Chapter 3 Functions of a Real Variable

Arthur Chen

February 10, 2022

Problem 1

Assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(t) - f(x)| \leq |t - x|^2$ for all t, x. Prove that f is constant.

Proof: The assumption implies that for all t, x,

$$0 \leq \left| \frac{f(t) - f(x)}{t - x} \right| = \frac{|f(t) - f(x)|}{|t - x|} \leq |t - x|$$

implies that $f'(t) = \lim_{x \to t} \frac{f(t)-f(x)}{t-x} = 0$ at all t. The only functions with derivatives that are zero everywhere are constant functions. □

Problem 2

A function $f : (a, b) \to \mathbb{R}$ satisfies a Holder condition of order α if $\alpha > 0$, and for some constant H and all $u, x \in (a, b)$ se have

$$|f(u) - f(x)| \leq H|u - x|^\alpha$$

The function is said to be α-Holder, with α-Holder constant H.

Part a

Prove that the α-Holder function defined on (a, b) is uniformly continuous and infer that it extends uniquely to a continuous function defined on $[a, b]$. Is the extended function α-Holder?

Proof: Let $\epsilon > 0$ and define $\delta = (\frac{\epsilon}{H})^{1/\alpha}$. Then for all $u, x \in (a, b)$ such that $|u - x| < \delta$, we have

$$|f(u) - f(x)| \leq H|u - x|^\alpha < \epsilon$$

since $\alpha > 0$. □

By Problem 54 in Chapter 2, a uniformly continuous function defined on a metric space S has a unique continuous extension on $\bar{S}$. Since $[a,b] = \overline{(a,b)}$, $f : (a,b) \to \mathbb{R}$ being uniformly continuous implies that f extends uniquely to $g : [a,b] \to \mathbb{R}$, where g is continuous. In fact, g is uniformly continuous because it is continuous on a compact.

We claim that g is α-Holder on $[a,b]$. Let $x, y \in [a,b]$. If $x, y \in (a,b)$, this just follows because g extends f.

Without loss of generality, let $x = a$ and let $y \in (a,b)$. Let $\epsilon > 0$ be fixed and arbitrary, and let $\delta > 0$ be the corresponding continuity condition. Then

$$|g(c) - g(a)| \leq |g(c) - g(a+\delta)| + |g(a) - g(a+\delta)|$$

by the Triangle inequality. For the first term, because c and $a+\delta$ are in the interval (a,b), the Holder condition from f extends to g, so

$$|g(c) - gf(a+\delta)| \leq H|c-a-\delta|^\alpha \leq H|c-a|^\alpha$$

because $\alpha > 0$ and $\delta > 0$. For the second term, continuity of g means $|g(a) - g(a+\delta)| < \epsilon$. Thus

$$|g(c) - g(a)| \leq H|c-a|^\alpha + \epsilon$$

and ϵ can be made arbitrarily small. The case where $y = b$, and the case where $x = a$ and $y = b$ simultaneously, are essentially the same.

Part b

What does α-Holder continuity mean when $\alpha = 1$?

When $\alpha = 1$, α-Holder continuity simplifies to Lipschitz continuity.

Part c

Prove that α-Holder continuity when $\alpha > 1$ implies that f is constant.

Let x in the domain of f be arbitrary. Dividing both sides by $|u-x|$,

$$0 \leq \frac{|f(u) - f(x)|}{|u-x|} \leq H|u-x|^{\alpha-1}$$

Let $u \to x$. Since $\alpha > 1$ the right side goes to 0, implying $\frac{|f(u)-f(x)|}{|u-x|} \to 0$ and that $f'(x) = 0$ for all x in f's domain. The only functions with this property are constant functions.

Problem 3

Assume that $f : (a,b) \to \mathbb{R}$ is differentiable.

Part a

If $f'(x) > 0$ for all x, prove that f is strictly monotone increasing.

