# Chapter 3 Functions of a Real Variable

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## Problem 1

Assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(t) - f(x)| \leq |t - x|^2$ for all $t, x$. Prove that $f$ is constant.

**Proof:** The assumption implies that for all $t, x$,

$$0 \leq \left| \frac{f(t) - f(x)}{t - x} \right| = \frac{|f(t) - f(x)|}{|t - x|} \leq |t - x|$$

implies that $f'(t) = \lim_{x \to t} \frac{f(t)-f(x)}{t-x} = 0$ at all $t$. The only functions with derivatives that are zero everywhere are constant functions. □

## Problem 2

A function $f : (a, b) \to \mathbb{R}$ satisfies a Holder condition of order $\alpha$ if $\alpha > 0$, and for some constant $H$ and all $u, x \in (a, b)$ se have

$$|f(u) - f(x)| \leq H|u - x|^\alpha$$

The function is said to be $\alpha$-Holder, with $\alpha$-Holder constant H.

### Part a

Prove that the $\alpha$-Holder function defined on $(a, b)$ is uniformly continuous and infer that it extends uniquely to a continuous function defined on $[a, b]$. Is the extended function $\alpha$-Holder?

**Proof:** Let $\epsilon > 0$ and define $\delta = (\frac{\epsilon}{H})^{1/\alpha}$. Then for all $u, x \in (a, b)$ such that $|u - x| < \delta$, we have

$$|f(u) - f(x)| \leq H|u - x|^\alpha < \epsilon$$

since $\alpha > 0$. □

By Problem 54 in Chapter 2, a uniformly continuous function defined on a metric space $S$ has a unique continuous extension on $\bar{S}$. Since $[a, b] = \overline{(a, b)}$, $f : (a, b) \to \mathbb{R}$ being uniformly continuous implies that $f$ extends uniquely to $g : [a, b] \to \mathbb{R}$, where $g$ is continuous. In fact, $g$ is uniformly continuous because it is continuous on a compact.

We claim that $g$ is $\alpha$-Holder on $[a, b]$. Let $x, y \in [a, b]$. If $x, y \in (a, b)$, this just follows because $g$ extends $f$.

Without loss of generality, let $x = a$ and let $y \in (a, b)$. Let $\epsilon > 0$ be fixed and arbitrary, and let $\delta > 0$ be the corresponding continuity condition. Then

$$|g(c) - g(a)| \leq |g(c) - g(a + \delta)| + |g(a) - g(a + \delta)|$$

by the Triangle inequality. For the first term, because $c$ and $a + \delta$ are in the interval $(a, b)$, the Holder condition from $f$ extends to $g$, so

$$|g(c) - gf(a + \delta)| \leq H|c - a - \delta|^\alpha \leq H|c - a|^\alpha$$

because $\alpha > 0$ and $\delta > 0$. For the second term, continuity of $g$ means $|g(a) - g(a + \delta)| < \epsilon$. Thus

$$|g(c) - g(a)| \leq H|c - a|^\alpha + \epsilon$$

and $\epsilon$ can be made arbitrarily small. The case where $y = b$, and the case where $x = a$ and $y = b$ simultaneously, are essentially the same.

### Part b

What does $\alpha$-Holder continuity mean when $\alpha = 1$?

When $\alpha = 1$, $\alpha$-Holder continuity simplifies to Lipschitz continuity.

### Part c

Prove that $\alpha$-Holder continuity when $\alpha > 1$ implies that $f$ is constant.

Let $x$ in the domain of $f$ be arbitrary. Dividing both sides by $|u - x|$,

$$0 \leq \frac{|f(u) - f(x)|}{|u - x|} \leq H|u - x|^{\alpha - 1}$$

Let $u \to x$. Since $\alpha > 1$ the right side goes to 0, implying $\frac{|f(u)-f(x)|}{|u-x|} \to 0$ and that $f'(x) = 0$ for all $x$ in $f$'s domain. The only functions with this property are constant functions.

## Problem 3

Assume that $f : (a, b) \to \mathbb{R}$ is differentiable.

## Part a

If $f'(x) > 0$ for all $x$, prove that $f$ is strictly monotone increasing.

**Proof:** Let $c, d \in (a, b), c < d$. Then because $f$ is differentiable on its domain, the Mean Value Theorem indicates that there is a point $\theta \in (c, d)$ such that

$$f(c) - f(d) = f'(\theta)(d - c)$$

Since $f'$ is always strictly positive and $c < d$, the right side is strictly positive. $\square$

## Part b

If $f'(x) \geq 0$ for all $x$, what can you prove?

