What are Combinatorial Species?

(Lagrange Inversion Formula if there's time)

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Intuitive Idea

What is it?

We have the idea of a type of discrete structure, such as graphs, endofunctions, etc. There are many combinatorial methods to count the number of discrete structures of a given type on a set with n elements, but this can get very difficult. Using Combinatorial Species, we can count more complicated discrete structures on some given set using generating functions and operations on them.

What is a discrete structure

Graphs

Linear orderings

Endofunctions

Subsets

Permutations

What is a discrete structure

Graphs
Linear orderings
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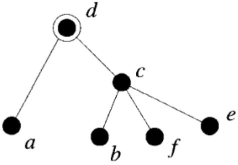
Permutations
Rooted Trees



A **species** is a rule $\mathcal{F}$ which maps finite sets to finite sets where;

For any set U, there is a finite set $\mathcal{F}[U]$ (the set of $\mathcal{F}$-structures on U).

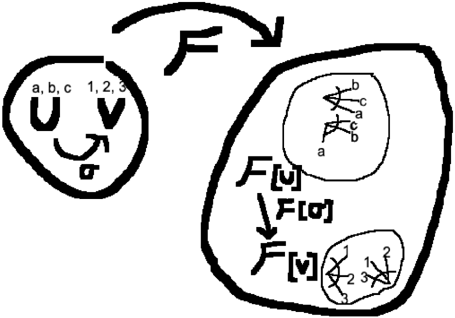
For any bijection $\sigma : U \to V$, there is a function $\mathcal{F}[\sigma] : \mathcal{F}[U] \to \mathcal{F}[V]$

and it satisfies the (functorial) properties

$\sigma : U \to V$ and $\tau : V \to W$ implies $\mathcal{F}[\tau \circ \sigma] = \mathcal{F}[\tau] \circ \mathcal{F}[\sigma]$

For the identity map $\mathrm{Id}_U : U \to U$, $\mathcal{F}[\mathrm{Id}_U] = \mathrm{Id}_{\mathcal{F}[U]}$

An element $s \in F[U]$ is called an F-**Structure** on U.



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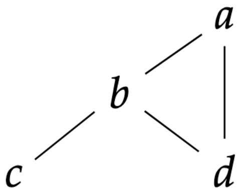
Example

Consider the following graph G:



We can describe G as

$$(\gamma, U) = (\{\{a, b\}, \{a, d\}, \{b, c\}, \{b, d\}\}, \{a, b, c, d\})$$

Examples

The species of linear orderings $\mathcal{L}$:

- The construction γ of a $\mathcal{L}$-structure on the set $[n]$ is some total order on it's elements, i.e.
$$1 < 2 < \cdots < n$$
- Such $\mathcal{L}$-structure consists of the pair $(\gamma, [n])$
- There is a bijection from the induced total order on $[n]$ by $\mathbb{N}$ to any other order on this set.

The set $\mathcal{L}[2]$ has two elements $1 < 2$ and $2 < 1$

Examples

Species of permutations $\mathcal{S}$

The construction of a $\mathcal{S}$-structure on the set $[n]$ is an element of S_n, the symmetric group.

$((1\ 3\ 2)(5\ 4), [5])$ and $((3\ 2\ 1)(5\ 4), [5])$ are the same structure on $[5]$.

Applying a permutation σ on a finite set U corresponds to the function $\mathrm{Id}_{\mathcal{S}[U]}$.

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For our purposes, the sequence (b_n) is the number of discrete ($\mathcal{F}$-)structures on some set with n elements, i.e., the cardinality of $\mathcal{F}[n]$.

A generating function is a clothesline on which we hang up a sequence of numbers for display.
Herbert Wilf, Generatingfunctionology

How many permutations are there on a set of n elements? The first element can go to n different places, the second has $n-1$ choices, and so on, so we get

$$\mathcal{S}(x) = \sum_{n-0}^{\infty} n! \frac{x^n}{n!} = \frac{1}{1-x}$$

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How many linear orderings are there on a set of n elements? There are n choices for the smallest element, $n-1$ for the second smallest, and so on and so similarly,

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We aim to define the number of $\mathcal{F}+\mathcal{G}$ structures on n elements such that the resulting generating functions is the sum of the series $\mathcal{F}(x)+\mathcal{G}(x)$. Then we say

$$(\mathcal{F}+\mathcal{G})[U]=\mathcal{F}[U]+\mathcal{G}[U]$$

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Example

As an example, consider the species of k-element sets E_k. It's generating series is

$$E_k(x)=0+\cdots+0+\frac{x^k}{k!}+0+\cdots$$

Then the sum

$$\sum E_k=E$$

where E is the species of sets, with $E[U]=\{U\}$, the singleton set containing U.

