Math 142: Elementary Algebraic Topology Notes as taught by John Lott

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Contents

Abstract

These are my notes for my math class: Math 142 @UC Berkeley as taught by Dr. John Lott. Textbook used is Basic Topology, By M.A. Armstrong.

1 Metric Spaces

1.1 Introduction to Metric Spaces

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Main Idea of This Class
"Assign numbers to spaces"

Of course, by space, we mean a topological space, and by number we mean algebraic objects.
The layout of this course is as follows. We will start out by covering general topology, before moving on to algebraic topology. We will start out with studying the real numbers.

Definition 1.1. Given $x, y \in \mathbb{R}$, let $D(x,y) = |x-y|$. D has the following properties:

1. $D(x,y) \geq 0$
2. $D(x,y) = 0$ if and only if $x = y$
3. D is symmetric
4. Triangle Inequality

Proof. 1. $\mathbb{R}$ is closed, so $x - y \in \mathbb{R}$. And $|z| > 0$ for all $z \in \mathbb{R}$. So 1 holds.[1]

2. Suppose $D(x,y) = 0$ Then $|x-y| = 0$, so $x - y = 0$, so x=y. The reverse direction is similar

□

Definition 1.2. A **metric** on a set X is a function $d : X \times X \to \mathbb{R}$, such that

1. $d(x,y)$ is nonegative
2. $d(x,y) = 0$ if and only if $x = y$
3. d is symmetric
4. Triangle Inequality

Example 1.3 (ℓ_p norm)**.** For a real number p, the ℓ_p norm of a vector $x = \{x_1, \ldots, x_n\}$ is

$$\|x\|_p := (\sum_{i=1}^{n} x_i^p)^{\frac{1}{p}}$$

So for $p = 1$, we get the taxicab norm, $p = 2$, we get the euclidean norm, and when $p \to \infty$ we get the ℓ_∞ norm, where $d(x,y) = \max(|x_i - y_i|)$

Definition 1.4. Let X be a metric space. The ball with center $a \in C$ and radius $\delta > 0$ is $B(a,\delta) = \{x \in X | d(a,x) < \delta\}$

Definition 1.5. $G \subset X$ is open if for all $x \in G$, there exists δ such that $B(x,\delta) \subset G$

Definition 1.6. A set is **closed** if its complement is open.

Proposition 1.7. 1. $\emptyset$ and X are always open

2. $B(a,r)$ is open

[1] Hopefully this proof was a waste of my time

3. $\{x \in X | d(a,x) > r\}$ is open

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Proposition 1.8 (Properties of Open and Closed Sets)**.**

$$\text{The union } \bigcup_{\alpha \in A} G_\alpha \text{ of open sets is open.}$$

$$\text{The finite intersection } \bigcap_{i=1}^{\infty} G_i \text{ of open sets is open.}$$

$$\text{The finite union } \bigcup_{i=1}^{\infty} G_i \text{ of closed sets is closed.}$$

$$\text{The intersection } \bigcap_{\alpha \in A} G_\alpha \text{ of closed sets is closed.}$$

Proof.

1. Let $x \in \bigcup_{\alpha \in A} G_\alpha$. Then there exists α such that $x \in G_\alpha$. And G_α is open, so there exists some $B(x,\delta) \subset G_\alpha \subseteq \bigcup_{\alpha \in A} G_\alpha$, so $B(x,\delta) \subset \bigcup_{\alpha \in A} G_\alpha$, so $\bigcup_{\alpha \in A} G_\alpha$ is open.
2. Let $x \in \bigcap_{i=1}^{\infty} G_i$. For all i, there exists $\delta_i > 0$ such that $B(x,\delta_i) \subset G_i$. Set $\delta = \min_i(\delta_i)$. Then $B(x,\delta) \subseteq B(x,\delta_i) \subset G_i$, therefore $B(x,\delta) \subset \bigcap_{i=1}^{n} G_i$
3. and 4. follow from DeMorgan's Laws:
   (a) $X - (A \cup B) = (X - A) \cap (X - B)$
   (b) $X - (A \cap B) = (X - A) \cup (X - B)$

   $\forall A, B, X$

□

Say (X,d) is a metric space and $X_1 \subset X$. Then there is a canonical metric $d_1(a,b) = d(a,b)$ for $a, b \in X_1$.

$$B_1(a,r) = B(a,r) \cap X_1, \text{ for } a \in X_1$$

where $B_1(a,r)$ is the metric ball with respect to d_1 and $B_1(a,r) \cap X_1$ is the metric ball with respect to d.

Proposition 1.9. If $G_1 \subset X_1$ is open then $G_1 = G \cap X_1$ for some G open in X.
If $F_1 \subset X_1$ is closed then $F_1 = F \cap X$ for some F closed in X

Proof.

1. Say $G_1 \subset C_1$ is open. For all $a \in G_1$, there exists $\delta_a > 0$ such that $B(a,d_a) \subset G_1 = \bigcup_{a \in G_1} B_1(a,\delta_a)$. But $G = \subset_a \in GB(a,\delta_a)$ in X. Then G is open in X and $G_1 = G \cap X$.

□

Products in Metric Spaces: Say (X_1,d_1) and (X_2,d_2) are metric spaces.

$$X_1 \times X_2 = \{(x_1,x_2) : x_1 \in X_1, x_2 \in X_2\}$$

Possible metrics on $X_1 \times X_2$ include

1. The euclidean norm $= \sqrt{d_1(x_1,x_2')^2 + d_2(x_2,x_2')^2}$
2. The taxicab norm
3. The ℓ_∞ norm

Fact: These 3 metrics all have the same open sets.

Remark 1.10. First, we generalized from $\mathbb{R}$ to metric spaces. Now we generalize to topological spaces. (Using the Armstrong book)

2 Continuity

2.1 Introduction to Topology

Definition 2.1. If X is a set, a **topology** on X is a collection of "open" subsets such that

1. $\emptyset$ and X are open
2. Any union of open sets is open
3. A finite intersection of open sets is open

A set together with a topology is called a **topological space** So a topological space (X, τ) is a set X, and a topology τ, whose members are open sets (open subsets of X) which satisfy the properties. So we can't define a nontrivial topological space without assigning the property of being open to at least multiple subsets of X.

