Problem 62

Part a

Show that the gamma distribution is a conjugate prior for the exponential distribution.

If $X \sim Exp(\lambda)$, then $f_X(x) = \lambda e^{-\lambda x}, x \geq 0$. If $\Lambda \sim Gamma(\alpha, \theta)$, then $f_\Lambda(\lambda) = \frac{\theta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\theta\lambda}, \lambda \geq 0$. Thus

$$f_{\Lambda|X}(\lambda|x) \propto \lambda^{\alpha-1} e^{-\theta\lambda}(\lambda e^{-\lambda x}) = \lambda^\alpha e^{-(\theta+x)\lambda}, \lambda \geq 0$$

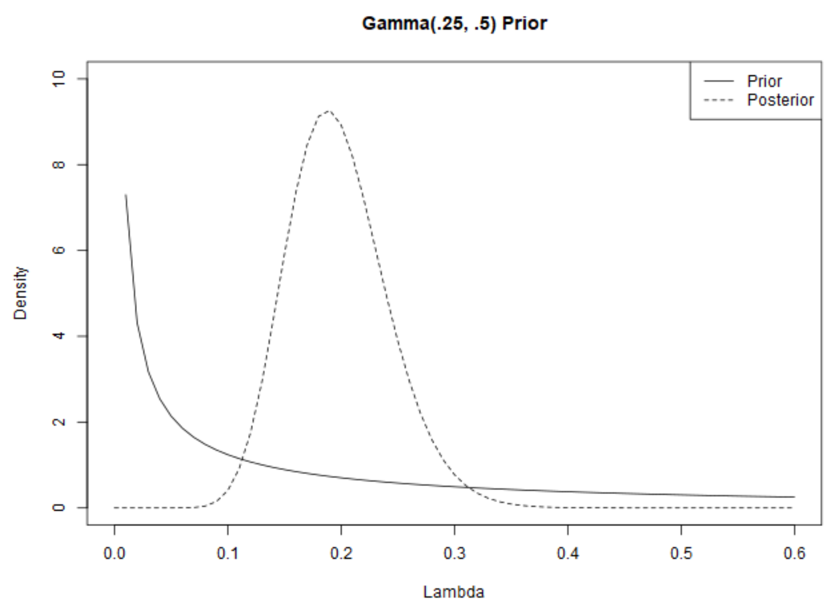
Which shows that $\Lambda|X \sim Gamma(\alpha+1, \theta+x)$. Repeated application shows that when n iid samples of X are drawn, $\Lambda|\mathbf{X} \sim Gamma(\alpha+n, \theta+\sum X_i)$.

Part b

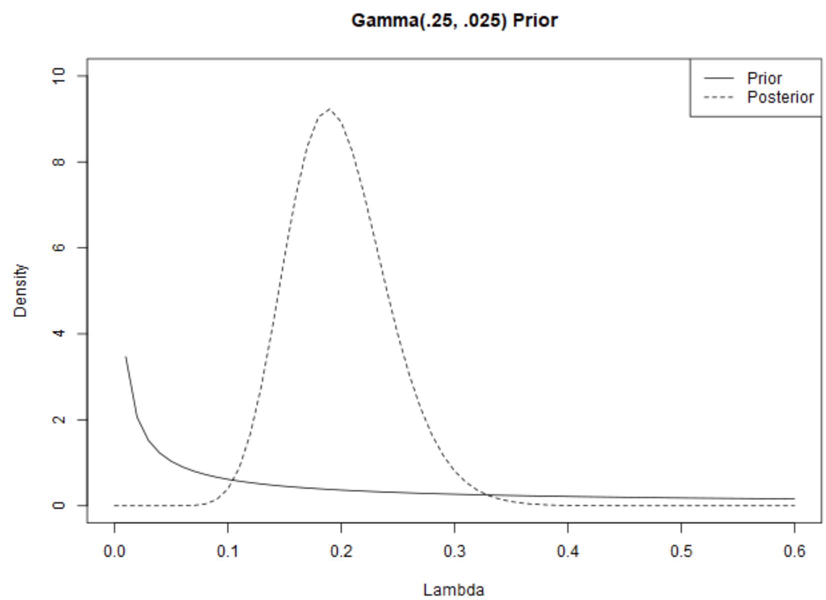
Suppose that the waiting time of customers in a queue is exponential with unknown parameter λ, and that the average time to serve a random sample of 20 customers is 5.1 minutes. A gamma distribution is used as the prior.

Plot the posterior distributions and find the means when 1. the prior has a mean of .5 and a standard deviation of 1 and 2. the prior has a mean of 10 and a standard deviation of 20.

Using the properties of the Gamma distribution, the first prior has $\alpha = .25$ and $\beta = .5$. Thus the posterior is distributed $Gamma(20.25, 102.5)$ with a mean of $E(\Lambda|\sum_{i=1}^{20} X_i = 102) = 0.1650$.



The second prior has $\alpha = .25$ and $\beta = .025$. Thus the posterior is distributed $Gamma(20.25, 102.025)$ with a mean of $E(\Lambda | \sum_{i=1}^{20} X_i = 102) = 0.1656$.



Although the second prior is flatter, the differences are insignificant compared to the data.