Problem 63

The proportion of items that are defective in a production run is θ. 100 items are randomly sampled from the items, of which 3 are found to be defective. Let X denote the number of defective items in the sample. Thus $X|\theta \sim Bin(100, \theta)$.

Uniform(0, 1) Prior

In this part, the prior distribution of θ is assumed to be Beta(1, 1), which is equivalent to a Uniform(0, 1) distribution. Thus the joint distribution is

$$f_{\Theta|X}(\theta|x) \propto f_{X|\Theta}(x|\theta)f_{\Theta}(\theta) = \theta^x(1-\theta)^{100-x}, \theta \in [0, 1]$$

Using the knowledge that $X = 3$,

$$f_{\Theta|X}(\theta|x=3) \propto \theta^3(1-\theta)^{97}, \theta \in [0, 1]$$

which we recognize as a Beta(4, 98) variable.

Beta(1/2, 5) Prior

Letting the prior of θ be Beta(1/2, 5),

$$f_{\Theta|X}(\theta|x) \propto \theta^{x-\frac{1}{2}}(1-\theta)^{104-x}, \theta \in [0, 1]$$

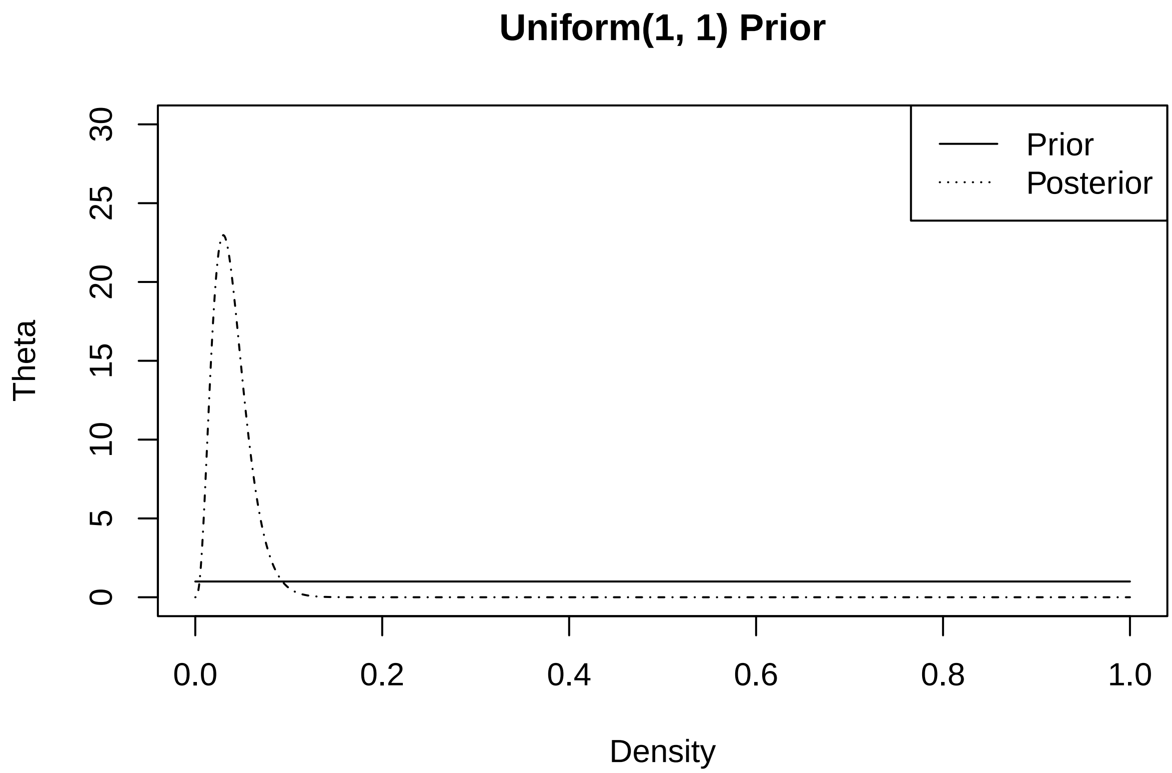
Evaluating at $x = 3$,

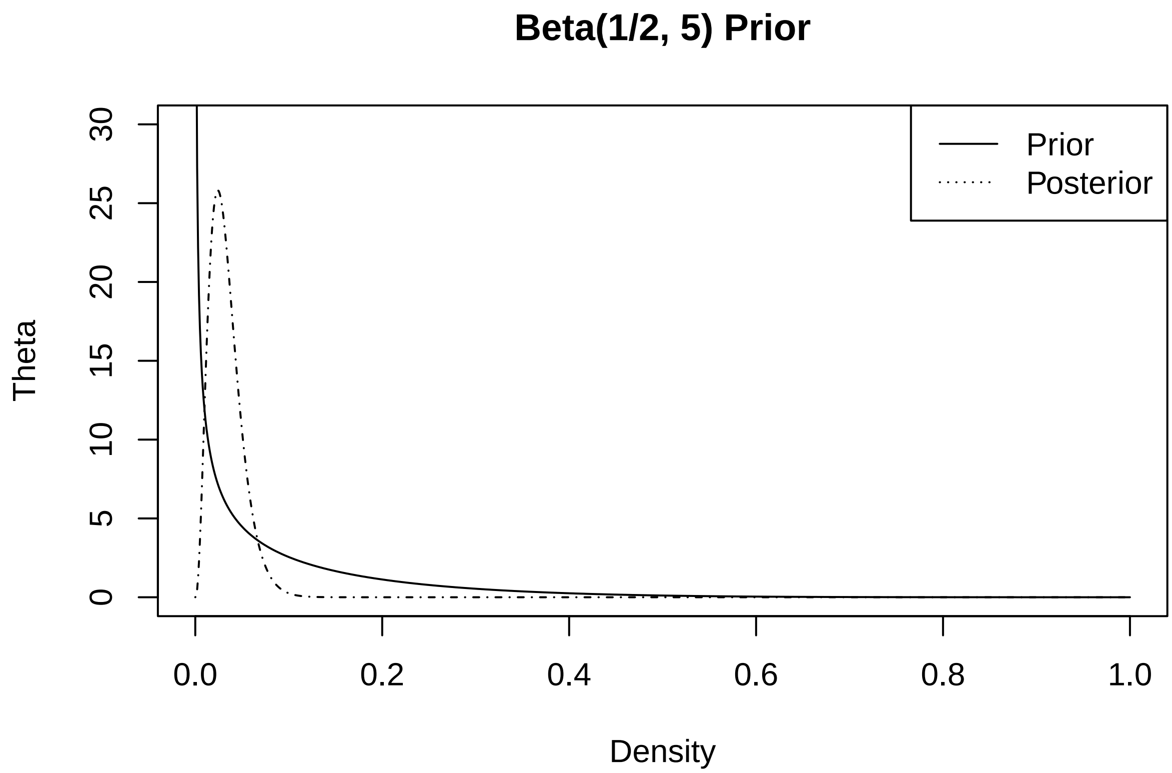
$$f_{\Theta|X}(\theta|x=3) \propto \theta^{\frac{5}{2}}(1-\theta)^{101}, \theta \in [0, 1]$$

Which we recognize as a Beta(7/2, 102) variable.

Graphs and Analysis

The posterior mean of theta with a Uniform(1, 1) prior is 0.0392157, while the posterior mean of theta with a Beta(1/2, 5) prior is 0.0343137.


Uniform(1, 1) Prior
Prior
Posterior
Theta
30
25
20
15
10
5
0
0.0
0.2
0.4
0.6
0.8
1.0
Density


Beta(1/2, 5) Prior
Prior
Posterior
Theta
0
5
10
15
20
25
30
0.0
0.2
0.4
0.6
0.8
1.0
Density