a lower bound for the integral. Thus

$$-\int_{\frac{1}{2^k}}^1 \frac{1}{t} dt \leq -\frac{k}{2}$$

which implies that the integral goes to negative infinity as  $k$  goes to infinity. Thus

$$-\int_0^1 \frac{1}{t} dt = -\infty$$

which implies that as  $x$  approaches zero,  $\log(x)$  approaches negative infinity. Thus on  $(0, 1]$ ,  $\log(x)$  takes on all values in  $(-\infty, 0]$ . Putting the two statements together implies that the range of  $\log(x)$  is all of  $\mathbb{R}$ .

## Problem 17

Define  $E : \mathbb{R} \rightarrow \mathbb{R}$  by

$$E(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

### Part a

Prove that  $E(x)$  is smooth; that is,  $E$  has derivatives of all orders at all points  $x$ .

For  $x < 0$ , smoothness is trivial. A quick application of the chain rule shows that on  $x > 0$ ,

$$E'(x) = \frac{1}{x^2} e^{-\frac{1}{x}}$$

**Theorem 1** For  $x > 0$ ,  $E^{(n)}(x)$  has the form

$$(a_{n+1}x^{-(n+1)} + a_{n+2}x^{-(n+2)} + \dots + a_{2n}x^{-2n})e^{-\frac{1}{x}}$$

for all  $n \in \mathbb{N}$ .



**Proof:** The base case  $n = 1$  has been established above. Assume that the hypothesis holds for  $n - 1$ . Then

$$E^{(n-1)}(x) = (a_n x^{-n} + a_{n+1} x^{-(n+1)} + \dots + a_{2n-2} x^{-(2n-2)}) e^{-\frac{1}{x}}$$

Using the Product Rule,

$$\begin{aligned} E^{(n)}(x) &= [(-n a_n x^{-(n+1)} - (n+1) a_{n+1} x^{-(n+2)} - \dots - (2n+2) a_{2n-2} x^{-(2n-1)}) \\ &\quad + (a_n x^{-(n+2)} + a_{n+1} x^{-(n+3)} + \dots + a_{2n-2} x^{-2n})] e^{-\frac{1}{x}} \\ &= (b_{n+1} x^{-(n+1)} + b_{n+2} x^{-(n+2)} + \dots + b_{2n} x^{-2n}) e^{-\frac{1}{x}} \end{aligned}$$

since  $n$  is a constant.  $\square$

**Lemma 2**  $\lim_{x \rightarrow 0} E(x) = E(0) = 0$ . Thus  $E(x) \in C^0$ .

**Proof:** The left limit is trivially zero. On the right, as  $x$  approaches zero from the positive direction,  $-\frac{1}{x}$  approaches negative infinity, so  $e^{-\frac{1}{x}}$  approaches zero.  $\square$

**Lemma 3**  $\lim_{x \rightarrow 0^+} \frac{1}{x} e^{-\frac{1}{x}} = 0$ .

**Proof:**

$$\lim_{x \rightarrow 0^+} \frac{e^{-\frac{1}{x}}}{x} = \lim_{x \rightarrow 0^+} \frac{x^{-1}}{e^{\frac{1}{x}}} = \lim_{x \rightarrow 0^+} \frac{-x^{-2}}{-x^{-2} e^{\frac{1}{x}}} = \lim_{x \rightarrow 0^+} e^{-\frac{1}{x}} = 0$$

by using l'Hopital's rule on the second expression.  $\square$

**Lemma 4**  $\lim_{x \rightarrow 0^+} \frac{1}{x^n} e^{-\frac{1}{x}} = 0$  for  $n \in \mathbb{N}$ .

**Proof:** The base case has been established. For the inductive case, assume that  $\lim_{x \rightarrow 0^+} \frac{1}{x^{n-1}} e^{-\frac{1}{x}} = 0$ . Then

