**Problem 6.R:13.** Let $f(x) = \int_{t=x}^{t=x+1} \sin(t^2)dt$.

*a.* Show that when $x > 0$, $|f(x)| < \frac{1}{x}$.

*Proof.* Note that $x > 0$ implies that the limits of integration are correct. Make the substitution $t^2 = u$ to get

$$f(x) = \frac{1}{2}\int_{u=x^2}^{u=(x+1)^2} u^{-\frac{1}{2}} \sin(u)du$$

Integrate by parts with $a = u^{\frac{1}{2}}$ and $db = \sin(u)$ to get

$$\begin{aligned} f(x) &= \frac{1}{2}\left[-u^{-\frac{1}{2}}\cos(u)]_{x^2}^{(x+1)^2} - \frac{1}{2}\int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}}\cos(u)du\right] \\ &= \frac{\cos(x^2)}{2x} - \frac{\cos((x+1)^2)}{2(x+1)} - \frac{1}{4}\int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}}\cos(u)du \end{aligned}$$

Evaluating the integral on the right, $\cos(x) \geq -1$, so

$$\int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}}\cos(u)du \geq -\int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}}du = 2\left(\frac{1}{x+1} - \frac{1}{x}\right)$$

Substituting,

$$\begin{aligned} f(x) &\leq \frac{\cos(x^2)}{2x} - \frac{\cos((x+1)^2)}{2(x+1)} - \frac{1}{2(x+1)} + \frac{1}{2x} \\ &= \frac{\cos(x^2)+1}{2x} - \frac{\cos((x+1)^2)+1}{2(x+1)} \\ &\leq \frac{2}{2x} = \frac{1}{x} \end{aligned}$$

Since $\cos(t) \leq 1$. To show that $f(x) \geq -\frac{1}{x}$, it suffices to show that $f(x) \leq \frac{1}{x}$.

$$-f(x) = \frac{\cos((x+1)^2)}{2(x+1} - \frac{\cos(x^2)}{2x} + \frac{1}{4}\int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}}\cos(u)du$$

By a similar argument as before, $\cos(x) \leq 1$, so

$$\int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}}\cos(u)du \leq \int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}}du = 2\left(\frac{1}{x} - \frac{1}{x+1}\right)$$

Substituting,

$$\begin{aligned} -f(x) &\leq \frac{\cos((x+1)^2)}{2(x+1} - \frac{\cos(x^2)}{2x} + \frac{1}{2x} - \frac{1}{2(x+1)} \\ &\leq \frac{1}{2(x+1)} + \frac{1}{2x} + \frac{1}{2x} - \frac{1}{2(x+1)} \\ &= \frac{1}{x} \end{aligned}$$

$\square$

*b.* Prove that there exists constant $c$ and function $r(x)$ with $|r(x)| < \frac{c}{x}$ such that

$$2xf(x) = \cos(x^2) - \cos((x+1)^2) + r(x)$$

*Proof.* I will assume that $x > 0$, as the expression $\frac{c}{x}$ doesn't make much sense for $x = 0$, and it's impossible for $|r(x)| < \frac{c}{x}$ to be true for both positive and negative $x$ while $c$ is constant.

From results in Part a,

$$\begin{aligned} 2xf(x) &= \cos(x^2) - \frac{x}{x+1}\cos((x+1)^2) - \frac{x}{2}\int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}}\cos(u)du \\ &= \cos(x^2) - \cos((x+1)^2) + \frac{1}{x+1}\cos((x+1)^2) - \frac{x}{2}\int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}}\cos(u)du \end{aligned}$$

From the above, it's clear that $r(x) = \frac{1}{x+1}\cos((x+1)^2) - \frac{x}{2}\int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}}\cos(u)du$. Now it remains to show that $|r(x)| < \frac{c}{x}$. Using that $\cos(t)$ and $-\cos(t)$ is are bounded above by 1,

$$r(x) \leq \frac{1}{x+1} + \frac{x}{2}\int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}}du = \frac{1}{x+1} - x\left[\frac{1}{x+1} - \frac{1}{x}\right] = \frac{2}{x+1} < \frac{2}{x}$$

Similarly,

$$\begin{aligned} -r(x) &= -\frac{1}{x+1}\cos((x+1)^2) + \frac{x}{2}\int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}}\cos(u)du \\ &\geq -\frac{1}{x+1} - \frac{x}{2}\int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}}du = -\frac{2}{x+1} \geq -\frac{2}{x} \end{aligned}$$

Thus $|r(x)| < \frac{3}{x}$.

□

*c.* Find the upper and lower limits of $xf(x)$ as $x$ approaches infinity.

