Problem 6.6:1. On $[a,b]$, let α be a strictly increasing function and f a continuous function, and for $x \in [a,b]$ define $F(x) = \int_a^x f(t)d\alpha(t)$. Show that for all $x \in [a,b]$, $\frac{dF(x)}{d\alpha(x)} = f(x)$, where the left-hand side is defined as $\lim_{t\to x} \frac{F(x)-F(t)}{\alpha(x)-\alpha(t)}$, and the equality includes the assertion that the limit exists.

Proof. First, we note that because $f \in \mathfrak{R}(\alpha)$ on $[a,b]$

$$F(x) - F(t) = \int_a^x f(s)d\alpha(s) - \int_a^t f(s)d\alpha(s) = \int_t^x f(x)d\alpha(s)$$

For all partitions P,

$$\int_t^x f(s)d\alpha(s) \leq \sum_{x_i \in P} M_i \Delta\alpha_i \leq \left(\sup_{s\in[x,t]} f(s)\right) \sum_{x_i\in P} \Delta\alpha_i = \left(\sup_{s\in[x,t]} f(s)\right)(\alpha(x)-\alpha(t))$$

Since $\alpha(x)$ is strictly increasing, $\alpha(x) - \alpha(t) > 0$ when $x \neq t$, so

$$\frac{F(x)-F(t)}{\alpha(x)-\alpha(t)} \leq \sup_{s\in[x,t]} f(s)$$

Taking the limit as t approaches x on both sides gives

$$\lim_{t\to x} \frac{F(x)-F(t)}{\alpha(x)-\alpha(t)} \leq \lim_{t\to x} \sup_{s\in[x,t]} f(s)$$

Lemma 0.0.0.1.

$$\lim_{t\to x} \sup_{s\in[x,t]} f(s) = f(x)$$

Proof. Let x_n be an arbitrary sequence such that $\forall n \in \mathbb{N}, x_n > x$ and $\lim_{n\to\infty} x_n = x$. Since $[x, x_n]$ is a closed, bounded interval on $\mathbb{R}$ and f is continuous, there exists a sequence of points $p_n \in [x, x_n]$ such that $f(p_n) = \sup_{s\in[x,x_n]} f(s)$. $x_n \to x$ implies $p_n \to x$ by the Squeeze Theorem, and the continuity of f implies that $f(p_n) \to f(x)$. □

Thus

$$\lim_{t\to x} \frac{F(x)-F(t)}{\alpha(x)-\alpha(t)} \leq f(x)$$

The analogous result for the lower integral and the Squeeze Theorem complete the proof.

□

Problem 6.6:2.

(a). Show that if f is continuous, then

$$\int_{t=a}^b \left(\int_{s=a}^t f(s)ds\right) dt = \int_{t=a}^b (b-t)f(t)dt$$

Proof. Let $x \in [a, b]$. Define $P(x) = \int_{t=a}^{x} \left(\int_{s=a}^{t} f(s) ds \right) dt$ and $Q(x) = \int_{t=a}^{x} (x - t) f(t) dt$.

$f(t)$ being continuous on $[a, b]$ implies that it is Riemann-integrable. This implies that $f^*(t) = \int_{s=a}^{t} f(s) ds$ is continuous, and that $P(x) = \int_{a}^{x} f^*(t) dt$ is continuous and differentiable. Similarly, $(b-t) f(t)$ is continuous on $[a, b]$, so $Q(x)$ is continuous and differentiable.

By the Fundamental Theorem of Calculus,

$$P'(x) = \int_{s=a}^{x} f(s) ds$$

For $Q(x)$, since t and $tf(t)$ are Riemann-integrable,

$$Q(x) = x \int_{t=a}^{x} f(t) dt - \int_{t=a}^{x} t f(t) dt$$

x is trivially differentiable. Since t and $tf(t)$ are continuous,

$$Q'(x) = \int_{t=a}^{x} f(t) dt + x f(x) - x f(x) = \int_{t=a}^{x} f(t) dt$$

Thus, $P'(x) = Q'(x)$. Integrating both sides from a to c, then setting $c = b$, produces the desired result.

□

(c). Show that the result of Part (a) continues to hold if f is merely assumed Riemann-integrable, but not necessarily continuous.

Proof. $P(x)$ has the same derivative as in Part (a), as the derivation only assumed that $P(x)$ is Riemann-integrable. Similarly, for $x_0 \in [a, b]$ where $f(x_0)$ is continuous, the above derivations hold for $Q(x)$.

Let x_0 be a point where $f(x_0)$ is discontinuous. First, we will prove two lemmas.