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When we multiply two species, the resulting species $F \cdot G$ is defined as follows

$$(F \cdot G)[U] := \sum_{\substack{U_1, U_2: \\ U_1 \cap U_2 = \emptyset \\ U_1 \cup U_2 = U}} F[U_1] \times G[U_2]$$

This counts the number of $\mathcal{F}$-structures times the number of $\mathcal{G}$-structures on all pairs of subsets which partition U into two parts.

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The species $\wp$ of subsets is the product $(E \cdot E)$.

$$\begin{aligned} E^2[n] &= 2^n \\ &= \wp[n] \end{aligned}$$

Which is consistent with the generating series

$$\begin{aligned} \wp(x) &= (E \cdot E)(x) \\ &= \sum \frac{x^n}{n!} \cdot \sum \frac{x^n}{n!} \\ &= e^x \cdot e^x \\ &= e^{2x} \end{aligned}$$

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Consider two species $\mathcal{F}, \mathcal{G}$. An $\mathcal{F} \circ \mathcal{G}$ structure on U is an $\mathcal{F}$-structure on some partition π of U, with a $\mathcal{G}$ structure on each $p \in \pi$.
The composition of structures is defined by

$$(\mathcal{F} \circ \mathcal{G})[U] := \sum_{\pi(U)} \mathcal{F}[\pi] \times \prod_{p \in \pi} \mathcal{G}[p]$$

with π a partition of U.
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where $X = E_1$, the species of singletons.
X chooses the root, and the the composition of $\mathcal{A}$ with E partitions the remaining elements, each of which has a $\mathcal{A}$ structure on it. Each of these rooted trees is connected to the main root, and we have another rooted tree.

Theorem (Lagrange Inversion Formula)

Let f be a formal power series. Assume $f(0) = 0, f'(0) \neq 0$. Let $f^{(-1)}$ denote the composition inverse of f, that is, $f^{(-1)} \circ f = f \circ f^{(-1)} = z$. One can write explicitly the coefficients of the inverse formal power series $f^{(-1)})(z)$ of $f(z)$ as

$$f^{-1}(z) = \sum_{n \geq 1} \left(\frac{d}{dt}\right)^{n-1} \left(\frac{t}{f(t)}\right)^n \Big|_{t=0} \frac{z^n}{n!}$$

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We want to interpret this as combinatorial species, so first we want a situation in which we are applying operations to species. We see this $\frac{t}{f(t)}$ term, which isn't an operation we can apply to species, so we want to get rid of it. Towards this goal, let $A(z)$ denote $f^{(-1)}(z)$ and set

$$R(z) = \frac{z}{f(z)}$$

. Then

$$R(A(z)) = \frac{A(z)}{f(A(z))}$$

Given this formula:

$$
\begin{aligned}
R(A(z)) &= \frac{A(z)}{f(A(z))} \\
&= \frac{A(z)}{z} \\
&\implies zR(A(z)) = A(z)
\end{aligned}
$$

we can now rewrite the Lagrange inversion formula as

$$A(z) = \sum_{n \geq 1} a_n \frac{z^n}{n!} \tag{1}$$

where

$$a_n = \left(\frac{d}{dt}\right)^{n-1} R(t)^n \Big|_{t=0} \tag{2}$$

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we can now rewrite the Lagrange inversion formula as

$$A(z) = \sum_{n \geq 1} a_n \frac{z^n}{n!} \tag{3}$$

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It is possible to show that when R is a species, we can construct a species A such that

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Earlier, we showed that the species of rooted trees $\mathcal{A}$, is defined recursively by

$$\mathcal{A} = X \cdot E(\mathcal{A})$$

which looks similar.