We know from previous results that

$$xf(x) = \frac{\cos(x^2))}{2} - \frac{\cos((x+1)^2)}{2} + s(x)$$

where $|s(x)| \leq \frac{1}{x}$. $\lim_{x\to\infty} s(x) = 0$, so the upper and lower limits of $s(x)$ are also 0.

Note that by the periodicity of cosine, for $n \in N$, $\cos(\sqrt{2\pi n}^2) = 1$. We now show that there exist infinite $n \in N$ such that $\cos((\sqrt{2\pi n}+1)^2) = \cos(2\pi n + 2\sqrt{2\pi n}+1) = -1$, thus implying that $\limsup_{n\to\infty} xf(x) = 1$.

**Theorem 1.** *Let $\delta > 0$. Then there exist infinite natural numbers $a$ such that $|2\sqrt{2a\pi} + 1 - \pi| < \delta \pmod{2\pi}$. In other words, $2\sqrt{2a\pi} + 1$ becomes arbitrarily close to a number of the form $b\pi$, where $b$ is an odd number.*

To prove this, we will need to analyze the behavior of $g(x) = 2\sqrt{2\pi}\sqrt{x} + 1$, then evaluate it at specifically chosen $a$'s. We first start by analyzing the Taylor series of $\sqrt{x}$.

**Lemma 2.** *The Taylor series of $\sqrt{x}$ about $x_0 > 0$ is*

$$\sqrt{x_0} + \frac{1}{2(1!)}x_0^{-\frac{1}{2}}(x-n) + \sum_{i=2}^{\infty} \frac{(-1)^{i-1}}{2^i}\frac{1(3)(5)...(2i-3)}{i!}x_0^{-i+\frac{1}{2}}(x-x_0)^i$$

*with radius of convergence* $R = x_0$.

*Proof.* The first few terms of the Taylor series are

$$T(x) = \sqrt{x_0} + \frac{1}{2(1!)}x_0^{-\frac{1}{2}}(x-x_0) - \frac{1}{2^2(2!)}x_0^{-\frac{3}{2}}(x-x_0)^2 + \frac{3}{2^3(3!)}x_0^{-\frac{5}{2}}(x-x_0)^3 \dots$$

To find the radius of convergence, note that

$$\frac{1(3)(5)...(2i-3)}{2^i} < \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\dots\frac{2i-3}{2} < \frac{1}{2}1(2)(3)\dots i = \frac{1}{2}i!$$

implying that when $x > x_0$,

$$\frac{1(3)(5)...(2i-3)}{2^i(i!)}x_0^{-i+\frac{1}{2}}(x-x_0)^i < \frac{\sqrt{x_0}}{2}\left(\frac{x}{x_0}-1\right)^i$$

Since $T(x)$ is alternating, it converges when $\frac{x}{x_0} - 1 < 1 \to x \in [x_0, 2x_0)$.

When $x < x_0$,

$$T(x) = \sqrt{x_0} + \frac{1}{2(1!)}x_0^{-\frac{1}{2}}(x-n) - \sqrt{x_0}\sum_{i=2}^{\infty}\frac{1(3)(5)...(2i-3)}{2^i(i!)}\left(1-\frac{x}{x_0}\right)^i$$

Since

$$\frac{1(3)(5)...(2i-3)}{2^i(i!)}\left(1-\frac{x}{x_0}\right)^i < \frac{1}{2}\left(1-\frac{x}{x_0}\right)^i$$

and the right series is a convergent geometric series under the assumption that $x \in (0, x_0]$, $T(x)$ converges.

□

We next show that viewed as a sequence over the natural numbers, $(g_n)$'s differences between terms become arbitrarily small.

**Lemma 3.** *Let* $(g_n) = 2\sqrt{2\pi}\sqrt{n} + 1$ *for* $n \in N$, *and* $\Delta g_n = g_{n+1} - g_n = 2\sqrt{2}(\sqrt{n+1} - \sqrt{n})$. *Then* $\lim_{n\to\infty} \Delta g_n = 0$.

*Proof.* It suffices to show that $\lim_{n\to\infty}\sqrt{n+1} - \sqrt{n} = 0$. From the results in Theorem 2, the Taylor series of $\sqrt{x}$ at $n$ is

$$\sqrt{n+x} = \sqrt{n} + \frac{1}{2(1!)}n^{-\frac{1}{2}}(x) + \sum_{i=2}^{\infty}\frac{(-1)^{i-1}}{2^i}\frac{1(3)(5)...(2i-3)}{i!}n^{-i+\frac{1}{2}}(x)^i$$

Since the series is alternating and convergent, its partial sums that have a positive term as their highest power are larger than the series. Thus

$$\sqrt{n+1} < \sqrt{n} + \frac{1}{2\sqrt{n}}$$

implying

$$0 \le \sqrt{n+1} - \sqrt{n} < \frac{1}{2\sqrt{n}}$$

implying that $\lim_{n\to\infty}\sqrt{n+1} - \sqrt{n} = 0$.