We can prove that $f$ is weakly monotone increasing. The proof is the same, except that $f'(\theta)(d - c)$ can be zero.

# Problem 4

Prove that $\sqrt{n+1} - \sqrt{n} \to 0$ as $n \to \infty$.

Consider the function $f(x) = \sqrt{x}$, and take a Taylor approximation of degree zero around $x = n$, where $n$ is a positive natural number. Then $P_0(x) = \sqrt{n}$. Use the Taylor approximation to approximate $x = n+1$. The Taylor remainder term is

$$R(1) = \sqrt{n+1} - \sqrt{n}$$

$\sqrt{x}$ is smooth when $x > 0$, and $n \geq 1$. Therefore, $f$ is smooth on $(n, n+1)$, and the Taylor approximation theorem states that there exists $\theta \in (n, n+1)$ such that

$$R(1; n) = \sqrt{n+1} - \sqrt{n} = \frac{f'(\theta)}{1!}(1)^1 = \frac{1}{2}\theta^{-\frac{1}{2}}$$

As $n \to \infty, \theta > n$ implies $\theta \to \infty$ implies $R(1; n) \to 0$ implies $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0$.

# Problem 8

## Part b

Find a formula for a continuous function defined on $[0, 1]$ that is differentiable on the interval $(0, 1)$, but not at the endpoints.

Consider the function

$$f(x) = \begin{cases} x\sin(\frac{1}{x}) & x \in (0,1] \\ 0 & \text{else} \end{cases}$$

$f$ is the composition of continuous functions on $(0,1]$, so it is continuous on that interval. At $x = 0$, we noting that for all $x \in (0,1]$, we have

$$-x \leq x\sin(\frac{1}{x}) \leq x$$

implying that $\lim_{x\to 0^+} f(x) = 0 = f(0)$ by the Squeeze theorem. This implies that $f(x)$ is continuous at $x = 0$, and thus $[0,1]$. $\frac{1}{x}$ is differentiable on $\mathbb{R} - 0$, so $f(x)$ is differentiable on $(0,1]$.

Taking the definition of derivative to attempt to evaluate $f'(0)$,

$$f'(0) = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \sin(\frac{1}{x})$$

which does not exist. Thus $f(x)$ is differentiable on $(0,1]$.

Consider the function

$$g(x) = f(x) + f(1-x)$$

This consists of $f$ and $f$ reflected about the line $x = \frac{1}{2}$ added together. From the above, $g$ is continuous on $[0,1]$, and differentiable on $(0,1)$, but not 0 or 1.

### Part c

Does the Mean Value Theorem apply to such a function?

Yes, since the Mean Value Theorem only requires the function to be differentiable on the open interval. In this case, the Mean Value Theorem states there is a point $\theta \in (0,1)$ such that $g'(\theta) = 0$. We can probably prove that a point exists by using the Intermediate Value Theorem on $g'(x)$ since it's continuous on $(0,1)$, but I'm too lazy at the moment.

## Problem 10

Concoct a function $f : \mathbb{R} \to \mathbb{R}$ with a discontinuity of the second kind at $x = 0$ such that $f$ does not have the intermediate value property there. Infer that it is incorrect to assert that functions without jumps are Darboux continuous.

Consider the function

$$f(x) = \begin{cases} x & x \in \mathbb{R} - \mathbb{Q} \\ 1 & \text{else} \end{cases}$$

$f$ is continuous at $x = 1$ and discontinuous everywhere else. These discontinuities are discontinuities of the second kind, since left and right limits don't exist when $x$ is not 1. $f(x)$ clearly does not have the intermediate value property, as except for 1, $f$ assumes no rational values. Since this is a function without jump discontinuities but does not possess the intermediate value property, functions without jumps are not necessarily Darboux continuous.

# Problem 11

Let $f : (a, b) \to \mathbb{R}$ be given.

## Part a

If $f''(x)$ exists, prove that

$$\lim_{h\to 0} \frac{f(x-h) - 2f(x) + f(x+h)}{h^2} = f''(x)$$

Denote $F(x) = \lim_{h\to 0} \frac{f(x-h)-2f(x)+f(x+h)}{h^2}$. Since $f$ is twice differentiable, we take take a second-order Taylor expansion of $f$ around $x$, getting

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + R(x)$$

where $R(x)$ is second-order flat at $h = 0$, i.e. $\lim_{h\to 0} R(x)/h^2 = 0$. Similarly,

$$f(x-h) = f(x) - hf'(x) + \frac{1}{2}h^2 f''(x) + S(x)$$

where $S(x)$ is second-order flat at $h = 0$. Substituting,

$$F(x) = \lim_{h\to 0} \frac{h^2 f''(x) + R(x) + S(x)}{h^2} = f''(x)$$

since the $f(x)$ and $hf'(x)$ terms cancel, and $R(x)$ and $S(x)$ are second-order flat.

