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Broken and Unbroken Supersymmetry in Quantum Mechanics

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1 Introduction

Quantum Mechanics is the most accurate theory we have right now to describe the subatomic world. One of the most important equation of Quantum Mechanics is the Schrodinger's equation. It describes how a system evolves with time. In coordinate representation, it is a second order differential equation of an entity called Wavefunction (Note -this term is used only for physical solutions). The modulus square of the Wavefunction $|\psi(x)|^2$ of a system basically describes the probability of the system to be at a position x. Schrodinger's equation is usually defined in 3 dimensions but in certain cases it can be converted into 1 dimension (for example in radial harmonic oscillator) by separation of coordinates. It is also possible to separate the time variable and obtain a time independent (stationary) Schrodinger's equation. This time independent version is the one used for the thesis. Solutions of this equation which describe a physical system must obey some boundary conditions. For example, the physical solution must be normalisable. Sometimes solutions can be described in a closed analytical form. This can be achieved my using special functions such as *Orthogonal polynomials*. The method to do this is described in the thesis by using Laguerre Polynomials $L_n^{(\alpha)}(g)$.

Symmetries play a major role in characterising quantum systems. Symmetries of a system leads to conserved quantities, which can be used to solved problems. A system is said to be symmetric with respect to an operator Q if [H,Q] = HQ - QH = 0. SUSYQM [1] is a special type of symmetry, which was found to be useful to obtain new solvable potentials problems from known ones. The corresponding operator is a 2x2 operator containing differential operator. This operator will be discussed in detail in the later part of the text. Based on this a system can either have *Unbroken* or *Broken* supersymmetry. In case of Unbroken SUSY, the ground state solution follows the same symmetry as the system, ie, Q $\psi(x) = 0$. While in case of Broken SUSY it is not true. This is further discussed in the text using a solution with different parameters to generate broken and unbroken symmetry.

SUSYQM is used to study cases of broken supersymmetry and also to find solvable potentials. In the first part of the thesis a method to find solvable potentials is shown (using Laguerre polynomials). And in the second part, a general discussion of examples of broken and unbroken cases of SUSYQM is carried out. This is done with the help of radial harmonic potential and its partner potentials.

Supersymmetry was first formulated as a solution to unite gravitational force with other forces of the standard model. According to it there should be fermionic partner for every boson and a bosonic partner for every fermion. This however is yet to be observed experimentally. If supersymmetry has something to do with nature, it must be broken, because bosons and fermions with the same mass are not observed. At the level of Lagranagian, supersymmetry transformation rules mix bosonic and fermionic fields, leaving the action invariant. Nevertheless broken supersymmetry has been observed in nuclear physics (two pion exchange processes) [2]. Supersymmetry is also helpful in explaining solutions of orbitals (s,p,d,f) for different atoms. For instance, the s levels of the lithium atom may be interpreted as the supersymmetric partner of the hydrogen atom s levels (assuming absence of electron - electron interaction and valence electron is far from the core) [3].

2 Brief Summary of SUSYQM

It is often not recognised that we can get potential of a Hamiltonian (up to a constant shift) if we know the system's ground state wave function. In this part of the text, the theory of SUSYQM will be built by assuming that the ground state wave function is known. It should be noted that this function should be nodeless and the eigenvalue for this function (ground state energy) can be taken as 0. Thus following equation holds - [1]

$$\begin{aligned} H\psi(x) = E_0\psi(x) = 0 \\ \left(-\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x)\right) = 0 \\ V(x) = \frac{\psi''}{\psi} \end{aligned} \tag{1}$$

This allows us to factorise the Hamiltonian -

$$H = A^\dagger A \tag{2}$$

where

$$A = \frac{d}{dx} + W(x) \qquad A^\dagger = -\frac{d}{dx} + W(x) \tag{3}$$

W(x) to be specified afterwards.
In order to construct a super algebra, we define following super charges Q and $Q^\dagger$

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \tag{4}$$

$$Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix} \tag{5}$$

These matrices generate symmetry transformation. And a supersymmetric Hamiltonian can be defined in the following way -

$$H = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} \tag{6}$$

where

$$\begin{aligned} H_- = A^\dagger A \\ H_+ = AA^\dagger \end{aligned} \tag{7}$$

On further calculations, we can see that H_- and H_+ can be defined in terms of $W(x)$.
To have the standard form of Hamiltonian, we define V_- and V_+ in the following way -

$$\begin{aligned} H_- = -\frac{d^2}{dx^2} + V_-(x). \\ V_-(x) = W^2(x) - W'(x) \end{aligned} \tag{8}$$

$$H_+ = -\frac{d^2}{dx^2} + V_+(x)$$
$$V_+(x) = W^2(x) + W^{'}(x) \tag{9}$$

The supercharges and the supersymmetric Hamiltonian satisfy the following commutation and anti - commutation relations -

$$\left\{Q, Q^\dagger\right\} = H, \quad [Q, H] = \left[Q^\dagger, H\right] = 0 \tag{10}$$

The fact that supercharges commute with the Hamiltonian implies that it is a symmetry. It also means that if ψ is an eigenfunction of H with some eignevalue then it will also be an eigenfunction of $Q\psi$ and $Q^\dagger\psi$ with the same eigenvalue. As we shall see the energy eigenvalues of H_- and H_+ are related.

$$H_-\psi_n^{(-)}(x) = A^\dagger A\psi_n^{(-)}(x) = E_n^-\psi_n^{(-)}(x) \tag{11}$$

implies

$$H_+(A\psi_n^{(-)}(x)) = AA^\dagger A\psi_n^{(-)}(x) = E_n^{(-)}(A\psi_n^{(-)}(x)) \tag{12}$$

Thus $A\psi_n^{(-)}(x)$ can be considered as the eigenfunction of H_+, ie $\psi_n^{(+)}(x)$

Similarly, applying Schrodinger's equation for H_+ -

$$H_+\psi_n^{(+)}(x) = AA^\dagger\psi_n^{(+)}(x) = E_n^{(+)}\psi_n^{(+)}(x) \tag{13}$$

implies

$$H_-(A^\dagger\psi_n^{(+)}(x)) = A^\dagger AA^\dagger\psi_n^{(+)}(x) = E_n^{(+)}(A^\dagger\psi_n^{(+)}(x)) \tag{14}$$

Therefore eigenfunctions of H_- are related to each other by A and $A^\dagger$. By using the fact that $E_0^{(-)} = 0$ and the equations above, following can be calculated -

$$E_n^{(+)} = E_{n+1}^{(-)}, \quad E_0^{(-)} = 0$$
$$\psi_n^{(+)}(x) = (E_{n+1}^{(-)})^{-\frac{1}{2}}A\psi_n^{(-)}(x) \tag{15}$$
$$\psi_n^{(-)}(x) = A^\dagger\psi_n^{(+)}(x)$$

As mentioned above $H_-\psi_0^{(-)}(x) = A^\dagger AE_0^{(-)}\psi_0^{(-)}(x) = 0$, it is required that A annihilates the ground state wave function of H_-

$$A\psi_0^{(-)}(x) = 0 \tag{16}$$

Therefore the spectrum of H_+ will lack one energy value. Except this difference, both H_- and H_+ will have identical spectrum. This is due to the symmetry followed by the Hamiltonian.

If the eigenfunctions of the Hamiltonian follow this symmetry as well then the system is said to have **Unbroken supersymmetry**. That is -

$$Q\psi_0^{(-)}(x) = 0 \tag{17}$$

holds for unbroken supersymmetry.

In case of **Broken Supersymmetry**, the Hamiltonian follows the symmetry but its eigenfunction does not. Therefore

$$Q\psi_0^{(-)}(x) \neq 0 \tag{18}$$

for broken supersymmetry. This implies that in case of broken supersymmetry, the ground state eigenvalue of H_- will be non vanishing ($E_0^{(-)} \neq 0$).

Equation (16) can be used to calculate W(x) in terms of ground state wave function,

$$W(x) = -\frac{d}{dx} ln\ \psi_0^{(-)}(x) \tag{19}$$

Lets consider a more generalised solution for H_- (not necessarily physical solutions).

$$H_-\Gamma(x) = \epsilon\Gamma(x) \tag{20}$$

here $\Gamma(x)$ is a nodeless therefore from equation (8) we can obtain a continuous function for potential -

$$\begin{aligned} V_-(x) &= \frac{\Gamma''(x)}{\Gamma(x)} + \epsilon \\ &= \left(-\frac{\Gamma'(x)}{\Gamma(x)}\right)^2 + \left(-\frac{\Gamma'(x)}{\Gamma(x)}\right)' + \epsilon \end{aligned} \tag{21}$$

To match the format as before we define a new superpotential -

$$\tilde{W} = -\frac{\Gamma'(x)}{\Gamma(x)} = -\frac{d}{dx} ln\ \Gamma(x) \tag{22}$$

Equation (21) thus becomes -

$$V_-(x) = \tilde{W}^2 - \frac{d}{dx}\tilde{W} + \epsilon \tag{23}$$

Similarly we can define V_+ -

$$V_+(x) = \tilde{W}^2 + \frac{d}{dx}\tilde{W} + \epsilon \tag{24}$$

Often $\Gamma(x)$ can be written as the product of a nodeless function with the ground state eigenfunction.

$$\Gamma(x) = \psi_0^{(-)}(x)\xi(x) \tag{25}$$

From equation (22) -

$$\tilde{W} = W - \frac{d}{dx} ln\xi(x) \tag{26}$$

It can be observed that when supersymmetry is unbroken, ϵ would be zero. Thus we get the special case $\Gamma(x) = \psi_0^{(-)}(x)$. While for $\epsilon \neq 0$, we will get broken supersymmetry.

