PHYS 390: Very Low Frequency (VLF) Electromagnetic Wave Phenomena in Near-Earth Space - End of Term Report

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The whistler wave phenomena has been studied for more than a century now, and a large body of literature exists on the topic. Being a very low frequency (VLF) electromagnetic wave generated by ordinary lightning discharges, a substantial portion of the whistler waves travel into the ionosphere where they interact with free electrons and ions. These wave-particle interactions in the ionospheric plasma occur under the effect of the magnetic field of the Earth, allowing researchers to understand the structure of the near-Earth space. This end of term report aims to provide the fundamental related concepts by explaining the wave propagation conditions in the magnetized and unmagnetized ionospheric plasma. It is a modest product of a one-term research under the supervision of Professor Umran S. Inan and the assistance of Senior Research Engineer David S. Lauben.

INTRODUCTION

Earth's atmosphere is consisted of many layers. The lower altitudes do not contain significant amount of free electrons and ions. On the other hand, the densities of ions and free electrons rise as the altitude is increased. Around 100 km (at the end of mesosphere and the beginning of thermosphere) ionosphere begins. The ionosphere is also divided into layers. F layer (around 200 km) is where the density of electrons has its largest value. The reason why there are more free electrons and ions in the ionosphere than other layers is mostly due to the ionization effect of the ultraviolet solar radiation. Consequently, the density of electrons and ions in ionosphere highly depend on the solar activity. Further solar activity increases ionization, such that the densities of electrons and ions in the layers of the ionosphere change from day to night.

As a consequence of continuous and in high quantity ionization process, ionosphere is consisted of gas of ions and free electrons, namely *plasma*. While investigating the interactions in the ionosphere, it is modeled as an (overall) neutral plasma and equations regarding plasma physics are used. Specifically, the wave propagation inside the ionospheric plasma is of interest of this paper. Because of Earth's constant magnetic field around itself, the effect of the magnetic fields must also be accounted. In the following sections, the equations for the wave propagation in the ionosphere is explained. For simplicity, the static magnetic field due to Earth is neglected first. Afterwards, a more detailed analysis by considering Earth's magnetic field is provided.

BASIS

Before investigating wave-particle interactions in the ionosphere, assumptions for the analysis must be clarified and some fundamental equations for electromagnetic waves and plasma physics must be provided.

Waves in this analysis are purely sinusoidal plane waves, and oscillations are *small.* This allows us to linearize fluid equations, so that higher order terms can be neglected. Medium (ionosphere) is unbounded, homogeneous, and time-independent. "Unbounded" assumption is valid since ionosphere is a large layer where waves can be assumed to be coming from infinity and going towards infinity. "Homogenous" and "time-independent" assumptions are valid for short intervals of analysis, which is the case.

$$\vec{J}(\vec{r},t) = \sum_i q_i N_i(\vec{r},t)\vec{u}_i(\vec{r},t) \tag{1}$$

$\vec{J}$ in the above equation is current density, $\vec{u}$ is the fluid velocity, q is the charge of the particles, and N is the density. As an example, electron current density in a given space and time is equal to the sum of each electron charge, times electron density, times electron velocity.

$$\rho(\vec{r},t) = \sum_i q_i N_i(\vec{r},t) \tag{2}$$

ρ in the above equation is charge density.

Ambient values in this report is denoted by the subscript "0" and perturbation quantities are denoted by the subscript "1". The analysis here considers when there is no steady fluid motion and external electric field. Therefore, $\vec{u}_0 = \vec{E}_0 = 0$, and $\vec{J}_0 = 0$ as a consequence of $\vec{u}_0$ being 0 (1). These quantities are directly denoted with $\vec{u}, \vec{E}, \vec{J}$.

Using small oscillations approximation, linearized versions of the fluid equations (in Phasor form) are as follows:

$$j\omega N_1 - jN_0\vec{k}\cdot\vec{u} = 0 \tag{3}$$

$$jN_0 m\omega\vec{u} = qN_0(\vec{E} + \vec{u}\times\vec{B}_0) - \nabla\rho_1 \tag{4}$$

$$p_1 = \gamma k_B T N_1 \tag{5}$$

In the above equations, m is the mass of the particle, p is the pressure, k_B is the Boltzmann constant, and T is the temperature. Each species are are assumed to be perfect isothermal gases. Note that (3) is derived from the continuity equation and (4) is derived from momentum transport [1].

Maxwell's equations in Phasor form are as follows:

$$\vec{k} \times \vec{B_1} = j\mu_0 \vec{J} - \omega\epsilon_0\mu_0\vec{E} \tag{6}$$

$$\vec{k} \times \vec{E} = \omega\vec{B_1} \tag{7}$$

$$\vec{k} \cdot \vec{E} = j\frac{\rho_1}{\epsilon_0} \tag{8}$$

$$k \cdot \vec{B_1} = 0 \tag{9}$$

In the above equations, ω is the wave angular frequency, and k is the wave vector.

Phase velocity and group velocity of the plane electromagnetic waves are as follows:

$$v_p = \frac{\omega}{k} \tag{10}$$

$$v_g = \frac{d\omega}{dk} \tag{11}$$

From equation (10) and (11), it is clear that the relation between ω and k is of importance. Indeed, the relation between ω and k is known as *dispersion relation* and is very important while investigating the behavior of waves inside plasmas.

WAVES IN NON-MAGNETIZED PLASMAS

Throughout this section, ambient (static) magnetic field is completely neglected. In other words, $\vec{B_0} = 0$, so that $\vec{B} = \vec{B_1}$. The waves that can propagate in non-magnetized plasma are plasma oscillations, transverse electron plasma waves and longitudinal electrostatic electron and ion waves.

Plasma Oscillations

With cold plasma assumption ($T = 0$), ρ_1 becomes 0 by (5). Using this result and plasma being non-magnetized, (4) becomes:

$$jm_e\omega\vec{u}_e = q_e\vec{E} \tag{12}$$

where the motion of electrons are considered only by assuming ions are stationary[1]. Taking its divergence (or dot product the equation with $\vec{k}$ from left) gives:

$$jm_e\omega\vec{k} \cdot \vec{u}_e = q_e\vec{k} \cdot \vec{E} \tag{13}$$

Using (3) for the term $\vec{k} \cdot \vec{u}_e$ and (8) for the term $\vec{k} \cdot \vec{E}$ then gives:

$$jm_e\omega\left(\frac{\omega N_1}{N_0}\right) = q_e\left(j\frac{\rho_1}{\epsilon_0}\right) \tag{14}$$

Finally, using charge density equation (2) for the term ρ_1 gives the following expression for ω.

$$\omega_{pe}^2 := \omega^2 = \frac{N_0 q_e^2}{m_e\epsilon_0} \tag{15}$$

ω_{pe} is known as *electron plasma frequency.* Since ω_{pe} is not a function of $\vec{k}$, this oscillation is localized by (11) (i.e., $d\omega/dk = v_g = 0$).

