6.6 Integration and Differentiation

Problem 6.6:1. On $[a, b]$, let α be a strictly increasing function and f a continuous function, and for $x \in [a, b]$ define $F(x) = \int_a^x f(t)d\alpha(t)$. Show that for all $x \in [a, b]$, $\frac{dF(x)}{d\alpha(x)} = f(x)$, where the left-hand side is defined as $\lim_{t\to x} \frac{F(x)-F(t)}{\alpha(x)-\alpha(t)}$, and the equality includes the assertion that the limit exists.

Proof. First, we note that because $f \in \mathfrak{R}(\alpha)$ on $[a, b]$

$$F(x) - F(t) = \int_a^x f(s)d\alpha(s) - \int_a^t f(s)d\alpha(s) = \int_t^x f(x)d\alpha(s)$$

For all partitions P,

$$\int_t^x f(s)d\alpha(s) \leq \sum_{x_i \in P} M_i \Delta\alpha_i \leq \left(\sup_{s\in[x,t]} f(s)\right) \sum_{x_i\in P} \Delta\alpha_i = \left(\sup_{s\in[x,t]} f(s)\right)(\alpha(x)-\alpha(t))$$

Since $\alpha(x)$ is strictly increasing, $\alpha(x) - \alpha(t) > 0$ when $x \neq t$, so

$$\frac{F(x) - F(t)}{\alpha(x) - \alpha(t)} \leq \sup_{s\in[x,t]} f(s)$$

Taking the limit as t approaches x on both sides gives

$$\lim_{t\to x} \frac{F(x) - F(t)}{\alpha(x) - \alpha(t)} \leq \lim_{t\to x} \sup_{s\in[x,t]} f(s)$$

Lemma 0.0.0.1.

$$\lim_{t\to x} \sup_{s\in[x,t]} f(s) = f(x)$$

Proof. Let x_n be an arbitrary sequence such that $\forall n \in \mathbb{N}, x_n > x$ and $\lim_{n\to\infty} x_n = x$. Since $[x, x_n]$ is a closed, bounded interval on $\mathbb{R}$ and f is continuous, there exists a sequence of points $p_n \in [x, x_n]$ such that $f(p_n) = \sup_{s\in[x,x_n]} f(s)$. $x_n \to x$ implies $p_n \to x$ by the Squeeze Theorem, and the continuity of f implies that $f(p_n) \to f(x)$. □

Thus

$$\lim_{t\to x} \frac{F(x) - F(t)}{\alpha(x) - \alpha(t)} \leq f(x)$$

The analogous result for the lower integral and the Squeeze Theorem complete the proof.

□

Problem 6.6:2.

(a). Show that if f is continuous, then

$$\int_{t=a}^b \left(\int_{s=a}^t f(s)ds\right) dt = \int_{t=a}^b (b-t)f(t)dt$$

Proof. Let $x \in [a,b]$. Define $P(x) = \int_{t=a}^{x} \left(\int_{s=a}^{t} f(s)ds \right) dt$ and $Q(x) = \int_{t=a}^{x} (x-t)f(t)dt$.

$f(t)$ being continuous on $[a,b]$ implies that it is Riemann-integrable. This implies that $f^*(t) = \int_{s=a}^{t} f(s)ds$ is continuous, and that $P(x) = \int_{a}^{x} f^*(t)dt$ is continuous and differentiable. Similarly, $(b-t)f(t)$ is continuous on $[a,b]$, so $Q(x)$ is continuous and differentiable.

By the Fundamental Theorem of Calculus,

$$P'(x) = \int_{s=a}^{x} f(s)ds$$

For $Q(x)$, since t and $tf(t)$ are Riemann-integrable,

$$Q(x) = x \int_{t=a}^{x} f(t)dt - \int_{t=a}^{x} tf(t)dt$$

x is trivially differentiable. Since t and $tf(t)$ are continuous,

$$Q'(x) = \int_{t=a}^{x} f(t)dt + xf(x) - xf(x) = \int_{t=a}^{x} f(t)dt$$

Thus, $P'(x) = Q'(x)$. Integrating both sides from a to c, then setting $c = b$, produces the desired result.

□

(c). Show that the result of Part (a) continues to hold if f is merely assumed Riemann-integrable, but not necessarily continuous.

