Properties of MC:

1.) Transformations: for $\theta^{(k)} \sim [\theta]$, $k=1,\ldots,K$

we can get $E(g(\theta)) = \int g(\theta)[\theta]\,d\theta$

$$\approx \frac{\sum_{k=1}^{K} g(\theta^{(k)})}{K}$$

2.) Marginalization: for $\underline{\theta}^{(k)} = (\theta_1^{(k)}, \theta_2^{(k)})' \sim [\underline{\theta}]$

we can get $E(\theta_1) = \iint \theta_1 [\underline{\theta}]\,d\theta_1\,d\theta_2$

$$\approx \frac{\sum_{k=1}^{K} \theta_1^{(k)}}{K}$$

Example 1: Suppose $\theta \sim \text{Unif}(0,1) = 1$

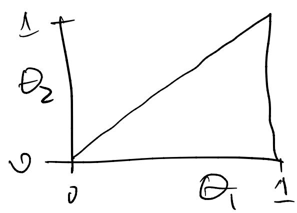
$$E(\theta) = \int_0^1 \theta[\theta]\,d\theta = \int_0^1 \theta\,d\theta = \left(\frac{\theta^2}{2}\right)\Big|_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

$$E(\theta^2) = \int_0^1 \theta^2[\theta]\,d\theta = \int_0^1 \theta^2\,d\theta = \left(\frac{\theta^3}{3}\right)\Big|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$$

$$= \int_0^1 \theta^2[\theta^2]\,d\theta^2$$

$[\theta_1] = \text{Unif}(0,1)$

Example 2: $[\underline{\theta}] = [\theta_1, \theta_2] = [\theta_2|\theta_1][\theta_1]$, $[\theta_2|\theta_1] = \text{Unif}(0,\theta_1)$



$$E(\theta_1) = \int_0^1\int_0^{\theta_1} \theta_1 [\theta_2|\theta_1][\theta_1]\,d\theta_2\,d\theta_1$$

$$= \int_0^1 \theta_1[\theta_1]\int_0^{\theta_1}[\theta_2|\theta_1]\,d\theta_2\,d\theta_1$$

$$= \int_0^1 \theta_1[\theta_1]\int_0^{\theta_1}\frac{1}{\theta_1}\,d\theta_2\,d\theta_1 = \int_0^1 \theta_1[\theta_1]\left(\frac{\theta_2}{\theta_1}\right)\Big|_0^{\theta_1} d\theta_1$$

$$= \int_0^1 \theta_1[\theta_1](1-0)\,d\theta_1 = \int_0^1 \theta_1[\theta_1]\,d\theta_1 = \frac{1}{2}$$

$$E(\theta_2|\theta_1) = \int_0^{\theta_1} \theta_2 [\theta_2|\theta_1] \, d\theta_2$$

$$= \int_0^{\theta_1} \theta_2 \frac{1}{\theta_1} \, d\theta_2 = \left(\frac{\theta_2^2}{2\theta_1} \right) \Bigg|_0^{\theta_1}$$

$$= \frac{\theta_1}{2} - 0 = \frac{\theta_1}{2}$$

$$E(\theta_2) = \int_0^1 \int_0^{\theta_1} \theta_2 [\theta_2|\theta_1][\theta_1] \, d\theta_2 \, d\theta_1$$

$$= \int_0^1 [\theta_1] \int_0^{\theta_1} \theta_2 [\theta_2|\theta_1] \, d\theta_2 \, d\theta_1$$

$$= \int_0^1 \frac{\theta_1}{2} \, d\theta_1 = \left(\frac{\theta_1^2}{4} \right) \Bigg|_0^1 = \frac{1}{4} - 0 = \frac{1}{4}$$