Note that if one starts with ions instead of electrons in (12), the procedure is the same and a similar result is found for ion plasma frequency, i.e., ω_{pi}:

$$\omega_{pi}^2 := \omega^2 = \frac{N_0 q_e^2}{m_i\epsilon_0} \tag{16}$$

At this point, it is straightforward to see that the mass of the charged particle (electron, ion) is different, and N_0 may also differ, according to (15) and (16). Since $m_i \gg m_e$, plasma oscillation ω is sometimes described by ω_{pe} only. The complete plasma oscillations is; however, described as $\omega_p^2 = \omega_{pe}^2 + \omega_{pi}^2$.

Transverse Electromagnetic Waves

The motion of transverse electromagnetic waves (TEM) in plasma is very important since electromagnetic waves are transverse in nature, that is, they travel with perpendicular varying electric and magnetic fields. To see how the plasma medium effect the behavior of TEM waves, the refractive index μ should be investigated. It is defined as $\mu := kc/\omega$, where k can be imaginary or real. From this, it is clear that μ depends on the characteristics of the plasma, mainly dielectric constant ϵ, and permeability μ. In the analysis μ is taken to be μ_0, but ϵ is different than ϵ_0 since plasma is not free space and contains plasma convection current[1]. To obtain the new ϵ, or ϵ_{eff}, Maxwell's equations are used.

Cross product (7) from left with $\vec{k}$ gives:

$$\vec{k} \times \vec{k} \times \vec{E} = \omega\vec{k} \times \vec{B_1} = -k^2\vec{E} \tag{17}$$

[1] This is generally a good enough approximation since $\omega_{pe} \gg \omega_{pi}$ due to $m_i \gg m_e$.

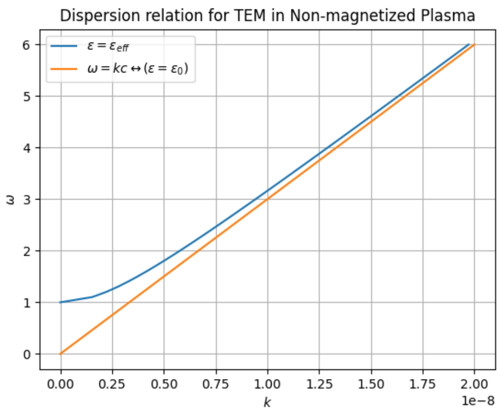

FIG. 1: *Dispersion ($\omega - k$) relation for Transverse Electromagnetic Waves in Non-magnetized Plasma (ω is normalized by ω_{pe} so that $\omega = 1$ corresponds to ω_{pe})*

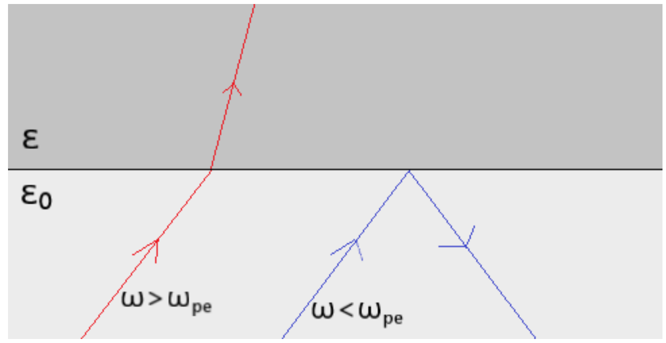

FIG. 2: *Sketch of wave propagation into the ionosphere, coming from the surface of Earth. Red wave propagates into the ionosphere since its frequency is larger than electron plasma frequency. It is also reflected (not shown). Blue wave cannot propagate inside the ionosphere and is reflected back towards the Earth.*

where the property $\vec{a} \times \vec{b} \times \vec{c} = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$ is used by noting $\vec{k}$ and $\vec{E}$ are perpendicular for the TEM waves. The term $\vec{k} \times \vec{B}$ in (6) can be replaced with the expression in (17) now to get:

$$-k^2 \vec{E} = -\frac{\omega^2}{c^2} \vec{E} = -\omega \mu_0 \epsilon \vec{E} = -\omega \mu_0 \vec{D} = j \mu_0 \vec{J} - \omega \epsilon_0 \mu_0 \vec{E} \tag{18}$$

Finally, replacing $\vec{J}$ in (18) by using (1) and (4) for considering cold and non-magnetized plasma assumptions (i.e., $\vec{J} = N_0 q_e \vec{u}_e = N_0 q_e (q_e \vec{E}/m_e j\omega)$), one gets:

$$-\omega \mu_0 \vec{D} = \left(\mu_0 \frac{N_0 q_e^2}{m_e \omega} - \omega \epsilon_0 \mu_0 \right) \vec{E} \tag{19}$$

which can be simplified into the following form

$$\vec{D} = \epsilon_0 \left(1 - \frac{N_0 q_e^2}{\epsilon_0 m_e \omega^2} \right) \vec{E} \tag{20}$$

Using the definition in (15), the effective dielectric constant can be defined as follows:

$$\epsilon_{\text{eff}} = \epsilon_0 \left(1 - \frac{\omega_{pe}^2}{\omega^2} \right) \tag{21}$$

Finally, refractive index becomes

$$\mu = \frac{kc}{\omega} = \frac{c}{v_p} = \frac{\sqrt{\mu_0 \epsilon_{\text{eff}}}}{\sqrt{\mu_0 \epsilon_0}} = \sqrt{1 - \frac{\omega_{pe}^2}{\omega^2}} \tag{22}$$

For $\omega < \omega_{pe}$, refractive index is imaginary. When refractive index is imaginary, the wave cannot enter the plasma. In other words, it is completely reflected and cannot propagate into the plasma. Mathematically, imaginary k creates *evanescent* waves inside the plasma, physically meaning a very rapid attenuation of the part of the wave that is able to propagate into the plasma. By the same token, TEM waves with $\omega > \omega_{pe}$ can propagate inside the plasma. When $\omega \gg \omega_{pe}$, propagation of the wave is similar to the one in free space. The dispersion relation is provided in (1). See (2) for reflection-transmission conditions for waves sent from Earth to the ionosphere.

Effect of Collisions

Until here, the effect of collisions between the molecules, ions, and electrons inside the plasma is ignored. In fact, those collisions also transform electromagnetic power into heat. To take them into account, an additional collision term with ν (effective collision frequency) is added to (4) by preserving non-magnetized, cold plasma assumptions.

$$q_e \vec{E} = j\omega m_e \vec{u}_e + m_e \nu \vec{u}_e = j\omega m_e \left(1 - j\frac{\nu}{\omega} \right) \vec{u}_e \tag{23}$$

The rest is similar before: (6) is used where $\vec{J}$ is written in terms of $\vec{u}_e$ by using (1) and $\vec{u}_e$ is written in terms of $\vec{E}$ by using (23) to obtain effective dielectric constant ϵ_{eff} from the relation $\vec{D} = \epsilon \vec{E}$. The procedure is straightforward and results with the following effective dielectric constant for plasma with collisions:

$$\epsilon_{\text{eff}} = \epsilon_0 \left(1 - \frac{\frac{\omega_p^2}{\omega^2}}{1 - j\frac{\nu}{\omega}} \right) \tag{24}$$

Note that ϵ_{eff} is imaginary now, where the imaginary part represents attenuation of the wave. One can verify when there is no collision, ν become 0 and (24) reduces to (21).

