

## 6.6 INTEGRATION AND DIFFERENTIATION

**Problem 6.R:13.** Let  $f(x) = \int_{t=x}^{t=x+1} \sin(t^2) dt$ .

a. Show that when  $x > 0$ ,  $|f(x)| < \frac{1}{x}$ .

*Proof.* Note that  $x > 0$  implies that the limits of integration are correct. Make the substitution  $t^2 = u$  to get

$$f(x) = \frac{1}{2} \int_{u=x^2}^{u=(x+1)^2} u^{-\frac{1}{2}} \sin(u) du$$

Integrate by parts with  $a = u^{\frac{1}{2}}$  and  $db = \sin(u)$  to get

$$\begin{aligned} f(x) &= \frac{1}{2} \left[ -u^{-\frac{1}{2}} \cos(u) \right]_{x^2}^{(x+1)^2} - \frac{1}{2} \int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}} \cos(u) du \\ &= \frac{\cos(x^2)}{2x} - \frac{\cos((x+1)^2)}{2(x+1)} - \frac{1}{4} \int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}} \cos(u) du \end{aligned}$$

Evaluating the integral on the right,  $\cos(x) \geq -1$ , so

$$\int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}} \cos(u) du \geq - \int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}} du = 2 \left( \frac{1}{x+1} - \frac{1}{x} \right)$$

Substituting,

$$\begin{aligned} f(x) &\leq \frac{\cos(x^2)}{2x} - \frac{\cos((x+1)^2)}{2(x+1)} - \frac{1}{2(x+1)} + \frac{1}{2x} \\ &= \frac{\cos(x^2) + 1}{2x} - \frac{\cos((x+1)^2) + 1}{2(x+1)} \\ &\leq \frac{2}{2x} = \frac{1}{x} \end{aligned}$$

Since  $\cos(t) \leq 1$ . To show that  $f(x) \geq -\frac{1}{x}$ , it suffices to show that  $f(x) \leq \frac{1}{x}$ .

$$-f(x) = \frac{\cos((x+1)^2)}{2(x+1)} - \frac{\cos(x^2)}{2x} + \frac{1}{4} \int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}} \cos(u) du$$

By a similar argument as before,  $\cos(x) \leq 1$ , so

$$\int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}} \cos(u) du \leq \int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}} du = 2 \left( \frac{1}{x} - \frac{1}{x+1} \right)$$

Substituting,

$$\begin{aligned} -f(x) &\leq \frac{\cos((x+1)^2)}{2(x+1)} - \frac{\cos(x^2)}{2x} + \frac{1}{2x} - \frac{1}{2(x+1)} \\ &\leq \frac{1}{2(x+1)} + \frac{1}{2x} + \frac{1}{2x} - \frac{1}{2(x+1)} \\ &= \frac{1}{x} \end{aligned}$$

□

b. Prove that there exists constant  $c$  and function  $r(x)$  with  $|r(x)| < \frac{c}{x}$  such that

$$2xf(x) = \cos(x^2) - \cos((x+1)^2) + r(x)$$

*Proof.* I will assume that  $x > 0$ , as the expression  $\frac{c}{x}$  doesn't make much sense for  $x = 0$ , and it's impossible for  $|r(x)| < \frac{c}{x}$  to be true for both positive and negative  $x$  while  $c$  is constant.

From results in Part a,

$$\begin{aligned} 2xf(x) &= \cos(x^2) - \frac{x}{x+1} \cos((x+1)^2) - \frac{x}{2} \int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}} \cos(u) du \\ &= \cos(x^2) - \cos((x+1)^2) + \frac{1}{x+1} \cos((x+1)^2) - \frac{x}{2} \int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}} \cos(u) du \end{aligned}$$

From the above, it's clear that  $r(x) = \frac{1}{x+1} \cos((x+1)^2) - \frac{x}{2} \int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}} \cos(u) du$ . Now it remains to show that  $|r(x)| < \frac{c}{x}$ . Using that  $\cos(t)$  and  $-\cos(t)$  is are bounded above by 1,

$$r(x) \leq \frac{1}{x+1} + \frac{x}{2} \int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}} du = \frac{1}{x+1} - x \left[ \frac{1}{x+1} - \frac{1}{x} \right] = \frac{2}{x+1} < \frac{2}{x}$$

