Problem 6.R:13. Let $f(x) = \int_{t=x}^{t=x+1} \sin(t^2)dt$.

a. Show that when $x > 0$, $|f(x)| < \frac{1}{x}$.

Proof. Note that $x > 0$ implies that the limits of integration are correct. Make the substitution $t^2 = u$ to get

$$f(x) = \frac{1}{2}\int_{u=x^2}^{u=(x+1)^2} u^{-\frac{1}{2}} \sin(u)du$$

Integrate by parts with $a = u^{\frac{1}{2}}$ and $db = \sin(u)$ to get

$$\begin{aligned} f(x) &= \frac{1}{2}\left[-u^{-\frac{1}{2}}\cos(u)]_{x^2}^{(x+1)^2} - \frac{1}{2}\int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}}\cos(u)du\right] \\ &= \frac{\cos(x^2)}{2x} - \frac{\cos((x+1)^2)}{2(x+1)} - \frac{1}{4}\int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}}\cos(u)du \end{aligned}$$

Evaluating the integral on the right, $\cos(x) \geq -1$, so

$$\int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}}\cos(u)du \geq -\int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}}du = 2\left(\frac{1}{x+1} - \frac{1}{x}\right)$$

Substituting,

$$\begin{aligned} f(x) &\leq \frac{\cos(x^2)}{2x} - \frac{\cos((x+1)^2)}{2(x+1)} - \frac{1}{2(x+1)} + \frac{1}{2x} \\ &= \frac{\cos(x^2)+1}{2x} - \frac{\cos((x+1)^2)+1}{2(x+1)} \\ &\leq \frac{2}{2x} = \frac{1}{x} \end{aligned}$$

Since $\cos(t) \leq 1$. To show that $f(x) \geq -\frac{1}{x}$, it suffices to show that $f(x) \leq \frac{1}{x}$.

$$-f(x) = \frac{\cos((x+1)^2)}{2(x+1} - \frac{\cos(x^2)}{2x} + \frac{1}{4}\int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}}\cos(u)du$$

By a similar argument as before, $\cos(x) \leq 1$, so

$$\int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}}\cos(u)du \leq \int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}}du = 2\left(\frac{1}{x} - \frac{1}{x+1}\right)$$

Substituting,

$$\begin{aligned} -f(x) &\leq \frac{\cos((x+1)^2)}{2(x+1} - \frac{\cos(x^2)}{2x} + \frac{1}{2x} - \frac{1}{2(x+1)} \\ &\leq \frac{1}{2(x+1)} + \frac{1}{2x} + \frac{1}{2x} - \frac{1}{2(x+1)} \\ &= \frac{1}{x} \end{aligned}$$

□

b. Prove that there exists constant c and function $r(x)$ with $|r(x)| < \frac{c}{x}$ such that

$$2xf(x) = \cos(x^2) - \cos((x+1)^2) + r(x)$$

Proof. I will assume that $x > 0$, as the expression $\frac{c}{x}$ doesn't make much sense for $x = 0$, and it's impossible for $|r(x)| < \frac{c}{x}$ to be true for both positive and negative x while c is constant.

From results in Part a,

$$\begin{aligned} 2xf(x) &= \cos(x^2) - \frac{x}{x+1}\cos((x+1)^2) - \frac{x}{2}\int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}}\cos(u)du \\ &= \cos(x^2) - \cos((x+1)^2) + \frac{1}{x+1}\cos((x+1)^2) - \frac{x}{2}\int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}}\cos(u)du \end{aligned}$$

From the above, it's clear that $r(x) = \frac{1}{x+1}\cos((x+1)^2) - \frac{x}{2}\int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}}\cos(u)du$. Now it remains to show that $|r(x)| < \frac{c}{x}$. Using that $\cos(t)$ and $-\cos(t)$ is are bounded above by 1,

$$r(x) \leq \frac{1}{x+1} + \frac{x}{2}\int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}}du = \frac{1}{x+1} - x\left[\frac{1}{x+1} - \frac{1}{x}\right] = \frac{2}{x+1} < \frac{2}{x}$$

Similarly,

$$\begin{aligned} -r(x) &= -\frac{1}{x+1}\cos((x+1)^2) + \frac{x}{2}\int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}}\cos(u)du \\ &\geq -\frac{1}{x+1} - \frac{x}{2}\int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}}du = -\frac{2}{x+1} \geq -\frac{2}{x} \end{aligned}$$

Thus $|r(x)| < \frac{3}{x}$.

□

c. Find the upper and lower limits of $xf(x)$ as x approaches infinity.