If our topological space has a subset, we have an **induced topology**:

Definition 2.2 (The subspace topology on Y)**.** $U \subset Y$ is open if $U = \mathcal{O} \cap Y$ for some open set $\mathcal{O} \subseteq X$

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Proposition 2.3. The subspace topology is a topology on Y

Proof. We just need to check that it satisfies the properties of a topology:

1. $\emptyset$ is open because $\emptyset = \emptyset \cap Y$, and $Y = X \cap Y$, therefore Y is open.
2. Say the family of subsets $\{U_\alpha\}_{\alpha \in A}$ are open in Y. We can write $U_\alpha = \mathcal{O}_\alpha \cap Y$, where $\mathcal{O}_\alpha$ is open in X. Then $\bigcup_{\alpha \in A} U_\alpha = (\bigcup_{\alpha \in A} \mathcal{O}_\alpha) \cap Y$, and since the union of open sets is open in X, its intersection with Y is open in Y by definition of the subspace topology, so we have that the union of open sets in Y is open in Y.
3. Say $\{U_i\}_{i \in I}^n$ are open in Y. Like before, we can write $U_i = \mathcal{O}_i \cap Y$ with $\mathcal{O}_i$ open in X. Then $\cap_{i=1}^n U_i = (\bigcap_{i=1}^n \mathcal{O}_i) \cap Y$, which an open set in X intersection with Y, which is open, as desired.

$\square$

Example 2.4. $X = \mathbb{R}^2$, $Y = [-1,1] \times [-1,1]$. Let $U = (0,1] \times [-1,1]$. As a subset of Y, U is open in Y (but not in X). $U = \mathcal{O} \cap Y$, and $\mathcal{O}$ open in X.

What are some examples of a topology defined on a set X?

Example 2.5 (Discrete and Indiscrete)**.** In the **Discrete topology**, every set is an open set.
In the **Indiscrete topology**, every set is no sets are open except for X and $\emptyset$. (Notice that this is also the trivial topology, since we can't define 2 distinct elements of X without them both being closed)

So topological spaces come from the metric space axioms, but we have been abstracting them into some ugly topologies. The question stands:

Question 2.6

Does every topological space come from a metric space?

Let's consider a few examples.

Example 2.7. Say X is a finite set $(x_1, \dots, x_n)$ with some metric d. What is the induced topology? Say $\delta = \frac{1}{2}\min_{i \neq j} \delta(x_i, x_j)$. Then $B(x_i, \delta)$ is open, and is equal to $\{x_i\}$. Every union of these open sets (which are all the subsets of X) are open, and the intersection of any of these singleton sets are $\emptyset$, which is open. So our finite set induces the discrete topology!

If the discrete topology comes from a metric space, what about the indiscrete topology?

Example 2.8. Let $X = \{x_1, x_2\}$ with the indiscrete topology. This does not come from a metric, since there is no way to define distance between the points. (also, by the example above, every metric induces the discrete topology, which is obviously not the indiscrete topology).

So the answer to our question is no.
In metric spaces we have an idea of distance, so lets define a topological analog of "nearness".

Definition 2.9. A set N is a **neighborhood** of x if there exists an open set $\mathcal{O}$ such that $x \in O \subseteq N$. Let $\mathring{N}$ denote the set of points z such that N is a neighborhood of z.

Definition 2.10. $F \subset X$ is **closed** if $F \backslash X$ is open.

Definition 2.11. Let $A \subset X$. Then $p \in X$ is a **limit point** of A if every neighborhood of p intersects A in some point other than p.

Example 2.12. For $X = \mathbb{R}$, $A = \{\frac{1}{n}\}_{n=1}^{\infty}$, the only limit point is 0.

Example 2.13. For $A = [0, 1)$, however, every point in $[0, 1)$ is a limit point. In fact, the set of limit points is larger: $[0, 1]$.

Theorem 2.14. A set $A \subset X$ is closed if and only if it contains all of its limit points.

Proof. For the forward direction, suppose A is closed. Then $X \backslash A$ is open. Suppose $\exists p \notin A$. Then p is not a limit point. $X \backslash A$ is open, so $X \backslash A$ is a neighborhood of p that doesn't intersect A, hence p is not a limit point. For the reverse direction, suppose A contains all its limit points. We want to show that $X \backslash A$ is open. Suppose $p \in X \backslash A$. Then p isn't a limit point, so there exists some open $\mathcal{O}$ with $p \in \mathcal{O}$ such that $\mathcal{O} \cap A = \emptyset$, i.e., $p \in \mathcal{O} \subset X \backslash A$, hence $X \backslash A = \bigcup_{p \in X \backslash A} \mathcal{O}_p$ is open. □

Definition 2.15. If $A \subset X$, $\overline{A} = A \cup \{\text{limit points of } A\}$ is denoted the **closure** of A.

Theorem 2.16. $\overline{A}$ is the smallest closed set containing A. i.e., $\overline{A} = \bigcap_{B \supset A} \overline{B}$

Proof.
1. First, we will show that $\overline{A}$ is closed. If $p \in X \backslash \overline{A}$, then there exists some open set U_p such that $p \in U_p \subset X$. But $U_p \subset X \backslash \overline{A}$, since any $q \in U_p$ is not in A and is not a limit point either. Hence the union $\bigcup_{p \in X \backslash \overline{A}} U_p$ is open. Hence intersection of closed sets containing A $\subseteq \overline{A}$

2. Suppose B is a closed set containing A. Any limit point of A is also a limit of B. So $\overline{A} \subset \overline{B} = B$, hence $\overline{A} = \bigcap_{\text{closed } B \supset A}$

□

In real analysis, we heard about $\mathbb{Q}$ being dense in $\mathbb{R}$. This meant that for all $x, y \in \mathbb{R}$, there exists $z \in \mathbb{Q}$ such that $x < z < y$. We may also recall the construction of $\mathbb{R}$ from $\mathbb{Q}$ via Dedekind cuts.

Question 2.17
Can we generalize the idea of density to topological spaces in general?

Definition 2.18. $A \subset X$ is **dense** in X if $\overline{A} = X$

Since we defined the closure of a subset, let us define the analogue for open-ness

Definition 2.19. The **interior** of a subset A is the largest open set in A. The interior $\text{int}(A) := \{p \in A \mid \exists \text{ open } \mathcal{O} \text{ such that } p \in \mathcal{O} \subseteq A\}$.

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Definition 2.20. The **frontier** of $A = \overline{A} \cap \overline{X \backslash A}$. You can think of it as the crust.

Example 2.21. For $X = \mathbb{R}^2$, $A = \{x \in \mathbb{R}^2 || x| \leq 1\} = \overline{A}$

$$X \backslash A = \{x \in \mathbb{R}^2 || x| > 1\}$$

$$\overline{X \backslash A} = \{x \in \mathbb{R}^2 | |x| \geq 1\}$$

which implies the frontier $\overline{A} \cap \overline{X \backslash A} \subset A$

When we learned about vector spaces, we defined the basis vectors $\{v_1, \ldots, v_n\}$ to be the vetors such that any other vector $x \in V$ is a linear combination $x = \sum_{i=1}^{n} a_i v_i$. This notion of an object having basis elements being those which create all the other elements appears in other mathematical objects.

