Chapter 4 Function Spaces

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In these exercises, $C^0 = C^0([a, b], \mathbb{R})$ is the space of continuous real-valued functions defined on the closed interval $[a, b]$. It is equipped with the usp norm, $||f|| = \sup\{|f(x)| : x \in [a, b]\}$.

Problem 1

Let M, N be metric spaces.

Part a

Formulate the concepts of pointwise convergence and uniform convergence for sequences of functions $f_n : M \to N$.

A sequence of functions $f_n : M \to N$ converges pointwise to a limit function $f : M \to N$ if for all $x \in M$ we have

$$\lim_{n\to\infty} d_n(f_n(x), f(x)) = 0$$

A sequence of functions converges uniformly to a limit function if for all $\epsilon > 0$, there is an N such that for all $n \geq N$ and all $x \in M$,

$$d_N(f_n(x), f(x)) < \epsilon$$

Part b

For which metric spaces are these concepts equivalent?

TODO. The immediate thing that springs to mind are trivial metric spaces with only one point,

Problem 3

Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of piecewise continuous functions, each of which is continuous at the point $x_0 \in [a, b]$. Assume that $f_n \rightrightarrows f$.

Part a

Prove that f is continuous at x_0.

The proof is as similar to Theorem 1 in the book. Let $\epsilon > 0$ be given. By uniform convergence, there exists an N such that for all $n \geq N$ and $x \in [a, b]$ we have

$$|f_n(x) - f(x)| < \frac{\epsilon}{3}$$

All the f_n are continuous at x_0, so f_N is continuous at x_0. This implies that there is a $\delta > 0$ such that $|x - x_0| < \delta$ implies

$$|f_N(x) - f_N(x_0)| < \frac{\epsilon}{3}$$

Thus, if $|x - x_0| < \delta$, then by the Triange inequality,

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

which shows that f is continuous at x_0.

Part b

Prove or disprove that f is piecewise continuous.

f is not piecewise continuous. A function $f : [a, b] \to \mathbb{R}$ is piecewise continuous if it has finitely many discontinuities.

Let $f : [0, 1] \to \mathbb{R}$ be the following function:

$$f(x) = \begin{cases} 0 & x \in \mathbb{R} - \mathbb{Q} \\ \frac{1}{q} & x = \frac{p}{q} : \gcd(p, q) = 1; p, q \in \mathbb{Z} \\ 1 & x = 0 \end{cases}$$

Let $f_n : [0, 1] \to \mathbb{R}$ be the rational ruler function. Specifically, for $n = 1, 2 \ldots$

$$f_n(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} : \gcd(p, q) = 1; p, q \in 1, 2 \ldots n \\ 1 & x = 0 \\ \frac{1}{n} & \text{else} \end{cases}$$

Thus f_1 is 1 everywhere, f_2 is 1 at 0 and 1 and 1/2 everywhere else, f_4 is 1 at 0 and 1, 1/2 at 1/2, 1/3 at 1/3 and 2/3, 1/4 everywhere else, etc.

$f_n(x) = f(x)$ when x is a rational number in reduced form with denominator $\leq n$. Everywhere else, $f(x) \geq 0$, and $f_n(x) = \frac{1}{n}$ imply $f_n(x) - f(x) \leq \frac{1}{n}$, which approaches zero as n goes to infinity. Thus $f_n \rightrightarrows f$. Similarly, f_n is piecewise continuous, since it only has $1 + 2 + 3 \cdots + n - 1$ discontinuities, which is finite. However, f is discontinuous at all rational numbers, and is thus is not piecewise continuous.

Problem 4

Part a

If $f_n : \mathbb{R} \to \mathbb{R}$ is uniformly continuous for each $n \in \mathbb{N}$ and if $f_n \rightrightarrows f$ as $n \to \infty$, prove or disprove that f is uniformly continuous.

f is uniformly continuous. Let $\epsilon > 0$ be arbitrary. Then by uniform convergence, there exists N such that $n \geq N$ implies that $||f - f_n||_{\sup} < \frac{\epsilon}{3}$. By the uniform continuity of f_n, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f_n(x) - f_n(y)| < \frac{\epsilon}{3}$, which is equivalent to $\max_{a \in [x,y]} f_n(a) - \min_{a \in [x,y]} f_n(a) < \frac{\epsilon}{3}$. Because $||f - f_n||_{\sup} < \frac{\epsilon}{3}$, this implies that for $|x - y| < \delta$,

$$\max_{|x-y|<\delta} f(y) - \min_{|x-y|<\delta} f(a) < \epsilon$$

which is equivalent to uniform continuity.

