Chapter 3 Functions of a Real Variable

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Problem 1

Assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(t) - f(x)| \leq |t - x|^2$ for all t, x. Prove that f is constant.

Proof: The assumption implies that for all t, x,

$$0 \leq \left| \frac{f(t) - f(x)}{t - x} \right| = \frac{|f(t) - f(x)|}{|t - x|} \leq |t - x|$$

implies that $f'(t) = \lim_{x \to t} \frac{f(t)-f(x)}{t-x} = 0$ at all t. The only functions with derivatives that are zero everywhere are constant functions. □

Problem 2

A function $f : (a, b) \to \mathbb{R}$ satisfies a Holder condition of order α if $\alpha > 0$, and for some constant H and all $u, x \in (a, b)$ se have

$$|f(u) - f(x)| \leq H|u - x|^\alpha$$

The function is said to be α-Holder, with α-Holder constant H.

Part a

Prove that the α-Holder function defined on (a, b) is uniformly continuous and infer that it extends uniquely to a continuous function defined on $[a, b]$. Is the extended function α-Holder?

Proof: Let $\epsilon > 0$ and define $\delta = (\frac{\epsilon}{H})^{1/\alpha}$. Then for all $u, x \in (a, b)$ such that $|u - x| < \delta$, we have

$$|f(u) - f(x)| \leq H|u - x|^\alpha < \epsilon$$

since $\alpha > 0$. □

By Problem 54 in Chapter 2, a uniformly continuous function defined on a metric space S has a unique continuous extension on $\bar{S}$. Since $[a, b] = \overline{(a, b)}$, $f : (a, b) \to \mathbb{R}$ being uniformly continuous implies that f extends uniquely to $g : [a, b] \to \mathbb{R}$, where g is continuous. In fact, g is uniformly continuous because it is continuous on a compact.

We claim that g is α-Holder on $[a, b]$. Let $x, y \in [a, b]$. If $x, y \in (a, b)$, this just follows because g extends f.

Without loss of generality, let $x = a$ and let $y \in (a, b)$. Let $\epsilon > 0$ be fixed and arbitrary, and let $\delta > 0$ be the corresponding continuity condition. Then

$$|g(c) - g(a)| \leq |g(c) - g(a + \delta)| + |g(a) - g(a + \delta)|$$

by the Triangle inequality. For the first term, because c and $a + \delta$ are in the interval (a, b), the Holder condition from f extends to g, so

$$|g(c) - gf(a + \delta)| \leq H|c - a - \delta|^\alpha \leq H|c - a|^\alpha$$

because $\alpha > 0$ and $\delta > 0$. For the second term, continuity of g means $|g(a) - g(a + \delta)| < \epsilon$. Thus

$$|g(c) - g(a)| \leq H|c - a|^\alpha + \epsilon$$

and ϵ can be made arbitrarily small. The case where $y = b$, and the case where $x = a$ and $y = b$ simultaneously, are essentially the same.

Part b

What does α-Holder continuity mean when $\alpha = 1$?

When $\alpha = 1$, α-Holder continuity simplifies to Lipschitz continuity.

Part c

Prove that α-Holder continuity when $\alpha > 1$ implies that f is constant.

Let x in the domain of f be arbitrary. Dividing both sides by $|u - x|$,

$$0 \leq \frac{|f(u) - f(x)|}{|u - x|} \leq H|u - x|^{\alpha - 1}$$

Let $u \to x$. Since $\alpha > 1$ the right side goes to 0, implying $\frac{|f(u)-f(x)|}{|u-x|} \to 0$ and that $f'(x) = 0$ for all x in f's domain. The only functions with this property are constant functions.

Problem 3

Assume that $f : (a, b) \to \mathbb{R}$ is differentiable.

Part a

If $f'(x) > 0$ for all x, prove that f is strictly monotone increasing.

Proof: Let $c, d \in (a, b), c < d$. Then because f is differentiable on its domain, the Mean Value Theorem indicates that there is a point $\theta \in (c, d)$ such that

$$f(c) - f(d) = f'(\theta)(d - c)$$

Since f' is always strictly positive and $c < d$, the right side is strictly positive. $\square$

Part b

If $f'(x) \geq 0$ for all x, what can you prove?

We can prove that f is weakly monotone increasing. The proof is the same, except that $f'(\theta)(d - c)$ can be zero.

Problem 4

Prove that $\sqrt{n+1} - \sqrt{n} \to 0$ as $n \to \infty$.

Consider the function $f(x) = \sqrt{x}$, and take a Taylor approximation of degree zero around $x = n$, where n is a positive natural number. Then $P_0(x) = \sqrt{n}$. Use the Taylor approximation to approximate $x = n+1$. The Taylor remainder term is

$$R(1) = \sqrt{n+1} - \sqrt{n}$$

$\sqrt{x}$ is smooth when $x > 0$, and $n \geq 1$. Therefore, f is smooth on $(n, n+1)$, and the Taylor approximation theorem states that there exists $\theta \in (n, n+1)$ such that

$$R(1; n) = \sqrt{n+1} - \sqrt{n} = \frac{f'(\theta)}{1!}(1)^1 = \frac{1}{2}\theta^{-\frac{1}{2}}$$

As $n \to \infty, \theta > n$ implies $\theta \to \infty$ implies $R(1; n) \to 0$ implies $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0$.

Problem 8

Part b

Find a formula for a continuous function defined on $[0, 1]$ that is differentiable on the interval $(0, 1)$, but not at the endpoints.

Consider the function

$$f(x) = \begin{cases} x\sin(\frac{1}{x}) & x \in (0,1] \\ 0 & \text{else} \end{cases}$$

f is the composition of continuous functions on $(0,1]$, so it is continuous on that interval. At $x = 0$, we noting that for all $x \in (0,1]$, we have

$$-x \leq x\sin(\frac{1}{x}) \leq x$$

implying that $\lim_{x\to 0^+} f(x) = 0 = f(0)$ by the Squeeze theorem. This implies that $f(x)$ is continuous at $x = 0$, and thus $[0,1]$. $\frac{1}{x}$ is differentiable on $\mathbb{R} - 0$, so $f(x)$ is differentiable on $(0,1]$.