Lemma 0.0.0.2. *If $f(x)$ is bounded, then $(x - x_0) f(x)$ is continuous at x_0.*

Proof. Let $M = \sup |f(x)|$. Then $(x - x_0) f(x) \leq |(x - x_0) f(x)| \leq |(x - x_0)| M$, which can be made arbitrarily small.

□

Lemma 0.0.0.3. *If $f(x)$ is continuous, then $(x - x_0) f(x)$ is differentiable at x_0 with derivative $f(x_0)$.*

Proof. By the definition of differentiability,

$$\lim_{x \to x_0} \frac{(x - x_0) f(x) - (x_0 - x_0) f(x_0)}{x - x_0} = \lim_{x \to x_0} f(x) = f(x_0)$$

by continuity.

□

We can rewrite $Q(x)$ as

$$Q(x) = \int_{t=a}^{x} ((x - x_0) + (x_0 - t)) f(t) dt = (x - x_0) \int_{t=a}^{x} f(t) dt + \int_{t=a}^{x} (x_0 - t) f(t) dt$$

because the sub-functions are trivially Riemann-integrable. $\int_{t=a}^{x} f(t)$ is a continuous function, so by Lemma 0.0.0.3 $(x - x_0)\int_{t=a}^{x} f(t)dt$ is differentiable at $x = x_0$ with derivative $\int_{t=a}^{x_0} f(t)dt$. Similarly, $f(t)$ is bounded because it is Riemann-integrable, so by Lemma 0.0.0.2 $(x_0 - t)f(t)$ is continuous at $t = x_0$. Therefore $\int_{t=a}^{x}(x_0 - t)f(t)dt$ is differentiable at $x = x_0$, with derivative 0.

Therefore, $Q(x)$ is differentiable at $x = x_0$, and $Q'(x_0) = \int_{t=a}^{x} f(t)dt$. The proof then follows using the same logic as in Part (a).

□

To build intuition for Problem 6.6:4, we first solve a special case.

Problem 6.6:4 Special Case. Let α, β be monotonically increasing nonnegative functions on $[a, b]$ such that $\alpha \in \mathfrak{R}(\beta)$, and $\beta \in \mathfrak{R}(\alpha)$. Prove that

$$\int d(\alpha\beta) = \int \alpha d(\beta) + \int \beta d(\alpha)$$

Proof. Note that the function 1 is trivially Riemann-Stieltjes integrable with respect to $\alpha\beta$.

Due to Riemann-integrability, there are partitions P, P_1, and P_2 such that

$$0 \leq |\int d(\alpha\beta) - \sum_{i\in P} \Delta(\alpha\beta)_i| \leq \frac{\epsilon}{3}$$
$$0 \leq |\int \alpha d(\beta) - \sum_{i\in P_1} \alpha_i \Delta\beta| \leq \frac{\epsilon}{3}$$
$$0 \leq |\int \beta d(\alpha) - \sum_{i\in P_2} \beta_{i-1} \Delta\alpha| \leq \frac{\epsilon}{3}$$

Taking the common partition $P^* = P \cup P_1 \cup P_2$ and using the Triangle Inequality,

$$0 \leq |\int \alpha d(\beta) + \int \beta d(\alpha) - \int d(\alpha\beta) + \sum_{i\in P^*} (-\Delta(\alpha\beta)_i + \alpha_i\Delta\beta + \beta_{i-1}\Delta\alpha)| \leq \epsilon$$

For any partition P,

$$\begin{aligned}
&\alpha_i\Delta\beta_i + \beta_{i-1}\Delta\alpha_i \\
=&\alpha_i\beta_i - \alpha_i\beta_{i-1} + \alpha_i\beta_{i-1} - \alpha_{i-1}\beta_{i-1} \\
=&\alpha_i\beta_i - \alpha_{i-1}\beta_{i-1} \\
=&\Delta(\alpha\beta)_i
\end{aligned}$$

Substituting,

$$0 \leq |\int \alpha d(\beta) + \int \beta d(\alpha) - \int d(\alpha\beta)| \leq \epsilon$$

Since ϵ can be made arbitrarily small, the proof is complete.

□

Problem 6.6:4. Let f be a function on $[a, b]$, and α, β monotonically increasing nonnegative functions on $[a, b]$ such that $f \in \mathfrak{R}(\alpha) \cap \mathfrak{R}(\beta)$, $\alpha \in \mathfrak{R}(\beta)$, and $\beta \in \mathfrak{R}(\alpha)$. Prove that

$$\int f d(\alpha\beta) = \int f \alpha d(\beta) + \int f \beta d(\alpha)$$

Proof. $\square$

TODO