We know from previous results that

$$xf(x) = \frac{\cos(x^2))}{2} - \frac{\cos((x+1)^2)}{2} + s(x)$$

where $|s(x)| \leq \frac{1}{x}$. $\lim_{x\to\infty} s(x) = 0$, so the upper and lower limits of $s(x)$ are also 0.

Note that by the periodicity of cosine, for $n \in N$, $cos(\sqrt{2\pi n}^2) = 1$. We now show that there exist infinite $n \in N$ such that $\cos((\sqrt{2\pi n}+1)^2) = \cos(2\pi n + 2\sqrt{2\pi n}+1) = -1$, thus implying that $\limsup_{n\to\infty} xf(x) = 1$.

Lemma 0.0.0.1. *The Taylor series of $\sqrt{x}$ about $x_0 > 0$ is*

$$\sqrt{x_0} + \frac{1}{2(1!)}x_0^{-\frac{1}{2}}(x-n) + \sum_{i=2}^{\infty}\frac{(-1)^{i-1}}{2^i}\frac{1(3)(5)...(2i-3)}{i!}x_0^{-i+\frac{1}{2}}(x-x_0)^i$$

with radius of convergence $R = x_0$.

Proof. The first few terms of the Taylor series are

$$T(x) = \sqrt{x_0} + \frac{1}{2(1!)}x_0^{-\frac{1}{2}}(x-x_0) - \frac{1}{2^2(2!)}x_0^{-\frac{3}{2}}(x-x_0)^2 + \frac{3}{2^3(3!)}x_0^{-\frac{5}{2}}(x-x_0)^3 \ldots$$

To find the radius of convergence, note that

$$\frac{1(3)(5)...(2i-3)}{2^i} < \frac{1}{4}\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\dots\frac{2i-3}{2} < \frac{1}{4}1(2)(3)\dots i = \frac{1}{4}i!$$

implying

$$|T(x)| < \sqrt{x_0} + \frac{1}{2(1!)}x_0^{-\frac{1}{2}}(x-n) + \frac{1}{4\sqrt{x_0}}\sum_{i=2}^{\infty}(-1)^{i-1}\left(\frac{x}{x_0}-1\right)^i$$

The last series is a geometric series, and converges when $|1-\frac{x}{x_0}| < 1$, implying that the radius of convergence is $R = x_0$. □

Theorem 0.0.1. *Let* $(g_n) = 2\sqrt{2\pi}\sqrt{n}+1$ *for* $n \in N$*, and* $\Delta g_n = g_{n+1} - g_n = 2\sqrt{2}(\sqrt{n+1}-\sqrt{n})$*. Then* $\lim_{n\to\infty}\Delta g_n = 0$*.*

Proof. It suffices to show that $\lim_{n\to\infty}\sqrt{n+1}-\sqrt{n} = 0$. The Taylor series of $\sqrt{x}$ around n is

$$\begin{aligned} T(x) =& \sqrt{n} + \frac{1}{2(1!)}n^{-\frac{1}{2}}(x-n) - \frac{1}{2^2(2!)}n^{-\frac{3}{2}}(x-n)^2 + \frac{3}{2^3(3!)}n^{-\frac{5}{2}}(x-n)^3\dots \\ =& \sqrt{n} + \frac{1}{2(1!)}n^{-\frac{1}{2}}(x-n) + \sum_{i=2}^{\infty}\frac{(-1)^{i-1}}{2^i}\frac{1(3)(5)...(2i-3)}{i!}n^{-i+\frac{1}{2}}(x-n)^i \end{aligned}$$

Noting that

$$\frac{1(3)(5)...(2i-3)}{2^i} < \frac{1}{4}\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\dots\frac{2i-3}{2} < \frac{1}{4}1(2)(3)\dots i = \frac{1}{4}i!$$

and using the Alternating Series Test, whenever $|\frac{x}{n}-1| < 1$, the Taylor series converges. Thus the radius of convergence of $T(x)$ is n. When $x > n$ and $T(x)$ is convergent, $T(x)$ is alternating, implying that the partial Taylor polynomials truncated at a positive term are larger than the actual value of $\sqrt{x}$. Thus when $x > n$,

$$\sqrt{n+x} \le \sqrt{n} + \frac{1}{2(1!)}n^{-\frac{1}{2}}(x-n)$$

□

Theorem 0.0.2. *Let* $\delta > 0$*. Then there exist infinite natural numbers* a *such that* $2\sqrt{2a\pi}+1-\pi \mod 2\pi < \delta$*.*

Proof. Let (g_n) be the sequence defined by $g_i = 2\sqrt{2\pi}\sqrt{i}+1-\pi$. □

Problem 6.6:1. On $[a,b]$, let α be a strictly increasing function and f a continuous function, and for $x \in [a,b]$ define $F(x) = \int_a^x f(t)d\alpha(t)$. Show that for all $x \in [a,b]$, $\frac{dF(x)}{d\alpha(x)} = f(x)$, where the left-hand side is defined as $\lim_{t\to x}\frac{F(x)-F(t)}{\alpha(x)-\alpha(t)}$, and the equality includes the assertion that the limit exists.