Part b

What happens for functions from one metric space to another instead of $\mathbb{R}$ to $\mathbb{R}$?

The same things happen. Let $f : M \to N$. The supremum norm is well defined for functions from M to N. For uniform continuity, there exists $\delta > 0$ such that $d_M(x, y) < \delta$ implies $d_N(f_n(x), f_n(y)) < \frac{\epsilon}{3}$, which is equivalent to

$$\sup_{d_M(x,y)<\delta} f_n(y) - \inf_{d_M(x,y)<\delta} f_n(y) < \frac{\epsilon}{3}$$

Combined with $||f - f_n||_{\sup} < \frac{\epsilon}{3}$, this implies that

$$\sup_{d_M(x,y)<\delta} f(y) - \inf_{d_M(x,y)<\delta} f(y) < \epsilon$$

which implies uniform continuity.

Problem 5

Suppose that $f_n : [a, b] \to \mathbb{R}$ and $f_n \rightrightarrows f$ as $n \to \infty$. Which of the following discontinuity properties of the functions f_n carry over to the limit function?

Part a

No discontinuities.

This is immediate. By the theorems in the book, the uniform limit of continuous functions is continuous.

Part b

At most ten discontinuities.

We begin with a lemma on how discontinuities in the limiting function imply discontinuities in the sequence.

Lemma 1 *Let f_n, f be as described in the problem, and let f be discontinuous at x_0. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, f_n is discontinuous at x_0.*

Proof: *Suppose not. Then for all $k \in \mathbb{N}$, there exists an $a > k$ such that f_a is continuous at x_0. By uniform convergence, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|f_n - f|_{\sup} < \frac{\epsilon}{3}$. Choose $n \geq N$ such that f_n is continuous at x_0.*

Let $\epsilon > 0$ be arbitrary. By the continuity of f_n, there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f_n(x) - f_n(x_0)| < \frac{\epsilon}{3}$. Because $n \geq N$, $|f_n - f|_{\sup} < \frac{\epsilon}{3}$ implies that for all $x \in (x_0 - \delta, x_0 + \delta)$,

$$|f(x) - f(x_0)| < \epsilon$$

implies that f is continuous at x_0, contradicting the assumption that f is discontinuous at x_0. Thus there is some $k \in \mathbb{N}$ such that for all $n \geq k$, f_n is discontinuous at x_0. $\square$

The statement is true by the contrapositive. If f has more than ten discontinuities, then by the above lemma, there exists some $N \in \mathbb{N}$ such that for all $n \geq N$, f_n has discontinuities at the discontinuities of f. Thus f having more than ten discontinuities implies the tail of f_n has more than ten discontinuities. Taking contrapositives, this implies that if the tail of f_n has at most ten discontinuities, f has at most ten discontinuities.

Part c

At least ten discontinuities.

No. Let the interval be $[0, 1]$ and f_n be the function

$$f_n = \begin{cases} \frac{1}{n} & \text{when } x \text{ in reduced form has denominator } 10^n \\ 0 & \text{else} \end{cases}$$

f_n has at least ten discontinuities for all n, but uniformly converges to the zero function, which has no discontinuities.

Part d

Finitely many discontinuities.

No. From Problem 3 Part b above, there are functions with finitely discontinuities but uniformly converge to a function with infinite discontinuities.

Problem 8

Is the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \cos(n + x) + \log(1 + \frac{1}{\sqrt{n+2}} \sin^2(n^n x))$$

equicontinuous? Prove or disprove.

IN PROGRESS.

We first start with a lemma on C^1 functions and equicontinuity.

