6.6 Integration and Differentiation

Problem 6.6:1 On $[a, b]$, let α be a strictly increasing function and f a continuous function, and for $x \in [a, b]$ define $F(x) = \int_\alpha^x f(t)d\alpha(t)$. Show that for all $x \in [a, b]$, $\frac{dF(x)}{d\alpha(x)} = f(x)$, where the left-hand side is defined as $\lim_{t\to x} \frac{F(x)-F(t)}{\alpha(x)-\alpha(t)}$, and the equality includes the assertion that the limit exists.

Proof. First, we note that because $f \in \mathfrak{R}(\alpha)$ on $[a, b]$

$$F(x) - F(t) = \int_a^x f(s)d\alpha(s) - \int_a^t f(s)d\alpha(s) = \int_t^x f(x)d\alpha(s)$$

For all partitions P,

$$\int_t^x f(x)d\alpha(s) \leq \overline{\int_t^x} f(x)d\alpha(s) = \sum_{x_i \in P} M_i \Delta\alpha_i \leq (\sup_{s\in[x,t]} f(s)) \sum_{x_i \in P} \Delta\alpha_i = (\sup_{s\in[x,t]} f(s))(\alpha(x)-\alpha(t))$$

Since $\alpha(x)$ is strictly increasing, $\alpha(x) - \alpha(t) \neq 0$ when $x \neq t$, so

$$\frac{F(x) - F(t)}{\alpha(x) - \alpha(t)} \leq \sup_{s\in[x,t]} f(s)$$

Taking the limit as x approaches t on both sides and using the continuity of f gives

$$\lim_{t\to x} \frac{F(x) - F(t)}{\alpha(x) - \alpha(t)} \leq f(x)$$

The analogous result for the lower integral and the Squeeze Theorem complete the proof.

□