$$\lim_{x \rightarrow 0^+} \frac{e^{-\frac{1}{x}}}{x^n} = \lim_{x \rightarrow 0^+} \frac{x^{-n}}{e^{\frac{1}{x}}} = \lim_{x \rightarrow 0^+} \frac{-n x^{-n-1}}{-x^{-2} e^{\frac{1}{x}}} = n \lim_{x \rightarrow 0^+} \frac{e^{-\frac{1}{x}}}{x^{n-1}} = 0$$

by the inductive hypothesis.  $\square$

**Corollary 5**  $\lim_{x \rightarrow 0^+} E^{(n)}(x) = 0$  when  $x > 0$  for all  $n \in \mathbb{N}$ .

**Proof:** By Theorem 1,  $E^{(n)}(x)$  is the sum of various terms of the form  $\frac{1}{x^n} e^{-\frac{1}{x}}$ , where  $n \in \mathbb{N}$ . By Lemma 4, each of these terms has right limit zero. Since  $\frac{1}{x^n} e^{-\frac{1}{x}}$  is the sum of a finite number of these terms, it has right limit zero.  $\square$

**Theorem 6**  $E^{(n)}(0)$  exists and it equals zero for all  $n \in \mathbb{N}$ .  $E^{(n)}(x)$  is continuous at  $x = 0$  for all  $n \in \mathbb{N}$ , thus making  $E(x)$  smooth.

**Proof:** For  $n \in \mathbb{N}$ , we need to evaluate

$$\lim_{x \rightarrow 0} \frac{E^{(n-1)}(x) - 0}{x - 0} = \lim_{x \rightarrow 0} \frac{E^{(n-1)}(x)}{x}$$

The left limit is zero, since  $E^{(n-1)}(x) = 0$  for  $x \leq 0$ . For the right limit, by Theorem 1,  $\frac{1}{x}E^{(n-1)}(x)$  has the form

$$(a_{n+2}x^{-(n+2)} + a_{n+3}x^{-(n+3)} + \dots + a_{2n+1}x^{-2n+1})e^{-\frac{1}{x}}$$

By repeated application of Lemma 4, we see that this has right limit zero. Thus

$$E^{(n)}(0) = \lim_{x \rightarrow 0} \frac{E^{(n-1)}(x)}{x} = 0$$

Because  $E^{(n)}(x) = 0$  on  $x \leq 0$ ,  $E^{(n)}(0) = 0$ , and  $E^{(n)}(x)$  is continuous at  $x = 0$  for all  $n \in \mathbb{N}$ . Combined with smoothness everywhere else, this implies that  $E(x)$  is smooth everywhere.  $\square$

## Part b

Is  $E(x)$  analytic?

No. A function  $f$  defined on an open interval  $(a, b)$  is analytic at  $x \in (a, b)$  if it equals its power series in a neighborhood of  $x$ . More specifically, for every  $x$  there exists a  $\delta > 0$  such that  $|h| < \delta$  implies that

$$f(x+h) = \sum_{r=0}^{\infty} a_r h^r$$

$$\text{where } a_r = \frac{f^{(r)}(x)}{r!}.$$

For  $E(x)$  at  $x = 0$ ,  $E^{(n)}(0) = 0$  for all whole numbers  $n$ , as established in Part a. Thus  $a_r = 0$  for all whole numbers  $r$ , and the power series is just 0. However, for  $h > 0$ ,  $f(h) \neq 0$  since  $e^{-\frac{1}{x}}$  is a strictly positive function. Thus  $E(x)$  is not analytic.

## Problem 29

Prove that the interval  $[a, b]$  is not a zero set.

### Part a

Explain why the following observation is not a solution to the problem: "Every open interval that contains  $[a, b]$  has length  $> b - a$ ."

This 'solution' does not consider the possibility that there is a union of open sets that cover  $[a, b]$  such that their sum of their lengths can be made arbitrarily small.

## Part b

Instead, suppose there is a "bad" covering of  $[a, b]$  by open intervals  $\{I_i\}$  whose total length is  $< b - a$ , and justify the following steps in the proof by contradiction.