Proof: Let $c, d \in (a, b), c < d$. Then because f is differentiable on its domain, the Mean Value Theorem indicates that there is a point $\theta \in (c, d)$ such that

$$f(c) - f(d) = f'(\theta)(d - c)$$

Since f' is always strictly positive and $c < d$, the right side is strictly positive. $\square$

Part b

If $f'(x) \geq 0$ for all x, what can you prove?

We can prove that f is weakly monotone increasing. The proof is the same, except that $f'(\theta)(d - c)$ can be zero.

Problem 4

Prove that $\sqrt{n+1} - \sqrt{n} \to 0$ as $n \to \infty$.

Consider the function $f(x) = \sqrt{x}$, and take a Taylor approximation of degree zero around $x = n$, where n is a positive natural number. Then $P_0(x) = \sqrt{n}$. Use the Taylor approximation to approximate $x = n + 1$. The Taylor remainder term is

$$R(1) = \sqrt{n+1} - \sqrt{n}$$

$\sqrt{x}$ is smooth when $x > 0$, and $n \geq 1$. Therefore, f is smooth on $(n, n+1)$, and the Taylor approximation theorem states that there exists $\theta \in (n, n+1)$ such that

$$R(1; n) = \sqrt{n+1} - \sqrt{n} = \frac{f'(\theta)}{1!}(1)^1 = \frac{1}{2}\theta^{-\frac{1}{2}}$$

As $n \to \infty, \theta > n$ implies $\theta \to \infty$ implies $R(1; n) \to 0$ implies $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0$.

Problem 8

Part b

Find a formula for a continuous function defined on $[0, 1]$ that is differentiable on the interval $(0, 1)$, but not at the endpoints.

Consider the function

$$f(x) = \begin{cases} x\sin(\frac{1}{x}) & x \in (0,1] \\ 0 & \text{else} \end{cases}$$

f is the composition of continuous functions on $(0,1]$, so it is continuous on that interval. At $x = 0$, we noting that for all $x \in (0,1]$, we have

$$-x \leq x\sin(\frac{1}{x}) \leq x$$

implying that $\lim_{x\to 0^+} f(x) = 0 = f(0)$ by the Squeeze theorem. This implies that $f(x)$ is continuous at $x = 0$, and thus $[0,1]$. $\frac{1}{x}$ is differentiable on $\mathbb{R} - 0$, so $f(x)$ is differentiable on $(0,1]$.

Taking the definition of derivative to attempt to evaluate $f'(0)$,

$$f'(0) = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \sin(\frac{1}{x})$$

which does not exist. Thus $f(x)$ is differentiable on $(0,1]$.

Consider the function

$$g(x) = f(x) + f(1-x)$$

This consists of f and f reflected about the line $x = \frac{1}{2}$ added together. From the above, g is continuous on $[0,1]$, and differentiable on $(0,1)$, but not 0 or 1.

Part c

Does the Mean Value Theorem apply to such a function?

Yes, since the Mean Value Theorem only requires the function to be differentiable on the open interval. In this case, the Mean Value Theorem states there is a point $\theta \in (0,1)$ such that $g'(\theta) = 0$. We can probably prove that a point exists by using the Intermediate Value Theorem on $g'(x)$ since it's continuous on $(0,1)$, but I'm too lazy at the moment.

Problem 10

Concoct a function $f : \mathbb{R} \to \mathbb{R}$ with a discontinuity of the second kind at $x = 0$ such that f does not have the intermediate value property there. Infer that it is incorrect to assert that functions without jumps are Darboux continuous.

Consider the function

$$f(x) = \begin{cases} x & x \in \mathbb{R} - \mathbb{Q} \\ 1 & \text{else} \end{cases}$$

f is continuous at $x = 1$ and discontinuous everywhere else. These discontinuities are discontinuities of the second kind, since left and right limits don't exist when x is not 1. $f(x)$ clearly does not have the intermediate value property, as except for 1, f assumes no rational values. Since this is a function without jump discontinuities but does not possess the intermediate value property, functions without jumps are not necessarily Darboux continuous.