□

We are now in the position to prove Theorem 1.

*Proof.* Let $(g_n)$ be the sequence defined by $g_n = 2\sqrt{2\pi}\sqrt{n}+1-\pi$. Since $\lim_{n\to\infty} \Delta g_n = 0$, there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies that $\Delta g_n < \delta$. This, combined with $(g_n)$ being strictly increasing and $\lim_{n\to\infty} g_n = \infty$, imply that $(g_n)$ passes through all numbers greater than $g_{n_0}$ while increasing to infinity, while taking step sizes less than $\delta$.

Specifically, for any natural number $b$ such that $2\pi b > g_{n_0}$, there exists an $a \geq n_0$ in the natural numbers such that

$$g_a \leq 2\pi b < g_{a+1}$$

$\Delta g_a < \delta$ implies

$$|\Delta g_a - 2\pi b| < \delta$$

which is equivalent to

$$|2\sqrt{2\pi}\sqrt{a} + 1 - (2b+1)\pi| < \delta$$

The theorem follows because for distinct $b$, $(g_n)$ being strictly increasing implies that the $a$'s are distinct. $\square$

The results of the main problem now follow.

**Lemma 4.** $\limsup_{n\to\infty} xf(x) = \frac{\cos(x^2))}{2} - \frac{\cos((x+1)^2)}{2} + s(x) = 1$

*Proof.* From Theorem 1 and the continuity of $\cos(x)$, for all $\epsilon > 0$, there exist infinite $n \in \mathbb{N}$ such that $\cos(x^2) = \cos(\sqrt{2\pi n}^2) = 1$ and $|\cos\big((x+1)^2\big) + 1| = |\cos\big((\sqrt{2\pi n}+1)^2\big) + 1| < \epsilon$. As previously established, $\limsup_{x\to\infty} s(x) = 0$. $\square$

**Corollary 5.** $\liminf_{n\to\infty} xf(x) = \frac{\cos(x^2))}{2} - \frac{\cos((x+1)^2)}{2} + s(x) = -1$

*Proof.* Add $\pi$ to the $a$'s in Lemma 4. $\square$

*d.* Does $\int_0^\infty \sin(t^2)dt$ converge?

**Theorem 6.** *$\int_0^\infty \sin(t^2)dt$ converges.*

*Proof.* For finite $n$,

$$\begin{aligned}
\int_0^n \sin(t^2)dt &= \int_0^1 \sin(t^2)dt + f(1) + f(2) \cdots + f(n-1) \\
&= \int_0^1 \sin(t^2)dt + \sum_{i=1}^{n-1} f(i) \\
&= \int_0^1 \sin(t^2)dt + \frac{1}{2}\sum_{i=1}^{n-1}\left[\frac{\cos(i^2)}{i} - \frac{\cos\big((i+1)^2\big)}{i}\right] + \frac{1}{2}\sum_{i=1}^{n-1}\frac{r(i)}{i}
\end{aligned}$$

Taking limits as n approaches infinity, $\sum_{i=1}^{\infty} \frac{r(i)}{i}$ converges due to a comparison with $\sum_{i=1}^{\infty} \frac{1}{i^2}$. To show convergence, we have to show that $\sum_{i=1}^{\infty} \frac{\cos(i^2)}{i} - \frac{\cos\big((i+1)^2\big)}{i}$

converges. Writing out the first few terms of the partial sum,

$$\begin{aligned}\sum_{i=1}^{n-1}\frac{\cos(i^2)}{i}-\frac{\cos\left((i+1)^2\right)}{i} &= \frac{\cos(1)}{1}-\frac{\cos(4)}{1}+\frac{\cos(4)}{2}-\frac{\cos(9)}{2}+\frac{\cos(9)}{3}-\frac{\cos(16)}{3}\cdots\\ &= \frac{\cos(1)}{1}-\frac{\cos(4)}{1*2}-\frac{\cos(9)}{2*3}\cdots-\frac{\cos\left((n-1)^2\right)}{(n-2)(n-1)}-\frac{\cos(n^2)}{n-1}\\ &= \frac{\cos(1)}{1}-\frac{\cos(n^2)}{n-1}-\sum_{i=1}^{n-2}\frac{\cos\left((i+1)^2\right)}{i(i+1)}\end{aligned}$$

Taking the limits as n goes to infinity, $\frac{cos(n^2)}{n-1}$ goes to zero. The sum on the right is absolutely convergent by comparing it with $\sum_{i=1}^{\infty}\frac{1}{i^2}$, so the sum on the right is convergent. Thus $\sum_{i=1}^{\infty}\frac{\cos(i^2)}{i}-\frac{\cos\left((i+1)^2\right)}{i}$, and by extension $\int_0^\infty \sin(t^2)dt$, converge. $\square$

**Problem 6.R.14.** let $f(x)=\int_x^{x+1}\sin(e^t)dt$.

*a.* Show that $e^x|f(x)|<2$.

