6.6 Integration and Differentiation

Problem 6.6:1. On $[a, b]$, let α be a strictly increasing function and f a continuous function, and for $x \in [a, b]$ define $F(x) = \int_a^x f(t) d\alpha(t)$. Show that for all $x \in [a, b]$, $\frac{dF(x)}{d\alpha(x)} = f(x)$, where the left-hand side is defined as $\lim_{t \to x} \frac{F(x)-F(t)}{\alpha(x)-\alpha(t)}$, and the equality includes the assertion that the limit exists.

Proof. First, we note that because $f \in \mathfrak{R}(\alpha)$ on $[a, b]$

$$F(x) - F(t) = \int_a^x f(s) d\alpha(s) - \int_a^t f(s) d\alpha(s) = \int_t^x f(x) d\alpha(s)$$

For all partitions P,

$$\int_t^x f(s) d\alpha(s) \leq \sum_{x_i \in P} M_i \Delta\alpha_i \leq \left(\sup_{s \in [x,t]} f(s) \right) \sum_{x_i \in P} \Delta\alpha_i = \left(\sup_{s \in [x,t]} f(s) \right) (\alpha(x) - \alpha(t))$$

Since $\alpha(x)$ is strictly increasing, $\alpha(x) - \alpha(t) > 0$ when $x \neq t$, so

$$\frac{F(x) - F(t)}{\alpha(x) - \alpha(t)} \leq \sup_{s \in [x,t]} f(s)$$

Taking the limit as t approaches x on both sides gives

$$\lim_{t \to x} \frac{F(x) - F(t)}{\alpha(x) - \alpha(t)} \leq \lim_{t \to x} \sup_{s \in [x,t]} f(s)$$

Lemma 0.0.0.1.

$$\lim_{t \to x} \sup_{s \in [x,t]} f(s) = f(x)$$

Proof. Let x_n be an arbitrary sequence such that $\forall n \in \mathbb{N}, x_n > x$ and $\lim_{n \to \infty} x_n = x$. Since $[x, x_n]$ is a closed, bounded interval on $\mathbb{R}$ and f is continuous, there exists a sequence of points $p_n \in [x, x_n]$ such that $f(p_n) = \sup_{s \in [x, x_n]} f(s)$. $x_n \to x$ implies $p_n \to x$ by the Squeeze Theorem, and the continuity of f implies that $f(p_n) \to f(x)$. □

Thus

$$\lim_{t \to x} \frac{F(x) - F(t)}{\alpha(x) - \alpha(t)} \leq f(x)$$

The analogous result for the lower integral and the Squeeze Theorem complete the proof.

□

Problem 6.6:2.

(a). Show that if f is continuous, then

$$\int_{t=a}^b \left(\int_{s=a}^t f(s) ds \right) dt = \int_{t=a}^b (b-t) f(t) dt$$

Proof. Let $x \in [a,b]$. Define $P(x) = \int_{t=a}^{x} \left(\int_{s=a}^{t} f(s)ds \right) dt$ and $Q(x) = \int_{t=a}^{x} (x-t)f(t)dt$.

$f(t)$ being continuous on $[a,b]$ implies that it is Riemann-integrable. This implies that $f^*(t) = \int_{s=a}^{t} f(s)ds$ is continuous, and that $P(x) = \int_a^x f^*(t)dt$ is continuous and differentiable. Similarly, $(b-t)f(t)$ is continuous on $[a,b]$, so $Q(x)$ is continuous and differentiable.

By the Fundamental Theorem of Calculus,

$$P'(x) = \int_{s=a}^{x} f(s)ds$$

For $Q(x)$, since t and $tf(t)$ are Riemann-integrable,

$$Q(x) = x\int_{t=a}^{x} f(t)dt - \int_{t=a}^{x} tf(t)dt$$

x is trivially differentiable. Since t and $tf(t)$ are continuous,

$$Q'(x) = \int_{t=a}^{x} f(t)dt + xf(x) - xf(x) = \int_{t=a}^{x} f(t)dt$$

Thus, $P'(x) = Q'(x)$. Integrating both sides from a to c, then setting $c = b$, produces the desired result.

□

(c). Show that the result of Part (a) continues to hold if f is merely assumed Riemann-integrable, but not necessarily continuous.

Proof. $P(x)$ has the same derivative as in Part (a), as the derivation only assumed that $P(x)$ is Riemann-integrable. Similarly, for $x_0 \in [a,b]$ where $f(x_0)$ is continuous, the above derivations hold for $Q(x)$.

Let x_0 be a point where $f(x_0)$ is discontinuous. First, we will prove two lemmas.

Lemma 0.0.0.2. *If $f(x)$ is bounded, then $(x-x_0)f(x)$ is continuous at x_0.*

Proof. Let $M = \sup|f(x)|$. Then $(x-x_0)f(x) \leq |(x-x_0)f(x)| \leq |(x-x_0)|M$, which can be made arbitrarily small.

□

Lemma 0.0.0.3. *If $f(x)$ is continuous, then $(x-x_0)f(x)$ is differentiable at x_0 with derivative $f(x_0)$.*

Proof. By the definition of differentiability,

$$\lim_{x\to x_0} \frac{(x-x_0)f(x) - (x_0-x_0)f(x_0)}{x-x_0} = \lim_{x\to x_0} f(x) = f(x_0)$$

by continuity.

