Chapter 2 A Taste of Topology

Problem 6

Determine whether $d_x(p, q) = \sin|p - q|$ on $[0, \frac{\pi}{2})$ is a metric.

Proof: It is a metric. Positive definiteness follows from the fact that the absolute value function is a metric over $\mathbb{R}$, and sine being one-to-one over the range of possible functions. Symmetry follows for the same reason. The triangle inequality follows from sine being increasing and concave over $[0, \frac{\pi}{2})$.

Specifically, let $p, r \in [0, \frac{\pi}{2})$, and without loss of generality, let $p \leq r$. If $q = p$ or $q = r$, the triangle inequality is trivial.

Let $q \notin [p, r]$. If $q > r$, then $q - p > r - p$ implies

$$d_s(p, q) + d_s(q, r) \geq d_s(p, r) + 0 =\geq d_s(p, r)$$

A similar result holds if $q < p$. If $q \in (p, r)$, imagine p, q, and r arranged on a line, with p at the origin. As x increases from q to r, the increase in sine is less than the corresponding increase from 0 to r-q, because sine is concave. Thus $\sin(r - p) < \sin(r - q) + \sin(q - p)$.

□

Problem 12

Let (p_n) be a sequence and $f : \mathbb{N} \to \mathbb{N}$ be a bijection. The sequence $(q_k)_{k\in\mathbb{N}}$ is a rearrangement of (p_n) with $q_k = p_{f(k)}$.

Part a

Are the limits of a sequence unaffected by rearrangement?

The limits of a sequence are unaffected by rearrangement.

Suppose that $(p_n) \to p$. Then for arbitrary $\epsilon > 0$, there exists an $n \in \mathbb{N}$ such that $i > n$ implies $d(p_i, p) < \epsilon$. This implies that there are at most finite elements of the sequence (p_n) such that $d(p_n, p) \geq \epsilon$. In the rearrangement (q_n), let M be the smallest integer such that $f^{-1}(k) \leq n$. M exists because $\{1, 2, 3 \ldots n\}$ is finite. Then for all $i > M$, $d(q_i, p) < \epsilon$, and thus $(q_i) \to p$.

On the other hand, suppose that (p_n) has no limit. Then for arbitrary p, there exists an $\epsilon > 0$ such that for all $n \in \mathbb{N}$, there exists $i > n$ such that $d(p_i, p) > \epsilon$. Letting $A(p)$ be the set of these points. A is infinite, because A being finite implies that A has a maximum element.

By contradiction, assume that that (q_n) has a limit of q. Then for all $\epsilon_q > 0$, there exists $n(\epsilon_q) \in \mathbb{N}$ such that $i > n(\epsilon_q)$ implies $d(q_i, q) < \epsilon_q$.

Let $\epsilon_q = \epsilon$. Then the set of points A_q such that $d(q_i, q) > \epsilon$ is a subset of $\{1, 2 \ldots n(\epsilon)\}$, and is thus finite. But because (q_n) is a rearrangement of (p_n), $A_q = A$. Thus A_q is both finite and infinite, which is a contradiction.

Part b

What if f is an injection?

At first glance, this question seems nonsensical, as for $n \in \mathbb{N}$ which are not in the preimage of $f(\mathbb{N})$, q_n is undefined. If we define such points to have $q_n = 0$, this question becomes intelligible.

Note that if A is an arbitrary set in $\mathbb{N}$, then $f(A)$ has the same cardinality as A, because the restriction of f on A is a bijection between the two.

The first result in Part a does not hold. Let $(p_n) = 1$, and let $f(x) = 2x$. The second result still holds, since the cardinality of the sets remains unchanged.

Part c

What if f is a surjection?

The first result in Part a holds. The cardinality of the image of a function must be less than or equal to the cardinality of the domain of the function. Letting $A = \{1, 2 \dots n\}$ and $B = \{n+1, n+2, n+3, \dots\}$, $f(A)$ is finite. Since f is surjective, it covers $\mathbb{N}$, which is infinite. $A + B = \mathbb{N}$ which is the domain of f. If $f(B)$ is finite, then $f(A) + f(B) = f(A + B) = f(\mathbb{N})$ is finite, which contradicts the assumption that f is surjective. Therefore $f(B)$ is infinite, and the argument in Part a holds.

The second result in Part b does not hold. Let $(p_n) = \{0, 0, 0, 1, 0, 1, 0, 1 \dots\}$ and let $f(x) = 1$ if x is odd or equals 2, and $\frac{1}{2}x$ if x is even and greater than 3. f is clearly surjective, and $(q_n) = \{0, 1, 1, 1, 1 \dots\}$ clearly has a limit of 1.

Problem 13

Show that if $f : M \to N$ is a function such that (p_n) converging in M implies that $(f(p_n))$ converges in N, then f is continuous.

Note that this is almost the definition of convergence, except for the requirement that $(f(p_n)) \to f(p)$. Thus showing that f is continuous is equivalent to showing the above requirement.

Proof: Let $p \in M$ be arbitrary. Let (a_n) be the uniform sequence $a_n = p$ for all $n \in \mathbb{N}$. From the properties of f, $(f(a_n))$ converges in N, and it converges to $f(p)$. If (a_n) is the only sequence in M that converges to p for all p in M (such as when the discrete metric is used), then f satisfies the sequential convergence condition and is thus continuous.

Let (b_n) be an arbitrary sequence in M such that $b_n \to p$. Construct the sequence (c_n) such that $c_n = a_{\lceil n/2 \rceil}$ if n is odd and $c_n = b_{n/2}$ if n is even. (c_n) clearly converges to p.

By the convergence preservation condition of f, $(f(c_n))$ converges in N. The subsequence of $(f(c_n))$ consisting of the odd numbers converges to $f(p)$. All subsequences of a convergent sequence converge to the same limit as the main sequence, so $f(c_n) \to f(p)$, and since $(f(b_n))$ is a subsequence of $(f(c_n))$, $f(b_n) \to f(p)$. Thus f preserves sequential convergence, and is thus continuous.

□

Problem 14

Let $f : M \to N$ be a bijection from one metric space to another that preserves distance, i.e. for all $p, q \in M$

$$d_N(fp, fq) = d_M(p, q)$$

Then f is called an isometry from M to N, and M and N are said to be isometric, $M \equiv N$.

Part a

Prove that every isometry is continuous.

Proof: Let $(p_n) \in M$ be an arbitrary sequence such that $p_n \to p \in M$. Then for arbitrary $\epsilon > 0$, there exists an $A \in \mathbb{N}$ such that $a > A$ implies that $d_M(p_a, p) < \epsilon$. Moving to the sequence $(f(p_n))$, by the distance preservation property we have that for the same $\epsilon > 0$, for the same $a > A$, we have $d_N(f(p_a), f(p)) < \epsilon$. Thus $f(p_n) \to f(p) \in N$, so f preserves sequential convergence and is thus continuous.

□

Part b

Prove that every isometry is a homeomorphism.

Proof: Since f is a bijection, its inverse f^{-1} exists. Since we proved that f is continuous, if we prove that f^{-1} is continuous, then f is a homeomorphism between M and N, and thus M and N are homeomorphic.

Let $(p_n) \in N$ be arbitrary such that $p_n \to p \in N$. Then because f is a bijection, f^{-1}, exists, and so $(f^{-1}(p_n))$ and $f^{-1}(p)$ are well defined.

Suppose f^{-1} is not continuous. Then $(f^{-1}(p_n))$ does not converge to $f^{-1}(p)$, and since f is continuous and the inverse of f^{-1}, this implies that $(f(f^{-1}(p_n))) = (p_n)$ does not converge to $f(f^{-1}(p)) = p$. But this contradicts the assumption that $p_n \to p$. Thus f^{-1} is continuous, and M and N are homeomorphic.

□

Part c

Prove that $[0, 1]$ is not isometric to $[0, 2]$.

Proof: Consider the set of pairs of points in $[0, 2]$ that are distance 1 from each other. Because of the total ordering that $[0, 2]$ inherits from $\mathbb{R}$, these pairs can be uniquely identified by their left endpoints. Letting A be the set of left endpoints of such pairs of points, A takes the form $A = \{a : [a, a + 1] \subset [0, 2]\} = [0, 1]$.

Let f be an isometry from $[0, 1]$ to $[0, 2]$. By the distance preservation condition of f, for all $a \in A$, the preimage of a under f is a point $b \in [0, 1]$ such that the interval $[b, b + 1] \subset [0, 1]$. However, the only interval of such form is $[0, 1]$. $f(0)$ can not correspond to $[0, 1]$, as this violates f being a function. Thus an isometry from $[0, 1]$ to $[0, 2]$ does not exist.

□

Problem 15

Prove that isometry is an equivalence relation.

Proof: Let M be isometric to N, and $f : M \to N$ be the isometry. For arbitrary $p, q \in N$, consider $f^{-1}(p), f^{-1}(p) \in M$. By the distance preserving condition of f, $d_M(f^{-1}(p), f^{-1}(q)) = d_N(f(f^{-1}(p)), f(f^{-1}(q))) = d_N(p, q)$. Thus f^{-1} is a bijection that preserves distances from N to M, and thus N is isometric to M.

Let M be a metric space and f be the identity function. f is clearly a bijection from M to M and distance-preserving, so f is an isometry. Therefore M is isometric to itself.

Let $f : M \to N$ and $g : N \to P$ be isometries. Consider their composition $H : M \to P = (g \circ f)(m)$. H is the composition of bijective functions, and thus is bijective itself, and it's clear that H preserves distances between M and P. Thus M is isometric to P. □

Problem 16

Is the perimeter of a square isometric to the circle? Homeomorphic?

Assuming that the square and the circle are embedded in $\mathbb{R}^2$ with the usual metric, the two are not isometric. If the length diagonal of the square does not equal the diameter of the circle, the proof is immediate. If the length of the diagonal of the square equals the diameter of the circle, call their common distance d. There are only two pairs of points on the square with distance d from each other, while there are an infinite number of pairs on the circle distance d from each other. Therefore f can not map between them while remaining a function.