Similarly,

$$\begin{aligned} -r(x) &= -\frac{1}{x+1} \cos((x+1)^2) + \frac{x}{2} \int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}} \cos(u) du \\ &\geq -\frac{1}{x+1} - \frac{x}{2} \int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}} du = -\frac{2}{x+1} \geq -\frac{2}{x} \end{aligned}$$

Thus  $|r(x)| < \frac{3}{x}$ .

□

c. Find the upper and lower limits of  $xf(x)$  as  $x$  approaches infinity.

We know from previous results that

$$xf(x) = \frac{\cos(x^2)}{2} - \frac{\cos((x+1)^2)}{2} + s(x)$$

where  $|s(x)| \leq \frac{1}{x}$ .  $\lim_{x \rightarrow \infty} s(x) = 0$ , so the upper and lower limits of  $s(x)$  are also 0.

Note that by the periodicity of cosine, for  $n \in \mathbb{N}$ ,  $\cos(\sqrt{2\pi n}) = 1$ . We now show that there exist infinite  $n \in \mathbb{N}$  such that  $\cos((\sqrt{2\pi n}+1)^2) = \cos(2\pi n + 2\sqrt{2\pi n}+1) = -1$ , thus implying that  $\limsup_{n \rightarrow \infty} xf(x) = 1$ .

**Theorem 1.** *Let  $\delta > 0$ . Then there exist infinite natural numbers  $a$  such that  $|2\sqrt{2a\pi} + 1 - \pi| < \delta \pmod{2\pi}$ . In other words,  $2\sqrt{2a\pi} + 1$  becomes arbitrarily close to a number of the form  $b\pi$ , where  $b$  is an odd number.*

To prove this, we will need to analyze the behavior of  $g(x) = 2\sqrt{2\pi}\sqrt{x} + 1$ , then evaluate it at specifically chosen  $a$ 's. We first start by analyzing the Taylor series of  $\sqrt{x}$ .

**Lemma 2.** *The Taylor series of  $\sqrt{x}$  about  $x_0 > 0$  is*

$$\sqrt{x_0} + \frac{1}{2(1!)} x_0^{-\frac{1}{2}} (x - x_0) + \sum_{i=2}^{\infty} \frac{(-1)^{i-1}}{2^i} \frac{1(3)(5)\dots(2i-3)}{i!} x_0^{-i+\frac{1}{2}} (x - x_0)^i$$

with radius of convergence  $R = x_0$ .

*Proof.* The first few terms of the Taylor series are

$$T(x) = \sqrt{x_0} + \frac{1}{2(1!)}x_0^{-\frac{1}{2}}(x - x_0) - \frac{1}{2^2(2!)}x_0^{-\frac{3}{2}}(x - x_0)^2 + \frac{3}{2^3(3!)}x_0^{-\frac{5}{2}}(x - x_0)^3 \dots$$

To find the radius of convergence, note that

$$\frac{1(3)(5)\dots(2i-3)}{2^i} < \frac{1}{2} \left(\frac{1}{2}\right) \left(\frac{3}{2}\right) \left(\frac{5}{2}\right) \dots \frac{2i-3}{2} < \frac{1}{2} 1(2)(3) \dots i = \frac{1}{2} i!$$

implying that when  $x > x_0$ ,

$$\frac{1(3)(5)\dots(2i-3)}{2^i(i!)} x_0^{-i+\frac{1}{2}} (x - x_0)^i < \frac{\sqrt{x_0}}{2} \left(\frac{x}{x_0} - 1\right)^i$$

Since  $T(x)$  is alternating, it converges when  $\frac{x}{x_0} - 1 < 1 \rightarrow x \in [x_0, 2x_0)$ .