3 Method of obtaining solvable potentials

Consider Schrodinger's equation for a wave function $\Psi(x)$ - [4]

$$\frac{d^2\Psi(x)}{dx^2} + (E - V(x))\Psi(x) = 0 \tag{27}$$

The solution of this equation generally takes the form -

$$\Psi(x) = F(g(x))f(x) \tag{28}$$

where F(g) is a special function which satisfies the following equation -

$$\frac{d^2F}{dg^2} + Q(g)\frac{dF}{dg} + R(g)F(g) = 0 \tag{29}$$

The form of Q(g) and R(g) is well defined by the special function F(g). Substituting equation (28) in the Schrodinger's equation -

$$\frac{d^2F}{dg^2} + \frac{dF}{dg}\left(\frac{g''}{(g')^2} + \frac{2f'}{fg'}\right) + F\left(\frac{f''}{f(g')^2} + \frac{E - V(x)}{(g')^2}\right) = 0 \tag{30}$$

Comparing with equation(29) :

$$\begin{aligned} Q(g(x)) &= \frac{g''}{(g')^2} + \frac{2f'}{fg'} \\ R(g(x)) &= \frac{f''}{f(g')^2} + \frac{E - V(x)}{(g')^2} \end{aligned} \tag{31}$$

Solving equation (31) for V(x) we get -

$$E - V(x) = \frac{g'''}{2g'} - \frac{3}{4} * \left(\frac{g''}{g'}\right) + (g')^2\left[R(g(x)) - \frac{1}{2}\frac{dQ}{dg} - \frac{1}{4}Q^2(g(x))\right] \tag{32}$$

This means that once we specify the special function F(g), we can know the form of Q and R. Thus we can experiment with different internal functions g(x) to find the corresponding potentials.

f(x) can also be easily derived from equation (31) -

$$f(x) = (g')^{-\frac{1}{2}} exp\left(\frac{1}{2}\int^{g(x)} Q(g)dg\right) \tag{33}$$

4 Using Generalised Laguerre polynomials as the special function

It was observed that orthogonal polynomials satisfy the second order differential equation (equation (29)). In this section we will use Laguerre polynomials as the special function F(g) and carry the calculations. Where g is the internal function. The domain of definition of generalised Laguerre polynomial is $0 \to \infty$. Thus the range of g should fall in this interval.

$$\Psi(x) = f(x) * L_n^{(\alpha)}(g(x)) \tag{34}$$

A generalised Laguerre polynomial satisfies :

$$g\frac{d^2 L_n^{(\alpha)}(g)}{dg^2} + (\alpha + 1 - g)\frac{dL_n^{(\alpha)}(g)}{dg} + nL_n^{(\alpha)}(g) = 0 \tag{35}$$

Therefore it can be deduced that the functions in equation (29) are -

$$\begin{aligned} Q(g) &= \frac{\alpha + 1 - g}{g} \\ R(g) &= \frac{n}{g} \end{aligned} \tag{36}$$

In the following equations, derivation with respect to x is denoted by f'(x) or g'(x)
Substituting in the Schrodinger's equation :

$$\frac{d^2\Psi}{dx^2} + (E - V(x)) * \Psi(x) = 0 \tag{37}$$

$$\begin{aligned} \frac{d^2\Psi}{dx^2} = \frac{d^2}{dx^2}(f(x) * L_n^{(\alpha)}(g)) &= \frac{d}{dx}(\frac{d}{dx}(f(x) * L_n^{(\alpha)}(g)) \\ &= \frac{d}{dx}(f'(x) * L_n^{(\alpha)}(g) + f(x) * \frac{d}{dg}L_n^{\alpha}(g)\frac{dg}{dx}) \end{aligned} \tag{38}$$

$$\frac{d^2 L_n^{(\alpha)}(g)}{dg^2} + \frac{dL_n^{(\alpha)}(g)}{dg}\left(\frac{g''}{(g')^2} + \frac{2f'}{fg'}\right) + L_n^{(\alpha)}(g)\left(\frac{f''}{f(g')^2} + \frac{E - V(x)}{(g')^2}\right) = 0 \tag{39}$$

Comparing equation (39) to equation (31) ;

$$\begin{aligned} Q(g) = \frac{\alpha + 1 - g(x)}{g(x)} &= \left(\frac{g''}{(g')^2} + \frac{2f'}{fg'}\right) \\ R(g) = \frac{n}{g(x)} &= \left(\frac{f''}{f * (g')^2} + \frac{E - V(x)}{(g')^2}\right) \end{aligned} \tag{40}$$

By substituting this in equation (32) we get -

$$E - V(x) = \frac{g'''}{2g'} - \frac{3}{4}\left(\frac{g''}{g'}\right)^2 + (g')^2\left[\frac{(2n + \alpha + 1)}{2g} + \frac{1}{g^2}\left(\frac{1 - \alpha^2}{4}\right) - \frac{1}{4}\right] \tag{41}$$

It can be seen that one of the three terms in the square bracket of equation (41) can be that of energy. Therefore that term should be independent of x. We will separately take each term as constant and derive g(x) from that condition.

4.1 Case I (Harmonic Potential)

Considering the fist term as the energy term, therefore assuming it is constant -

$$\frac{\mathbf{g'^2}}{\mathbf{g}} = \mathbf{c^2} \tag{42}$$

$$g(x) = \frac{1}{4}c^2x^2 \tag{43}$$

The constant of integration was chosen to be 0 because g(x) must go from 0 to ∞.

Substituting equation (43) in equation (41)-

$$E - V(x) = 0 - \frac{3}{4}\left(\frac{1}{x}\right)^2 + \left[c^2\frac{(2n+\alpha+1)}{2} + \frac{1-\alpha^2}{x^2} - \frac{c^4}{16}x^2\right] \tag{44}$$

It can be deduced from equation (44) that -

$$E = c^2\left(n + \frac{\alpha+1}{2}\right) \tag{45}$$

$$\begin{aligned} V(x) &= \frac{c^4}{16}x^2 + \frac{3}{4}\frac{1}{x^2} - \frac{1-\alpha^2}{x^2} \\ V(x) &= \frac{c^4}{16}x^2 + \frac{1}{x^2}\left(\alpha+\frac{1}{2}\right)\left(\alpha-\frac{1}{2}\right) \end{aligned} \tag{46}$$

To see the similarity with the harmonic oscillator potential, we can consider the following -
$c^2 = 2\omega \;\; and \;\; \alpha = l + \frac{1}{2}$

where l is the angular momentum quantum number and ω is the frequency of the oscillator. V(x) now becomes -

$$\boxed{V(x) = \frac{1}{4}\omega^2x^2 + \frac{l(l+1)}{x^2}} = v(x,l) \tag{47}$$

The harmonic potential will be denoted by v(x,l) in the following part of the text.
It can be seen that if we substitute l with $-l-1$, potential would remain the same. ($v(x,l) = v(x,-l-1)$)

It can be said that n = 0, will give a nodeless solution of the Schrodinger's equation. Therefore ground state energy can be calculated by substituting n = 0 in equation (45) -

$$\boxed{E_0 = \omega\left(l + \frac{3}{2}\right)} \tag{48}$$

From equation (33), f(x) can be calculated in the following way -

$$\begin{aligned} f(x) &= (cx)^{\frac{-1}{2}} exp\left(\frac{1}{2}\int^{g(x)} Q(g)dg\right) \\ &= (cx)^{\frac{-1}{2}} exp\left(\frac{1}{2}[(\alpha+1)lng - g]\right) \\ &= \frac{1}{\sqrt{2}}\left(\frac{c}{2}\right)^{\alpha} x^{\alpha+\frac{1}{2}} * exp\left(\frac{-c^2x^2}{8}\right) \end{aligned} \tag{49}$$

$$f'(x) = \frac{1}{\sqrt{2}}\left(\frac{c}{2}\right)^{\alpha}\left[\left(\alpha+\frac{1}{2}\right)x^{\alpha-\frac{1}{2}} - \frac{1}{4}c^2x^{\alpha+\frac{3}{2}}\right]exp\left(\frac{-c^2x^2}{8}\right) \tag{50}$$

Finding super potential using equation (19) -

$$\begin{aligned} W(x) = -\frac{f'(x)}{f(x)} &= -\left(\alpha+\frac{1}{2}\right)\frac{1}{x} + \frac{1}{4}c^2x \\ &= -(l+1)\frac{1}{x} + \frac{1}{2}\omega x \end{aligned} \tag{51}$$

