Example Model:

$$y_i \sim N(\mu, \sigma^2), \quad i = 1, \dots, n$$
$$\mu \sim N(\mu_0, \sigma_0^2) = (2\pi\sigma_0^2)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\mu-\mu_0)^2\right\}$$
$$\sigma^2 \sim IG(q, r) = \frac{1}{r^q \Gamma(q)} (\sigma^2)^{-(q+1)} e^{-\frac{1}{r\sigma^2}}$$

Posterior:

$$[\mu, \sigma^2 \mid y] \propto \left(\prod_{i=1}^{n} [y_i \mid \mu, \sigma^2]\right) [\mu][\sigma^2]$$

Full-conditionals:

$$[\mu \mid \cdot] \propto \left(\prod_{i=1}^{n} [y_i \mid \mu, \sigma^2]\right) [\mu]$$

$$\propto \left(\prod_{i=1}^{n} \exp\left\{-\frac{1}{2\sigma^2}(y_i-\mu)^2\right\}\right) \exp\left\{-\frac{1}{2\sigma_0^2}(\mu-\mu_0)^2\right\}$$

$$\propto \exp\left\{-\frac{1}{2\sigma^2}\left(\cancel{\sum_i y_i^2} - 2\mu\sum_i y_i + n\mu^2\right)\right\} \exp\left\{-\frac{1}{2\sigma_0^2}\left(\mu^2 - 2\mu\mu_0 + \cancel{\mu_0^2}\right)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left(-2\underbrace{\left(\frac{\sum_i y_i}{\sigma^2} + \frac{\mu_0}{\sigma_0^2}\right)}_{b}\mu + \mu^2 \underbrace{\left(\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}\right)}_{a}\right)\right\}$$

$$= N(a^{-1}b, a^{-1})$$

Why? Complete the square! Note:

$$\frac{(\mu - a^{-1}b)^2}{a^{-1}} = \frac{\mu^2}{a^{-1}} - \frac{2\mu a^{-1} b}{a^{-1}} + \frac{(a^{-1}b)^2}{a^{-1}} = a\mu^2 - 2\mu b + a^{-1}b^2$$

$$a^{-1}b^2 = \left(\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}\right)^{-1} \left(\frac{\sum_i y_i}{\sigma^2} + \frac{\mu_0}{\sigma_0^2}\right)^2$$

doesn't involve μ, so we can multiply $e^{a^{-1}b^2}$ by $[\mu \mid \cdot]$ proportionality and it will have a Gaussian form (i.e., conjugate)

$$[\sigma^2 \mid \cdot] \propto \left(\prod_{i=1}^{n} [y_i \mid \mu, \sigma^2]\right)[\sigma^2]$$

$$\propto \left(\prod_{i=1}^{n} (\cancel{2\pi}\sigma^2)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma^2}(y_i-\mu)^2\right\}\right) \cancel{\frac{1}{r^q\Gamma(q)}} (\sigma^2)^{-(q+1)} \exp\left\{-\frac{1}{r\sigma^2}\right\}$$

$$\propto (\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{\sigma^2} \frac{\sum_i (y_i-\mu)^2}{2}\right\} (\sigma^2)^{-(q+1)} \exp\left\{-\frac{1}{\sigma^2}\cdot\frac{1}{r}\right\}$$

$$\propto (\sigma^2)^{-(\underbrace{\frac{n}{2}+q+1}_{\tilde{q}})} \exp\left\{-\frac{1}{\sigma^2}\left(\underbrace{\frac{\sum_i (y_i-\mu)^2}{2}+\frac{1}{r}}_{\tilde{r}^{-1}}\right)\right\}$$

$$= \mathrm{IG}(\tilde{q}, \tilde{r})$$

<u>why?</u> Because we can multiply by $\frac{1}{\tilde{r}^{\tilde{q}}\Gamma(\tilde{q})}$ and $[\sigma^2 \mid \cdot]$ will take the form of another inverse gamma (i.e., conjugate)

<u>MCMC algorithm:</u>

1.) Set $\mu^{(0)} = \bar{y}$
2.) $k=1$
3.) Sample $\sigma^{2(k)} \sim [\sigma^{2(k)} \mid y, \mu^{(k-1)}]$
4.) Sample $\mu^{(k)} \sim [\mu^{(k)} \mid y, \sigma^{2(k)}]$
5.) $k = k+1$
6.) Goto 3, repeat for $k = 1, 2, \ldots, K$

This is called a <u>Gibbs sampler</u>.