Problem 11

Let $f : (a, b) \to \mathbb{R}$ be given.

Part a

If $f''(x)$ exists, prove that

$$\lim_{h\to 0} \frac{f(x-h) - 2f(x) + f(x+h)}{h^2} = f''(x) \tag{1}$$

I will assume that '$f''(x)$ exists' means that $f''(x)$ exists everywhere. Denote the left side of this equation as $G(x)$. We begin by reparameterizing the derivative.

Lemma 1 *If $f'(x)$ exists, then*

$$f'(x) = \lim h \to 0 \frac{f(x+h) - f(x)}{h}$$

Proof: *Taking the definition of derivative as*

$$f'(x) = \lim_{t\to x} \frac{f(t) - f(x)}{t - x}$$

substitute $t = x + h$. □

Rearranging the terms in Equation 1, we have

$$G(x) = \lim_{h\to 0} \frac{\frac{f(x+h)-f(x)}{h} - \frac{f(x)-f(x-h)}{h}}{h} \tag{2}$$

We now show a quick lemma.

Lemma 2 *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. If $\lim_{x\to a} f(g(a))$ exists and g is continuous near a, then*

$$\lim_{x\to a} f(g(a)) = \lim_{x\to a} f(\lim_{y\to x} g(y))$$

Proof: *Since in the limit x is close to a, $g(x)$ is continuous, so sequential preservation holds.* □

Corollary 3

$$G(x) = \lim_{h\to 0} \frac{\lim_{i\to h} \frac{f(x+i)-f(x)}{i} - \frac{f(x)-f(x-i)}{i}}{h}$$

Proof: *$f''(x)$ exists everywhere, so $f'(x)$ is continuous everywhere. We then use the above lemma.* $\square$

TO BE COMPLETED.

Problem 29

Prove that the interval $[a, b]$ is not a zero set.

Part a

Explain why the following observation is not a solution to the problem: "Every open interval that contains $[a, b]$ has length $> b - a$."

This 'solution' does not consider the possibility that there is a union of open sets that cover $[a, b]$ such that their sum of their lengths can be made arbitrarily small.

Part b

Instead, suppose there is a "bad" covering of $[a, b]$ by open intervals $\{I_i\}$ whose total length is $< b - a$, and justify the following steps in the proof by contradiction.

I will define a good covering as a covering of $[a, b]$ by open intervals $\{J\}$ such that the total length of the intervals in $\{J\}$ is greater than or equal to $b - a$.

i

It is enough to deal with finite bad coverings.

Let $\{I\}$ be an infinite bad covering of $[a, b]$. Because $\{I\}$ is an open cover of compact $[a, b]$, it reduces to a finite subcovering $\{I_i\}$. Thus, either $\{I\}$ reduces to a finite bad covering, or it reduces to a good covering. If $\{I\}$ reduces to a good covering $\{J_i\}$, then $\{J_i\} \subset \{I\}$ and the sum of the intervals in $\{J_i\}$ being $\geq b - a$ implies that the sum of the intervals in $\{I\}$ is $\geq b - a$. Thus $\{I\}$ is an infinite good covering, which contradicts the assumption that $\{I\}$ is a bad covering.

Thus, if $\{I\}$ is an infinite bad covering, it reduces to a finite bad covering. Contrapositively, if there are no finite bad coverings, then there are no infinite bad coverings, and the theorem is proven.

ii

Let $\mathbb{B} = \{I_1, \ldots I_n\}$ be a bad covering such that n is minimal among all bad coverings.

There is at least one finite bad covering, by assumption. $n = 1$ is a lower bound for the size of bad coverings. Then because $\mathbb{R}$ is complete, there exists a greatest lower bound for the sizes of the bad coverings, denoted c.