Taking the definition of derivative to attempt to evaluate $f'(0)$,

$$f'(0) = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \sin(\frac{1}{x})$$

which does not exist. Thus $f(x)$ is differentiable on $(0,1]$.

Consider the function

$$g(x) = f(x) + f(1-x)$$

This consists of f and f reflected about the line $x = \frac{1}{2}$ added together. From the above, g is continuous on $[0,1]$, and differentiable on $(0,1)$, but not 0 or 1.

Part c

Does the Mean Value Theorem apply to such a function?

Yes, since the Mean Value Theorem only requires the function to be differentiable on the open interval. In this case, the Mean Value Theorem states there is a point $\theta \in (0,1)$ such that $g'(\theta) = 0$. We can probably prove that a point exists by using the Intermediate Value Theorem on $g'(x)$ since it's continuous on $(0,1)$, but I'm too lazy at the moment.

Problem 10

Concoct a function $f : \mathbb{R} \to \mathbb{R}$ with a discontinuity of the second kind at $x = 0$ such that f does not have the intermediate value property there. Infer that it is incorrect to assert that functions without jumps are Darboux continuous.

Consider the function

$$f(x) = \begin{cases} x & x \in \mathbb{R} - \mathbb{Q} \\ 1 & \text{else} \end{cases}$$

f is continuous at $x = 1$ and discontinuous everywhere else. These discontinuities are discontinuities of the second kind, since left and right limits don't exist when x is not 1. $f(x)$ clearly does not have the intermediate value property, as except for 1, f assumes no rational values. Since this is a function without jump discontinuities but does not possess the intermediate value property, functions without jumps are not necessarily Darboux continuous.

Problem 11

Let $f : (a, b) \to \mathbb{R}$ be given.

Part a

If $f''(x)$ exists, prove that

$$\lim_{h\to 0} \frac{f(x-h) - 2f(x) + f(x+h)}{h^2} = f''(x)$$

Denote $F(x) = \lim_{h\to 0} \frac{f(x-h)-2f(x)+f(x+h)}{h^2}$. Since f is twice differentiable, we take take a second-order Taylor expansion of f around x, getting

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + R(x)$$

where $R(x)$ is second-order flat at $h = 0$, i.e. $\lim_{h\to 0} R(x)/h^2 = 0$. Similarly,

$$f(x-h) = f(x) - hf'(x) + \frac{1}{2}h^2 f''(x) + S(x)$$

where $S(x)$ is second-order flat at $h = 0$. Substituting,

$$F(x) = \lim_{h\to 0} \frac{h^2 f''(x) + R(x) + S(x)}{h^2} = f''(x)$$

since the $f(x)$ and $hf'(x)$ terms cancel, and $R(x)$ and $S(x)$ are second-order flat.

Part b

Find an example that this limit can exist even when $f''(x)$ fails to exist.

Let $f(x) = x|x|$. Taking the first derivative, when $x > 0$, $f(x) = x^2$, so $f'(x) = 2x$. Similarly, when $x < 0$, $f'(x) = -2x$. When $x = 0$,

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h|h|}{h} = \lim_{h\to 0} |h| = 0$$

Thus

$$f'(x) = \begin{cases} 2x & x \geq 0 \\ -2x & x < 0 \end{cases}$$

As previously stated, $f''(0)$ does not exist, since

$$f''(0) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h\to 0} \frac{f'(h)}{h}$$

which does not exist, since the limit from the positive direction is 2 and the limit from the negative direction is -2.

Despite this, the partial difference approximation exists at $x = 0$. The partial difference approximation from the right is

$$\lim_{h\to 0^+} \frac{f(-h) + f(h)}{h^2} = \lim_{h\to 0^+} \frac{-h| - h| + h|h|}{h^2} = \lim_{h\to 0^+} \frac{0}{h^2} = \infty$$

Similarly,

$$\lim_{h\to 0^-} \frac{f(-h) + f(h)}{h^2} = \lim_{h\to 0^-} \frac{h|h| + -h| - h|}{h^2} = \lim_{h\to 0^-} \frac{0}{h^2} = \infty$$

Thus the difference approximation exists at $x = 0$, even though $f''(0)$ does not exist.

Problem 15

Define $f(x) = x^2$ if $x < 0$ and $f(x) = x + x^2$ if $x \geq 0$. Differentiation gives $f''(x) = 2$. This is bogus. Why?

By the Fundamental Theorem of Calculus, if G is an antiderivative of g, then g equals the derivative of G where g is continuous. In this case, the standard power rule only applies when $x \neq 0$, since there is a discontinuity there.

Specifically, we have $f''(0)$ does not exist, since $f'(x) = 2x$ when $x \geq 0$, and $f'(x) = 2x + 1$ when $x < 0$. $f'(x)$ is discontinuous at $x = 0$, so its derivative does not exist there.

Problem 16

$\log x$ is defined to be $\int_1^x 1/tdt$ for $x > 0$. Using only the mathematics explained in this chapter,

Part a

Prove that log is a smooth function.

By the Fundamental Theorem of Calculus, the indefinite integral of a Riemann integrable function is continuous with respect to x. Thus, $\log x$ is continuous. Its derivative, again by the Fundamental Theorem of Calculus, is $\frac{d}{dx}\int_1^x 1/tdx = 1/x$ when $x > 0$, which is continuous. $1/x$ itself is smooth, so it has derivatives of all orders, which are continuous. Thus $\log x$ is smooth.

Part b

Prove that $\log(xy) = \log x + \log y$ for all $x, y > 0$.

For any given $y > 0$, define $f(x) = \log xy - \log x - \log y$. By definition,

$$\begin{aligned} f(x) &= \int_1^{xy} 1/tdt - \int_1^x 1/tdt - \int_1^y 1/tdt \\ &= \int_x^{xy} 1/tdt - \int_1^y 1/tdt \end{aligned}$$

When $x = 1$, $f(x) = \int_1^y 1/tdt - \int_1^y 1/tdt = 0$.