Proof. First, we note that because $f \in \mathfrak{R}(\alpha)$ on $[a,b]$

$$F(x) - F(t) = \int_a^x f(s)d\alpha(s) - \int_a^t f(s)d\alpha(s) = \int_t^x f(x)d\alpha(s)$$

For all partitions P,

$$\int_t^x f(s)d\alpha(s) \le \sum_{x_i \in P} M_i \Delta\alpha_i \le \left(\sup_{s\in[x,t]} f(s)\right) \sum_{x_i\in P} \Delta\alpha_i = \left(\sup_{s\in[x,t]} f(s)\right)(\alpha(x)-\alpha(t))$$

Since $\alpha(x)$ is strictly increasing, $\alpha(x) - \alpha(t) > 0$ when $x \neq t$, so

$$\frac{F(x) - F(t)}{\alpha(x) - \alpha(t)} \le \sup_{s\in[x,t]} f(s)$$

Taking the limit as t approaches x on both sides gives

$$\lim_{t\to x} \frac{F(x) - F(t)}{\alpha(x) - \alpha(t)} \le \lim_{t\to x} \sup_{s\in[x,t]} f(s)$$

Lemma 0.0.0.2.

$$\lim_{t\to x} \sup_{s\in[x,t]} f(s) = f(x)$$

Proof. Let x_n be an arbitrary sequence such that $\forall n \in \mathbb{N}, x_n > x$ and $\lim_{n\to\infty} x_n = x$. Since $[x, x_n]$ is a closed, bounded interval on $\mathbb{R}$ and f is continuous, there exists a sequence of points $p_n \in [x, x_n]$ such that $f(p_n) = \sup_{s\in[x,x_n]} f(s)$. $x_n \to x$ implies $p_n \to x$ by the Squeeze Theorem, and the continuity of f implies that $f(p_n) \to f(x)$. □

Thus

$$\lim_{t\to x} \frac{F(x) - F(t)}{\alpha(x) - \alpha(t)} \le f(x)$$

The analogous result for the lower integral and the Squeeze Theorem complete the proof.

□

Problem 6.6:2.

(a). Show that if f is continuous, then

$$\int_{t=a}^b \left(\int_{s=a}^t f(s)ds\right) dt = \int_{t=a}^b (b-t)f(t)dt$$

Proof. Let $x \in [a, b]$. Define $P(x) = \int_{t=a}^x \left(\int_{s=a}^t f(s)ds\right) dt$ and $Q(x) = \int_{t=a}^x (x - t)f(t)dt$.

$f(t)$ being continuous on $[a, b]$ implies that it is Riemann-integrable. This implies that $f^*(t) = \int_{s=a}^t f(s)ds$ is continuous, and that $P(x) = \int_a^x f^*(t)dt$ is continuous and differentiable. Similarly, $(b-t)f(t)$ is continuous on $[a, b]$, so $Q(x)$ is continuous and differentiable.

By the Fundamental Theorem of Calculus,

$$P'(x) = \int_{s=a}^x f(s)ds$$

For $Q(x)$, since t and $tf(t)$ are Riemann-integrable,

$$Q(x) = x\int_{t=a}^x f(t)dt - \int_{t=a}^x tf(t)dt$$

x is trivially differentiable. Since t and $tf(t)$ are continuous,

$$Q'(x) = \int_{t=a}^{x} f(t)dt + xf(x) - xf(x) = \int_{t=a}^{x} f(t)dt$$

Thus, $P'(x) = Q'(x)$. Integrating both sides from a to c, then setting $c = b$, produces the desired result.

□

(c). Show that the result of Part (a) continues to hold if f is merely assumed Riemann-integrable, but not necessarily continuous.

Proof. $P(x)$ has the same derivative as in Part (a), as the derivation only assumed that $P(x)$ is Riemann-integrable. Similarly, for $x_0 \in [a, b]$ where $f(x_0)$ is continuous, the above derivations hold for $Q(x)$.

Let x_0 be a point where $f(x_0)$ is discontinuous. First, we will prove two lemmas.

