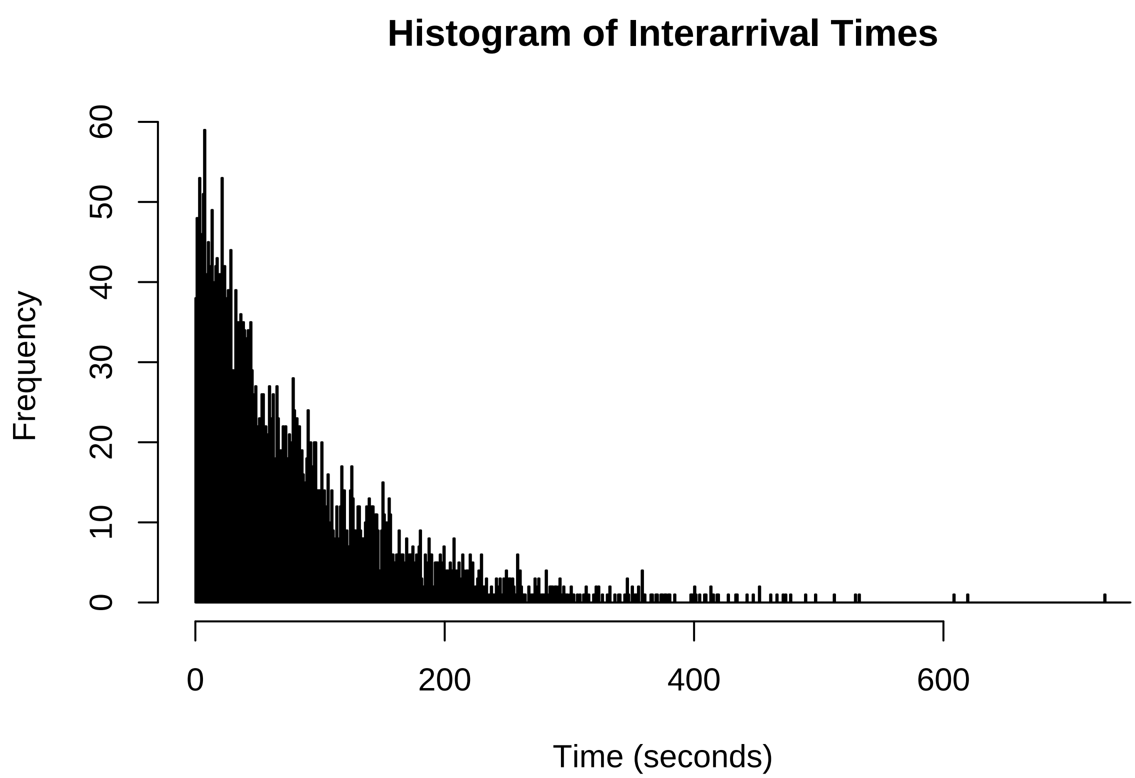
Problem 43

Gamma Ray Interarrival Times

Initial Histogram

The data consists of 3935 observations of interarrival times for gamma rays, with units in seconds.

```
hist(data, breaks = seq(from = 0, to = 750, by = 1), main = 'Histogram of Interarrival Times', xlab = '
```



The interarrival times look roughly exponential, or at least gamma, since the exponential distribution is a special case of the gamma distribution. For the moment, we will assume that interarrival times are distributed via a gamma distribution.

Parameter Estimates

Method of Moments

The method of moments estimators for α and γ are

$$\hat{\alpha}_{MoM} = \frac{\bar{X}^2}{\bar{\sigma}^2}$$

$$\hat{\lambda}_{MoM} = \frac{\bar{X}}{\bar{\sigma}^2}$$

```
#Computes the MoM estimates of alpha and lambda for a gamma distribution, respectively
gammaMoM <- function(data) {
  output <- list()
  output$alpha <- (mean(data)^2) / var(data)
  output$lambda <- mean(data) / var(data)
  return(output)
}
foo <- gammaMoM(data)
aMoM <- foo$alpha
lMoM <- foo$lambda
rm(foo)
```

Thus the method of moments estimates are $\hat{\alpha}_{MoM} = 1.0120949$ and $\hat{\lambda}_{MoM} = 0.0126614$.

Maximum Likelihood

The maximum likelihood estimator for α satisfies the following equation

$$nln(\hat{\alpha}) - nln(\bar{X}) + \sum_{i=1}^{n} ln(X_i) - n\psi(\hat{\alpha}) = 0$$

Where $\psi(x)$ is the digamma function, $\frac{\Gamma'(x)}{\Gamma(x)}$. Then an estimator for λ is