I will define a good covering as a covering of  $[a, b]$  by open intervals  $\{J\}$  such that the total length of the intervals in  $\{J\}$  is greater than or equal to  $b - a$ .

**i**

It is enough to deal with finite bad coverings.

Let  $\{I\}$  be an infinite bad covering of  $[a, b]$ . Because  $\{I\}$  is an open cover of compact  $[a, b]$ , it reduces to a finite subcovering  $\{I_i\}$ . Thus, either  $\{I\}$  reduces to a finite bad covering, or it reduces to a good covering. If  $\{I\}$  reduces to a good covering  $\{J_i\}$ , then  $\{J_i\} \subset \{I\}$  and the sum of the intervals in  $\{J_i\}$  being  $\geq b - a$  implies that the sum of the intervals in  $\{I\}$  is  $\geq b - a$ . Thus  $\{I\}$  is an infinite good covering, which contradicts the assumption that  $\{I\}$  is a bad covering.

Thus, if  $\{I\}$  is an infinite bad covering, it reduces to a finite bad covering. Contrapositively, if there are no finite bad coverings, then there are no infinite bad coverings, and the theorem is proven.

**ii**

Let  $\mathbb{B} = \{I_1, \dots, I_n\}$  be a bad covering such that  $n$  is minimal among all bad coverings.

There is at least one finite bad covering, by assumption.  $n = 1$  is a lower bound for the size of bad coverings. Then because  $\mathbb{R}$  is complete, there exists a greatest lower bound for the sizes of the bad coverings, denoted  $c$ .

There must be a finite bad covering  $\{C\}$  such that the size of  $\{C\} = c$ . Suppose not. Then all bad coverings have size  $> c$ , and since the sizes of the bad coverings must be integers, all bad coverings have size  $\geq c + 1$ . This contradicts the assumption that  $c$  is a greatest lower bound. This bad covering  $\{C\}$  is the bad covering with minimal  $n$  among all bad coverings.

**iii**

Show that no bad covering has  $n = 1$  so we have  $n \geq 2$ .

This follows from the observation in Part a.

**iv**

Show that it is no loss of generality to assume  $a \in I_1$  and  $I_1 \cap I_2 \neq \emptyset$ .

There exists at least one interval such that  $a \in I_j$ , and we are free to denote that interval  $I_1$ .

There must exist an interval that intersects  $I_1$ . Suppose not. Let  $d_1$  be the right endpoint of  $I_1$ , and let  $c_2, c_3 \dots c_n$  be the left endpoints of the other

intervals in the bad covering, and let  $c = \min\{c_1 \dots c_n\}$ . Then  $\frac{c-d}{2}$  is not covered by the bad covering, contradicting the assumption that  $\{I\}$  is a covering. Thus, there exists an interval in  $\{I\}$  that intersects  $I_1$ . Denote it  $I_2$ . By construction,  $I_1 \cap I_2$  is nonempty.

**v**

Show that  $I = I_1 \cup I_2$  is an open interval and  $|I| < |I_1| + |I_2|$ .

If  $I_1 \subset I_2$  or  $I_2 \subset I_1$ ,  $I_1 \cup I_2$  is trivially an open interval. Otherwise,  $I_1 \cup I_2$  is the open because it is the union of open sets, connected because it is the union of two connected sets with a common point, and bounded because it is the finite union of bounded sets. Therefore  $I_1 \cup I_2$  is a open, connected, and bounded subset of  $\mathbb{R}$ , and by the theorems shown in Chapter 2 Problem 31, open, connected, and bounded subsets of  $\mathbb{R}$  are open intervals.