## Part b

Find an example that this limit can exist even when $f''(x)$ fails to exist.

Let $f(x) = x|x|$. Taking the first derivative, when $x > 0$, $f(x) = x^2$, so $f'(x) = 2x$. Similarly, when $x < 0$, $f'(x) = -2x$. When $x = 0$,

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h|h|}{h} = \lim_{h\to 0} |h| = 0$$

Thus

$$f'(x) = \begin{cases} 2x & x \geq 0 \\ -2x & x < 0 \end{cases}$$

As previously stated, $f''(0)$ does not exist, since

$$f''(0) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0} \frac{f'(h)}{h}$$

which does not exist, since the limit from the positive direction is 2 and the limit from the negative direction is $-2$.

Despite this, the partial difference approximation exists at $x = 0$. The partial difference approximation from the right is

$$\lim_{h \to 0^+} \frac{f(-h) + f(h)}{h^2} = \lim_{h \to 0^+} \frac{-h| - h| + h|h|}{h^2} = \lim_{h \to 0^+} \frac{0}{h^2} = \infty$$

Similarly,

$$\lim_{h \to 0^-} \frac{f(-h) + f(h)}{h^2} = \lim_{h \to 0^-} \frac{h|h| + -h| - h|}{h^2} = \lim_{h \to 0^-} \frac{0}{h^2} = \infty$$

Thus the difference approximation exists at $x = 0$, even though $f''(0)$ does not exist.

# Problem 15

Define $f(x) = x^2$ if $x < 0$ and $f(x) = x + x^2$ if $x \geq 0$. Differentiation gives $f''(x) = 2$. This is bogus. Why?

By the Fundamental Theorem of Calculus, if $G$ is an antiderivative of $g$, then $g$ equals the derivative of $G$ where $g$ is continuous. In this case, the standard power rule only applies when $x \neq 0$, since there is a discontinuity there.

Specifically, we have $f''(0)$ does not exist, since $f'(x) = 2x$ when $x \geq 0$, and $f'(x) = 2x + 1$ when $x < 0$. $f'(x)$ is discontinuous at $x = 0$, so its derivative does not exist there.

# Problem 16

$\log x$ is defined to be $\int_1^x 1/tdt$ for $x > 0$. Using only the mathematics explained in this chapter,

## Part a

Prove that log is a smooth function.

By the Fundamental Theorem of Calculus, the indefinite integral of a Riemann integrable function is continuous with respect to $x$. Thus, $\log x$ is continuous. Its derivative, again by the Fundamental Theorem of Calculus, is $\frac{d}{dx} \int_1^x 1/tdx = 1/x$ when $x > 0$, which is continuous. $1/x$ itself is smooth, so it has derivatives of all orders, which are continuous. Thus $\log x$ is smooth.

## Part b

Prove that $\log(xy) = \log x + \log y$ for all $x, y > 0$.

For any given $y > 0$, define $f(x) = \log xy - \log x - \log y$. By definition,

$$\begin{aligned} f(x) &= \int_1^{xy} 1/t dt - \int_1^x 1/t dt - \int_1^y 1/t dt \\ &= \int_x^{xy} 1/t dt - \int_1^y 1/t dt \end{aligned}$$

When $x = 1$, $f(x) = \int_1^y 1/t dt - \int_1^y 1/t dt = 0$.

We now evaluate $f'(x)$. Splitting the integrals, for all $x > 0$, we can find a constant $0 < c < x$. Then

$$f(x) = \int_c^{xy} 1/t dt - \int_c^x 1/t dt - \int_1^y 1/t dt$$

By the Fundamental Theorem of Calculus, $\frac{d}{dx}\int_c^x 1/t dt = 1/x$ since $1/t$ is continuous on $[c, \infty)$. By the Chain Rule, $\frac{d}{dx}\int_c^{xy} 1/t dt = y\frac{1}{xy} = 1/x$. $\int_1^y 1/t dt$ is constant with regards to $x$, and thus has derivative zero. Thus, $f'(x) = 0$ for all $x > 0$. The only functions with derivatives equal to zero everywhere are constant functions, and since $f(1) = 0$, this implies that $f(x) = 0$. Thus $\log xy = \log x + \log y$.

