Chapter 2 A Taste of Topology

Problem 6

Determine whether $d_x(p,q) = \sin|p-q|$ on $[0, \frac{\pi}{2})$ is a metric.

Proof: It is a metric. Positive definiteness follows from the fact that the absolute value function is a metric over $\mathbb{R}$, and sine being one-to-one over the range of possible functions. Symmetry follows for the same reason. The triangle inequality follows from sine being increasing and concave over $[0, \frac{\pi}{2})$.

Specifically, let $p, r \in [0, \frac{\pi}{2})$, and without loss of generality, let $p \leq r$. If $q = p$ or $q = r$, the triangle inequality is trivial.

Let $q \notin [p, r]$. If $q > r$, then $q - p > r - p$ implies

$$d_s(p,q) + d_s(q,r) \geq d_s(p,r) + 0 =\geq d_s(p,r)$$

A similar result holds if $q < p$. If $q \in (p, r)$, imagine p, q, and r arranged on a line, with p at the origin. As x increases from q to r, the increase in sine is less than the corresponding increase from 0 to r-q, because sine is concave. Thus $sin(r-p) < sin(r-q) + sin(q-p)$.

$\square$

Problem 12

Let (p_n) be a sequence and $f : \mathbb{N} \to \mathbb{N}$ be a bijection. The sequence $(q_k)_{k\in\mathbb{N}}$ is a rearrangement of (p_n) with $q_k = p_{f(k)}$.

Part a

Are the limits of a sequence unaffected by rearrangement?

The limits of a sequence are unaffected by rearrangement.

Suppose that $(p_n) \to p$. Then for arbitrary $\epsilon > 0$, there exists an $n \in \mathbb{N}$ such that $i > n$ implies $d(p_i, p) < \epsilon$. This implies that there are at most finite elements of the sequence (p_n) such that $d(p_n, p) \geq \epsilon$. In the rearrangement (q_n), let M be the smallest integer such that $f^{-1}(k) \leq n$. M exists because $\{1, 2, 3 \ldots n\}$ is finite. Then for all $i > M$, $d(q_i, p) < \epsilon$, and thus $(q_i) \to p$.

On the other hand, suppose that (p_n) has no limit. Then for arbitrary p, there exists an $\epsilon > 0$ such that for all $n \in \mathbb{N}$, there exists $i > n$ such that $d(p_i, p) > \epsilon$. Letting $A(p)$ be the set of these points. A is infinite, because A being finite implies that A has a maximum element.

By contradiction, assume that that (q_n) has a limit of q. Then for all $\epsilon_q > 0$, there exists $n(\epsilon_q) \in \mathbb{N}$ such that $i > n(\epsilon_q)$ implies $d(q_i, q) < \epsilon_q$.

Let $\epsilon_q = \epsilon$. Then the set of points A_q such that $d(q_i, q) > \epsilon$ is a subset of $\{1, 2 \ldots n(\epsilon)\}$, and is thus finite. But because (q_n) is a rearrangement of (p_n), $A_q = A$. Thus A_q is both finite and infinite, which is a contradiction.

Part b

What if f is an injection?

At first glance, this question seems nonsensical, as for $n \in \mathbb{N}$ which are not in the preimage of $f(\mathbb{N})$, q_n is undefined. If we define such points to have $q_n = 0$, this question becomes intelligible.

Note that if A is an arbitrary set in $\mathbb{N}$, then $f(A)$ has the same cardinality as A, because the restriction of f on A is a bijection between the two.

The first result in Part a does not hold. Let $(p_n) = 1$, and let $f(x) = 2x$. The second result still holds, since the cardinality of the sets remains unchanged.

Part c

What if f is a surjection?

The first result in Part a holds. The cardinality of the image of a function must be less than or equal to the cardinality of the domain of the function. Letting $A = \{1, 2 \dots n\}$ and $B = \{n+1, n+2, n+3, \dots\}$, $f(A)$ is finite. Since f is surjective, it covers $\mathbb{N}$, which is infinite. $A + B = \mathbb{N}$ which is the domain of f. If $f(B)$ is finite, then $f(A) + f(B) = f(A + B) = f(\mathbb{N})$ is finite, which contradicts the assumption that f is surjective. Therefore $f(B)$ is infinite, and the argument in Part a holds.

The second result in Part b does not hold. Let $(p_n) = \{0, 0, 0, 1, 0, 1, 0, 1 \dots\}$ and let $f(x) = 1$ if x is odd or equals 2, and $\frac{1}{2}x$ if x is even and greater than 3. f is clearly surjective, and $(q_n) = \{0, 1, 1, 1, 1 \dots\}$ clearly has a limit of 1.

Problem 13

Show that if $f : M \to N$ is a function such that (p_n) converging in M implies that $(f(p_n))$ converges in N, then f is continuous.

Note that this is almost the definition of convergence, except for the requirement that $(f(p_n)) \to f(p)$. Thus showing that f is continuous is equivalent to showing the above requirement.

Proof: Let $p \in M$ be arbitrary. Let (a_n) be the uniform sequence $a_n = p$ for all $n \in \mathbb{N}$. From the properties of f, $(f(a_n))$ converges in N, and it converges to $f(p)$. If (a_n) is the only sequence in M that converges to p for all p in M (such as when the discrete metric is used), then f satisfies the sequential convergence condition and is thus continuous.

Let (b_n) be an arbitrary sequence in M such that $b_n \to p$. Construct the sequence (c_n) such that $c_n = a_{\lceil n/2 \rceil}$ if n is odd and $c_n = b_{n/2}$ if n is even. (c_n) clearly converges to p.

By the convergence preservation condition of f, $(f(c_n))$ converges in N. The subsequence of $(f(c_n))$ consisting of the odd numbers converges to $f(p)$. All subsequences of a convergent sequence converge to the same limit as the main sequence, so $f(c_n) \to f(p)$, and since $(f(b_n))$ is a subsequence of $(f(c_n))$, $f(b_n) \to f(p)$. Thus f preserves sequential convergence, and is thus continuous.

$\square$