Electrostatic Electron - Ion Waves

In case $T \neq 0$, (5) gives non-zero pressure, and in the case $\nabla p_1 \neq 0$, i.e., $\nabla N_1 \neq 0$ for constant T, pressure gradient term in (4) does not vanish. By using (3), ∇N_1 in (5) can be written in terms of $\vec{u}$ and then (5) can be used in (4). Thus, start with:

$$\nabla p_1 = \gamma k_B T \nabla N_1 = \gamma k_B T \nabla \left(\frac{N_0 \vec{k} \cdot \vec{u}}{\omega} \right) \tag{25}$$

Then, using this result in (4) gives:

$$j\omega N_0 m \vec{u} = q N_0 \vec{E} - \gamma k_B T \nabla \left(\frac{N_0 \vec{k} \cdot \vec{u}}{\omega} \right) \tag{26}$$

which can be simplified by diving the equation by $j\omega m N_0$ to give

$$\vec{u} = \frac{q}{j\omega m}\vec{E} + j\frac{\gamma k_B T}{m\omega^2}\nabla(\vec{k}\cdot\vec{u}) \tag{27}$$

Therefore, the effect of nonzero temperature is the second term in the RHS of (27). Note that for TEM waves, this term is 0 since $\vec{k}$ and $\vec{u}$ are perpendicular. So, nonzero temperature has no effect on TEM waves' fluid velocity $\vec{u}$ (for both ions and electrons).

For longitudinal waves; however, $\vec{k} \cdot \vec{u} = ku$. Since $\vec{E}$ and $\vec{k}$ are parallel, $\vec{B_1} = 0$ by (7). Thus, $\vec{E} = j\frac{\vec{J}}{\epsilon_0 \omega}$ by (6). Replacing $\vec{J}$ with $\vec{u}$ by using (1) gives:

$$\vec{E} = \frac{jq_e}{\omega\epsilon_0}(\vec{u}) \tag{28}$$

where this relation holds for both ions and electrons. To consider the effect of both electrons and ions, $\vec{u}$ in (28) must be replaced with $\vec{u}_e - \vec{u}_i$, where the minus sign is due to electrons' and ions' having opposite charges (oppositely directed $\vec{J}$ vectors). One can get coupled $\vec{u}_i$ and $\vec{u_e}$ relations by using $\vec{E}$ in (28) inside (27). Then, there are two important results that come out by assuming small k and exploiting the fact that $m_i \gg m_e$:

$$\begin{aligned} \lambda_D &= \sqrt{\frac{\epsilon_0 k_B T_e}{N_0 q_e^2}} \\ \omega^2 &\approx \omega_{pe}^2(1+\gamma\lambda_D^2 k^2) \\ v_p &\approx \sqrt{\frac{\gamma k_B (T_e + T_i)}{m_i}} \end{aligned} \tag{29}$$

T_i, T_e are ion and electron temperatures, respectively. λ_D is known as *Debye Length*. It is important because it is required to have $L \gg \lambda_D$ and $N\lambda_D^3 \gg 1$ for the plasma state to model movement of the charged particles as fluids [2]. Around 350 km, $N_e \approx 10.5 \times 10^{-19} m^{-3}$ [3] and $T_e \approx 2500K$ [4]. This results with a Debye length of $3.368 \times 10^{-3} m$. This is much smaller than the system (ionosphere) size L, and $N_e\lambda_D^3 \approx 40112 \gg 1$. Therefore, satisfaction of this criteria allowed us to model motion of electrons as fluids from the beginning. This value λ_D is encountered in the second equation in (29), where the second term moves plasma oscillations along with an acoustic wave. The third equation gives the phase velocity v_p, which is ion acoustic wave and determined by the ion mass and electron, ion temperatures.

WAVES IN MAGNETIZED PLASMAS

Unlike non-magnetized plasma, magnetized plasma has an ambient magnetic field, i.e., $B_0 \neq 0$. It is straightforward to see that nonzero B_0 changes the equation (4) since the term with B_0 is non-vanishing now. Therefore, analysis in the previous section is not enough and a more detailed analysis is required now.

As in the previous section, refractive index μ is of importance here since it determines the behavior of the wave inside the plasma. Therefore, aim is to determine a general refractive index μ for non-magnetized plasma, which is also known as Appleton-Hartree equation. As in the previous chapter, since permeability is $\mu = \mu_0$, it is effective dielectric constant which is $\epsilon \neq \epsilon_0$ that affects the refractive index μ significantly.

One can derive the general formula for effective permittivity from Maxwell's equations. Consider (18), but this time replace $\vec{J}$ with $\sigma\vec{E}$ instead of (1) to get a more general expression. Then, after dividing both sides of the equation with $-\omega\mu_0$ one gets:

$$\vec{D} = \epsilon_0 \left(1 + \frac{\sigma}{j\omega\epsilon_0}\right)\vec{E} \tag{30}$$

Therefore, effective permittivity in isotropic (non-magnetized) plasma is:

$$\overleftrightarrow{\epsilon_p} = 1 + \frac{\overleftrightarrow{\sigma_p}}{j\omega\epsilon_0} \tag{31}$$

where the subscript "p" denotes these properties are related to the plasma, and both $\overleftrightarrow{\epsilon_p}$ and $\overleftrightarrow{\sigma_p}$ are tensors (with x, y, z coordinates). In the following parts, electric field will be considered since all waves have nonzero electric fields, but it is straightforward to calculate corresponding magnetic fields by using Maxwell's equations once the electric field is known. The dispersion relation will be of interest, which one obtains after getting refractive index μ. Take plasma infinite, cold ($T = 0$), collisionless ($\nu = 0$), and homogeneous. Assume ions to be stationary.

