6.6 Integration and Differentiation

Problem 6.6:1. On $[a, b]$, let α be a strictly increasing function and f a continuous function, and for $x \in [a, b]$ define $F(x) = \int_a^x f(t)d\alpha(t)$. Show that for all $x \in [a, b]$, $\frac{dF(x)}{d\alpha(x)} = f(x)$, where the left-hand side is defined as $\lim_{t\to x} \frac{F(x)-F(t)}{\alpha(x)-\alpha(t)}$, and the equality includes the assertion that the limit exists.

Proof. First, we note that because $f \in \mathfrak{R}(\alpha)$ on $[a, b]$

$$F(x) - F(t) = \int_a^x f(s)d\alpha(s) - \int_a^t f(s)d\alpha(s) = \int_t^x f(x)d\alpha(s)$$

For all partitions P,

$$\int_t^x f(x)d\alpha(s) \leq \overline{\int_t^x} f(x)d\alpha(s) = \sum_{x_i \in P} M_i \Delta\alpha_i \leq \left(\sup_{s\in[x,t]} f(s) \right) \sum_{x_i \in P} \Delta\alpha_i = \left(\sup_{s\in[x,t]} f(s) \right) (\alpha(x)-\alpha(t))$$

Since $\alpha(x)$ is strictly increasing, $\alpha(x) - \alpha(t) \neq 0$ when $x \neq t$, so

$$\frac{F(x) - F(t)}{\alpha(x) - \alpha(t)} \leq \sup_{s\in[x,t]} f(s)$$

Taking the limit as x approaches t on both sides and using the continuity of f gives

$$\lim_{t\to x} \frac{F(x) - F(t)}{\alpha(x) - \alpha(t)} \leq f(x)$$

The analogous result for the lower integral and the Squeeze Theorem complete the proof.

□

Problem 6.6:2.

(a). Show that if f is continuous, then

$$\int_{t=1}^b \left(\int_{s=a}^t f(s)ds \right) dt = \int_{t=a}^b (b-t)f(t)dt$$

Proof. Let $x \in [a, b]$. Define $P(x) = \int_{t=a}^x \left(\int_{s=a}^t f(s)ds \right) dt$ and $Q(x) = \int_{t=a}^x (x - t)f(t)dt$.

$f(t)$ being continuous on $[a, b]$ implies that it is Riemann-integrable. This implies that $f^*(t) = \int_{s=a}^t f(s)ds$ is continuous, and that $P(x)$ is continuous and differentiable. Similarly, $(b - t)f(t)$ is continuous on $[a, b]$, so $Q(x)$ is continuous and differentiable.

By the Fundamental Theorem of Calculus,

$$P'(x) = \int_{s=a}^x f(s)ds$$

For $Q(x)$, since t and $tf(t)$ are Riemann-integrable,

$$Q(x) = x \int_{t=a}^x f(t)dt - \int_{t=a}^x tf(t)dt$$

x is trivially differentiable. Since t and $tf(t)$ are continuous,

$$Q'(x) = \int_{t=a}^{x} f(t)dt + xf(x) - xf(x) = \int_{t=a}^{x} f(t)dt$$

Thus, $P'(x) = Q'(x)$. Integrating both sides from a to c, then setting $c = b$, produces the desired result.

□

(c). Show that the result of Part (a) continues to hold if f is merely assumed Riemann-integrable, but not necessarily continuous.

Proof. $P(x)$ has the same derivative as in Part (a), as the derivation only assumed that $P(x)$ is Riemann-integrable. Similarly, for $x_0 \in [a, b]$ where $f(x_0)$ is continuous, the above derivations hold for $Q(x)$.

Let x_0 be a point where $f(x_0)$ is discontinuous. First, we will prove two lemmas.

Lemma 0.0.1. *If $f(x)$ is bounded, then $(x - x_0)f(x)$ is continuous at x_0.*

Proof. Let $M = \sup |f(x)|$. Then $(x - x_0)f(x) \leq |(x - x_0)f(x)| \leq |(x - x_0)|M$, which can be made arbitrarily small.

□

Lemma 0.0.2. *If $f(x)$ is continuous, then $(x - x_0)f(x)$ is differentiable at x_0 with derivative $f(x_0)$.*

Proof. By the definition of differentiability,

$$\lim_{x \to x_0} \frac{(x - x_0)f(x) - (x_0 - x_0)f(x_0)}{x - x_0} = \lim_{x \to x_0} f(x) = f(x_0)$$

by continuity.

□

We can rewrite $Q(x)$ as

$$Q(x) = \int_{t=a}^{x} ((x - x_0) + (x_0 - t))f(t)dt = (x - x_0)\int_{t=a}^{x} f(t)dt + \int_{t=a}^{x} (x_0 - t)f(t)dt$$

because the sub-functions are trivially Riemann-integrable. $\int_{t=a}^{x} f(t)$ is a continuous function, so by Lemma 0.0.2 $(x - x_0)\int_{t=a}^{x} f(t)dt$ is differentiable at $x = x_0$ with derivative $\int_{t=a}^{x_0} f(t)dt$. Similarly, $f(t)$ is bounded because it is Riemann-integrable, so by Lemma 0.0.1 $(x_0 - t)f(t)$ is continuous at $t = x_0$. Therefore $\int_{t=a}^{x}(x_0 - t)f(t)dt$ is differentiable at $x = x_0$, with derivative 0.

Therefore, $Q(x)$ is differentiable at $x = x_0$, and $Q'(x_0) = \int_{t=a}^{x} f(t)dt$. The proof then follows using the same logic as in Part (a).

□