Chapter 7 Lebesgue Measure

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7.2 Outer Measure

Exercise 7.2.1

Prove Lemma 7.2.5. Show that outer measure has the following six properties:

v - Empty set

The empty set $\emptyset$ has outer measure $m^*(\emptyset) = 0$.

Let $\epsilon > 0$, and let B_i be an open box with volume less than ϵ. B_i covers the empty set, for all values of ϵ, so $m^*(\emptyset) \leq \epsilon$ for all $\epsilon > 0$. Thus 0 is a lower bound for the volume of the boxes covering $\emptyset$. In fact, it is a least lower bound, because the volume of a box is nonnegative. Thus $0 = \inf\{\text{vol}(B_i) : B_i \text{ covers } \emptyset\} = m^*(\emptyset)$.

vi - Positivity

We have $0 \leq m^8(\Omega) \leq +\infty$ for every measurable set Ω.

The outer measure is the infimum of the volume of open boxes that cover a set. The volume of open boxes is nonnegative, meaning that the infimum is ≥ 0.

vii - Monotonicity

If $A \subset B \subset \mathbb{R}^n$, then $m^*(A) \leq m^*(B)$.

Let (B_j) cover B. Then (B_j) covers A, and thus the infimum of the volume of boxes that cover A is less than or equal to the infimum of the volume of boxes that cover B.

x - Countable Sub-additivity

If $(A_j)_{j \in J}$ are a countable collection of subsets of $\mathbb{R}^n$, then $m^*(\cup_{j \in J} A_j) \leq \sum_{j \in J} m^*(A_j)$.

Fix $\epsilon > 0$. Because A can be covered by a countable number of boxes, the index set can be relabeled as $J = \{j_1, j_2 \ldots\}$.

Lemma 1 *Let B be an outer measurable set. Then for all $\epsilon > 0$, there exists a countable covering of B by open boxes $(A_j)_{j \in J}$ such that $\sum_{j \in J} vol(A_j) < m^*(B) + \epsilon$.*

Proof: *Suppose not. Then there exists an $\epsilon > 0$ such that for all countable coverings of B by open boxes $(A_j)_{j \in J}$, we have $\sum_{j \in J} vol(A_j) \geq m^*(B) + \epsilon$. Then $m^*(B) + \epsilon$ is a lower bound for the set*

$$\{\sum_{j \in J} vol(A_j) : (A_j)_{j \in J} \text{ covers } B; J \text{ at most countable}\}$$

which contradicts $m^*(B)$ being the greatest lower bound for this set. $\square$

Using the lemma, for all $j \in N$, we can cover A_j with countable open boxes with volume less than $m^*(A_j) + \frac{\epsilon}{2^j}$. Denoting these boxes B_{jk}, the B_{jk} cover $\cup_{j \in J} A_j$). Because the A_j and B_k are countable, the B_{jk} are countable. Thus

$$m^*(\cup_{j \in J} A_j) \leq \sum_{j \in \mathbb{N}, k \in \mathbb{N}} \text{vol}(B_{jk}) < \sum_{j \in \mathbb{N}} (m^*(A_j) + \frac{\epsilon}{2^j}) \leq \sum_{j \in \mathbb{N}} (m^*(A_j)) + \epsilon$$

Because $\epsilon > 0$ was arbitrary, the result is proved.

viii - Finite Sub-additivity

If $(A_j)_{j \in J}$ are a finite collection of subsets of $\mathbb{R}^n$, then $m^*(\cup_{j \in J} A_j) \leq \sum_{j \in J} m^*(A_j)$.

This just follows from x.

xiii - Translation Invariance

If Ω is a subset of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$, then $m^*(x + \Omega) = m^*(\Omega)$.

Let $(A_j)_{j \in J}$ be a countable cover of Ω. Consider $A_j + x$. It has volume

$$\text{vol}(A_j + x) = \prod_{i=1}^{n} ((b_i + x_i) - (a_i + x_i)) = \prod_{i=1}^{n} (b_i - a_i) = \text{vol}(A_j)$$

A covering means that for all $p \in \Omega$, there exists A_j such that $p \in A_j$. For all points $p+x \in \Omega+x$, there exists A_j+x such that $p+x \in A_j+x$. Thus, $(A_j+x)_{j \in J}$ cover $\Omega + x$. Since the volume of A_j is unchanged through translation, the infimum of their union remains unchanged, so $m^*(\Omega + x) = m^*(\Omega)$.