*Proof.* Making the substitution $u=e^t$,

$$f(x)=\int_{e^x}^{e^{x+1}}\frac{\sin(u)}{u}du$$

Integrating by parts with $a=u^{-1}$ and $db=\sin(u)$,

$$\begin{aligned}f(x) &= -u^{-1}\cos(u)\Big]_{e^x}^{e^{x+1}}-\int_{e^x}^{e^{x+1}}u^{-2}\cos(u)du\\ &= \frac{\cos(e^x)}{e^x}-\frac{\cos(e^{x+1})}{e^{x+1}}-\int_{e^x}^{e^{x+1}}u^{-2}\cos(u)du\end{aligned}$$

implies

$$\begin{aligned}xf(x) &= \cos(e^x)-e^{-1}\cos(e^{x+1})-e^x\int_{e^x}^{e^{x+1}}u^{-2}\cos(u)du\\ &\le 1+\frac{1}{e}+e^x\int_{e^x}^{e^{x+1}}u^{-2}du\\ &= 1+\frac{1}{e}-e^x\left[\frac{1}{u}\right]_{e^x}^{e^{x+1}}\\ &= 1+\frac{1}{e}+1-\frac{1}{e}=2\end{aligned}$$

Similarly,

$$xf(x)\ge -1-\frac{1}{e}-e^x\int_{e^x}^{e^{x+1}}u^{-2}du=-2$$

$\square$

*b.* Show that $e^x f(x) = \cos(e^x) - e^{-1}\cos(e^{x+1}) + r(x)$, where $|r(x)| < Ce^{-x}$, for some constant $C$.

*Proof.* From the form above, it's clear that $r(x) = -e^x \int_{e^x}^{e^{x+1}} u^{-2}\cos(u)du$. Integrating by parts with $a = u^{-2}$ and $db = \cos(u)$,

$$\begin{aligned}
r(x) &= -e^x \left[ u^{-2}\sin(u)\Big|_{e^x}^{e^{x+1}} + 2\int_{e^x}^{e^{x+1}} u^{-3}\sin(u)du \right] \\
&= \frac{\sin(e^x)}{e^x} - \frac{\sin(e^{x+1})}{e^{x+2}} - 2e^x \int_{e^x}^{e^{x+1}} u^{-3}\sin(u)du \\
&\leq \frac{1}{e^x} + \frac{1}{e^{x+2}} + 2e^x \int_{e^x}^{e^{x+1}} u^{-3}du \\
&= \frac{1}{e^x} + \frac{1}{e^{x+2}} - e^x \left[ \frac{1}{e^{2x+2}} - \frac{1}{e^x} \right] = \frac{2}{e^x}
\end{aligned}$$

Similarly,

$$r(x) \geq -\frac{1}{e^x} - \frac{1}{e^{x+2}} - 2e^x \int_{e^x}^{e^{x+1}} u^{-3}du = \frac{-2}{e^x}$$

□

**Problem 6.R.15.** Let $f$ be real and continuously differentiable on $[a, b]$, with $f(a) = f(b) = 0$ and $\int_a^b f^2(x)dx = 1$. Prove that $\int_a^b xf(x)f'(x)dx = -\frac{1}{2}$ and $\int_a^b [f'(x)]^2 dx \int_a^b x^2 f^2(x)dx > \frac{1}{4}$.

*Proof.* Integrating $\int_a^b f^2(x)dx$ by parts with $a = f^2(x)$ and $db = 1$,

$$1 = \int_a^b f^2(x)dx = xf^2(x)\Big|_a^b - 2\int_a^b xf(x)f'(x)dx = -2\int_a^b xf(x)f'(x)dx$$

Similarly, by the Cauchy-Schwartz inequality,

$$\frac{1}{2} = \left| \int_a^b xf(x)f'(x)dx \right| \leq \left[ \int_a^b (f'(x))^2 dx \right]^{\frac{1}{2}} \left[ \int_a^b x^2 f^2(x)dx \right]^{\frac{1}{2}}$$

**Lemma 7.** $\int_a^b (f'(x))^2 dx > 0$

*Proof.* $f(a) = f(b) = 0$ implies that if $f$ is a constant function, it must be the zero function. $\int_a^b f^2(x)dx \neq 0$ implies that $f$ is not the zero function. Thus, $f'$ is not the zero function. Since $f'$ is continuous, $f'^2$ is continuous and nonzero, implying that $\int_a^b (f'(x))^2 dx > 0$. □

**Lemma 8.** $\int_a^b x^2 f^2(x)dx > 0$

*Proof.* $f$ is continuous and nonzero implies that $f^2$ and $x^2 f^2$ are continuous and nonzero. □

The Cauchy-Schwartz inequality only holds with equality when at least one of the vectors has norm zero. Since $\int_a^b (f'(x))^2 dx$ and $\int_a^b x^2 f^2(x)dx$ are nonzero, the inequality is strict.