The partner potentials can be calculated using -

$$V_+(x) = W^2(x) + W'(x) = (l+1)(l+2)\frac{1}{x^2} + \frac{\omega^2}{4}x^2 - \left(l+\frac{1}{2}\right)\omega \tag{52}$$

$$V_-(x) = W^2(x) - W'(x) = l(l+1)\frac{1}{x^2} + \frac{\omega^2}{4}x^2 - \left(l+\frac{3}{2}\right)\omega \tag{53}$$

Or

$$\begin{aligned} \boxed{V_- = v(x,l) - E_0} \\ \boxed{V_+ = v(x,l+1) - E_0 + \omega} \end{aligned} \tag{54}$$

4.2 Case II (Morse Potential)

Considering the second term of equation (41) to be the energy term, thus assuming it does not depend upon x -

$$\mathbf{\frac{g'^2}{g^2} = c^2} \tag{55}$$

$$g(x) = exp(-cx) \tag{56}$$

We choose -c because this will give the range of g between 0 and ∞.

$$\begin{aligned} g(x \to -\infty) = 0 \\ g(x \to \infty) = \infty \end{aligned} \tag{57}$$

Substituting g(x) in equation (41) -

$$E - V(x) = \frac{c^2}{2} - \frac{3}{4}c^2 + \left[\frac{c^2}{2}exp(-cx)(2n+\alpha+1) + c^2\frac{1-\alpha^2}{4} - \frac{c^2exp(2cx)}{4}\right] \tag{58}$$

To remove the n dependence from the first term, a constant τ was considered such that -

$$n + \frac{\alpha+1}{2} = \tau \Rightarrow \alpha_n = 2\tau - 2n - 1 \tag{59}$$

Energy is related to the second term and the first two constant terms -

$$\begin{aligned} E &= \frac{c^2}{4}(1-\alpha^2) - \frac{c^2}{4} \\ &= -\frac{c^2}{4}(2\tau - 2n - 1)^2 \\ &\boxed{E = -c^2\left(\tau - n - \frac{1}{2}\right)^2} \end{aligned} \tag{60}$$

$$\boxed{V(x) = -c^2\tau * exp(-cx) + \frac{c^2}{4}exp(-2cx)} \tag{61}$$

This is the Morse potential.

Following similar steps as before, f(x) was calculated to be -

$$f(x) = \frac{-i}{\sqrt{c}}exp\left(\frac{-cx(2\tau-2n-1)}{2}\right)exp\left(-\frac{exp(-cx)}{2}\right) \tag{62}$$

For W(x) we use f(x) with n =0, i.e. with $\alpha_n = \alpha_0 = 2\tau - 1$

$$\begin{aligned} W(x) = -\frac{f'}{f} &= \frac{c(2\tau-1)}{2} - \frac{c * exp(-cx)}{2} \\ W'(x) &= \frac{c^2}{2}exp(-cx) \end{aligned} \tag{63}$$

$$\boxed{V_+(x) = c^2\frac{(2\tau-1)^2}{4} + \frac{c^2}{4}exp(-2cx) - c^2(\tau-1)exp(-cx)} \tag{64}$$

$$\boxed{V_-(x) = c^2\frac{(2\tau-1)^2}{4} + \frac{c^2}{4}exp(-2cx) - c^2(\tau)exp(-cx)} \tag{65}$$

4.3 Case III (Coulomb Potential)

Considering the last term of equation (41) to be the energy term, thus it should be a constant -

$$\mathbf{g'^2 = c^2} \tag{66}$$

$$g(x) = cx \tag{67}$$

here the constant of integration is 0 because g(x = 0) = 0.

$$E - V(x) = \left[\frac{c}{2x}(2n + \alpha + 1) + \frac{1 - \alpha^2}{4x^2} - \frac{c^2}{4}\right] \tag{68}$$

Removing n dependance from the first term -

$$\begin{gathered} c * \left(n + \frac{\alpha + 1}{2}\right) = \tau \\ c_n = \tau * \left(n + \frac{\alpha + 1}{2}\right)^{-1} \end{gathered} \tag{69}$$

$$\therefore E = -\frac{\tau^2}{(2n + \alpha + 1)^2} \tag{70}$$

$$V(x) = -\frac{\tau}{x} - \frac{1 - \alpha^2}{4x^2} \tag{71}$$

This is the Coulomb potential. We can use more conventional notations such that -
$\tau = e^2 \quad and \quad \alpha = 2l + 1$

$$\begin{gathered} \boxed{V(x) = -\frac{e^2}{x} + \frac{l(l+1)}{x^2}} \\ \boxed{E = -\frac{e^2}{4(n + l + 1)^2}} \end{gathered} \tag{72}$$

$$f(x) = (e^2(n + l + 1)^{-1})^{(\frac{2l+1}{2})} x^{(l+1)} exp\left(\frac{-e^2(n + l + 1)^{-1} x}{2}\right) \tag{73}$$

W(x) is calculated at n =0, therefore we can write the following using equation (19) -

$$W(x) = \frac{e^2}{2(l + 1)} - (l + 1)\frac{1}{x} \tag{74}$$

$$\boxed{V_{-}(x) = \frac{l(l+1)}{x^2} - \frac{e^2}{x} + \frac{e^4}{4(l+1)^2}}$$

$$\boxed{V_{+}(x) = \frac{(l+1)(l+2)}{x^2} - \frac{e^2}{x} + \frac{e^4}{4(l+1)^2}} \tag{75}$$

5 Unbroken and Broken Supersymmetry

In this section, a thorough study of a given solution of the Schrodinger's equation was made. The solution is -[5]

$$f = x^t \exp\left(\frac{1}{4}swx^2\right)\left(a + wx^2\right)^k \tag{76}$$

Following effects of the parameters can be observed -
If $s < 0$ then the function will tend to 0 as x tends to ∞. And at $x \to 0$, x^t will dominate the shape of the function. Therefore to get a function which tends to 0 at $x = 0$ and $x \to \infty$, we require $t > 0$ and $s < 0$. These set of conditions will lead to a physical solution (as discussed in the Introduction). Additionally, $t = 0$ is not allowed as it will lead to a finite value of $f(x = 0)$. Finally, k is an integer number.
If $s > 0$ then the function will tend to ∞ at $x \to \infty$. This cannot correspond to a physical solution.
If $t < 0$ then the function will tend to ∞ at $x \to 0$. This again cannot correspond to a physical solution.
Additionally SUSQM requires the function to be nodeless thus $a > 0$ should be satisfied.

From Schrodinger's equation we can say that -

$$\begin{aligned} E - V(x) &= -\frac{f''(x)}{f(x)} \\ &= \frac{2k(a\mathrm{sw} - 2kw - 2tw + w)}{a + wx^2} + \frac{4a(k-1)kw}{\left(a + wx^2\right)^2} - \frac{1}{2}\mathrm{sw}(4k + 2t + 1) - \frac{1}{4}\mathrm{sw}^2x^2 - \frac{(t-1)t}{x^2} \end{aligned} \tag{77}$$

Assuming that if the following is a wave function then it describes the ground state wave function. Therefore -

$$\begin{aligned} E_0 &= -\frac{1}{2}sw(4k + 2t + 1) \\ V(x) &= -\frac{2k(a\mathrm{sw} - 2kw - 2tw + w)}{a + wx^2} - \frac{4a(k-1)kw}{\left(a + wx^2\right)^2} + \frac{w^2x^2}{4} + \frac{(t-1)t}{x^2} \end{aligned} \tag{78}$$

where E_0 is the ground state energy
By using the concepts of SUSY, the Superpotential can be calculated in the following way -

$$W(x) = -\frac{f'}{f} = -\frac{2kwx}{a + wx^2} - \frac{\mathrm{sw}x}{2} - \frac{t}{x} \tag{79}$$

Using the W, partner potentials can be calculated -

$$
\begin{aligned}
V_- = W^2 - W' &= -\frac{2k(asw-2kw-2tw+w)}{a+wx^2} - \frac{4a(k-1)kw}{\left(a+wx^2\right)^2} + \frac{1}{2}sw(4k+2t+1) + \frac{(sw)^2x^2}{4} + \frac{(t-1)t}{x^2} \\
V_+ = W^2 + W' &= -\frac{2k(asw-2kw-2tw-w)}{a+wx^2} - \frac{4ak(k+1)w}{\left(a+wx^2\right)^2} + \frac{1}{2}sw(4k+2t-1) + \frac{(sw)^2x^2}{4} + \frac{t(t+1)}{x^2}
\end{aligned}
\tag{80}
$$

V_- is equal to $V(x)$ equation (78) with a constant shift of E_0.
In the following part of the thesis, SUSY partners of radial harmonic oscillator ($v(x,l)$)are explored. Thus parameters are chosen in such a way that we can reproduce $v(x,l)$ for either V_- or V_+ (wherever possible).

