Theorem 1. *If f_n converges uniformly to f and each f_n is continuous at x_0, then f is continuous at x_0.*

Proof. Let $\frac{\epsilon}{3} > 0$ be arbitrary. Since the f_n are continuous at x_0, for all n, there exists a $\delta_n > 0$ such that for all $x \in (x_0 - \delta_n, x_0 + \delta_n)$,

$$|f_n(x) - f_n(x_0)| < \frac{\epsilon}{3}$$

Since the f_n converge uniformly to f, for all x in the f's domain, there exists N such that for all n greater than N,

$$|f_n(x) - f(x)| < \frac{\epsilon}{3}$$

Choose arbitrary $n_0 \geq N$. Since f_{n_0} is continuous, for all $x \in (x_0 - \delta_n, x_0 + \delta_n)$,

$$|f_{n_0}(x) - f_{n_0}(x_0)| < \frac{\epsilon}{3}$$

Using the second equation and the Triangle Inequality shows that for all $x \in (x_0 - \delta_n, x_0 + \delta_n)$,

$$|f(x) - f(x_0)| < \epsilon$$

Which shows that f is continuous.

□

Theorem 4. *C_b is a complete metric space. That is, every Cauchy sequence converges to a limit in C_b.*

Have: Let (p_n) be a Cauchy sequence in C_b. Then for all $\epsilon > 0$, there exists an N such that for all $m, n \geq N$,

$$|p_n - p_m| = \sup(|p_n(x) - p_m(x)| : x \in [a, b]) < \epsilon$$

Want: Let (p_n) be a Cauchy sequence in C_b. Then there exists $p \in C_b$ such that (p_n) converges to p. In other words, for all $\epsilon > 0$, there exists an N such that for all $n \geq N$,

$$|p_n - p| = \sup(|p_n(x) - p(x)| : x \in [a, b]) < \epsilon$$