

6.6 INTEGRATION AND DIFFERENTIATION

Problem 6.R:13. Let $f(x) = \int_{t=x}^{t=x+1} \sin(t^2) dt$.

a. Show that when $x > 0$, $|f(x)| < \frac{1}{x}$.

Proof. Note that $x > 0$ implies that the limits of integration are correct. Make the substitution $t^2 = u$ to get

$$f(x) = \frac{1}{2} \int_{u=x^2}^{u=(x+1)^2} u^{-\frac{1}{2}} \sin(u) du$$

Integrate by parts with $a = u^{\frac{1}{2}}$ and $db = \sin(u)$ to get

$$\begin{aligned} f(x) &= \frac{1}{2} \left[-u^{-\frac{1}{2}} \cos(u) \right]_{x^2}^{(x+1)^2} - \frac{1}{2} \int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}} \cos(u) du \\ &= \frac{\cos(x^2)}{2x} - \frac{\cos((x+1)^2)}{2(x+1)} - \frac{1}{4} \int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}} \cos(u) du \end{aligned}$$

Evaluating the integral on the right, $\cos(x) \geq -1$, so

$$\int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}} \cos(u) du \geq - \int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}} du = 2 \left(\frac{1}{x+1} - \frac{1}{x} \right)$$

Substituting,

$$\begin{aligned} f(x) &\leq \frac{\cos(x^2)}{2x} - \frac{\cos((x+1)^2)}{2(x+1)} - \frac{1}{2(x+1)} + \frac{1}{2x} \\ &= \frac{\cos(x^2) + 1}{2x} - \frac{\cos((x+1)^2) + 1}{2(x+1)} \\ &\leq \frac{2}{2x} = \frac{1}{x} \end{aligned}$$

Since $\cos(t) \leq 1$. To show that $f(x) \geq -\frac{1}{x}$, it suffices to show that $f(x) \leq \frac{1}{x}$.

$$-f(x) = \frac{\cos((x+1)^2)}{2(x+1)} - \frac{\cos(x^2)}{2x} + \frac{1}{4} \int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}} \cos(u) du$$

By a similar argument as before, $\cos(x) \leq 1$, so

$$\int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}} \cos(u) du \leq \int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}} du = 2 \left(\frac{1}{x} - \frac{1}{x+1} \right)$$

Substituting,

$$\begin{aligned} -f(x) &\leq \frac{\cos((x+1)^2)}{2(x+1)} - \frac{\cos(x^2)}{2x} + \frac{1}{2x} - \frac{1}{2(x+1)} \\ &\leq \frac{1}{2(x+1)} + \frac{1}{2x} + \frac{1}{2x} - \frac{1}{2(x+1)} \\ &= \frac{1}{x} \end{aligned}$$

□

b. Prove that there exists constant c and function $r(x)$ with $|r(x)| < \frac{c}{x}$ such that

$$2xf(x) = \cos(x^2) - \cos((x+1)^2) + r(x)$$

Proof. I will assume that $x > 0$, as the expression $\frac{c}{x}$ doesn't make much sense for $x = 0$, and it's impossible for $|r(x)| < \frac{c}{x}$ to be true for both positive and negative x while c is constant.

From results in Part a,

$$\begin{aligned} 2xf(x) &= \cos(x^2) - \frac{x}{x+1} \cos((x+1)^2) - \frac{x}{2} \int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}} \cos(u) du \\ &= \cos(x^2) - \cos((x+1)^2) + \frac{1}{x+1} \cos((x+1)^2) - \frac{x}{2} \int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}} \cos(u) du \end{aligned}$$

From the above, it's clear that $r(x) = \frac{1}{x+1} \cos((x+1)^2) - \frac{x}{2} \int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}} \cos(u) du$. Now it remains to show that $|r(x)| < \frac{c}{x}$. Using that $\cos(t)$ and $-\cos(t)$ is are bounded above by 1,

$$r(x) \leq \frac{1}{x+1} + \frac{x}{2} \int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}} du = \frac{1}{x+1} - x \left[\frac{1}{x+1} - \frac{1}{x} \right] = \frac{2}{x+1} < \frac{2}{x}$$

Similarly,

$$\begin{aligned} -r(x) &= -\frac{1}{x+1} \cos((x+1)^2) + \frac{x}{2} \int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}} \cos(u) du \\ &\geq -\frac{1}{x+1} - \frac{x}{2} \int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}} du = -\frac{2}{x+1} \geq -\frac{2}{x} \end{aligned}$$

Thus $|r(x)| < \frac{3}{x}$.