The above derived generalised will be now used in specific cases -

5.1 Case (I) : s = -1 and t > 0

$$
f = x^t \exp\left(-\frac{1}{4}wx^2\right)\left(a+wx^2\right)^k \tag{81}
$$

It can be seen that this function is node less if $a > 0$ and the function goes to 0 at x = 0 and x = ∞. Therefore the function can be considered as a physical wave function. From equation (78) -

$$
\begin{aligned}
E_0 &= \frac{1}{2}w(4k+2t+1) \\
V(x) &= -\frac{2k(-aw-2kw-2tw+w)}{a+wx^2} - \frac{4a(k-1)kw}{\left(a+wx^2\right)^2} + \frac{(w)^2x^2}{4} + \frac{(t-1)t}{x^2}
\end{aligned}
\tag{82}
$$

We can see that if k = 0, and t = l+1 then we get back the 3 dimensional oscillator potential -

$$
\begin{aligned}
V(x) &= \frac{l(l+1)}{x^2} + \frac{1}{4}wx^2 \\
&= v(x,l)
\end{aligned}
\tag{83}
$$

And

$$
E_0 = \frac{1}{2}w(2l+3) \tag{84}
$$

The super potential was calculated to be -

$$
W(x) = -\frac{2kwx}{a+wx^2} + \frac{\mathrm{w}x}{2} - \frac{t}{x} \tag{85}
$$

And finally partner potentials were found to be -

$$
\begin{aligned}
V_- &= -\frac{2k(-aw-2kw-2tw+w)}{a+wx^2} - \frac{4a(k-1)kw}{\left(a+wx^2\right)^2} - \frac{1}{2}w(4k+2t+1) + \frac{w^2x^2}{4} + \frac{(t-1)t}{x^2} \\
V_+ &= -\frac{2k(-aw-2kw-2tw-w)}{a+wx^2} - \frac{4ak(k+1)w}{\left(a+wx^2\right)^2} - \frac{1}{2}w(4k+2t-1) + \frac{w^2x^2}{4} + \frac{t(t+1)}{x^2}
\end{aligned}
\tag{86}
$$

To see the resemblance with the harmonic potential, the following assumptions were made - $k = 0$ and $t = l + 1$

It can be seen that V_- is exactly the potential calculated from the Schrodinger's equation and V_+ has the same form with some shift -

$$\begin{aligned} V_-(x) &= v(x,l) - \frac{1}{2}w(2l+3) = v(x,l) - E_0 \\ V_+(x) &= v(x,l+1) - \frac{1}{2}w(2l+1) = v(x,l+1) - E_0 + w \end{aligned} \tag{87}$$

The shift in the constant term (energy) is of w. We define H_1 and H_2 in the following way -

$$\begin{aligned} H_- &= -\frac{d^2}{dx^2} + V_-(x) \\ H_+ &= -\frac{d^2}{dx^2} + V_+(x) \end{aligned} \tag{88}$$

It can be seen, from equation (87) and (83), that the eigenfunction of H_-,ie $\psi_0^{(-)}(x)$,would be the same as the function f(x) defined in the beginning of the section.

$$\psi_0^{(-)}(x) = f(x) \tag{89}$$

$$\begin{aligned} H_-\psi_0^{(-)}(x) &= E_0^{(-)}\psi_0^{(-)}(x) \\ -\frac{d^2}{dx^2}\psi_0^{(-)}(x) + [v(x,l) - \frac{1}{2}w(2l+3)]\psi_0^{(-)}(x) &= E_0^{(-)}\psi_0^{(-)}(x) \end{aligned} \tag{90}$$

From equation (82), with assumptions that k = 0 and t = l+1, we get -

$$\begin{aligned} -\frac{d^2}{dx^2}\psi_0^{(-)}(x) &= [-v(x,l) + \frac{1}{2}w(2l+3)]\psi_0^{(-)}(x) \\ \therefore E_0^{(-)} &= \frac{1}{2}w(2l+3) - \frac{1}{2}w(2l+3) \\ &\therefore \boxed{E_0^{(-)} = 0} \end{aligned} \tag{91}$$

For H_+ -

It is clear that the eigenfunction of H_- will not be an eigenfunction of H_+ because $v(x,l) \neq v(x,l+1)$.But if t is replaced by t+1 in f(x) then, the new function could act as the eigenfunction of H_+

$$\psi_0^{(+)}(x) = x^{t+1} \exp\left(-\frac{1}{4}wx^2\right)\left(a + wx^2\right)^k \tag{92}$$

(s = -1 is applied in the above equation)

Writing the Schrodinger's equation -

$$\begin{aligned}H_+\psi_0^{(+)}(x) &= E_0^{(+)}\psi_0^{(+)}(x)\\ \frac{d^2}{dx^2}\psi_0^{(+)}(x)+[v(x,l+1)-\frac{1}{2}w(2l+1)]\psi_0^{(+)}(x) &= E_0^{(+)}\psi_0^{(+)}(x)\end{aligned} \tag{93}$$

By following similar steps as before it is clear that -

$$\begin{aligned}-\frac{d^2}{dx^2}\psi_0^{(+)}(x) &= [-v(x,l+1)+\frac{1}{2}w(2(l+1)+3)]\psi_0^{(+)}(x)\\ E_0^{(+)} &= \frac{1}{2}w(2(l+1)+3)-\frac{1}{2}w(2l+1)\\ \therefore &\boxed{E_0^{(+)}=2w}\end{aligned} \tag{94}$$

This case has zero energy for the ground state of V_- and positive energy for the ground state of V_+. Thus it is a case of **Unbroken Supersymmetry**

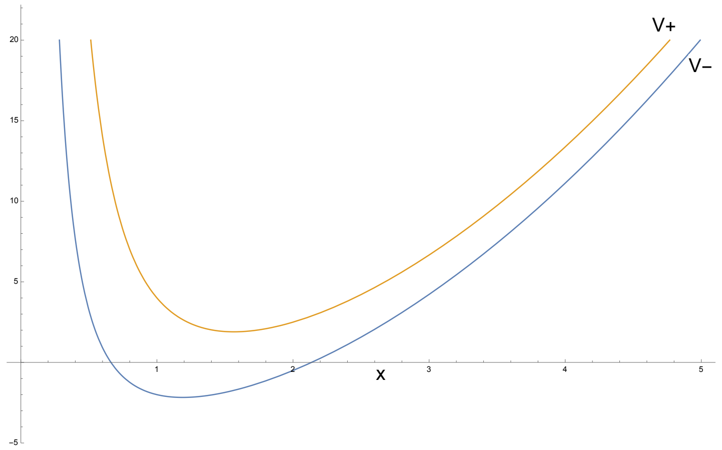

Figure 1: V_-(x) and V_+(x) from equation (87) defined for w = 2 and l =1.

The potential minima fall below the respective ground state energies, $E_0^{(-)}=0$ and $E_0^{(+)}=4$ obtained from (91) and (94).

5.1.1 Assuming "k = 1"

If we assume that k is not 0 but 1 then following changes can be seen -

$$\begin{aligned}E_0 &= \left(t+\frac{5}{2}\right)w\\ V(x) &= \frac{2w(a+2t+1)}{a+wx^2}+\frac{(t-1)t}{x^2}+\frac{w^2x^2}{4}\end{aligned} \tag{95}$$

To again get harmonic potential, we can put $t=l+1$ and $a=-2t-1$

But this would mean that $a<0$, as $t>0$ for every physical l. If $a<0$ then the solution will not be node less.

$$f\left(x=\sqrt{\frac{-a}{w}}\right)=0 \tag{96}$$

It can be seen from equation (77) that the potential will be singular at the node. Thus this case is not acceptable and harmonic potential cannot be constructed with "k = 1" .

Calculating Superpotential using the same assumptions -

$$W(x)=-\frac{2wx}{a+wx^2}+\frac{\mathrm{w}x}{2}-\frac{t}{x} \tag{97}$$

Finally calculating partner potentials -

$$V_-=\frac{2(aw+w+2tw)}{a+wx^2}-\frac{1}{2}w(5+2t)+\frac{w^2x^2}{4}+\frac{(t-1)t}{x^2} \tag{98}$$

$$V_+=\frac{2(aw+3w+2tw)}{a+wx^2}-\frac{8aw}{(a+wx^2)^2}-\frac{1}{2}w(3+2t)+\frac{w^2x^2}{4}+\frac{t(t+1)}{x^2} \tag{99}$$

To get harmonic potential for V_-, a should be $-2t-1$. This would cause V_+ to contain a singularity at a finite x value[eq (99)]. Thus Harmonic Oscillator cannot be realised in this case.

$$V_-=v(x,l)-\frac{1}{2}w(7+2l) \tag{100}$$

5.1.2 Assuming "k = -1"

Following the same steps as before -

$$\begin{gathered} E_0=\frac{1}{2}w(2t-3) \\ V(x)=\frac{2w(-a-2t+3)}{a+wx^2}-\frac{8aw}{(a+wx^2)^2}+\frac{(t-1)t}{x^2}+\frac{w^2x^2}{4} \end{gathered} \tag{101}$$

In this case $V(x)$ cannot be reduced to Harmonic potential for non zero values of a.