The two are homeomorpic, as one can be stretched into the other.

Problem 18

Is $\mathbb{R}$ homeomorphic to $\mathbb{Q}$?

No, as the two have different cardinalities, there can not be a bijection between them.

Problem 19

Is $\mathbb{Q}$ homeomorphic to $\mathbb{N}$?

No. Let $p \in \mathbb{N}$. All convergent sequences $p_n \to p$ in $\mathbb{N}$ eventually end in repeating p. On the other hand, for $q \in \mathbb{Q}$, there exist sequences $(q_n), (r_n) \in \mathbb{Q}$ such that for all n, $q_n \neq r_n \neq q$.

Suppose that f is a homeomorphism from $\mathbb{Q}$ to $\mathbb{N}$ such that $f(q) = p \in \mathbb{N}$. Since f is bicontinuous, it preserves sequential convergence. Thus $(f(q_n))$ and $(f(r_n))$ both converge to p. Because of the nature of continuous sequences in $\mathbb{N}$, there exists an $N \in \mathbb{N}$ such that $n > N$ implies that $f(q_n) = f(r_n) = p$. But this implies that f is not injective, and thus not a bijection, which contradicts the assumption that f is a homeomorphism. Thus there exists no homeomorphism between $\mathbb{Q}$ and $\mathbb{N}$.

Problem 20

What function is a homeomorphism from $(-1, 1)$ to $\mathbb{R}$? Is every open interval homeomorphic to $(0, 1)$?

The function $\tan(\frac{\pi}{2}x)$ is a homeomorphism from $(-1, 1)$ to $\mathbb{R}$. It is bijective, continuous, and its inverse $\frac{2}{\pi}\tan^{-1}(x)$ is continuous.

Every open interval (a, b) is homeomorphic to $(0, 1)$. The function $\frac{x-a}{b-a}$ is bijective and bicontinuous.

Problem 22

If every closed and bounded subset of a metric space M is compact, does it follow that M is complete?

No. Let $M = x \in \mathbb{R} : x > 0$ with the usual metric. The closed and bounded subsets of M are compact, by the same reasoning as in $\mathbb{R}$. However, the Cauchy sequence $(p_n) = \frac{1}{n}$ has no limit in M.

Problem 24

For which intervals $[a, b]$ in $\mathbb{R}$ is the intersection $[a, b] \cup \mathbb{Q}$ a clopen subset of the metric space $\mathbb{Q}$?

The intervals where $a \leq b$ and a, b are irrational numbers. Since $\mathbb{Q}$ is a metric subspace of $\mathbb{R}$, it inherits its open and closed sets from $\mathbb{R}$. Since all intervals of the form $[a, b]$ are closed in $\mathbb{R}$, their intersection with $\mathbb{Q}$ is closed.

These intersections are also open. In $\mathbb{R}$, the interval (a, b) is open, and because a, b are irrational, $a, b \notin \mathbb{Q}$, so $[a, b] \cup \mathbb{Q} = (a, b) \cup \mathbb{Q}$. Thus $[a, b] \cup \mathbb{Q}$ is open.

The above does not hold if a or b is rational. Without any loss of generality, suppose a is rational. Then $a \in [a, b] \cup \mathbb{Q}$, and there is no open ball that contains a and is a subset of $[a, b] \cup \mathbb{Q}$. Thus $[a, b] \cup \mathbb{Q}$ is not open.

Problem 26

Prove that a set $U \subset M$ is open if and only if none of its points are the limits of its complement.

For the forward, suppose $x \in U$ is a limit point of U^C. Then $x \in \bar{U^C}$, and since for all sets $A \subset \bar{A}$, $x \in U^C$, which contradicts the assumption that $x \in U$. Thus U contains none of the limits of its complement.

For the reverse, suppose that U is not open. Then there exists a $x \in U$ such that for all $r > 0$, the open ball $B_U(x, r)$ is not fully contained in U. Thus, for $r = \frac{1}{n}$, we can choose a sequence $x_n \in B_U(x, r) \cup U^C$. $x_n \to x$ and $x_n \in U^C$ for all n, so x is a limit point of U^C. This contradicts the assumption that none of U's points are the limit points of its complement, so U is open.

Problem 27

If $S, T \subset M$, a metric space, and $S \subset T$, prove that

Part a

$\bar{S} \subset \bar{T}$

Proof: Let $x \in \bar{S}$. Then there exists a sequence $(x_n) \in S$ such that $x_n \to x$. Since $S \subset T$, $(x_n)\ inT$, so $x_n \to x$ implies that $x \in \bar{T}$. □

Part b

$\text{int}(S) \subset \text{int}(T)$

Proof: $\text{int}(S) \subset S \subset T$. Because $\text{int}(S)$ is open and a subset of T, it must be contained in the largest open subset of T, which is $\text{int}(T)$. □

Problem 28

A map $f : M \to N$ is open if U being an open subset of M implies that $f(U)$ is an open subset of N.

Part a

If f is open, is it continuous?

No. Let f be the floor function from $\mathbb{R}$ to $\mathbb{Z}$. Every set in Z is open, since singleton sets are open in Z and the union of open sets is open, whether finite or infinite. However, the floor function is not continuous, as the example of $p_n = 1 - \frac{1}{n}$ shows.

Part b

If f is a homeomorphism, is it open?

Yes. Let $f : M \to N$. Let $A \subset M$ be an arbitrary open set, and $f(A) \subset N$ be its image. Because f is a bijection, it has a continuous inverse function f^{-1}. Since $f(A)$ is the preimage of A, $f^{-1}(f(A)) = A$ being open implies $f(A)$ is open. Thus open sets are mapped to open sets, and f is open.

Part c

If f is an open, continuous bijection, is it a homeomorphism?

Yes. By similar reasoning to Part b, let $A \subset M$ be an arbitrary open set. Because f is a bijection, f^{-1} exists, and A is the image of some set $f(A) \subset N$. Because f is open, $f(A)$ is open. Thus the preimage of open sets of f^{-1} is an open set, and f^{-1} is continuous. Since f is a bicontinuous bijection, it is a homeomorphism.

Part d

If $f : \mathbb{R} \to \mathbb{R}$ is a continuous surjection, must it be open?

No. Let f be defined as $f(x) = x$ if $x \in \mathbb{Z}$. For $x \in \mathbb{R} - \mathbb{Z}$, define f as

$$f(x) = \begin{cases} x & x \in \mathbb{Z} \\ \lfloor x \rfloor + 2\text{Remainder}(x) & \text{Remainder}(x) \in (0, \frac{1}{2}) \\ \lceil x \rceil & \text{Remainder}(x) \in [\frac{1}{2}, 1) \end{cases}$$

where $\text{Remainder}(x)$ is the non-integer part of x. $f(x)$ is essentially a piecewise staircase with a slope of 2 on $[0, \frac{1}{2})$ pieces and a slope of 0 on $(\frac{1}{2}, 1)$ pieces. The function is continuous and surjective, but not open. For example, the open set $(\frac{1}{2}, \frac{3}{4})$ maps to the singleton 1, which is not open.

Part e

If $f : \mathbb{R} \to \mathbb{R}$ is a continuous, open surjection, must it be a homeomorphism?

Yes. Note that if f is injective, then f is a bijection and Part c implies that it is a homeomorphism.

Suppose that f is not injective. Then there exists $a, b \in \mathbb{R}, a \neq b$ such that $f(a) = f(b)$. Without loss of generality, let $a < b$. Then by continuity, f achieves a maximum value on $[a, b]$.

If $f(a) = M = m$, then f is constant on $[a, b]$. This contradicts the assumption that f is open. For example, the image of the open ball $(a + \frac{b-a}{4}, b - \frac{b-a}{4})$ is a singleton, and thus not open.

If $f(a) < M$, then there exists $c \in (a, b)$ such that $f(c) = M$. Thus there exists $\delta = \min(\frac{c-a}{2}, \frac{b-a}{2}) > 0$. Consider the open ball $(c - \delta, c + \delta)$ and its image under f. Because f is bounded, the image is bounded, and because f is continuous, the image obtains all intermediate values. Thus $f((c - \delta, c + \delta))$ is an interval. Let x, y be the left and right endpoints of the interval, respectively.

Since $f(c)$ is the maximum of f, the right endpoint of the interval is closed. Thus $f((c-\delta, c+\delta))$ has the form $(x, y]$ or $[x, y]$, neither of which is open. Thus, the image of an open set under f is not open, which contradicts the assumption that f is open. A similar argument holds for the minimum. This, f is injective, bijective, and a homeomorphism.

Part f

What happens in Part e if $\mathbb{R}$ is replaced by the unit circle S^1?

The result does not hold. Parameterize S^1 by the angle $\theta = \arctan(x)$, and let $f : S^1 \to S^1$ map a point x to the corresponding point at 2θ. f is obviously surjective, and it is continuous because it preserves sequential limits. It is open because open sets in S^1 consist of open line segments or their unions, and the image of line segments is either another union of line segments, or the entire metric space S^1. However, f is not injective. Letting a be the point at $\theta = 0$ and b the point at $\theta = \pi$, $a \neq b$, but $f(a) = f(b)$.

Problem 30

Consider a two-point set $M = \{a, b\}$ whose topology consists of two sets, M and the empty set. Why does this topology not arise from a metric on M?

Let d be a metric on M. By the properties of metrics, $d(a, b) = d(b, a) = c > 0$, and $d(a, a) = d(b, b) = 0$. The singleton sets a, b are open, since the open ball $B_d(a, c/2) = a$ and $B_d(b, c/2) = b$ are contained within themselves. However, the singleton sets are not contained in the topology.

Problem 31

Prove the following.

Part a

If U is an open subset of $\mathbb{R}$ then it consists of countably many disjoint intervals $U = \sqcup U_i$. (Unbounded intervals $(-\infty, b)$, (a, ∞), and $(-\infty, \infty)$ are allowed).

