Continuing off of Problem 63, let Y be a random variable indicating whether a randomly sampled item is defective or not. In all cases, $Y|\Theta$ is a Bernoulli random variable with parameter θ.

Part a

Find the marginal distribution of Y with Θ having a Uniform(0, 1) prior when X is unknown, and when $X = 3$.

When X is unknown, the marginal distribution of Θ is Uniform(0, 1). Thus

$$P(Y = 0) = \int P(Y = 0|\Theta = \theta)f_\Theta(\theta)d\theta = \int_0^1 1 - \theta d\theta = \frac{1}{2}$$

From the previous results, $\Theta|X = 3$ follows a Beta(4, 98) distribution. Thus

$$P(Y = 0|X = 3) = \int P(Y = 0|\Theta = \theta, X = 3)f_{\Theta|X}(\theta|X = 3)d\theta$$
$$\frac{102!}{98!4!}\int_0^1 (1 - \theta)\theta^3(1 - \theta)^{97}d\theta = \frac{1}{4}$$

Thus $P(Y = 1|X = 3) = \frac{3}{4}$.

Part b

Find the marginal distribution of Y with Θ having a Beta(1/2, 5) prior when X is unknown, and when $X = 3$.

Defining $\alpha = \frac{\Gamma(11/2)}{\Gamma(1/2)\Gamma(5)}$,

$$P(Y = 0) = \int P(Y = 0|\Theta = \theta)f_\Theta(\theta)d\theta = \alpha\int_0^1 (1 - \theta)\theta^{-\frac{1}{2}}(1 - \theta)^4 d\theta$$