When  $x < x_0$ ,

$$T(x) = \sqrt{x_0} + \frac{1}{2(1!)}x_0^{-\frac{1}{2}}(x - x_0) - \sqrt{x_0} \sum_{i=2}^{\infty} \frac{1(3)(5)\dots(2i-3)}{2^i(i!)} \left(1 - \frac{x}{x_0}\right)^i$$

Since

$$\frac{1(3)(5)\dots(2i-3)}{2^i(i!)} \left(1 - \frac{x}{x_0}\right)^i < \frac{1}{2} \left(1 - \frac{x}{x_0}\right)^i$$

and the right series is a convergent geometric series under the assumption that  $x \in (0, x_0]$ ,  $T(x)$  converges. □

We next show that viewed as a sequence over the natural numbers,  $(g_n)$ 's differences between terms become arbitrarily small.

**Lemma 3.** *Let  $(g_n) = 2\sqrt{2\pi}\sqrt{n} + 1$  for  $n \in \mathbb{N}$ , and  $\Delta g_n = g_{n+1} - g_n = 2\sqrt{2}(\sqrt{n+1} - \sqrt{n})$ . Then  $\lim_{n \rightarrow \infty} \Delta g_n = 0$ .*

*Proof.* It suffices to show that  $\lim_{n \rightarrow \infty} \sqrt{n+1} - \sqrt{n} = 0$ . From the results in Theorem 2, the Taylor series of  $\sqrt{x}$  at  $n$  is

$$\sqrt{n+x} = \sqrt{n} + \frac{1}{2(1!)}n^{-\frac{1}{2}}(x) + \sum_{i=2}^{\infty} \frac{(-1)^{i-1}}{2^i} \frac{1(3)(5)\dots(2i-3)}{i!} n^{-i+\frac{1}{2}}(x)^i$$

Since the series is alternating and convergent, its partial sums that have a positive term as their highest power are larger than the series. Thus

$$\sqrt{n+1} < \sqrt{n} + \frac{1}{2\sqrt{n}}$$

implying

$$0 \leq \sqrt{n+1} - \sqrt{n} < \frac{1}{2\sqrt{n}}$$

implying that  $\lim_{n \rightarrow \infty} \sqrt{n+1} - \sqrt{n} = 0$ . □

We are now in the position to prove Theorem 1.

*Proof.* Let  $(g_n)$  be the sequence defined by  $g_n = 2\sqrt{2\pi}\sqrt{n} + 1 - \pi$ . Since  $\lim_{n \rightarrow \infty} \Delta g_n = 0$ , there exists  $n_0 \in \mathbb{N}$  such that  $n \geq n_0$  implies that  $\Delta g_n < \delta$ . This, combined with  $(g_n)$  being strictly increasing and  $\lim_{n \rightarrow \infty} g_n = \infty$ , imply that  $(g_n)$  passes through all numbers greater than  $g_{n_0}$  while increasing to infinity, while taking step sizes less than  $\delta$ .

Specifically, for any natural number  $b$  such that  $2\pi b > g_{n_0}$ , there exists an  $a \geq n_0$  in the natural numbers such that

$$g_a \leq 2\pi b < g_{a+1}$$

$\Delta g_a < \delta$  implies

$$|\Delta g_a - 2\pi b| < \delta$$

which is equivalent to

$$|2\sqrt{2\pi}\sqrt{a} + 1 - (2b + 1)\pi| < \delta$$

The theorem follows because for distinct  $b$ ,  $(g_n)$  being strictly increasing implies that the  $a$ 's are distinct.  $\square$

The results of the main problem now follow.

