Chapter 4 Function Spaces

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In these exercises, $C^0 = C^0([a, b], \mathbb{R})$ is the space of continuous real-valued functions defined on the closed interval $[a, b]$. It is equipped with the usp norm, $||f|| = \sup\{|f(x)| : x \in [a, b]\}$.

Problem 1

Let M, N be metric spaces.

Part a

Formulate the concepts of pointwise convergence and uniform convergence for sequences of functions $f_n : M \to N$.

A sequence of functions $f_n : M \to N$ converges pointwise to a limit function $f : M \to N$ if for all $x \in M$ we have

$$\lim_{n\to\infty} d_n(f_n(x), f(x)) = 0$$

A sequence of functions converges uniformly to a limit function if for all $\epsilon > 0$, there is an N such that for all $n \geq N$ and all $x \in M$,

$$d_N(f_n(x), f(x)) < \epsilon$$

Part b

For which metric spaces are these concepts equivalent?

TODO. The immediate thing that springs to mind are trivial metric spaces with only one point,

Problem 3

Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of piecewise continuous functions, each of which is continuous at the point $x_0 \in [a, b]$. Assume that $f_n \rightrightarrows f$.

Part a

Prove that f is continuous at x_0.

The proof is as similar to Theorem 1 in the book. Let $\epsilon > 0$ be given. By uniform convergence, there exists an N such that for all $n \geq N$ and $x \in [a, b]$ we have

$$|f_n(x) - f(x)| < \frac{\epsilon}{3}$$

All the f_n are continuous at x_0, so f_N is continuous at x_0. This implies that there is a $\delta > 0$ such that $|x - x_0| < \delta$ implies

$$|f_N(x) - f_N(x_0)| < \frac{\epsilon}{3}$$

Thus, if $|x - x_0| < \delta$, then by the Triange inequality,

$$\begin{aligned}|f(x) - f(x_0)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon\end{aligned}$$

which shows that f is continuous at x_0.

Part b

Prove or disprove that f is piecewise continuous.

f is not piecewise continuous. A function $f : [a, b] \to \mathbb{R}$ is piecewise continuous if it has finitely many discontinuities.

Let $f : [0, 1] \to \mathbb{R}$ be the following function:

$$f(x) = \begin{cases} 0 & x \in \mathbb{R} - \mathbb{Q} \\ \frac{1}{q} & x = \frac{p}{q} : \gcd(p, q) = 1; p, q \in \mathbb{Z} \\ 1 & x = 0 \end{cases}$$

Let $f_n : [0, 1] \to \mathbb{R}$ be the rational ruler function. Specifically, for $n = 1, 2 \ldots$

$$f_n(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} : \gcd(p, q) = 1; p, q \in 1, 2 \ldots n \\ 1 & x = 0 \\ \frac{1}{n} & \text{else} \end{cases}$$

Thus f_1 is 1 everywhere, f_2 is 1 at 0 and 1 and 1/2 everywhere else, f_4 is 1 at 0 and 1, 1/2 at 1/2, 1/3 at 1/3 and 2/3, 1/4 everywhere else, etc.

$f_n(x) = f(x)$ when x is a rational number in reduced form with denominator $\leq n$. Everywhere else, $f(x) \geq 0$, and $f_n(x) = \frac{1}{n}$ imply $f_n(x) - f(x) \leq \frac{1}{n}$, which approaches zero as n goes to infinity. Thus $f_n \rightrightarrows f$. Similarly, f_n is piecewise continuous, since it only has $1 + 2 + 3 \cdots + n - 1$ discontinuities, which is finite. However, f is discontinuous at all rational numbers, and is thus is not piecewise continuous.

Problem 4

Part a

If $f_n : \mathbb{R} \to \mathbb{R}$ is uniformly continuous for each $n \in \mathbb{N}$ and if $f_n \rightrightarrows f$ as $n \to \infty$, prove or disprove that f is uniformly continuous.

f is uniformly continuous. Let $\epsilon > 0$ be arbitrary. Then by uniform convergence, there exists N such that $n \geq N$ implies that $||f - f_n||_{\text{sup}} < \frac{\epsilon}{3}$. By the uniform continuity of f_n, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f_n(x) - f_n(y)| < \frac{\epsilon}{3}$, which is equivalent to $\max_{a\in[x,y]} f_n(a) - \min_{a\in[x,y]} f_n(a) < \frac{\epsilon}{3}$. Because $||f - f_n||_{\text{sup}} < \frac{\epsilon}{3}$, this implies that for $|x - y| < \delta$,

$$\max_{|x-y|<\delta} f(y) - \min_{|x-y|<\delta} f(a) < \epsilon$$

which is equivalent to uniform continuity.

