Markov Chain Monte Carlo (MCMC):

We would like to get MC sample from posterior so we can use MC integration to compute post. mean, variance, etc..

But, we usually don't know the full-posterior!

We have: $[\underline{\theta} | y] \propto [y | \underline{\theta}][\underline{\theta}]$

However, we can get a correlated sample from the posterior dist'n using this algorithm:

1.) Assign initial value for $\underline{\theta}^{(0)}$
2.) Set $k=1$
3.) Sample proposal for $\underline{\theta}^{(*)} \sim [\underline{\theta}^{(*)}]_*$
4.) Calculate Metropolis-Hastings ratio:

$$mh = \frac{[y | \underline{\theta}^{(*)}][\underline{\theta}^{(*)}][\underline{\theta}^{(k-1)}]_*}{[y | \underline{\theta}^{(k-1)}][\underline{\theta}^{(k-1)}][\underline{\theta}^{(*)}]_*}$$

5.) Let $\underline{\theta}^{(k)} = \underline{\theta}^{(*)}$ w.p. $\min(mh, 1)$, otherwise let $\underline{\theta}^{(k)} = \underline{\theta}^{(k-1)}$

6.) Set $k=k+1$, goto step 3, repeat for $k=1,\ldots,K$, until K large enough to approx. posterior quantities well.

Note: This involves block updates for $\underline{\theta}$.

We could use sequential updates for parameters conditionally $\longrightarrow$

for model: $y \sim [y \mid \theta_1, \theta_2]$
$\theta_1 \sim [\theta_1]$
$\theta_2 \sim [\theta_2]$

The posterior is: $[\theta_1, \theta_2 \mid y] \propto [y \mid \theta_1, \theta_2][\theta_1][\theta_2]$

MCMC algorithm w/ sequential updates:

1.) Set $\theta_2^{(0)}$ at init value
2.) $k = 1$
3.) Sample $\theta_1^{(k)}$ from its full-conditional distn
$\Longrightarrow \theta_1^{(k)} \sim [\theta_1 \mid \theta_2^{(k-1)}, y]$
4.) Sample $\theta_2^{(k)}$ from its full-cond. distn
$\Longrightarrow \theta_2^{(k)} \sim [\theta_2 \mid \theta_1^{(k)}, y]$
5.) let $k = k+1$
6.) Repeat steps 3-5 for $k = 1, \ldots, K$.

<u>Notes:</u>

Steps 3, 4 could be done using <u>Metropolis-Hastings updates</u>.

If we can sample from the full-cond. distn directly, then it's a <u>Gibbs update</u>.

(The resulting MCMC algorithm is called a Gibbs sampler).

Example Model: $y_i \sim \text{Bern}(p)$, $i=1,\dots,n$

$p \sim \text{Beta}(\alpha, \beta)$

Posterior:

$$[p|y] \propto [y|p][p]$$

$$\propto \left(\prod_{i=1}^{n} [y_i|p]\right)[p]$$

$$\propto \left(\prod_{i=1}^{n} p^{y_i}(1-p)^{1-y_i}\right) p^{\alpha-1}(1-p)^{\beta-1}$$

$$\propto p^{\sum_i y_i + \alpha - 1}(1-p)^{\sum_i (1-y_i) + \beta - 1}$$

$$= \text{Beta}\Big(\underbrace{\sum_{i=1}^{n} y_i + \alpha}_{\tilde{\alpha}}, \underbrace{\sum_{i=1}^{n} (1-y_i) + \beta}_{\tilde{\beta}}\Big)$$

Note: when form of prior matches posterior (or full-conditional)) it's called "conjugate."

We don't need stochastic computing to fit this model, but we could use MC:

$$p^{(k)} \sim \text{Beta}(\tilde{\alpha}, \tilde{\beta}), \quad k=1,\dots,K$$

or MCMC:

1.) set $p^{(0)} = \frac{\sum_i y_i}{n}$

2.) set $k=1$

3.) sample $p^{(*)} \sim \text{Unif}(0,1) = [p]_* \propto 1$ (proposal)

4.) $mh = \dfrac{[y|p^{(*)}][p^{(*)}]\cancel{[p^{(k-1)}]_*}}{[y|p^{(k-1)}][p^{(k-1)}]\cancel{[p^{(*)}]_*}} = \dfrac{p^{(*)\tilde{\alpha}-1}(1-p^{(*)})^{\tilde{\beta}-1}}{p^{(k-1)\tilde{\alpha}-1}(1-p^{(k-1)})^{\tilde{\beta}-1}}$

5.) let $p^{(k)} = p^{(*)}$ w.p. $\min(mh, 1)$, else $p^{(k)} = p^{(k-1)}$

6.) let $k=k+1$, goto 3. until $k=K$.