Definition 2.22 (Basis)**.** Let X be a topological space. A **basis** β is a collection of open sets such that any open set is a union of basis elements

Example 2.23. We could take $\beta = \{\text{open sets}\}$, but this is useless

Example 2.24. We could also take $\beta = \{(a, b) | a, b, \in \mathbb{Q}\}$ for $X = \mathbb{R}$

Proposition 2.25. A collection of open sets is a basis if and only if given $p \in X$ and an open neighborhood N of p, there exists $B \in \beta$ such that $p \in B \subset N$

Proof. For the forwards direction, suppose β is a basis. Then for all $N \ni p \in X$, N is a union of elements of β, therefore there exists some $B \ni p$, and $B \subset N$.
For the backwards direction, say N is an open set. Then for all $p \in N$, there exists $B_p \subset N$ where $p \in B_p$. So $\bigcup_{p \in N} B_p = N$, so β is a basis. $\square$

2.2 Continuous functions

In real analysis we defined continuous functions $f : X \to Y$, where X, Y are subsets of $\mathbb{R}$. Now, let $f : X \to Y$, where X, Y are topological spaces. What makes f continuous?

Definition 2.26. f is continuous if for all open sets $\mathcal{O} \subseteq Y$, $f^{-1}(\mathcal{O}) = \{x \in X | f(x) \in \mathcal{O}\}$ is open in X

Definition 2.27. A **map** is a continuous function.

Proposition 2.28. If $f : X \to Y$ and $g : Y \to Z$ are maps, then so is $g \circ f$

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

Proof. If $\mathcal{O} \subset Z$ is open, then $g^{-1}(\mathcal{O})$ is open, which means $f^{-1}g^{-1}(\mathcal{O})$ is open, so $f \circ g$ is a continuous function. □

Proposition 2.29. If $f : X \to Y$ is continuous and $A \subset X$, then $f|_A : A \to Y$ is continuous, where A gets the subspace topology from X.

Proof. Say $\mathcal{O} \subset Y$ is open. $f|_A^{-1}(\mathcal{O}) = A \cap f^{-1}(\mathcal{O})$ is open in the relative (subspace) topology on A. □

Example 2.30. The identity map $1_X : X \to X$ is continuous. If $A \subset X$, then $1_X|_A : A \to X$ is continuous. Here, $1_X|_A$ is the inclusion map of A into X

Theorem 2.31. The following are equivalent.

1. $f : X \to Y$ is a map.
2. If β is a basis of Y, and $B \in \beta$, then $f^{-1}(B)$ is open.
3. $f(\overline{A}) \subseteq \overline{f(A)}$
4. $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for all $B \subset Y$
5. The preimage of a closed set is closed.

Proof. **1** $\implies$ **2** Any basis element B is open, therefore $f^{-1}(B)$ is open.

2 $\implies$ **3** We want to use (2) to show that the closure stays in the closure. Suppose $A \subset X$. Then $f(A) \subset \overline{f(A)}$. Let $x \in \overline{A}$. We want to show that $f(x) \in \overline{f(A)}$. There are two cases:

1. If $f(x)$ is in $f(A)$, then we are done
2. Now suppose $f(x)$ is only in the frontier of $f(A)$. Take a neighborhood N of $f(x)$. We can then find an element B of the basis such that $f(x) \in B \subseteq N$. By (2), the preimage is also open, so $f^{-1}(B)$ is a neighborhood of x. But x is a limit point of A, so $f^{-1}(B) \cap A \backslash x \neq 0$, which means there exists some point $a \in f^{-1}(B) \cap A$. Then $f(a) \in B$. So there exists some $f(a) \in B \subseteq N$ such that $f(a) \in A$ for every neighborhood N around x, which is just the definition of a limit point. Then $f(x) \in \overline{f(A)}$, as desired.

3 $\implies$ **4** Suppose we have $B \subset Y$ Put $A = f^{-1}(B)$. By (3), $f(\overline{f^{-1}(B)}) \subset \overline{f(f^{-1}(B))} \subset \overline{B}$. So $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$.

4 $\implies$ **5** First, note that if B is a closed subset, then $B = \overline{B}$. Say $B \subseteq Y$ is closed. $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) = f^{-1}(B)$. But $f^{-1}(B) \subseteq \overline{f^{-1}(B)}$, so $f^{-1}(B) = \overline{f^{-1}(B)}$, so $f^{-1}(B)$ is closed.

5 $\implies$ **1** Say $\mathcal{O} \subseteq Y$ is open. $Y \backslash \mathcal{O}$ is closed. By (5), its inverse $(Y \backslash \mathcal{O})$ is closed in X, but $f^{-1}(\mathcal{O}) = X \backslash f^{-1}(Y \backslash \mathcal{O})$ is open in X.

□

Example 2.32 (The $\epsilon - \delta$ definition). Say X, Y are metric spaces and $f : X \to Y$ is continuous. Given $a \in X$ and $\epsilon > 0$, look at $B(f(a), \epsilon)$. $f^{-1}(B(f(a), \epsilon))$ is open in X and $a \in f^{-1}(B(f(a), \epsilon))$. We can find some $\delta > 0$ such that

$$a \in B(a, \delta) \subset f^{-1}(B(f(a), \epsilon))$$

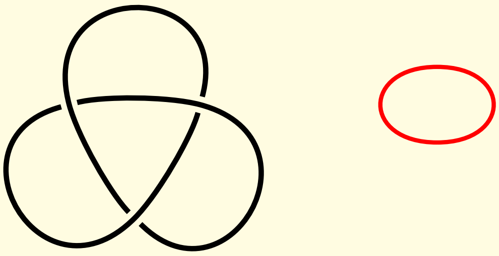
. $f(B(a, \delta)) \subseteq B(f(a), \epsilon)$, which is the $\epsilon - \delta$ definition. If $x \in B(a, \delta)$, then $f(x) \in B(f(a), \epsilon)$, that is,

$d(a, x) < \delta \implies d(f(a), f(x)) < \epsilon$

Definition 2.33. A **homeomorphism** (coming from the latin "homeo" for same, and "morphism" for form/shape) is a function $h : X \to Y$ that is injective, surjective, continuous, and has a continuous inverse.

Remark 2.34. Homeomorphisms are isomorphisms in the Category Top. The morphisms are maps.

Example 2.35. Consider the knots k_1 (the unknot) and k_2



Give k_1 and k_2 the subspace topology from $\mathbb{R}^3$. Then they are homeomorphic. (Imagine traveling around the both at a constant speed)

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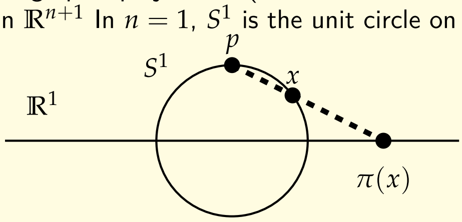
Recall the definitions of a continuous function and a homeomorphism from last class. How are these different? How are they the same? Let's look at an example of a continuous function that is not a homeomorphism.

