Chapter 5 Multivariable Calculus

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Problem 1

Let $T : V \to W$ be a linear transformation, and let $p \in V$ be given. Prove that the following are equivalent:

(a) T is continuous at the origin.

(b) T is continuous at p.

(c) T is continuous at at least one point of V.

For a to b, let $p_n \to p \in V$ be arbitrary. Consider $a_n = p_n - p$, which converges to 0. Then

$$T(p_n) = T(a_n + p) = T(a_n) + T(p) \to T(p)$$

by the linearity of T and continuity at the origin.

b to c is trivial. For c to a, let T be continuous at $q \in V$ and let $a_n \to 0$ be arbitrary. Then $q_n = q + a_n \to q$, and

$$T(a_n) = T(q_n - q) = T(q_n) - T(q) \to T(q) - T(q) = 0$$

Problem 22

If Y is a metric space and $f : [a, b] \times Y \to \mathbb{R}$ is continuous, then show that

$$F(y) = \int_a^b f(x, y)dx$$

is continuous.

Let $y_0 \in Y$ be arbitrary; we will show continuity of $F(y_0)$. Let P be a partition of $[a, b]$, and let $a = x_1 < \cdots < x_n = b$ be the points of the partition. Since f is continuous, for an arbitrary $\epsilon > 0$, there exist $\delta_1, \ldots \delta_n$ such that

$$|y - y_0| < \delta_i \Rightarrow |f(x_i, y) - f(x_i, y_0)| < \frac{\epsilon}{n \times \text{mesh } P}$$

Let $\delta = \min(\delta_i)$. Thus for $|y - y_0| < \delta$, letting $g(x) = f(x, y)$ and $h(x) = f(x, y_0)$,

$$|R(g, P, T) - R(h, P, T)| = \sum_{i=1}^{n} |f(x_i, y_0) - f(x_i, y)|\Delta t_i < \epsilon$$

and letting the mesh go to zero,

$$|F(y) - F(y_0)| = \left| \int_a^b f(x, y)dx - \int_a^b f(x, y_0)dx \right| \leq \epsilon$$

as desired.