We now evaluate $f'(x)$. Splitting the integrals, for all $x > 0$, we can find a constant $0 < c < x$. Then

$$f(x) = \int_c^{xy} 1/tdt - \int_c^x 1/tdt - \int_1^y 1/tdt$$

By the Fundamental Theorem of Calculus, $\frac{d}{dx}\int_c^x 1/tdt = 1/x$ since $1/t$ is continuous on $[c, \infty)$. By the Chain Rule, $\frac{d}{dx}\int_c^{xy} 1/tdt = y\frac{1}{xy} = 1/x$. $\int_1^y 1/tdt$ is constant with regards to x, and thus has derivative zero. Thus, $f'(x) = 0$ for all $x > 0$. The only functions with derivatives equal to zero everywhere are constant functions, and since $f(1) = 0$, this implies that $f(x) = 0$. Thus $\log xy = \log x + \log y$.

Part c

Prove that log is strictly monotone increasing and its range is all of $\mathbb{R}$.

$\frac{d}{dx}\log x = 1/x$, which is strictly positive for all $x > 0$. Thus $\log x$ is strictly monotone increasing.

We know that $\log(1) = 0$. Going to the right, let $a_k = \frac{1}{k}$. Because $\frac{1}{t}$ is decreasing, for all $t \in [k, k+1]$, $\frac{1}{t} \leq a_{k+1}$. Thus because $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{k}$ diverges to infinity, by the Integral Test, $\int_1^{\infty} \frac{1}{t}dt$ diverges to infinity. This means that there is for large x, $\log(x) = \int_1^x \frac{1}{t}dt$ can be made arbitrarily large. This implies that when $x \geq 0$, $\log(x)$ takes on all values in $[0, \infty)$.

Going to the left, for $x \in (0, 1]$, $\log(x) = -\int_x^1 \frac{1}{t}dt$. Let $k \in \mathbb{N}$ and consider $\log(\frac{1}{2^k}) = -\int_{\frac{1}{2^k}}^1 \frac{1}{t}dt$.

To evaluate $\int_{\frac{1}{2^k}}^1 \frac{1}{t}dt$, consider the partition P such that $x_i = \frac{1}{2^i}$ for $i \in \mathbb{N}$. Thus $x_0 = 1$, $x_1 = \frac{1}{2}$, $x_2 = \frac{1}{4}$, etc. Because $\frac{1}{t}$ is strictly decreasing, the minimum of $\frac{1}{t}$ occurs at the right endpoint of the interval. Thus the lower integral is greater than or equal to

$$
\begin{aligned}
&1(\frac{1}{2}) + 2(\frac{1}{4}) + 4(\frac{1}{8}) \ldots \\
=&\frac{1}{2} + \frac{1}{2} + \frac{1}{2} \ldots \\
=&\frac{k}{2}
\end{aligned}
$$

because there are k intervals. Since $\frac{1}{t}$ is Riemann integrable on $(0,1]$, $\frac{k}{2}$ is a lower bound for the integral. Thus

$$
-\int_{\frac{1}{2^k}}^{1} \frac{1}{t} dt \leq -\frac{k}{2}
$$

which implies that the integral goes to negative infinity as k goes to infinity. Thus

$$
-\int_{0}^{1} \frac{1}{t} dt = -\infty
$$

which implies that as x approaches zero, $\log(x)$ approaches negative infinity. Thus on $(0,1]$, $\log(x)$ takes on all values in $(-\infty, 0]$. Putting the two statements together implies that the range of $\log(x)$ is all of $\mathbb{R}$.

Problem 17

Define $E : \mathbb{R} \to \mathbb{R}$ by

$$
E(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}
$$

Part a

Prove that $E(x)$ is smooth; that is, E has derivatives of all orders at all points x.

For $x < 0$, smoothness is trivial. A quick application of the chain rule shows that on $x > 0$,

$$
E'(x) = \frac{1}{x^2} e^{-\frac{1}{x}}
$$

Theorem 1 *For $x > 0$, $E^{(n)}(x)$ has the form*

$$
(a_{n+1}x^{-(n+1)} + a_{n+2}x^{-(n+2)} + \cdots + a_{2n}x^{-2n})e^{-\frac{1}{x}}
$$

for all $n \in \mathbb{N}$.

Proof: *The base case $n = 1$ has been established above. Assume that the hypothesis holds for $n-1$. Then*

$$E^{(n-1)}(x) = (a_n x^{-n} + a_{n+1}x^{-(n+1)} + \cdots + a_{2n-2}x^{-(2n-2)})e^{-\frac{1}{x}}$$

Using the Product Rule,

$$\begin{aligned} E^{(n)}(x) &= [(-na_n x^{-(n+1)} - (n+1)a_{n+1}x^{-(n+2)} - \cdots - (2n+2)a_{2n-2}x^{-(2n-1)}) \\ &\quad + (a_n x^{-(n+2)} + a_{n+1}x^{-(n+3)} + \cdots + a_{2n-2}x^{-2n})]e^{-\frac{1}{x}} \\ &= (b_{n+1}x^{-(n+1)} + b_{n+2}x^{-(n+2)} + \cdots + b_{2n}x^{-2n})e^{-\frac{1}{x}} \end{aligned}$$

since n is a constant. □

Lemma 2 $\lim_{x\to 0} E(x) = E(0) = 0$. *Thus $E(x) \in C^0$.*

Proof: *The left limit is trivially zero. On the right, as x approaches zero from the positive direction, $-\frac{1}{x}$ approaches negative infinity, so $e^{-\frac{1}{x}}$ approaches zero.* □

Lemma 3 $\lim_{x\to 0^+} \frac{1}{x}e^{-\frac{1}{x}} = 0$.