**Lemma 4.**  $\limsup_{n \rightarrow \infty} x f(x) = \frac{\cos(x^2)}{2} - \frac{\cos((x+1)^2)}{2} + s(x) = 1$

*Proof.* From Theorem 1 and the continuity of  $\cos(x)$ , for all  $\epsilon > 0$ , there exist infinite  $n \in \mathbb{N}$  such that  $\cos(x^2) = \cos(\sqrt{2\pi n}) = 1$  and  $|\cos((x+1)^2) + 1| = |\cos((\sqrt{2\pi n} + 1)^2) + 1| < \epsilon$ . As previously established,  $\limsup_{x \rightarrow \infty} s(x) = 0$ .  $\square$

**Corollary 5.**  $\liminf_{n \rightarrow \infty} x f(x) = \frac{\cos(x^2)}{2} - \frac{\cos((x+1)^2)}{2} + s(x) = -1$

*Proof.* Add  $\pi$  to the  $a$ 's in Lemma 4.  $\square$

d. Does  $\int_0^\infty \sin(t^2) dt$  converge?

**Theorem 6.**  $\int_0^\infty \sin(t^2) dt$  converges.

*Proof.* For finite  $n$ ,

$$\begin{aligned} \int_0^n \sin(t^2) dt &= \int_0^1 \sin(t^2) dt + f(1) + f(2) \cdots + f(n-1) \\ &= \int_0^1 \sin(t^2) dt + \sum_{i=1}^{n-1} f(i) \\ &= \int_0^1 \sin(t^2) dt + \frac{1}{2} \sum_{i=1}^{n-1} \left[ \frac{\cos(i^2)}{i} - \frac{\cos((i+1)^2)}{i} \right] + \frac{1}{2} \sum_{i=1}^{n-1} \frac{r(i)}{i} \end{aligned}$$

Taking limits as  $n$  approaches infinity,  $\sum_{i=1}^\infty \frac{r(i)}{i}$  converges due to a comparison with  $\sum_{i=1}^\infty \frac{1}{i^2}$ . To show convergence, we have to show that  $\sum_{i=1}^\infty \frac{\cos(i^2)}{i} - \frac{\cos((i+1)^2)}{i}$

converges. Writing out the first few terms of the partial sum,

$$\begin{aligned} \sum_{i=1}^{n-1} \frac{\cos(i^2)}{i} - \frac{\cos((i+1)^2)}{i} &= \frac{\cos(1)}{1} - \frac{\cos(4)}{1} + \frac{\cos(4)}{2} - \frac{\cos(9)}{2} + \frac{\cos(9)}{3} - \frac{\cos(16)}{3} \dots \\ &= \frac{\cos(1)}{1} - \frac{\cos(4)}{1*2} - \frac{\cos(9)}{2*3} \dots - \frac{\cos((n-1)^2)}{(n-2)(n-1)} - \frac{\cos(n^2)}{n-1} \\ &= \frac{\cos(1)}{1} - \frac{\cos(n^2)}{n-1} - \sum_{i=1}^{n-2} \frac{\cos((i+1)^2)}{i(i+1)} \end{aligned}$$

Taking the limits as  $n$  goes to infinity,  $\frac{\cos(n^2)}{n-1}$  goes to zero. The sum on the right is absolutely convergent by comparing it with  $\sum_{i=1}^{\infty} \frac{1}{i^2}$ , so the sum on the right is convergent. Thus  $\sum_{i=1}^{\infty} \frac{\cos(i^2)}{i} - \frac{\cos((i+1)^2)}{i}$ , and by extension  $\int_0^{\infty} \sin(t^2)dt$ , converge.  $\square$

**Problem 6.R.14.** let  $f(x) = \int_x^{x+1} \sin(e^t)dt$ .

a. Show that  $e^x |f(x)| < 2$ .

*Proof.* Making the substitution  $u = e^t$ ,

$$f(x) = \int_{e^x}^{e^{x+1}} \frac{\sin(u)}{u} du$$

Integrating by parts with  $a = u^{-1}$  and  $db = \sin(u)$ ,

$$\begin{aligned} f(x) &= -u^{-1} \cos(u) \Big|_{e^x}^{e^{x+1}} - \int_{e^x}^{e^{x+1}} u^{-2} \cos(u) du \\ &= \frac{\cos(e^x)}{e^x} - \frac{\cos(e^{x+1})}{e^{x+1}} - \int_{e^x}^{e^{x+1}} u^{-2} \cos(u) du \end{aligned}$$