## Part c

Prove that log is strictly monotone increasing and its range is all of $\mathbb{R}$.

$\frac{d}{dx}\log x = 1/x$, which is strictly positive for all $x > 0$. Thus $\log x$ is strictly monotone increasing.

We know that $\log(1) = 0$. Going to the right, let $a_k = \frac{1}{k}$. Because $\frac{1}{t}$ is decreasing, for all $t \in [k, k+1]$, $\frac{1}{t} \leq a_{k+1}$. Thus because $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{k}$ diverges to infinity, by the Integral Test, $\int_1^{\infty} \frac{1}{t} dt$ diverges to infinity. This means that there is for large $x$, $\log(x) = \int_1^x \frac{1}{t} dt$ can be made arbitrarily large. This implies that when $x \geq 0$, $\log(x)$ takes on all values in $[0, \infty)$.

Going to the left, for $x \in (0, 1]$, $\log(x) = -\int_x^1 \frac{1}{t} dt$. Let $k \in \mathbb{N}$ and consider $\log(\frac{1}{2^k}) = -\int_{\frac{1}{2^k}}^1 \frac{1}{t} dt$.

To evaluate $\int_{\frac{1}{2^k}}^1 \frac{1}{t} dt$, consider the partition $P$ such that $x_i = \frac{1}{2^i}$ for $i \in \mathbb{N}$. Thus $x_0 = 1$, $x_1 = \frac{1}{2}$, $x_2 = \frac{1}{4}$, etc. Because $\frac{1}{t}$ is strictly decreasing, the minimum of $\frac{1}{t}$ occurs at the right endpoint of the interval. Thus the lower integral is greater than or equal to

$$
\begin{aligned}
&1(\frac{1}{2}) + 2(\frac{1}{4}) + 4(\frac{1}{8}) \dots \\
=&\frac{1}{2} + \frac{1}{2} + \frac{1}{2} \dots \\
=&\frac{k}{2}
\end{aligned}
$$

because there are $k$ intervals. Since $\frac{1}{t}$ is Riemann integrable on $(0,1]$, $\frac{k}{2}$ is a lower bound for the integral. Thus

$$-\int_{\frac{1}{2^k}}^{1} \frac{1}{t} dt \leq -\frac{k}{2}$$

which implies that the integral goes to negative infinity as $k$ goes to infinity. Thus

$$-\int_{0}^{1} \frac{1}{t} dt = -\infty$$

which implies that as $x$ approaches zero, $\log(x)$ approaches negative infinity. Thus on $(0,1]$, $\log(x)$ takes on all values in $(-\infty, 0]$. Putting the two statements together implies that the range of $\log(x)$ is all of $\mathbb{R}$.

# Problem 17

Define $E : \mathbb{R} \to \mathbb{R}$ by

$$E(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

## Part a

Prove that $E(x)$ is smooth; that is, $E$ has derivatives of all orders at all points $x$.

For $x < 0$, smoothness is trivial. A quick application of the chain rule shows that on $x > 0$,

$$E'(x) = \frac{1}{x^2} e^{-\frac{1}{x}}$$

**Theorem 1** *For $x > 0$, $E^{(n)}(x)$ has the form*

$$(a_{n+1}x^{-(n+1)} + a_{n+2}x^{-(n+2)} + \dots + a_{2n}x^{-2n})e^{-\frac{1}{x}}$$

*for all $n \in \mathbb{N}$.*

**Proof:** *The base case $n = 1$ has been established above. Assume that the hypothesis holds for $n - 1$. Then*

$$E^{(n-1)}(x) = (a_n x^{-n} + a_{n+1} x^{-(n+1)} + \cdots + a_{2n-2} x^{-(2n-2)}) e^{-\frac{1}{x}}$$

*Using the Product Rule,*

$$\begin{aligned} E^{(n)}(x) &= [(-n a_n x^{-(n+1)} - (n+1) a_{n+1} x^{-(n+2)} - \cdots - (2n+2) a_{2n-2} x^{-(2n-1)}) \\ &\quad + (a_n x^{-(n+2)} + a_{n+1} x^{-(n+3)} + \cdots + a_{2n-2} x^{-2n})] e^{-\frac{1}{x}} \\ &= (b_{n+1} x^{-(n+1)} + b_{n+2} x^{-(n+2)} + \cdots + b_{2n} x^{-2n}) e^{-\frac{1}{x}} \end{aligned}$$

