Problem 12

Let $X_1 \dots X_n$ be iid from an exponential distribution with $f(x|\theta) = \theta e^{-\theta x}, x \geq 0$. Derive a likelihood test for $H_0 : \theta = \theta_0$ versus $H_A : \theta \neq \theta_0$ and show that the rejection region is of the form $\bar{X} e^{-\theta_0 \bar{X}} \leq c$.

The likelihood ratio is

$$\Lambda = \frac{\max_{\theta=\theta_0} lik(\theta)}{\max_{\theta \in \mathbb{R}^+} lik(\theta)}$$

where $lik(\theta|x) = \theta^n e^{-\theta \sum X_i}$. The maximum likelihood estimator of an exponential distribution is $\theta^* = \frac{1}{\bar{X}}$. Since small values of Λ are evidence against the null, the test rejects the null when

$$\Lambda = \frac{\theta_0^n e^{-\theta_0 n \bar{X}}}{(\frac{1}{\bar{X}})^n e^n} < c$$

θ_0^n is a constant and can be immediately absorbed into c. For the exponent, $\bar{X}^n e^{-\theta_0 n \bar{X} - n} = (\bar{X} e^{-\theta_0 \bar{X}} e^{-1})^n$, and since Λ is positive, taking the nth root won't introduce more inequalities. Thus by taking the nth root of both sides, e^{-1} can be absorbed into c, the rejection region has the form

$$\bar{X} e^{-\theta_0 \bar{X}} < c$$

as desired.