# Problem 46

## Overview

Chromatin folding is process in biology. We assume that chromatin polymers follow a random walk model. In this model, the two-dimensional distance between two chromatin fibers, $R$, follows a Raleigh distribution, with density

$$f(r|\theta) = \frac{r}{\theta^2} e^{\frac{-r^2}{2\theta^2}}$$

where $r \geq 0$ and $\theta > 0$.

## Estimating Theta

### Maximum Likelihood

#### Derivation of the Estimator

The likelihood function is

$$lik(\theta) = \frac{1}{\theta^{2n}} e^{\frac{-1}{2\theta^2} \sum_{i=1}^{n} r_i^2} \prod_{i=1}^{n} r_i$$

The log likelihood is thus

$$l(\theta) = -2n \ln(\theta) + \sum_{i=1}^{n} \ln(r_i) - \frac{1}{2\theta^2} \sum_{i=1}^{n} r_i^2$$

The first-order condition for maximizing the likelihood satisfies

$$0 = \frac{-2n}{\hat{\theta}} + \frac{1}{\hat{\theta}^3} (\sum_{i=1}^{n} r_i^2)$$

Which, after rearranging, gives the maximum likelihood estimator for $\theta$, $\hat{\theta}_{MLE}$

$$\hat{\theta}_{MLE} = \frac{1}{\sqrt{2}} \sqrt{\frac{1}{n} \sum_{i=1}^{n} r_i^2}$$

### Asymptotic Variance

As usual, the asymptotic variance of a maximum likelihood estimator is roughly

$$Var(\hat{\theta}_{MLE}) \approx \frac{1}{nI(\theta)}$$

where under sufficient smoothness conditions that at the level of this textbook we will assume hold,

$$I(\theta) = E\left[\frac{\partial}{\partial\theta}\log(f(x|\theta))\right]^2 = -E\left[\frac{\partial^2}{\partial\theta^2}\log(f(x|\theta))\right]$$

$f(r|\theta) = \frac{r}{\theta^2}e^{\frac{-r^2}{2\theta^2}}$, so

$$\begin{aligned}\frac{\partial}{\partial\theta}\log f(r|\theta) &= -\frac{2}{\theta} + \frac{r^2}{\theta^3},\\ \frac{\partial^2}{\partial\theta^2}\log f(r|\theta) &= \frac{2}{\theta^2} - 3\frac{r^2}{\theta^4},\\ I(\theta) &= -\frac{2}{\theta^2} + 3\frac{E(R^2)}{\theta^4}\end{aligned}$$

The expectation of $R^2$ is

$$E(R^2) = \frac{1}{\theta^2}\int_0^\infty r^3 e^{\frac{-r^2}{2\theta^2}}dr = 4\theta^2\int_0^\infty x^3e^{-x^2}dx = 2\theta^2\int_0^\infty ue^{-u}du$$

by making the substitutions $x = \frac{r}{\sqrt{2}\theta}$ and $u = x^2$. Integrating by parts,

$$E(R^2) = 2\theta^2\left[-ue^{-u}|_0^\infty + \int_0^\infty e^{-u}du\right] = -2\theta^2(e^{-u})|_0^\infty = 2\theta^2$$

Thus

$$\begin{aligned}I(\theta) &= -\frac{4}{\theta^2},\\ Var(\hat{\theta}_{MLE}) &\approx \frac{\theta^2}{4n}\end{aligned}$$

## Method of Moments

### Derivation of the Estimator

The expectation of $R$ is

$$E(R) = \frac{1}{\theta^2}\int_0^\infty r^2e^{\frac{-r^2}{2\theta^2}}dr = (2\sqrt{2})\theta\int_0^\infty x^2e^{-x^2}dx$$

by making the substitution $x = \frac{r}{\sqrt{2}\theta}$. Integrating by parts,

$$E(r) = (2\sqrt{2})\theta\left[\frac{1}{2}xe^{-x^2}|_0^\infty + \frac{1}{2}\int_0^\infty e^{-x^2}dx\right] = \sqrt{2}\theta\int_0^\infty e^{-x^2}dx = \theta\sqrt{\frac{\pi}{2}}$$

since $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}$, and the integrand is an even function. Thus

$$\hat{\theta}_{MoM} = \mu_1 \sqrt{\frac{2}{\pi}}$$

**Asymptotic Variance**