□

**Problem 6.R.16.** For $1 < s < \infty$, define

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

*a.* Prove that

$$\zeta(s) = s \int_1^{\infty} \frac{[x]}{x^{s+1}} dx$$

where $[x]$ denotes the greatest integer $\leq x$.

*Proof.* Denote $f_N(s) = s \int_1^N \frac{[x]}{x^{s+1}} dx$ for $N \in \mathbb{N}$, $N > 1$. Splitting the integral and evaluating the greatest integer function,

$$\begin{aligned}
f_N(s) &= s \int_1^N \frac{[x]}{x^{s+1}} dx \\
&= s \left( \int_1^2 \frac{1}{x^{s+1}} dx + \int_2^3 \frac{2}{x^{s+1}} dx + \cdots + \int_{N-1}^N \frac{N-1}{x^{s+1}} dx \right) \\
&= s \sum_{i=1}^{N-1} i \int_i^{i+1} \frac{1}{x^{s+1}} dx = - \sum_{i=1}^{N-1} i(x)_i^{i+1} = \sum_{i=1}^{N-1} i \left( \frac{1}{i^s} - \frac{1}{(i+1)^s} \right) \\
&= 1 \left( \frac{1}{1^s} - \frac{1}{2^s} \right) + 2 \left( \frac{1}{2^s} - \frac{1}{3^s} \right) + 3 \left( \frac{1}{3^s} - \frac{1}{4^s} \right) + \cdots + (N-1) \left( \frac{1}{(N-1)^s} - \frac{1}{N^s} \right) \\
&= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots + \frac{1}{(N-1)^s} - \frac{N-1}{N^s} \\
&= \sum_{i=1}^{N-1} \frac{1}{i^s} - \frac{N-1}{N^s}
\end{aligned}$$

Let $\zeta_N(s) = \sum_{i=1}^{N} \frac{1}{i^s}$ be the nth partial sum of $\zeta(s)$. Taking the difference between $f_N(s)$ and $\zeta_N(s)$,

$$\begin{aligned}
|f_N(s) - \zeta_N(s)| &= \left| \sum_{i=1}^{N-1} \frac{1}{i^s} - \sum_{i=1}^{N} \frac{1}{i^s} - \frac{N-1}{N^s} \right| \\
&= \frac{1}{N^{s-1}}
\end{aligned}$$

Since $s > 1$, the difference goes to 0 as N approaches infinity.

□

*b.* Prove that

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{\infty} \frac{x - [x]}{x^{s+1}} dx$$

*Proof.* Taking the integral to $N$ and splitting the integral,

$$\frac{s}{s-1} - s \int_1^N \frac{x - [x]}{x^{s+1}} dx = \frac{s}{s-1} - s \int_1^N \frac{1}{x^s} dx + f_N(s)$$

Since $f_N s \to \zeta(s)$ as n approaches infinity, we need to show that $\frac{s}{s-1} - s \int_1^N \frac{1}{x^s} dx \to 0$. Integrating,

$$s\int_1^N \frac{1}{x^s}dx = -\frac{s}{s-1}\left(\frac{1}{x^{s-1}}\right)_1^N = -\frac{s}{s-1}\left(\frac{1}{N^{s-1}} - 1\right)$$

As n approaches infinity, this approaches $-\frac{s}{s-1}$, so $\frac{s}{s-1} - s\int_1^N \frac{1}{x^s}dx \to 0$. □

*c.* Prove that the integral in Part b converges for $s > 0$.

*Proof.*

$$\int_1^N \frac{x-[x]}{x^{s+1}}dx \le \int_1^N \frac{1}{x^{s+1}}dx = -\frac{1}{s}\left(\frac{1}{x^s}\right)_1^N = -\frac{1}{s}\left[\frac{1}{N^s} - 1\right]$$

When $s > 0$, this converges to $\frac{1}{s}$ as n approaches infinity. □

**Problem 6.6:1.** On $[a, b]$, let $\alpha$ be a strictly increasing function and $f$ a continuous function, and for $x \in [a, b]$ define $F(x) = \int_a^x f(t)d\alpha(t)$. Show that for all $x \in [a, b]$, $\frac{dF(x)}{d\alpha(x)} = f(x)$, where the left-hand side is defined as $\lim_{t\to x} \frac{F(x)-F(t)}{\alpha(x)-\alpha(t)}$, and the equality includes the assertion that the limit exists.

