

Problem 46

Overview

Chromatin folding is process in biology. We assume that chromatin polymers follow a random walk model. In this model, the two-dimensional distance between two chromatin fibers, R , follows a Raleigh distribution, with density

$$f(r|\theta) = \frac{r}{\theta^2} e^{-\frac{r^2}{2\theta^2}}$$

where $r \geq 0$ and $\theta > 0$.

Estimating Theta

Maximum Likelihood

Derivation of the Estimator

The likelihood function is

$$lik(\theta) = \frac{1}{\theta^{2n}} e^{-\frac{1}{2\theta^2} \sum_{i=1}^n r_i^2} \prod_{i=1}^n r_i$$

The log likelihood is thus

$$l(\theta) = -2n \ln(\theta) + \sum_{i=1}^n \ln(r_i) - \frac{1}{2\theta^2} \sum_{i=1}^n r_i^2$$

The first-order condition for maximizing the likelihood satisfies

$$0 = \frac{-2n}{\hat{\theta}} + \frac{1}{\hat{\theta}^3} \left(\sum_{i=1}^n r_i^2 \right)$$

Which, after rearranging, gives the maximum likelihood estimator for θ , $\hat{\theta}_{MLE}$

$$\hat{\theta}_{MLE} = \frac{1}{\sqrt{2}} \sqrt{\frac{1}{n} \sum_{i=1}^n r_i^2}$$

Asymptotic Variance

As usual, the asymptotic variance of a maximum likelihood estimator is roughly

$$\text{Var}(\hat{\theta}_{MLE}) \approx \frac{1}{nI(\theta)}$$

where under sufficient smoothness conditions that at the level of this textbook we will assume hold,

$$I(\theta) = E \left[\frac{\partial}{\partial \theta} \log(f(x|\theta)) \right]^2 = -E \left[\frac{\partial^2}{\partial \theta^2} \log(f(x|\theta)) \right]$$

$f(r|\theta) = \frac{r}{\theta^2} e^{-\frac{r^2}{2\theta^2}}$, so

$$\begin{aligned} \frac{\partial}{\partial \theta} \log f(r|\theta) &= -\frac{2}{\theta} + \frac{r^2}{\theta^3}, \\ \frac{\partial^2}{\partial \theta^2} \log f(r|\theta) &= \frac{2}{\theta^2} - 3\frac{r^2}{\theta^4}, \\ I(\theta) &= -\frac{2}{\theta^2} + 3\frac{E(R^2)}{\theta^4} \end{aligned}$$

The expectation of R^2 is

$$E(R^2) = \frac{1}{\theta^2} \int_0^\infty r^3 e^{-\frac{r^2}{2\theta^2}} dr = 4\theta^2 \int_0^\infty x^3 e^{-x^2} dx = 2\theta^2 \int_0^\infty u e^{-u} du$$

by making the substitutions $x = \frac{r}{\sqrt{2}\theta}$ and $u = x^2$. Integrating by parts,

$$E(R^2) = 2\theta^2 \left[-u e^{-u} \Big|_0^\infty + \int_0^\infty e^{-u} du \right] = -2\theta^2 (e^{-u}) \Big|_0^\infty = 2\theta^2$$

Thus

$$\begin{aligned} I(\theta) &= \frac{4}{\theta^2}, \\ \text{Var}(\hat{\theta}_{MLE}) &\rightarrow \frac{\theta^2}{4n} \end{aligned}$$

Method of Moments

Derivation of the Estimator

The expectation of R is

$$E(R) = \frac{1}{\theta^2} \int_0^\infty r^2 e^{-\frac{r^2}{2\theta^2}} dr = (2\sqrt{2})\theta \int_0^\infty x^2 e^{-x^2} dx$$

by making the substitution $x = \frac{r}{\sqrt{2}\theta}$. Integrating by parts,

$$E(R) = (2\sqrt{2})\theta \left[\frac{1}{2} x e^{-x^2} \Big|_0^\infty + \frac{1}{2} \int_0^\infty e^{-x^2} dx \right] = \sqrt{2}\theta \int_0^\infty e^{-x^2} dx = \theta \sqrt{\frac{\pi}{2}}$$

since $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}$, and the integrand is an even function. Thus

$$\hat{\theta}_{MoM} = \bar{X} \sqrt{\frac{2}{\pi}}$$

Asymptotic Variance

By the Central Limit Theorem, \bar{X} converges to a normal random variable with mean $E(R)$ and variance $\frac{Var(R)}{n}$ as n approaches infinity. From previous results.

$$Var(R) = E(R^2) - E(R)^2 = 2\theta^2 - \frac{\pi}{2}\theta^2 = \frac{4-\pi}{2}\theta^2$$