**Lemma 7** *Let  $C, D \subset \mathbb{R}$  be (bounded) intervals that intersect, and let  $E = C + D$ . Then  $|E| < |C| + |D|$ .*

**Proof:** *If  $C$  is a subset of  $D$  or vice versa, the proof is trivial. Without loss of generality, let the left endpoint of  $C$  be less than the left endpoint of  $D$ . Denote  $c$  as the right endpoint of  $C$ , and  $d$  the left endpoint of  $D$ .  $d < c$ , otherwise the two intervals do not intersect. Letting  $\epsilon = c - d > 0$ , the total length of  $E$  is  $|C| + |D| - \epsilon$ , which is strictly less than  $|C| + |D|$ .  $\square$*

By using the above Lemma, we see that  $|I| < |I_1| + |I_2|$ .

**vi**

Show that  $\mathbb{B}' = \{I, I_3, \dots, I_n\}$  is a bad covering of  $[a, b]$  with fewer intervals, contradicting the minimality of  $n$ .

Let  $x \in [a, b]$ . Since  $\mathbb{B}$  is a covering of  $[a, b]$ , there exists  $i \in 1, 2, \dots, n$  such that  $x \in I_i$ . If  $i \geq 3$ , then because  $I_i \in \mathbb{B}'$ ,  $x$  is also covered by  $\mathbb{B}'$ . If  $i = 1, 2$ , then  $x \in I = I_1 \cup I_2$ , so  $x$  is still covered by  $\mathbb{B}'$ .  $\mathbb{B}'$  is a covering by open intervals, because  $I$  is an open interval.  $\mathbb{B}'$  is a bad covering.  $|I| < |I_1| + |I_2|$  implies that  $|I| + \sum_{j=3}^n |I_j| < \sum_{i=1}^n |I_i| < b - a$ , implying that the total length of  $\mathbb{B}'$  is less than the total length of  $\mathbb{B}$ . Thus  $\mathbb{B}'$  is a bad covering with fewer intervals than  $\mathbb{B}$ , contradicting the assumption that  $\mathbb{B}$  is the minimal bad covering. Thus, there are no bad coverings of  $[a, b]$ , coverings of  $[a, b]$  can not have arbitrarily small length, and  $[a, b]$  is not a zero set.

## Problem 34

Assume that  $\psi : [a, b] \rightarrow \mathbb{R}$  is continuously differentiable. A critical point of  $\psi$  is an  $x$  such that  $\psi'(x) = 0$ . A critical value is a number  $y$  such that for at least one critical point  $x$  we have  $y = \psi(x)$ .

## Part a

Prove that the set of critical values is a zero set. (This is the Morse-Sard Theorem in dimension one.)

I will first introduce some notation. Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous. I will define a **zero of type 1** of  $f$  to be a zero of  $f$  such that  $f$  is uniformly zero in an open neighborhood of the root. In other words, if  $f(x) = 0$ , then there exists  $\epsilon > 0$  such that for all  $y$  such that  $|x - y| < \epsilon$ ,  $f(y) = 0$ . I will denote a **zero point of type 2** of  $f$  as all other zeros of  $f$ . It's clear that the disjoint union of zeros of types 1 and 2 make up all zeros of  $f$ .

Let  $\psi$  be continuously differentiable. We will characterize the critical values of  $\psi$  based on the zeros of type 1 and 2 of  $\psi'$ . If  $\psi(x) = y$  is a critical value and  $\psi'(x)$  is a zero of type 1, we say that  $y$  is a **critical value of type 1** of  $f$ . Similarly, if  $\psi(x) = y$  is a critical value and  $\psi'(x)$  is a zero of type 2, we say that  $y$  is a **critical value of type 2** of  $f$ . Since the zeros of type 1 and 2 partition the set of zeros of  $\psi'$ , the critical values of type 1 and 2 partition the set of critical values of  $\psi$ .

I have no idea if this characterization is standard, but that's what I've come up with.

The immediate characterization for zeros of type 2 is stated below.

**Lemma 8** *Let  $f$  be continuous, and let  $x$  be a zero of type 2 of  $f$ . Then for all  $\epsilon > 0$ , there exists  $y \in (x - \epsilon, x + \epsilon)$  such that  $f(y) \neq 0$ .*

**Proof:** *If this is not true, then  $x$  is a zero of type 1.* □

To begin with zero points of type 2, we next state a lemma on non-zero points of continuous functions implying an interval with no zeros. This can be thought of as non-zero points of continuous functions creating 'exclusion zones' with a delta-radius that contain no zeros.