Lemma 1 *The bounded, connected sets on $\mathbb{R}$ are intervals.*

Proof: *Let A be bounded and connected. By connectedness, it has the intermediate value property. By boundedness, it has a l.u.b. and g.l.b. Letting a be the g.l.b. and b be the l.u.b. and using the intermediate value property, $(a, b) \subset A$. By the l.u.b., $A \subset [a, b]$. Thus A is an interval. The reverse was shown in the book.* $\square$

Corollary 2 *The bounded, connected, open sets on $\mathbb{R}$ are open intervals.*

Proof: *The connected sets on $\mathbb{R}$ are intervals, and the (topologically) open intervals are (colloquial) open intervals.* $\square$

Lemma 3 *Let A and B be overlapping bounded open intervals. Then $A \cup B$ is an open interval.*

Proof: *A and B are connected and share a common point, so $A \cup B$ is connected. Since $A \cup B$ is bounded, $A \cup B$ is an interval. Since $A \cup B$ is open, $A \cup B$ is an open interval.* $\square$

Corollary 4 *Let A_i be a collection of disjoint bounded open intervals of $\mathbb{R}$. Then there exists disjoint open intervals B_i such that $\cup A_i = \sqcup B_i$*

Proof: *If the A_i are disjoint, the proof is obvious. If not, then there exist i, j such that A_i and A_j overlap. By Lemma 3, $B_i = A_i \cup A_j$ is an open interval. Substitute A_i and A_j with B_i, and repeat while there are still overlapping intervals in A.* $\square$

(I'm not sure that this proof is rigorous. It seems like I'm implicitly assuming that A_i is countable, but I don't know enough to be sure.)

Lemma 5 *If A is an open subset in $\mathbb{R}$, then there exist bounded open intervals A_i such that $\cup A_i = A$.*

Proof: *Because A is open, for all $a \in A$, there exists $r_a > 0$ such that $(a - r_a, a + r_a) \subset A$. Do this for all points in A.* $\square$

Lemma 6 *Let $\epsilon > 0$. Let A_i be a collection of disjoint open intervals such that length$(A_i) > \epsilon$ for all i. Then A_i is countable.*

Proof: *Let B_i be the intervals of A_i dilated by a factor of $1/\epsilon$. That is, if $A_i = (x, y)$, $B_i = (x/\epsilon, y/\epsilon)$. Because all of the A_is have length greater than $1/\epsilon$, the B_i have length greater than 1. Thus the B_i can be uniquely identified with the natural numbers by associating each B_i with the floor of its left endpoint. Since the natural numbers are countable, B_i is at most countable, and there exists a homeomorphism between the A_i and B_i, the A_i are countable.* $\square$

Theorem 7 *If A is an open subset of $\mathbb{R}$, then it consists of disjoint open intervals $A = \sqcup A_i$ (unbounded open intervals are acceptable).*

Proof: *If A is bounded, then by Lemma 5, A can be expressed as the union of open intervals, and by Corollary 4, A can be expressed as the disjoint union of bounded open intervals.*

If A is unbounded and $A = \mathbb{R}$, the proof is trivial. Suppose A is unbounded in the positive direction. For simplicity, suppose that A is bounded in the negative direction. Since A does not equal $\mathbb{R}$, A^C is nonempty, and by A being unbounded in the positive direction, A^C is bounded above. Thus A^C has a least upper bound.

Denote the l.u.b. of A^C as a. By definition, $(a, \infty) \subset A$. Because A is open, A^C is closed, and since a is a limit point of A^C, $a \in A^C$.

Let $C = A - (a, \infty)$. For all $c \in C$, $c < a$. If $c = a$, then $C \subset A$ implies $c \in A$, which contradicts that $a \in A^C$. If $c > a$, then $c \in C$ and $c \in (a, \infty)$, which is a contradiction because by construction C and (a, ∞) are disjoint. Thus C is bounded above. C is bounded below because $C \subset A$ and A is bounded below. Thus C is bounded.

C is open. $C^C = A^C \cup (a, \infty) = A^C \cup [a, \infty)$ because $a \in A^C$. Thus C^C is the finite union of closed sets, and is closed.

Thus C is a bounded open subset of $\mathbb{R}$, and can be expressed as the disjoint union of bounded open intervals C_i. Adding back in (a, ∞) gives A as the disjoint union of open intervals A_i. The cases where A is bounded above and unbounded below, and case when A is unbounded in both directions, follow similarly. □

Lemma 8 *Let A_i be a collection of disjoint intervals in $\mathbb{R}$. Then there is at most one unbounded interval in the positive direction, and and most one (possibly the same) unbounded interval in the negative direction.*

Proof: *Let A_m and A_n be distinct intervals in A that are unbounded above. Because they are intervals, they are non-empty, so there exist $m \in A_m$ and $n \in A_n$. By the trichotomy property on $\mathbb{R}$, either $m = n$, $m > n$, or $n > m$. In any of the cases, one of m or n is in both sets, violating the assumption that A_i is disjoint.* □

Theorem 9 *If A_i is a collection of disjoint open intervals in $\mathbb{R}$, then A_i is countable.*

Proof: *Create subcollections B_i, where $i = 0, 1, 2 \ldots \mathbb{N}$. Let B_0 consist of the unbounded intervals of A_i, of which there are at most 2, B_1 the intervals of A_i with finite length greater than 1, and B_i the intervals of A_i with length greater than $1/i$ and less than or equal to $1/(i-1)$. By Lemma 6, B_i contains at most countable many intervals. Continuing this process for all $\mathbb{N}$, we see that the B_is contains all the intervals of A_i. Because B_i contains a countable number of sets, each with a countable number of elements, B_i, and thus A_i is countable.* □

Corollary 10 *If U is an open subset of $\mathbb{R}$ then it consists of countably many disjoint intervals $U = \sqcup U_i$. (Unbounded intervals $(-\infty, b)$, (a, ∞), and $(-\infty, \infty)$ are allowed).*

Proof: *Follows immediately from Theorems 7 and 9.* □

Part b

Prove that the intervals U_i are uniquely determined by U. In other words, there is only one way to express U as a disjoint union of open intervals.

I will assume that the U_i have to be open, otherwise the statement is false. For example, $(0, 1)$ can also be written as $(0, .5] \cup (.5, 1)$.

Lemma 11 *Let A_i be a disjoint union of open intervals and $A = \sqcup A_i$. Let A_k be a particular interval with left and right endpoints b and c, respectively. Then $b, c \in A^C$.*

Proof: *Suppose $b \in A$. Since $b \in A_k^C$, b must be in a different open interval. Denote the open interval A_j. Since A_j is open, there exists $r > 0$ such that $(b-r, b+r) \subset A_j$. Consider $b - r/2$. $b - r/2 < b$, so $b - r/2 \in A_k$. $b - r/2 \in (b-r, b+r) \subset A_j$, so $b - r/2 \in A_j$. Thus A_k and A_j intersect, which contradicts the assumption that the A_i are disjoint. Thus $b \in A^C$. The exact same argument holds for c.* □

Now to prove the theorem. Let A be an open subset and B_i and C_j be disjoint collections of open intervals such that $A = \sqcup B_i = \sqcup C_j$. If A is unbounded in both directions, the proof is trivial.

Consider an arbitrary point $a \in A$. Because the B_i's are disjoint and the C_j's are disjoint, there exist unique B_a and C_a such that $a \in B_a$ and $a \in C_a$.

Letting x be the right endpoint of B_a and y the right endpoint of C_a, $x = y$. Suppose it's not. Without loss of generality, let $x < y$. Then $x \in A$ because $x \in C_a$, but $x \in A^C$ because x is the right endpoint of $B_a \subset A$, and so $x \in B_a$. Repeating this for $x > y$ shows that $x = y$.

A similar argument holds for the left endpoint. Repeat this argument for all points in A to show that for all $a \in A$, the endpoints of B_a equal the endpoints of C_a. Thus, $B_a = C_a$ and the two collections B and C are equal.

Part c

If $U, V \subset \mathbb{R}$ are both open, so $U = \sqcup U_i$ and $V = \sqcup V_j$ where U_i and V_j are open intervals, show that U and V are homeomorphic if and only if there are equally many U_i and V_j.

Proof: Let m be the number of element in U_i, and n be the number of elements in V_j. Note that m, n may be infinity.

For the forward, suppose there are not equally many U_i and V_j. Without loss of generality, let $m < n$. Let f be a homeomorphism between U_i and V_j. For arbitrary $k \in 1, 2 \ldots m$, consider the restriction of f on U_k. Because U_k is an interval, it is connected, and so its image is connected. Repeating this for all U_i shows that V is the union of at most m connected sets. However, V consists of n connected intervals, so V can not be the image of U. Thus the homeomorphism f does not exist. A similar argument in reverse if $m > n$.

For the reverse, construct a homeomorphism as follows. By Part a above, U_i and V_j are countable. Label the sets. Let the homeomorphism be f such that each U_k is shifted and dilated so that $f(U_k) = V_k$. f is trivially a bijection. Since on each segment, f is a non-singular linear transformation, f and f^{-1} are continuous. □

Problem 33

Part a

Find a metric space in which the boundary of $M_r p$ is not equal to the sphere of radius r at p, $\partial(M_r p) \neq x \in M : d(x,p) = r$.

Consider the discrete metric on a space with at least two elements. Note that since all sets under the discrete metric are clopen, the boundry of all sets is the null set. However, the sphere of radius 0 centered at p is p itself, and the sphere of radius 1 centered at p is the entire metric space, neither of which are the null set.

Part b

Need the boundary be contained in the sphere?

Yes. Suppose not. For an arbitrary set A in a metric space M, $\partial A \subset \overline{A}$. Thus the boundary not being contained in the sphere implies that $CB_r p = \{x \in M | d(x,p) \leq r\}$ is a proper subset of $\overline{M_r p}$, so that the boundary can potentially be outside of $CB_r p$. Because $\overline{M_r p}$ is the smallest closed set that contains $M_r p$ and $M_r p \subset CB_r p$, $CB_r p$ being a proper subset of $\bar{M_r} p$ implies that $CB_r p$ is not closed. This implies that $CB_r p^C = \{x \in M : d(x,p) > r\}$ is not open. If $CB_r p^C$ is the null set, this is a contradiction.