*Proof.* First, we note that because $f \in \mathfrak{R}(\alpha)$ on $[a, b]$

$$F(x) - F(t) = \int_a^x f(s)d\alpha(s) - \int_a^t f(s)d\alpha(s) = \int_t^x f(x)d\alpha(s)$$

For all partitions $P$,

$$\begin{aligned}\int_t^x f(s)d\alpha(s) &\le \sum_{x_i\in P} M_i\Delta\alpha_i \\ &\le \left(\sup_{s\in[x,t]} f(s)\right)\sum_{x_i\in P}\Delta\alpha_i \\ &= \left(\sup_{s\in[x,t]} f(s)\right)(\alpha(x) - \alpha(t))\end{aligned}$$

Since $\alpha(x)$ is strictly increasing, $\alpha(x) - \alpha(t) > 0$ when $x \ne t$, so

$$\frac{F(x) - F(t)}{\alpha(x) - \alpha(t)} \le \sup_{s\in[x,t]} f(s)$$

Taking the limit as $t$ approaches $x$ on both sides gives

$$\lim_{t\to x}\frac{F(x) - F(t)}{\alpha(x) - \alpha(t)} \le \lim_{t\to x}\sup_{s\in[x,t]} f(s)$$

**Lemma 9.**

$$\lim_{t\to x}\sup_{s\in[x,t]} f(s) = f(x)$$

*Proof.* Let $x_n$ be an arbitrary sequence such that $\forall n \in \mathbb{N}, x_n > x$ and $\lim_{n\to\infty} x_n = x$. Since $[x, x_n]$ is a closed, bounded interval on $\mathbb{R}$ and $f$ is continuous, there exists a sequence of points $p_n \in [x, x_n]$ such that $f(p_n) = \sup_{s\in[x,x_n]} f(s)$. $x_n \to x$ implies $p_n \to x$ by the Squeeze Theorem, and the continuity of $f$ implies that $f(p_n) \to f(x)$. □

Thus

$$\lim_{t\to x}\frac{F(x)-F(t)}{\alpha(x)-\alpha(t)}\le f(x)$$

The analogous result for the lower integral and the Squeeze Theorem complete the proof.

□

**Problem 6.6:2.**

*(a).* Show that if $f$ is continuous, then

$$\int_{t=a}^{b}\left(\int_{s=a}^{t}f(s)ds\right)dt=\int_{t=a}^{b}(b-t)f(t)dt$$

*Proof.* Let $x\in[a,b]$. Define $P(x)=\int_{t=a}^{x}\left(\int_{s=a}^{t}f(s)ds\right)dt$ and $Q(x)=\int_{t=a}^{x}(x-t)f(t)dt$.

$f(t)$ being continuous on $[a,b]$ implies that it is Riemann-integrable. This implies that $f^*(t)=\int_{s=a}^{t}f(s)ds$ is continuous, and that $P(x)=\int_a^x f^*(t)dt$ is continuous and differentiable. Similarly, $(b-t)f(t)$ is continuous on $[a,b]$, so $Q(x)$ is continuous and differentiable.

By the Fundamental Theorem of Calculus,

$$P'(x)=\int_{s=a}^{x}f(s)ds$$

For $Q(x)$, since $t$ and $tf(t)$ are Riemann-integrable,

$$Q(x)=x\int_{t=a}^{x}f(t)dt-\int_{t=a}^{x}tf(t)dt$$

$x$ is trivially differentiable. Since $t$ and $tf(t)$ are continuous,

$$Q'(x)=\int_{t=a}^{x}f(t)dt+xf(x)-xf(x)=\int_{t=a}^{x}f(t)dt$$

Thus, $P'(x)=Q'(x)$. Integrating both sides from $a$ to $c$, then setting $c=b$, produces the desired result.

□

*(c).* Show that the result of Part (a) continues to hold if $f$ is merely assumed Riemann-integrable, but not necessarily continuous.

*Proof.* $P(x)$ has the same derivative as in Part (a), as the derivation only assumed that $P(x)$ is Riemann-integrable. Similarly, for $x_0\in[a,b]$ where $f(x_0)$ is continuous, the above derivations hold for $Q(x)$.

Let $x_0$ be a point where $f(x_0)$ is discontinuous. First, we will prove two lemmas.

**Lemma 10.** *If $f(x)$ is bounded, then $(x-x_0)f(x)$ is continuous at $x_0$.*

*Proof.* Let $M=\sup|f(x)|$. Then $(x-x_0)f(x)\le|(x-x_0)f(x)|\le|(x-x_0)|M$, which can be made arbitrarily small.

