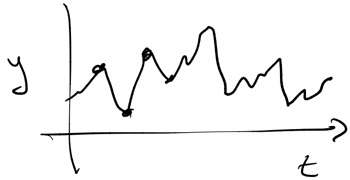
Temporal Models:

Suppose y_t for $t = 1, \ldots, T$ are observed over time:

To study dynamics of a time series:



First $z_t = y_t - \bar{y}$, then specify an autoregressive model:

$$z_t \sim N(\alpha z_{t-1}, \sigma^2), \quad t = 2, \ldots, T$$

If $\underline{z}$ is stationary then $|\alpha| < 1$

$$\alpha \sim N(\mu_\alpha, \sigma^2_\alpha)$$
$$\sigma^2 \sim IG(q, r)$$

Posterior:

$$[\alpha, \sigma^2 | \underline{z}] \propto \left(\prod_{t=2}^{T} [z_t | z_{t-1}, \alpha, \sigma^2] \right) [\alpha][\sigma^2]$$

Full-conditional distributions:

$$[\alpha | \cdot] \propto \left(\prod_{t=2}^{T} [z_t | z_{t-1}, \alpha, \sigma^2] \right) [\alpha]$$

$$\propto \exp\left\{ -\frac{1}{2} \frac{\sum_t (z_t - \alpha z_{t-1})^2}{\sigma^2} \right\} \exp\left\{ -\frac{1}{2} \frac{(\alpha - \mu_\alpha)^2}{\sigma^2_\alpha} \right\}$$

$$\propto \exp\left\{ -\frac{1}{2} \left(-2 \underbrace{\left(\frac{\sum_t z_t z_{t-1}}{\sigma^2} + \frac{\mu}{\sigma^2_\alpha} \right)}_{b} \alpha + \alpha^2 \underbrace{\left(\frac{\sum_t z^2_{t-1}}{\sigma^2} + \frac{1}{\sigma^2_\alpha} \right)}_{a} \right) \right\}$$

$$= N(a^{-1} b, a^{-1})$$

$$[\sigma^2 | \cdot] \propto \left(\prod_{t=2}^{T} [z_t | z_{t-1}, \alpha, \sigma^2] \right) [\sigma^2]$$

$$\propto (\sigma^2)^{-\frac{(T-1)}{2}} \exp\left\{ -\frac{1}{\sigma^2} \frac{\sum_t (z_t - \alpha z_{t-1})^2}{2} \right\} (\sigma^2)^{-(q+1)} \exp\left\{ -\frac{1}{\sigma^2 r} \right\}$$

$$\propto (\sigma^2)^{-(\underbrace{\frac{T-1}{2} + q}_{\tilde{q}} + 1)} \exp\left\{ -\frac{1}{\sigma^2} \underbrace{\left(\frac{\sum_t (z_t - \alpha z_{t-1})^2}{2} + \frac{1}{r} \right)}_{\tilde{r}^{-1}} \right\}$$

$$= IG(\tilde{q}, \tilde{r})$$

Bayesian forecasting: Use posterior pred. distn (PPD)

$$[z_{T+1} | \underline{z}] = \int\int [z_{T+1}, \alpha, \sigma^2 | \underline{z}] \, d\alpha \, d\sigma^2$$

$$= \int\int \underbrace{[z_{T+1} | \alpha, \sigma^2, \underline{z}]}_{\text{pred. full-cond.}} \underbrace{[\alpha, \sigma^2 | \underline{z}]}_{\text{posterior}} \, d\alpha \, d\sigma^2$$

$$= \int\int [z_{T+1} | \alpha, \sigma^2, z_T] [\alpha, \sigma^2 | \underline{z}] \, d\alpha \, d\sigma^2$$

Note: $[z_{T+1} | \alpha, \sigma^2, z_T] = N(\alpha z_T, \sigma^2)$

Procedure:

1.) Sample $\alpha^{(k)}, \sigma^{2(k)}$ using MCMC, $k = 1, \ldots, K$

2.) Sample $z_{T+1}^{(k)} \sim N(\alpha^{(k)} z_T, \sigma^{2(k)})$ for $k = 1, \ldots, K$

3.) $\hat{z}_{T+1} = E(z_{T+1} | \underline{z}) = \frac{\sum_{k=1}^{K} z_{T+1}^{(k)}}{K}$

"point forecast"