□

c. Find the upper and lower limits of $xf(x)$ as x approaches infinity.

We know from previous results that

$$xf(x) = \frac{\cos(x^2)}{2} - \frac{\cos((x+1)^2)}{2} + s(x)$$

where $|s(x)| \leq \frac{1}{x}$. $\lim_{x \rightarrow \infty} s(x) = 0$, so the upper and lower limits of $s(x)$ are also 0.

Note that by the periodicity of cosine, for $n \in \mathbb{N}$, $\cos(\sqrt{2\pi n}) = 1$. We now show that there exist infinite $n \in \mathbb{N}$ such that $\cos((\sqrt{2\pi n}+1)^2) = \cos(2\pi n + 2\sqrt{2\pi n}+1) = -1$, thus implying that $\limsup_{n \rightarrow \infty} xf(x) = 1$.

Theorem 1. *Let $\delta > 0$. Then there exist infinite natural numbers a such that $|2\sqrt{2a\pi} + 1 - \pi| < \delta \pmod{2\pi}$. In other words, $2\sqrt{2a\pi} + 1$ becomes arbitrarily close to a number of the form $b\pi$, where b is an odd number.*

To prove this, we will need to analyze the behavior of $g(x) = 2\sqrt{2\pi}\sqrt{x} + 1$, then evaluate it at specifically chosen a 's. We first start by analyzing the Taylor series of \sqrt{x} .

Lemma 2. *The Taylor series of \sqrt{x} about $x_0 > 0$ is*

$$\sqrt{x_0} + \frac{1}{2(1!)} x_0^{-\frac{1}{2}} (x - x_0) + \sum_{i=2}^{\infty} \frac{(-1)^{i-1}}{2^i} \frac{1(3)(5)\dots(2i-3)}{i!} x_0^{-i+\frac{1}{2}} (x - x_0)^i$$

with radius of convergence $R = x_0$.

Proof. The first few terms of the Taylor series are

$$T(x) = \sqrt{x_0} + \frac{1}{2(1!)}x_0^{-\frac{1}{2}}(x - x_0) - \frac{1}{2^2(2!)}x_0^{-\frac{3}{2}}(x - x_0)^2 + \frac{3}{2^3(3!)}x_0^{-\frac{5}{2}}(x - x_0)^3 \dots$$

To find the radius of convergence, note that

$$\frac{1(3)(5)\dots(2i-3)}{2^i} < \frac{1}{2} \left(\frac{1}{2}\right) \left(\frac{3}{2}\right) \left(\frac{5}{2}\right) \dots \frac{2i-3}{2} < \frac{1}{2} 1(2)(3) \dots i = \frac{1}{2} i!$$

implying that when $x > x_0$,

$$\frac{1(3)(5)\dots(2i-3)}{2^i(i!)} x_0^{-i+\frac{1}{2}} (x - x_0)^i < \frac{\sqrt{x_0}}{2} \left(\frac{x}{x_0} - 1\right)^i$$

Since $T(x)$ is alternating, it converges when $\frac{x}{x_0} - 1 < 1 \rightarrow x \in [x_0, 2x_0)$.

When $x < x_0$,

$$T(x) = \sqrt{x_0} + \frac{1}{2(1!)}x_0^{-\frac{1}{2}}(x - x_0) - \sqrt{x_0} \sum_{i=2}^{\infty} \frac{1(3)(5)\dots(2i-3)}{2^i(i!)} \left(1 - \frac{x}{x_0}\right)^i$$

Since

$$\frac{1(3)(5)\dots(2i-3)}{2^i(i!)} \left(1 - \frac{x}{x_0}\right)^i < \frac{1}{2} \left(1 - \frac{x}{x_0}\right)^i$$

and the right series is a convergent geometric series under the assumption that $x \in (0, x_0]$, $T(x)$ converges. □

We next show that viewed as a sequence over the natural numbers, (g_n) 's differences between terms become arbitrarily small.

