Chapter 3 Functions of a Real Variable

Arthur Chen

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Problem 1

Assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(t) - f(x)| \leq |t - x|^2$ for all t, x. Prove that f is constant.

Proof: The assumption implies that for all t, x,

$$0 \leq \left| \frac{f(t) - f(x)}{t - x} \right| = \frac{|f(t) - f(x)|}{|t - x|} \leq |t - x|$$

implies that $f'(t) = \lim_{x \to t} \frac{f(t)-f(x)}{t-x} = 0$ at all t. The only functions with derivatives that are zero everywhere are constant functions. $\square$

Problem 2

A function $f : (a, b) \to \mathbb{R}$ satisfies a Holder condition of order α if $\alpha > 0$, and for some constant H and all $u, x \in (a, b)$ se have

$$|f(u) - f(x)| \leq H|u - x|^\alpha$$

The function is said to be α-Holder, with α-Holder constant H.

Part a

Prove that the α-Holder function defined on (a, b) is uniformly continuous and infer that it extends uniquely to a continuous function defined on $[a, b]$. Is the extended function α-Holder?

Proof: Let $\epsilon > 0$ and define $\delta = (\frac{\epsilon}{H})^{1/\alpha}$. Then for all $u, x \in (a, b)$ such that $|u - x| < \delta$, we have

$$|f(u) - f(x)| \leq H|u - x|^\alpha < \epsilon$$

since $\alpha > 0$. $\square$

By Problem 54 in Chapter 2, a uniformly continuous function defined on a metric space S has a unique continuous extension on $\bar{S}$. Since $[a,b] = \overline{(a,b)}$, $f : (a,b) \to \mathbb{R}$ being uniformly continuous implies that f extends uniquely to $g : [a,b] \to \mathbb{R}$, where g is continuous. In fact, g is uniformly continuous because it is continuous on a compact.

We claim that g is α-Holder on $[a,b]$. Let $x, y \in [a,b]$. If $x, y \in (a,b)$, this just follows because g extends f.

Without loss of generality, let $x = a$ and let $y \in (a,b)$. Let $\epsilon > 0$ be fixed and arbitrary, and let $\delta > 0$ be the corresponding continuity condition. Then

$$|g(c) - g(a)| \leq |g(c) - g(a+\delta)| + |g(a) - g(a+\delta)|$$

by the Triangle inequality. For the first term, because c and $a + \delta$ are in the interval (a,b), the Holder condition from f extends to g, so

$$|g(c) - gf(a+\delta)| \leq H|c - a - \delta|^\alpha \leq H|c-a|^\alpha$$

because $\alpha > 0$ and $\delta > 0$. For the second term, continuity of g means $|g(a) - g(a+\delta)| < \epsilon$. Thus

$$|g(c) - g(a)| \leq H|c-a|^\alpha + \epsilon$$

and ϵ can be made arbitrarily small. The case where $y = b$, and the case where $x = a$ and $y = b$ simultaneously, are essentially the same.

Part b

What does α-Holder continuity mean when $\alpha = 1$?

When $\alpha = 1$, α-Holder continuity simplifies to Lipschitz continuity.

Part c

Prove that α-Holder continuity when $\alpha > 1$ implies that f is constant.

Let x in the domain of f be arbitrary. Dividing both sides by $|u - x|$,

$$0 \leq \frac{|f(u) - f(x)|}{|u - x|} \leq H|u-x|^{\alpha - 1}$$

Let $u \to x$. Since $\alpha > 1$ the right side goes to 0, implying $\frac{|f(u)-f(x)|}{|u-x|} \to 0$ and that $f'(x) = 0$ for all x in f's domain. The only functions with this property are constant functions.

Problem 3

Assume that $f : (a,b) \to \mathbb{R}$ is differentiable.

Part a

If $f'(x) > 0$ for all x, prove that f is strictly monotone increasing.

Proof: Let $c, d \in (a, b), c < d$. Then because f is differentiable on its domain, the Mean Value Theorem indicates that there is a point $\theta \in (c, d)$ such that

$$f(c) - f(d) = f'(\theta)(d - c)$$

Since f' is always strictly positive and $c < d$, the right side is strictly positive.
□

Part b

If $f'(x) \geq 0$ for all x, what can you prove?

We can prove that f is weakly monotone increasing. The proof is the same, except that $f'(\theta)(d - c)$ can be zero.

Problem 4

Prove that $\sqrt{n+1} - \sqrt{n} \to 0$ as $n \to \infty$.

