

# Chapter 4 Function Spaces

Arthur Chen

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In these exercises,  $C^0 = C^0([a, b], \mathbb{R})$  is the space of continuous real-valued functions defined on the closed interval  $[a, b]$ . It is equipped with the sup norm,  $\|f\| = \sup\{|f(x)| : x \in [a, b]\}$ .

## Problem 1

Let  $M, N$  be metric spaces.

### Part a

Formulate the concepts of pointwise convergence and uniform convergence for sequences of functions  $f_n : M \rightarrow N$ .

A sequence of functions  $f_n : M \rightarrow N$  converges pointwise to a limit function  $f : M \rightarrow N$  if for all  $x \in M$  we have

$$\lim_{n \rightarrow \infty} d_n(f_n(x), f(x)) = 0$$

A sequence of functions converges uniformly to a limit function if for all  $\epsilon > 0$ , there is an  $N$  such that for all  $n \geq N$  and all  $x \in M$ ,

$$d_N(f_n(x), f(x)) < \epsilon$$

### Part b

For which metric spaces are these concepts equivalent?

TODO. The immediate thing that springs to mind are trivial metric spaces with only one point,

## Problem 3

Let  $f_n : [a, b] \rightarrow \mathbb{R}$  be a sequence of piecewise continuous functions, each of which is continuous at the point  $x_0 \in [a, b]$ . Assume that  $f_n \rightrightarrows f$ .

## Part a

Prove that  $f$  is continuous at  $x_0$ .

The proof is as similar to Theorem 1 in the book. Let  $\epsilon > 0$  be given. By uniform convergence, there exists an  $N$  such that for all  $n \geq N$  and  $x \in [a, b]$  we have

$$|f_n(x) - f(x)| < \frac{\epsilon}{3}$$

All the  $f_n$  are continuous at  $x_0$ , so  $f_N$  is continuous at  $x_0$ . This implies that there is a  $\delta > 0$  such that  $|x - x_0| < \delta$  implies

$$|f_N(x) - f_N(x_0)| < \frac{\epsilon}{3}$$

Thus, if  $|x - x_0| < \delta$ , then by the Triangle inequality,

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

which shows that  $f$  is continuous at  $x_0$ .

## Part b

Prove or disprove that  $f$  is piecewise continuous.

$f$  is not piecewise continuous. A function  $f : [a, b] \rightarrow \mathbb{R}$  is piecewise continuous if it has finitely many discontinuities.

Let  $f : [0, 1] \rightarrow \mathbb{R}$  be the following function:

$$f(x) = \begin{cases} 0 & x \in \mathbb{R} - \mathbb{Q} \\ \frac{1}{q} & x = \frac{p}{q} : \gcd(p, q) = 1; p, q \in \mathbb{Z} \\ 1 & x = 0 \end{cases}$$

Let  $f_n : [0, 1] \rightarrow \mathbb{R}$  be the rational ruler function. Specifically, for  $n = 1, 2, \dots$

$$f_n(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} : \gcd(p, q) = 1; p, q \in 1, 2, \dots, n \\ 1 & x = 0 \\ \frac{1}{n} & \text{else} \end{cases}$$

Thus  $f_1$  is 1 everywhere,  $f_2$  is 1 at 0 and 1 and 1/2 everywhere else,  $f_4$  is 1 at 0 and 1, 1/2 at 1/2, 1/3 at 1/3 and 2/3, 1/4 everywhere else, etc.

$f_n(x) = f(x)$  when  $x$  is a rational number in reduced form with denominator  $\leq n$ . Everywhere else,  $f(x) \geq 0$ , and  $f_n(x) = \frac{1}{n}$  imply  $f_n(x) - f(x) \leq \frac{1}{n}$ , which approaches zero as  $n$  goes to infinity. Thus  $f_n \rightrightarrows f$ . Similarly,  $f_n$  is piecewise continuous, since it only has  $1 + 2 + 3 \dots + n - 1$  discontinuities, which is finite. However,  $f$  is discontinuous at all rational numbers, and is thus not piecewise continuous.

## Problem 4

### Part a

If  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  is uniformly continuous for each  $n \in \mathbb{N}$  and if  $f_n \rightrightarrows f$  as  $n \rightarrow \infty$ , prove or disprove that  $f$  is uniformly continuous.