Lemma 3. *Let $(g_n) = 2\sqrt{2\pi}\sqrt{n} + 1$ for $n \in \mathbb{N}$, and $\Delta g_n = g_{n+1} - g_n = 2\sqrt{2}(\sqrt{n+1} - \sqrt{n})$. Then $\lim_{n \rightarrow \infty} \Delta g_n = 0$.*

Proof. It suffices to show that $\lim_{n \rightarrow \infty} \sqrt{n+1} - \sqrt{n} = 0$. From the results in Theorem 2, the Taylor series of \sqrt{x} at n is

$$\sqrt{n+x} = \sqrt{n} + \frac{1}{2(1!)}n^{-\frac{1}{2}}(x) + \sum_{i=2}^{\infty} \frac{(-1)^{i-1}}{2^i} \frac{1(3)(5)\dots(2i-3)}{i!} n^{-i+\frac{1}{2}}(x)^i$$

Since the series is alternating and convergent, its partial sums that have a positive term as their highest power are larger than the series. Thus

$$\sqrt{n+1} < \sqrt{n} + \frac{1}{2\sqrt{n}}$$

implying

$$0 \leq \sqrt{n+1} - \sqrt{n} < \frac{1}{2\sqrt{n}}$$

implying that $\lim_{n \rightarrow \infty} \sqrt{n+1} - \sqrt{n} = 0$. □

We are now in the position to prove Theorem 1.

Proof. Let (g_n) be the sequence defined by $g_n = 2\sqrt{2\pi}\sqrt{n} + 1 - \pi$. Since $\lim_{n \rightarrow \infty} \Delta g_n = 0$, there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies that $\Delta g_n < \delta$. This, combined with (g_n) being strictly increasing and $\lim_{n \rightarrow \infty} g_n = \infty$, imply that (g_n) passes through all numbers greater than g_{n_0} while increasing to infinity, while taking step sizes less than δ .

Specifically, for any natural number b such that $2\pi b > g_{n_0}$, there exists an $a \geq n_0$ in the natural numbers such that

$$g_a \leq 2\pi b < g_{a+1}$$

$\Delta g_a < \delta$ implies

$$|\Delta g_a - 2\pi b| < \delta$$

which is equivalent to

$$|2\sqrt{2\pi}\sqrt{a} + 1 - (2b + 1)\pi| < \delta$$

The theorem follows because for distinct b , (g_n) being strictly increasing implies that the a 's are distinct. \square

The results of the main problem now follow.

Lemma 4. $\limsup_{n \rightarrow \infty} x f(x) = \frac{\cos(x^2)}{2} - \frac{\cos((x+1)^2)}{2} + s(x) = 1$

Proof. From Theorem 1 and the continuity of $\cos(x)$, for all $\epsilon > 0$, there exist infinite $n \in \mathbb{N}$ such that $\cos(x^2) = \cos(\sqrt{2\pi n}^2) = 1$ and $|\cos((x+1)^2) + 1| = |\cos((\sqrt{2\pi n} + 1)^2) + 1| < \epsilon$. As previously established, $\limsup_{x \rightarrow \infty} s(x) = 0$. \square

Corollary 5. $\liminf_{n \rightarrow \infty} x f(x) = \frac{\cos(x^2)}{2} - \frac{\cos((x+1)^2)}{2} + s(x) = -1$

Proof. Add π to the a 's in Lemma 4. \square

Problem 6.6:1. On $[a, b]$, let α be a strictly increasing function and f a continuous function, and for $x \in [a, b]$ define $F(x) = \int_a^x f(t) d\alpha(t)$. Show that for all $x \in [a, b]$, $\frac{dF(x)}{d\alpha(x)} = f(x)$, where the left-hand side is defined as $\lim_{t \rightarrow x} \frac{F(x) - F(t)}{\alpha(x) - \alpha(t)}$, and the equality includes the assertion that the limit exists.