*since $n$ is a constant.* □

**Lemma 2** $\lim_{x\to 0} E(x) = E(0) = 0$. *Thus $E(x) \in C^0$.*

**Proof:** *The left limit is trivially zero. On the right, as $x$ approaches zero from the positive direction, $-\frac{1}{x}$ approaches negative infinity, so $e^{-\frac{1}{x}}$ approaches zero.* □

**Lemma 3** $\lim_{x\to 0^+} \frac{1}{x} e^{-\frac{1}{x}} = 0$.

**Proof:**

$$\lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}}{x} = \lim_{x\to 0^+} \frac{x^{-1}}{e^{\frac{1}{x}}} = \lim_{x\to 0^+} \frac{-x^{-2}}{-x^{-2} e^{\frac{1}{x}}} = \lim_{x\to 0^+} e^{-\frac{1}{x}} = 0$$

*by using l'Hopital's rule on the second expression.* □

**Lemma 4** $\lim_{x\to 0^+} \frac{1}{x^n} e^{-\frac{1}{x}} = 0$ *for $n \in \mathbb{N}$.*

**Proof:** *The base case has been established. For the inductive case, assume that $\lim_{x\to 0^+} \frac{1}{x^{n-1}} e^{-\frac{1}{x}} = 0$. Then*

$$\lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}}{x^n} = \lim_{x\to 0^+} \frac{x^{-n}}{e^{\frac{1}{x}}} = \lim_{x\to 0^+} \frac{-n x^{-n-1}}{-x^{-2} e^{\frac{1}{x}}} = n \lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}}{x^{n-1}} = 0$$

*by the inductive hypothesis.* □

**Corollary 5** $\lim_{x\to 0^+} E^{(n)}(x) = 0$ *when $x > 0$ for all $n \in \mathbb{N}$.*

**Proof:** *By Theorem 1, $E^{(n)}(x)$ is the sum of various terms of the form $\frac{1}{x^n} e^{-\frac{1}{x}}$, where $n \in N$. By Lemma 4, each of these terms has right limit zero. Since $\frac{1}{x^n} e^{-\frac{1}{x}}$ is the sum of a finite number of these terms, it has right limit zero.* □

**Theorem 6** $E^{(n)}(0)$ *exists and it equals zero for all $n \in \mathbb{N}$. $E^{(n)}(x)$ is continuous at $x = 0$ for all $n \in \mathbb{N}$, thus making $E(x)$ smooth.*

**Proof:** *For $n \in N$, we need to evaluate*

$$\lim_{x\to 0} \frac{E^{(n-1)}(x) - 0}{x - 0} = \lim_{x\to 0} \frac{E^{(n-1)}(x)}{x}$$

*The left limit is zero, since $E^{(n-1)}(x) = 0$ for $x \leq 0$. For the right limit, by Theorem 1, $\frac{1}{x}E^{(n-1)}(x)$ has the form*

$$(a_{n+2}x^{-(n+2)} + a_{n+3}x^{-(n+3)} + \cdots + a_{2n+1}x^{-2n+1})e^{-\frac{1}{x}}$$

*By repeated application of Lemma 4, we see that this has right limit zero. Thus*

$$E^{(n)}(0) = \lim_{x\to 0} \frac{E^{(n-1)}(x)}{x} = 0$$

*Because $E^{(n)}(x) = 0$ on $x \leq 0$, $E^{(n)}(0) = 0$, and $E^{(n)}(x)$ is continuous at $x = 0$ for all $n \in \mathbb{N}$. Combined with smoothness everywhere else, this implies that $E(x)$ is smooth everywhere.* $\square$

## Part b

Is $E(x)$ analytic?

No. A function $f$ defined on an open interval $(a, b)$ is analytic at $x \in (a, b)$ if it equals its a power series in a neighborhood of $x$. More specifically, for every $x$ there exists a $\delta > 0$ such that $|h| < \delta$ implies that

$$f(x + h) = \sum_{r=0}^{\infty} a_r h^r$$

where $a_r = \frac{f^{(r)}(x)}{r!}$.

For $E(x)$ at $x = 0$, $E^{(n)}(0) = 0$ for all whole numbers $n$, as established in Part a. Thus $a_r = 0$ for all whole numbers $r$, and the power series is just 0. However, for $h > 0$, $f(h) \neq 0$ since $e^{-\frac{1}{x}}$ is a strictly positive function. Thus $E(x)$ is not analytic.

# Problem 29

Prove that the interval $[a, b]$ is not a zero set.

## Part a

Explain why the following observation is not a solution to the problem: "Every open interval that contains $[a, b]$ has length $> b - a$."

This 'solution' does not consider the possibility that there is a union of open sets that cover $[a, b]$ such that their sum of their lengths can be made arbitrarily small.