□

**Lemma 11.** *If $f(x)$ is continuous, then $(x - x_0)f(x)$ is differentiable at $x_0$ with derivative $f(x_0)$.*

*Proof.* By the definition of differentiability,

$$\lim_{x \to x_0} \frac{(x - x_0)f(x) - (x_0 - x_0)f(x_0)}{x - x_0} = \lim_{x \to x_0} f(x) = f(x_0)$$

by continuity.

□

We can rewrite $Q(x)$ as

$$Q(x) = \int_{t=a}^{x} ((x - x_0) + (x_0 - t))f(t)dt = (x - x_0)\int_{t=a}^{x} f(t)dt + \int_{t=a}^{x} (x_0 - t)f(t)dt$$

because the sub-functions are trivially Riemann-integrable. $\int_{t=a}^{x} f(t)$ is a continuous function, so by Lemma 11 $(x - x_0)\int_{t=a}^{x} f(t)dt$ is differentiable at $x = x_0$ with derivative $\int_{t=a}^{x_0} f(t)dt$. Similarly, $f(t)$ is bounded because it is Riemann-integrable, so by Lemma 10 $(x_0 - t)f(t)$ is continuous at $t = x_0$. Therefore $\int_{t=a}^{x}(x_0 - t)f(t)dt$ is differentiable at $x = x_0$, with derivative 0.

Therefore, $Q(x)$ is differentiable at $x = x_0$, and $Q'(x_0) = \int_{t=a}^{x} f(t)dt$. The proof then follows using the same logic as in Part (a).

□

**Problem 6.6:4.** Let $f$ be a function on $[a, b]$, and $\alpha$, $\beta$ monotonically increasing nonnegative functions on $[a, b]$ such that $f \in \mathfrak{R}(\alpha) \cap \mathfrak{R}(\beta)$, $\alpha \in \mathfrak{R}(\beta)$, and $\beta \in \mathfrak{R}(\alpha)$. Prove that

$$\int f d(\alpha\beta) = \int f\alpha d(\beta) + \int f\beta d(\alpha)$$

*Proof.* First note that $f\alpha \in \mathfrak{R}(\beta)$ and $f\beta \in \mathfrak{R}(\alpha)$. Also note that $\alpha\beta$ is monotonically increasing, so $\alpha\beta$ is a valid integrator.

**Theorem 12.** *Under the assumptions of Problem 6.6.4, $f \in \mathfrak{R}(\alpha\beta)$.*

*Proof.* First note that by expanding the terms,

$$\alpha_i\beta_i - \alpha_{i-1}\beta_{i-1} = (\alpha_i - \alpha_{i-1})(\beta_i - \beta_{i-1}) + \alpha_{i-1}(\beta_i - \beta_{i-1}) + \beta_{i-1}(\alpha_i - \alpha_{i-1})$$

Writing out the difference between the upper and lower Riemann sums,

$$\begin{aligned}
&\sum_{i=1}^{n}(M_i - m_i)(\alpha_i\beta_i - \alpha_{i-1}\beta_{i-1}) \\
=&\sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i\Delta\beta_i + \sum_{i=1}^{n}(M_i - m_i)\alpha_{i-1}\Delta\beta_i + \sum_{i=1}^{n}(M_i - m_i)\beta_{i-1}\Delta\alpha_i \\
\leq&\alpha(b)\sum_{i=1}^{n}(M_i - m_i)\Delta\beta_i + \alpha(b)\sum_{i=1}^{n}(M_i - m_i)\Delta\beta_i + \beta(b)\sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i
\end{aligned}$$

The third line follows because all terms are positive, and $\alpha$ and $\beta$ are monotonically increasing. Because $f \in \mathfrak{R}(\alpha) \cap \mathfrak{R}(\beta)$, there exist partitions that make the third line arbitrarily small.

$\square$

Now we prove the main result. For arbitrary $\epsilon > 0$, let $P_1$, $P_2$, and $P_3$ be partitions of $[a,b]$ such that

(1) $U(P_1, f, \alpha\beta) - L(P_1, f, \alpha\beta) < \epsilon$
(2) $U(P_2, f\alpha, \beta) - L(P_2, f\alpha, \beta) < \epsilon$
(3) $U(P_3, f\beta, \alpha) - L(P_3, f\beta, \alpha) < \epsilon$

Let $P$ be their common partition. Let $x_0 < x_1 ... < x_n$ denote the points of $P$. For all $i$ in $(1, 2...n)$, let $t_i \in (x_{i-1}, x_i)$ be arbitrary, fixed points, and let $P^* = (x_0, t_1, x_1 ... x_{i-1}, t_i, x_i ... t_n, x_n)$. Trivially, $P^*$ partitions $[a,b]$ and is a refinement of $P$.