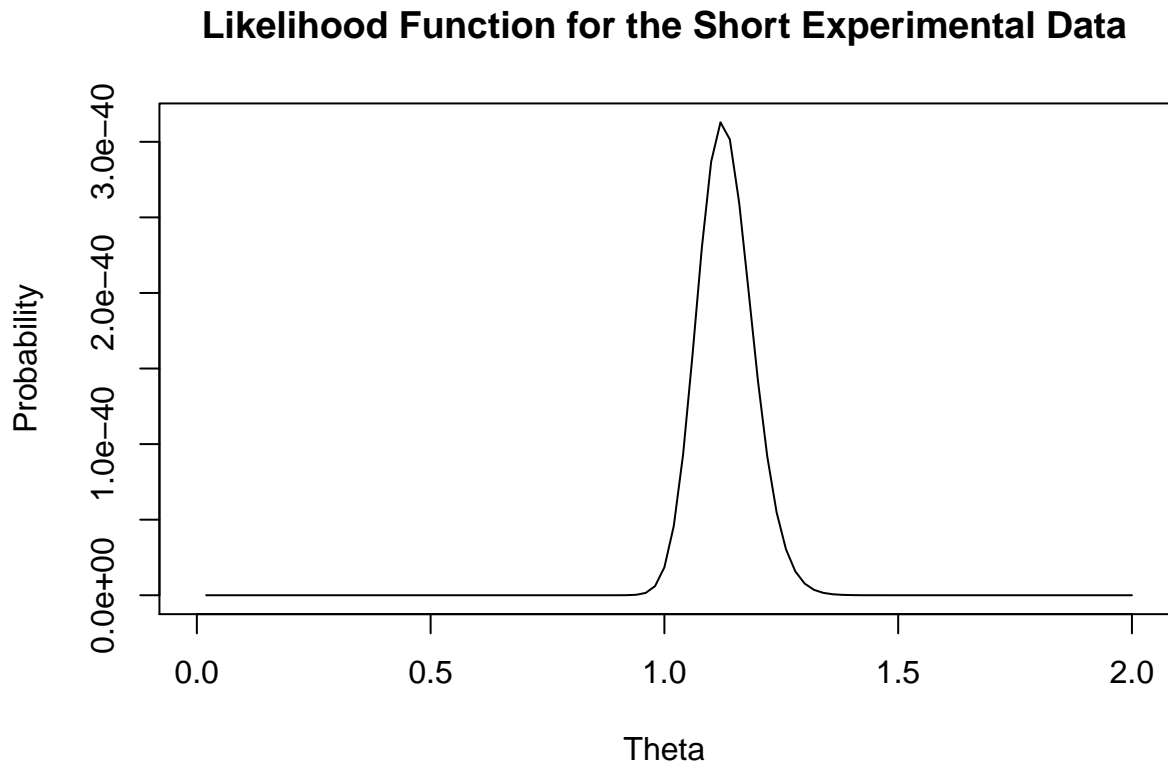
and the asymptotic variance of the method of moments estimator for θ is

$$Var(\hat{\theta}_{MoM}) \rightarrow \left(\frac{4}{\pi} - 1\right) \frac{\theta^2}{n}$$

