Homework Example

Prepare a written response to the following using a variety of LaTeX (Overleaf recommended). The main body of the solutions shouldn't be longer than 10 pages (double-spaced), not including title page and appendices. Due Thursday, Jan. 20, at the beginning of the class period.

We are interested in conducting a Bayesian analysis using the data set in 'faithful.csv' to infer the average length of time between eruptions of the Old Faithful geyser in Yellowstone National Park.

1. Write a Bayesian model with an exponential likelihood, assuming conditional independence among eruptions, and assume an exponential distribution for the rate parameter in the data model.

2. Derive the posterior distribution for the rate parameter analytically.

3. Develop an MCMC algorithm to fit the model above that uses a Metropolis-Hastings update for the rate parameter. Use an exponential distribution $\lambda^{(*)} \sim \text{Exp}(1/\lambda^{(k-1)})$ for the proposal.

4. Specify and justify the hyperparameter associated with the prior for the rate parameter.

5. Fit the model to the Old Faithful data set using the MCMC algorithm and report the following:

   (a) An MCMC trace plot for the rate parameter.

   (b) Metropolis-Hastings acceptance probability for the rate parameter.

   (c) Posterior mean, standard deviation, and 95% equal-tailed credible interval of the rate parameter.

(d) Posterior mean, standard deviation, and 95% equal-tailed credible interval of the scale (reciprocal of rate) parameter.

6. What can we infer about the average wait time between Old Faithful eruptions under our assumed model?

7. Create a graphic comparing an MCMC-based posterior histogram and the posterior density function for the rate parameter based on the analytical solution.

8. Qualitatively, discuss the appropriateness of the chosen likelihood for the Old Faithful wait time data.

9. Describe an alternative proposal distribution for the Metropolis-Hastings updates.