Lemma 0.0.0.3. *If $f(x)$ is bounded, then $(x - x_0)f(x)$ is continuous at x_0.*

Proof. Let $M = \sup|f(x)|$. Then $(x - x_0)f(x) \leq |(x - x_0)f(x)| \leq |(x - x_0)|M$, which can be made arbitrarily small.

□

Lemma 0.0.0.4. *If $f(x)$ is continuous, then $(x - x_0)f(x)$ is differentiable at x_0 with derivative $f(x_0)$.*

Proof. By the definition of differentiability,

$$\lim_{x \to x_0} \frac{(x - x_0)f(x) - (x_0 - x_0)f(x_0)}{x - x_0} = \lim_{x \to x_0} f(x) = f(x_0)$$

by continuity.

□

We can rewrite $Q(x)$ as

$$Q(x) = \int_{t=a}^{x} ((x - x_0) + (x_0 - t))f(t)dt = (x - x_0) \int_{t=a}^{x} f(t)dt + \int_{t=a}^{x} (x_0 - t)f(t)dt$$

because the sub-functions are trivially Riemann-integrable. $\int_{t=a}^{x} f(t)$ is a continuous function, so by Lemma 0.0.0.4 $(x - x_0) \int_{t=a}^{x} f(t)dt$ is differentiable at $x = x_0$ with derivative $\int_{t=a}^{x_0} f(t)dt$. Similarly, $f(t)$ is bounded because it is Riemann-integrable, so by Lemma 0.0.0.3 $(x_0 - t)f(t)$ is continuous at $t = x_0$. Therefore $\int_{t=a}^{x}(x_0 - t)f(t)dt$ is differentiable at $x = x_0$, with derivative 0.

Therefore, $Q(x)$ is differentiable at $x = x_0$, and $Q'(x_0) = \int_{t=a}^{x} f(t)dt$. The proof then follows using the same logic as in Part (a).

□

Problem 6.6:4. Let f be a function on $[a, b]$, and α, β monotonically increasing nonnegative functions on $[a, b]$ such that $f \in \mathfrak{R}(\alpha) \cap \mathfrak{R}(\beta)$, $\alpha \in \mathfrak{R}(\beta)$, and $\beta \in \mathfrak{R}(\alpha)$. Prove that

$$\int f d(\alpha\beta) = \int f \alpha d(\beta) + \int f \beta d(\alpha)$$

Proof. First note that $f\alpha \in \mathfrak{R}(\beta)$ and $f\beta \in \mathfrak{R}(\alpha)$. Also note that $\alpha\beta$ is monotonically increasing, so $\alpha\beta$ is a valid integrator.

Theorem 0.0.3. *Under the assumptions of Problem 6.6.4, $f \in \mathfrak{R}(\alpha\beta)$.*

Proof. First note that by expanding the terms,

$$\alpha_i\beta_i - \alpha_{i-1}\beta_{i-1} = (\alpha_i - \alpha_{i-1})(\beta_i - \beta_{i-1}) + \alpha_{i-1}(\beta_i - \beta_{i-1}) + \beta_{i-1}(\alpha_i - \alpha_{i-1})$$

Writing out the difference between the upper and lower Riemann sums,

$$\begin{aligned}
&\sum_{i=1}^{n}(M_i - m_i)(\alpha_i\beta_i - \alpha_{i-1}\beta_{i-1})\\
=&\sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i\Delta\beta_i + \sum_{i=1}^{n}(M_i - m_i)\alpha_{i-1}\Delta\beta_i + \sum_{i=1}^{n}(M_i - m_i)\beta_{i-1}\Delta\alpha_i\\
\leq&\alpha(b)\sum_{i=1}^{n}(M_i - m_i)\Delta\beta_i + \alpha(b)\sum_{i=1}^{n}(M_i - m_i)\Delta\beta_i + \beta(b)\sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i
\end{aligned}$$

The third line follows because all terms are positive, and α and β are monotonically increasing. Because $f \in \mathfrak{R}(\alpha) \cap \mathfrak{R}(\beta)$, there exist partitions that make the third line arbitrarily small.

□

Now we prove the main result. For arbitrary $\epsilon > 0$, let P_1, P_2, and P_3 be partitions of $[a, b]$ such that

(1) $U(P_1, f, \alpha\beta) - L(P_1, f, \alpha\beta) < \epsilon$
(2) $U(P_2, f\alpha, \beta) - L(P_2, f\alpha, \beta) < \epsilon$
(3) $U(P_3, f\beta, \alpha) - L(P_3, f\beta, \alpha) < \epsilon$

Let P be their common partition. Let $x_0 < x_1 ... < x_n$ denote the points of P. For all i in $(1, 2...n)$, let $t_i \in (x_{i-1}, x_i)$ be arbitrary, fixed points, and let $P^* = (x_0, t_1, x_1...x_{i-1}, t_i, x_i...t_n, x_n)$. Trivially, P^* partitions $[a, b]$ and is a refinement of P.