Proof:

$$\lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}}{x} = \lim_{x\to 0^+} \frac{x^{-1}}{e^{\frac{1}{x}}} = \lim_{x\to 0^+} \frac{-x^{-2}}{-x^{-2}e^{\frac{1}{x}}} = \lim_{x\to 0^+} e^{-\frac{1}{x}} = 0$$

by using l'Hopital's rule on the second expression. □

Lemma 4 $\lim_{x\to 0^+} \frac{1}{x^n}e^{-\frac{1}{x}} = 0$ *for $n \in \mathbb{N}$.*

Proof: *The base case has been established. For the inductive case, assume that $\lim_{x\to 0^+} \frac{1}{x^{n-1}}e^{-\frac{1}{x}} = 0$. Then*

$$\lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}}{x^n} = \lim_{x\to 0^+} \frac{x^{-n}}{e^{\frac{1}{x}}} = \lim_{x\to 0^+} \frac{-nx^{-n-1}}{-x^{-2}e^{\frac{1}{x}}} = n \lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}}{x^{n-1}} = 0$$

by the inductive hypothesis. □

Corollary 5 $\lim_{x\to 0^+} E^{(n)}(x) = 0$ *when $x > 0$ for all $n \in \mathbb{N}$.*

Proof: *By Theorem 1, $E^{(n)}(x)$ is the sum of various terms of the form $\frac{1}{x^n}e^{-\frac{1}{x}}$, where $n \in N$. By Lemma 4, each of these terms has right limit zero. Since $\frac{1}{x^n}e^{-\frac{1}{x}}$ is the sum of a finite number of these terms, it has right limit zero.* □

Theorem 6 $E^{(n)}(0)$ *exists and it equals zero for all $n \in \mathbb{N}$. $E^{(n)}(x)$ is continuous at $x = 0$ for all $n \in \mathbb{N}$, thus making $E(x)$ smooth.*

Proof: *For $n \in N$, we need to evaluate*

$$\lim_{x\to 0} \frac{E^{(n-1)}(x) - 0}{x - 0} = \lim_{x\to 0} \frac{E^{(n-1)}(x)}{x}$$

The left limit is zero, since $E^{(n-1)}(x) = 0$ for $x \le 0$. For the right limit, by Theorem 1, $\frac{1}{x}E^{(n-1)}(x)$ has the form

$$(a_{n+2}x^{-(n+2)} + a_{n+3}x^{-(n+3)} + \cdots + a_{2n+1}x^{-2n+1})e^{-\frac{1}{x}}$$

By repeated application of Lemma 4, we see that this has right limit zero. Thus

$$E^{(n)}(0) = \lim_{x\to 0} \frac{E^{(n-1)}(x)}{x} = 0$$

Because $E^{(n)}(x) = 0$ on $x \le 0$, $E^{(n)}(0) = 0$, and $E^{(n)}(x)$ is continuous at $x = 0$ for all $n \in \mathbb{N}$. Combined with smoothness everywhere else, this implies that $E(x)$ is smooth everywhere. □

Part b

Is $E(x)$ analytic?

No. A function f defined on an open interval (a, b) is analytic at $x \in (a, b)$ if it equals its a power series in a neighborhood of x. More specifically, for every x there exists a $\delta > 0$ such that $|h| < \delta$ implies that

$$f(x + h) = \sum_{r=0}^{\infty} a_r h^r$$

where $a_r = \frac{f^{(r)}(x)}{r!}$.

For $E(x)$ at $x = 0$, $E^{(n)}(0) = 0$ for all whole numbers n, as established in Part a. Thus $a_r = 0$ for all whole numbers r, and the power series is just 0. However, for $h > 0$, $f(h) \ne 0$ since $e^{-\frac{1}{x}}$ is a strictly positive function. Thus $E(x)$ is not analytic.

Problem 29

Prove that the interval $[a, b]$ is not a zero set.

Part a

Explain why the following observation is not a solution to the problem: "Every open interval that contains $[a, b]$ has length $> b - a$."

This 'solution' does not consider the possibility that there is a union of open sets that cover $[a, b]$ such that their sum of their lengths can be made arbitrarily small.

Part b

Instead, suppose there is a "bad" covering of $[a, b]$ by open intervals $\{I_i\}$ whose total length is $< b-a$, and justify the following steps in the proof by contradiction.

I will define a good covering as a covering of $[a, b]$ by open intervals $\{J\}$ such that the total length of the intervals in $\{J\}$ is greater than or equal to $b - a$.

i

It is enough to deal with finite bad coverings.

Let $\{I\}$ be an infinite bad covering of $[a, b]$. Because $\{I\}$ is an open cover of compact $[a, b]$, it reduces to a finite subcovering $\{I_i\}$. Thus, either $\{I\}$ reduces to a finite bad covering, or it reduces to a good covering. If $\{I\}$ reduces to a good covering $\{J_i\}$, then $\{J_i\} \subset \{I\}$ and the sum of the intervals in $\{J_i\}$ being $\geq b - a$ implies that the sum of the intervals in $\{I\}$ is $\geq b - a$. Thus $\{I\}$ is an infinite good covering, which contradicts the assumption that $\{I\}$ is a bad covering.

Thus, if $\{I\}$ is an infinite bad covering, it reduces to a finite bad covering. Contrapositively, if there are no finite bad coverings, then there are no infinite bad coverings, and the theorem is proven.

ii

Let $\mathbb{B} = \{I_1, \ldots I_n\}$ be a bad covering such that n is minimal among all bad coverings.

There is at least one finite bad covering, by assumption. $n = 1$ is a lower bound for the size of bad coverings. Then because $\mathbb{R}$ is complete, there exists a greatest lower bound for the sizes of the bad coverings, denoted c.

The must be a finite bad covering $\{C\}$ such that the size of $|\{C\}| = c$. Suppose not. Then all bad coverings have size $> c$, and size the sizes of the bad coverings must be integers, all bad coverings have size $\geq c+1$. This contradicts the assumption that c is a greatest lower bound. This bad covering $\{C\}$ is the bad covering with minimal n among all bad coverings.

iii

Show that no bad covering has $n = 1$ so we have $n \geq 2$.

This follows from the observation in Part a.

iv

Show that it is no loss of generality to assume $a \in I_1$ and $I_1 \cap I_2 \neq \emptyset$.

There exists at least one interval such that $a \in I_j$, and we are free to denote that interval I_1.

There must exist an interval that intersects I_1. Suppose not. Let d_1 be the right endpoint of I_1, and let $c_2, c_3 \ldots c_n$ be the left endpoints of the other intervals in the bad covering, and let $c = \min\{c_1 \dots c_n\}$. Then $\frac{c-d}{2}$ is not covered by the bad covering, contradicting the assumption that $\{I\}$ is a covering. Thus, there exists an interval in $\{I\}$ that intersects I_1. Denote it I_2. By construction, $I_1 \cap I_2$ is nonempty.

v

Show that $I = I_1 \cup I_2$ is an open interval and $|I| < |I_1| + |I_2|$.