The must be a finite bad covering $\{C\}$ such that the size of $|\{C\}| = c$. Suppose not. Then all bad coverings have size $> c$, and size the sizes of the bad coverings must be integers, all bad coverings have size $\geq c+1$. This contradicts the assumption that c is a greatest lower bound. This bad covering $\{C\}$ is the bad covering with minimal n among all bad coverings.

iii

Show that no bad covering has $n = 1$ so we have $n \geq 2$.

This follows from the observation in Part a.

iv

Show that it is no loss of generality to assume $a \in I_1$ and $I_1 \cap I_2 \neq \emptyset$.

There exists at least one interval such that $a \in I_j$, and we are free to denote that interval I_1.

There must exist an interval that intersects I_1. Suppose not. Let d_1 be the right endpoint of I_1, and let $c_2, c_3 \ldots c_n$ be the left endpoints of the other intervals in the bad covering, and let $c = \min\{c_1 \ldots c_n\}$. Then $\frac{c-d}{2}$ is not covered by the bad covering, contradicting the assumption that $\{I\}$ is a covering. Thus, there exists an interval in $\{I\}$ that intersects I_1. Denote it I_2. By construction, $I_1 \cap I_2$ is nonempty.

v

Show that $I = I_1 \cup I_2$ is an open interval and $|I| < |I_1| + |I_2|$.

If $I_1 \subset I_2$ or $I_2 \subset I_1$, $I_1 \cup I_2$ is trivially an open interval. Otherwise, $I_1 \cup I_2$ is the open because it is the union of open sets, connected because it is the union of two connected sets with a common point, and bounded because it is the finite union of bounded sets. Therefore $I_1 \cup I_2$ is a open, connected, and bounded subset of $\mathbb{R}$, and by the theorems shown in Chapter 2 Problem 31, open, connected, and bounded subsets of $\mathbb{R}$ are open intervals.

Lemma 4 *Let $C, D \subset \mathbb{R}$ be (bounded) intervals that intersect, and let $E = C + D$. Then $|E| < |C| + |D|$.*

Proof: *If C is a subset of D or vice versa, the proof is trivial. Without loss of generality, let the left endpoint of C be less than the left endpoint of D. Denote c as the right endpoint of C, and d the left endpoint of D. $d < c$, otherwise the two intervals do not intersect. Letting $\epsilon = c - d > 0$, the total length of E is $|C| + |D| - \epsilon$, which is strictly less than $|C| + |D|$.* $\square$

By using the above Lemma, we see that $|I| < |I_1| + |I_2|$.

vi

Show that $\mathbb{B}' = \{I, I_3, \dots I_n\}$ is a bad covering of $[a, b]$ with fewer intervals, contradicting the minimality of n.

Let $x \in [a, b]$. Since $\mathbb{B}$ is a covering of $[a, b]$, there exists $i \in 1, 2 \dots n$ such that $x \in I_i$. If $i \geq 3$, then because $I_i \in \mathbb{B}'$, x is also covered by $\mathbb{B}'$. If $i = 1, 2$, then $x \in I = I_1 \cup I_2$, so x is still covered by $\mathbb{B}'$. $\mathbb{B}'$ is a covering by open intervals, because I is an open interval. $\mathbb{B}'$ is a bad covering. $|I| < |I_1| + |I_2|$ implies that $|I| + \sum_{j=3}^{n} I_j < \sum_{i=1}^{n} I_i < b - a$, implying that the total length of $\mathbb{B}'$ is less than the total length of $\mathbb{B}$. Thus $\mathbb{B}'$ is a bad covering with fewer intervals than $\mathbb{B}$, contradicting the assumption that $\mathbb{B}$ is the minimal bad covering. Thus, there are no bad coverings of $[a, b]$, coverings of $[a, b]$ can not have arbitrarily small length, and $[a, b]$ is not a zero set.