QR Householder Function Descriptions

Arthur Chen

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Main Functions

reflector.m

Given $x \in \mathbb{R}^{n\times 1}$, implicitly computes a reflector $Q \in \mathbb{R}^{n\times n}$ such that $Qx = \alpha e_1$, where e_1 is the first standard basis vector in $\mathbb{R}^n$ and $\alpha \in \mathbb{R} \neq 0$.

Specifically, given x, reflector.m outputs $\tau, \gamma \in \mathbb{R}$, $u \in R^{n\times 1}$ such that $Q = I - \gamma u u^T$ is a reflector such that $Qx = -\tau e^1$.

Given $B \in \mathbb{R}^{n\times m}$, the efficient way to compute QB is as follows:

$$
\begin{aligned}
&v^T \leftarrow \gamma u^T \\
&v^T \leftarrow v^T B \\
&B \leftarrow B - u v^T
\end{aligned}
$$

qr decomposition.m

Given a full-rank real matrix $A \in \mathbb{R}^{n\times m}$, $n \geq m$, implicitly computes the (condensed) QR decomposition. Specifically, implicitly generates $Q \in \mathbb{R}^{n\times m}$, Q is isometric, and $R \in \mathbb{R}^{m\times m}$, R is upper triangular, such that $A = QR$.

The output matrix A contains R on its upper diagonal, and contains the m normalized u vectors as columns in the strict lower triangle, minus the initial 1. Thus, if

$$A = \begin{pmatrix} 1 & 2 \\ .5 & 3 \\ .2 & .3 \end{pmatrix}$$

then

$$R = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

and

$$u_1 = \begin{pmatrix} 1 \\ .5 \\ .2 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ .3 \end{pmatrix}$$

least squares.m

Uses QR Decomposition to solve the least squares problem. Given $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^{m\times 1}$, $m \geq n$, A is full rank, finds $x \in \mathbb{R}^{n\times 1}$ such that x solves the least squares problem, i.e. minimizes the 2-norm of the residual. Alternatively, using the normal equations, $A^T Ax = A^T b$.

Because $A = QR$ and Q is an isometry, $A^T A = (R^T Q^T)(QR) = R^T R$. Thus

$$A^T Ax = A^T b \to R^T Rx = R^T Q^T b \to Rx = Q^T b$$

because we assumed A has full rank, meaning R is invertible. Thus we can use reflectors to compute $y = Q^T b$. Since R is upper-triangular, $Rx = y$ can be solved by backwards substitution.

Helper Functions

generate q.m

Given a matrix A and vector γ from from qr decomposition.m, generates Q explicitly as a matrix. Useful for debugging.

Each u_k, γ_k, τ_k implicitly defines a Q_k, and $Q = Q_1 Q_2 \dots Q_{\text{end}}$. generate q.m simply applies these reflectors right to left to an identity matrix to generate Q.

backsubs.m

Given $Ax = b$, $A \in \mathbb{R}^{n\times n}$ and upper triangular, uses backwards substitution to calculate x.

Test Functions

linsolve qr.m

Solves a linear system $Ax = b$, $A \in R^{n\times n}$, $b \in \mathbb{R}^{n\times 1}$ using QR decomposition. This is inefficient, but useful for debugging purposes.

References

More details can be found in Watkin's Fundamentals of Matrix Computations, 2nd Edition.

(Ideally I'd have a Works Cited section here, but I haven't figured out how to configure that yet.)