6.6 Integration and Differentiation

Problem 6.6:1. On $[a, b]$, let α be a strictly increasing function and f a continuous function, and for $x \in [a, b]$ define $F(x) = \int_a^x f(t)d\alpha(t)$. Show that for all $x \in [a, b]$, $\frac{dF(x)}{d\alpha(x)} = f(x)$, where the left-hand side is defined as $\lim_{t\to x} \frac{F(x)-F(t)}{\alpha(x)-\alpha(t)}$, and the equality includes the assertion that the limit exists.

Proof. First, we note that because $f \in \mathfrak{R}(\alpha)$ on $[a, b]$

$$F(x) - F(t) = \int_a^x f(s)d\alpha(s) - \int_a^t f(s)d\alpha(s) = \int_t^x f(x)d\alpha(s)$$

For all partitions P,

$$\int_t^x f(s)d\alpha(s) \le \sum_{x_i\in P} M_i\Delta\alpha_i \le \left(\sup_{s\in[x,t]} f(s)\right) \sum_{x_i\in P} \Delta\alpha_i = \left(\sup_{s\in[x,t]} f(s)\right) (\alpha(x)-\alpha(t))$$

Since $\alpha(x)$ is strictly increasing, $\alpha(x) - \alpha(t) > 0$ when $x \ne t$, so

$$\frac{F(x) - F(t)}{\alpha(x) - \alpha(t)} \le \sup_{s\in[x,t]} f(s)$$

Taking the limit as t approaches x on both sides gives

$$\lim_{t\to x} \frac{F(x) - F(t)}{\alpha(x) - \alpha(t)} \le \lim_{t\to x} \sup_{s\in[x,t]} f(s)$$

Lemma 0.0.0.1.

$$\lim_{t\to x} \sup_{s\in[x,t]} f(s) = f(x)$$

Proof. Let x_n be an arbitrary sequence such that $\forall n \in \mathbb{N}, x_n > x$ and $\lim_{n\to\infty} x_n = x$. Since $[x, x_n]$ is a closed, bounded interval on $\mathbb{R}$ and f is continuous, there exists a sequence of points $p_n \in [x, x_n]$ such that $f(p_n) = \sup_{s\in[x,x_n]} f(s)$. $x_n \to x$ implies $p_n \to x$ by the Squeeze Theorem, and the continuity of f implies that $f(p_n) \to f(x)$. □

Thus

$$\lim_{t\to x} \frac{F(x) - F(t)}{\alpha(x) - \alpha(t)} \le f(x)$$

The analogous result for the lower integral and the Squeeze Theorem complete the proof.

□

Problem 6.6:2.

(a). Show that if f is continuous, then

$$\int_{t=a}^{b} \left(\int_{s=a}^{t} f(s)ds\right) dt = \int_{t=a}^{b} (b - t)f(t)dt$$

Proof. Let $x \in [a,b]$. Define $P(x) = \int_{t=a}^{x} \left(\int_{s=a}^{t} f(s)ds \right) dt$ and $Q(x) = \int_{t=a}^{x} (x - t) f(t)dt$.

$f(t)$ being continuous on $[a,b]$ implies that it is Riemann-integrable. This implies that $f^*(t) = \int_{s=a}^{t} f(s)ds$ is continuous, and that $P(x) = \int_a^x f^*(t)dt$ is continuous and differentiable. Similarly, $(b-t)f(t)$ is continuous on $[a,b]$, so $Q(x)$ is continuous and differentiable.

By the Fundamental Theorem of Calculus,

$$P'(x) = \int_{s=a}^{x} f(s)ds$$

For $Q(x)$, since t and $tf(t)$ are Riemann-integrable,

$$Q(x) = x \int_{t=a}^{x} f(t)dt - \int_{t=a}^{x} tf(t)dt$$

x is trivially differentiable. Since t and $tf(t)$ are continuous,

$$Q'(x) = \int_{t=a}^{x} f(t)dt + xf(x) - xf(x) = \int_{t=a}^{x} f(t)dt$$

Thus, $P'(x) = Q'(x)$. Integrating both sides from a to c, then setting $c = b$, produces the desired result.

□

(c). Show that the result of Part (a) continues to hold if f is merely assumed Riemann-integrable, but not necessarily continuous.

Proof. $P(x)$ has the same derivative as in Part (a), as the derivation only assumed that $P(x)$ is Riemann-integrable. Similarly, for $x_0 \in [a,b]$ where $f(x_0)$ is continuous, the above derivations hold for $Q(x)$.

Let x_0 be a point where $f(x_0)$ is discontinuous. First, we will prove two lemmas.

Lemma 0.0.0.2. *If $f(x)$ is bounded, then $(x - x_0)f(x)$ is continuous at x_0.*

Proof. Let $M = \sup |f(x)|$. Then $(x - x_0)f(x) \leq |(x - x_0)f(x)| \leq |(x - x_0)|M$, which can be made arbitrarily small.