Lemma 2 *Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of C^1 functions. Suppose that there exists $\epsilon > 0$ such that for all $\delta > 0$, there exists an interval (s, t) with $t - s < \delta$ and an $n \in \mathbb{N}$ such that*

$$f_n(x) \geq \frac{\epsilon}{t - s}$$

or

$$f_n(x) \leq -\frac{\epsilon}{t - s}$$

on the interval (s, t). Then (f_n) is not equicontinuous.

Proof: *Because $f_n \in C^1$ for all $n \in \mathbb{N}$, by the Fundamental Theorem of Calculus*

$$f_n(t) = f_n(s) + \int_s^t f_n'(x)dx$$

In the case where $f_n(x) > 0$,

$$|f_n(t) - f_n(s)| = f_n(t) - f_n(s) = \int_s^t f_n'(x)dx \geq \int_s^t \frac{\epsilon}{t - s} dx = \epsilon$$

which violates equicontinuity. The case where $f_n(x) < 0$ is analogous. □

We now give a result analogous to convergent sequences. The sum of a convergent and divergent sequence is divergent. Similarly, the sum of an equicontinuous sequence and a non-equicontinuous sequence is not equicontinuous.

Lemma 3 *If $f_n, g_n : \mathbb{R} \to \mathbb{R}$ be sequences of functions in C^0. If the f_n are equally continuous but the g_n are not equally continuous, the $(f + g)_n$ are not equally continuous.*

Proof: *Since the g_n are not equally continuous, there exists an $\epsilon > 0$ such that for all $\delta_g > 0$, there is an $n \in \mathbb{N}$ and interval $|s - t| < \delta_g$ such that $|g_n(s) - g_n(t)| > \epsilon$. Fix that epsilon and n. Because the f_n are equally continuous, there is a $\delta_f > 0$ such that for all intervals $|s - t| < \delta_f$, we have $|f_n(s) - f_n(t)| < \frac{\epsilon}{2}$.*

Let $\delta = \min(\delta_f, \delta_g)$. Fix the interval $|s - t| < \delta$ such that $|g_n(s) - g_n(t)| > \epsilon$. Since $\delta \leq \delta_f$, we also have $|f_n(s) - f_n(t)| < \frac{\epsilon}{2}$ on this interval. Thus on this interval,

$$|f_n(s) + g_n(s) - f_n(t) - g_n(t)| > \frac{\epsilon}{2}$$

Thus there exists $\epsilon/2 > 0$ *such that for all* $\delta > 0$, *there is an* $n \in N$ *and interval* (s,t) *with* $t-s < \delta$ *such that* $|f_n(s)+g_n(s)-f_n(t)-g_n(t)| > \epsilon/2$. *Thus the* $(f+g)_n$ *are not equally continuous.* $\square$

The derivative of f_n is

$$\begin{aligned} f_n'(x) &= -\sin(n+x) + \frac{\frac{1}{\sqrt{n+2}} n^n 2\sin(n^n x)\cos(n^n x)}{1 + \frac{1}{\sqrt{n+2}}\sin^2(n^n x)} \\ &= -\sin(n+x) + \frac{n^n \sin(2n^n x)}{\sqrt{n+2} + \sin^2(n^n x)} \end{aligned}$$

We now show that the cos part of the expression is equally continuous, but the log part is not equally continuous.

Theorem 4 $g(x) = \cos(n+x)$ *is equally continuous.*

Proof: *Let* $\epsilon = \delta$. $\cos(n+x) = C^1$, *so by the Fundamental Theorem of Calculus*

$$\begin{aligned} |\cos(n+t) - \cos(n+s)| = |-\int_s^t \sin(n+x)dx| &\le \int_s^t |\sin(n+x)|dx \\ &\le \int_s^t 1dx = t-s < \delta \end{aligned}$$

for all $n \in \mathbb{N}$. $\square$

Theorem 5 $h(x) = \log(1 + \frac{1}{\sqrt{n+2}}\sin^2(n^n x))$ *is not equally continuous.*

Insert something about how the log term is not equicontinuous due to the periodic nature, and how its period tends to zero as n goes to infinity. Since the cos term is equicontinuous (or at least tentatively it is), the whole things is not equicontinuous.