**Lemma 9** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous, and let  $x \in [a, b]$  be a point such that  $f(x) \neq 0$ . Then there exists a  $\delta > 0$  such that  $f$  has no zeros in  $(x - \delta, x + \delta)$ .*

**Proof:** *Because  $f$  is uniformly continuous, there exists a  $\delta > 0$  such that for all  $y$  such that  $|x - y| < \delta$ ,  $|f(x) - f(y)| < |f(x)|$ . This implies that  $y$  is not a zero of  $f$ .* □

**Lemma 10** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous, and let  $x \in [a, b]$  such that  $f(x) \neq 0$ . Then  $f$  is not a clustering point of zeros. In other words, we can reasonably speak of the nearest zero of  $f$  greater than  $x$ , and the nearest zero of  $f$  less than  $x$ .*

**Proof:** *Suppose not. Then there exists a sequence  $(x_n) \rightarrow x$  of zeros of  $f$ .  $\lim_{n \rightarrow \infty} f(x_n) = 0$ , but  $f(x) \neq 0$ , violating the continuity of  $f$ .* □

We now introduce some useful terminology (that I have no idea whether is standard, but I am going to use it). Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous, and let  $x$  be a point such that  $f(x) \neq 0$ . The **covering interval of  $x$**  is the open interval between the nearest zero of  $f$  to the left of  $x$ , and the nearest zero of  $f$  to the right of  $x$ . This interval covers  $x$ . By Lemma 10, this is a well-defined construction.

I will denote the interval as  $C$ . If there are no zeros of  $f$  to the left of  $x$ , then the left endpoint of  $C$  is  $a$ , and if there are no zeros to the right of  $x$ , then the right endpoint of  $C$  is  $b$ .

**Lemma 11** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous. Then the covering intervals of  $f$  are disjoint.*

**Proof:** *Any two covering intervals are separated by at least one zero of  $f$ .*  $\square$

We now show that the number of covering intervals is closely related to the number of zeros of type 2.

**Lemma 12** *Let  $x$  be a zero of type 2. Then there is a covering interval such that  $x$  is the endpoint.*

**Proof:** *Suppose not. Then  $x$  is not the nearest zero of type 2 to any nonzero point of  $f$ . This means that  $x$  is a clustering point of zeros of type 2, which contradicts Lemma 10.*  $\square$

**Corollary 13** *The set of zeros of type 2 is of equal or lesser cardinality to the set of covering intervals.*

**Proof:** *Each zero of type 2 belongs to at least one covering interval.*  $\square$

The main results for zeros of type 2 follows.

**Corollary 14** *Let  $f$  be continuous on  $[a, b]$ . Then  $f$  has at most countable zeros of type 2.*

**Proof:** *Let  $Q$  be the set of covering intervals for  $f$ . Because  $Q$  is the disjoint union of intervals,  $Q$  is countable. By Lemma 13, the zeros of type 2 of  $f$  are countable.*  $\square$

**Corollary 15** *Let  $f$  be continuously differentiable on  $[a, b]$ . Then  $f$  has at most countable critical values of type 2.*

**Proof:**  $f'$  is continuous, implying that  $f$  has at most countable zeros of type 2. Each zero of type 2 of  $f'$  maps to at most one critical value of type 2 of  $f$ .  $\square$

We now turn to critical points of type 1. We first state a useful characterization of zeros of type 1.