Proof. First, we note that because $f \in \mathfrak{R}(\alpha)$ on $[a, b]$

$$F(x) - F(t) = \int_a^x f(s) d\alpha(s) - \int_a^t f(s) d\alpha(s) = \int_t^x f(s) d\alpha(s)$$

For all partitions P ,

$$\begin{aligned} \int_t^x f(s) d\alpha(s) &\leq \sum_{x_i \in P} M_i \Delta\alpha_i \\ &\leq \left(\sup_{s \in [x, t]} f(s) \right) \sum_{x_i \in P} \Delta\alpha_i \\ &= \left(\sup_{s \in [x, t]} f(s) \right) (\alpha(x) - \alpha(t)) \end{aligned}$$

Since $\alpha(x)$ is strictly increasing, $\alpha(x) - \alpha(t) > 0$ when $x \neq t$, so

$$\frac{F(x) - F(t)}{\alpha(x) - \alpha(t)} \leq \sup_{s \in [x, t]} f(s)$$

Taking the limit as t approaches x on both sides gives

$$\lim_{t \rightarrow x} \frac{F(x) - F(t)}{\alpha(x) - \alpha(t)} \leq \lim_{t \rightarrow x} \sup_{s \in [x, t]} f(s)$$

Lemma 6.

$$\lim_{t \rightarrow x} \sup_{s \in [x, t]} f(s) = f(x)$$

Proof. Let x_n be an arbitrary sequence such that $\forall n \in \mathbb{N}, x_n > x$ and $\lim_{n \rightarrow \infty} x_n = x$. Since $[x, x_n]$ is a closed, bounded interval on \mathbb{R} and f is continuous, there exists a sequence of points $p_n \in [x, x_n]$ such that $f(p_n) = \sup_{s \in [x, x_n]} f(s)$. $x_n \rightarrow x$ implies $p_n \rightarrow x$ by the Squeeze Theorem, and the continuity of f implies that $f(p_n) \rightarrow f(x)$. \square

Thus

$$\lim_{t \rightarrow x} \frac{F(x) - F(t)}{\alpha(x) - \alpha(t)} \leq f(x)$$

The analogous result for the lower integral and the Squeeze Theorem complete the proof. \square

Problem 6.6:2.

(a). Show that if f is continuous, then

$$\int_{t=a}^b \left(\int_{s=a}^t f(s) ds \right) dt = \int_{t=a}^b (b-t)f(t) dt$$

Proof. Let $x \in [a, b]$. Define $P(x) = \int_{t=a}^x \left(\int_{s=a}^t f(s) ds \right) dt$ and $Q(x) = \int_{t=a}^x (x-t)f(t) dt$.

$f(t)$ being continuous on $[a, b]$ implies that it is Riemann-integrable. This implies that $f^*(t) = \int_{s=a}^t f(s) ds$ is continuous, and that $P(x) = \int_a^x f^*(t) dt$ is continuous and differentiable. Similarly, $(b-t)f(t)$ is continuous on $[a, b]$, so $Q(x)$ is continuous and differentiable.

By the Fundamental Theorem of Calculus,

$$P'(x) = \int_{s=a}^x f(s) ds$$

For $Q(x)$, since t and $tf(t)$ are Riemann-integrable,

$$Q(x) = x \int_{t=a}^x f(t) dt - \int_{t=a}^x tf(t) dt$$

x is trivially differentiable. Since t and $tf(t)$ are continuous,

$$Q'(x) = \int_{t=a}^x f(t) dt + xf(x) - xf(x) = \int_{t=a}^x f(t) dt$$

Thus, $P'(x) = Q'(x)$. Integrating both sides from a to c , then setting $c = b$, produces the desired result. \square

(c). Show that the result of Part (a) continues to hold if f is merely assumed Riemann-integrable, but not necessarily continuous.

Proof. $P(x)$ has the same derivative as in Part (a), as the derivation only assumed that $P(x)$ is Riemann-integrable. Similarly, for $x_0 \in [a, b]$ where $f(x_0)$ is continuous, the above derivations hold for $Q(x)$.

Let x_0 be a point where $f(x_0)$ is discontinuous. First, we will prove two lemmas.

Lemma 7. *If $f(x)$ is bounded, then $(x - x_0)f(x)$ is continuous at x_0 .*

Proof. Let $M = \sup |f(x)|$. Then $(x - x_0)f(x) \leq |(x - x_0)f(x)| \leq |x - x_0|M$, which can be made arbitrarily small. \square

Lemma 8. *If $f(x)$ is continuous, then $(x - x_0)f(x)$ is differentiable at x_0 with derivative $f(x_0)$.*