Otherwise, for all $x \in CB_r p^C$, let $d(x,p) = y > r$. Then the open ball $M_{\frac{y-r}{2}} x \subset CB_r p^C$ by the triangle inequality, and $CB_r p^C$ is open, implying $CB_r p$ is closed. Thus there is a closed set containing $M_r p$ that is strictly smaller than the closure of $M_r p$, a contradiction.

Problem 34

Assume that N is a metric subspace of M and is also a closed subset of M. Show that $L \subset N$ is closed in N if and only if it is closed in M. Similarly, if N is a metric subspace of M and also is an open subset of M then $U \subset N$ is open in N if and only if it is open in M.

Proof: For the closed version, we start with the forward. Because L is closed in N, by the Inheritance Principle, $L = N \cap P$, where P is a closed set in M. Viewing N and P as subsets of M, since N is closed by assumption, $L = N \cap P$ is closed in M. Conversely, let L be closed in M. Then by the Inheritance Principle, $L \cap N$ is a closet set in N. Since $L \subset N$, $L \cap N = L$ is closed in N. The open version is essentially the same, with the word 'open' replacing the word 'closed.' $\square$

Problem 35

Prove that S clusters at p if and only if for each $r > 0$ there is a point $q \in M_r p \cap S$ such that $q \neq p$.

This should be a direct consequence of Theorem 52, that p being a cluster point of S is characterized by each neighborhood of p containing at least one point of S other than p.

Problem 36

Construct a set with exactly three cluster points.

In $\mathbb{R}^2$, let $A_i = \{(x, y) \in \mathbb{R}^2 | (x, y) = (i, 1/n) \text{ for some } n \in \mathbb{N}\}$. Let $A = A_1 \cup A_2 \cup A_3$. The three cluster points are $\{(0, 0), (1, 0), (2, 0)\}$.

Problem 37

Construct a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous only at points of $\mathbb{Z}$.

First we consider the function which is x on the rationals and $-x$ on the irrationals. It is continuous only at $x = 0$. Take miniature copies of this function, centered at the integers, all only on the interval $(n-1/2, n+1/2]$. For all integers z, $f(z) = 0$. The function is continuous at the integers, and nowhere else.

Problem 38

Let X, Y be metric spaces with metrics d_X, d_Y, and let $M = X \times Y$ be their Cartesian product. Prove that the three natural metrics d_E, d_{max}, and d_{sum} on M are actually metrics.

Symmetry is obvious. The positive definiteness of d_{max} follows from the positive definiteness of the d_X and d_Y. The positive definiteness of d_{sum} is similar, and the positive definiteness of d_E follows because the square root function only has a root at 0.

For the triangle inequality on d_{sum},

$$\begin{aligned} d_{sum}(p, p^*) + d_{sum}(p^*, p') &= d_X(x, x^*) + d_X(x^*, x') + d_Y(y, y^*) + d_Y(y^*, y') \\ &> d_X(x, x') + d(y, y') = d_{sum}(p, p') \end{aligned}$$

by the triangle inequality on d_X and d_Y. For d_{max}, $d_{max}(p, p') = \max\{d_X(x, x'), d_Y(y, y')\}$. Without loss of generality, suppose that $d_X(x, x') \geq d_Y(y, y')$. Then

$$\begin{aligned} d_{max}(p, p^*) + d_{max}(p^*, p') &= \max\{d_X(x, x^*), d_Y(y, y^*)\} + \max\{d_X(x^*, x'), d_Y(y, y')\} \\ &\geq d_X(x, x^*) + d_X(x^*, x') \geq d_X(x, x') = d_{max}(p, p') \end{aligned}$$

The argument if $d_X(x, x') < d_Y(y, y')$ is similar. For the triangle inequality on d_E,

Problem 40

Let M be a metric space with metric d. Prove that the following are equivalent.

1. M is homeomorphic to M equipped with the discrete metric.
2. Every function $f : M \to M$ is continuous.
3. Every bijection $g : M \to M$ is a homeomorphism.
4. M has no cluster points.
5. Every subset of M is clopen.
6. Every compact subset of M is finite.

If written with no other qualifiers, M is equipped with its metric d.

For $1 \to 2$, for any arbitrary set equipped with the discrete metric, if p_n is a sequence such that $p_n \to p$, then the tail of $p_n = p$. Because $M_{discrete}$ is homeomorphic to M_d, this is true on M_d. Thus every convergent sequence has a constant tail, which is trivially preserved under an arbitrary function. Thus arbitrary functions are continuous.

Thus for any function $f : M_1 \to M_2$, sequential limits are trivially preserved, so f is continuous.

For $2 \to 3$, this follows because g^{-1} exists due to g being a bijection, and $g : M \to M$ and $g^{-1} : M \to M$ are both continuous.

For $3 \to 4$, let $p \in M$ be a cluster point of M. Because p is a cluster point, there exists a sequence of distinct points $p_n \in M$ such that $p_n \to p$. Let $g : M \to M$ be a function that swaps points p and p_1, and is the identity function everywhere else. g is trivially a bijection, and thus it is a homeomorphism, and thus g is continuous and preserves sequential limits. However, $g(p_n) \to p$, but $g(p) = p_1 \neq p$, contradicting sequential limit preservation. Thus M has no cluster points.

For $4 \to 5$, because M has no cluster points, for all $p \in M$, there exists some $r > 0$ such that the neighborhood of M about r contains only p. Thus the open ball with radius r is contained in p, so all singleton sets are open. The arbitrary union of open sets is open, so all subsets of M are open. By taking compliments, all subsets of M are closed, and thus clopen.

For $5 \to 6$, let A be an infinite, compact subset of M, and A_n be an open covering of singletons. Then by compactness, there exists a finite subcover A_{n_k} of A. But it's clear that a finite union union of singletons can not cover an infinite set. Thus A can not be infinite.

For $6 \to 1$, if d is the discrete metric, the proof is trivial. Otherwise, let d be the metric on M. I will show that M has no cluster points under metric d. From above, that means that all singletons are clopen in M with d, thus meaning that all subsets are clopen in M with d. Then the identity function between M with d and M with the discrete metric is trivially a homemorphism.

Let p be a cluster point of M. Then for every open neighborhood of p, there exists a point in M that is not p. Consider the open neighborhoods $C_n = B_{r=1/n}(p)$ for all $n \in \mathbb{Z}^+$. Then for all n, there exists x_n such that $x_n \in C_n$.

Consider the set $C = (\cup_{n=1}^{\infty} x_n) \cup p$, and let A be an open cover of C. Because A is an open cover, there exists $A_p \in A$ such that $p \in A_p$. Since A_p is open, there exists a neighborhood with $r > 0$ centered at p such that $B_r(p) \subset A_p$. By the construction of C, $B_r(p)$ covers all of the x_n with $n > 1/r$. Thus the finite subcover $A_1 \cup A_2 \ldots A_p$ covers A, making A compact. But this contradicts the assumption that all compact subsets of M are finite. Thus M has no cluster points.

Problem 41

Let $\|\cdot\|$ be a norm on $\mathbb{R}^m$, and let $B = \{x \in \mathbb{R}^m : \|x\| \leq 1\}$. Prove that B is compact.

Let $\|\cdot\|_E$ be the Euclidean norm on $\mathbb{R}^m$. If $\|\cdot\|$ equals the Euclidean norm, this is trivial.

We first want to show that the identity transformation from $(\mathbb{R}^m, \|\cdot\|_E)$ to $(\mathbb{R}^m, \|\cdot\|)$ is continuous. Due to the translation invariance of norms, if we show that the identity transformation is continuous at the origin, we have shown it for all $\mathbb{R}^m$. We begin with some lemmas. Let $\epsilon > 0$, and $C = \{x \in \mathbb{R}^m : \|x\| = \epsilon\}$.

Lemma 12 *The identity function from $(\mathbb{R}^m, \|\cdot\|_E)$ to $(\mathbb{R}^m, \|\cdot\|)$ is continuous.*

Proof: *All norms on a finite-dimensional vector space are equivalent, which means that open sets under $\|\cdot\|$ are open sets under $\|\cdot\|_E$, and vice versa. Thus the preimage of open sets under the identity transformation is trivially open.* □

Corollary 13 *If $A \subset \mathbb{R}^m$ is compact with respect to the Euclidean norm, then it is compact with respect to $\|\cdot\|$.*

Proof: *The identity function is continuous, so $A \subset (\mathbb{R}^m, \|\cdot\|)$ is the image of a compact with respect to the Euclidean norm. By the theorems in the book, this implies that A is compact with respect to the $\|\cdot\|$ norm.* □

Thus, if we can prove that B is compact with respect to the Euclidean norm, it is compact with respect to the norm $\|\cdot\|$. By the Heine-Borel theorem, this is equivalent to B being closed and bounded under the Euclidean norm.

For being closed, norms are continuous functions from $\mathbb{R}^m$ to $\mathbb{R}$. It's clear that B is the preimage of the closed set $[0, 1] \subset \mathbb{R}$, implying that B is closed with respect to $\|\cdot\|$. Since the identity transformation from the Euclidean norm to the $\|\cdot\|$ norm is continuous, this implies that B is closed with respect to the Euclidean norm.

For boundedness, let $A = \{x \in \mathbb{R}^m : \|x\|_E = 1\}$ be the surface of the unit ball in $\mathbb{R}^m$ under the Euclidean norm. A is clearly closed and bounded, so because it is a subset of $\mathbb{R}^m$, it is a compact. Since all norms are continuous, $\|\cdot\| : \mathbb{R}^m \to \mathbb{R}$ is a continuous function, and so the image of A under $\|\cdot\|$ obtains maximum and minimum values. Let $y = \min \|A\|$ be the minimum value. Because norms are positive definite, the zero vector is not an element of A, so $y > 0$.

Define $B' = \{x \in \mathbb{R}^m : \|x\| \leq y\}$, and A' be the unit ball in $\mathbb{R}^m$. By construction, $B' \subset A'$ and is thus bounded under the Euclidean metric. Dilate B' and A' by a factor of $1/y$. By the norm property that $\|\alpha x\| = |\alpha| \|x\|$, after dilation, B' maps to B, and A' maps to a ball with radius 1/x centered at the origin. It's clear that B is a subset of this ball, thus making B bounded under the Euclidean metric.