Consider the function $f(x) = \sqrt{x}$, and take a Taylor approximation of degree zero around $x = n$, where n is a positive natural number. Then $P_0(x) = \sqrt{n}$. Use the Taylor approximation to approximate $x = n+1$. The Taylor remainder term is

$$R(1) = \sqrt{n+1} - \sqrt{n}$$

$\sqrt{x}$ is smooth when $x > 0$, and $n \geq 1$. Therefore, f is smooth on $(n, n+1)$, and the Taylor approximation theorem states that there exists $\theta \in (n, n+1)$ such that

$$R(1; n) = \sqrt{n+1} - \sqrt{n} = \frac{f'(\theta)}{1!}(1)^1 = \frac{1}{2}\theta^{-\frac{1}{2}}$$

As $n \to \infty, \theta > n$ implies $\theta \to \infty$ implies $R(1; n) \to 0$ implies $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0$.

Problem 8

Part b

Find a formula for a continuous function defined on $[0, 1]$ that is differentiable on the interval $(0, 1)$, but not at the endpoints.

Consider the function

$$f(x) = \begin{cases} x\sin(\frac{1}{x}) & x \in (0,1] \\ 0 & \text{else} \end{cases}$$

f is the composition of continuous functions on $(0,1]$, so it is continuous on that interval. At $x = 0$, we noting that for all $x \in (0,1]$, we have

$$-x \leq x\sin(\frac{1}{x}) \leq x$$

implying that $\lim_{x\to 0^+} f(x) = 0 = f(0)$ by the Squeeze theorem. This implies that $f(x)$ is continuous at $x = 0$, and thus $[0,1]$. $\frac{1}{x}$ is differentiable on $\mathbb{R} - 0$, so $f(x)$ is differentiable on $(0,1]$.

Taking the definition of derivative to attempt to evaluate $f'(0)$,

$$f'(0) = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \sin(\frac{1}{x})$$

which does not exist. Thus $f(x)$ is differentiable on $(0,1]$.

Consider the function

$$g(x) = f(x) + f(1-x)$$

This consists of f and f reflected about the line $x = \frac{1}{2}$ added together. From the above, g is continuous on $[0,1]$, and differentiable on $(0,1)$, but not 0 or 1.

Part c

Does the Mean Value Theorem apply to such a function?

Yes, since the Mean Value Theorem only requires the function to be differentiable on the open interval. In this case, the Mean Value Theorem states there is a point $\theta \in (0,1)$ such that $g'(\theta) = 0$. We can probably prove that a point exists by using the Intermediate Value Theorem on $g'(x)$ since it's continuous on $(0,1)$, but I'm too lazy at the moment.

Problem 10

Concoct a function $f : \mathbb{R} \to \mathbb{R}$ with a discontinuity of the second kind at $x = 0$ such that f does not have the intermediate value property there. Infer that it is incorrect to assert that functions without jumps are Darboux continuous.

Consider the function

$$f(x) = \begin{cases} x & x \in \mathbb{R} - \mathbb{Q} \\ 1 & \text{else} \end{cases}$$

f is continuous at $x = 1$ and discontinuous everywhere else. These discontinuities are discontinuities of the second kind, since left and right limits don't exist when x is not 1. $f(x)$ clearly does not have the intermediate value property, as except for 1, f assumes no rational values. Since this is a function without jump discontinuities but does not possess the intermediate value property, functions without jumps are not necessarily Darboux continuous.

Problem 11

Let $f : (a, b) \to \mathbb{R}$ be given.

Part a

If $f''(x)$ exists, prove that

$$\lim_{h\to 0} \frac{f(x-h) - 2f(x) + f(x+h)}{h^2} = f''(x)$$

Denote $F(x) = \lim_{h\to 0} \frac{f(x-h)-2f(x)+f(x+h)}{h^2}$. Since f is twice differentiable, we take take a second-order Taylor expansion of f around x, getting

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + R(x)$$

where $R(x)$ is second-order flat at $h = 0$, i.e. $\lim_{h\to 0} R(x)/h^2 = 0$. Similarly,

$$f(x-h) = f(x) - hf'(x) + \frac{1}{2}h^2 f''(x) + S(x)$$

where $S(x)$ is second-order flat at $h = 0$. Substituting,

$$F(x) = \lim_{h\to 0} \frac{h^2 f''(x) + R(x) + S(x)}{h^2} = f''(x)$$

since the $f(x)$ and $hf'(x)$ terms cancel, and $R(x)$ and $S(x)$ are second-order flat.

Part b

Find an example that this limit can exist even when $f''(x)$ fails to exist.

