Chapter 5 Multivariable Calculus

Arthur Chen

September 20, 2022

Problem 1

Let $T : V \to W$ be a linear transformation, and let $p \in V$ be given. Prove that the following are equivalent:

(a) T is continuous at the origin.

(b) T is continuous at p.

(c) T is continuous at at least one point of V.

For (a) to (b), let $p_n \in V$ be an arbitrary sequence such that $p_n \to p$. I claim that this implies $T(p_n) \to T(p)$ in W, meaning that T is continuous at p.

Although we can not use the translation invariance of the norm, we can derive a similar property for sequences when X is a vector space.

Lemma 1 *Let V be a vector space. Translation is a continuous function.*

Proof: *Let $T : V \to V$ be defined such that $T(x) = x + y$, $x, y \in V$ are arbitrary, y is fixed. Let $\epsilon > 0$ be arbitrary, and let $\delta = \frac{\epsilon}{2}$. Then the translation of the δ-ball around x to the δ-ball around $x + y$ lies inside the ϵ-ball around $x + y$.*

Specifically, $B_(x, \delta)$ is defined as

TO FINISH

$\square$

Lemma 2 *Let V be a vector space and d be a metric. Let $p_n \to p$ and $x \in X$ be arbitrary. Then $p_n + x \to p + x$.*

Proof: *Translation is continuous, and the product of continuous functions is continuous. All metrics are continuous, and the composition of continuous functions is continuous. Thus $p_n \to p$ implies*

$$\lim_{n\to\infty} d(p_n + x, p + x) = d(p + x, p + x) = 0$$

by continuity. $\square$

Letting d_W be the metric on W, we have

$$d_W(T(x_n + p), T(p)) = d_W(T(x_n) + T(p)$$