

Problem 12

Let $X_1 \dots X_n$ be iid from an exponential distribution with $f(x|\theta) = \theta e^{-\theta x}, x \geq 0$. Derive a likelihood test for $H_0 : \theta = \theta_0$ versus $H_A : \theta \neq \theta_0$ and show that the rejection region is of the form $\bar{X} e^{-\theta_0 \bar{X}} \leq c$.

The likelihood ratio is

$$\Lambda = \frac{\max_{\theta=\theta_0} \text{lik}(\theta)}{\max_{\theta \in \mathbb{R}^+} \text{lik}(\theta)}$$

where $\text{lik}(\theta|x) = \theta^n e^{-\theta \sum X_i}$. The maximum likelihood estimator of an exponential distribution is $\theta^* = \frac{1}{\bar{X}}$. Since small values of Λ are evidence against the null, the test rejects the null when

$$\Lambda = \frac{\theta_0^n e^{-\theta_0 n \bar{X}}}{(\frac{1}{\bar{X}})^n e^n} < c$$

θ_0^n is a constant and can be immediately absorbed into c . For the exponent, $\bar{X}^n e^{-\theta_0 n \bar{X} - n} = (\bar{X} e^{-\theta_0 \bar{X} e^{-1}})^n$, and since Λ is positive, taking the n th root won't introduce more inequalities. Thus by taking the n th root of both sides, e^{-1} can be absorbed into c , the rejection region has the form

$$\bar{X} e^{-\theta_0 \bar{X}} < c$$

as desired.

Problem 13

Continuing from the previous problem, suppose that $\theta_0 = 1$, $n = 10$, and $\alpha = .05$.

Part a

Show that the rejection region is of the form $\{\bar{X} \leq x_0\} \cup \{\bar{X} \geq x_1\}$, where x_0, x_1 are determined by c .

First note that $f(\bar{X}) = \bar{X} e^{-\theta_0 \bar{X}}$ is continuously differentiable. Note that $f(\bar{X})$ is always nonnegative, $f(0) = 0$, and $\lim_{\bar{X} \rightarrow \infty} f(\bar{X}) = 0$. Since $f'(\bar{X})$ is continuous, $f'(\bar{X}) = 0$ is a necessary condition for a local extrema. $f(\bar{X}) = 0$ when $\bar{X}^* = \frac{1}{\theta_0}$, and since this is the only inflection point of f , $f(\bar{X})$ must have a local extrema at \bar{X}^* . It's obvious that this is a maximum, since the endpoint limits go to zero.

Therefore, small values of $f(\bar{X})$ correspond to areas towards 0 and infinity, and away from the central hump. This corresponds to $\{\bar{X} \leq x_0\} \cup \{\bar{X} \geq x_1\}$.

Part b

Explain why c should be chosen so that $P(\bar{X}e^{-\bar{X}} \leq c) = .05$ when $\theta_0 = 1$.

Under the null hypothesis of $\theta_0 = 1$, the test statistic $\bar{X}e^{-\bar{X}}$ being small is evidence against the null. We want a 95 percent confidence set, so under the null, $P(rejectnull) = .05$. We reject the null when the test statistic is below a level c , so we need to choose c such that $(rejectnull) = P(\bar{X}e^{-\bar{X}} \leq c) = .05$.

Part c

Explain why $\sum_{i=1}^{10} X_i$ and \bar{X} follow gamma distributions. Use this information to choose c .

It's well established that when X and Y are iid Gamma with the same rate parameters, their sum is a Gamma random variable with its rate parameter equal to X and Y 's common rate parameter, and its shape parameter equal to the sum of X and Y 's shape parameters. Since an exponential random variable is a Gamma random variable with shape parameter 1, $\sum_{i=1}^{10} X_i$ is distributed $Gamma(10, \theta_0)$.

Let X be a Gamma random variable, and $Y = \frac{X}{m}$, with m a positive constant. Then $F_Y y = F_X cy$, and

$$f_Y(y) = cf_X cy = m \frac{\theta^\alpha}{\Gamma(\alpha)} (my)^{\alpha-1} e^{-\theta(my)} = \frac{(\theta m)^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-(\theta m)y}, y \geq 0$$

which is a $Gamma(\alpha, \theta m)$ random variable.

PUT STUFF HERE ABOUT THE DISTRIBUTION OF \bar{X} .