Let $f(x) = x|x|$. Taking the first derivative, when $x > 0$, $f(x) = x^2$, so $f'(x) = 2x$. Similarly, when $x < 0$, $f'(x) = -2x$. When $x = 0$,

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h|h|}{h} = \lim_{h\to 0} |h| = 0$$

Thus

$$f'(x) = \begin{cases} 2x & x \geq 0 \\ -2x & x < 0 \end{cases}$$

As previously stated, $f''(0)$ does not exist, since

$$f''(0) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h\to 0} \frac{f'(h)}{h}$$

which does not exist, since the limit from the positive direction is 2 and the limit from the negative direction is -2.

Despite this, the partial difference approximation exists at $x = 0$. The partial difference approximation from the right is

$$\lim_{h\to 0^+} \frac{f(-h) + f(h)}{h^2} = \lim_{h\to 0^+} \frac{-h| - h| + h|h|}{h^2} = \lim_{h\to 0^+} \frac{0}{h^2} = \infty$$

Similarly,

$$\lim_{h\to 0^-} \frac{f(-h) + f(h)}{h^2} = \lim_{h\to 0^-} \frac{h|h| + -h| - h|}{h^2} = \lim_{h\to 0^-} \frac{0}{h^2} = \infty$$

Thus the difference approximation exists at $x = 0$, even though $f''(0)$ does not exist.

Problem 15

Define $f(x) = x^2$ if $x < 0$ and $f(x) = x + x^2$ if $x \geq 0$. Differentiation gives $f''(x) = 2$. This is bogus. Why?

By the Fundamental Theorem of Calculus, if G is an antiderivative of g, then g equals the derivative of G where g is continuous. In this case, the standard power rule only applies when $x \neq 0$, since there is a discontinuity there.

Specifically, we have $f''(0)$ does not exist, since $f'(x) = 2x$ when $x \geq 0$, and $f'(x) = 2x + 1$ when $x < 0$. $f'(x)$ is discontinuous at $x = 0$, so its derivative does not exist there.

Problem 16

$\log x$ is defined to be $\int_1^x 1/tdt$ for $x > 0$. Using only the mathematics explained in this chapter,

Part a

Prove that log is a smooth function.

By the Fundamental Theorem of Calculus, the indefinite integral of a Riemann integrable function is continuous with respect to x. Thus, $\log x$ is continuous. Its derivative, again by the Fundamental Theorem of Calculus, is $\frac{d}{dx}\int_1^x 1/tdx = 1/x$ when $x > 0$, which is continuous. $1/x$ itself is smooth, so it has derivatives of all orders, which are continuous. Thus $\log x$ is smooth.

Part b

Prove that $\log(xy) = \log x + \log y$ for all $x, y > 0$.

For any given $y > 0$, define $f(x) = \log xy - \log x - \log y$. By definition,

$$\begin{aligned} f(x) &= \int_1^{xy} 1/tdt - \int_1^x 1/tdt - \int_1^y 1/tdt \\ &= \int_x^{xy} 1/tdt - \int_1^y 1/tdt \end{aligned}$$

When $x = 0$, $f(x) = \int_1^y 1/tdt - \int_1^y 1/tdt = 0$.

We now evaluate $f'(x)$. Splitting the integrals, for all $x > 0$, we can find a constant $0 < c < x$. Then

$$f(x) = \int_c^{xy} 1/tdt - \int_c^x 1/tdt - \int_1^y 1/tdt$$

By the Fundamental Theorem of Calculus, $\frac{d}{dx}\int_c^x 1/tdt = 1/x$ since $1/t$ is continuous on $[c, \infty)$. By the Chain Rule, $\frac{d}{dx}\int_c^{xy} 1/tdt = y\frac{1}{xy} = 1/x$. Thus, $f'(x) = 0$ for all $x > 0$. $\int_1^y 1/tdt$ is constant with regards to x, and thus has derivative zero. The only functions with derivatives equal to zero everywhere are constant functions, and since $f(1) = 0$, this implies that $f(x) = 0$. Thus $\log xy = \log x + \log y$.

Part c

Prove that log is strictly monotone increasing and its range is all of $\mathbb{R}$.

$\frac{d}{dx}\log x = 1/x$, which is strictly positive for all $x > 0$. Thus $\log x$ is strictly monotone increasing.

TO FINISH.

Problem 29

Prove that the interval $[a, b]$ is not a zero set.

Part a

Explain why the following observation is not a solution to the problem: "Every open interval that contains $[a, b]$ has length $> b - a$."

This 'solution' does not consider the possibility that there is a union of open sets that cover $[a, b]$ such that their sum of their lengths can be made arbitrarily small.