implies

$$\begin{aligned} xf(x) &= \cos(e^x) - e^{-1} \cos(e^{x+1}) - e^x \int_{e^x}^{e^{x+1}} u^{-2} \cos(u) du \\ &\leq 1 + \frac{1}{e} + e^x \int_{e^x}^{e^{x+1}} u^{-2} du \\ &= 1 + \frac{1}{e} - e^x \left[ \frac{1}{u} \right]_{e^x}^{e^{x+1}} \\ &= 1 + \frac{1}{e} + 1 - \frac{1}{e} = 2 \end{aligned}$$

Similarly,

$$xf(x) \geq -1 - \frac{1}{e} - e^x \int_{e^x}^{e^{x+1}} u^{-2} du = -2$$

$\square$

b. Show that  $e^x f(x) = \cos(e^x) - e^{-1} \cos(e^{x+1}) + r(x)$ , where  $|r(x)| < Ce^{-x}$ , for some constant  $C$ .

*Proof.* From the form above, it's clear that  $r(x) = -e^x \int_{e^x}^{e^{x+1}} u^{-2} \cos(u) du$ . □

**Problem 6.6:1.** On  $[a, b]$ , let  $\alpha$  be a strictly increasing function and  $f$  a continuous function, and for  $x \in [a, b]$  define  $F(x) = \int_a^x f(t) d\alpha(t)$ . Show that for all  $x \in [a, b]$ ,  $\frac{dF(x)}{d\alpha(x)} = f(x)$ , where the left-hand side is defined as  $\lim_{t \rightarrow x} \frac{F(x) - F(t)}{\alpha(x) - \alpha(t)}$ , and the equality includes the assertion that the limit exists.

*Proof.* First, we note that because  $f \in \mathfrak{R}(\alpha)$  on  $[a, b]$

$$F(x) - F(t) = \int_a^x f(s) d\alpha(s) - \int_a^t f(s) d\alpha(s) = \int_t^x f(s) d\alpha(s)$$

For all partitions  $P$ ,

$$\begin{aligned} \int_t^x f(s) d\alpha(s) &\leq \sum_{x_i \in P} M_i \Delta\alpha_i \\ &\leq \left( \sup_{s \in [x, t]} f(s) \right) \sum_{x_i \in P} \Delta\alpha_i \\ &= \left( \sup_{s \in [x, t]} f(s) \right) (\alpha(x) - \alpha(t)) \end{aligned}$$

Since  $\alpha(x)$  is strictly increasing,  $\alpha(x) - \alpha(t) > 0$  when  $x \neq t$ , so

$$\frac{F(x) - F(t)}{\alpha(x) - \alpha(t)} \leq \sup_{s \in [x, t]} f(s)$$

Taking the limit as  $t$  approaches  $x$  on both sides gives

$$\lim_{t \rightarrow x} \frac{F(x) - F(t)}{\alpha(x) - \alpha(t)} \leq \lim_{t \rightarrow x} \sup_{s \in [x, t]} f(s)$$

**Lemma 7.**

$$\lim_{t \rightarrow x} \sup_{s \in [x, t]} f(s) = f(x)$$

*Proof.* Let  $x_n$  be an arbitrary sequence such that  $\forall n \in \mathbb{N}, x_n > x$  and  $\lim_{n \rightarrow \infty} x_n = x$ . Since  $[x, x_n]$  is a closed, bounded interval on  $\mathbb{R}$  and  $f$  is continuous, there exists a sequence of points  $p_n \in [x, x_n]$  such that  $f(p_n) = \sup_{s \in [x, x_n]} f(s)$ .  $x_n \rightarrow x$  implies  $p_n \rightarrow x$  by the Squeeze Theorem, and the continuity of  $f$  implies that  $f(p_n) \rightarrow f(x)$ . □

Thus

$$\lim_{t \rightarrow x} \frac{F(x) - F(t)}{\alpha(x) - \alpha(t)} \leq f(x)$$

The analogous result for the lower integral and the Squeeze Theorem complete the proof. □

**Problem 6.6:2.**

(a). Show that if  $f$  is continuous, then

$$\int_{t=a}^b \left( \int_{s=a}^t f(s) ds \right) dt = \int_{t=a}^b (b-t)f(t) dt$$

*Proof.* Let  $x \in [a, b]$ . Define  $P(x) = \int_{t=a}^x \left( \int_{s=a}^t f(s) ds \right) dt$  and  $Q(x) = \int_{t=a}^x (x-t)f(t) dt$ .

