

Chapter 6 Lebesgue Theory

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Problem 1

Part a

Show that the definition of linear outer measure is unaffected if we demand that the intervals I_k in the coverings be closed instead of open.

Let m^* be the open linear outer measure (the linear outer measure with open intervals), and n^* be the closed linear outer measure (the same with closed intervals).

We start with a lemma.

Lemma 1 *A point is a zero set in m^* and n^* .*

Proof: *The same $(x - \frac{1}{n}, x + \frac{1}{n})$ proof works for both closed and open intervals.*
 \square

Theorem 2 *The closed linear outer measure of an open interval is its length.*

Proof: *Let the open interval be (a, b) . Since the closed interval $[a, b]$ covers (a, b) and we take infimums of closed covers, $n^*A \leq b - a$.*

*To show that $n^*a \geq b - a$, we use induction on k , the size of the closed cover. This is essentially the proof of Theorem 2. If we can show that for all closed covers I_k of (a, b) , $\sum |I_k| \geq b - a$, the properties of the infimum imply that $n^*A \geq b - a$.*

For $k = 1$, $[a, b]$ covers (a, b) . Moving the left endpoint to the right means that the cover no longer contains a , with a similar statement when moving the right endpoint left. Thus all other closed covers of (a, b) have length greater than $b - a$.

Suppose we have shown that for all open intervals (c, d) and closed covers I_k of size n , we have $\sum_{k=1}^n |I_k| \geq b - a$. We now have (a, b) which is covered by J_l with size $n + 1$. We want to show that $\sum_{l=1}^{n+1} |J_l| \geq (b - a)$.

a is contained in a closed interval, say J_1 . Thus $a_1 \leq a$. If $b_1 \geq b$, then

$$\sum_{l=1}^{n+1} |J_l| \geq |J_1| \geq b_1 - a_1 \geq b - a$$

and the inductive hypothesis is proven. On the other hand, if $b_1 < b$, then the open interval divides into

$$(a, b) = (a, b_1] \cup (b_1, b)$$

and $\{J_2 \dots J_{n+1}\}$ covers (b_1, b) . Thus

$$\sum_{l=1}^{n+1} |J_l| = |J_1| + \sum_{l=2}^{n+1} |J_l| \geq (b_1 - a) + (b - b_1) = b - a$$

by the inductive hypothesis. Thus the theorem is proved by induction. \square

Lemma 3 *The open and closed linear measures of a half-open interval are both its length.*

Proof: Without loss of generality, let $[a, b)$ be the interval. Then there is a point $c \in (a, b)$, and

$$[a, b) = [a, c] \cup (c, b)$$

and by Theorem 2 above and Theorem 3 in the book, the closed and open linear outer measures of each sub-interval equal the lengths of the subintervals. \square

Lemma 4 *The closed and open linear outer measures of intervals with a side going to infinity are infinite.*

Proof: Suppose not. Then there exists a countable covering of the interval by closed intervals such that the sum of their lengths is finite. This is impossible. The same holds for open linear outer measure. \square

Theorem 5 *For all subsets A of \mathbb{R} , $m^*A = n^*A$.*

Proof: All subsets of \mathbb{R} are the disjoint union of countable points and countable intervals. The countable points are zero sets and do not affect the measure. If the intervals have ends going to infinity, the open and closed linear outer measures are both infinity. Otherwise, the disjoint intervals all have equal open and closed linear and outer measure, and thus the sums of those measures are equal. \square

Part b

Why does this immediately imply that the middle-thirds Cantor set has linear outer measure zero?

At each stage C_n of the middle-thirds Cantor set, C_n is a subset of 2^n closed intervals of length $\frac{1}{3^n}$, with total length $(\frac{2}{3})^n$. This immediately means that the closed linear outer measure, and thus the open linear outer measure, of C_n is $(\frac{2}{3})^n$, which goes to zero.

Problem 3

A line in the plane that is parallel to one of the coordinate axes is a planar zero set because it is the Cartesian product of a point (it's a linear zero set) and \mathbb{R} .

Part a

What about a line that is not parallel to the coordinate axis?

A line that is not parallel to a coordinate axis is also a planar zero set. Since the line is not a coordinate axis, it eventually intersects the x-axis with angle $\theta \in (0, \pi/2)$. The rectangles have height $\epsilon \sin(\theta)$, length $\epsilon \cos(\theta)$, and an area of $\epsilon^2 \sin(\theta) \cos(\theta) = \epsilon^2/2 \sin(2\theta)$.

Consider a section of the line of unit length. This section of the line can be covered by $1/\epsilon$ boxes with area $\epsilon^2 \sin(\theta) \cos(\theta)$, with total area $\epsilon \sin(\theta) \cos(\theta)$. Since this can be made arbitrarily small, a non-parallel line segment of unit length has planar measure zero. Since the line that is not parallel to a coordinate axis is a countable union of such segments, it also has planar measure zero.

Part b

What about the situation in higher dimensions?

It should essentially be analogous. Consider a plane in \mathbb{R}^3 . Divide the plane into sections with area one, and cover each section with boxes in which the plane intersects the corners of the boxes, and the intersection of the box and the plane has area $\epsilon > 0$. The volume of the boxes is proportional to ϵ^3 , while the total area is proportional to $1/\epsilon^2$. Thus the section of the plane with area one can be covered by boxes with volume proportional to ϵ , and is thus a zero set. Since \mathbb{Z}^2 is countable, the entire plane is a zero set.