## Part b

Instead, suppose there is a "bad" covering of $[a, b]$ by open intervals $\{I_i\}$ whose total length is $< b - a$, and justify the following steps in the proof by contradiction.

I will define a good covering as a covering of $[a, b]$ by open intervals $\{J\}$ such that the total length of the intervals in $\{J\}$ is greater than or equal to $b - a$.

### i

It is enough to deal with finite bad coverings.

Let $\{I\}$ be an infinite bad covering of $[a, b]$. Because $\{I\}$ is an open cover of compact $[a, b]$, it reduces to a finite subcovering $\{I_i\}$. Thus, either $\{I\}$ reduces to a finite bad covering, or it reduces to a good covering. If $\{I\}$ reduces to a good covering $\{J_i\}$, then $\{J_i\} \subset \{I\}$ and the sum of the intervals in $\{J_i\}$ being $\geq b - a$ implies that the sum of the intervals in $\{I\}$ is $\geq b - a$. Thus $\{I\}$ is an infinite good covering, which contradicts the assumption that $\{I\}$ is a bad covering.

Thus, if $\{I\}$ is an infinite bad covering, it reduces to a finite bad covering. Contrapositively, if there are no finite bad coverings, then there are no infinite bad coverings, and the theorem is proven.

### ii

Let $\mathbb{B} = \{I_1, \ldots I_n\}$ be a bad covering such that $n$ is minimal among all bad coverings.

There is at least one finite bad covering, by assumption. $n = 1$ is a lower bound for the size of bad coverings. Then because $\mathbb{R}$ is complete, there exists a greatest lower bound for the sizes of the bad coverings, denoted $c$.

The must be a finite bad covering $\{C\}$ such that the size of $|\{C\}| = c$. Suppose not. Then all bad coverings have size $> c$, and size the sizes of the bad coverings must be integers, all bad coverings have size $\geq c + 1$. This contradicts the assumption that $c$ is a greatest lower bound. This bad covering $\{C\}$ is the bad covering with minimal $n$ among all bad coverings.

### iii

Show that no bad covering has $n = 1$ so we have $n \geq 2$.

This follows from the observation in Part a.

### iv

Show that it is no loss of generality to assume $a \in I_1$ and $I_1 \cap I_2 \neq \emptyset$.

There exists at least one interval such that $a \in I_j$, and we are free to denote that interval $I_1$.

There must exist an interval that intersects $I_1$. Suppose not. Let $d_1$ be the right endpoint of $I_1$, and let $c_2, c_3 \ldots c_n$ be the left endpoints of the other intervals in the bad covering, and let $c = \min\{c_1 \dots c_n\}$. Then $\frac{c-d}{2}$ is not covered by the bad covering, contradicting the assumption that $\{I\}$ is a covering. Thus, there exists an interval in $\{I\}$ that intersects $I_1$. Denote it $I_2$. By construction, $I_1 \cap I_2$ is nonempty.

**v**

Show that $I = I_1 \cup I_2$ is an open interval and $|I| < |I_1| + |I_2|$.

If $I_1 \subset I_2$ or $I_2 \subset I_1$, $I_1 \cup I_2$ is trivially an open interval. Otherwise, $I_1 \cup I_2$ is the open because it is the union of open sets, connected because it is the union of two connected sets with a common point, and bounded because it is the finite union of bounded sets. Therefore $I_1 \cup I_2$ is a open, connected, and bounded subset of $\mathbb{R}$, and by the theorems shown in Chapter 2 Problem 31, open, connected, and bounded subsets of $\mathbb{R}$ are open intervals.

**Lemma 7** *Let $C, D \subset \mathbb{R}$ be (bounded) intervals that intersect, and let $E = C + D$. Then $|E| < |C| + |D|$.*

**Proof:** *If $C$ is a subset of $D$ or vice versa, the proof is trivial. Without loss of generality, let the left endpoint of $C$ be less than the left endpoint of $D$. Denote $c$ as the right endpoint of $C$, and $d$ the left endpoint of $D$. $d < c$, otherwise the two intervals do not intersect. Letting $\epsilon = c - d > 0$, the total length of $E$ is $|C| + |D| - \epsilon$, which is strictly less than $|C| + |D|$.* □

By using the above Lemma, we see that $|I| < |I_1| + |I_2|$.

**vi**

Show that $\mathbb{B}' = \{I, I_3, \dots I_n\}$ is a bad covering of $[a, b]$ with fewer intervals, contradicting the minimality of $n$.