Converting (30) into the form with $\vec{J}$ is convenient here, so using the relation $\vec{J} = \sigma\vec{E}$, (30) can be written as follows:

$$\overleftrightarrow{\epsilon_p} \cdot \vec{E} = \frac{\vec{J}}{j\omega\epsilon_0} + \vec{E} \tag{32}$$

Using (1) to write $\vec{J}$ in terms of $\vec{u}$, one gets $\vec{J} = q_e N_0 \vec{u}_e$, so

$$\vec{u}_e = \frac{\vec{J}}{q_e N_0} \tag{33}$$

Using (33 for the $\vec{u}_e$ at the left hand side of the (4) (by noting $T = 0$):

$$\begin{aligned} j\frac{\vec{J} m_e \omega}{q_e} &= q_e N_0 (\vec{E} + \vec{u}_e \times \vec{B}_0) \\ \rightarrow \frac{\vec{J}}{j\omega\epsilon_0} &= -\frac{q_e^2 N_0}{m_e \omega^2 \epsilon_0}\left(\vec{E} + \vec{u}_e \times \vec{B}_0\right) \end{aligned} \tag{34}$$

Using (15) for ω_{pe}, taking static magnetic field to be in the z-direction and $\omega_c = q_e B_0 / m_e$ (electron cyclotron frequency)[2], (34) becomes:

$$\frac{\vec{J}}{j\omega\epsilon_0} = -\left(\frac{\omega_p^2}{\omega^2}\right)\vec{E} + \frac{q_e N_0 \vec{u}_e \times \hat{z}\omega_c}{\epsilon_0 \omega^2} \tag{35}$$

At last, it is possible to write $\vec{J}$ instead of $q_e N_0 \vec{u_e}$, which is the second term above. Then, finally, (35) becomes:

$$\frac{\vec{J}}{j\omega\epsilon_0} = -\left(\frac{\omega_p^2}{\omega^2}\right)\vec{E} - \vec{J} \times \hat{z}\left(\frac{\omega_c}{\epsilon_0 \omega^2}\right) \tag{36}$$

The equation (36) above is to be solved for $\vec{E}$ to write $\vec{J}$ in terms of electric field components and then put into the equation (32). It should be noted that $\vec{E} = \hat{x}E_x + \hat{y}E_y + \hat{z}E_z$ and $\vec{J} = \hat{x}J_x + \hat{y}J_y + \hat{z}J_z$. The following cross product rules in the right-handed Cartesian Coordinate System are used: $\hat{x} \times \hat{y} = \hat{z}$, $\hat{y} \times \hat{z} = \hat{x}$, and $\hat{z} \times \hat{x} = \hat{y}$. Then, since $\vec{E}$ and $\vec{J}$ has three coordinates (x, y, z), the following three equations needs to be considered.

$$\begin{aligned} \frac{J_x}{j\omega\epsilon_0} &= -\left(\frac{\omega_p^2}{\omega^2}\right)E_x + \left(j\frac{\omega_c}{\omega}\right)\frac{J_y}{j\omega\epsilon_0} \\ \frac{J_y}{j\omega\epsilon_0} &= -\left(\frac{\omega_p^2}{\omega^2}\right)E_y - \left(j\frac{\omega_c}{\omega}\right)\frac{J_x}{j\omega\epsilon_0} \\ \frac{J_z}{j\omega\epsilon_0} &= -\left(\frac{\omega_p^2}{\omega^2}\right)E_z \end{aligned} \tag{37}$$

In (37), J_z is uncoupled. However, J_x and J_y are coupled; thus, required to be solved simultaneously. Using the right hand side of the second equation for the last term of the first equation, one can determine $J_x/(j\omega\epsilon_0)$. Once it is determined, $J_y/(j\omega\epsilon_0)$ can be determined from the second equation in (37). Therefore, all J_x, J_y, J_z is written in terms of electric field components, so that $\vec{J}$ in (32) can be replaced with a term involving $\vec{E}$, but not $\vec{J}$. This is nice, because now (32) becomes a linear equation of single variable $\vec{E}$. The result of $\overleftrightarrow{\epsilon_p} \cdot \vec{E}$ is then as follows:

$$\overleftrightarrow{\epsilon_p} \cdot \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} \epsilon_\perp & -j\epsilon_\times & 0 \\ j\epsilon_\times & \epsilon_\perp & 0 \\ 0 & 0 & \epsilon_\parallel \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \tag{38}$$

where $\epsilon_\perp, \epsilon_\times$ and $\epsilon_\parallel$ are noted below.

$$\begin{aligned} \epsilon_\perp &= 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} \\ \epsilon_\times &= \left(\frac{\omega_c}{\omega}\right)\frac{\omega_p^2}{\omega^2 - \omega_c^2} \\ \epsilon_\parallel &= 1 - \frac{\omega_p^2}{\omega^2} \end{aligned} \tag{39}$$

Since J_x and J_y were coupled in (37), E_x and E_y are coupled in (38). Recall that there was no coupling for non-magnetized case. Thus, coupling here is due to the static magnetic field B_0, which is inherited in ω_c in the $\epsilon_\perp$ and $\epsilon_\times$ terms in (39). It is also possible to observe that as $B_0 \rightarrow 0$, then $\omega_c \rightarrow 0$, resulting with $\epsilon_\times = 0$ and $\epsilon_\perp = \epsilon_\parallel$, without any coupling between E_x, E_y, E_z.

The Wave Equation

To obtain dispersion relation, or equivalently the refractive index μ, the wave equation must be used. It is obtained from Maxwell's equations. Here, instead of (17), the general wave equation is to be calculated without making the simplification $\vec{k} \cdot \vec{E} = 0$ for TEM waves. Therefore, $\vec{k} \times \vec{k} \times \vec{E} = \vec{k}(\vec{k} \cdot \vec{E}) - k^2 E = \omega\vec{k} \times \vec{B}_1$. Using this in (6) gives:

[2] Electron cyclotron frequency is in fact a very important quantity since circularly polarized VLF waves at this specific Doppler-shifted frequency interact with the free electrons (and ions) in the ionosphere significantly.

$$\vec{k}(\vec{k}\cdot\vec{E}) - k^2\vec{E} = \omega(j\mu_0\overleftrightarrow{\sigma} - \omega\epsilon_0\mu_0)\cdot\vec{E}$$
$$= -\omega^2\epsilon_0\mu_0 \underbrace{\left(1 + \frac{\sigma}{j\omega\epsilon_0}\right)}_{\overleftrightarrow{\epsilon_p}} \cdot\vec{E} \qquad (40)$$

Noting $c^2 = (\epsilon_0\mu_0)^{-1}$, the wave equation becomes:

$$\vec{k}(\vec{k}\cdot\vec{E}) - k^2\vec{E} + \frac{\omega^2}{c^2}\overleftrightarrow{\epsilon_p}\cdot\vec{E} = 0 \qquad (41)$$

It should be notes that since $\vec{k}$ has three dimensions, k^2 in (41) is equal to $k_x^2 + k_y^2 + k_z^2$. The good news is the complicated tensor-vector product in (41) is already calculated in (38). Other terms must be expanded further.