If $I_1 \subset I_2$ or $I_2 \subset I_1$, $I_1 \cup I_2$ is trivially an open interval. Otherwise, $I_1 \cup I_2$ is the open because it is the union of open sets, connected because it is the union of two connected sets with a common point, and bounded because it is the finite union of bounded sets. Therefore $I_1 \cup I_2$ is a open, connected, and bounded subset of $\mathbb{R}$, and by the theorems shown in Chapter 2 Problem 31, open, connected, and bounded subsets of $\mathbb{R}$ are open intervals.

Lemma 7 *Let $C, D \subset \mathbb{R}$ be (bounded) intervals that intersect, and let $E = C + D$. Then $|E| < |C| + |D|$.*

Proof: *If C is a subset of D or vice versa, the proof is trivial. Without loss of generality, let the left endpoint of C be less than the left endpoint of D. Denote c as the right endpoint of C, and d the left endpoint of D. $d < c$, otherwise the two intervals do not intersect. Letting $\epsilon = c - d > 0$, the total length of E is $|C| + |D| - \epsilon$, which is strictly less than $|C| + |D|$.* □

By using the above Lemma, we see that $|I| < |I_1| + |I_2|$.

vi

Show that $\mathbb{B}' = \{I, I_3, \dots I_n\}$ is a bad covering of $[a, b]$ with fewer intervals, contradicting the minimality of n.

Let $x \in [a, b]$. Since $\mathbb{B}$ is a covering of $[a, b]$, there exists $i \in 1, 2 \dots n$ such that $x \in I_i$. If $i \geq 3$, then because $I_i \in \mathbb{B}'$, x is also covered by $\mathbb{B}'$. If $i = 1, 2$, then $x \in I = I_1 \cup I_2$, so x is still covered by $\mathbb{B}'$. $\mathbb{B}'$ is a covering by open intervals, because I is an open interval. $\mathbb{B}'$ is a bad covering. $|I| < |I_1| + |I_2|$ implies that $|I| + \sum_{j=3}^{n} I_j < \sum_{i=1}^{n} I_i < b - a$, implying that the total length of $\mathbb{B}'$ is less than the total length of $\mathbb{B}$. Thus $\mathbb{B}'$ is a bad covering with fewer intervals than $\mathbb{B}$, contradicting the assumption that $\mathbb{B}$ is the minimal bad covering. Thus, there are no bad coverings of $[a, b]$, coverings of $[a, b]$ can not have arbitrarily small length, and $[a, b]$ is not a zero set.

Problem 30

The standard **middle-quarters Cantor set** Q is formed by removing the middle quarter from $[0, 1]$, then removing the middle quarter from each of the remaining two intervals, then removing the middle quarter from each of the remaining four intervals, and so on.

Part a

Prove that Q is a zero set.

Let Q_n be the nth stage of the construction of Q, and let $|Q_n|$ be the total length of the intervals in Q_n.

Q_1 consists of two intervals with total length of 3/8, so $|Q_1| = 3/4$. Q_2 consists of four intervals with length 9/64 so $|Q_2| = 9/16$. In general, Q_n consists of 2^n closed intervals of length $(\frac{3}{8})^n$, thus implying that $|Q_n| = (\frac{3}{4})^n$.

Let $\epsilon > 0$ be arbitrary, and choose n such that $|Q_n| = (\frac{3}{4})^n < \epsilon$. This implies that $(\frac{3}{8})^n < \frac{\epsilon}{2^n}$. Since the length of an interval in Q_n is $(\frac{3}{8})^n$, this means that we can replace each interval I_i in Q_n with a slightly larger open interval $I_i \subset (a_i, b_i)$ such that $b_i - a_i < \frac{\epsilon}{2^n}$. Since there are 2^n intervals in Q_n, the total length of all these open intervals is ϵ. Since Q is a subset of Q_n, we have thus covered Q with open intervals of arbitrary small length. Thus Q is a zero set.

Part b

Formulate the natural definition of the middle-β Cantor set.

I will define the standard middle-β Cantor set, for $\beta \in (0, 1)$, as the set formed by removing the middle β from $[0, 1]$, then removing the middle β from each of the remaining two intervals, then removing the middle β from each of the remaining four intervals, and so on.

Part c

Is this also a zero set? Prove or disprove.

This is also a zero set. The proof is a generalization of Part a. Let B be the middle-β Cantor set, and let $\beta \in (0, 1)$ be fixed. Let B_n and $|B_n|$ be as in Part a.

B_1 consists of two intervals with total length of $\frac{1}{2}(1-\beta)$, so $|B_1| = 1 - \beta$. Q_2 consists of four intervals with length $\frac{1}{4}(1-\beta)^2$ so $|B_2| = (1-\beta)^2$. In general, B_n consists of 2^n closed intervals of length $(\frac{1-\beta}{2})^n$, thus implying that $|B_n| = (1-\beta)^n$.

Let $\epsilon > 0$ be arbitrary, and choose n such that $|B_n| = (1-\beta)^n < \epsilon$. This is possible because $\beta \in (0, 1)$ implies that the sequence $(1-\beta)^n \to 0$. Multiplying both sides by $1/2^n$ shows that that $(\frac{1-\beta}{2})^n < \frac{\epsilon}{2^n}$. Since the length of an interval in B_n is $(\frac{1-\beta}{2})^n$, this means that we can replace each interval I_i in B_n with a slightly larger open interval $I_i \subset (a_i, b_i)$ such that $b_i - a_i < \frac{\epsilon}{2^n}$. Since there are 2^n intervals in B_n, the total length of all these open intervals is ϵ. For similar reasons to Part a, this implies that B is a zero set.

Problem 31

Define a Cantor set by removing from $[0, 1]$ the middle interval of length 1/4. From the remaining two intervals F^1 remove the middle intervals of length 1/16.