Nonexample 2.36. $1_X : (X, \text{ Discrete}), \to (X, \text{ Indiscrete})$ does not satisfy all of the properties of a homeomorphism. It is a continuous function, since the two open sets in the indiscrete topology are open in the discrete one, but it is plain to see that many sets in the discrete topology map to closed sets in the indiscrete one.

Intuitively, we can imagine a homeomorphism from a circle excluding a single point to a line, by "unfolding" the circle and just getting a line. But this isn't very "scientific"...

Example 2.37 (S^n is homeomorphic to $\mathbb{R}^n$). Look at $S^n = \{\overline{x} \in \mathbb{R}^{n+1} ||\overline{X}| = 1$, with ths subspace topology from $\mathbb{R}^{n+1}$. This is a unit sphere. Consider $p \in S^n$. $S^n \backslash \{p\} \subset S^n$, so give it the subspace topology from S^n. We claim that $S^n \backslash \{p\}$ is a homeomorphism to $\mathbb{R}^n$.

We will show this using a stereographic projection. (the same one from Complex Analysis). Say p is the north pole. Look at S^n and $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$ In $n = 1$, S^1 is the unit circle on the horizontal line $\mathbb{R}$.



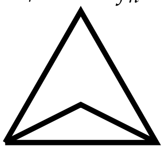
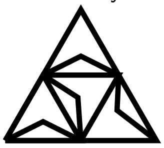
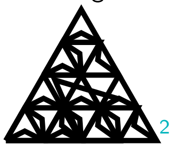
Let

$$\pi : S^n \backslash \{p\} \to \mathbb{R}^n \qquad x \mapsto \text{where } \overline{px} \text{ intersects } \mathbb{R}^n$$

be a projection from the circle to the line, which takes $x \in S^n \backslash \{p\}$ to $\pi(x)$ where the line from the north pole intersects $\mathbb{R}^n$. Then $\mathbb{S}^n \backslash \{p\}$ is homeomorphic to $\mathbb{R}^n$.

2.3 Space Filling Curve

In this subsection we will explore a curve that can fill up a polygon. Let $\triangle$ denote the solid triangle with edge lengths $\frac{1}{2}$. We claim that there exists a continuous map $t : [0,1] \to \triangle$ which is onto. Define $f_n : [0,1] \to n = 1,2,\ldots,$ where f_n is parameterized by normalized arc length.

$f(1)$  , $f(2)$  , $f(3)$ [2] , ...

Question 2.38

Does this converge? If this does converge, what does this mean for dimensionality?

We will show that it does converge.

Proof. For any two points $x, y \in \mathbb{E}^2$, let $||x-y||$ denote the distance between them. Suppose $n \geq m$. Then for any $t \in [0,1]$, $f_n(t)$ and $f_m(t)$ are both in a subtriangle with edge lengths $\frac{1}{2^m}$. Therefore

$$||f_m(t) - f_n(t)|| \leq \frac{1}{2^m}$$

for every value of t in $[0,1]$, which proves that the sequence $\{f_n\}$ is uniformly convergent. This matters because we know that each f_n is continuous, so, letting $f : [0,1] \to \triangle$ denote the limit function, f is continuous. Now we must show that f does in fact completely cover $\triangle$.

First, we will show that the image $\operatorname{Im}(f)$ is dense in $\triangle$. Suppose $x \in \triangle$. We want to show that $x \in \overline{\operatorname{Im}(f)}$. If x is in $\operatorname{Im}(f)$, then we are done. If x is not, then it is a limit point of f. Remember that for any n, the image of f_n comes within 2^{-n} of every point of $\triangle$. Say U is a neighborhood of x. Choose N large enough so that the disc $B(x, 2^{-(N-2)})$ lies inside U. Choose some $t_0 \in [0,1]$ such that $||x - f_N(t_0)|| \leq \frac{1}{2^N} = 2^{-N}$. Since $||f_N(t) - f(t)|| \leq 2^{-N}$ for every $t \in [0,1]$, the triangle inequality gives

$$\begin{aligned} ||x - f(t_0)|| &\leq ||x - f_N(t_0)|| + ||f_N(t_0) - f(t_0)|| \\ &\leq 2^{-N} + 2^{-N} = 2^{-(N-1)} \\ &< 2^{-(N-2)} \end{aligned}$$

so $f(t_0) \in B(x, 2^{-(N-2)})$. Hence x is a limit point of $\operatorname{Im}(f)$.

This means that every neighborhood U_x we can choose a smaller open ball which has an intersection point with the image of f. So we've shown that $\overline{\operatorname{Im}(f)} = \triangle$. So we know now that there is some curve that is dense in $\triangle$. In the future, we will show that $\operatorname{Im}(f) = \overline{\operatorname{Im}(f)}$, but we will leave it as is ☐

2.4 More on Continuity

What makes $\mathbb{Q}$ so nice? It's that there's no limit to how small you can go. In between any two numbers, there is a middle one.

Definition 2.39. A topological space is **Hausdorff** if for all points x, y where $x \neq y$, there exists open sets U, V such that $x \in U$, $y \in V$, and $U \cap V \neq \emptyset$.

Example 2.40 (Metric Space)**.** Let $r = \frac{1}{2}d(x,y)$, $U = B(x,r)$, $V = B(y,r)$. Therefore every metric space is Hausdorff.

Nonexample 2.41. $X = \mathbb{Z}$, equipped with the indiscrete topology. Let $y = x+1$. The only open set that contains y is the entire space X, which also contains y. This is not Hausdorff.

It's very popular to try and find the distance between a point and a set in $\mathbb{E}^n$. One may recall finding the perpendicular distance between a point and a plane in multivariable Calculus. That was the shortest distance between the point and any point on the plane. We have something similar when dealing with metric spaces.

[2] I did not draw this third one correctly but you get the gist

Definition 2.42. If x is a metric space and $A \subset X$, define $d_A : X \to \mathbb{R}$ by

$$d_A(x) = \inf_{a \in A} d(x,a)$$

Proposition 2.43. $d_A : X \to \mathbb{R}$ is continuous.