Appleton-Hartree Equation

By using the wave equation (41) and (38), it is possible to calculate the dispersion relation, thus, refractive index. Dividing (41) by ω^2/c^2, one gets:

$$\overleftrightarrow{\epsilon_p}\cdot\vec{E} - \frac{c^2k^2}{\omega^2}\vec{E} + \frac{c^2}{\omega^2}\vec{k}(\vec{k}\cdot\vec{E}) = 0 \qquad (42)$$

The first term is given in (38) in product form. However, it can be opened up to get:

$$\overleftrightarrow{\epsilon_p}\cdot\begin{bmatrix}E_x\\E_y\\E_z\end{bmatrix} = \begin{bmatrix}\epsilon_\perp E_x - j\epsilon_\times E_y\\ \epsilon_\times E_x - j\epsilon_\perp E_y\\ \epsilon_\parallel E_z\end{bmatrix} \qquad (43)$$

The second term in (38) is as follows:

$$-\frac{c^2k^2}{\omega^2}\vec{E} = -\frac{c^2}{\omega^2}(k_x^2 + k_y^2 + k_z^2)\begin{bmatrix}E_x\\E_y\\E_z\end{bmatrix} \qquad (44)$$

The third term in (38) is as follows:

$$\frac{c^2}{\omega^2}\vec{k}(\vec{k}\cdot\vec{E}) = \frac{c^2}{\omega^2}\begin{bmatrix}(k_xE_x + k_yE_y + k_zE_z)k_x\\(k_xE_x + k_yE_y + k_zE_z)k_y\\(k_xE_x + k_yE_y + k_zE_z)k_z\end{bmatrix}$$
$$= \frac{c^2}{\omega^2}\begin{bmatrix}k_x^2E_x + k_xk_yE_y + k_xk_zE_z\\k_xk_yE_x + k_y^2E_y + k_yk_zE_z\\k_xk_zE_x + k_yk_zE_y + k_z^2E_z\end{bmatrix} \qquad (45)$$

So, (42) gives the following three equations:

$$\begin{aligned}
&\epsilon_\perp E_x - j\epsilon_\times E_y - \frac{c^2}{\omega^2}(k_x^2 + k_y^2 + k_z^2)E_x\\
&\quad + \frac{c^2}{\omega^2}(k_x^2E_x + k_xk_yE_y + k_xk_zE_z) = 0\\
&\epsilon_\times E_x - j\epsilon_\perp E_y - \frac{c^2}{\omega^2}(k_x^2 + k_y^2 + k_z^2)E_y\\
&\quad + \frac{c^2}{\omega^2}(k_xk_y + k_y^2E_y + k_yk_zE_z) = 0\\
&\epsilon_\parallel E_z - \frac{c^2}{\omega^2}(k_x^2 + k_y^2 + k_z^2)E_z\\
&\quad + \frac{c^2}{\omega^2}(k_xk_zE_x + k_yk_zE_y + k_z^2E_z) = 0
\end{aligned} \qquad (46)$$

To simplify (46), coordinate system is oriented in a way to make propagation on the $x-z$ plane. Therefore, y component of $\vec{k}$ vanishes, i.e., $k_y = 0$. Then, (46) simplifies to:

$$\begin{aligned}
&\epsilon_\perp E_x - j\epsilon_\times E_y - \frac{c^2}{\omega^2}(k_x^2 + k_z^2)E_x + \frac{c^2}{\omega^2}(k_x^2E_x + k_xk_zE_z) = 0\\
&\epsilon_\times E_x - j\epsilon_\perp E_y - \frac{c^2}{\omega^2}(k_x^2 + k_z^2)E_y = 0\\
&\epsilon_\parallel E_z - \frac{c^2}{\omega^2}(k_x^2 + k_z^2)E_z + \frac{c^2}{\omega^2}(k_xk_zE_x + k_z^2E_z) = 0
\end{aligned} \qquad (47)$$

Equivalently, this can be written in the tensor-vector product form.

$$\begin{bmatrix}\epsilon_\perp - \frac{k_z^2c^2}{\omega^2} & -j\epsilon_\times & \frac{k_xk_zc^2}{\omega^2}\\ j\epsilon_\times & \epsilon_\perp - (k_x^2 + k_z^2)\frac{c^2}{\omega^2} & 0\\ \frac{k_zk_zc^2}{\omega^2} & 0 & \epsilon_\perp - \frac{k_x^2c^2}{\omega^2}\end{bmatrix}\begin{bmatrix}E_x\\E_y\\E_z\end{bmatrix} = 0 \qquad (48)$$

Letting θ be the angle between $\vec{k}$ and $\hat{z}$, k_z becomes $k\cos\theta$ and k_x becomes $k\sin\theta$. Finally, one can replace kc/ω with refractive index μ. Then:

$$\begin{aligned}
k_x &= k\sin\theta = \frac{\omega\mu}{c}\sin\theta\\
k_z &= k\cos\theta = \frac{\omega\mu}{c}\cos\theta\\
k_y &= 0
\end{aligned} \qquad (49)$$

Writing k_x, k_y in terms of μ, θ in (48) then results:

$$\underbrace{\begin{bmatrix}\epsilon_\perp - \mu^2\cos^2\theta & -j\epsilon_\times & \mu^2\cos\theta\sin\theta\\ j\epsilon_\times & \epsilon_\perp - \mu^2 & 0\\ \mu^2\cos\theta\sin\theta & 0 & \epsilon_\parallel - \mu^2\cos^2\theta\end{bmatrix}}_{\overleftrightarrow{A}}\underbrace{\begin{bmatrix}E_x\\E_y\\E_z\end{bmatrix}}_{\vec{E}} = 0 \qquad (50)$$

This is a linear equation of single variable $\vec{E}$. For non-trivial solutions ($E_x, E_y, E_z \neq 0$), determinant of (50) must be equal to 0. Then, determinant of the matrix $\overleftrightarrow{A}$ is set to 0 to get the following equality.

$$\tan^2\theta = -\frac{\epsilon_\parallel(\mu^2-\epsilon_\perp-\epsilon_\times)(\mu^2-\epsilon_\perp+\epsilon_\times)}{(\mu^2-\epsilon_\parallel)(\epsilon_\perp\mu-\epsilon_\perp^2+\epsilon_\times^2)} \tag{51}$$

Letting μ^2 alone in (51) and putting the values in (39) in place of $\epsilon_\parallel, \epsilon_\perp, \epsilon_\times$, one gets the Appleton-Hartree equation, which also gives the relation between ω and k (dispersion relation).

$$\mu^2 = \frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_p^2/\omega^2}{1-\frac{\omega_c^2\sin^2\theta}{2(\omega^2-\omega_p^2)} \pm \sqrt{\left(\frac{\omega_c^2\sin^2\theta}{2(\omega^2-\omega_p^2)}\right)^2+\frac{\omega_c^2}{\omega^2}\cos^2\theta}} \tag{52}$$

Wave Propagation Parallel to B_0

When the wave propagation is parallel to B_0, then $\theta = 0\,\text{deg}$. Imposing this condition in (50), one gets 4 different $\omega - k$ relations. One of them is plasma oscillations, where $\epsilon_\parallel = 0$. The other three are left- and right-handed circularly polarized waves (2 RH, 1 LH). Their dispersion relation can be easily obtained from Appleton-Hartree equation by letting θ to be 0. Then, (52) becomes:

$$\mu^2 = \frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega(\omega\pm\omega_c)} \tag{53}$$

Using parameters at $L = 1$ and $MLAT = 20\,\text{deg}$, dispersion relation is plotted in (3). Note that $B_0 = 5.267\times10^{-5}$ T, $\omega_c = 9.262\times10^6$ rad/s and $\omega_p = 8.304\times10^8$ rad/s in (3).