$f(t)$  being continuous on  $[a, b]$  implies that it is Riemann-integrable. This implies that  $f^*(t) = \int_{s=a}^t f(s) ds$  is continuous, and that  $P(x) = \int_a^x f^*(t) dt$  is continuous and differentiable. Similarly,  $(b-t)f(t)$  is continuous on  $[a, b]$ , so  $Q(x)$  is continuous and differentiable.

By the Fundamental Theorem of Calculus,

$$P'(x) = \int_{s=a}^x f(s) ds$$

For  $Q(x)$ , since  $t$  and  $tf(t)$  are Riemann-integrable,

$$Q(x) = x \int_{t=a}^x f(t) dt - \int_{t=a}^x tf(t) dt$$

$x$  is trivially differentiable. Since  $t$  and  $tf(t)$  are continuous,

$$Q'(x) = \int_{t=a}^x f(t) dt + xf(x) - xf(x) = \int_{t=a}^x f(t) dt$$

Thus,  $P'(x) = Q'(x)$ . Integrating both sides from  $a$  to  $c$ , then setting  $c = b$ , produces the desired result.  $\square$

(c). Show that the result of Part (a) continues to hold if  $f$  is merely assumed Riemann-integrable, but not necessarily continuous.

*Proof.*  $P(x)$  has the same derivative as in Part (a), as the derivation only assumed that  $P(x)$  is Riemann-integrable. Similarly, for  $x_0 \in [a, b]$  where  $f(x_0)$  is continuous, the above derivations hold for  $Q(x)$ .

Let  $x_0$  be a point where  $f(x_0)$  is discontinuous. First, we will prove two lemmas.

**Lemma 8.** *If  $f(x)$  is bounded, then  $(x - x_0)f(x)$  is continuous at  $x_0$ .*

*Proof.* Let  $M = \sup |f(x)|$ . Then  $(x - x_0)f(x) \leq |(x - x_0)f(x)| \leq |x - x_0|M$ , which can be made arbitrarily small.  $\square$

**Lemma 9.** *If  $f(x)$  is continuous, then  $(x - x_0)f(x)$  is differentiable at  $x_0$  with derivative  $f(x_0)$ .*

*Proof.* By the definition of differentiability,

$$\lim_{x \rightarrow x_0} \frac{(x - x_0)f(x) - (x_0 - x_0)f(x_0)}{x - x_0} = \lim_{x \rightarrow x_0} f(x) = f(x_0)$$

by continuity.  $\square$

We can rewrite  $Q(x)$  as

$$Q(x) = \int_{t=a}^x ((x - x_0) + (x_0 - t))f(t)dt = (x - x_0) \int_{t=a}^x f(t)dt + \int_{t=a}^x (x_0 - t)f(t)dt$$

because the sub-functions are trivially Riemann-integrable.  $\int_{t=a}^x f(t)dt$  is a continuous function, so by Lemma 9  $(x - x_0) \int_{t=a}^x f(t)dt$  is differentiable at  $x = x_0$  with derivative  $\int_{t=a}^{x_0} f(t)dt$ . Similarly,  $f(t)$  is bounded because it is Riemann-integrable, so by Lemma 8  $(x_0 - t)f(t)$  is continuous at  $t = x_0$ . Therefore  $\int_{t=a}^x (x_0 - t)f(t)dt$  is differentiable at  $x = x_0$ , with derivative 0.

