Problem 42

Gamma Ray Arrivals

Estimating the Parameter of Poisson Arrivals with Variable Time Lengths

```
data <- read.csv("../Excel Comma/Chapter 8/gamma-ray.csv")
```

We will assume that gamma ray arrivals follow a Poisson process with parameter λ arrivals per second. Under this assumption, let the 100 observations be indexed as $i = 1, 2, ...n$. Let t_i and X_i be the length of the ith time interval and the number of observed gamma rays during the ith time interval, respectively. I will assume that the time intervals are disjoint.

According to the model, X_i follows a Poisson distribution with parameter λt_i. Due to the assumption that the time intervals are disjoint, the X_is are independent of each other.

The distribution of X_i is thus

$$P(X_i = k_i | \lambda, t_i) = \frac{(\lambda t_i)^{k_i} e^{-\lambda t_i}}{k_i!}$$

The likelihood function of λ is thus

$$lik(\lambda | \mathbf{t}, \mathbf{X}\ ) = \frac{(\lambda t_i)^{\sum_{i=1}^{n} X_i} e^{-\lambda \sum_{i=1}^{n} t_i}}{\prod_{i=1}^{n}(X_i!)}$$

The log likelihood is thus

$$l(\lambda) = \ln(\lambda) \sum_{i=1}^{n} X_i + \sum_{i=1}^{n} X_i \ln(t_i) - \lambda \sum_{i=1}^{n} t_i - \sum_{i=1}^{n} \ln(X_i!)$$

The maximum likelihood estimator satisfies the first-order condition

$$0 = \frac{1}{\hat{\lambda}} \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} t_i$$

Which after some rearranging, yields the maximum likelihood estimator

$$\hat{\lambda} = \frac{\sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} t_i}$$

Setting all t_i's to 1 yields the usual estimator for λ in an iid sample from a Poisson distribution.

```
lambdaHat <- sum(data$count) / sum(data$seconds)
```

Thus $\hat{\lambda} = 0.0038809$.

The Standard Error of the Estimate

Define $S = \sum_{i=1}^{n} X_i$ and $T = \sum_{i=1}^{n} t_i$. Then the estimator $\hat{\lambda}$ can be rewritten as

$$\hat{\lambda} = \frac{S}{T}$$

Since S is the sum of independent Poisson variables with parameter λt_i, S follows a Poisson distribution with parameter λT. Thus $Var(S) = \lambda T$. Since $\hat{\lambda} = \frac{S}{T}$, $Var(\hat{\lambda}) = \frac{\lambda}{T}$, and $SD(\hat{\lambda}) = \sqrt{\frac{\lambda}{T}}$. This gives us an estimator

$$s_{\hat{\lambda}} = \sqrt{\frac{\hat{\lambda}}{T}}$$

```
s_lambdaHat <- sqrt(lambdaHat / sum(data$seconds))
```

This produces $s_{\hat{\lambda}} = 4.9689212 \times 10^{-4}$.

Posterior Distribution Using an Improper Gamma

Assume that the prior distribution of λ follows an improper gamma distribution, $f_\Lambda(\lambda) \propto \lambda^{-1}$. Then the posterior distribution is proportional to

$$f_\Lambda(\lambda|\mathbf{X}, \mathbf{t}) \propto \lambda^{\sum_{i=1}^{n} x_i - 1} \prod_{i=1}^{n} t_i^{x_i} \frac{e^{-\lambda \sum_{i=1}^{n} t_i}}{\prod_{i=1}^{n} x_i} \propto \lambda^{\sum_{i=1}^{n} x_i - 1} e^{-\lambda \sum_{i=1}^{n} t_i}$$

Which is a gamma distribution with parameters $\sum_{i=1}^{n} x_i$ and $\sum_{i=1}^{n} t_i$.