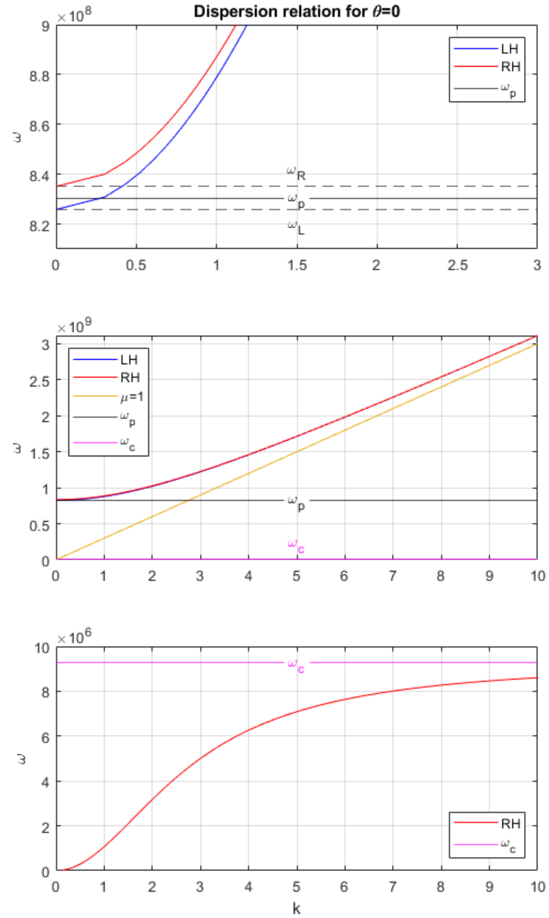

FIG. 3: *Dispersion relation for wave propagation parallel to the static magnetic field* B_0

Wave Propagation Perpendicular to B_0

When the wave propagation is perpendicular to B_0, then $\theta = 90\,\text{deg}$. Imposing this condition in (50), one gets 3 different $\omega - k$ relations. One of them is called *extraordinary* mode while the other two are *ordinary* modes. Their dispersion relation can be easily obtained from Appleton-Hartree equation by letting θ to be $\pi/2$. Then, (52) becomes:

$$\begin{aligned}\mu^2 &= \frac{k^2c^2}{\omega^2} = 1-\frac{\omega_p^2}{\omega^2}\\ \mu^2 &= \frac{k^2c^2}{\omega^2} = 1-\frac{\omega_p^2}{\omega^2}\left(\frac{\omega^2-\omega_p^2}{\omega^2-\omega_p^2-\omega_c^2}\right)\end{aligned} \tag{54}$$

where the first relation in (54) is for ordinary mode and the second relation is for extraordinary modes.

Using the same parameters ($L = 1$, $MLAT = 20\,\text{deg}$), dispersion relation is plotted in (4).

Complete Refractive Index

The refractive index formula given in (52) does not contain collisions of electrons with other electrons or heavy particles in the plasma. To take the effect of collisions into account, an extra term with ν should be added, as done before in the previous sections. Assuming the wave normal is in the positive z-direction, the refractive index in a homogeneous medium is given by[5]:

$$n^2 = 1 - \frac{X}{1-iZ-\frac{\frac{1}{2}Y_T^2}{1-X-iZ}\pm\frac{\sqrt{\frac{1}{4}Y_T^4+Y_L^2(1-X-iZ)^2}}{1-X-iZ}} \tag{55}$$

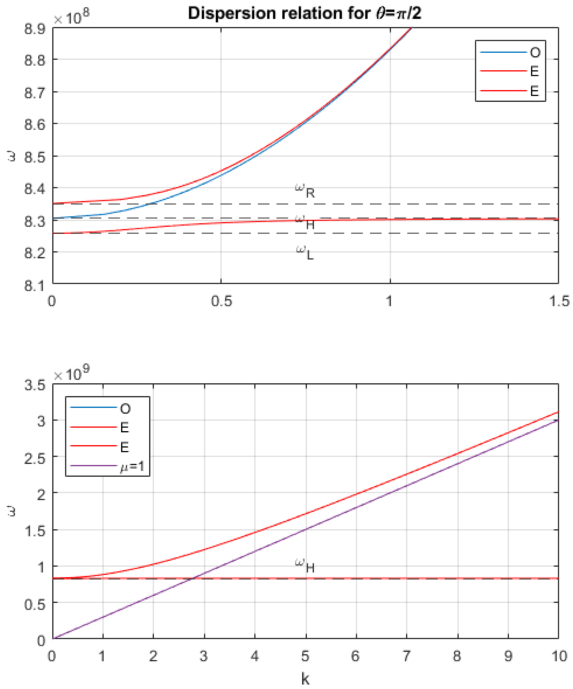

FIG. 4: *Dispersion relation for wave propagation perpendicular to the static magnetic field* B_0

where n is the complete refractive index consisting of real and imaginary parts: $\mu - i\chi$. The parameters in (55) are as follows:

$$
\begin{aligned}
X &= \frac{\omega_p^2}{\omega^2} \\
Y_T &= \frac{\omega_c \sin\theta}{\omega} \\
Y_L &= \frac{\omega_c \cos\theta}{\omega} \\
Y &= \frac{\omega_c}{\omega} = \sqrt{Y_L^2 + Y_T^2} \\
Z &= \frac{\nu}{\omega}
\end{aligned}
\tag{56}
$$

It is straightforward to observe that when the effects of collisions are neglected[3] ($\nu = 0$), then Z in (56) is 0. Then, (55) reduces to the following form:

$$n^2 = 1 - \frac{X}{1 - \frac{\frac{1}{2}Y_T^2}{1-X} \pm \frac{\sqrt{\frac{1}{4}Y_T^4 + Y_L^2(1-X)^2}}{1-X}} \tag{57}$$

[3] It is reasonable to neglect the effect of collisions when considering wave-particle interactions for the waves with very low frequencies. For example, VLF Whistler-mode propagation takes place where $\nu \ll \omega$.

Expressing (57) in terms of the variables in (56) then gives the equation (52). Note that since $Z = 0$, n^2 is real. Therefore, $n^2 = \mu^2$, which agrees with the Appleton-Hartree equation given in (52).

Further approximations[4] can be done for outer ionosphere conditions [5], where $\omega_p \gg \omega_c$ for whistler mode wave propagation. Then, refractive index reduces to:

$$\mu^2 = \frac{\omega_p^2}{\omega(\omega_c \cos\theta - \omega)} \tag{58}$$

Overall, n is a complex number. If it is purely imaginary ($n = -i\chi$), then the wave is evanescent, meaning it cannot propagate inside the plasma medium. On the contrary, if n is completely real ($n = \mu$), then it propagates in the plasma without any attenuation. When n is instead complex ($n = \mu - i\chi$), the wave propagates inside the plasma medium while being attenuated over distances.