Proof. Let $X, \mathbb{R}$ be metric spaces. Given $x \in X$ and $\epsilon > 0$, we want to define δ such that $d(x,z) < \delta \implies |d_A(x) - d_A(z)| < \epsilon$. Take $\delta = \frac{\epsilon}{2}$. Choose $a \in A$ such that $d(x,a) < d_A(x) + \frac{\epsilon}{2}$. Then

$$\begin{aligned} d_A(z) &\leq d(z,a) \\ &\leq d(z,x) + d(x,a) \\ &< \frac{\epsilon}{2} + d_A(x) + \frac{\epsilon}{2} \\ &< d_A(x) + \epsilon \end{aligned}$$

Also, $d_A(x) < d_A(z) + \epsilon$. So we have $|d_A(x) - d_A(z)| < \epsilon$. □

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Lemma 2.44. Let X be a metric space, and A a subset. Then $d(x,A) = 0$ if and only if $x \in \overline{A}$

Proof. For the forwards direction, suppose $d(x,A)0$. For all $\epsilon > 0$, there exists $a \in A$ such that $d(x,a) < \epsilon$. Hence $B(x,\epsilon) \cap A \neq 0$, so $x \in \overline{X}$.
For the backwards direction, suppose $x \in \overline{A}$, then any $B(x,\epsilon) \cap \overline{A} \neq \emptyset$. Choose $a \in B(x,\epsilon) \cap A$. $d(a,x) < \epsilon$. Hence $d_A(x) = 0$. □

Proposition 2.45. Say $A, B \subset X$ are closed and $A \cap B \neq \emptyset$. Then there exists a continuous function $f : X \to [-1,1]$ such that $f|_A = 1$ and $f|_B = -1$.

Proof. But $f(x) = \frac{d(x,B) - d(x,A)}{d(x,B) + d(x,A)}$. Plug in values and you will see this works. But wait!! What if the denominator is zero? Well, since $d(x,B)$ and $d(x,A)$ are both nonnegative, both of them must be 0. But by the lemma 2.44, $x \in \overline{A}$ and $x \in \overline{B}$. Then $x \in A$ and B since they are both closed, but $A \cap B = \emptyset$, so the denominator can never be 0. □

Theorem 2.46 (Tietze Extension Theorem)**.** Let X be a metric space. $A \subset X$ is closed if $f : A \to \mathbb{R}$ is continuous, then there exists a continuous function $F : X \to \mathbb{R}$ such that $F|_A = f$.

"I just mentioned this so you can impress people at parties, but we won't really be using this."
- *John Lott*

3 Compactness and Completeness

3.1 Closed and Bounded Subsets

Definition 3.1 (Open Cover)**.** A family of open sets $\mathcal{F}$ is called an **open cover** if $\bigcap \mathcal{F} = X$

Definition 3.2. X is **compact** if every open cover of X has a finite subcover.

Nonexample 3.3. Let $X = \mathbb{R}$, $\mathcal{F} = \{i, i+2 | i \in \mathbb{R}\}$. Since there is no finite subcover, $\mathbb{R}$ is not compact.

The rest of today's notes are just about compactness of subspaces.

Theorem 3.4 (Heine-Borel)**.** For all $a, b \in \mathbb{R}$, $[a, b]$ is compact.

Proof. Let $\mathcal{F}$ be a open cover of $[a, b]$. Define a subset X of $[a, b]$ as follows:

$$X = \{x \in [a, b] | [a, x] \text{ is contained in the union of some finite subfamily of } \mathcal{F}\}$$

Then X is nonempty and is bounded above by b. So X has a supremum s. We will claim that $s \in X$ and that $s = b$. Let $O \in \mathcal{F}$ contain s. Since O is open we can choose some $\epsilon > 0$ such that $(s - \epsilon, s] \subseteq \mathcal{O}$, and if s is less than b we can assume $(s - \epsilon, s + \epsilon) \subseteq \mathcal{O}$.

So s is the least upper bound of X and thus there are points of X arbitrarily close to s. And X has the property that $x \in X$ and $a \leq y \leq x$, then $y \in X$. Therefore we can say $s - \frac{\epsilon}{2} \in X$. By the definition of X, the interval $[a, s - \frac{\epsilon}{2}]$ is contained in the union of some finite subfamily $\mathcal{F}'$ of $\mathcal{F}$. The union $\bigcup \mathcal{F}' \cup O$ is a finite subfamily of $\mathcal{F}$ whose union contains $[a, s]$. Then $s \in X$. If s is less than b then $\bigcup \mathcal{F}' O \cup$ contains $[a, s + \frac{\epsilon}{2}]$, which means $s + \frac{\epsilon}{2} \in X$ and contradicting the fact that s is an upper bound for X. Therefore $s = b$ and all of $[a, b]$ is contained in $\bigcup \mathcal{F}' \cup O$. $\square$

Intuitively, what we have done is we have taken some "longest" finite subcover, and shown that any epsilon ball around the supremum of that finite subcover must be a finite subcover in the closed interval, which means either the supremum isn't the supremum, or it is just b.

Warning 3.5

If X is a metric space and $A \subseteq X$ is compact, then it will be closed and bounded. The converse, however, does not hold in general. Let $f : X \to Y$ be a function.
$A \subseteq X \implies f^{-1}(f(A)) \supseteq A$, which has equality if f is injective.
$B \subset Y implies f(f^{-1}(B)) \subseteq B$, which has equality if f is surjective.

3.2 Properties of Compact Spaces

Theorem 3.6. If $f : X \to Y$ is a map and X is compact, then $f(X)$ is compact. (With the subspace topology)

Proof. We can replace Y by $f(X)$, to assume f is surjective. We want to show that Y is compact. Then $\{f^{-1}\}(\mathcal{O})_{\mathcal{O} \in \mathcal{F}}$ is an open cover of X. Since X is compact, we can extract a finite subcover $X = \cup_{i=1}^{n} f^{-1}(\mathcal{O}_i)$. Since f is onto, $Y = f(x) = \cup_{i=1}^{n} f(f^{-1}(\mathcal{O}_i)) = \cup_{i=1}^{n} \mathcal{O}_i$. $\square$

Definition 3.7. A subset $C \subset X$ is compact if it's compact in the subspace topology.

Theorem 3.8. A closed subset C of a compact space is compact.

Proof. Take a collection $\mathcal{F}$ of open sets in X that cover C. Then $\mathcal{F} = \{\mathcal{O}_\alpha\}_{\alpha \in C}$. Add $X \backslash C$ to the collection. Then we get an open cover of X. Since X is compact, extract a finite subcover $\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_n$.

So we get a finite cover $X \backslash C \cup \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_n$ of X. Then $C \subseteq \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_n$, that is, a finite cover of C, hence it is compact. $\square$

Today we have shown a few things:

1. Any closed subset of $\mathbb{R}$ is compact
2. The codomain of a map from a compact set is a compact set
3. Any closed subset of a compact space is compact.

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Theorem 3.9. If X is Hausdorff, $A \subset X$ is compact, and $x \in X \backslash A$, then $\exists$ disjoint neighborhoods of $\{x\}$ and A. Particularly, $X \backslash A$ is open, so A is closed.

