Problem 64

Continuing off of Problem 63, let Y be a random variable indicating whether a randomly sampled item is defective or not. In all cases, $Y|\Theta$ is a Bernoulli random variable with parameter θ.

Part a

Find the marginal distribution of Y with Θ having a Uniform(0, 1) prior when X is unknown, and when $X = 3$.

When X is unknown, the marginal distribution of Θ is Uniform(0, 1). Thus

$$P(Y = 0) = \int P(Y = 0|\Theta = \theta) f_\Theta(\theta) d\theta = \int_0^1 1 - \theta d\theta = \frac{1}{2}$$

From the previous results, $\Theta|X = 3$ follows a Beta(4, 98) distribution. Thus

$$\begin{aligned}P(Y = 0|X = 3) &= \int P(Y = 0|\Theta = \theta, X = 3) f_{\Theta|X}(\theta|X = 3) d\theta \\ &\frac{101!}{97!3!} \int_0^1 (1-\theta)\theta^3(1-\theta)^{97} d\theta = \frac{49}{51} \approx 0.9608\end{aligned}$$

Part b

Find the marginal distribution of Y with Θ having a Beta(1/2, 5) prior when X is unknown, and when $X = 3$.

$$\begin{aligned}P(Y = 0) &= \int P(Y = 0|\Theta = \theta) f_\Theta(\theta) d\theta \\ &= \frac{\Gamma(11/2)}{\Gamma(1/2)\Gamma(5)} \int_0^1 (1-\theta)\theta^{-\frac{1}{2}}(1-\theta)^4 d\theta = 0.9091\end{aligned}$$

From previous results, $\Theta|X = 3$ follows a Beta(7/2, 102) distribution. Thus

$$\begin{aligned}P(Y = 0|X = 3) &= \int P(Y = 0|\Theta = \theta, X = 3) f_{\Theta|X}(\theta|X = 3) d\theta \\ &= \frac{\Gamma(211/2)}{\Gamma(7/2)\Gamma(102)} \int_0^1 (1-\theta)\theta^{\frac{5}{2}}(1-\theta)^{101} d\theta = 0.9668\end{aligned}$$

The results make intuitive sense, as the distribution in Part b is more clumped around 0 than the distribution in Part a. Note that after updating, the results of the two priors are the same up to two decimal places.