Refractive-Index Surface Plot

Refractive index n is a function of ω, B, N_i, N_e and θ. By fixing L-shell and magnetic latitude (MLAT), it is possible to obtain B, N_i, and N_e. Then, for different ω (frequency of the wave propagating in the plasma) values, it becomes possible to investigate the change of refractive index n by changing θ. The resulting $n(\theta)$ vs θ plot is called *refractive-index surface plot.* Because imaginary and real parts of refractive index inherit information about the wave propagation and attenuation in the plasma, the behavior of n with θ becomes important. For example, Whistler modes propagating parallel to Earth's static magnetic field B_0 with frequencies ω_c can effect the free electrons in the ionosphere significantly to cause electron precipitation from ionosphere to lower layers of the atmosphere.

In the following figures, the configuration $L = 4$, $MLAT = 0$ is used. ω_H is defined as 13.5 kHz, and ω is varied between $0.1\omega_H$ and $0.5\omega_H$. The corresponding refractive-index surfaces are plotted in polar coordinate system.

For $L = 4$, $MLAT = 0$, magnetic field strength can be calculated with the following formula[6]:

$$B_0 = 0.312 \times 10^{-4} \left(\frac{R_0}{R}\right)^3 \sqrt{1 + 3\sin^2\lambda} \quad [Wb \cdot m^{-2}] \tag{59}$$

where λ is geomagnetic latitude (MLAT), R is geocentric distance, and R_0 is the mean radius of Earth (6370

[4] This approximation results with a maximum error of 6%[5]

km). Dipole field approximation of Earth gives the relation $R = R_0 \cos^2\lambda/\cos^2\lambda_0$. Finally, $1/\cos^2\lambda_0$ can be replaced with L, where L is the L-shell value. Therefore, B_0 can be calculated with (59) by using L and $MLAT$ values only.

Electron and ion density values N_e and N_i are calculated by using the MATLAB program tracedens1.m. Electron gyro-frequency ω_c, particle plasma frequency ω_p and static magnetic field strength B_0 are calculated by using (59), (15) and the equality $\omega_c = q_e B_0/m_e$. Their results are presented below.

$$\begin{aligned} B_0 &= 4.875 \times 10^{-7} \quad [Wb \cdot m^{-2}] \\ \omega_c &= 8.57314 \times 10^{4} \quad [rad \cdot s^{-1}] \\ \omega_p &= 1.54886 \times 10^{6} \quad [rad \cdot s^{-1}] \\ N_e &= 7.5354 \quad [m^{-3}] \\ N_i &= 7.5351 \quad [m^{-3}] \end{aligned} \tag{60}$$

For the plots, loss is assumed to be 0 ($Z = 0$), and equation (57) -or equivalently (52)- is used. Since n is a complex value, real part of n (μ), imaginary part of n (χ) and magnitude of n ($|n|$) are plotted separately. Also, since there is $\pm$ sign in (57) and in (52), there exists two possible positive n values, denoted by n^+ and n^- in the plots. The polar plots in figures (6) and (5) are zoomed in between $r \in [0, 150]$ for better visibility. Note that the direction $\theta = 0°$ is also the direction of the static magnetic field $\vec{B_0}$. So, $\theta = 0°$ corresponds to a wave propagating in the same direction with $\vec{B_0}$, while $\theta = 180°$ in the opposite direction and $\theta = 90°$ and $\theta = 270°$ in the perpendicular directions.

As discussed before, the real part of the refractive index (μ) is related to the wave propagation and imaginary part of the refractive index (χ) is related to the wave attenuation. In figure (5) for n^-, it is observed that $\mu = 0$ for all θ values. It means that it is not possible for the wave to propagate for the solution n^-. On the other hand, $\chi \neq 0$, so that the wave attenuates. This is very much expected since a wave that cannot propagate inside a medium is expected to be attenuated in that medium. In the second polar plot in (5), there exists attenuation in all directions. Although attenuation exists in all directions, it is sharper as $\theta \to 90°$ and $\theta \to 180°$. In other words, the wave attenuates faster if it travels perpendicular to the static magnetic field $\vec{B_0}$, and a bit slower if it travels parallel to the static magnetic field $\vec{B_0}$. Since the real part of n (μ) is 0, the magnitude of n is equal to the imaginary part of n. Therefore, second and third polar plots in (5) are the same. Overall, since $\mu = 0$ and $\chi \neq 0$, the wave is purely imaginary; therefore, evanescent.

As ω increases from $0.1\omega_H$ to $0.5\omega_H$ in (5), real part μ of the wave keeps being zero. So, the wave is evanescent for all ω values and it cannot propagate in the medium. On the other hand, attenuation is not constant. It is observed that attenuation decreases with increasing ω. Since the wave is evanescent, the wave cannot propagate inside the medium and dies out pretty fast anyways. However, it should be dying out faster for the waves of higher frequencies, according to the result of the figure (5).

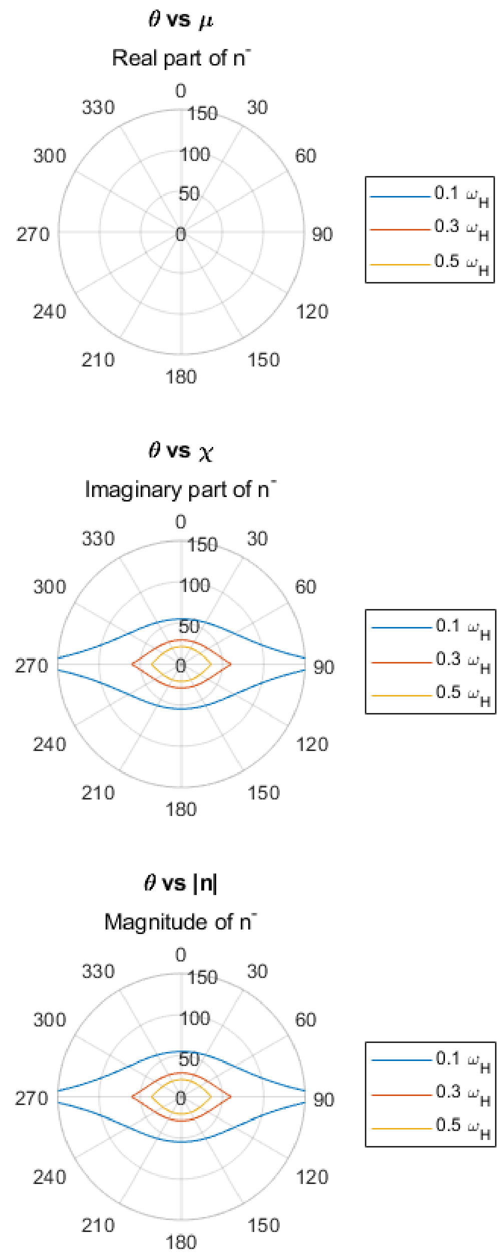

FIG. 5: *Refractive-index surface polar plot for the (-) signed version of Appleton-Hartree equation, i.e.,* n^-