Proof. Take $z \in A$. X is Hausdorff, so we can find disjoint open sets U_z and V_z such that $x \in U_z$ and $z \in V_z$. $\{V_z\}_{z \in A}$ is an open cover of A, and there exists a finite subcover $\{V_{z_1}, \dots, V_{z_k}\}$ of A.

Put $V = \cup_{i=1}^{k} V_{z_i}$ and $U = \cap_{i=1}^{k} U_{z_i}$. Then U and V are open disjoint sets, $\{x\} \subset U$ and $A \subset V$.
In particular, $\{x\} \subset U \subset X \backslash A$, for any $x \notin A$, so $X \backslash A$ is open. □

Theorem 3.10. Suppose $f : X \to Y$ is injective, surjective, continuous, X is compact, and Y is Hausdorff. Then f is a homeomorphism.

Proof. Suppose $g = f^{-1}$. We want to show that g is continuous, i.e., If $C \subseteq X$ is closed, then $g^{-1}(C)$ is closed in Y.
Take C closed in X. Since X is compact, C is compact. Then $f(C)$ is compact in Y. Since Y is Hausdorff, $f(c)$ is closed in Y. Thus f is a homeomorphism. □

Example 3.11. Consider the space filling curve $f : [0, 1] \to \triangle$. These are not homeomorphic. Since f is continuous, surjective, X is compact, and Y is Hausdorff, we have that f must not be injective

While the curve isn't injective (one-to-one), it can be shown that the space filling curve can actually be four-to-one, i.e., no more than four points in the domain map to a point in the codomain

Theorem 3.12 (Bolzano Weierstrass Theorem)**.** If X is compact then any infinite subset of X has a limit point.

Proof. Suppose S is any subset of X. Suppose S doesn't have any of it's limit points. Then for all $x \in X$, x is not a limit point of S. Then there exists some neighborhood U_x such that

$$U_x \cap S = \begin{cases} \{x\} \text{ if } x \in S \\ \varnothing \text{ if } x \notin S \end{cases}$$

$\{U_x\}_{x \in X}$ covers X. Since X is compact, there exists a finite subcover $\{U_{x_i}\}_{i=1}^{k}$. Each U_{x_i} intersects S in at most one point, so the cardinality of S is at most K. So a lack of limit points implies the subset is finite, therefore the contrapositive holds. □

Note that in this scenario, we are imagining S as a set of isolated points, rather than a "blob" of points. This makes a lot of sense if you imagine some sequence (s_n). Bolzano Weierstrass tells us that no matter what subsequence you choose, it will accumulate at some place, and (if you remember from Real Analysis) it has at some convergent subsequence.

Theorem 3.13. A compact subset of $\mathbb{R}^n$ is closed and bounded

Proof. $\mathbb{R}^n$ is Hausdorff, so any compact subset is closed. $\mathbb{R}^n = \cup_{\alpha=1}^{\infty} B(0, \alpha)$. Any compact subset C can be written $C = \cup_{\alpha=1}^{\infty} B(0, \alpha) \cap C$. Then there exists a finite subcover, so $C \subset B(0, N)$ for some $N \in \mathbb{Z}$ □

We can imagine just taking bigger and bigger spheres, until one of them encapsulates our compact subset.

Theorem 3.14. If $f : X \to \mathbb{R}$ is a map and X is compact, then f is bounded above and below, and the infimum and the supremum are realized.

Proof. Since X is compact, $f(X)$ must be compact in $\mathbb{R}$. From theorem 3.13, $f(X)$ is bounded in $\mathbb{R}$. It's also closed in $\mathbb{R}$, so $\sup(f(X)) \in f(x)$. The infimum case follows similarly. □

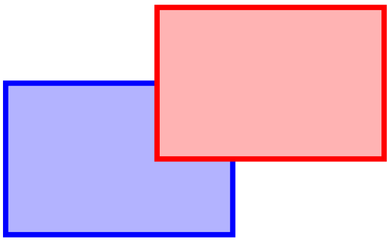
3.3 Product Spaces

Say we have two topological spaces X and Y.

Question 3.15

Can we define topology on the product of their sets $X \times Y$?

Why don't we just try it? Lets say open sets in $X \times Y$ are defined as the product of open sets, i.e., $\{U \times V \mid U$ open in X and V in $Y\}$. But life isn't fair, and this is false.



Imagine the red rectangle being some open set $U \times V$ and the blue rectangle being a different open set $W \times Z$. There are no open sets $A \in X$ and $B \in Y$ where $A \times B = U \times V \cup W \times Z$.

A collection B of subsets of a set X is the basis for some topoloy on X if and only if whenever $x \in B_1 \cap B_2$, there exists $x \in B_3 \subseteq B_1 \cap B_2$. This allows us to say that open sets are unions of elements of B. We make the following claim:

Theorem 3.16. $\beta = \{U \times V \mid U$ is open in $X \wedge V$ is open in $Y\}$ forms a basis for a topology.

Proof. Say $B_1 = U_1 \times V_1$, $B_2 = U_2 \times V_2$. Then we can take $B_3 = (U_1 \cap U_2) \times (V_1 \cap V_2)$ □

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Theorem 3.17 (Space Filling Curve Coninued)**.** Now we can prove that the space filling is closed, which if you recall before means that the curve is space filling in $\triangle$.

Proof. We know that $f : [0,1] \to \triangle$ is continuous and $\mathrm{Im}(f)$ is dence in $\triangle$. $[0,1]$is compact, so $\mathrm{Im}(f)$ is compact in $\triangle$. Since $\triangle$ is Hausdorff, (since it is a closed subset of $\mathbb{R}^2$) $\mathrm{Im}(f)$ is closed, so $\mathrm{Im}(f) = \overline{\mathrm{Im}(f)} = \triangle$ □

Recall from last week that we said that $\beta = \{U \times V \mid U$ is open in $X \wedge V$ is open in $Y\}$ forms a basis for a topology.

Example 3.18. Let $X = Y = \mathbb{R}$. $X \times Y = \mathbb{R}^2 = \mathbb{E}^2$ has the euclidean topology.

Definition 3.19 (Projections)**. Projections** $P_1 : X \times Y \to X$ and $P_2 : X \times Y \to Y$ are functions where $P_1(x,y) = x$ and $P_2(x,y) = y$

Proposition 3.20. P_1 and P_2 are continuous, and send open sets to open sets.