Proof. By the definition of differentiability,

$$\lim_{x \rightarrow x_0} \frac{(x - x_0)f(x) - (x_0 - x_0)f(x_0)}{x - x_0} = \lim_{x \rightarrow x_0} f(x) = f(x_0)$$

by continuity. \square

We can rewrite $Q(x)$ as

$$Q(x) = \int_{t=a}^x ((x - x_0) + (x_0 - t))f(t)dt = (x - x_0) \int_{t=a}^x f(t)dt + \int_{t=a}^x (x_0 - t)f(t)dt$$

because the sub-functions are trivially Riemann-integrable. $\int_{t=a}^x f(t)dt$ is a continuous function, so by Lemma 8 $(x - x_0) \int_{t=a}^x f(t)dt$ is differentiable at $x = x_0$ with derivative $\int_{t=a}^{x_0} f(t)dt$. Similarly, $f(t)$ is bounded because it is Riemann-integrable, so by Lemma 7 $(x_0 - t)f(t)$ is continuous at $t = x_0$. Therefore $\int_{t=a}^x (x_0 - t)f(t)dt$ is differentiable at $x = x_0$, with derivative 0.

Therefore, $Q(x)$ is differentiable at $x = x_0$, and $Q'(x_0) = \int_{t=a}^{x_0} f(t)dt$. The proof then follows using the same logic as in Part (a). \square

Problem 6.6:4. Let f be a function on $[a, b]$, and α, β monotonically increasing nonnegative functions on $[a, b]$ such that $f \in \mathfrak{R}(\alpha) \cap \mathfrak{R}(\beta)$, $\alpha \in \mathfrak{R}(\beta)$, and $\beta \in \mathfrak{R}(\alpha)$. Prove that

$$\int f d(\alpha\beta) = \int f \alpha d(\beta) + \int f \beta d(\alpha)$$

Proof. First note that $f\alpha \in \mathfrak{R}(\beta)$ and $f\beta \in \mathfrak{R}(\alpha)$. Also note that $\alpha\beta$ is monotonically increasing, so $\alpha\beta$ is a valid integrator.

Theorem 9. *Under the assumptions of Problem 6.6.4, $f \in \mathfrak{R}(\alpha\beta)$.*

Proof. First note that by expanding the terms,

$$\alpha_i \beta_i - \alpha_{i-1} \beta_{i-1} = (\alpha_i - \alpha_{i-1})(\beta_i - \beta_{i-1}) + \alpha_{i-1}(\beta_i - \beta_{i-1}) + \beta_{i-1}(\alpha_i - \alpha_{i-1})$$

Writing out the difference between the upper and lower Riemann sums,

$$\begin{aligned} & \sum_{i=1}^n (M_i - m_i)(\alpha_i \beta_i - \alpha_{i-1} \beta_{i-1}) \\ &= \sum_{i=1}^n (M_i - m_i) \Delta \alpha_i \Delta \beta_i + \sum_{i=1}^n (M_i - m_i) \alpha_{i-1} \Delta \beta_i + \sum_{i=1}^n (M_i - m_i) \beta_{i-1} \Delta \alpha_i \\ &\leq \alpha(b) \sum_{i=1}^n (M_i - m_i) \Delta \beta_i + \alpha(b) \sum_{i=1}^n (M_i - m_i) \Delta \beta_i + \beta(b) \sum_{i=1}^n (M_i - m_i) \Delta \alpha_i \end{aligned}$$

The third line follows because all terms are positive, and α and β are monotonically increasing. Because $f \in \mathfrak{R}(\alpha) \cap \mathfrak{R}(\beta)$, there exist partitions that make the third line arbitrarily small. \square

Now we prove the main result. For arbitrary $\epsilon > 0$, let P_1 , P_2 , and P_3 be partitions of $[a, b]$ such that

- (1) $U(P_1, f, \alpha\beta) - L(P_1, f, \alpha\beta) < \epsilon$
- (2) $U(P_2, f\alpha, \beta) - L(P_2, f\alpha, \beta) < \epsilon$
- (3) $U(P_3, f\beta, \alpha) - L(P_3, f\beta, \alpha) < \epsilon$

Let P be their common partition. Let $x_0 < x_1 \dots < x_n$ denote the points of P . For all i in $(1, 2 \dots n)$, let $t_i \in (x_{i-1}, x_i)$ be arbitrary, fixed points, and let $P^* = (x_0, t_1, x_1 \dots x_{i-1}, t_i, x_i \dots t_n, x_n)$. Trivially, P^* partitions $[a, b]$ and is a refinement of P .

