Chapter 2 A Taste of Topology

Problem 6

Determine whether $d_x(p,q) = \sin|p-q|$ on $[0, \frac{\pi}{2}$ is a metric.

Proof: It is a metric. Positive definiteness follows from the fact that the absolute value function is a metric over $\mathbb{R}$, and sine being one-to-one over the range of possible functions. Symmetry follows for the same reason. The triangle inequality follows from sine being increasing and concave over $[0, \frac{\pi}{2})$.

Specifically, let $p, r \in [0, \frac{\pi}{2})$, and without loss of generality, let $p \leq r$. If $q = p$ or $q = r$, the triangle inequality is trivial.

Let $q \notin [p, r]$. If $q > r$, then $q - p > r - p$ implies

$$d_s(p,q) + d_s(q,r) \geq d_s(p,r) + 0 =\geq d_s(p,r)$$

A similar result holds if $q < p$. If $q \in (p, r)$, imagine p, q, and r arranged on a line, with p at the origin. As x increases from q to r, the increase in sine is less than the corresponding increase from 0 to r-q, because sine is concave. Thus $sin(r-p) < sin(r-q) + sin(q-p)$.

$\square$