Problem 43

Assume that the Cartesian product of two nonempty sets $A \subset M$ and $B \subset M$ is compact in $M \times N$. Prove that A and B are compact.

Proof: Let $(a_n) \in M$ and $(b_n) \in N$ be arbitrary sequences. Consider their Cartesian product $(a,b)_n \in M \times N$. By compactness of $M \times N$, there exists a convergent subsequence $(a,b)_{n_k} \to (a,b)$. Since all metrics on a Cartesian product are equivalent, let's use the sum metric, d_{sum}. Then for all $\epsilon > 0$ and large n, $d_{sum}((a,b)_{n_k}, (a,b)) = d_M(a_{n_k}, a) + d_N(b_{n_k}, b) < \epsilon$. Since distances are nonnegative, this implies that for large n $d_M(a_{n_k}, a) + d_N(b_{n_k}, b) < \epsilon$ and $d_N(b_{n_k}, b) < \epsilon$. Thus (a_n) and (b_n) have convergent subsequences, indicating that M and N are compact. $\square$

Problem 44

Consider a function $f : M \to \mathbb{R}$. Its graph is the set

$$G(f) = \{(p,y) \in M \times \mathbb{R} : y = f(p)\}$$

(Note the notation is my own invention; I have no idea if it's standard or not).

Part a

Prove that if f is continuous then its graph is closed as a subset of $M \times \mathbb{R}$.

Proof: By the theorems in the book, the metrics on a Cartesian product are equivalent. I will use d_{sum} for simplicity. Let d_M be the metric on M, and d_E be the Euclidean metric on $\mathbb{R}$.

Let $((a, f(a))_n) \in G(f)$ be a convergent sequence such that $(a, f(a))_n \to (a,b) \in M \times \mathbb{R}$. Because of convergence, this implies that for all $\epsilon > 0$, for large n, $d_{sum}((a, f(a))_n, (a,b)) = d_M(a_n, a) + d_E(f(a_n), b) < \epsilon$. Note that this implies that $a_n \to a$ and $f(a) \to b$. By continuity, $a_n \to a$ implies that $f(a_n) \to f(a)$. Since limits are unique, this implies $f(a) = b$, and so $(a,b) = (a, f(a)) \in M \times \mathbb{R}$. Thus $M \times \mathbb{R}$ is closed. $\square$

Part b

Prove that if f is continuous and M is compact then its graph is compact.

Proof: Because f is continuous and M is compact, $f(M) \subset \mathbb{R}$ is a compact. Therefore $G(f) = M \times f(M)$ is the Cartesian product of two compacts, and thus compact. $\square$

Part c

Prove that if the graph of f is compact then f is continuous.

If M is empty, then f is trivially continuous. If M is nonempty but p is an isolated point of M, then f is trivially continuous at p.

Otherwise, let p be a cluster point of M and suppose f is discontinuous at p. Then there exists $\epsilon > 0$ such that for all $\delta > 0$, there exists a $x \neq p \in M$: $d_M(x,p) < \delta$ such that $|f(p) - f(x)| > \epsilon$.

Therefore for all $n \in \mathbb{N}$, there exists a point $a_n \neq p \in M$ such that $d_M(a_n, p) < \frac{1}{n}$ and $|f(a_n) - f(p)| > \epsilon$.

Consider the sequence $(a_n) \in M$, and its associated sequence $((a_n, f(a_n))) \in M \times \mathbb{R}$. By construction, it's clear that all subsequences of $((a_n, f(a_n)))$ converge to a point with first coordinate p. However, because f is a function, the only point in $G(f)$ with first coordinate p is $(p, f(p))$. Also by construction, $f(a_n)$ not converging to $f(p)$ implies that $((a_n, f(a_n)))$ does not converge to $(p, f(p))$. Thus the sequence $((a_n, f(a_n))) \in G(f)$ has no convergent subsequence, which contradicts the assumption that $G(f)$ is compact. Therefore f is continuous at x. Since all points in M are either isolated points or cluster points, f is continuous on M.

Part d

Give an example of a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ whose graph is closed.

Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{x} & \text{else} \end{cases}$$

The singleton at $(0,0)$ is trivially closed. The two hyperbola arms are closed because they contain all of their limits. Thus, the graph if f is closed. However, the f is obviously discontinuous.

Problem 45

Draw a Cantor set C on the circle and consider the set A of all chords between points of C.

Without loss of generality, I will assume the circle has circumference 1. I will arbitrarily choose a point p to be 0 and count off distances going counter-clockwise.

Part a

Prove that A is compact.

For an interval I_a, denote the left and right endpoints of I_A as A_L and A_R, respectively.

Lemma 14 *Let I_A and I_B be intervals in C_n, the nth step of the Cantor set. Without loss of generality, let $A_R < B_L$. Then the set of all chords between the intervals is the shape bounded by the curve $A_L A_R$, the line segment $\overline{A_R B_L}$, the curve $B_L B_R$, and the line segment $\overline{A_R B_L}$. Denote this shape S_{AB}.*

Proof: *All chords must have one of their endpoints on I_A and the other on I_B. Since the intervals are disjoint, mental playing around with the picture will show that the curve draws the desired shape.* □

Lemma 15 *Let I_A and I_B be intervals as defined in 14, and let I_C, I_D be subintervals such that $I_C, I_D \subset I_A \cup I_B$ Then $S_{CD} \subset S_{AB}$.*

Proof: *For $p \in S_{CD}$, p is on a chord with one endpoint on I_C and the other on I_D. By subintervals, the endpoints are in $I_A \cup I_B$, so the chord that connects those endpoints is contained in S_{AB}. Thus $p \in S_{AB}$.* □

Corollary 16 *S_{AB} as defined in Lemma 14 is compact.*

Proof: *The shape is closed and bounded in $\mathbb{R}^2$.* □

Lemma 17 *Denote A_n the set of all chords that are drawn between intervals in C_n. Then A_n is compact.*

Proof: *$A_n = \cup_{i,j=1}^{n} S_{ij}$. In other words, A_n is the union of all S_{ij}, where i, j are two intervals in C_n. A_n is bounded, and because it is the union of finite closed sets, A_n is closed. Since $A_n \subset \mathbb{R}^2$, A_n is compact.* □

Lemma 18 *A_{n+1} is a subset of A_n.*

Proof: *Let $p \in A_{n+1}$. Then p is in an S between two intervals of C_{n+1}, which are subintervals of an interval or intervals in C_n. By Lemma 15, p is in the corresponding S in C_n, so $p \in A_n$.* □

Lemma 19 *A_n is nonempty for all n.*

Proof: $1/3, 2/3$ *are always points in A_n, and their chord is nonempty.* □

Lemma 20 *A is compact.*

Proof: *$A = \cap_{i=1}^{\infty} A_i$. By Lemmas 17, 18, and 19, A is the intersection of a nested series of nonempty compacts, and is thus compact.* □

Problem 46

Assume that A and B are compact, disjoint, nonempty subsets of M. Prove that there are $a_0 \in A$ and $b_0 \in B$ such that for all $a \in A$ and $b \in B$ we have

$$d(a_0, b_0) \leq d(a, b)$$

In other words, a_0 and b_0 are closest together.

The metric $d : A \times b \to \mathbb{R}$ is a function from the Cartesian product of A and B to $\mathbb{R}$. As shown in the book, A and B being compact implies $A \times B$ is compact. Furthermore, all metrics are continuous functions. Therefore, d achieves a maximum and minimum value on $\mathbb{R}$. Denote m as the minimum. Because $A \times B$ is nonempty, $m < \infty$, and because A and B are disjoint, $m > 0$. Let (a_0, b_0) be an arbitrary point in the preimage of m. Then the desired property follows immediately.

Problem 47

Suppose $A, B \subset \mathbb{R}^2$.

Part a

If A and B are homeomorphic, are their complements homeomorphic? No. Let $A = (0, 1)$ and B be the x-axis. $(0, 1)$ is homeomorphic to $\mathbb{R}$, so A is homeomorphic to B. However, A^C is connected while B^C is not.

Problem 48

Prove that there is an embedding of a line as a closed subset of the plane, and an embedding of a line as a bounded subset of the plane, but there is not an embedding of a line as a closed and bounded subset of the plane.

For closed subset, embed the line onto the x-axis on the plane. This is trivially a homeomorphism, and the x-axis is trivially closed.

For the bounded subset, a line is homeomorphic to the interval $(0, 1)$. Embed $(0, 1)$ on the x-axis on the plane. This is trivially bounded.

For the closed and bounded subset, if a line can be embedded onto a closed and bounded subset of the plane, then there exists a homeomorphism $f : \mathbb{R} \to \mathbb{R}^2$ such that $f(\mathbb{R})$ is closed and bounded. By the Heine-Borel theorem, this implies that $f(\mathbb{R})$ is compact. Since compactness is a topological property, this implies that the line is compact. But this is a contradiction, as $\mathbb{R}$ is not compact.

Problem 52

Let (A_n) be a nested decreasing sequence of nonempty closed sets in the metric space M

Part a

Show that if M is complete and $\text{diam}(A_n) \to 0$ as $n \to \infty$, show that $A = \cap A_n$ is exactly one point.

Let a_n be an arbitrary point in A_n. Because the (A_n) are decreasing, for $n \geq m$, $A_n \subset A_m$. Because $\text{diam}(A_n) \to 0$, for all $\epsilon > 0$, there exists an $m \in \mathbb{N}$ such that $\text{diam}(A_m) < \epsilon$. Thus for all $n \geq m$, $a_m, a_n \in A_m$, so $d(a_m, a_n) \leq \text{diam}(A_m) < \epsilon$, implying that (a_n) is Cauchy.