□

We can rewrite $Q(x)$ as

$$Q(x) = \int_{t=a}^{x} ((x-x_0)+(x_0-t))f(t)dt = (x-x_0)\int_{t=a}^{x} f(t)dt + \int_{t=a}^{x} (x_0-t)f(t)dt$$

because the sub-functions are trivially Riemann-integrable. $\int_{t=a}^{x} f(t)$ is a continuous function, so by Lemma 0.0.0.3 $(x - x_0)\int_{t=a}^{x} f(t)dt$ is differentiable at $x = x_0$ with derivative $\int_{t=a}^{x_0} f(t)dt$. Similarly, $f(t)$ is bounded because it is Riemann-integrable, so by Lemma 0.0.0.2 $(x_0 - t)f(t)$ is continuous at $t = x_0$. Therefore $\int_{t=a}^{x}(x_0 - t)f(t)dt$ is differentiable at $x = x_0$, with derivative 0.

Therefore, $Q(x)$ is differentiable at $x = x_0$, and $Q'(x_0) = \int_{t=a}^{x} f(t)dt$. The proof then follows using the same logic as in Part (a).

□

Problem 6.6:4. Let f be a function on $[a, b]$, and α, β monotonically increasing nonnegative functions on $[a, b]$ such that $f \in \mathfrak{R}(\alpha) \cap \mathfrak{R}(\beta)$, $\alpha \in \mathfrak{R}(\beta)$, and $\beta \in \mathfrak{R}(\alpha)$. Prove that

$$\int f d(\alpha\beta) = \int f\alpha d(\beta) + \int f\beta d(\alpha)$$

Proof. We first begin with two lemmas.

Lemma 0.0.0.4. *Let α and β be monotonically increasing non-negative functions with $\alpha \in \mathfrak{R}(\beta)$. Then for all $\epsilon > 0$, there exists a partition P such that on intervals $[x_{i-1}, x_i]$ where $\Delta\alpha_i > \epsilon$, $\Delta\beta_i < \epsilon$, and on intervals $[x_{i-1}, x_i]$ where $\Delta\beta_i > \epsilon$, $\Delta\alpha_i < \epsilon$.*

Proof. Because $\alpha \in \mathfrak{R}(\beta)$ and α is monotonically increasing, for all $\epsilon > 0$, there exists a partition P such that

$$\sum_{i \in P}[\sup_{x \in [x_{i-1}, x_i]} \alpha(x) - \inf_{x \in [x_{i-1}, x_i]} \alpha(x)]\Delta\beta_i = \sum_{i \in P} \Delta\alpha_i \Delta\beta_i < \epsilon$$

Because α and β are monotonic and increasing, the terms in the summation are nonnegative. Thus for all i in partition P, $\Delta\alpha_i\Delta\beta_i < \epsilon$. The lemma follows by letting $\epsilon = \epsilon^2$.

□

Lemma 0.0.0.5. *Let f and α be real-valued functions on $[a, b]$, with α non-negative and weakly monotonically increasing. Then the following are true.*

- *If* $\sup f > 0$, *then* $\sup(f\alpha) \leq \sup(f)\alpha(b)$
- *If* $\sup f \leq 0$, *then* $\sup(f\alpha) \leq \sup(f)\alpha(a)$

Proof. TODO □

We now find an upper bound for the difference.

Theorem 0.0.1.

$$\int f\alpha d\beta + \int f\beta d\alpha - \int f d(\alpha\beta) \leq 0$$

Proof. Let $\epsilon > 0$ and partition P be a partition that satisfies Lemma 0.0.0.4. By the Lemma, we can divide the interval $[a, b]$ into two sets

$$C := \{i \leq N | \Delta\alpha_i < \epsilon\}$$

$$D := \{i \leq N | \Delta\alpha_i \geq \epsilon, \Delta\beta_i \leq \epsilon\}$$

Restricting our attention to C, by upper and lower integrals,

$$\int f\alpha d\beta + \int f\beta d\alpha - \int fd(\alpha\beta) \leq \sum_{i\in C} \sup_{x\in[x_i,x_{i-1}]}(f\alpha)\Delta\beta_i + \sup_{x\in[x_i,x_{i-1}]}(f\beta)\Delta\alpha_i - m_i\Delta(\alpha\beta)_i$$

By Lemma 0.0.0.5,

$$\sum_{i\in C} \sup_{x\in[x_i,x_{i-1}]}(f\alpha)\Delta\beta_i \leq \sum_{i\in C} M_i\alpha(x_i)\Delta\beta_i$$

By assumption on C,

$$\sum_{i\in C} \sup_{x\in[x_i,x_{i-1}]}(f\beta)\Delta\alpha_i < \sum_{i\in C} \sup_{x\in[x_i,x_{i-1}]}(f\beta)\epsilon$$

Similarly,

$$m_i\Delta(\alpha\beta)_i = m_i(\alpha_i\beta_i - \alpha_{i-1}\beta_{i-1}) \geq m_i\Delta(\alpha\beta)_i = m_i(\alpha_i\beta_i - (\alpha_i - \epsilon)\beta_{i-1}) = m_i\alpha_i\Delta\beta + m_i\beta_{i-1}\epsilon$$

Putting it all together,

$$\begin{aligned}
&\sum_{i\in C} \sup_{x\in[x_i,x_{i-1}]}(f\alpha)\Delta\beta_i + \sup_{x\in[x_i,x_{i-1}]}(f\beta)\Delta\alpha_i - m_i\Delta(\alpha\beta)_i \\
\leq &\sum_{i\in C} M_i\alpha(x_i)\Delta\beta_i + \sup_{x\in[x_i,x_{i-1}]}(f\beta)\epsilon - m_i\alpha_i\Delta\beta + m_i\beta_{i-1}\epsilon \\
= &\sum_{i\in C}(M_i - m_i)\alpha(x_i)\Delta\beta_i + \epsilon\sum_{i\in C}(\sup_{x\in[x_i,x_{i-1}]}(f\beta) + m_i\beta_{i-1})
\end{aligned}$$

□

□