1.3 Triangular Systems

Exercise 1.3.15

Develop row-oriented back substitution for upper-triangular matrices.

Let the system be $Ux = y$, where U is upper triangular. Writing out the equations,

$$\begin{aligned} u_{11}x_1 + u_{12}x_2 \dots u_{1,n-1}x_{n-1} + u_{1n}x_n &= y_1 \\ u_{22}x_2 \dots u_{2,n-1}x_{n-1} + u_{2n}x_n &= y_2 \\ &\vdots \\ u_{n-1,n-1}x_{n-1} + u_{n-1,n}x_n &= y_{n-1} \\ u_{nn}x_n &= y_n \end{aligned}$$

1.7 Gaussian Elimination and the LU Decomposition

Exercise 1.7.2

Prove Proposition 1.7.1: If $\hat{A}x = \hat{b}$ is obtained from $Ax = b$ by an elementary operation of type 1, 2, or 3, then the systems $Ax = b$ and $\hat{A}x = \hat{b}$ are equivalent. That is, they have the same solution set.

Proof: We begin with showing operations of type 1, i.e., adding a multiple of one equation to another equation. Let A_i be the ith row of matrix A.

For the forward, we need to show that if x solves $Ax = b$, then x solves $\hat{A}x = \hat{b}$. Suppose that $\hat{A}$ was created by adding m times row p to row q of A. Then $\hat{A}_i = A_i$ and $\hat{b}_i = b_i$ for $i \neq q$, and $\hat{A}_q = mA_p + A_q$, $\hat{b}_q = mb_p + b_q$. x solves $\hat{A}x = \hat{b}$ for rows $i \neq q$ trivially. For row q, $\hat{A}_q x = (mA_p + A_q)x = mA_p x + A_q x = mb_p + b_q = \hat{b}_q$. Thus x solves $\hat{A}x = \hat{b}$.

For the reverse, we need to show that if x solves $\hat{A}x = \hat{b}$, then x solves $Ax = b$. Again, suppose that $\hat{A}$ was created by adding m times row p to row q of A. Since $\hat{A}_p = A_p$, by subtracting m times row p from row q of $\hat{A}$, we get back A_q and b_q. Since this is an elementary row operation of type 1, the theorem proved above holds.

Interchanging two rows and multiplying an equation by a non-zero constant are trivial. $\square$