Part b

What happens for functions from one metric space to another instead of $\mathbb{R}$ to $\mathbb{R}$?

The same things happen. Let $f : M \to N$. The supremum norm is well defined for functions from M to N. For uniform continuity, there exists $\delta > 0$ such that $d_M(x, y) < \delta$ implies $d_N(f_n(x), f_n(y)) < \frac{\epsilon}{3}$, which is equivalent to

$$\sup_{d_M(x,y)<\delta} f_n(y) - \inf_{d_M(x,y)<\delta} f_n(y) < \frac{\epsilon}{3}$$

Combined with $||f - f_n||_{\text{sup}} < \frac{\epsilon}{3}$, this implies that

$$\sup_{d_M(x,y)<\delta} f(y) - \inf_{d_M(x,y)<\delta} f(y) < \epsilon$$

which implies uniform continuity.

Problem 5

Suppose that $f_n : [a, b] \to \mathbb{R}$ and $f_n \rightrightarrows f$ as $n \to \infty$. Which of the following discontinuity properties of the functions f_n carry over to the limit function?

Part a

No discontinuities.

This is immediate. By the theorems in the book, the uniform limit of continuous functions is continuous.

Part b

At most ten discontinuities.

We begin with a lemma on how discontinuities in the limiting function imply discontinuities in the sequence.

Lemma 1 *Let f_n, f be as described in the problem, and let f be discontinuous at x_0. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, f_n is discontinuous at x_0.*

Proof: *Suppose not. Then for all $k \in \mathbb{N}$, there exists an $a > k$ such that f_a is continuous at x_0. By uniform convergence, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|f_n - f|_{\text{sup}} < \frac{\epsilon}{3}$. Choose $n \geq N$ such that f_n is continuous at x_0.*

Let $\epsilon > 0$ be arbitrary. By the continuity of f_n, there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f_n(x) - f_n(x_0)| < \frac{\epsilon}{3}$. Because $n \geq N$, $|f_n - f|_{\text{sup}} < \frac{\epsilon}{3}$ implies that for all $x \in (x_0 - \delta, x_0 + \delta)$,

$$|f(x) - f(x_0)| < \epsilon$$

implies that f is continuous at x_0, contradicting the assumption that f is discontinuous at x_0. Thus there is some $k \in \mathbb{N}$ such that for all $n \geq k$, f_n is discontinuous at x_0. $\square$

The statement is true by the contrapositive. If f has more than ten discontinuities, then by the above lemma, there exists some $N \in \mathbb{N}$ such that for all $n \geq N$, f_n has discontinuities at the discontinuities of f. Thus f having more than ten discontinuities implies the tail of f_n has more than ten discontinuities. Taking contrapositives, this implies that if the tail of f_n has at most ten discontinuities, f has at most ten discontinuities.

Part c

At least ten discontinuities.

No. Let the interval be $[0, 1]$ and f_n be the function

$$f_n = \begin{cases} \frac{1}{n} & \text{when } x \text{ in reduced form has denominator } 10^n \\ 0 & \text{else} \end{cases}$$

f_n has at least ten discontinuities for all n, but uniformly converges to the zero function, which has no discontinuities.

Part d

Finitely many discontinuities.

No. From Problem 3 Part b above, there are functions with finitely discontinuities but uniformly converge to a function with infinite discontinuities.

Problem 8

Is the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \cos(n + x) + \log(1 + \frac{1}{\sqrt{n+2}} \sin^2(n^n x))$$

equicontinuous? Prove or disprove.

Yes, the sequence is equicontinuous. We first show that sequences of equicontinuous functions form a vector space.

Lemma 2 *Let $f_n, g_n : \mathbb{R} \to \mathbb{R}$ be sequences of functions such that the f_n are equally continuous and the g_n are equally continuous. Then for all $\alpha, \beta \in \mathbb{R}$, $(\alpha f + \beta g)_n$ are equally continuous. In other words, equicontinuous sequences form a vector space.*