Proof. $P(x)$ has the same derivative as in Part (a), as the derivation only assumed that $P(x)$ is Riemann-integrable. Similarly, for $x_0 \in [a,b]$ where $f(x_0)$ is continuous, the above derivations hold for $Q(x)$.

Let x_0 be a point where $f(x_0)$ is discontinuous. First, we will prove two lemmas.

Lemma 0.0.0.2. *If $f(x)$ is bounded, then $(x-x_0)f(x)$ is continuous at x_0.*

Proof. Let $M = \sup |f(x)|$. Then $(x-x_0)f(x) \leq |(x-x_0)f(x)| \leq |(x-x_0)|M$, which can be made arbitrarily small.

□

Lemma 0.0.0.3. *If $f(x)$ is continuous, then $(x-x_0)f(x)$ is differentiable at x_0 with derivative $f(x_0)$.*

Proof. By the definition of differentiability,

$$\lim_{x \to x_0} \frac{(x-x_0)f(x) - (x_0-x_0)f(x_0)}{x-x_0} = \lim_{x \to x_0} f(x) = f(x_0)$$

by continuity.

□

We can rewrite $Q(x)$ as

$$Q(x) = \int_{t=a}^{x} ((x-x_0) + (x_0-t))f(t)dt = (x-x_0) \int_{t=a}^{x} f(t)dt + \int_{t=a}^{x} (x_0-t)f(t)dt$$

because the sub-functions are trivially Riemann-integrable. $\int_{t=a}^{x} f(t)$ is a continuous function, so by Lemma 0.0.0.3 $(x - x_0)\int_{t=a}^{x} f(t)dt$ is differentiable at $x = x_0$ with derivative $\int_{t=a}^{x_0} f(t)dt$. Similarly, $f(t)$ is bounded because it is Riemann-integrable, so by Lemma 0.0.0.2 $(x_0 - t)f(t)$ is continuous at $t = x_0$. Therefore $\int_{t=a}^{x}(x_0 - t)f(t)dt$ is differentiable at $x = x_0$, with derivative 0.

Therefore, $Q(x)$ is differentiable at $x = x_0$, and $Q'(x_0) = \int_{t=a}^{x} f(t)dt$. The proof then follows using the same logic as in Part (a).

□

To build intuition for Problem 6.6:4, we first solve a special case.

Problem 6.6:4 Special Case. Let α, β be monotonically increasing nonnegative functions on $[a, b]$ such that $\alpha \in \mathfrak{R}(\beta)$, and $\beta \in \mathfrak{R}(\alpha)$. Prove that

$$\int d(\alpha\beta) = \int \alpha d(\beta) + \int \beta d(\alpha)$$

Proof. Note that the function 1 is trivially Riemann-Stieltjes integrable with respect to $\alpha\beta$.

Due to Riemann-integrability, there are partitions P, P_1, and P_2 such that

$$0 \leq |\int d(\alpha\beta) - \sum_{j\in P} \Delta(\alpha\beta)_j| \leq \frac{\epsilon}{3}$$
$$0 \leq |\int \alpha d(\beta) - \sum_{k\in P_1} \alpha_k \Delta\beta| \leq \frac{\epsilon}{3}$$
$$0 \leq |\int \beta d(\alpha) - \sum_{l\in P_2} \beta_{l-1} \Delta\alpha| \leq \frac{\epsilon}{3}$$

Taking the common partition $P^* = P \cup P_1 \cup P_2$ and using the Triangle Inequality,

$$0 \leq |\int \alpha d(\beta) + \int \beta d(\alpha) - \int d(\alpha\beta) + \sum_{i\in P^*} (-\Delta(\alpha\beta)_i + \alpha_i \Delta\beta + \beta_{i-1}\Delta\alpha)| \leq \epsilon$$

For any partition P,

$$\begin{aligned}
&\alpha_i\Delta\beta_i + \beta_{i-1}\Delta\alpha_i \\
=&\alpha_i\beta_i - \alpha_i\beta_{i-1} + \alpha_i\beta_{i-1} - \alpha_{i-1}\beta_{i-1} \\
=&\alpha_i\beta_i - \alpha_{i-1}\beta_{i-1} \\
=&\Delta(\alpha\beta)_i
\end{aligned}$$

Substituting,

$$0 \leq |\int \alpha d(\beta) + \int \beta d(\alpha) - \int d(\alpha\beta)| \leq \epsilon$$

Since ϵ can be made arbitrarily small, the proof is complete.