$f$  is uniformly continuous. Let  $\epsilon > 0$  be arbitrary. Then by uniform convergence, there exists  $N$  such that  $n \geq N$  implies that  $\|f - f_n\|_{\sup} < \frac{\epsilon}{3}$ . By the uniform continuity of  $f_n$ , there exists  $\delta > 0$  such that  $|x - y| < \delta$  implies  $|f_n(x) - f_n(y)| < \frac{\epsilon}{3}$ , which is equivalent to  $\max_{a \in [x, y]} f_n(a) - \min_{a \in [x, y]} f_n(a) < \frac{\epsilon}{3}$ . Because  $\|f - f_n\|_{\sup} < \frac{\epsilon}{3}$ , this implies that for  $|x - y| < \delta$ ,

$$\max_{|x-y|<\delta} f(y) - \min_{|x-y|<\delta} f(a) < \epsilon$$

which is equivalent to uniform continuity.

### Part b

What happens for functions from one metric space to another instead of  $\mathbb{R}$  to  $\mathbb{R}$ ?

The same things happen. Let  $f : M \rightarrow N$ . The supremum norm is well defined for functions from  $M$  to  $N$ . For uniform continuity, there exists  $\delta > 0$  such that  $d_M(x, y) < \delta$  implies  $d_N(f_n(x), f_n(y)) < \frac{\epsilon}{3}$ , which is equivalent to

$$\sup_{d_M(x,y)<\delta} f_n(y) - \inf_{d_M(x,y)<\delta} f_n(y) < \frac{\epsilon}{3}$$

Combined with  $\|f - f_n\|_{\sup} < \frac{\epsilon}{3}$ , this implies that

$$\sup_{d_M(x,y)<\delta} f(y) - \inf_{d_M(x,y)<\delta} f(y) < \epsilon$$

which implies uniform continuity.

## Problem 5

Suppose that  $f_n : [a, b] \rightarrow \mathbb{R}$  and  $f_n \rightrightarrows f$  as  $n \rightarrow \infty$ . Which of the following discontinuity properties of the functions  $f_n$  carry over to the limit function?

### Part a

No discontinuities.

This is immediate. By the theorems in the book, the uniform limit of continuous functions is continuous.

## Part b

At most ten discontinuities.

We begin with a lemma on how discontinuities in the limiting function imply discontinuities in the sequence.

**Lemma 1** *Let  $f_n, f$  be as described in the problem, and let  $f$  be discontinuous at  $x_0$ . Then there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $f_n$  is discontinuous at  $x_0$ .*

**Proof:** *Suppose not. Then for all  $k \in \mathbb{N}$ , there exists an  $a > k$  such that  $f_a$  is continuous at  $x_0$ . By uniform convergence, there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|f_n - f|_{\sup} < \frac{\epsilon}{3}$ . Choose  $n \geq N$  such that  $f_n$  is continuous at  $x_0$ .*

*Let  $\epsilon > 0$  be arbitrary. By the continuity of  $f_n$ , there exists  $\delta > 0$  such that  $|x - x_0| < \delta$  implies  $|f_n(x) - f_n(x_0)| < \frac{\epsilon}{3}$ . Because  $n \geq N$ ,  $|f_n - f|_{\sup} < \frac{\epsilon}{3}$  implies that for all  $x \in (x_0 - \delta, x_0 + \delta)$ ,*

$$|f(x) - f(x_0)| < \epsilon$$

*implies that  $f$  is continuous at  $x_0$ , contradicting the assumption that  $f$  is discontinuous at  $x_0$ . Thus there is some  $k \in \mathbb{N}$  such that for all  $n \geq k$ ,  $f_n$  is discontinuous at  $x_0$ .  $\square$*

The statement is true by the contrapositive. If  $f$  has more than ten discontinuities, then by the above lemma, there exists some  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $f_n$  has discontinuities at the discontinuities of  $f$ . Thus  $f$  having more than ten discontinuities implies the tail of  $f_n$  has more than ten discontinuities. Taking contrapositives, this implies that if the tail of  $f_n$  has at most ten discontinuities,  $f$  has at most ten discontinuities.

## Part c

At least ten discontinuities.

No. Let the interval be  $[0, 1]$  and  $f_n$  be the function

$$f_n = \begin{cases} \frac{1}{n} & \text{when } x \text{ in reduced form has denominator } 10^n \\ 0 & \text{else} \end{cases}$$

$f_n$  has at least ten discontinuities for all  $n$ , but uniformly converges to the zero function, which has no discontinuities.

## Part d

Finitely many discontinuities.

No. From Problem 3 Part b above, there are functions with finitely discontinuities but uniformly converge to a function with infinite discontinuities.