Therefore,  $Q(x)$  is differentiable at  $x = x_0$ , and  $Q'(x_0) = \int_{t=a}^{x_0} f(t)dt$ . The proof then follows using the same logic as in Part (a).  $\square$

**Problem 6.6:4.** Let  $f$  be a function on  $[a, b]$ , and  $\alpha, \beta$  monotonically increasing nonnegative functions on  $[a, b]$  such that  $f \in \mathfrak{R}(\alpha) \cap \mathfrak{R}(\beta)$ ,  $\alpha \in \mathfrak{R}(\beta)$ , and  $\beta \in \mathfrak{R}(\alpha)$ . Prove that

$$\int f d(\alpha\beta) = \int f \alpha d(\beta) + \int f \beta d(\alpha)$$

*Proof.* First note that  $f\alpha \in \mathfrak{R}(\beta)$  and  $f\beta \in \mathfrak{R}(\alpha)$ . Also note that  $\alpha\beta$  is monotonically increasing, so  $\alpha\beta$  is a valid integrator.

**Theorem 10.** Under the assumptions of Problem 6.6.4,  $f \in \mathfrak{R}(\alpha\beta)$ .

*Proof.* First note that by expanding the terms,

$$\alpha_i \beta_i - \alpha_{i-1} \beta_{i-1} = (\alpha_i - \alpha_{i-1})(\beta_i - \beta_{i-1}) + \alpha_{i-1}(\beta_i - \beta_{i-1}) + \beta_{i-1}(\alpha_i - \alpha_{i-1})$$

Writing out the difference between the upper and lower Riemann sums,

$$\begin{aligned} & \sum_{i=1}^n (M_i - m_i)(\alpha_i \beta_i - \alpha_{i-1} \beta_{i-1}) \\ &= \sum_{i=1}^n (M_i - m_i) \Delta \alpha_i \Delta \beta_i + \sum_{i=1}^n (M_i - m_i) \alpha_{i-1} \Delta \beta_i + \sum_{i=1}^n (M_i - m_i) \beta_{i-1} \Delta \alpha_i \\ &\leq \alpha(b) \sum_{i=1}^n (M_i - m_i) \Delta \beta_i + \alpha(b) \sum_{i=1}^n (M_i - m_i) \Delta \beta_i + \beta(b) \sum_{i=1}^n (M_i - m_i) \Delta \alpha_i \end{aligned}$$

The third line follows because all terms are positive, and  $\alpha$  and  $\beta$  are monotonically increasing. Because  $f \in \mathfrak{R}(\alpha) \cap \mathfrak{R}(\beta)$ , there exist partitions that make the third line arbitrarily small.  $\square$

Now we prove the main result. For arbitrary  $\epsilon > 0$ , let  $P_1$ ,  $P_2$ , and  $P_3$  be partitions of  $[a, b]$  such that

- (1)  $U(P_1, f, \alpha\beta) - L(P_1, f, \alpha\beta) < \epsilon$
- (2)  $U(P_2, f\alpha, \beta) - L(P_2, f\alpha, \beta) < \epsilon$
- (3)  $U(P_3, f\beta, \alpha) - L(P_3, f\beta, \alpha) < \epsilon$



Let  $P$  be their common partition. Let  $x_0 < x_1 \dots < x_n$  denote the points of  $P$ . For all  $i$  in  $(1, 2 \dots n)$ , let  $t_i \in (x_{i-1}, x_i)$  be arbitrary, fixed points, and let  $P^* = (x_0, t_1, x_1 \dots x_{i-1}, t_i, x_i \dots t_n, x_n)$ . Trivially,  $P^*$  partitions  $[a, b]$  and is a refinement of  $P$ .

