Problem 17

Let $X \sim N(0, \sigma^2)$, and let $H_0 : \sigma = \sigma_0$, $H_A : \sigma = \sigma_1$, with $\sigma_0 < \sigma_1$. The values of σ_0 and σ_A are fixed.

Part b

Let $X_1 \dots X_n$ be an iid sample from X. Find the likelihood ratio as a function of $X_1 \dots X_n$. What values favor the null? What is the rejection region of a α level test?

The likelihood ratio is the likelihood under the null divided by the likelihood under the alternative. Thus

$$\Lambda = \frac{\frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{1}{2\sigma_0^2}\sum_{i=1}^n X_i^2}}{\frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{1}{2\sigma_1^2}\sum_{i=1}^n X_i^2}} = \frac{\sigma_0}{\sigma_1} e^{-\frac{1}{2}\sum_{i=1}^n X_i^2 (\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2})}$$

High values of Λ favor the null. Since $\sigma_1 > \sigma_0$, $\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} > 0$, and so small values of $\sum X_i^2$ lead to large values of Λ. This intuitively makes sense. Since the mean of the distribution is assumed to be zero, small values of $\sum X_i^2$ indicate a smaller sample variance, and thus support the smaller variance of the null.

Under the null, $\frac{X}{\sigma_0} \sim Z$, so $\frac{X^2}{\sigma_0^2} \sim \chi_1^2$ and $\frac{1}{\sigma_0^2} \sum_{i=1}^n X_i^2 \sim \chi_n^2$. Thus, the rejection region at significance level α is when $\frac{1}{\sigma_0^2} \sum_{i=1}^n X_i^2 > \chi_n^2(\alpha)$.