Let $x \in [a, b]$. Since $\mathbb{B}$ is a covering of $[a, b]$, there exists $i \in 1, 2 \dots n$ such that $x \in I_i$. If $i \geq 3$, then because $I_i \in \mathbb{B}'$, $x$ is also covered by $\mathbb{B}'$. If $i = 1, 2$, then $x \in I = I_1 \cup I_2$, so $x$ is still covered by $\mathbb{B}'$. $\mathbb{B}'$ is a covering by open intervals, because $I$ is an open interval. $\mathbb{B}'$ is a bad covering. $|I| < |I_1| + |I_2|$ implies that $|I| + \sum_{j=3}^{n} I_j < \sum_{i=1}^{n} I_i < b - a$, implying that the total length of $\mathbb{B}'$ is less than the total length of $\mathbb{B}$. Thus $\mathbb{B}'$ is a bad covering with fewer intervals than $\mathbb{B}$, contradicting the assumption that $\mathbb{B}$ is the minimal bad covering. Thus, there are no bad coverings of $[a, b]$, coverings of $[a, b]$ can not have arbitrarily small length, and $[a, b]$ is not a zero set.

# Problem 34

Assume that $\psi : [a, b] \to \mathbb{R}$ is continuously differentiable. A critical point of $\psi$ is an $x$ such that $\psi'(x) = 0$. A critical value is a number $y$ such that for at least one critical point $x$ we have $y = \psi(x)$.

## Part a

Prove that the set of critical values is a zero set. (This is the Morse-Sard Theorem in dimension one.)

I will first introduce some notation. Let $f : [a, b] \to \mathbb{R}$ be continuous. I will define a **zero of type 1** of $f$ to be a zero of $f$ such that $f$ is uniformly zero in an open neighborhood of the root. In other words, if $f(x) = 0$, then there exists $\epsilon > 0$ such that for all $y$ such that $|x - y| < \epsilon$, $f(y) = 0$. I will denote a **zero point of type 2** of $f$ as all other zeros of $f$. It's clear that the disjoint union of zeros of types 1 and 2 make up all zeros of $f$.

Let $\psi$ be continuously differentiable. We will characterize the critical values of $\psi$ based on the zeros of type 1 and 2 of $\psi'$. If $\psi(x) = y$ is a critical value and $\psi'(x)$ is a zero of type 1, we say that $y$ is a **critical value of type 1** of $f$. Similarly, if $\psi(x) = y$ is a critical value and $\psi'(x)$ is a zero of type 2, we say that $y$ is a **critical value of type 2** of $f$. Since the zeros of type 1 and 2 partition the set of zeros of $\psi'$, the critical values of type 1 and 2 partition the set of critical values of $\psi$.

I have no idea if this characterization is standard, but that's what I've come up with.

The immediate characterization for zeros of type 2 is stated below.

**Lemma 8** *Let $f$ be continuous, and let $x$ be a zero of type 2 of $f$. Then for all $\epsilon > 0$, there exists $y \in (x - \epsilon, x + \epsilon)$ such that $f(y) \neq 0$.*

**Proof:** *If this is not true, then $x$ is a zero of type 1.* □

To begin with zero points of type 2, we next state a lemma on non-zero points of continuous functions implying an interval with no zeros. This can be thought of as non-zero points of continuous functions creating 'exclusion zones' with a delta-radius that contain no zeros.

**Lemma 9** *Let $f : [a, b] \to \mathbb{R}$ be continuous, and let $x \in [a, b]$ be a point such that $f(x) \neq 0$. Then there exists a $\delta > 0$ such that $f$ has no zeros in $(x - \delta, x + \delta)$.*

**Proof:** *Because $f$ is uniformly continuous, there exists a $\delta > 0$ such that for all $y$ such that $|x - y| < \delta$, $|f(x) - f(y)| < |f(x)|$. This implies that $y$ is not a zero of $f$.* □

**Lemma 10** *Let $f : [a, b] \to \mathbb{R}$ be continuous, and let $x \in [a, b]$ such that $f(x) \neq 0$. Then $f$ is not a clustering point of zeros. In other words, we can reasonably speak of the nearest zero of $f$ greater than $x$, and the nearest zero of $f$ less than $x$.*

**Proof:** *Suppose not. Then there exists a sequence $(x_n) \to x$ of zeros of $f$. $\lim_{n\to\infty} f(x_n) = 0$, but $f(x) \neq 0$, violating the continuity of $f$.* □

We now introduce some useful terminology (that I have no idea whether is standard, but I am going to use it). Let $f : [a, b] \to \mathbb{R}$ be continuous, and let $x$ be a point such that $f(x) \neq 0$. The **covering interval of x** is the open interval between the nearest zero of $f$ to the left of $x$, and the nearest zero of $f$ to the right of $x$. This interval covers $x$. By Lemma 10, this is a well-defined construction.