□

Lemma 0.0.0.3. *If $f(x)$ is continuous, then $(x - x_0)f(x)$ is differentiable at x_0 with derivative $f(x_0)$.*

Proof. By the definition of differentiability,

$$\lim_{x \to x_0} \frac{(x - x_0)f(x) - (x_0 - x_0)f(x_0)}{x - x_0} = \lim_{x \to x_0} f(x) = f(x_0)$$

by continuity.

□

We can rewrite $Q(x)$ as

$$Q(x) = \int_{t=a}^{x} ((x - x_0) + (x_0 - t))f(t)dt = (x - x_0) \int_{t=a}^{x} f(t)dt + \int_{t=a}^{x} (x_0 - t)f(t)dt$$

because the sub-functions are trivially Riemann-integrable. $\int_{t=a}^{x} f(t)$ is a continuous function, so by Lemma 0.0.0.3 $(x - x_0) \int_{t=a}^{x} f(t)dt$ is differentiable at $x = x_0$ with derivative $\int_{t=a}^{x_0} f(t)dt$. Similarly, $f(t)$ is bounded because it is Riemann-integrable, so by Lemma 0.0.0.2 $(x_0 - t)f(t)$ is continuous at $t = x_0$. Therefore $\int_{t=a}^{x}(x_0 - t)f(t)dt$ is differentiable at $x = x_0$, with derivative 0.

Therefore, $Q(x)$ is differentiable at $x = x_0$, and $Q'(x_0) = \int_{t=a}^{x} f(t)dt$. The proof then follows using the same logic as in Part (a).

□

Problem 6.6:4. Let f be a function on $[a, b]$, and α, β monotonically increasing nonnegative functions on $[a, b]$ such that $f \in \mathfrak{R}(\alpha) \cap \mathfrak{R}(\beta)$, $\alpha \in \mathfrak{R}(\beta)$, and $\beta \in \mathfrak{R}(\alpha)$. Prove that

$$\int f d(\alpha\beta) = \int f\alpha d(\beta) + \int f\beta d(\alpha)$$

Proof. First note that $f\alpha \in \mathfrak{R}(\beta)$ and $f\beta \in \mathfrak{R}(\alpha)$, so the right-side integrals exist. Also note that $\alpha\beta$ is monotonically increasing, so $\alpha\beta$ is a valid integrator.

We first begin by proving that the left-hand integral exists.

Lemma 0.0.0.4. *Let α and β be monotonically increasing non-negative functions with $\alpha \in \mathfrak{R}(\beta)$. Then for all $\epsilon > 0$, there exists a partition P such that on all intervals $[x_{i-1}, x_i]$, at least one of $\Delta\alpha_i$ or $\Delta\beta_i$ is less than or equal to ϵ.*

Proof. Because $\alpha \in \mathfrak{R}(\beta)$ and α is monotonically increasing, for all $\epsilon > 0$, there exists a partition P such that

$$\sum_{i \in P}[\sup_{x \in [x_{i-1}, x_i]} \alpha(x) - \inf_{x \in [x_{i-1}, x_i]} \alpha(x)]\Delta\beta_i = \sum_{i \in P} \Delta\alpha_i\Delta\beta_i < \epsilon$$

Because α and β are monotonic and increasing, the terms in the summation are nonnegative. Thus for all i in partition P, $\Delta\alpha_i\Delta\beta_i < \epsilon$.

Let $\epsilon = \epsilon^2$. Then $\Delta\alpha_i > \epsilon$ implies $\Delta\beta_i < \epsilon$, and vice-versa, implying that at least one of $\Delta\alpha_i$ or $\Delta\beta_i$ is less than or equal to ϵ.

□

Lemma 0.0.0.5. *Let f be a function on $[a, b]$, α, β monotonically increasing nonnegative functions on $[a, b]$, and $f \in \mathfrak{R}(\alpha)$. Then $f \in \mathfrak{R}(\alpha\beta)$.*

Proof. For all partitions P of $[a, b]$,

$$\sum_{i=1}^{n}(M_i - m_i)(\alpha_i\beta_i - \alpha_{i-1}\beta_{i-1}) < \sum_{i=1}^{n}(M_i - m_i)\beta_i(\alpha_i\beta_i - \alpha_{i-1}\beta_{i-1})$$

□

Now we work on evaluating the left-hand integral. For arbitrary $\epsilon > 0$, let P_1, P_2, and P_3 be partitions of $[a, b]$ such that

(1) $U(P_1, f, \alpha\beta) - L(P_1, f, \alpha\beta) < \epsilon$
(2) $U(P_2, f\alpha, \beta) - L(P_2, f\alpha, \beta) < \epsilon$
(3) $U(P_3, f\beta, \alpha) - L(P_3, f\beta, \alpha) < \epsilon$

Let P be their common partition. Let $x_0 < x_1 ... < x_n$ denote the points of P. For all i in $(1, 2...n)$, let $t_i \in (x_{i-1}, x_i)$ be arbitrary, fixed points, and let $P^* = (x_0, t_1, x_1 ... x_{i-1}, t_i, x_i ... t_n, x_n)$. Trivially, P^* partitions $[a, b]$ and is a refinement of P.