But to get a nodeless solution $a>0$ is required. Thus it is not possible to construct harmonic potential.

Calculating the superpotential -

$$W=\frac{2wx}{a+wx^2}-\frac{t}{x}+\frac{wx}{2} \tag{102}$$

Calculating partner potentials -

$$V_-=\frac{2(-aw+3w-2tw)}{a+wx^2}-\frac{8aw}{(a+wx^2)^2}-\frac{1}{2}w(2t-3)+\frac{w^2x^2}{4}+\frac{(t-1)t}{x^2} \tag{103}$$

$$V_+ = \frac{1}{2}w\left(-\frac{4(a+2t-1)}{a+wx^2} - 2t + 5\right) + \frac{t(t+1)}{x^2} + \frac{w^2x^2}{4} \tag{104}$$

In this case, V_+ can be made like a harmonic potential. Following assumptions should be made for that -

$$\begin{aligned} a &= -2t+1 \\ t &= l+1 \end{aligned} \tag{105}$$

This assumption of t would be valid for all values of l.

But it was argued before that "a" should be greater than 0. This is would imply that l should be negative which is not allowed. Therefore we cannot produce harmonic potential. Nevertheless it is preferred to write V_- and V_+ as a function of l instead of t.

$$\begin{aligned} V_- &= \frac{2w(-a+1-2l)}{a+wx^2} - \frac{8aw}{(a+wx^2)^2} - \frac{1}{2}w(2l-1) + \frac{w^2x^2}{4} + \frac{(l+1)l}{x^2} \\ V_+ &= -\frac{2w(a+2l+1)}{a+wx^2} + \frac{1}{2}w(-2l+3) + \frac{(l+1)(l+2)}{x^2} + \frac{w^2x^2}{4} \end{aligned} \tag{106}$$

Here V_+ can be made into harmonic potential but it will not be a meaningful solution as a must be negative for that. Just to get rid of a dependance from V_+, we assume $a = -2l-1$. However this will lead to a singular term in $V_-(x)$ as discussed before

$$V_+ = v(x, l+1) + \frac{1}{2}w(-2l+3) \tag{107}$$

5.2 Case (II) : s = 1, t > 0

$$f = x^t \exp\left(\frac{wx^2}{4}\right)\left(a+wx^2\right)^k \tag{108}$$

It can be seen that the function goes to infinity when $x = \infty$ as the exponential is positive and at $x = 0$ the function equals to 0 as $t > 0$.

Following similar steps as before -

$$\begin{aligned} E &= -\frac{1}{2}w(4k+2t+1) \\ V(x) &= -\frac{2k(aw-2kw-2tw+w)}{a+wx^2} - \frac{4a(k-1)kw}{(a+wx^2)^2} + \frac{(w)^2x^2}{4} + \frac{(t-1)t}{x^2} \end{aligned} \tag{109}$$

Putting "k = 0" and "t = l+1 " to get the harmonic potential. Note that for all physical values of l, $t > 0$ is satisfied.

$$
\begin{gathered}
E = -\frac{1}{2}w(2l+3) \\
V(x) = \frac{l(l+1)}{x^2} + \frac{w^2x^2}{4} = v(x,l)
\end{gathered}
\tag{110}
$$

Superpotential is -

$$
W(x) = -\frac{2kwx}{a+wx^2} - \frac{wx}{2} - \frac{t}{x} \tag{111}
$$

And corresponding partner potentials are -

$$
V_-(x) = -\frac{2k(aw-2kw-2tw+w)}{a+wx^2} - \frac{4a(k-1)kw}{(a+wx^2)^2} + \frac{1}{2}w(4k+2t+1) + \frac{w^2x^2}{4} + \frac{(t-1)t}{x^2} \tag{112}
$$

$$
V_+(x) = -\frac{2k(aw-2kw-2tw-w)}{a+wx^2} - \frac{4ak(k+1)w}{(a+wx^2)^2} + \frac{1}{2}w(4k+2t-1) + \frac{w^2x^2}{4} + \frac{t(t+1)}{x^2} \tag{113}
$$

Again putting k = 0 and t = l+1; (note that $t > 0$ is satisfied by all physical values of l)

$$
\begin{gathered}
V_-(x) = v(x,l) + \frac{1}{2}w(2l+3) = v(x,l) - E \\
V_+(x) = v(x,l+1) + \frac{1}{2}w(2l+1) = v(x,l+1) - E - w
\end{gathered}
\tag{114}
$$

Following similar steps as the previous case -

$$
\begin{aligned}
H_-\psi_0^{(-)}(x) &= E_0^{(-)}\psi_0^{(-)}(x) \\
-\frac{d^2}{dx^2}\psi_0^{(-)}(x) + [v(x,l) + \frac{1}{2}w(2l+3)]\psi_0^{(-)}(x) &= E_0^{(-)}\psi_0^{(-)}(x)
\end{aligned}
\tag{115}
$$

From equation (110) -

$$
\begin{aligned}
-\frac{d^2}{dx^2}\psi_0^{(-)}(x) = [-v(x,l) + \frac{1}{2}w(2l+3)]\psi_0^{(-)}(x) \\
\therefore \boxed{E_0^{(-)} = w(2l+3)}
\end{aligned}
\tag{116}
$$

For H_+ -

Using the similar trick as before and defining $\psi_0^{(+)}(x)$ as -

$$
\begin{aligned}
H_+\psi_0^{(+)}(x) &= E_0^{(+)}\psi_0^{(+)}(x) \\
-\frac{d^2}{dx^2}\psi_0^{(+)}(x) + [v(x,l+1) + \frac{1}{2}w(2l+1)]\psi_0^{(+)}(x) &= E_0^{(+)}\psi_0^{(+)}(x)
\end{aligned}
\tag{117}
$$

$$-\frac{d^2}{dx^2}\psi_0^{(+)}(x) = [-v(x, l+1) + \frac{1}{2}w(2(l+1)+3)]\psi_0^{(-)}(x)$$
$$\therefore \boxed{E_0^{(+)} = w(2l+3)} \tag{118}$$

It can be seen that energy eigenvalues for H_1 and H_2 are the same and are greater than the original E_0(which was the ground state energy of potential V(x)). This says that the solution ψ_0 will be nodeless and will have same value for the entire range $-\infty < x < \infty$. The eigenvalues have non zero value, unlike case (I), they do not exhibit same symmetry as the hamiltonian.

This is the case of **Broken Supersymmetry**.

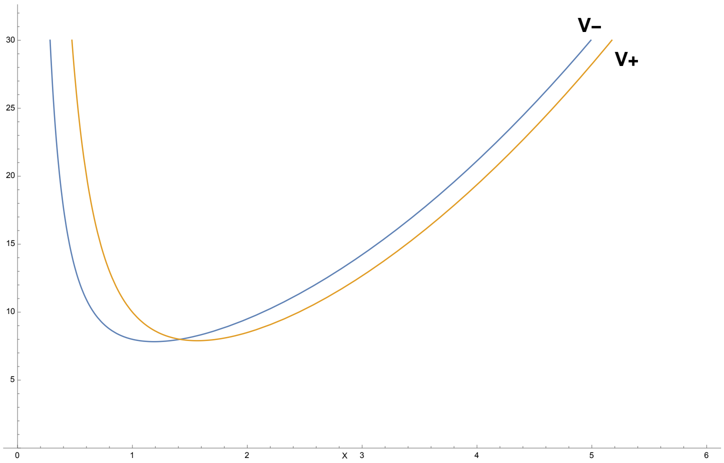

Figure 2: V_-(x) and V_+ (x) from equation (114) defined at $w = 2$ and $l = 1$

Using $w = 2$ and $l = 1$, the value of energies can be calculated. $E_0^{(-)} = E_0^{(+)} = 10$. Thus it can be observed from the graph that these values of energies is more than the minima of both potentials (as expected).

5.2.1 Assuming "k = 1"

$$E = -\frac{1}{2}(2t+5)w$$
$$V(x) = -\frac{2w(a-2t-1)}{a+wx^2} + \frac{(t-1)t}{x^2} + \frac{1}{4}w^2x^2 \tag{119}$$

To get harmonic potential, assuming that - " a = 2t + 1, t = l+1"

$$E = -\frac{1}{2}(2l+7)w$$
$$V(x) = v(x, l) \tag{120}$$

Superpotential was calculated using the same assumptions-

$$W(x) = \frac{1}{2}wx\left(-\frac{4}{2l+wx^2+3} - 1\right) - \frac{l+1}{x} \tag{121}$$

Partner potentials were found to be -

$$\begin{aligned} V_{-}(x) &= (7/2+l)w + (l(1+l))/x^2 + (w^2x^2)/4 \\ &= v(x,l) - E \end{aligned} \tag{122}$$

$$\begin{aligned} V_{+}(x) = \frac{4w}{2l+wx^2+3} - \frac{8(2l+3)w}{(2l+wx^2+3)^2} + \left(l+\frac{5}{2}\right)w + \frac{(l+1)(l+2)}{x^2} + \frac{w^2x^2}{4} \\ = v(x,l+1) + \frac{4w}{2l+wx^2+3} - \frac{8(2l+3)w}{(2l+wx^2+3)^2} - E - w \end{aligned} \tag{123}$$

It can be observed that the V_{+} potential is different from harmonic potential $v(x,l)$.