I will denote the interval as $C$. If there are no zeros of $f$ to the left of $x$, then the left endpoint of $C$ is $a$, and if there are no zeros to the right of $x$, then the right endpoint of $C$ is $b$.

**Lemma 11** *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then the covering intervals of $f$ are disjoint.*

**Proof:** *Any two covering intervals are separated by at least one zero of $f$.* □

We now show that the number of covering intervals is closely related to the number of zeros of type 2.

**Lemma 12** *Let $x$ be a zero of type 2. Then there is a covering interval such that $x$ is the endpoint.*

**Proof:** *Suppose not. Then $x$ is not the nearest zero of type 2 to any nonzero point of $f$. This means that $x$ is a clustering point of zeros of type 2, which contradicts Lemma 10.* □

**Corollary 13** *The set of zeros of type 2 is of equal or lesser cardinality to the set of covering intervals.*

**Proof:** *Each zero of type 2 belongs to at least one covering interval.* □

The main results for zeros of type 2 follows.

**Corollary 14** *Let $f$ be continuous on $[a, b]$. Then $f$ has at most countable zeros of type 2.*

**Proof:** *Let $Q$ be the set of covering intervals for $f$. Because $Q$ is the disjoint union of intervals, $Q$ is countable. By Lemma 13, the zeros of type 2 of $f$ are countable.* □

**Corollary 15** *Let $f$ be continuously differentiable on $[a, b]$. Then $f$ has at most countable critical values of type 2.*

**Proof:** *$f'$ is continuous, implying that $f$ has at most countable zeros of type 2. Each zero of type 2 of $f'$ maps to at most one critical value of type 2 of $f$.* $\square$

We now turn to critical points of type 1. We first state a useful characterization of zeros of type 1.

**Lemma 16** *Let $f : [a, b] \to \mathbb{R}$ be continuous, and let $Z$ be the set of zeros of type 1. Then $Z$ is the disjoint union of countable open intervals, with perhaps one or two half-open intervals at the endpoints $a$ and $b$.*

**Proof:** *The definition of zeros of type 1 implies that $Z$ is an open set in $[a, b]$. By the Inheritance Principle, there exists a set $W \subset \mathbb{R}$ that is open in $\mathbb{R}$ such that $W \cap \mathbb{R} = Z$. By Problem 31 in Chapter 2, an open set in $\mathbb{R}$ can be expressed as the disjoint union of countably many open intervals. Taking $W \cap Z$, open intervals that do not contain the endpoints $a$ and $b$ are still open in $Z$, while the half-intervals that have their closed end at $a$ and $b$ become open in $[a, b]$.* $\square$

We next state a lemma on critical points of type 1.

**Lemma 17** *Let $x$ be a critical point of type 1 for $\psi'(x)$. Then on the neighborhood where $\psi'(x) = 0$, there is only one critical value. Specifically, if $\psi'(x) = 0$ on an interval $(c, d) \subset [a, b]$, then $\psi(c)$ is the only critical value on that interval.*

**Proof:** *By the Fundamental Theorem of Calculus, for $x \in [c, d]$, $\psi(x) = \psi(c) + \int_c^x \psi'(x)dx = \psi(c)$ since $\psi'(x) = 0$ on the interval.* $\square$

We now want to prove the main theorem for critical values corresponding to critical points of type 1.

**Theorem 18** *If $f : [a, b] \to \mathbb{R}$ is continuously differentiable, then $f$ has countably many critical values of type 1.*

**Proof:** *$f'$ is continuous by assumption. By Lemma 16, $Z$, the set of zeros of type 1 of $f'$, consists of countable disjoint open intervals, with perhaps one or two half-open intervals at $a$ and $b$. By Lemma 17, each (half)-open interval in $Z$ corresponds to one critical value in $f$. Thus $f$ has at most countably many critical values of type 1.* $\square$

**Theorem 19** *The critical values of $f$ form a zero set.*

**Proof:** *The union of countable sets is a countable set, which is a zero set.* $\square$

## Part b

Generalize this result to continuous functions on $\mathbb{R} \to \mathbb{R}$.

The result immediately generalizes. Divide $\mathbb{R}$ into countably many intervals of length 1. By Part a, there are countably many critical values of $f$ on each of these intervals, and the countable union of countable sets is a zero set. Thus the set of critical values of a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ is a zero set.