Consider $\int fd(\alpha\beta)$ and its associated Riemann-Stieltjes sum over P^*. Since P^* is a refinement of P_1, for arbitrary points $u_i \in [x_{i-1}, t_i]$ and $v_i \in [t_i, x_i]$,

$$\left|\sum_{i=1}^{n}\left[f(u_i)(\alpha_{t_i}\beta_{t_i} - \alpha_{x_{i-1}}\beta_{x_{i-1}}) + f(v_i)(\alpha_{x_i}\beta_{x_i} - \alpha_{t_i}\beta_{t_i})\right] - \int fd(\alpha\beta)\right| < \epsilon$$

Letting $u_i = x_{i-1}$ and $v_i = x_i$ for all i,

$$\left|\sum_{i=1}^{n}\left[f_{x_{i-1}}(\alpha_{t_i}\beta_{t_i} - \alpha_{x_{i-1}}\beta_{x_{i-1}}) + f_{x_i}(\alpha_{x_i}\beta_{x_i} - \alpha_{t_i}\beta_{t_i})\right] - \int fd(\alpha\beta)\right| < \epsilon$$

Now consider $\int f\alpha d(\beta)$ and its associated Riemann-Stieltjes sum over P. Letting $u_i = x_i$ for all i,

$$\left|\sum_{i=1}^{n}\left[f_{x_i}\alpha_{x_i}(\beta_{x_i} - \beta_{x_{i-1}})\right] - \int f\alpha d(\beta)\right| < \epsilon$$

Similarly, considering $\int f\beta d(\alpha)$ over P and letting $u_i = x_{i-1}$ for all i,

$$\left|\sum_{i=1}^{n}\left[f_{x_{i-1}}\beta_{x_{i-1}}(\alpha_{x_i}-\alpha_{x_{i-1}})\right]-\int f\beta d(\alpha)\right|<\epsilon$$

Adding the inequalities and using the Triangle Inequality gives

$$\begin{aligned}&\Bigg|\int fd(\alpha\beta)-\int f\alpha d(\beta)-\int f\beta d(\alpha)\\+\sum_{i=1}^{n}\Big[&-\left[f_{x_{i-1}}(\alpha_{t_i}\beta_{t_i}-\alpha_{x_{i-1}}\beta_{x_{i-1}})+f_{x_i}(\alpha_{x_i}\beta_{x_i}-\alpha_{t_i}\beta_{t_i})\right]\\&+f_{x_i}\alpha_{x_i}(\beta_{x_i}-\beta_{x_{i-1}})+f_{x_{i-1}}\beta_{x_{i-1}}(\alpha_{x_i}-\alpha_{x_{i-1}})\Big]\Bigg|<3\epsilon\end{aligned}$$

Simplifying,

$$\left|\int fd(\alpha\beta)-\int f\alpha d(\beta)-\int f\beta d(\alpha)+\sum_{i=1}^{n}\Big[(f_{x_i}-f_{x_{i-1}})(\alpha_{t_i}\beta_{t_{i-1}}-\alpha_{x_i}\beta_{x_{i-1}})\Big]\right|<3\epsilon$$

To analyze the sum on the right, note that

$$\begin{aligned}&\left|\sum_{i=1}^{n}\Big[(f_{x_i}-f_{x_{i-1}})(\alpha_{t_i}\beta_{t_{i-1}}-\alpha_{x_i}\beta_{x_{i-1}})\right|\\\leq&\sum_{i=1}^{n}\left|f_{x_i}-f_{x_{i-1}}\right|\left|\alpha_{t_i}\beta_{t_{i-1}}-\alpha_{x_i}\beta_{x_{i-1}}\right|\\\leq&\sum_{i=1}^{n}\left(\sup_{x\in[x_{i-1},x_i]}f(x)-\inf_{x\in[x_{i-1},x_i]}f(x)\right)\left(\alpha_{x_i}\beta_{x_i}-\alpha_{x_{i-1}}\beta_{x_{i-1}}\right)\\<&\epsilon\end{aligned}$$

The second inequality comes from noticing that t_i is in (x_{i-1},x_i), and that $\alpha\beta$ is weakly increasing. The third inequality comes from P being a refinement of P_1. Thus

$$\left|\int fd(\alpha\beta)-\int f\alpha d(\beta)-\int f\beta d(\alpha)\right|<4\epsilon$$

Which can be made arbitrarily close to 0.

□