□

Problem 6.6:4. Let f be a function on $[a, b]$, and α, β monotonically increasing nonnegative functions on $[a, b]$ such that $f \in \mathfrak{R}(\alpha) \cap \mathfrak{R}(\beta)$, $\alpha \in \mathfrak{R}(\beta)$, and $\beta \in \mathfrak{R}(\alpha)$. Prove that

$$\int f d(\alpha\beta) = \int f\alpha d(\beta) + \int f\beta d(\alpha)$$

Proof. Note that from previous results in Problem 6.2.1, $f \in \mathfrak{R}(\alpha) \cap \mathfrak{R}(\beta)$ implies $f \in \mathfrak{R}(\alpha\beta)$. Since all of the above Riemann integrals are well-defined, for arbitrary $\epsilon > 0$, let P_1, P_2, and P_3 be partitions of $[a, b]$ such that

(1) $U(P_1, f, \alpha\beta) - L(P_1, f, \alpha\beta) < \epsilon$
(2) $U(P_2, f\alpha, \beta) - L(P_2, f\alpha, \beta) < \epsilon$
(3) $U(P_3, f\beta, \alpha) - L(P_3, f\beta, \alpha) < \epsilon$

Let P be their common partition. Let $x_0 < x_1 ... < x_n$ denote the points of P. For all i in $(1, 2...n)$, let $t_i \in (x_{i-1}, x_i)$ be arbitrary, fixed points, and let $P^* = (x_0, t_1, x_1 ... x_{i-1}, t_i, x_i ... t_n, x_n)$. Trivially, P^* partitions $[a, b]$ and is a refinement of P.

Consider $\int f d(\alpha\beta)$ and its associated Riemann-Stieltjes sum over P^*. Since P^* is a refinement of P_1, for arbitrary points $u_i \in [x_{i-1}, t_i]$ and $v_i \in [t_i, x_i]$,

$$\left| \sum_{i=1}^{n} \left[f(u_i)(\alpha_{t_i}\beta_{t_i} - \alpha_{x_{i-1}}\beta_{x_{i-1}}) + f(v_i)(\alpha_{x_i}\beta_{x_i} - \alpha_{t_i}\beta_{t_i}) \right] - \int f d(\alpha\beta) \right| < \epsilon$$

Letting $u_i = x_{i-1}$ and $v_i = x_i$ for all i,

$$\left| \sum_{i=1}^{n} \left[f_{x_{i-1}}(\alpha_{t_i}\beta_{t_i} - \alpha_{x_{i-1}}\beta_{x_{i-1}}) + f_{x_i}(\alpha_{x_i}\beta_{x_i} - \alpha_{t_i}\beta_{t_i}) \right] - \int f d(\alpha\beta) \right| < \epsilon$$

Now consider $\int f\alpha d(\beta)$ and its associated Riemann-Stieltjes sum over P. Letting $u_i = x_i$ for all i,

$$\left| \sum_{i=1}^{n} \left[f_{x_i}\alpha_{x_i}(\beta_{x_i} - \beta_{x_{i-1}}) \right] - \int f\alpha d(\beta) \right| < \epsilon$$

Similarly, considering $\int f\beta d(\alpha)$ over P and letting $u_i = x_{i-1}$ for all i,

$$\left| \sum_{i=1}^{n} \left[f_{x_{i-1}}\beta_{x_{i-1}}(\alpha_{x_i} - \alpha_{x_{i-1}}) \right] - \int f\beta d(\alpha) \right| < \epsilon$$

Adding the inequalities and using the Triangle Inequality gives

$$\Bigg| \int f d(\alpha\beta) - \int f\alpha d(\beta) - \int f\beta d(\alpha)$$

$$\sum_{i=1}^{n} \Big[- \left[f_{x_{i-1}}(\alpha_{t_i}\beta_{t_i} - \alpha_{x_{i-1}}\beta_{x_{i-1}}) + f_{x_i}(\alpha_{x_i}\beta_{x_i} - \alpha_{t_i}\beta_{t_i}) \right]$$

$$+ f_{x_i}\alpha_{x_i}(\beta_{x_i} - \beta_{x_{i-1}}) + f_{x_{i-1}}\beta_{x_{i-1}}(\alpha_{x_i} - \alpha_{x_{i-1}}) \Big] \Bigg| < 3\epsilon$$

∎