Consider $\int f d(\alpha\beta)$ and its associated Riemann-Stieltjes sum over $P^*$. Since $P^*$ is a refinement of $P_1$, for arbitrary points $u_i \in [x_{i-1}, t_i]$ and $v_i \in [t_i, x_i]$,

$$\left| \sum_{i=1}^{n} \left[ f(u_i)(\alpha_{t_i}\beta_{t_i} - \alpha_{x_{i-1}}\beta_{x_{i-1}}) + f(v_i)(\alpha_{x_i}\beta_{x_i} - \alpha_{t_i}\beta_{t_i}) \right] - \int f d(\alpha\beta) \right| < \epsilon$$

Letting $u_i = x_{i-1}$ and $v_i = x_i$ for all $i$,

$$\left| \sum_{i=1}^{n} \left[ f_{x_{i-1}}(\alpha_{t_i}\beta_{t_i} - \alpha_{x_{i-1}}\beta_{x_{i-1}}) + f_{x_i}(\alpha_{x_i}\beta_{x_i} - \alpha_{t_i}\beta_{t_i}) \right] - \int f d(\alpha\beta) \right| < \epsilon$$

Now consider $\int f\alpha d(\beta)$ and its associated Riemann-Stieltjes sum over $P$. Letting $u_i = x_i$ for all $i$,

$$\left| \sum_{i=1}^{n} \left[ f_{x_i}\alpha_{x_i}(\beta_{x_i} - \beta_{x_{i-1}}) \right] - \int f\alpha d(\beta) \right| < \epsilon$$

Similarly, considering $\int f\beta d(\alpha)$ over $P$ and letting $u_i = x_{i-1}$ for all $i$,

$$\left| \sum_{i=1}^{n} \left[ f_{x_{i-1}}\beta_{x_{i-1}}(\alpha_{x_i} - \alpha_{x_{i-1}}) \right] - \int f\beta d(\alpha) \right| < \epsilon$$

Adding the inequalities and using the Triangle Inequality gives

$$\begin{aligned} \Bigg| \int f d(\alpha\beta) - \int f\alpha d(\beta) - \int f\beta d(\alpha) \\ + \sum_{i=1}^{n} \Big[ - \big[ f_{x_{i-1}}(\alpha_{t_i}\beta_{t_i} - \alpha_{x_{i-1}}\beta_{x_{i-1}}) + f_{x_i}(\alpha_{x_i}\beta_{x_i} - \alpha_{t_i}\beta_{t_i}) \big] \\ + f_{x_i}\alpha_{x_i}(\beta_{x_i} - \beta_{x_{i-1}}) + f_{x_{i-1}}\beta_{x_{i-1}}(\alpha_{x_i} - \alpha_{x_{i-1}}) \Big] \Bigg| < 3\epsilon \end{aligned}$$

Simplifying,

$$\left| \int f d(\alpha\beta) - \int f\alpha d(\beta) - \int f\beta d(\alpha) + \sum_{i=1}^{n} \Big[ (f_{x_i} - f_{x_{i-1}})(\alpha_{t_i}\beta_{t_i} - \alpha_{x_i}\beta_{x_{i-1}}) \Big] \right| < 3\epsilon$$

To analyze the sum on the right, note that

$$\begin{aligned}
&\left|\sum_{i=1}^{n}\left[(f_{x_i}-f_{x_{i-1}})(\alpha_{t_i}\beta_{t_i}-\alpha_{x_i}\beta_{x_{i-1}})\right|\right.\\
\leq&\sum_{i=1}^{n}\left|f_{x_i}-f_{x_{i-1}}\right|\left|\alpha_{t_i}\beta_{t_i}-\alpha_{x_i}\beta_{x_{i-1}}\right|\\
\leq&\sum_{i=1}^{n}\left(\sup_{x\in[x_{i-1},x_i]}f(x)-\inf_{x\in[x_{i-1},x_i]}f(x)\right)\left(\alpha_{x_i}\beta_{x_i}-\alpha_{x_{i-1}}\beta_{x_{i-1}}\right)\\
<&\epsilon
\end{aligned}$$

The second inequality comes from noticing that $t_i$ is in $(x_{i-1}, x_i)$, and that $\alpha$ and $\beta$ are weakly increasing. The third inequality comes from $P$ being a refinement of $P_1$. Thus

$$\left|\int fd(\alpha\beta)-\int f\alpha d(\beta)-\int f\beta d(\alpha)\right|<4\epsilon$$

Which can be made arbitrarily close to 0.

$\square$