Since M is complete, there exists $a \in M$ such that $a_n \to a$. Since for all n, $a_n \in A_1$ and A_1 is closed, $a \in A_1$. Letting m be fixed and considering the tail of (a_n), we see that $a_m \in A_m$ implies that $a \in A_m$, for all m. Thus $a \in A$. Because the diameter of a set with more than one distinct points is greater than zero, a is the only point in A.

Part b

To what assertions do the sets $[n, \infty)$ provide counterexamples to?

The sets are not closed, and their diameter does not go to zero.

Problem 53

Suppose that (K_n) is a nested sequence of compact nonempty sets, $K_1 \supset K_2 \supset \ldots$, and $K = \cap K_n$. If for some $\mu > 0$, $\text{diam}(K_n) \geq \mu$ for all n, is it true that $\text{diam}(K) \geq \mu$?

Yes.

Note that for all K_n, there exist $a_n, b_n \in K_n$ such that $d(a_n, b_n) \geq \mu$. To see this, note that for all n, the metric is a continuous function $d : K_n \times K_n \to \mathbb{R}$, and continuous functions achieve a maximum and minimum on a compact set. Since $\text{diam}(K_n)$ is the maximum of d on $K_n \times K_n$, there exists a point $(a_n, b_n) \in K_n \times K_n$ that achieves this maximum.

Repeat this for all K_n to create the sequence $((a_n, b_n))$. Because the sets are nested, $((a_n, b_n)) \in K_1 \times K_1$. Because K_1 is compact, $K_1 \times K_1$ is compact, so there exists a subsequence $((a_{n_k}, b_{n_k})) \in K_1 \times K_1$ and a point $(p, q) \in K_1 \times K_1$ such that $((a_{n_k}, b_{n_k})) \to (p, q)$. (p, q) is also in $K_2 \times K_2$, since all terms of $((a_{n_k}, b_{n_k}))$ except for possibly the first one are in K_2. The same is true for K_3, and so on. Thus, $(p, q) \in K \times K$.

Now noting that the metric is a continuous function, it preserves sequential limits, so $\lim_{n\to\infty} d((a_{n_k}, b_{n_k})) = d((p, q))$. Because $d((a_{n_k}, b_{n_k})) \geq \mu$ by definition, $\lim_{n\to\infty} d((a_{n_k}, b_{n_k})) \geq \mu$. Therefore $d((p, q)) \geq \mu$, and since $(p, q) \in K \times K$, this implies that $\text{diam}(K) \geq \mu$.

Problem 54

If $f : A \to B$ and $g : C \to B$ such that $A \subset C$ and for each $a \in A$ we have $f(a) = g(a)$, then g extends f, or f is extended by g. Assume that $f : S \to \mathbb{R}$ is a uniformly continuous function defined on a subset S of a metric space M.

Part a

Prove that f extends to a uniformly continuous function $\bar{f} : \bar{S} \to \mathbb{R}$.

If $S = \bar{S}$, then the result is trivial. Otherwise, from the theorems in the book, we know that $\bar{S} = S \cup S'$, where S is the set of cluster points of S. Note that by the definition of cluster point, if $x \in S'$, then there exists $(x_n) \in S$ such that $x_n \to x$.

We begin with a theorem.

Theorem 21 *Let $f : S \to T$ be uniformly continuous between two metric spaces. Then f preserves Cauchy sequences.*

Proof: *Let $(x_n) \in S$ be Cauchy. Fix $\epsilon > 0$ and let $\delta > 0$ be the associated distance in uniform continuity. Since (x_n) is Cauchy, there exists $N \in \mathbb{N}$ such that $c, d \geq N$ implies $d_S(x_c, x_d) < \delta$. By uniform continuity, this implies that for all $c, d \geq N$, $d_T(f(x_c), f(x_d)) < \epsilon$, which shows that $f(x_n)$ is Cauchy.* $\square$

Corollary 22 *Let $(x_n) \in S$ be a sequence with a limit in $\bar{S}$, that is, considered as a sequence in $\bar{S}$, $x_n \to x \in \bar{S}$. Let $f : S \to \mathbb{R}$ be uniformly continuous. Then $\lim_{n\to\infty} f(x_n)$ exists.*

Proof: *If $x \in S$, the proof is trivial due to continuity of f. Otherwise, (x_n) is Cauchy because all sequences with limits are Cauchy, and the uniform continuity of f implies that $f(x_n)$ is Cauchy. Since $\mathbb{R}$ is complete, $f(x_n)$ has a limit in $\mathbb{R}$.* $\square$

Lemma 23 *Let $(a_n), (b_n) \in S$ be sequences with the same limit in $\bar{S}$, that is, $a_n, b_n \to x$. Let $f : S \to \mathbb{R}$ be uniformly continuous. Then $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n)$.*

Proof: *Let $\lim_{n\to\infty} f(a_n) = L$, $\lim_{n\to\infty} f(b_n) = M$. Because a_n and b_n considered in $\bar{S}$ are approaching the same point, for large enough n, $d(a_n, b_n)$ can be made arbitrarily small. Because the elements of $(a_n), (b_n)$ are in S, by the uniform continuity of f, $|f(a_n) - f(b_n)|$ can be made arbitrarily small for large n. Similarly, because $\lim_{n\to\infty} f(a_n) = L$ and $\lim_{n\to\infty} f(b_n) = M$, $|f(a_n) - L|$ and $|f(b_n) - M|$ can be made arbitrarily small for large n. By the Triangle Inequality, $|L - M|$ can be made arbitrarily small for large n, implying $L = M$.* $\square$

Now we prove the main result. By the theorems in the book, $x \in S'$ implies that there exists a sequence $(x_n) \in S$ such that $x_n \to x$. Define $\bar{f} : \bar{S} \to \mathbb{R}$ as

$$\bar{f}(x) = \begin{cases} f(x) & x \in S \\ \lim_{n\to\infty} f(x_n) & \text{else} \end{cases}$$

where $(x_n) \in S, x_n \to x$ is arbitrary. As stated above, for all $x \in S$, at least one such (x_n) exists, and by Lemma 23, all such sequences share the same limit under f. Therefore, $\bar{f}$ is uniquely defined. Because $\bar{S} = S \cup S'$, $\bar{f}$ is properly defined. $\bar{f}$ trivially continues f.

Now to show uniform continuity. If $p, q \in S$, then the result is trivial. Fix $\epsilon > 0$, and choose δ for the uniform continuity of f such that for all $a, b \in S$, $d(a, b) < \delta$ implies $|f(b) - f(a)| < \epsilon$.

Let $p, q \in \bar{S}, p, q \in S^C$ such that $d(p, q) \leq \delta/3$. Then $p, q \in S'$. From the above, there exist sequences $(p_n), (q_n) \in S$ such that $p_n \to p$, $q_n \to q$. Because $p_n \to p$, there exists n_1 such that for all $n \geq n_1$, $d(p_{n_1}, p) < \delta/3$. Similarly, because $\bar{f}(p_n) \to \bar{f}(p)$ by construction, there exists n_2 such that for all $n \geq n_2$, $|\bar{f}(p_{n_2}) - \bar{f}(p)| \leq \epsilon$. Define $n^* = \max\{n_1, n_2\}$. Define $p' = p_{n^*} \in S$. Define $q' \in S$ similarly.

By construction of p' and q', $d(p', p) < \delta/3$ and $d(q', q) < \delta/3$, and by assumption $d(p, q) < \delta/3$. By the Triangle Inequality, this implies that $d(p', q') < \delta$. Since $p', q' \in S$, the uniform continuity of f implies that $|f(p') - f(q')| = |\bar{f}(p') - \bar{f}(q')| < \epsilon$. By construction of p' and q', $|\bar{f}(p') - \bar{f}(p)| < \epsilon$ and $|\bar{f}(q') - \bar{f}(q)| < \epsilon$. By the Triangle Inequality, this implies that $|\bar{f}(p) - \bar{f}(q)| < 3\epsilon$, which can be made arbitrarily small.

If only one of p or q is in $\bar{S} - S$, the proof is essentially the same. Thus, $p, q \in \bar{S}$ such that $d(p, q) < \delta/3$ implies that $|\bar{f}(p) - \bar{f}(q)| < 3\epsilon$, implying that $\bar{f}$ is uniformly continuous.

Part b

Prove that $\bar{f}$ is the unique continuous extension of f to a function defined on $\bar{S}$.

If $S = \bar{S}$ then uniqueness of the extension is trivial. Otherwise, let $g : \bar{S} \to \mathbb{R}$ be a continuous extension of $f : S \to \mathbb{R}$ such that $g \neq \bar{f}$. Then there exists $p \in \bar{S}$ such that $g(p) \neq \bar{f}(p)$. Because g and $\bar{f}$ are both extensions of f, for all $x \in S$, $\bar{f}(x) = g(x)$. Thus $p \in \bar{S} - S \subset S'$. Because p is a cluster point of S, there exists $(p_n) \in S$ such that $p_n \to p$.

Because $(p_n) \in S$ and g is an extension of f, for all n, $g(p_n) = f(p_n)$. Taking limits, $\lim_{n\to\infty} g(p_n) = \lim_{n\to\infty} f(p_n) = \bar{f}(p)$, by the definition of $\bar{f}$. By the continuity of g, $\lim_{n\to\infty} g(p_n) = g(p)$. Thus $g(p) = \bar{f}(p)$, contradicting the assumption that $g(p) \neq \bar{f}(p)$. Thus there is no continuous extension g that differs from $\bar{f}$, and $\bar{f}$ is the unique continuous extension of f to $\bar{S}$.

Part c

Prove the same things when $\mathbb{R}$ is replaced with a complete metric space N.

The proof is essentially the same. The only properties of $\mathbb{R}$ that we used in the above are that it is a metric space, and that it is complete. Replace all absolute value signs with the metric on N, and replace all references to d with references to the metric on S.

Problem 55

The distance from a point p in a metric space M to a nonempty subset $S \subset M$ is defined to be $\operatorname{dist}(p, S) = \inf\{d(p, s) : s \in S\}$.

Part a

Show that p is a limit of S if and only if $\operatorname{dist}(p, S) = 0$.