From the remaining four intervals F^2 remove the middle intervals of length $1/64$, and so on. At the nth step in the construction F^n consists of 2^n subintervals of F^{n-1}.

Part a

Prove that $F = \cap F^n$ is a Cantor set but not a zero set. It is referred to as a **fat Cantor set**.

To start, I will calculate the lengths of the intervals in F_n. Let l_n be the lengths of the intervals in F_n. $l_0 = 1$. For l_1, since we remove a length of $1/4$ and split the remaining length in half,

$$l_1 = \frac{1}{2}(1 - \frac{1}{4}) = \frac{3}{8}$$

For l_2, we remove $1/16$ from the length of l_1 and split the remainder in half, so

$$l_2 = \frac{1}{2}(l_1 - \frac{1}{16}) = \frac{1}{2}(\frac{1}{2}(1 - \frac{1}{4}) - \frac{1}{16}) = \frac{1}{2^2}(1 - \frac{1}{4}) - \frac{1}{32}$$

Similarly for l_3,

$$\begin{aligned} l_3 &= \frac{1}{2}(l_2 - \frac{1}{4^3}) = \frac{1}{2}(\frac{1}{2^2}(1 - \frac{1}{4}) - \frac{1}{2 \times 4^2} - \frac{1}{4^3}) \\ &= \frac{1}{2^3}(1 - \frac{1}{4}) - \frac{1}{2^2 \times 4^2} - \frac{1}{2 \times 4^3} \end{aligned}$$

Generally, for $n \geq 1$, we have the recursive relation

$$l_n = \frac{1}{2}(l_{n-1} - \frac{1}{4^n})$$

We now begin substituting. Substituting $l_{n-1} = \frac{1}{2}(l_{n-2} - \frac{1}{4^{n-1}})$,

$$l_n = \frac{1}{2}(\frac{1}{2}(l_{n-2} - \frac{1}{4^{n-1}}) - \frac{1}{4^n}) = \frac{1}{2^2}(l_{n-2} - \frac{1}{4^{n-1}}) - \frac{1}{2 \times 4^n}$$

Substituting $l_{n-2} = \frac{1}{2}(l_{n-3} - \frac{1}{4^{n-2}})$,

$$\begin{aligned} l_n &= \frac{1}{2^2}(\frac{1}{2}(l_{n-3} - \frac{1}{4^{n-2}}) - \frac{1}{4^{n-1}}) - \frac{1}{2 \times 4^n} \\ &= \frac{1}{2^3}(l_{n-3} - \frac{1}{4^{n-2}}) - \frac{1}{2^2 \times 4^{n-1}} - \frac{1}{2 \times 4^n} \end{aligned}$$

Continuing this $n - 1$ times until the the l_{n-1} term reduces to l_0,

$$
\begin{aligned}
l_n &= \frac{1}{2^n}(l_0 - \frac{1}{4}) - \frac{1}{2^{n-1} \times 4^2} - \frac{1}{2^{n-2} \times 4^3} - \cdots - \frac{1}{2^2 \times 4^{n-1}} - \frac{1}{2 \times 4^n} \\
&= \frac{3}{2^{n+2}} - \sum_{k=1}^{n-1} \frac{1}{2^k \times 4^{n+1-k}} = \frac{3}{2^{n+2}} - \frac{1}{4^{n+1}} \sum_{k=1}^{n-1} 2^k \\
&= \frac{3}{2^{n+2}} - \frac{1}{4^{n+1}}(2^n - 2) = \frac{1}{2^{n+1}} + \frac{1}{2^{2n+1}}
\end{aligned}
$$

This matches up with the heuristic evaluation of lengths. The total length of F_n for $n \geq 1$ is

$$
|F_n| = 1 - \frac{1}{4} - \frac{1}{8} - \cdots - \frac{1}{2^{n+1}} = 1 - \frac{1}{2} \sum_{k=1}^{n} \frac{1}{2^k}
$$

which approaches $\frac{1}{2}$ as n goes to infinity. Similarly, by summing up the 2^n intervals, the total length of F_n is $\frac{1}{2} + \frac{1}{2^{n+1}}$, which approaches to $\frac{1}{2}$ as n goes to infinity.

By the Moore-Kline Theorem, if I can show that F is compact, nonempty, perfect, and totally disconnected, then it is homeomorphic to the standard middle-thirds Cantor set. The discussion as follows essentially follows the proof that the standard Cantor set is compact, nonempty, perfect, and totally disconnected.

F is the intersection of compacts, so it is compact. It contains the point 0, so it is nonempty. Let E be the set of all endpoints that are contained in some F^n. E is infinite, since each iteration n except for the zeroth introduces 2^n new endpoints to E.

To show that F is perfect, take an arbitrary $x \in F$ and $\epsilon > 0$, and let n be large enough such that $\frac{1}{2^{n+1}} + \frac{1}{2^{2n+1}} < \epsilon$. Since $F \subset F^n$, x lies in one of the 2^n intervals of length $\frac{1}{2^{n+1}} + \frac{1}{2^{2n+1}}$ that make up F^n. Keeping I fixed, $E \cup I$ is infinite, since by blowing up I to $[0, 1]$ we can fit a miniature copy of F in I, and E is infinite. $E \cup I \subset F$, and $E \cup I \subset (x - \epsilon, x + \epsilon)$ imply that F clusters at x, and so F is perfect.

For totally disconnected, keep I fixed and consider F^n. I is a closed interval in F^n, and $J = F^n - I$ is the union of finite closed intervals, so it's closed. Thus I and J are clopen in F^n, and by the Inheritance Principle, $I \cup F$ is clopen in F. Since I is a subset of an arbitrary small epsilon-ball, $I \cup F$ is an arbitrarily small clopen subset. Thus F is totally disconnected. By the Moore-Kline Theorem, F is homeomorphic to the standard Cantor set.

I have no idea how to rigorously prove that F is not a zero set. Intuitively, F should have length (or outer measure) $\frac{1}{2}$, as shown in the above calculation. However, I don't know if that's rigorous enough, or if I have to show it using a proof by contradiction similar to Problem 29. Even if I can show that all of the F^n can not be covered by open intervals with total length of $\frac{1}{2}$, the implication that F can not be covered the same way does not necessarily follow.