**Lemma 16** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous, and let  $Z$  be the set of zeros of type 1. Then  $Z$  is the disjoint union of countable open intervals, with perhaps one or two half-open intervals at the endpoints  $a$  and  $b$ .*

**Proof:** The definition of zeros of type 1 implies that  $Z$  is an open set in  $[a, b]$ . By the Inheritance Principle, there exists a set  $W \subset \mathbb{R}$  that is open in  $\mathbb{R}$  such that  $W \cap [a, b] = Z$ . By Problem 31 in Chapter 2, an open set in  $\mathbb{R}$  can be expressed as the disjoint union of countably many open intervals. Taking  $W \cap Z$ , open intervals that do not contain the endpoints  $a$  and  $b$  are still open in  $Z$ , while the half-intervals that have their closed end at  $a$  and  $b$  become open in  $[a, b]$ .  $\square$

We next state a lemma on critical points of type 1.

**Lemma 17** *Let  $x$  be a critical point of type 1 for  $\psi'(x)$ . Then on the neighborhood where  $\psi'(x) = 0$ , there is only one critical value. Specifically, if  $\psi'(x) = 0$  on an interval  $(c, d) \subset [a, b]$ , then  $\psi(c)$  is the only critical value on that interval.*

**Proof:** By the Fundamental Theorem of Calculus, for  $x \in [c, d]$ ,  $\psi(x) = \psi(c) + \int_c^x \psi'(x)dx = \psi(c)$  since  $\psi'(x) = 0$  on the interval.  $\square$

We now want to prove the main theorem for critical values corresponding to critical points of type 1.

**Theorem 18** *If  $f : [a, b] \rightarrow \mathbb{R}$  is continuously differentiable, then  $f$  has countably many critical values of type 1.*

**Proof:**  $f'$  is continuous by assumption. By Lemma 16,  $Z$ , the set of zeros of type 1 of  $f'$ , consists of countable disjoint open intervals, with perhaps one or two half-open intervals at  $a$  and  $b$ . By Lemma 17, each (half)-open interval in  $Z$  corresponds to one critical value in  $f$ . Thus  $f$  has at most countably many critical values of type 1.  $\square$

**Theorem 19** *The critical values of  $f$  form a zero set.*

**Proof:** The union of countable sets is a countable set, which is a zero set.  $\square$

## Part b

Generalize this result to continuous functions on  $\mathbb{R} \rightarrow \mathbb{R}$ .

The result immediately generalizes. Divide  $\mathbb{R}$  into countably many intervals of length 1. By Part a, there are countably many critical values of  $f$  on each of these intervals, and the countable union of countable sets is a zero set. Thus the set of critical values of a continuously differentiable function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a zero set.

## Problem 36

We say that  $f : (a, b) \rightarrow \mathbb{R}$  has a **jump discontinuity** (or a discontinuity of the **first kind**) at  $c \in (a, b)$  if

$$f(c^-) = \lim_{x \rightarrow c^-} f(x) \text{ and } f(c^+) = \lim_{x \rightarrow c^+} f(x)$$

exist, but are either unequal or are unequal to  $f(c)$ . An **oscillating discontinuity** (or a discontinuity of the **second kind**) is any nonjump discontinuity).

## Part a

Show that  $f : \mathbb{R} \rightarrow \mathbb{R}$  has at most countably many jump discontinuities.

I will first start with showing that  $f : [0, 1] \rightarrow \mathbb{R}$  has countably many jump discontinuities. Let  $M(c)$  be defined as

$$M(c) = \begin{cases} \max\{|f(c^-) - f(c)|, |f(c^+) - f(c)|\} & \text{if } f \text{ has a jump discontinuity at } c \\ 0 & \text{else} \end{cases}$$

Let  $A_n$  be the set of points in  $\mathbb{R}$  such that  $M(c) > \frac{1}{n}$ . It's clear that the set of jump discontinuities of  $f$  is  $A = \bigcup_{n=1}^{\infty} A_n$ .

For all  $c \in A_n$ , because  $c$  is a jump discontinuity, there exists  $\delta_-, \delta_+ > 0$  such that  $x \in (c - \delta_-, c)$  implies that  $|f(x) - f(c^-)| < \frac{1}{2n}$ , with a similar result for  $\delta_+$ . Let  $\delta = \min(\delta_-, \delta_+)$ . Note that  $\delta$  depends on  $c$ .