Part b

Instead, suppose there is a "bad" covering of $[a, b]$ by open intervals $\{I_i\}$ whose total length is $< b - a$, and justify the following steps in the proof by contradiction.

I will define a good covering as a covering of $[a, b]$ by open intervals $\{J\}$ such that the total length of the intervals in $\{J\}$ is greater than or equal to $b - a$.

i

It is enough to deal with finite bad coverings.

Let $\{I\}$ be an infinite bad covering of $[a, b]$. Because $\{I\}$ is an open cover of compact $[a, b]$, it reduces to a finite subcovering $\{I_i\}$. Thus, either $\{I\}$ reduces to a finite bad covering, or it reduces to a good covering. If $\{I\}$ reduces to a good covering $\{J_i\}$, then $\{J_i\} \subset \{I\}$ and the sum of the intervals in $\{J_i\}$ being $\geq b - a$ implies that the sum of the intervals in $\{I\}$ is $\geq b - a$. Thus $\{I\}$ is an infinite good covering, which contradicts the assumption that $\{I\}$ is a bad covering.

Thus, if $\{I\}$ is an infinite bad covering, it reduces to a finite bad covering. Contrapositively, if there are no finite bad coverings, then there are no infinite bad coverings, and the theorem is proven.

ii

Let $\mathbb{B} = \{I_1, \ldots I_n\}$ be a bad covering such that n is minimal among all bad coverings.

There is at least one finite bad covering, by assumption. $n = 1$ is a lower bound for the size of bad coverings. Then because $\mathbb{R}$ is complete, there exists a greatest lower bound for the sizes of the bad coverings, denoted c.

The must be a finite bad covering $\{C\}$ such that the size of $|\{C\}| = c$. Suppose not. Then all bad coverings have size $> c$, and size the sizes of the bad coverings must be integers, all bad coverings have size $\geq c + 1$. This contradicts the assumption that c is a greatest lower bound. This bad covering $\{C\}$ is the bad covering with minimal n among all bad coverings.

iii

Show that no bad covering has $n = 1$ so we have $n \geq 2$.

This follows from the observation in Part a.

iv

Show that it is no loss of generality to assume $a \in I_1$ and $I_1 \cap I_2 \neq \emptyset$.

There exists at least one interval such that $a \in I_j$, and we are free to denote that interval I_1.

There must exist an interval that intersects I_1. Suppose not. Let d_1 be the right endpoint of I_1, and let $c_2, c_3 \ldots c_n$ be the left endpoints of the other intervals in the bad covering, and let $c = \min\{c_1 \dots c_n\}$. Then $\frac{c-d}{2}$ is not covered by the bad covering, contradicting the assumption that $\{I\}$ is a covering. Thus, there exists an interval in $\{I\}$ that intersects I_1. Denote it I_2. By construction, $I_1 \cap I_2$ is nonempty.

v

Show that $I = I_1 \cup I_2$ is an open interval and $|I| < |I_1| + |I_2|$.

If $I_1 \subset I_2$ or $I_2 \subset I_1$, $I_1 \cup I_2$ is trivially an open interval. Otherwise, $I_1 \cup I_2$ is the open because it is the union of open sets, connected because it is the union of two connected sets with a common point, and bounded because it is the finite union of bounded sets. Therefore $I_1 \cup I_2$ is a open, connected, and bounded subset of $\mathbb{R}$, and by the theorems shown in Chapter 2 Problem 31, open, connected, and bounded subsets of $\mathbb{R}$ are open intervals.

Lemma 1 *Let $C, D \subset \mathbb{R}$ be (bounded) intervals that intersect, and let $E = C + D$. Then $|E| < |C| + |D|$.*

Proof: *If C is a subset of D or vice versa, the proof is trivial. Without loss of generality, let the left endpoint of C be less than the left endpoint of D. Denote c as the right endpoint of C, and d the left endpoint of D. $d < c$, otherwise the two intervals do not intersect. Letting $\epsilon = c - d > 0$, the total length of E is $|C| + |D| - \epsilon$, which is strictly less than $|C| + |D|$.* □

By using the above Lemma, we see that $|I| < |I_1| + |I_2|$.

vi

Show that $\mathbb{B}' = \{I, I_3, \dots I_n\}$ is a bad covering of $[a, b]$ with fewer intervals, contradicting the minimality of n.