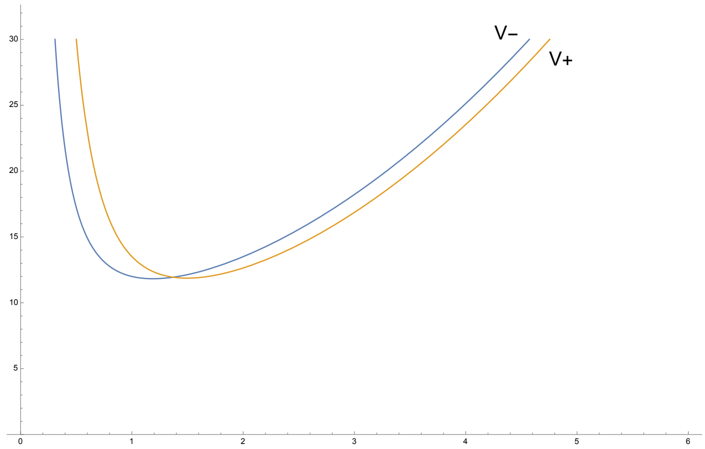

Figure 3: $V_{-}(x)$ and $V_{+}(x)$ from equation (122) and (123) defined at $w=2,\ a=5,\ l=1$

5.2.2 Assuming "k = -1"

Following similar steps as before -

$$\begin{aligned} E &= \frac{1}{2}(2t-3)w \\ V(x) = \frac{2w(a-2t+3)}{a+wx^2} - \frac{8aw}{(a+wx^2)^2} &+ \frac{(t-1)t}{x^2} + \frac{w^2x^2}{4} \end{aligned} \tag{124}$$

It can be seen that harmonic oscillator potential cannot be constructed for non zero values of a.

Calculating Super potential -

$$W(x) = \frac{2wx}{a+wx^2} - \frac{l+1}{x} - \frac{wx}{2} \tag{125}$$

Calculating partner potentials -

$$V_{-} = \frac{2w(a-2l+1)}{a+wx^2} - \frac{8aw}{(a+wx^2)^2} + \frac{1}{2}w(2l-1) + \frac{w^2x^2}{4} + \frac{(l+1)}{x^2} \tag{126}$$

$$V_+ = \frac{2w(a-2l-1)}{a+wx^2} + \frac{1}{2}w\,(2l-3) + \frac{(l+1)(l+2)}{x^2} + \frac{w^2x^2}{4} \tag{127}$$

To get harmonic potential for V_+, following assumptions should be made -

$$a = 2l+1 \tag{128}$$

The condition $a > 0$ is satisfied by all the physical values of l.

V_+ then becomes -

$$\begin{gathered} V_+ = w\left(l-\frac{3}{2}\right) + \frac{(l+2)(l+1)}{x^2} + \frac{w^2x^2}{4} \\ V_+ = v(x,l+1) + w\left(l-\frac{3}{2}\right) \end{gathered} \tag{129}$$

Calculating V_- using the same assumptions -

$$\begin{aligned} V_- &= \frac{4w}{2l+wx^2+1} - \frac{8(1+2l)w}{(2l+wx^2-1)^2} + \left(l-\frac{1}{2}\right)w + \frac{(l+1)l}{x^2} + \frac{w^2x^2}{4} \\ &= w\left(l-\frac{1}{2}\right) + v(x,l) + \frac{4w}{2l+wx^2+1} - \frac{8(2l+1)w}{(2l+wx^2+1)^2} \end{aligned} \tag{130}$$

5.3 Case (III) : s = -1, t< 0

$$f(x) = x^t \exp\left(\frac{-wx^2}{4}\right)\left(a+wx^2\right)^k \tag{131}$$

It is evident that at origin, the function will go to ∞ because $t < 0$ and at $x = \infty$ the function will tend to 0 as the exponential term is negative.

Following similar steps, -

$$\begin{gathered} E = \frac{1}{2}w(4k+2t+1) \\ V(x) = -\frac{2k(-aw-2kw-2tw+w)}{a+wx^2} - \frac{4a(k-1)kw}{(a+wx^2)^2} + + \frac{(w)^2x^2}{4} + \frac{(t-1)t}{x^2} \end{gathered} \tag{132}$$

This is exactly as the first case, except that t is less than 0.
Let t = -m, where m is a positive real number. energy and potential becomes -

$$E = \frac{1}{2}w(4k - 2m + 1)$$
$$V(x) = \frac{2kw(a + 2k - 2m - 1)}{a + wx^2} - \frac{4a(k-1)kw}{(a + wx^2)^2} + \frac{(m+1)m}{x^2} + \frac{1}{4}w^2x^2 \tag{133}$$

Taking the following assumptions - "k = 0 and m = l+1". This assures $t < 0$ for all values of l.

$$E = -\frac{1}{2}w(2l + 1)$$
$$V(x) = \frac{(l+1)(l+1)}{x^2} + \frac{1}{4}w^2x^2 = v(x, l+1) \tag{134}$$

The assumptions made above give the harmonic potential.

Superpotential is calculated to be -

$$W = -\frac{2kwx}{a + wx^2} + \frac{wx}{2} + \frac{l+1}{x} \tag{135}$$

$$V_- = -\frac{2k(-aw - 2kw + 2lw + 3w)}{a + wx^2} - \frac{4a(k-1)kw}{(a + wx^2)^2} - \frac{1}{2}w(4k - 2l - 1) + \frac{(w)^2x^2}{4} + \frac{(l+1)(l+2)}{x^2}$$
$$V_+ = -\frac{2k(-aw - 2kw + 2lw + w)}{a + wx^2} - \frac{4ak(k+1)w}{(a + wx^2)^2} - \frac{1}{2}w(4k - 2l - 3) + \frac{(w)^2x^2}{4} + \frac{l(l+1)}{x^2} \tag{136}$$

Putting k = 0;

$$V_- = v(x, l+1) + \frac{1}{2}w(2l + 1) = v(x, l+1) - E$$
$$V_+ = v(x, l) + \frac{1}{2}w(2l + 3) = v(x, l) - E + w \tag{137}$$

Following similar steps as the cases above -
(Note - In this case as V_- has potential of form $v(x, l+1)$, $\psi_0^{(-)}(x)$ should be considered

$$H_-\psi_0^{(-)}(x) = E_0^{(-)}\psi_0^{(-)}(x)$$
$$-\frac{d^2}{dx^2}\psi_0^{(-)}(x) + [v(x, l+1) + \frac{1}{2}w(2l + 1)]\psi_0^{(-)}(x) = E_0^{(-)}\psi_0^{(-)}(x) \tag{138}$$

Using equation (134) and following similar steps as before -

$$\boxed{E_0^{(-)} = w(2l + 3)} \tag{139}$$

For H_+ -

$\psi_0^{(+)}(x)$ is defined as equation (92) -

$$H_+\psi_0^{(+)}(x) = E_0^{(+)}\psi_0^{(+)}(x) - \frac{d^2}{dx^2}\psi_0^{(+)}(x) + [v(x,l) + \frac{1}{2}w(2l+3)]\psi_0^{(-)}(x) = E_0^{(-)}\psi_0^{(-)}(x) \tag{140}$$

Using equation (134) but substituting l with $l+1$ and following similar steps -

$$\boxed{E_0^{(+)} = w(2l+3)} \tag{141}$$

Using similar reasoning as at the end of Case 2. It can be said that this case is also an example of **Broken Supersymmetry**.

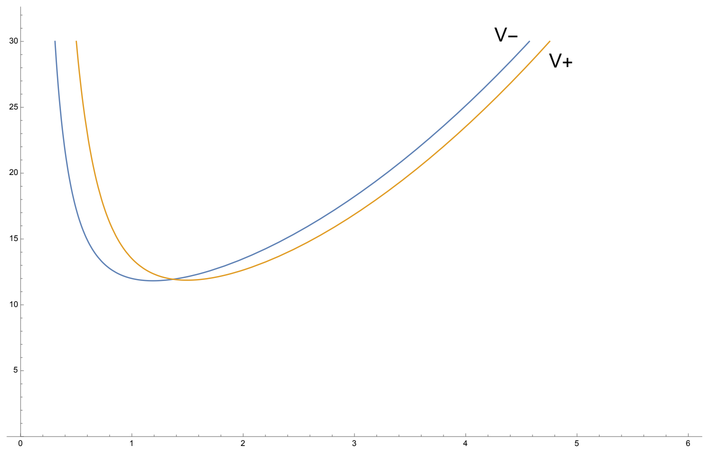

Figure 4: $V_-(x)$ and $V_+(x)$ from equation (137) defined for $l = 2,\ w = 2$

The common ground state energy with these parameters is 14. And it can be seen from the graph that the minima of potentials fall below this value, as expected.