Proof. Without loss of generality, consider P_1. Say $U \subseteq X$ is open. Then $P_1^{-1}(U) = U \times Y \in \beta$, which is open. If $U \times V \in \beta$, $P_1(U \times V) = U$. So $P_1(\cup_\alpha(U_\alpha \times V_\alpha) = \cup_\alpha P_1(U_\alpha \times V_\alpha) = \cup_\alpha U_\alpha$, which is open in X. □

So projections are canonical maps from the product topology. We will now explore the conditions these maps and the product topology preserve.

Theorem 3.21. A function $f : Z \to X \times Y$ is continuous if and only if $P_1 \circ f$ and $P_2 \circ f$ are continuous.

Proof. The forwards direction is clear because it is a composition of continuous functions. The backwards direction is less so. Suppose $P_1 \circ f$ and $P_2 \circ f$ are continuous. We want to show that $f^{-1}(U \times V)$ is open in Z for every open set $U \times V$ in the basis. Then

$$\begin{aligned} f^{-1}(U \times V) &= f^{-1}(P_1^{-1}(U) \cap P_2^{-1}(V)) \\ &= f^{-1}(P_1^{-1}(U)) \cap f^{-1}(P_2^{-1}(V)) \\ &= (P_1 \circ f)^{-1}(U) \cap (P_2 \circ f)(V) \end{aligned}$$

which is open in Z, as desired. □

Theorem 3.22. $X \times Y$ is Hausdorff if and only if X and Y are Hausdorff.

Proof. For the forwards direction, Suppose $X \times Y$ is Hausdorff. Without loss of generality we will show that this means X is Hausdorff. Look at (x_1, y_1) and (x_2, y_2). We can separate them by basis elements $U_1 \times V_1$ and $U_2 \times V_2$. U_1 and U_2 separate X_1 and X_2. For the backwards direction, suppose X and Y are Hausdorff. Choose $(x_1, y_1) \neq (x_2, y_2) \in X \times Y$. Without loss of generality, suppose $x_1 \neq x_2$. Choose open sets U_1, U_2 in X such that $U_1 \cap U_2 = \emptyset$ and $x_1 \in U_1$, and $x_2 \in U_2$. Then $U_1 \times Y$ and $U_2 \times Y$ separate (x_1, y_1) and (x_2, y_2). □

Lemma 3.23. Say X is a topological space and β is a basis. Then X is compact if and only if every open cover at X by basis elements has a finite subcover.

Proof. The forwards direction holds by definition of compactness. To prove the backwards direction, suppose $\mathcal{F}$ is an open cover of X. Each element $\mathcal{O} \in \mathcal{F}$ is a union of basis elements. Let $\mathcal{F}'$ be that collection of basis elements. $\mathcal{F}'$ is an open cover of X. By assumption, there is a finite subcover $\mathcal{F}''$. We will now use this to show that every cover has a finite subcover. For every $B \in \mathcal{F}''$, choose $\mathcal{O}_B \in \mathcal{F}$ so $B \subset \mathcal{O}$. Then $\{\mathcal{O}_B\}_{B \in \mathcal{F}''}$ is a finite subcover if $\mathcal{F}$, which means it is compact. □

Theorem 3.24. $X \times Y$ is compact if and only if X and Y are compact.

Proof. In the forwards direction, suppose $X \times Y$ is compact. Without loss of generality, $X = P_1(X \times Y)$ is compact. In the other direction, suppose X, Y are both compact. Say $\mathcal{F}$ is an open cover of $X \times Y$ by basis elements. We will show that $\mathcal{F}$ has a finite subcover, and using lemma 3.23 we will have that $X \times Y$ is compact.

Given $x \in X$, consider $P_1^{-1}(x) = \{x\} \times Y$. This is homeomorphic to Y, so Y is a compact subset of $X \times Y$. We can find a finite subset $\{U_{x,1} \times V_{x,1}, \dots, U_{x,n_x} \times V_{x,n_x}\}$ of $\mathcal{F}$ that covers $\{x\} \times Y$. Their union contains $U_x \times Y$, where $U_x = \cap_{i=1}^{n_x} U_{x,i}$. We can do this for all $x \in X$.

To visualize what we have done so far, let $X \subset \mathbb{R} \supset Y$ be compact, and imagine $\{x\} \times Y$ as a vertical line passing through $\{x\}$ that is Y. The finite subset of $\mathcal{F}$ imagine as a bunch of rectangles that cover this rectangle. Then the intersection of their X-components is just a thin strip that covers $\{x\}$.

Now we want to show that we can do this finitely many times to cover $X \times Y$. We have an open cover $\{U_x\}_{x \in X}$ of X. This has a finite subcover $\{U_{x_1}, \dots, U_{x_s}\}$. Since the union of this set is X, we have

$$X \times Y = \bigcup_{j=1}^{S} (U_{x_j} \times Y) \subseteq \bigcup_{j=1}^{S} \bigcup_{i=1}^{n_{x_j}} (U_{x_j,i} \times V_{x_j,i})$$

which is a finite subcover of $\mathcal{F}$, as desired. □

Now we can say some things about spaces we are more familiar with.

Theorem 3.25. A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof. The forwards direction we've already done. Suppose $C \subset \mathbb{R}^n$ is closed and bounded. We can find $S > 0$ so that $C \subset [-S, S]X \times \cdots \times [-S, S]$. We know that $[-S, S]$ is compact in $\mathbb{R}$, so $[-S, S]X \times \cdots \times [-S, S]$ is compact in $\mathbb{R}^n$. C is closed, so it is compact. □

Definition 3.26. Let X be a metric space, and $C \subset X$. Put the diameter $\text{diam}(C) = \sup_{x,y \in C} d(x,y)$. We say C is **bounded** if $\text{diam}(C) < \infty$

Theorem 3.27. A compact subset of a metric space is closed and bounded.

This holds the same way as $\mathbb{R}^n$. Note that in general, the converse is not true for metric spaces.

Example 3.28. Let $X = \mathbb{R}$. Define the distance metric

$$d(x,y) = \frac{|x-y|}{1+|x-y|}$$

This gives the usual topology. Take $X = \mathbb{R}$. C is closed, the diameter is 1, but C is not compact.

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3.4 Connectedness

Definition 3.29. A topological space X is **connected** if for all $X = A \cup B$ such that $A \neq \emptyset, B \neq \emptyset$, we have $\overline{A} \cap B \neq \emptyset$ or $A \cap \overline{B} \neq \emptyset$.

Example 3.30. Imagine $A = (a,b]$ and a different set $B = (b,c)$, where $a < b < c$. Then the topological space $A \cup B$ with the subspace topology is connected.

Nonexample 3.31. But consider $X = [0,1] \cup [2,3]$ with the subspace topology. This is disconnected.

In fact, $\mathbb{R}$ is connected, and a subspace with the subspace topology is connected if and only if it's an interval.