Consider  $\int f d(\alpha\beta)$  and its associated Riemann-Stieltjes sum over  $P^*$ . Since  $P^*$  is a refinement of  $P_1$ , for arbitrary points  $u_i \in [x_{i-1}, t_i]$  and  $v_i \in [t_i, x_i]$ ,

$$\left| \sum_{i=1}^n [f(u_i)(\alpha_{t_i}\beta_{t_i} - \alpha_{x_{i-1}}\beta_{x_{i-1}}) + f(v_i)(\alpha_{x_i}\beta_{x_i} - \alpha_{t_i}\beta_{t_i})] - \int f d(\alpha\beta) \right| < \epsilon$$

Letting  $u_i = x_{i-1}$  and  $v_i = x_i$  for all  $i$ ,

$$\left| \sum_{i=1}^n [f_{x_{i-1}}(\alpha_{t_i}\beta_{t_i} - \alpha_{x_{i-1}}\beta_{x_{i-1}}) + f_{x_i}(\alpha_{x_i}\beta_{x_i} - \alpha_{t_i}\beta_{t_i})] - \int f d(\alpha\beta) \right| < \epsilon$$

Now consider  $\int f \alpha d(\beta)$  and its associated Riemann-Stieltjes sum over  $P$ . Letting  $u_i = x_i$  for all  $i$ ,

$$\left| \sum_{i=1}^n [f_{x_i}\alpha_{x_i}(\beta_{x_i} - \beta_{x_{i-1}})] - \int f \alpha d(\beta) \right| < \epsilon$$

Similarly, considering  $\int f \beta d(\alpha)$  over  $P$  and letting  $u_i = x_{i-1}$  for all  $i$ ,

$$\left| \sum_{i=1}^n [f_{x_{i-1}}\beta_{x_{i-1}}(\alpha_{x_i} - \alpha_{x_{i-1}})] - \int f \beta d(\alpha) \right| < \epsilon$$

Adding the inequalities and using the Triangle Inequality gives

$$\begin{aligned} & \left| \int f d(\alpha\beta) - \int f \alpha d(\beta) - \int f \beta d(\alpha) \right. \\ & + \sum_{i=1}^n \left[ - [f_{x_{i-1}}(\alpha_{t_i}\beta_{t_i} - \alpha_{x_{i-1}}\beta_{x_{i-1}}) + f_{x_i}(\alpha_{x_i}\beta_{x_i} - \alpha_{t_i}\beta_{t_i})] \right. \\ & \left. \left. + f_{x_i}\alpha_{x_i}(\beta_{x_i} - \beta_{x_{i-1}}) + f_{x_{i-1}}\beta_{x_{i-1}}(\alpha_{x_i} - \alpha_{x_{i-1}}) \right] \right| < 3\epsilon \end{aligned}$$

Simplifying,

$$\left| \int f d(\alpha\beta) - \int f \alpha d(\beta) - \int f \beta d(\alpha) + \sum_{i=1}^n [(f_{x_i} - f_{x_{i-1}})(\alpha_{t_i}\beta_{t_i} - \alpha_{x_i}\beta_{x_{i-1}})] \right| < 3\epsilon$$

To analyze the sum on the right, note that

$$\begin{aligned}
& \left| \sum_{i=1}^n \left[ (f_{x_i} - f_{x_{i-1}})(\alpha_{t_i}\beta_{t_i} - \alpha_{x_i}\beta_{x_{i-1}}) \right] \right| \\
& \leq \sum_{i=1}^n |f_{x_i} - f_{x_{i-1}}| |\alpha_{t_i}\beta_{t_i} - \alpha_{x_i}\beta_{x_{i-1}}| \\
& \leq \sum_{i=1}^n \left( \sup_{x \in [x_{i-1}, x_i]} f(x) - \inf_{x \in [x_{i-1}, x_i]} f(x) \right) (\alpha_{x_i}\beta_{x_i} - \alpha_{x_{i-1}}\beta_{x_{i-1}}) \\
& < \epsilon
\end{aligned}$$

The second inequality comes from noticing that  $t_i$  is in  $(x_{i-1}, x_i)$ , and that  $\alpha$  and  $\beta$  are weakly increasing. The third inequality comes from  $P$  being a refinement of  $P_1$ . Thus

$$\left| \int f d(\alpha\beta) - \int f \alpha d(\beta) - \int f \beta d(\alpha) \right| < 4\epsilon$$

Which can be made arbitrarily close to 0.

□