Accounting for Dependence:

We already discussed temporal models that account for dependence in time series data that are naturally dynamic.

What about other forms of dependence?

For example: suppose we observe data $y = (y_1, y_2)'$ and we seek to learn about the population mean μ.

1.) A simple model: $y_i \overset{iid}{\sim} N(\mu, \sigma^2)$, $i = 1, 2$

assumes conditional independence for y.

2.) A joint model: $y \sim N(\mu \cdot \underline{1}, \Sigma)$, $\Sigma = \sigma^2 R$, $R = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$

Suppose we're interested in: $P(\mu > 0 \mid y)$
"Posterior exceedance probability"

If we use model 1 and fail to account for positive dependence when it exists, our inferred exceedance probability may be artificially large or small!

How to assess benefit of accommodating dependence?

Recall the posterior for μ (assuming σ^2, ρ known):

$$[\mu | y] \propto [y | \mu][\mu]$$

$$\propto \exp\left\{-\frac{1}{2}(y-\mu\underline{1})'\Sigma^{-1}(y-\mu\underline{1})\right\}\exp\left\{-\frac{1}{2}\frac{(\mu-\mu_0)^2}{\sigma_0^2}\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left(-2\underbrace{\left(y'\Sigma^{-1}\underline{1}+\frac{\mu_0}{\sigma_0^2}\right)}_{b}\mu+\mu^2\underbrace{\left(\underline{1}'\Sigma^{-1}\underline{1}+\frac{1}{\sigma_0^2}\right)}_{a}\right)\right\}$$

$$= N(a^{-1}b, a^{-1})$$

notice that as $\sigma_0^2 \to \infty$ (prior variance)

$$\mathrm{Var}(\mu | y) \to (\underline{1}'\Sigma^{-1}\underline{1})^{-1} = \frac{\sigma^2}{2}(1+\rho) = \frac{\sigma^2}{2}+\frac{\sigma^2\rho}{2}$$

$$\Sigma^{-1} = \begin{pmatrix}\sigma^2 & \sigma^2\rho \\ \sigma^2\rho & \sigma^2\end{pmatrix}^{-1}$$

$$= \frac{1}{(\sigma^2)^2-(\sigma^2\rho)^2}\begin{pmatrix}\sigma^2 & -\sigma^2\rho \\ -\sigma^2\rho & \sigma^2\end{pmatrix}$$

$$\underline{1}'\Sigma^{-1}\underline{1}' = \frac{1}{(\sigma^2)^2-(\sigma^2\rho)^2}\left(\sigma^2-\sigma^2\rho\,,\ -\sigma^2\rho+\sigma^2\right)\underline{1}$$

$$= \frac{2(\sigma^2-\sigma^2\rho)}{(\sigma^2)^2-(\sigma^2\rho)^2}$$

$$(\underline{1}'\Sigma^{-1}\underline{1})^{-1} = \frac{(\sigma^2)^2-(\sigma^2\rho)^2}{2(\sigma^2-\sigma^2\rho)} = \frac{(\sigma^2)^2(1-\rho^2)}{2\sigma^2(1-\rho)}$$

$$= \frac{\sigma^2}{2}\,\frac{1-\rho^2}{1-\rho} = \frac{\sigma^2}{2}\,\frac{1^2-\rho^2}{1-\rho} = \frac{\sigma^2\cancel{(1-\rho)}(1+\rho)}{2\,\cancel{1-\rho}} = \frac{\sigma^2}{2}(1+\rho)$$

Take Home message :

$$\mathrm{var}(\mu|y) \to \frac{\sigma^2}{2} + \frac{\sigma^2\rho}{2} > \frac{\sigma^2}{2}$$

for $\rho > 0$, thus the posterior variance will be too small if we don't account for dependence in y.