$$\hat{\lambda}_{MLE} = \frac{\hat{a}}{\bar{X}}$$

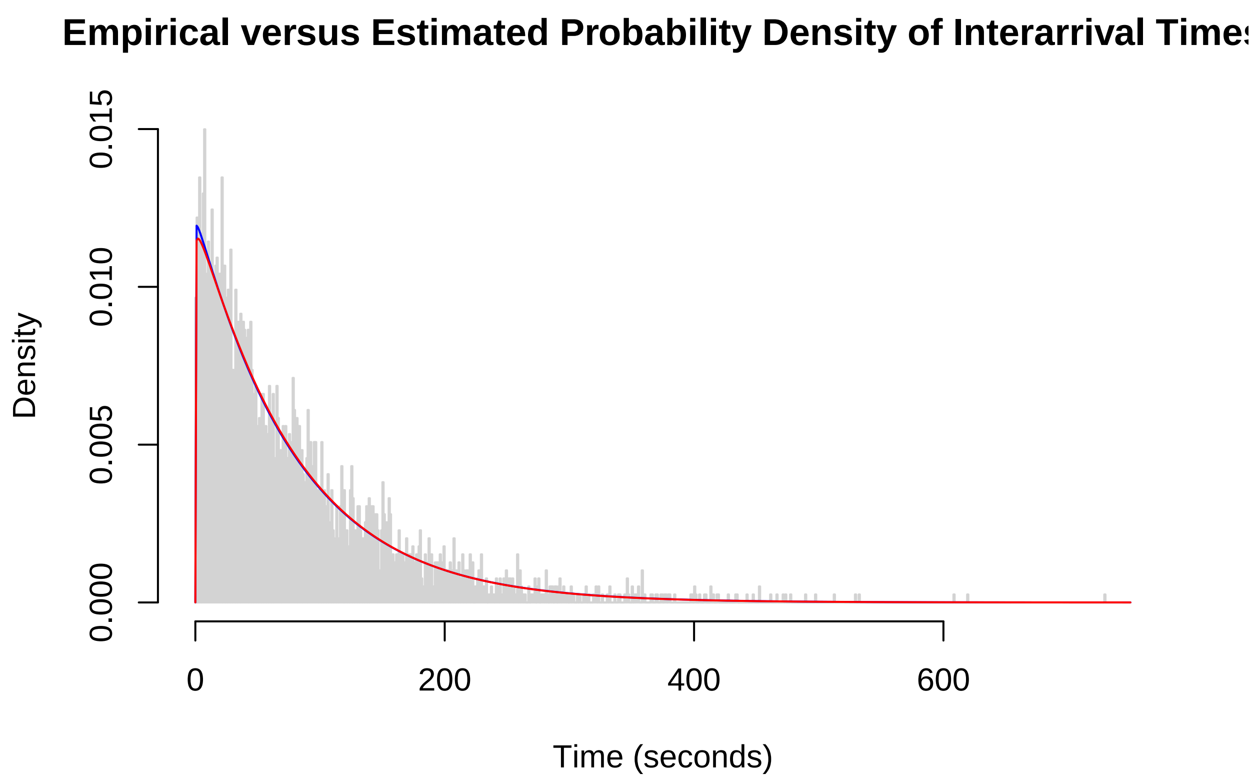
We will use the method of moments estimator as an initial guess to solve for $\hat{\alpha}_M LE$.

```
#Computes the MLE estimates of alpha and lambda for a gamma distribution, respectively
gammaMLE <- function(data) {
  aMoM <- gammaMoM(data)$alpha
  #Initialize variables
  n <- length(data)
  lnXBar <- log(mean(data))
  sumLnX <- sum(log(data))
  aMLEFun <- function(a) {
    n*log(a) - n*lnXBar + sumLnX - n*digamma(a)
  }

  #diagnostic plots
  #x <- seq(from = aMoM - .5, to = aMoM + .5, by = 0.01)
  #plot(x, aMLEFun(x))
  output = list()
  output$alpha <- uniroot(aMLEFun, interval = c(aMoM - .5, aMoM + .5))$root
  output$lambda <- output$alpha / mean(data)
  return(output)
```

```
}

foo <- gammaMLE(data)
aMLE <- foo$alpha
lMLE <- foo$lambda
rm(foo)
```

Thus the maximum likelihood estimates are $\hat{\alpha}_{MLE} = 1.026308$ and $\hat{\lambda}_{MLE} = 0.0128392$. We see that the two are nearly identical. Looking at the estimated distributions versus the histogram shows that the fit is good, and that the two estimated densities are nearly identical.

```
hist(data, freq = FALSE, breaks = seq(from = 0, to = 750, by = 1), density = 1, main = 'Empirical versu
xSeq <- seq(from = 0, to = 750, by = 1)
#MoM fit
lines(xSeq, dgamma(xSeq, shape = aMoM, rate = lMoM), col = 'blue', lwd = 1)
#MLE fit
lines(xSeq, dgamma(xSeq, shape = aMLE, rate = lMLE), col = 'red', lwd = 1)
```



Estimating Standard Errors via Bootstrap

```
#Bootstrap estimation of standard errors of alpha and lambda of a gamma distribution calculated uses ca
#Create dataSize iid draws from a gamma distribution, then estimates alpha and lambda using calcFun. Re
gammaSE <- function(dataSize, reps, alpha, lambda, calcFun) {
```

```r
  #Initialize vectors for storing the n calculated alpha and beta hats
  estAlpha <- vector(mode = 'numeric', length = reps)
  estLambda <- vector(mode = 'numeric', length = reps)

  #Generate the reps estimates of alpha and lambda
  for (i in 1:reps) {
    data <- rgamma(dataSize, shape = alpha, rate = lambda)
    ests <- calcFun(data)
    estAlpha[i] <- ests$alpha
    estLambda[i] <- ests$lambda
  }

  #Return the estimated standard errors
  output <- list()
  output$alphaSE <- sd(estAlpha)
  output$lambdaSE <- sd(estLambda)
  return(output)
}
```

```r
set.seed(1000)
MoMSE <- gammaSE(dataSize = length(data), reps = 1000, alpha = aMoM, lambda = lMoM, calcFun = gammaMoM)
MLESE <- gammaSE(dataSize = length(data), reps = 1000, alpha = aMLE, lambda = lMLE, calcFun = gammaMLE)
```

The estimates for $s_{\hat{\alpha}_{MoM}}$ and $s_{\hat{\lambda}_{MoM}}$ are thus 0.0311579 and 4.3464725×10^{-4}. The corresponding estimates for $s_{\hat{\alpha}_{MLE}}$ and $s_{\hat{\lambda}_{MLE}}$ are thus 0.0208755 and 3.2373483×10^{-4}. The estimated standard errors for the maximum likelihood estimates are significantly lower than the corresponding estimates for the method of moments.