I claim that  $(c - \delta, c + \delta) - \{c\} \subset A_n^C$ . Let  $x \in (c - \delta, c + \delta) - \{c\}$ . If  $f$  is continuous at  $x$ , or if  $x$  is an oscillating discontinuity of  $x$ , the result is trivial.

If  $x$  is a jump discontinuity of  $x$ , then the left and right limits of  $f$  exist at  $x$ . Let  $\gamma = \min\{|x - c|, |x - (c - \delta)|, |x - (c + \delta)|\}$ . If  $p_n$  is an arbitrary sequence such that  $p_n \rightarrow x$ , then eventually the tail of  $p_n$  will lie entirely in  $(x - \gamma, x + \gamma) \subset (c - \delta, c + \delta) - \{c\}$ . Thus

$$\text{diam} f((x - \gamma, x + \gamma)) < \frac{1}{n}$$

implies that  $x$  is not in  $A_n$ . Thus  $A_n$  is a zero set, implying that  $A$  is a zero set. Thus  $f : [0, 1] \rightarrow \mathbb{R}$  has at most countably many jump discontinuities. Repeating this argument for all  $\mathbb{Z}$  shows that  $f : \mathbb{R} \rightarrow \mathbb{R}$  has at most countably many jump discontinuities.



## Part b

What about the function

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

$f$  is continuous when  $x \neq 0$ , since on the left it is a constant function, and on the right it is the composition of continuous functions.  $f$  has an oscillating discontinuity at  $x = 0$ . Consider the sequences  $p_n = 1/(\frac{\pi}{2} + 2n)$  and  $q_n = 1/(\frac{3\pi}{2} + 2n)$ . Both of these sequences converge to zero, but  $f(p_n) = 1$  and  $f(q_n) = -1$  for all  $n \in \mathbb{N}$ . Thus the right limit of  $f$  does not exist at 0, and so  $x = 0$  is an oscillating discontinuity.

## Part c

What about the characteristic function of the rationals?

$f$  is oscillating discontinuous everywhere. For at any point  $x$ , there are a convergent sequence of rationals, and a convergent series of irrationals approaching  $x$ , implying that the limit of  $f$  at  $x$  does not exist.

## Problem 39

Consider the characteristic functions  $f(x)$  and  $g(x)$  of the intervals  $[1, 4]$  and  $[2, 5]$ . The derivatives  $f'$  and  $g'$  exist almost everywhere. The integration by parts formula says that

$$\int_0^3 f(x)g'(x)dx = f(3)g(3) - f(0)g(0) - \int_0^3 f'(x)g(x)dx$$

But both integrals are zero, while  $f(3)g(3) - f(0)g(0) = 1$ . Where is the error?

The textbook integration by parts formula assumes that  $f$  and  $g$  are continuous on  $[a, b]$ . In this case, the interval in question is  $[1, 5]$ . However,  $f$  is discontinuous at  $x = 4$ , and  $g$  is discontinuous at  $x = 2$ .

Specifically, the Leibniz formula states that if  $f, g$  are continuous at  $x$ , then  $(fg)'(x) = (f'g)(x) + (fg')(x)$ . If  $f, g$  are continuous on the interval in question then the Leibniz formula says that  $f'g + fg'$  is an antiderivative of  $fg$ . Then by the Antiderivative Theorem, the indefinite integral of  $f'g + fg'$  differs from the antiderivative by a constant.

However, in this case,  $fg'$  is zero on  $[1, 5]$ , except at  $x = 2$ , where it is undefined. A similar thing holds for  $f'g$  at  $x = 4$ . Since a potential antiderivative of  $fg'$  has to have its derivative equal  $fg'$  everywhere, not just almost everywhere,  $fg'$  and  $f'g$  do not have antiderivatives.

In fact, because  $fg'$  is zero almost everywhere,  $\int_0^3 f(x)g'(x)dx = 0$ .