For the forward, p being a limit of S implies that for all $\epsilon > 0$, there exists $s \in S$ such that $d(s, p) < \epsilon$. This is equivalent to the definition of the infimum. For the reverse, $\operatorname{dist}(p, S) = 0$ implies that for all $\epsilon > 0$, there exists $s \in S$ such that $d(s, p) < \epsilon$. Choose $\epsilon_n = 1/n$, and let the associated $s_n \in S$ form a sequence that converges to p. Thus p is a limit point of S.

Part b

Show that $p \to \operatorname{dist}(p, S)$ is a uniformly continuous function of $p \in M$.

Let $\delta = \epsilon$, and let $p, q \in M$. We first consider the case when p or q are in $\bar{S}$. Without loss of generality, let $p \in \bar{S} = S \cup S'$. Then $\operatorname{dist}(p, S) = 0$. Since $p \in \bar{S} = S \cup S'$, there exist points in S arbitrarily close to p, so $\inf\{d(q, s) : s \in S\} \leq d(p, q)$. Thus

$$|\operatorname{dist}(p, S) - \operatorname{dist}(q, S)| = \operatorname{dist}(q, S) = \inf\{d(q, s) : s \in S\} \leq d(p, q) < \epsilon$$

which proves uniform continuity.

If $p, q \in \bar{S}^C$, then we proceed through the equality trichotomy.

Lemma 24 *Let $p, q \in S$ such that $d(p, q) < \epsilon$. Then $dist(p, S) \leq dist(q, S) + \epsilon$*

Proof: *Suppose not. Then there exists $\xi > 0$ such that $dist(p, S) \geq dist(q, S) + \epsilon + \xi$. Because $dist(p, S) = \inf\{d(p, s) : s \in S\}$ is a greatest lower bound, there exists $s \in S$ such that $dist(p, S) \leq d(p, s) < dist(p, S) + \frac{\xi}{2}$. By the Triangle Inequality,*

$$dist(q, S) \leq d(q, s) \leq d(p, q) + d(p, s) \leq dist(p, S) + \epsilon + \frac{\xi}{2}$$

which contradicts the assumption that $dist(q, S) > dist(p, S) + \epsilon$. □

Thus $\operatorname{dist}(p, S) \leq \operatorname{dist}(q, S) + \epsilon$. By symmetry, this also implies that $\operatorname{dist}(q, S) \leq \operatorname{dist}(p, S) + \epsilon$. Combining the two inequalities gives $|\operatorname{dist}(p, S) - \operatorname{dist}(q, S)| \leq \epsilon$, as desired.

Problem 56

Show that the 2-sphere is not homeomorphic to the plane.

The 2-sphere is compact, because it is a closed and bounded subset of $\mathbb{R}^3$. If the 2-sphere is homeomorphic to the plane, then because compactness is a topological property, the plane is compact.

I claim that plane is not compact. Cover the plane with open disks with radius 1 centered at all points $\mathbb{Z}^2$. By compactness this should reduce to a finite subcover. However, we can not remove a single disk without losing the covering property. Say we remove the disk centered at the origin. Then there is no other disk that covers the origin, for the disks at $(0,1),(1,0)$, etc. are too far away, and the other disks are even further away. Thus this open cover does not reduce to a finite subcover, and the plane is not compact.

Problem 57

If S is connected, is the interior of S connected? Prove this or give a counterexample.

No. Let M be the $\mathbb{R}^2$ plane, and let S be the union of two disjoint closed disks, with a line drawn between them. S is connected, but the interior of S consists of two disjoint open disks, which are disconnected.

Problem 58

Theorem 49 in the book states that the closure of a connected set is connected.

Part a

Is the closure of a disconnected set disconnected?

No. Let M be $\mathbb{R}$, and let S be a punctured closed interval. S is disconnected but its closure, a closed interval, is connected.

Part b

What about the interior of a disconnected set?

Still no. Let M be $\mathbb{R}$, and let S be the union of an open interval with a point not contained in that interval. S is disconnected but its interior is just an open interval, which is connected.

Problem 59

Prove that every countable metric space (not empty and not a singleton) is disconnected.

Connectedness is a topological property. Thus a countable metric space M is either finite, making it homeomorphic to a finite (nonempty and non-singleton) subset of $\mathbb{R}$, or countably infinite, making it homeomorphic to $\mathbb{R}$. Thus this statement is equivalent to proving that $\mathbb{R}$ and finite subsets of $\mathbb{R}$ are disconnected. But this is easy, since all subsets of $\mathbb{R}$ are clopen. All singletons are open in $\mathbb{R}$, and the union of open sets is open, meaning all sets are open, meaning all sets are closed. Since $\mathbb{R}$ or its nonempty or nonsingleton subsets are nonempty, there always exists a partition of $\mathbb{R}$ or its nonempty/nonsingleton subsets into two proper clopen subsets. By homeomorphism, this implies M is disconnected.

Problem 60

Part a

Prove that a continuous function $f : M \to \mathbb{R}$, all of whose values are integers is a constant provided that M is connected.

I will assume that M is nonempty. If M is a singleton, the proof is trivial.

Assume that M has at least two elements. Because f is continuous and M is connected, the range of f is connected. Suppose that there are at least two distinct points in the range of f. Denote them $x, y \in \mathbb{Z}$. Then the range of f is disconnected. The set $\{x\}$ is proper clopen subset of the range of f. It is closed, because singleton sets are always closed in any metric space. It is open, because we can construct an open ball with radius $1/2$ centered at x that is a subset of the set. It is proper, because y is not in $\{x\}$. Thus the range of f is disconnected. Since this is a contradiction, there is only one point in the range of f. Thus f is a constant function.

Part b

What if all the values are irrational?

The same conclusion holds, that f is a constant function. Again, if M has only one element, the proof is trivial. Otherwise, let x, y be distinct irrational numbers in the range of f. By the denseness of rationals in $\mathbb{R}$, there exists a rational number $p \in (x, y)$.

Consider the metric space $X = \mathbb{R} - \{p\}$, the real line with p removed, and the subset $A = (-\infty, p)$. A and A^C are open in X, implying that A is a clopen subset of X. Letting F be the range of f, because $F \subset X$, we see that $A \cap F$ is a clopen subset of F by the Inheritance Principle. Because $p < y$, y is not in $A \cap F$, so $A \cap F$ is a proper clopen subset of F. Thus the range of f is disconnected, a contradiction. Thus there are no two distinct points x, y in the range of f, and f is a constant function.

Problem 61

Prove that the double cone $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = x^2\}$ is path-connected.

The double-cone consists of a union of lines that meet at the origin, and form a 45 degree angle with the xy plane. Each point in the double cone lies on one of those lines. If a and b are points on the double-cone, we can draw a path from a, up the line, until we hit the z coordinate of b. Then since for a given z coordinate, the double cone is a circle, we can travel along the circle until we reach b. This is a path. Since this holds for all points, the double cone is path-connected.

Problem 62

Prove that the annulus $A = \{z \in \mathbb{R}^2 : r \leq |z| \leq R\}$ is connected.

The easiest way to do this is via path connectedness. For a and b in the annulus, starting from a, draw a path along the radial direction of the annulus until the radius matches the radius of b. Then move along the circle with constant radius until you reach b. This is a path, implying that the annulus is path-connected, implying that it is connected.

Problem 63

A subset E of $\mathbb{R}^m$ is **starlike** if it contains a point p_0 (called a **center** for E) such that for each $q \in E$, the segment between p_0 and q lies in E.

Part a

If E is convex and nonempty prove that it is starlike.

If E consists of one point the proof is trivial. Otherwise, pick an arbitrary point of E and denote it the center, p_0. For all other points $x \in E$, by convexity, the line segment between x and p_0 lies in E.

Part b

Why is the converse false?

Consider a three-pointed star in $\mathbb{R}^2$. It is starlike, but it is not convex. The line between two points of the star does not lie in the star.

Part c

Is every starlike set connected?

Yes. Let $p, q \in E$ be arbitrary. Because E is starlike, there exists a path between p and p_0, and between p_0 and q. Putting those together gives a path between p and q, implying that E is path-connected, implying that E is connected.

Part d

Is every connected set starlike? Why or why not?

No. By Part c, being starlike implies path connectedness. The topologist's sine curve is connected, but not path-connected, and thus not starlike.

Problem 64

Suppose that $E \subset \mathbb{R}^m$ is open, bounded, and starlike, and p_0 is a center for E.

Part a

Is it true or false that all points p_1 in a small enough neighborhood of p_0 are also centers for E?

False. Consider $\mathbb{R}^2$, the open rectangles $A = (-1,1)\times(2,0)$, $B = (-1/2,-1/2)\times(1/2,1/2)$, and let $E = A \cup B$. E is clearly open and bounded, and a bit of thought will show that A is starlike with $p_0 = (0,0)$ as a center.

For all $0 < \epsilon < \frac{1}{2}$, consider the point $p_1 = (0,-\epsilon)$. $p_1 \in E$. The point $p_2 = (1,\epsilon)$ is also an element of E. By symmetry, the line segment between p_1 and p_2 runs through $(\frac{1}{2},0)$, which is not in E. Thus for all $0 < \epsilon < \frac{1}{2}$, p_1 is not a center for E.

Part b

Is the set of centers convex?

Yes. To start, we begin with the $\mathbb{R}^2$ case. We begin by showing that if $a, b \in E$ are centers and $c \in E$ is arbitrary, then the 'area' between a, b, c is a subset of E.