Problem 34

Assume that $\psi : [a, b] \to \mathbb{R}$ is continuously differentiable. A critical point of ψ is an x such that $\psi'(x) = 0$. A critical value is a number y such that for at least one critical point x we have $y = \psi(x)$.

Part a

Prove that the set of critical values is a zero set. (This is the Morse-Sard Theorem in dimension one.)

I will first introduce some notation. Let $f : [a, b] \to \mathbb{R}$ be continuous. I will define a **zero of type 1** of f to be a zero of f such that f is uniformly zero in an open neighborhood of the root. In other words, if $f(x) = 0$, then there exists $\epsilon > 0$ such that for all y such that $|x - y| < \epsilon$, $f(y) = 0$. I will denote a **zero point of type 2** of f as all other zeros of f. It's clear that the disjoint union of zeros of types 1 and 2 make up all zeros of f.

Let ψ be continuously differentiable. We will characterize the critical values of ψ based on the zeros of type 1 and 2 of ψ'. If $\psi(x) = y$ is a critical value and $\psi'(x)$ is a zero of type 1, we say that y is a **critical value of type 1** of f. Similarly, if $\psi(x) = y$ is a critical value and $\psi'(x)$ is a zero of type 2, we say that y is a **critical value of type 2** of f. Since the zeros of type 1 and 2 partition the set of zeros of ψ', the critical values of type 1 and 2 partition the set of critical values of ψ.

I have no idea if this characterization is standard, but that's what I've come up with.

The immediate characterization for zeros of type 2 is stated below.

Lemma 8 *Let f be continuous, and let x be a zero of type 2 of f. Then for all $\epsilon > 0$, there exists $y \in (x - \epsilon, x + \epsilon)$ such that $f(y) \neq 0$.*

Proof: *If this is not true, then x is a zero of type 1.* □

To begin with zero points of type 2, we next state a lemma on non-zero points of continuous functions implying an interval with no zeros. This can be thought of as non-zero points of continuous functions creating 'exclusion zones' with a delta-radius that contain no zeros.

Lemma 9 *Let $f : [a, b] \to \mathbb{R}$ be continuous, and let $x \in [a, b]$ be a point such that $f(x) \neq 0$. Then there exists a $\delta > 0$ such that f has no zeros in $(x - \delta, x + \delta)$.*

Proof: *Because f is uniformly continuous, there exists a $\delta > 0$ such that for all y such that $|x - y| < \delta$, $|f(x) - f(y)| < |f(x)|$. This implies that y is not a zero of f.* □

Lemma 10 *Let $f : [a, b] \to \mathbb{R}$ be continuous, and let $x \in [a, b]$ such that $f(x) \neq 0$. Then f is not a clustering point of zeros. In other words, we can reasonably speak of the nearest zero of f greater than x, and the nearest zero of f less than x.*

Proof: *Suppose not. Then there exists a sequence* $(x_n) \to x$ *of zeros of* f. $\lim_{n\to\infty} f(x_n) = 0$*, but* $f(x) \neq 0$*, violating the continuity of* f. □

We now introduce some useful terminology (that I have no idea whether is standard, but I am going to use it). Let $f : [a, b] \to \mathbb{R}$ be continuous, and let x be a point such that $f(x) \neq 0$. The **covering interval of x** is the open interval between the nearest zero of f to the left of x, and the nearest zero of f to the right of x. This interval covers x. By Lemma 10, this is a well-defined construction.

I will denote the interval as C. If there are no zeros of f to the left of x, then the left endpoint of C is a, and if there are no zeros to the right of x, then the right endpoint of C is b.

Lemma 11 *Let* $f : [a, b] \to \mathbb{R}$ *be continuous. Then the covering intervals of* f *are disjoint.*

Proof: *Any two covering intervals are separated by at least one zero of* f. □

We now show that the number of covering intervals is closely related to the number of zeros of type 2.

Lemma 12 *Let* x *be a zero of type 2. Then there is a covering interval such that* x *is the endpoint.*

Proof: *Suppose not. Then* x *is not the nearest zero of type 2 to any nonzero point of* f*. This means that* x *is a clustering point of zeros of type 2, which contradicts Lemma 10.* □

Corollary 13 *The set of zeros of type 2 is of equal or lesser cardinality to the set of covering intervals.*

Proof: *Each zero of type 2 belongs to at least one covering interval.* □

The main results for zeros of type 2 follows.

Corollary 14 *Let* f *be continuous on* $[a, b]$*. Then* f *has at most countable zeros of type 2.*

Proof: *Let* Q *be the set of covering intervals for* f*. Because* Q *is the disjoint union of intervals,* Q *is countable. By Lemma 13, the zeros of type 2 of* f *are countable.* □

Corollary 15 *Let* f *be continuously differentiable on* $[a, b]$*. Then* f *has at most countable critical values of type 2.*

Proof: *f' is continuous, implying that f has at most countable zeros of type 2. Each zero of type 2 of f' maps to at most one critical value of type 2 of f.* $\square$

We now turn to critical points of type 1. We first state a useful characterization of zeros of type 1.

Lemma 16 *Let $f : [a, b] \to \mathbb{R}$ be continuous, and let Z be the set of zeros of type 1. Then Z is the disjoint union of countable open intervals, with perhaps one or two half-open intervals at the endpoints a and b.*

Proof: *The definition of zeros of type 1 implies that Z is an open set in $[a, b]$. By the Inheritance Principle, there exists a set $W \subset \mathbb{R}$ that is open in $\mathbb{R}$ such that $W \cap \mathbb{R} = Z$. By Problem 31 in Chapter 2, an open set in $\mathbb{R}$ can be expressed as the disjoint union of countably many open intervals. Taking $W \cap Z$, open intervals that do not contain the endpoints a and b are still open in Z, while the half-intervals that have their closed end at a and b become open in $[a, b]$.* $\square$

We next state a lemma on critical points of type 1.

