Chapter 4 Function Spaces

Arthur Chen

March 28, 2022

In these exercises, $C^0 = C^0([a,b],\mathbb{R})$ is the space of continuous real-valued functions defined on the closed interval $[a,b]$. It is equipped with the usp norm, $||f|| = \sup\{|f(x)| : x \in [a,b]\}$.

Problem 1

Let M, N be metric spaces.

Part a

Formulate the concepts of pointwise convergence and uniform convergence for sequences of functions $f_n : M \to N$.

A sequence of functions $f_n : M \to N$ converges pointwise to a limit function $f : M \to N$ if for all $x \in M$ we have

$$\lim_{n\to\infty} d_n(f_n(x), f(x)) = 0$$

A sequence of functions converges uniformly to a limit function if for all $\epsilon > 0$, there is an N such that for all $n \geq N$ and all $x \in M$,

$$d_N(f_n(x), f(x)) < \epsilon$$

Part b

For which metric spaces are these concepts equivalent?

TODO. The immediate thing that springs to mind are trivial metric spaces with only one point,

Problem 3

Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of piecewise continuous functions, each of which is continuous at the point $x_0 \in [a,b]$. Assume that $f_n \rightrightarrows f$.

Part a

Prove that f is continuous at x_0.

The proof is as similar to Theorem 1 in the book. Let $\epsilon > 0$ be given. By uniform convergence, there exists an N such that for all $n \geq N$ and $x \in [a, b]$ we have

$$|f_n(x) - f(x)| < \frac{\epsilon}{3}$$

All the f_n are continuous at x_0, so f_N is continuous at x_0. This implies that there is a $\delta > 0$ such that $|x - x_0| < \delta$ implies

$$|f_N(x) - f_N(x_0)| < \frac{\epsilon}{3}$$

Thus, if $|x - x_0| < \delta$, then by the Triange inequality,

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

which shows that f is continuous at x_0.

Part b

Prove or disprove that f is piecewise continuous.

f is not piecewise continuous. A function $f : [a, b] \to \mathbb{R}$ is piecewise continuous if it has finitely many discontinuities.

Let $f : [0, 1] \to \mathbb{R}$ be the following function:

$$f(x) = \begin{cases} 0 & x \in \mathbb{R} - \mathbb{Q} \\ \frac{1}{q} & x = \frac{p}{q} : \gcd(p, q) = 1; p, q \in \mathbb{Z} \\ 1 & x = 0 \end{cases}$$

Let $f_n : [0, 1] \to \mathbb{R}$ be the following function for $n = 1, 2 \ldots$

$$f_n(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} : \gcd(p, q) = 1; p, q \in 1, 2 \ldots n \\ 1 & x = 0 \\ \frac{1}{n} & \text{else} \end{cases}$$

Thus f_1 is 1 everywhere, f_2 is 1 at 0 and 1 and 1/2 everywhere else, f_4 is 1 at 0 and 1, 1/2 at 1/2, 1/3 at 1/3 and 2/3, 1/4 everywhere else, etc.

$f_n(x) = f(x)$ when x is a rational number in reduced form with denominator $\leq n$. Everywhere else, $f(x) \geq 0$, and $f_n(x) = \frac{1}{n}$ imply $f_n(x) - f(x) \leq \frac{1}{n}$, which approaches zero as n goes to infinity. Thus $f_n \rightrightarrows f$. Similarly, f_n is piecewise continuous, since it only has $1 + 2 + 3 \cdots + n - 1$ discontinuities, which is finite. However, f is discontinuous at all rational numbers, and is thus is not piecewise continuous.