Lemma 25 *Let $E \subset \mathbb{R}^2$ be starlike. Let $a, b \in E$ be centers of E, and let $c \in E$ lie on the line $\bar{ab}$. Then the segment $\bar{cab}$ (or $\bar{cba}$ depending on the ordering) is a subset of E.*

Proof: *Without loss of generality, assume that the order is $\bar{cab}$. Because b is a center of E, the line segment $\bar{cb} \subset E$, and $\bar{cab} = \bar{cb}$.* $\square$

Theorem 26 *Let $E \subset \mathbb{R}^2$ be starlike. Let $a, b \in E$ be centers of E, and let $c \in E$ not lie on the line $\bar{ab}$. Then the triangle $\triangle abc \subset E$.*

Proof: *Because a and b are centers, the line segments $\bar{ab}$, $\bar{ac}$, and $\bar{bc}$ are subsets of E. These form the outline of a triangle $\triangle abc$. For any point d inside $\triangle abc$, extend the line segment $\bar{ad}$ until it intersects $\bar{bc}$, and denote that point e. Because $e \in \bar{bc}$, $e \in E$. Because a is a center of E, $\bar{ac} \subset E$, which implies $\bar{ad} \subset E$. Repeating this process for all points inside the outline of $\triangle abc$ shows that $\triangle abc \subset E$.* $\square$

Theorem 27 *Let $E \subset \mathbb{R}^2$ be starlike. Then the set of centers of E is convex. Specifically, if $a, b \in E$ are centers and c lies on the line segment $\bar{ab}$, then c is a center of E.*

Proof: *Fix a, b, c, and let $d \in E$ be arbitrary. By Theorem 26, $\triangle abd \subset E$. Since $c \in \bar{ab}$, $c \in \triangle abd$, and so the line segment $\bar{cd} \subset \triangle abd \subset E$. Repeating this for all points $d \in E$ shows that c is a center of E.* $\square$

The result extends to $\mathbb{R}^m$.

Theorem 28 *Let $E \subset \mathbb{R}^m$ be starlike. Then the set of centers is convex.*

Proof: *If $m = 1$, the proof is trivial, because the starlike subsets of $\mathbb{R}$ are intervals. The $m = 2$ case has been proven.*

For $m > 2$, let $a, b \in E$ be centers of E, $c \in \bar{ab}$ be the prospective center, and $d \in E$ be arbitrary. If d is collinear with $\bar{acb}$, then the proof is trivial. Otherwise, a line and a point define a plane. By Theorem , the triangle $\triangle abd$ in this plane is a subset of E, which implies that $\bar{dc} \subset E$. Thus c is a center for E, and the set of centers of E is convex. □

Here's a random lemma that I came up with. I don't need it yet, but I might need it later, so I'm saving it.

Lemma 29 *For all $q \in E$, there exists $\epsilon_q > 0$ such that for all points $p_1 \in E$ in the ϵ_q neighborhood of p_0, the segment between q and p_1 lies in E.*

Proof: *Let S be the segment between q and p_0, and let $\epsilon_0 > 0$ be the radius of the epsilon-ball around p_0 that lies completely in E. Since $S \subset E$ and E is open, there is a covering of S by open balls that are subsets of E centered at the points of S. Since E is bounded, S is bounded. Since S is a closed and bounded subset of $\mathbb{R}^m$, S is compact, and so the covering reduces to a finite subcovering. Add the epsilon-ball around p_0 to the finite subcovering.*

Let $\epsilon_q > 0$ be the minimum of those epsilons in the balls of the finite subcovering. By construction, all points that are closer than ϵ_q-away from S are in E. For all p_1 such that $d_E(p_0, p_1) < \epsilon_q$, let T be the line between p_1 and q. By construction, the epsilon-ball around p_0 was in the finite subcovering, so $\epsilon_q \leq \epsilon_0$, implying that $p_1 \in E$.

By the Pythagorean Theorem, the distance from p_1 to S is less than ϵ_q, and the distance between any point on T and the line S is also less than ϵ_q. Thus T lies within E. □

Problem 70

Prove that (a, b) and $[a, b)$ are not homeomorphic.

A single point, a, can be removed from $[a, b)$ without disconnecting it, but removing a single point from (a, b) disconnects it. Since connectedness is a topological property, the two are not homeomorphic.

Problem 81

The topologist's sine curve is the set

$$\{(x, y) : x = 0 \text{ and } |y| \leq 1 \text{ or } 0 < x \leq 1 \text{ and } y = \sin(\frac{1}{x})$$

The **topologist's sine circle** is is the union of a circular arc and the topologist's sine curve. Prove that it is path-connected but not locally path-connected. (M is **locally path connected** if for each $p \in M$ and each neighborhood U of p there is a path-connected subneighborhood V of p).

I will denote A to be the part of the topologist's sine curve that is a vertical line segment through the origin, and B to be the part that consists of the sine curve. It's clear that A and B individually are path-connected. The entire sine circle is path-connected because the circular arc connects the two sets.

Consider $p = (0, 1) \in A$ and the of $\epsilon_0 = \frac{1}{2}$ neighborhood around p. The neighborhood does not have a path-connected subneighborhood. Each cycle of the sine waves goes from -1 to 1 and back again, meaning that for all $\epsilon \leq \frac{1}{2}$, B leaves the ϵ_0 neighborhood of p. Thus the intersection of B and the ϵ_0 neighborhood of p is a series of disconnected peaks of the sine function, which implies that the epsilon neighborhood of p does not have a path connected subneighborhood.

Problem 96

If $A \subset B \subset C$, A is dense in B, and B is dense in C prove that A is dense in C.

Because A is dense in B, $B \subset \bar{A}$, and all points in B are limit points of A. Similarly, since C is dense in B, $C \subset \bar{B}$, and all points in C are limit points of B.

Let $c \in C$ be arbitrary. There exists $b_j \in B$ such that $b_j \to C$, and for each b_j, there exists $a_{i,j} \in A$ such that $a_{i,j} \to b_j$ with respect to i. $a_{i,j} \to b_j$ and $b_j \to c$ imply that $a_{i,i} \to c$ by the Triangle Inequality.

Specifically, there exists $I \in \mathbb{N}$ such that $i \geq I$ implies that $|a_{i,j} - b_j| < \epsilon/2$. Similarly, there exists $J \in \mathbb{N}$ such that $j \geq J$ implies that $|b_j - c| < \epsilon/2$. For $N = \max(I, J)$ and $n \geq N$,

$$|a_{n,n} - c| \leq |a_{n,n} - b_n| + |b_n - c| < \epsilon$$

which shows that c is a limit point of A, and thus in $\bar{A}$. Thus $C \subset \bar{A}$ and A is dense in C.

Problem 126

Suppose E is an uncountable subset of $\mathbb{R}$. Prove that there exists a point $p \in \mathbb{R}$ where E condenses.

I will start with some lemmas that will lead to using the decimal expansion.

Lemma 30 *There exists $n \in \mathbb{N}$ such that $[n, n + 1)$ contains uncountable elements of E.*

Proof: *Suppose not. Then for all $n \in \mathbb{N}$, $[n, n+1)$ contains countable elements of E. Taking unions over all $\mathbb{N}$, the countable union of countable elements is countable, so E is countable, a contradiction.* □

Lemma 31 *Given $n \in \mathbb{N}$ such that $[n, n + 1)$ contains uncountable elements of E, there exists a decimal subinterval of $[n, n + 1)$ such that the decimal subinterval contains uncountable E.*

(A decimal subinterval $[a, b)$ has endpoints $a = n + k/10, b = n + (k + 1)/10$ for some digit $k, 0 \leq k \leq 9$).

Proof: *The proof is similar to Lemma 30. Suppose not. Then there are at most countable elements of E in each subinterval. Since there are ten decimal subintervals, there are at most countable elements of E in the interval $[n, n+1)$, a contradiction.* $\square$

Let $n \in \mathbb{N}$ denote the interval $[n, n+1)$ that contains uncountable elements of E. Let $n.n_1 \in 0, 1 \ldots 9$ denote the decimal subinterval $[n.n_1, n.(n_1+1))$ of $[n, n+1)$ that contains uncountable elements of E. Specifically, $n.n_1$ denotes the interval $[n.n_1, n.(n_1+1))$ unless $n_1 = 9$, then $n.n_1$ denotes the interval $[n.n_1, (n+1).0)$.

This can be extended to all $k \in \mathbb{N}$. If $p_k \in \mathbb{R}$ is a real number with a terminating decimal expansion $n.n_1 n_2 \ldots n_k$ of length k, then p_k denotes the decimal subinterval of length 10^{-k}. Specifically, p_k denotes $[n.n_1 n_2 \ldots n_k, n.n_1 n_2 \ldots (n_k + 1))$ unless $n_k = 9$, in which the appropriate addition and carrying occurs. Denote I_k as the (decimal sub)-interval of length 10^{-k} that contains uncountably many points of E.

Theorem 32 *Let $p_0, p_1, p_2 \ldots p_k \cdots \in \mathbb{R}$ be the representations of the decimal subintervals I_k length 10^{-k} that contain uncountable elements of E. Then there exists $p = \cap_{k=0}^{\infty} I_k$ such that p is a condensing point of E.*

Proof: *The I_k are closed intervals in $\mathbb{R}$, so they are compact. They are nested and nonempty. The diameter of the sets I_k is the length of the interval 10^{-k}, which tends to 0 as $k \to \infty$. Thus, by the theorems in the book, $\cap_{k=0}^{\infty} I_k$ is a single point, and we can denote it p.*

For all $\epsilon > 0$, by the Archimedian property there is some $k \in \mathbb{N}$ such that $10^{-k} < \epsilon$. Taking the decimal expansion of p and truncating it at k, $p_0.p_1 p_2 \ldots p_k$ corresponds to an interval $I_k = [p_0.p_1 p_2 \ldots p_k, p_0.p_1 p_2 \ldots (p_k+1))$ that contains uncountably many elements of E. The length of this interval, I_k, is 10^{-k}, and since $p \in I_k$ and $\epsilon > 10^{-k}$, I_k is a subset of the epsilon-ball around p. Thus there are uncountable elements of E within distance epsilon of p for all positive epsilon, implying that p is a condensing point of E. $\square$

The argument very easily generalizes to $\mathbb{R}^n$. From a similar argument to above, we can find cubes $A_1, A_2 \ldots A_k$ with side lengths 10^{-k} such that there are uncountable elements of E within the cube. By the Heine-Borel property on $\mathbb{R}^n$, A_k is compact, and the intersection of all these cubes is a single point, p. For any epsilon-ball, we can find a cube A_k that fits within the epsilon-ball centered at p that contains uncountably many elements of E. Since this holds for all epsilon, it follows that p is a condensing point of E.