Lemma 17 *Let x be a critical point of type 1 for $\psi'(x)$. Then on the neighborhood where $\psi'(x) = 0$, there is only one critical value. Specifically, if $\psi'(x) = 0$ on an interval $(c, d) \subset [a, b]$, then $\psi(c)$ is the only critical value on that interval.*

Proof: *By the Fundamental Theorem of Calculus, for $x \in [c, d]$, $\psi(x) = \psi(c) + \int_c^x \psi'(x)dx = \psi(c)$ since $\psi'(x) = 0$ on the interval.* $\square$

We now want to prove the main theorem for critical values corresponding to critical points of type 1.

Theorem 18 *If $f : [a, b] \to \mathbb{R}$ is continuously differentiable, then f has countably many critical values of type 1.*

Proof: *f' is continuous by assumption. By Lemma 16, Z, the set of zeros of type 1 of f', consists of countable disjoint open intervals, with perhaps one or two half-open intervals at a and b. By Lemma 17, each (half)-open interval in Z corresponds to one critical value in f. Thus f has at most countably many critical values of type 1.* $\square$

Theorem 19 *The critical values of f form a zero set.*

Proof: *The union of countable sets is a countable set, which is a zero set.* $\square$

Part b

Generalize this result to continuous functions on $\mathbb{R} \to \mathbb{R}$.

The result immediately generalizes. Divide $\mathbb{R}$ into countably many intervals of length 1. By Part a, there are countably many critical values of f on each of these intervals, and the countable union of countable sets is a zero set. Thus the set of critical values of a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ is a zero set.

Problem 36

We say that $f : (a, b) \to \mathbb{R}$ has a **jump discontinuity** (or a discontinuity of the **first kind**) at $c \in (a, b)$ if

$$f(c^-) = \lim_{x \to c^-} f(x) \text{ and } f(c^+) = \lim_{x \to c^+} f(x)$$

exist, but are either unequal or are unequal to $f(c)$. An **oscillating discontinuity** (or a discontinuity of the **second kind** is any nonjump discontinuity).

Part a

Show that $f : \mathbb{R} \to \mathbb{R}$ has at most countably many jump discontinuities.

I will first start with showing that $f : [0, 1] \to \mathbb{R}$ has countably many jump discontinuities. Let $M(c)$ be defined as

$$M(c) = \begin{cases} \max\{|f(c^-) - f(c)|, |f(c^+) - f(c)|\} & \text{if } f \text{ has a jump discontinuity at } c \\ 0 & \text{else} \end{cases}$$

Let A_n be the set of points in $\mathbb{R}$ such that $M(c) > \frac{1}{n}$. It's clear that the set of jump discontinuities of f is $A = \cup_{n=1}^{\infty} A_n$.

For all $c \in A_n$, because c is a jump discontinuity, there exists $\delta_-, \delta_+ > 0$ such that $x \in (c - \delta_-, c)$ implies that $|f(x) - f(c^-)| < \frac{1}{2n}$, with a similar result for δ_+. Let $\delta = \min(\delta_-, \delta_+)$. Note that δ depends on c.

I claim that $(c - \delta, c + \delta) - \{c\} \subset A_n^C$. Let $x \in (c - \delta, c + \delta) - \{c\}$. If f is continuous at x, or if x is an oscillating discontinuity of x, the result is trivial.

If x is a jump discontinuity of x, then the left and right limits of f exist at x. Let $\gamma = \min\{|x - c|, |x - (c - \delta)|, |x - (c + \delta)|\}$. If p_n is an arbitrary sequence such that $p_n \to x$, then eventually the tail of p_n will lie entirely in $(x - \gamma, x + \gamma) \subset (c - \delta, c + \delta) - \{c\}$. Thus

$$\text{diam} f((x - \gamma, x + \gamma)) < \frac{1}{n}$$

implies that x is not in A_n. Thus A_n is a zero set, implying that A is a zero set. Thus $f : [0, 1] \to \mathbb{R}$ has at most countably many jump discontinuities. Repeating this argument for all $\mathbb{Z}$ shows that $f : \mathbb{R} \to \mathbb{R}$ has at most countably many jump discontinuities.

Part b

What about the function

$$f(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

f is continuous when $x \neq 0$, since it on the left it is a constant function, and on the right it is the composition of continuous functions. f has an oscillating discontinuity at $x = 0$. Consider the sequences $p_n = 1/(\frac{\pi}{2} + 2n)$ and $q_n = 1/(\frac{3\pi}{2} + 2n)$. Both of these sequences converge to zero, but $f(p_n) = 1$ and $f(q_n) = -1$ for all $n \in \mathbb{N}$. Thus the right limit of f does not exist at 0, and so $x = 0$ is an oscillating discontinuity.

Part c

What about the characteristic function of the rationals?

f is oscillating discontinuous everywhere. For at any point x, there are a convergent sequence of rationals, and a convergent series of irrationals approaching x, implying that the limit of f at x does not exist.

Problem 39

Consider the characteristic functions $f(x)$ and $g(x)$ of the intervals $[1, 4]$ and $[2, 5]$. The derivatives f' and g' exist almost everywhere. The integration by parts formula says that

$$\int_0^3 f(x)g'(x)dx = f(3)g(3) - f(0)g(0) - \int_0^3 f'(x)g(x)dx$$

But both integrals are zero, while $f(3)g(3) - f(0)g(0) = 1$. Where is the error?

The textbook integration by parts formula assumes that f and g are continuous on $[a, b]$. In this case, the interval in question is $[0, 3]$. However, f is discontinuous at $x = 1$, and g is discontinuous at $x = 2$.

Specifically, the Leibniz formula states that if f, g are continuous at x, then $(fg)'(x) = (f'g)(x) + (fg')(x)$. If f, g are continuous on the interval in question then the Leibniz formula says that $f'g + fg'$ is an antiderivative of fg, Then by the Antiderivative Theorem, the indefinite integral of $fg' + f'g$ differs from the antiderivative by a constant.

However, in this case, fg' is zero on $[0, 3]$, except at $x = 2$, where it is undefined. A similar thing holds for $f'g$ at $x = 1$. Since a potential antiderivative of fg' has to have its derivative equal fg' everywhere, not just almost everywhere, fg' and $f'g$ do not have antiderivatives.

In fact, because fg' is zero almost everywhere, $\int_0^3 f(x)g'(x)dx = 0$.