in which $\frac{4}{\pi} - 1 \approx .273295\dots$

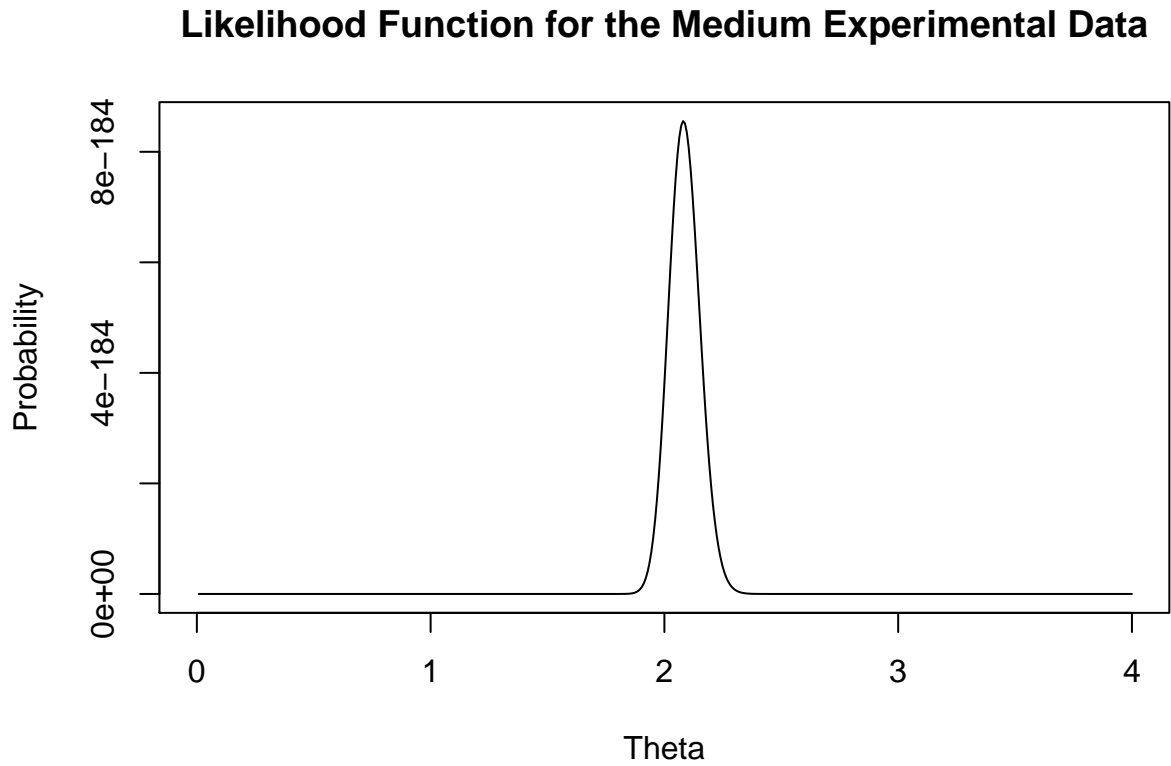
Data Analysis

Short Experiment



The MLE estimate is $\hat{\theta}_{MLE} = 1.1237776$, with an estimated variance of 0.0029265. The MoM estimate is $\hat{\theta}_{MoM} = 1.1705631$, with an estimated variance of 0.0039.

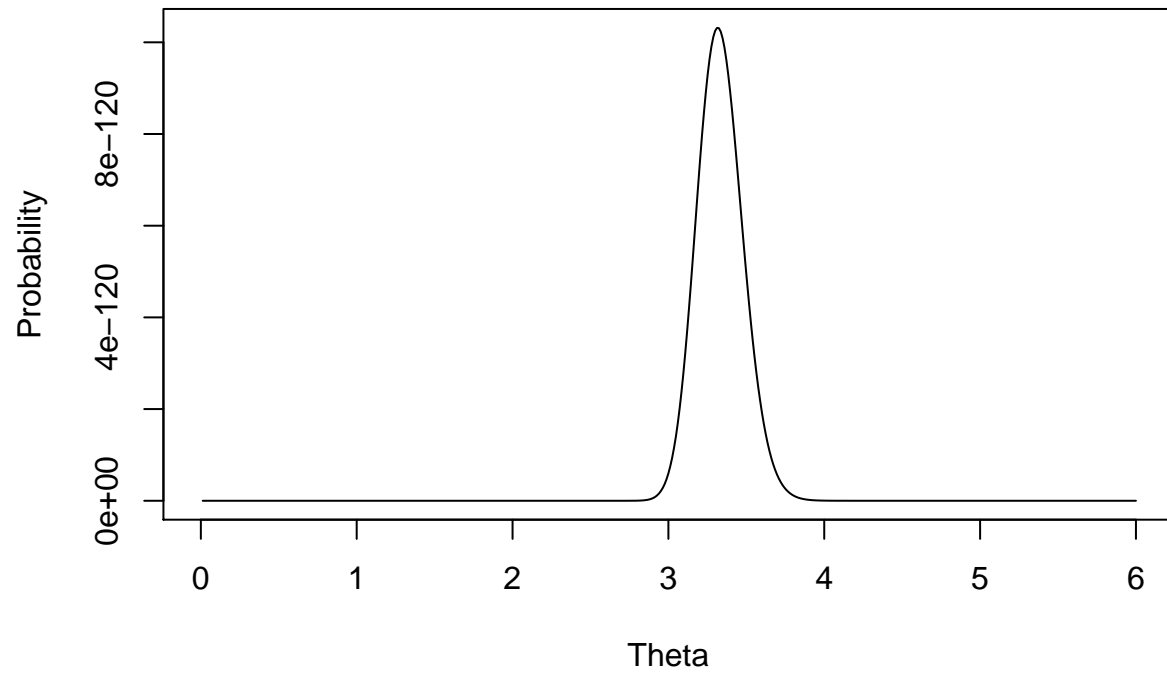
Medium Experiment



The MLE estimate is $\hat{\theta}_{MLE} = 2.0812133$, with an estimated variance of 0.0020896. The MoM estimate is $\hat{\theta}_{MoM} = 2.0748203$, with an estimated variance of 0.0047239.

Long Experiment

Likelihood Function for the Long Experimental Data



The MLE estimate is $\hat{\theta}_{MLE} = 3.3168463$, with an estimated variance of 0.0062819. The MoM estimate is $\hat{\theta}_{MoM} = 3.4044888$, with an estimated variance of 0.0239924.