5.3.1 Assuming "k = 1"

Following previous steps, -

$$\begin{gathered} E = \frac{1}{2}w(5+2t) \\ V(x) = \frac{-2w(a+2t+1)}{a+wx^2} + \frac{(t-1)t}{x^2} + \frac{w^2x^2}{4} \end{gathered} \tag{142}$$

Assuming "$t = -l-1, a = -2t-1$" to get harmonic potential -

$$\begin{gathered} E = \frac{1}{2}w(3-2l) \\ V(x) = \frac{(l+1)(l+2)}{x^2} + \frac{w^2x^2}{4} = v(x,l+1) \end{gathered} \tag{143}$$

Finding superpotential using the same assumptions -

$$W = -\frac{2\text{wx}}{2l + \text{wx}^2 + 1} + \frac{l+1}{x} + \frac{\text{wx}}{2} \tag{144}$$

Calculating partner potentials -

$$\begin{aligned} V_- &= \left(l - \frac{3}{2}\right) w + \frac{(l+2)(l+1)}{x^2} + \frac{w^2 x^2}{4} \\ &= v(x, l+1) - E \end{aligned} \tag{145}$$

$$\begin{aligned} V_+ = \frac{4w}{2l + wx^2 + 1} - \frac{8(2l+1)w}{(2l + wx^2 + 1)^2} + \left(l - \frac{1}{2}\right) w + \frac{(l+1)l}{x^2} + \frac{w^2 x^2}{4} \\ = v(x, l) + \frac{4w}{2l + wx^2 + 1} - \frac{8(2l+1)w}{(2l + wx^2 + 1)^2} - E + w \end{aligned} \tag{146}$$

It can be easily seen that the potentials obtained in this case are inverse of potentials obtained in Case (II) with k = -1.

Inverse transformation has following effects -

$$\begin{aligned} V_- &\to V_+ \\ V_+ &\to V_- \\ W &\to -W \end{aligned} \tag{147}$$

5.3.2 Assuming "k = -1"

By following similar steps - As $t < 0$, $t = -m$ will be used, where $m > 0$

$$\begin{aligned} E = \frac{1}{2}(-2m - 3)w \\ V(x) = \frac{2w(-a + 2m + 3)}{a + wx^2} - \frac{8aw}{(a + wx^2)^2} - \frac{(-m-1)m}{x^2} + \frac{w^2 x^2}{4} \end{aligned} \tag{148}$$

Harmonic Oscillator potential cannot be realised for non zero values of a.

Calculating Super potential -

$$W(x) = \frac{2wx}{a + wx^2} + \frac{m}{x} + \frac{wx}{2} \tag{149}$$

Calculating partner potentials -

$$V_- = \frac{2w(-a + 3 + 2m)}{a + wx^2} - \frac{8aw}{(a + wx^2)^2} + \frac{1}{2}w(2m+3) + \frac{w^2 x^2}{4} + \frac{(m+1)m}{x^2} \tag{150}$$

$$V_+ = -\frac{2w(a - 2m - 1)}{a + wx^2} + \left(m + \frac{5}{2}\right) w + \frac{(m-1)m}{x^2} + \frac{w^2 x^2}{4} \tag{151}$$

It is possible to get harmonic potential for V_+. Following assumptions are needed -

$$\begin{aligned} a &= 2m+1 \\ m &= l+1 \end{aligned} \tag{152}$$

These assumptions are valid for all values of l. After applying these, V_+ would become -

$$\begin{aligned} V_+ = \left(l+\frac{7}{2}\right)w + \frac{l(l+1)}{x^2} + \frac{w^2x^2}{4} \\ = v(x,l) + \left(l+\frac{7}{2}\right)w \end{aligned} \tag{153}$$

And V_- would become -

$$\begin{aligned} V_- = \frac{4w}{2l+wx^2+3} - \frac{8(2l+3)w}{(2l+wx^2+3)^2} + \left(l+\frac{5}{2}\right)w + \frac{(l+2)(l+1)}{x^2} + \frac{w^2x^2}{4} \\ = \frac{4w}{2l+wx^2+3} - \frac{8(2l+3)w}{(2l+wx^2+3)^2} + \left(l+\frac{5}{2}\right)w + v(x,l+1) \end{aligned} \tag{154}$$

This case can be considered as the inverse of Case II (k=1). As the partner potentials are inverse of each other.

5.4 Case (IV) : s = +1, t<0)

$$f(x) = x^t \exp\left(\frac{wx^2}{4}\right)\left(a+wx^2\right)^k \tag{155}$$

In this case, the function goes to ∞ at both $x=0$ and $x=\infty$.

Following similar steps,

$$\begin{aligned} E = -\frac{1}{2}w(4k+2t+1) \\ V(x) = -\frac{2k(aw-2kw-2tw+w)}{a+wx^2} - \frac{4a(k-1)kw}{(a+wx^2)^2} + \frac{w^2x^2}{4} + \frac{(t-1)t}{x^2} \end{aligned} \tag{156}$$

Letting $t=-l-1$, where $l \geq 0$ and assuming k = 0;

$$\begin{aligned} E = \frac{1}{2}w(2l+1) \\ V(x) = \frac{(l+1)(l+2)}{x^2} + \frac{w^2x^2}{4} = v(x,l+1) \end{aligned} \tag{157}$$

Superpotential was calculated to be -

$$W(x) = -\frac{2kwx}{a + wx^2} - \frac{wx}{2} + \frac{l+1}{x} \tag{158}$$

$$\begin{aligned} V_- &= -\frac{2k(aw - 2kw + 2lw + 3w)}{a + wx^2} - \frac{4a(k-1)kw}{(a + wx^2)^2} + \frac{1}{2}w(4k - 2l - 1) + \frac{w^2x^2}{4} + \frac{(l+1)(l+2)}{x^2} \\ V_+ &= -\frac{2k(aw - 2kw + 2lw + w)}{a + wx^2} - \frac{4ak(k+1)w}{(a + wx^2)^2} + \frac{1}{2}w(4k - 2l - 3) + \frac{w^2x^2}{4} + \frac{l(l+1)}{x^2} \end{aligned} \tag{159}$$

Putting k = 0;

$$\begin{aligned} V_- &= \frac{1}{2}w(-2l - 1) + v(x, l) = v(x, l+1) - E \\ V_+ &= \frac{1}{2}w(-2l - 3) + v(x, l-1) = v(x, l) - E - w \end{aligned} \tag{160}$$

Following similar steps as before -

$$\begin{aligned} H_-\psi_0^{(-)}(x) &= E_0^{(-)}\psi_0^{(-)}(x) \\ -\frac{d^2}{dx^2}\psi_0^{(-)}(x) + [v(x, l+1) - \frac{1}{2}w(2l+1)]\psi_0^{(-)}(x) &= E_0^{(-)}\psi_0^{(-)}(x) \end{aligned} \tag{161}$$

Using equation (157) and following similar steps as before -

$$\boxed{E_0^{(-)} = 2w} \tag{162}$$

For H_+ -

$$\begin{aligned} H_+\psi_0^{(+)}(x) &= E_0^{(+)}\psi_0^{(+)}(x) \\ -\frac{d^2}{dx^2}\psi_0^{(+)}(x) + [v(x, l) - \frac{1}{2}w(2l+1)]\psi_0^{(+)}(x) &= E_0^{(+)}\psi_0^{(+)}(x) \end{aligned} \tag{163}$$

Using equation (157) and subtituting l with $l + 1$ -

$$-\frac{d^2}{dx^2}\psi_0^{(+)}(x) = (v(x, l) + \frac{1}{2}w(2l+3))\psi_0^{(+)}(x) \tag{164}$$

Therefore -

$$\boxed{E_0^{(+)} = 0} \tag{165}$$

In this case an extra ground state is added to the spectrum of $V_+(x)$. Case (IV) can be considered as the inverse of Case (I). It can be observed that in Case (I) V_- has ground state level with zero energy and V_+ has positive eigenvalue. While in Case (IV), V_+ has ground state with zero energy (eigenvalue) and V_- has positive eigenvalue.

5.4.1 Assuming "k =1 "

By following similar calculations as above -

$$
\begin{gathered}
E=-\frac{1}{2}(2t+5) \\
V(x)=\frac{2(\mathrm{aw}-2tw-w)}{a+wx^2}+\frac{(t-1)t}{x^2}+\frac{w^2x^2}{4}
\end{gathered}
\tag{166}
$$

Assuming "a = 2t + 1, t = - l-1" to get harmonic potential -

$$
\begin{gathered}
E=\frac{1}{2}(2l-3)w \\
V=\frac{w^2x^2}{4}+\frac{(l+1)(l+2)}{x^2}=v(x,l)
\end{gathered}
\tag{167}
$$

But it can be argued that the assumptions would require $a<0$, for all the cases of l, thus it is not possible to construct Harmonic Oscillator for V(x) in this case.