Let $x \in [a, b]$. Since $\mathbb{B}$ is a covering of $[a, b]$, there exists $i \in 1, 2 \dots n$ such that $x \in I_i$. If $i \geq 3$, then because $I_i \in \mathbb{B}'$, x is also covered by $\mathbb{B}'$. If $i = 1, 2$, then $x \in I = I_1 \cup I_2$, so x is still covered by $\mathbb{B}'$. $\mathbb{B}'$ is a covering by open intervals, because I is an open interval. $\mathbb{B}'$ is a bad covering. $|I| < |I_1| + |I_2|$ implies that $|I| + \sum_{j=3}^{n} I_j < \sum_{i=1}^{n} I_i < b - a$, implying that the total length of $\mathbb{B}'$ is less than the total length of $\mathbb{B}$. Thus $\mathbb{B}'$ is a bad covering with fewer intervals than $\mathbb{B}$, contradicting the assumption that $\mathbb{B}$ is the minimal bad covering. Thus, there are no bad coverings of $[a, b]$, coverings of $[a, b]$ can not have arbitrarily small length, and $[a, b]$ is not a zero set.

Problem 34

Assume that $\psi : [a, b] \to \mathbb{R}$ is continuously differentiable. A critical point of ψ is an x such that $\psi'(x) = 0$. A critical value is a number y such that for at least one critical point x we have $y = \psi(x)$.

Part a

Prove that the set of critical values is a zero set. (This is the Morse-Sard Theorem in dimension one.)

Let f be differentiable. We first divide critical points into two types. I shall define a **critical point of type 1** of a function f as a critical point x such that $f'(x)$ is uniformly zero in an open neighborhood of x. In other words, there exists a $\epsilon > 0$ such that for all y such that $|x - y| < \epsilon$, $f'(y) = 0$. I will denote a **critical point of type 2** as all other critical points. It's clear that the disjoint union of these two types is the set of all critical points of f. I will define **critical values of type 1** to be critical values that correspond to critical points of type 1, and **critical values of type 2** similarly. It's clear that the disjoint union of these sets is the set of all critical values of f. I have no idea if this characterization is standard, but that's what I've come up with.

The immediate characterization for critical points of type 2 is stated below.

Lemma 2 *Let x be a critical point of type 2 of f. Then for all $\epsilon > 0$, there exists $y \in (x - \epsilon, x + \epsilon)$ such that $f'(y) \neq 0$.*

Proof: *If this is not true, then x is a critical point of type 1.* □

We state a lemma for critical points of type 2.

Lemma 3 *Let f be continuously differentiable with an infinite number of critical points of type 2. Then there exists at least one $x \in [a, b]$ such that x is a cluster point of critical points of type 2.*

Proof: *Suppose not. Then there is some $\epsilon > 0$ such that for all $x \in [a, b]$, $(x - \epsilon) \cup (x + \epsilon)$ contains no critical points of type 2, meaning $(x - \epsilon, x + \epsilon)$ contains at most one critical point of type 2. Spacing out test points with distance ϵ between them in $[a, b]$ shows that there are at most a finite number of critical points of type 2 in $[a, b]$, a contradiction.* □

We now want to prove the main theorem for critical points of type 2.

Theorem 4 *Let f be continuously differentiable on $[a, b]$. Then f has at most countable critical points of type 2.*

Proof: ***TODO***. □

Corollary 5 *Let f be continuously differentiable on $[a, b]$. Then f has at most countable critical values of type 2.*

Proof: *Each critical point of type 2 maps to at most one critical value of type 2.* □

We now turn to critical points of type 1. We now state a lemma on critical points of type 1.

Lemma 6 *Let x be a critical point of type 1 for $\psi'(x)$. Then on the neighborhood where $\psi'(x) = 0$, there is only one critical value. Specifically, if $\psi'(x) = 0$ on an interval $(c, d) \subset [a, b]$, then $\psi(c)$ is the only critical value on that interval.*

Proof: *By the Fundamental Theorem of Calculus, for $x \in [c, d]$, $\psi(x) = \psi(c) + \int_c^x \psi'(x)dx = \psi(c)$ since $\psi'(x) = 0$ on the interval.* □

We now want to prove the main theorem for critical values corresponding to critical points of type 1.

Theorem 7 *If $f : [a, b] \to \mathbb{R}$ is continuously differentiable, then f has countably many critical values of type 1.*

Proof: ***TODO**. Some sort of 1/k interval counting trick, since the critical points of type 1 can be grouped up into disjoint open intervals, except for some endpoint behavior. Then the countable intervals with critical points of type 1 map to at most countable critical values, using the lemma thingy with the fundamental theorem of calculus further up the page. Need to iron out the kinks.* □

Theorem 8 *The critical values of f form a zero set.*

Proof: *The union of a countable set with a finite set is a countable set, which is a zero set.* □