Consider $\int f d(\alpha\beta)$ and its associated Riemann-Stieltjes sum over P^* . Since P^* is a refinement of P_1 , for arbitrary points $u_i \in [x_{i-1}, t_i]$ and $v_i \in [t_i, x_i]$,

$$\left| \sum_{i=1}^n [f(u_i)(\alpha_{t_i} \beta_{t_i} - \alpha_{x_{i-1}} \beta_{x_{i-1}}) + f(v_i)(\alpha_{x_i} \beta_{x_i} - \alpha_{t_i} \beta_{t_i})] - \int f d(\alpha\beta) \right| < \epsilon$$

Letting $u_i = x_{i-1}$ and $v_i = x_i$ for all i ,

$$\left| \sum_{i=1}^n [f_{x_{i-1}}(\alpha_{t_i} \beta_{t_i} - \alpha_{x_{i-1}} \beta_{x_{i-1}}) + f_{x_i}(\alpha_{x_i} \beta_{x_i} - \alpha_{t_i} \beta_{t_i})] - \int f d(\alpha\beta) \right| < \epsilon$$

Now consider $\int f \alpha d(\beta)$ and its associated Riemann-Stieltjes sum over P . Letting $u_i = x_i$ for all i ,

$$\left| \sum_{i=1}^n [f_{x_i} \alpha_{x_i} (\beta_{x_i} - \beta_{x_{i-1}})] - \int f \alpha d(\beta) \right| < \epsilon$$

Similarly, considering $\int f \beta d(\alpha)$ over P and letting $u_i = x_{i-1}$ for all i ,

$$\left| \sum_{i=1}^n [f_{x_{i-1}} \beta_{x_{i-1}} (\alpha_{x_i} - \alpha_{x_{i-1}})] - \int f \beta d(\alpha) \right| < \epsilon$$

Adding the inequalities and using the Triangle Inequality gives

$$\begin{aligned} & \left| \int f d(\alpha\beta) - \int f \alpha d(\beta) - \int f \beta d(\alpha) \right. \\ & + \sum_{i=1}^n \left[-[f_{x_{i-1}}(\alpha_{t_i}\beta_{t_i} - \alpha_{x_{i-1}}\beta_{x_{i-1}}) + f_{x_i}(\alpha_{x_i}\beta_{x_i} - \alpha_{t_i}\beta_{t_i})] \right. \\ & \left. \left. + f_{x_i}\alpha_{x_i}(\beta_{x_i} - \beta_{x_{i-1}}) + f_{x_{i-1}}\beta_{x_{i-1}}(\alpha_{x_i} - \alpha_{x_{i-1}}) \right] \right| < 3\epsilon \end{aligned}$$

Simplifying,

$$\left| \int f d(\alpha\beta) - \int f \alpha d(\beta) - \int f \beta d(\alpha) + \sum_{i=1}^n [(f_{x_i} - f_{x_{i-1}})(\alpha_{t_i}\beta_{t_{i-1}} - \alpha_{x_i}\beta_{x_{i-1}})] \right| < 3\epsilon$$

To analyze the sum on the right, note that

$$\begin{aligned} & \left| \sum_{i=1}^n [(f_{x_i} - f_{x_{i-1}})(\alpha_{t_i}\beta_{t_{i-1}} - \alpha_{x_i}\beta_{x_{i-1}})] \right| \\ & \leq \sum_{i=1}^n |f_{x_i} - f_{x_{i-1}}| |\alpha_{t_i}\beta_{t_{i-1}} - \alpha_{x_i}\beta_{x_{i-1}}| \\ & \leq \sum_{i=1}^n \left(\sup_{x \in [x_{i-1}, x_i]} f(x) - \inf_{x \in [x_{i-1}, x_i]} f(x) \right) (\alpha_{x_i}\beta_{x_i} - \alpha_{x_{i-1}}\beta_{x_{i-1}}) \\ & < \epsilon \end{aligned}$$

The second inequality comes from noticing that t_i is in (x_{i-1}, x_i) , and that $\alpha\beta$ is weakly increasing. The third inequality comes from P being a refinement of P_1 . Thus

$$\left| \int f d(\alpha\beta) - \int f \alpha d(\beta) - \int f \beta d(\alpha) \right| < 4\epsilon$$

Which can be made arbitrarily close to 0.

□