Proof: *Fix $\epsilon > 0$. If $\alpha, \beta \neq 0$, then there exist $\delta_f, \delta_g > 0$ such that for all $n \in \mathbb{N}$, $|s - t| < \delta_f$ implies*

$$|f_n(s) - f_n(t)| < \frac{\epsilon}{2|\alpha|}$$

and $|u - v| < \delta_g$ implies

$$|f_n(s) - f_n(t)| < \frac{\epsilon}{2|\beta|}$$

Take $\delta = \min(\delta_f, \delta_g)$. Then for all $n \in \mathbb{N}$ and $|x - y| < \delta$,

$$\begin{aligned}|\alpha f_n(x) + \beta g_n(x) - \alpha f_n(y) - \beta g_n(y)| &\leq |\alpha||f_n(x) - f_n(y)| + |\beta||g_n(x) - g_n(y)| \\ &< \epsilon\end{aligned}$$

thus $(\alpha f + \beta g)_n$ is equicontinuous. If α or β equals zero then those terms drop out of the expression, and the inequality still holds. □

We now prove that the cosine and log terms are equicontinuous, which combined with the above lemma show that f is equicontinuous. In all that follows, let $g_n(x) = \cos(n + x)$ and $h_n(x) = \log(1 + \frac{1}{\sqrt{n+2}} \sin^2(n^n x))$.

Lemma 3 *The g_n are equally continuous.*

Proof: *Let $\epsilon = \delta$. $\cos(n + x) = C^1$, so by the Fundamental Theorem of Calculus*

$$\begin{aligned}|\cos(n + t) - \cos(n + s)| = |-\int_s^t \sin(n + x)dx| &\leq \int_s^t |\sin(n + x)|dx \\ \leq \int_s^t 1dx = t - s < \delta = \epsilon\end{aligned}$$

for all $n \in \mathbb{N}$. □

We now show that the h_n are equally continuous.

Lemma 4 *h_n has variation bounded by $\frac{1}{\sqrt{n+2}}$ for all $n \in \mathbb{N}$.*

Proof: *$h_n(x) = \log(1+\frac{1}{\sqrt{n+2}}\sin^2(n^n x))$. The $\sin^2(n^n x)$ term has a maximum of 1 and minimum of 0. $\log$ is an increasing function, so*

$$0 = \log(1) \leq h(x) \leq \log(1 + \frac{1}{\sqrt{n+2}}) \leq \frac{1}{\sqrt{n+2}}$$

by taking the Taylor series of $\log(1+x)$ and truncating after the first term. $\frac{1}{\sqrt{n+2}} < 1$ for $n \in \mathbb{N}$, so the Taylor series converges. $\square$

Theorem 5 *The h_n are equally continuous.*

Proof: *Let $\epsilon > 0$ be arbitrary. By Lemma 4, var$(h_n) \leq \frac{1}{\sqrt{n+2}}$. When n is large enough such that $\frac{1}{\sqrt{n+2}} < \epsilon$, the conditions for equicontinuity are automatically met, since for all intervals, var$(h_n) < \epsilon$. Let $N \in \mathbb{N}$ be the largest n such that $\frac{1}{\sqrt{n+2}} \geq \epsilon$. Since $\frac{1}{\sqrt{n+2}} \to 0$, N exists and is finite.*

Computing h_n',

$$h_n'(x) = \frac{\frac{1}{\sqrt{n+2}} n^n 2\sin(n^n x)\cos(n^n x)}{1 + \frac{1}{\sqrt{n+2}}\sin^2(n^n x)} = \frac{n^n \sin(2n^n x)}{\sqrt{n+2} + \sin^2(n^2 x)}$$

which implies that

$$|h_n'(x)| \leq \frac{n^n}{\sqrt{n+2}}$$

Since $h_n \in C^1$ for all $n \in \mathbb{N}$, by the Fundamental Theorem of Calculus

$$|h_n(t) - h_n(s)| = |\int_s^t h_n'(x)dx| \leq \int_s^t |h_n'(x)|dx \leq \int_s^t \frac{n^n}{\sqrt{n+2}}dx = \frac{(t-s)n^n}{\sqrt{n+2}}$$

Letting $\delta_n < \frac{\sqrt{n+2}}{n^n}\epsilon$ satisfies the bound needed for equicontinuity for a given n. Let $\delta = \min(\delta_1, \ldots \delta_n)$. Since the n are finite, this operation is well defined, and $\delta > 0$. For $n \leq N$, the above calculation shows that the variation in h_n is bounded by ϵ on intervals smaller than δ. For $n > N$, as shown above, h_n has variation less than ϵ everywhere. Thus the h_n are equicontinuous. $\square$

Thus, since the g_n are equicontinuous and the h_n are equicontinuous, $h_n = f_n + g_n$ is equicontinuous.