Calculating a general Superpotential with m = $l+1$ -

$$
W(x)=\frac{1}{2}wx\left(-\frac{4}{a+wx^2}-1\right)+\frac{l+1}{x}
\tag{168}
$$

Calculating partner potentials -

$$
\begin{aligned}
V_-&=\frac{1}{2}w\left(-\frac{4(a+2l+1)}{a+wx^2}\right)+\frac{1}{2}w\left(-2l+3\right)+\frac{(l+2)(l+1)}{x^2}+\frac{w^2x^2}{4} \\
&=v(x,l+1)+\frac{1}{2}w\left(-\frac{4(a+2l+1)}{a+wx^2}\right)+\frac{1}{2}w\left(-2l+3\right)
\end{aligned}
\tag{169}
$$

To get rid of a dependance, we put $a=-2l-1$. But this will cause V_+ to contain singularity for a finite x, as discussed before -

$$
V_-=v(x,l+1)+\frac{1}{2}w\left(-2l+3\right)
\tag{170}
$$

V_+ is a singular potential due to the reasons discussed before.

It can be observed that this case is inverse of Case 1, k = -1. Because V_+ of equation (107) is exactly the same as V_- that is calculated above equation (170). And in both the cases, the corresponding partner potential was singular.

5.4.2 Assuming "k = - 1 "

Following previous steps and assuming $t = -m$, where $m > 0$ -

$$\begin{aligned} E &= \frac{1}{2}(-2m-3)w \\ V &= \frac{2w(a+2m+3)}{a+wx^2} - \frac{8aw}{(a+wx^2)^2} - \frac{(-m-1)m}{x^2} + \frac{w^2x^2}{4} \end{aligned} \tag{171}$$

To get a harmonic potential, following assumptions should be made -

$$\begin{aligned} a &= 0 \\ m &= -\frac{3}{2} \end{aligned} \tag{172}$$

This is not possible as m and a should be greater than 0 . Therefore harmonic potential cannot be realised for V(x).

Calculating superpotential -

$$W(x) = \frac{2wx}{a+wx^2} + \frac{m}{x} - \frac{wx}{2} \tag{173}$$

Calculating partner potentials -

$$V_- = \frac{2w(2m+3+a)}{a+wx^2} - \frac{8aw}{(a+wx^2)^2} - \frac{1}{2}w(2m+3) + \frac{m(m+1)}{x^2} + \frac{w^2x^2}{4} \tag{174}$$

$$V_+ = \frac{2w(a+2m+1)}{a+wx^2} - \frac{1}{2}w(2m+5) + \frac{(m-1)m}{x^2} + \frac{w^2x^2}{4} \tag{175}$$

By putting $m = l+1$ and $a = -2m-1$ -

$$V_+ = v(x,l) - \frac{1}{2}w(2l+7) \tag{176}$$

In this case V_- is a singular potential.

it can be seen that these results are inverse transformations of Case (I) when k = 1, ie

$$\begin{aligned} V_- &\to V_+ \\ V_+ &\to V_- \\ W &\to -W \end{aligned} \tag{177}$$

6 Summary

In this section I will give a brief summary of the supersymmetric transformations and their effects on the potential. The main focus would be inverse transformation.
Following notations will be used in the upcoming part of the document -

$$v(x,l,\delta) = v(x,l) + \delta \tag{178}$$

δ is the constant shift which was observed in almost all the cases. Its value however would vary on the chosen parameters.
Another type of potential which was observed will be referred to in the following way

$$\hat{v}(x,l,\delta) = v(x,l,\delta) + \frac{4w}{2l+1+wx^2} - \frac{8w(2l+1)}{(2l+1+wx^2)^2} \tag{179}$$

Collecting all partner potentials in a tabular form -
(It should be noted that δ is not same for all the potentials, unless mentioned. It is merely used to show that the potentials are of the same type).

CASE I

Assumption	V_-	V_+
k = 0	$v\left(x,l,-\frac{1}{2}w(2l+3)\right)$ eq(87)	$v(x,l+1,-\frac{1}{2}w(2l+1))$ eq(87)
k = 1	$v\left(x,l,-\frac{1}{2}w(7+2l)\right)$ eq(100)	singular potential eq(100)
k = -1	singular potential eq (106)	$v\left(x,l+1,\frac{1}{2}w(-2l+3)\right)$ eq(107)

CASE II

Assumption	V_-	V_+
k = 0	$v\left(x,l,\frac{1}{2}w(2l+3)\right)$(114)	$v\left(x,l+1,\frac{1}{2}w(2l+1)\right)$ (112)
k = 1	$v\left(x,l,\frac{1}{2}w(2l+7)\right)$ (122)	$\hat{v}\left(x,l+1,\frac{1}{2}w(2l+5)\right)$ (123)
k = -1	$\hat{v}\left(x,l,\frac{1}{2}w(2l-1)\right)$ (130)	$v\left(x,l+1,\frac{1}{2}w(2l-3)\right)$ (129)

CASE III

Assumption	V_-	V_+
k = 0	$v\left(x,l+1,\frac{1}{2}w(2l+1)\right)$ (137)	$v\left(x,l,\frac{1}{2}w(2l+3)\right)$ (137)
k = 1	$v\left(x,l+1,\frac{1}{2}w(2l-3)\right)$ (145)	$\hat{v}\left(x,l,\frac{1}{2}w(2l-1)\right)$ (146)
k = -1	$\hat{v}\left(x,l+1,\frac{1}{2}(2l+5)\right)$ (155)	$v\left(x,l,\frac{1}{2}w(2l+7)\right)$ (154)

	Assumption	V_-	V_+
CASE IV	k = 0	$v\left(x, l+1, -\frac{1}{2}w(2l+1)\right)$ (161)	$v\left(x, l, -\frac{1}{2}w(2l+3)\right)$ (161)
	k = 1	$v\left(x, l+1, \frac{1}{2}w(-2l+3)\right)$ (170)	singular potential
	k = -1	singular potential	$v\left(x, l, -\frac{1}{2}w(2l+7)\right)$ (176)

The above tables will help to recognise inverse transformations -

CASE I (k=0) $\longleftrightarrow$ **Case IV (k=0)**	Potential
$V_- \longleftrightarrow V_+$	$v\left(x, l, -\frac{1}{2}w(2l+3)\right)$
$V_+ \longleftrightarrow V_-$	$v(x, l+1, -\frac{1}{2}w(2l+1))$

CASE I (k=1) $\longleftrightarrow$ **Case IV (k=-1)**	Potential
$V_- \longleftrightarrow V_+$	$v\left(x, l, -\frac{1}{2}w(7+2l)\right)$
$V_+ \longleftrightarrow V_-$	singular potential

CASE I (k=-1) $\longleftrightarrow$ **Case IV (k=1)**	Potential
$V_- \longleftrightarrow V_+$	singular potential
$V_+ \longleftrightarrow V_-$	$v\left(x, l+1, \frac{1}{2}w(-2l+3)\right)$

CASE II (k=1) $\longleftrightarrow$ **Case III (k=-1)**	Potential
$V_- \longleftrightarrow V_+$	$v\left(x, l, \frac{1}{2}w(2l+7)\right)$
$V_+ \longleftrightarrow V_-$	$\hat{v}\left(x, l+1, \frac{1}{2}w(2l+5)\right)$

CASE II (k=-1) $\longleftrightarrow$ **Case III (k=1)**	Potential
$V_- \longleftrightarrow V_+$	$\hat{v}\left(x, l, \frac{1}{2}w(2l-1)\right)$
$V_+ \longleftrightarrow V_-$	$v\left(x, l+1, \frac{1}{2}w(2l-3)\right)$

CASE II (k=0) $\longleftrightarrow$ **Case III (k=0)**	Potential
$V_- \longleftrightarrow V_+$	$v\left(x, l, \frac{1}{2}w(2l+3)\right)$
$V_+ \longleftrightarrow V_-$	$v\left(x, l+1, \frac{1}{2}w(2l+1)\right)$

Inverse Transformations for all the cases is presented in the tables above.

Inverse transformations are obtained by taking $\frac{1}{f}$ instead of f. We can make this transformation by changing the parameters in the following way -

$$\begin{aligned} s &\to -s \\ t &\to -t \\ k &\to -k \end{aligned} \tag{180}$$

It can be verified that parameters of cases which are inverse to each other follow the transformation laws from equation (180)
This transformation changes $W(x) \to -W(X)$. This can be readily checked from equation (19). This change in $W(x)$ causes SUSY operator A and $A^\dagger$ to transform into each other (equation (3)). Therefore H_- and H_+, ie V_- and V_+ are also interchanged.

$$\begin{aligned} A \to -A^\dagger \quad & A^\dagger \to -A \\ H_- \to H_+ \quad & H_+ \to H_- \end{aligned} \tag{181}$$

However super algebra constructed in equation (10) will be conserved after inverse transformation. Because $Q \to -(Q^\dagger)^T$, $Q^\dagger \to -Q^T$ and $H \to H^T$. Thus the commutation and anti commutation relations will still hold. Which implies that inverse transformation leaves the underlying SUSY invariant.

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