Consider $\int f d(\alpha\beta)$ and its associated Riemann-Stieltjes sum over P^*. Since P^* is a refinement of P_1, for arbitrary points $u_i \in [x_{i-1}, t_i]$ and $v_i \in [t_i, x_i]$,

$$\left| \sum_{i=1}^{n} \left[f(u_i)(\alpha_{t_i}\beta_{t_i} - \alpha_{x_{i-1}}\beta_{x_{i-1}}) + f(v_i)(\alpha_{x_i}\beta_{x_i} - \alpha_{t_i}\beta_{t_i}) \right] - \int f d(\alpha\beta) \right| < \epsilon$$

Letting $u_i = x_{i-1}$ and $v_i = x_i$ for all i,

$$\left| \sum_{i=1}^{n} \left[f_{x_{i-1}}(\alpha_{t_i}\beta_{t_i} - \alpha_{x_{i-1}}\beta_{x_{i-1}}) + f_{x_i}(\alpha_{x_i}\beta_{x_i} - \alpha_{t_i}\beta_{t_i}) \right] - \int f d(\alpha\beta) \right| < \epsilon$$

Now consider $\int f\alpha d(\beta)$ and its associated Riemann-Stieltjes sum over P. Letting $u_i = x_i$ for all i,

$$\left| \sum_{i=1}^{n} \left[f_{x_i}\alpha_{x_i}(\beta_{x_i} - \beta_{x_{i-1}}) \right] - \int f\alpha d(\beta) \right| < \epsilon$$

Similarly, considering $\int f\beta d(\alpha)$ over P and letting $u_i = x_{i-1}$ for all i,

$$\left| \sum_{i=1}^{n} \left[f_{x_{i-1}}\beta_{x_{i-1}}(\alpha_{x_i} - \alpha_{x_{i-1}}) \right] - \int f\beta d(\alpha) \right| < \epsilon$$

Adding the inequalities and using the Triangle Inequality gives

$$\Bigg| \int f d(\alpha\beta) - \int f\alpha d(\beta) - \int f\beta d(\alpha) + \sum_{i=1}^{n} \Big[- \left[f_{x_{i-1}}(\alpha_{t_i}\beta_{t_i} - \alpha_{x_{i-1}}\beta_{x_{i-1}}) + f_{x_i}(\alpha_{x_i}\beta_{x_i} - \alpha_{t_i}\beta_{t_i}) \right] + f_{x_i}\alpha_{x_i}(\beta_{x_i} - \beta_{x_{i-1}}) + f_{x_{i-1}}\beta_{x_{i-1}}(\alpha_{x_i} - \alpha_{x_{i-1}}) \Big] \Bigg| < 3\epsilon$$

Simplifying,

$$\left| \int f d(\alpha\beta) - \int f\alpha d(\beta) - \int f\beta d(\alpha) + \sum_{i=1}^{n} \Big[(f_{x_i} - f_{x_{i-1}})(\alpha_{t_i}\beta_{t_{i-1}} - \alpha_{x_i}\beta_{x_{i-1}}) \Big] \right| < 3\epsilon$$

To analyze the sum on the right, note that

$$\left|\sum_{i=1}^{n}\left[(f_{x_i}-f_{x_{i-1}})(\alpha_{t_i}\beta_{t_{i-1}}-\alpha_{x_i}\beta_{x_{i-1}})\right|\le$$

$$\sum_{i=1}^{n}\left|f_{x_i}-f_{x_{i-1}}\right|\left|\alpha_{t_i}\beta_{t_{i-1}}-\alpha_{x_i}\beta_{x_{i-1}}\right|\le$$

$$\sum_{i=1}^{n}\left(\sup_{x\in[x_{i-1},x_i]}f(x)-\inf_{x\in[x_{i-1},x_i]}f(x)\right)\left(\alpha_{x_i}\beta_{x_i}-\alpha_{x_{i-1}}\beta_{x_{i-1}}\right)<\epsilon$$

The second inequality comes from noticing that t_i is in (x_{i-1}, x_i), and that $\alpha\beta$ is weakly increasing. The third inequality comes from P being a refinement of P_1. Thus

$$\left|\int fd(\alpha\beta)-\int f\alpha d(\beta)-\int f\beta d(\alpha)\right|<4\epsilon$$

Which can be made arbitrarily close to 0.

$\square$