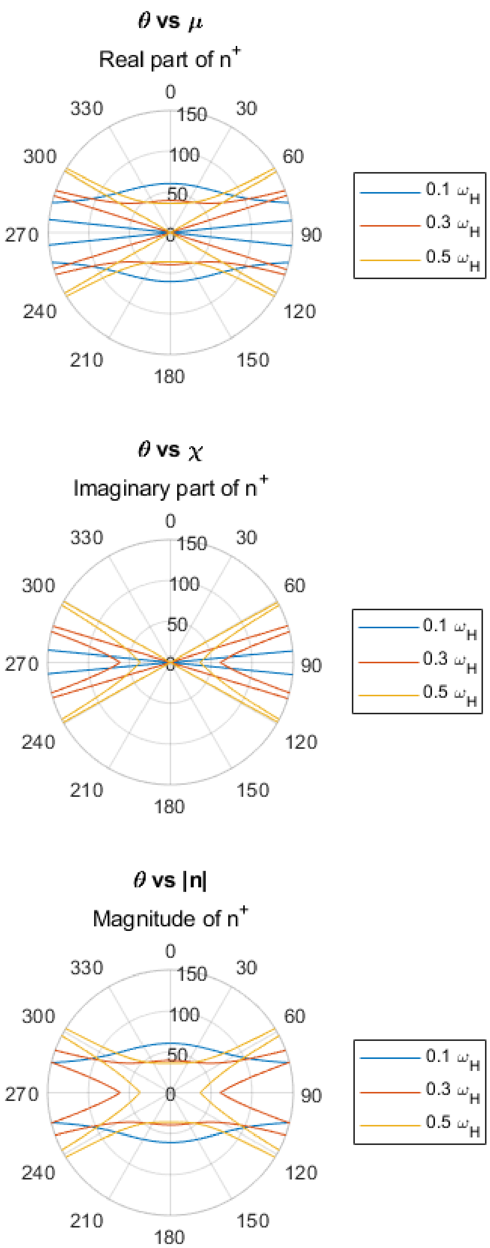

FIG. 6: *Refractive-index surface polar plot for the (+) signed version of Appleton-Hartree equation, i.e.,* n^+

For n^+, the wave is not purely imaginary, nor real. Instead, it is complex, meaning that the wave propagates over distances while being attenuated inside the ionospheric plasma. This result is presented in (6), where $n = \mu - i\chi$ for nonzero μ and χ values. The first figure in the (6) is the real part of n, which gives information of the wave propagation. It is observed that when $\theta = 0$ (wave propagation parallel to $\vec{B_0}$), wave propagates. As θ is increased, the value of μ increases. On the other hand, as $\theta \to \theta_0$ (where θ_0 changes for different ω values) χ also increases and becomes much larger than μ. For example, around 80°, both μ and χ increases for $\omega = 0.1\omega_H$. However, χ increases to a higher value, and between $\approx [80°, 100°]$, $\chi \gg \mu$. Physically, it means that between around 80° and 100° the wave with $\omega = 0.1\omega_H$ cannot propagate and attenuates very fast. This happens also in the conjugate angles between around 260° and 280°. In other words, the wave with $\omega = 0.1\omega_H$ cannot propagate in the ionosphere's $L = 4, MLAT = 0$ region if it travels with an angle of $\theta = [80°, 100°]$ or $\theta = [260°, 280°]$ away from the static magnetic field $\vec{B_0}$. From figure (6); however, it is observed that these allowed angles of propagation depend on the frequency of the wave. The trend for the frequencies in the figure shows that the range of the allowed angles of propagation are larger for the waves of smaller frequencies. The approximate prohibited angles of propagation between $\theta \in [0°, 180°]$ for $\omega \in [0.1, 0.3, 0.5]\omega_H$ are then as follows:

$$\begin{aligned}
&[80°, 100°] \to Total : 20°, \qquad (0.1)\omega_H \\
&[75°, 105°] \to Total : 30°, \qquad (0.3)\omega_H \\
&[60°, 120°] \to Total : 60°, \qquad (0.5)\omega_H
\end{aligned} \tag{61}$$

In general, it is observed that the range of allowed angles of propagation is larger for the waves of smaller frequencies. Asymptotically, it is observed that frequencies that are further decreased are likely to be prohibited only for the angles of propagation perpendicular to $\vec{B_0}$.

Another observation is that propagation μ is larger for waves of smaller frequencies. Consider $\theta = 0$ in the first figure in (5). It is observed that $\mu_{0.1} > \mu_{0.3} > \mu_{0.5}$. In short, the overall result is that waves of lower frequencies are able to propagate for a larger range of angles (away from $\vec{B_0}$) and with less attenuation (stronger signals) in the ionospheric plasma.

CONCLUSION

In this report, the basic formulations for the wave propagation inside (ionospheric) plasma are provided. Starting with unmagnetized plasma ($\vec{B_0} = 0$), equations for magnetized plasma ($\vec{B_0} \neq 0$) are derived by using Maxwell's equations and some fluid equations only. Such important concepts as plasma oscillation frequency (ω_p), electron gyrofrequency (ω_c), upper hybrid oscillation frequency (ω_H), effective dielectric constant (ϵ_{eff}), the wave equation are provided with their derivations. In the end, they all are combined to achieve Appleton-Hartree equation that describes the refractive index ($n =$

$\mu - i\chi$), and refractive-index surface polar plots are generated by using it. Dispersion relations for specific cases ($\theta = 0°, 180°$) are also provided with their corresponding $\omega - k$ plots. Overall, the refractive index of the medium turned out to be a very effective method to understand the conditions of wave propagation inside that medium. Whistler mode is verified to be an effective mode of propagation parallel to the static magnetic field $\vec{B_0}$ inside the ionospheric plasma with less attenuation and longer distances of propagation, compared to the waves of higher frequencies. Although wave-particle interactions are mentioned in this report, a more detailed analysis based on whistler mode wave-particle interactions in the ionospheric plasma can be carried out. For further analysis and experimental data related to this topic, the reader can refer to [5] and [6].

[1] U. S. Inan and M. Gołkowski, *Principles of Plasma Physics for Engineers and Scientists* (Cambridge University Press, 2010).

[2] S. Kuo, *Plasma physics in active wave ionosphere interaction* (World Scientific Publishing Co. Pte Ltd, 2017).

[3] R. Gillies, Modelling of transionospheric hf radio wave propagation for the isis ii and epop satellites (2006).

[4] R. W. Schunk, J. J. Sojka, and M. D. Bowline, Theoretical study of the electron temperature in the high-latitude ionosphere for solar maximum and winter conditions, Journal of Geophysical Research: Space Physics **91**, 12041 (1986).

[5] R. Helliwell, *Whistlers and related ionospheric phenomena* (Standford University Press,, California:, 1965).

[6] U. S. Inan, *Non-linear gyroresonant interactions of energetic particles and coherent VLF waves in the magnetosphere*, Ph.D. thesis, Stanford Univ., CA. (1977).