Theorem 3.32. The following are equivalent

1. X is connected
2. The only subsets of X that are open and closed (**clopen**) are X and $\emptyset$
3. X is not the union of 2 nonempty disjoint open subsets.
4. There is no surjective map from X to a discrete space with multiple points.

Proof. 1 $\implies$ 2 Suppose X is connected, and suppose $A \subset X$ is clopen. Then $X \backslash A$ is clopen. $\overline{A} = A$ and $\overline{B} = B$, so $\overline{A} \cap B = A \cap B = \emptyset$ and $A \cap \overline{B} = A \cap B = \emptyset$, which contradicts our definition of connected.

2 $\implies$ 3 Suppose our clopen subsets are $\emptyset$ and X. Suppose $X = U \cup V$ with U, V disjoint open sets. Since $U = X \backslash V$, U is closed, which means by our assumption U or V is empty.

3 $\implies$ 4 Suppose (3) holds, and Y is a discrete space with $|Y| \geq 2$, and some map $f : X \to Y$. Then write $Y = U \cup V$, with U and V being 2 nonempty disjoint open subsets. Then $X = f^{-1}(U) \cup f^{-1}(V)$. Now we have that $f^{-1}(U)$ and $f^{-1}(V)$ are open, disjoint, and nonempty. This contradicts (3).

4 $\implies$ 1 Suppose (4) holds and X is disconnected. We can write $X = A \cup B$ with A and B nonempty, $\overline{A} \cap B = \emptyset$ and $\overline{B} \cap A = \emptyset$. $B = X \backslash \overline{A}$ is open, as is A. Then define $f : X \to \{-1, 1\}$ by $f|_A = -1$, $f|B = 1$. Then f is continuous and surjective, which is a contradiction to (4)

□

Remark 3.33. This is where the old adage "If a clopen set is can be detected, then your metric space is disconnected" comes from.

Theorem 3.34. If X is connected and f is a map, then $f(x)$ is connected in the subspace topology.

Proof. We can reduce to the case where f is surjective. Suppose we can write $Y = U \cup V$, where U and V are disjoint open sets. Look at $f^{-1}(V)$ and $f^{-1}(U)$., which are open in X and $X = f^{-1}(U) \cup f^{-1}(V)$. Since X is connected, $f^{-1}(U) = \emptyset$ or $f^{-1}(V) = \emptyset$. Then $U = \emptyset$ or $V = \emptyset$, hence Y is connected. □

Example 3.35 (Intermediate Value Theorem)**.** Consider a function $f : [a,b] \to \mathbb{R}$. Say y lies between $f(a)$ and $f(b)$. Then there exists some $x \in [a,b]$ such that $f(x) = y$.

Proof. $[a,b]$ is connected, so the image is connected. If $y \notin \text{Im}(f)$, then $f([a,b]) \cap (-\infty, y)$ and $f([a,b] \cap (y, \infty)$ decomposes $f([a,b])$ into 2 distinct nonempty subsets. This is a contradiction, so $x \in [a,b]$ exists. □

Definition 3.36. Two subsets are called **separated** if the intersection of their closure is empty.

Theorem 3.37. If X, Y are connected, then $X \times Y$ is connected.

The proof of this is given later.

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Theorem 3.38. Let X be a topological space and Z a dense subset. If Z is connected then so is X.

Example 3.39. Let $X = \{x \in \mathbb{R}^2 | |x| \leq 1\}$ and $Z = \{x \in \mathbb{R}^2 | |x| < 1\}$. Then X is the closure of Z. Suppose Z is connected and suppose A is a nonempty clopen subset of X. Since Z is dense in X, Z intersects every nonempty open set. In particular, $Z \cap A \neq \emptyset$. $A \cap Z$ is open in Z (under the subspace topology) and $A \cap Z$ is closed in Z. So $A \cap Z$ is a nonempty clopen subset of Z, so it must be Z, i.e., $A \cap Z = Z$, so $Z \subseteq A$. Then $x = \overline{Z} \subseteq \overline{A}$ and $\overline{A} = A$ (since it is clopen), so it is all of X, which means X is connected.

Corollary 3.40. If Z is a connected subset of X, then $\overline{Z}$ is connected.

Proof. Z is dense in $\overline{Z}$ □

Example 3.41. $X = \mathbb{R}^2$, $z = \{x, \sin(\frac{\pi}{x}) | 0 < x \leq 1\}$. Z is connected since it is the image of $(0,1]$ under a map. But $\overline{Z} = Z \cup \{(0,y) | -1 \leq y \leq 1\}$

Theorem 3.42. Say $\mathcal{F}$ is a family of subsets of X, whose union is X. If each $Z \in \mathcal{F}$ is connected and no two $Z, Z' \in \mathcal{F}$ are separated, then X is also connected.

Proof. Suppose $A \subset X$ is clopen. We want to show that $A = \emptyset$ or $A = X$.
If $Z \in \mathcal{F}$, then Z is connected and $A \cap Z = \emptyset$ or $A \cap Z = Z$

1. If $A \cap Z = \emptyset$ for all $Z \in \mathcal{F}$, then $A = \emptyset$, since the union of Z's is X

2. Suppose $A \cap Z = Z$ for some $Z \in \mathcal{F}$, i.e., $Z \subseteq A$, then suppose $W \in \mathcal{F}$. If $W \cap A = \emptyset$, then Z and W are separated. This is impossible, so $W \cap A = W$, so $W \subseteq A$

Hence $\cup_{W \in \mathcal{F}} W \subseteq A$, so $A = X$ □

Theorem 3.43. If X and Y are connected, then $X \times Y$ is also connected.

Proof. If $x \in X$, then $\{x\} \times Y$ is connected, and if $y \in Y$, then $X \times \{y\}$ is connected. Then

$$(\{x\} \times Y) \cap (X \times \{y\}) = \{(x, y)\}$$

so $(\{x\} \times Y)$ and $(X \times \{y\})$ are not separated, so

$$Z(x, y) := (\{x\} \times Y) \cup (X \times \{y\}) \text{ is connected}$$

Then $X \times Y = \bigcup_{(x,y) \in X \times Y} Z(x, y)$, and $(x, y') \in Z(x, y) \cap Z(x', y')$. So by theorem 3.42, $X \times Y$ is connected. $\square$

Definition 3.44 (Component). A **component** of X is a maximal connected subset

This means that any two components are separated from each other. A lot of this terminology is similar to that of graph theory.

Example 3.45. $X = \mathbb{Q}$ with the subspace topology from $\mathbb{R}$. The components are just singleton sets. (yet notice this isn't the discrete topology)

Theorem 3.46. Each component of a topological space is closed and distinct components are separated from one another.

Proof. Suppose C is a component. Then C is connected. By corollary 3.40, $\overline{C}$ is connected. But C is maximal, so we must have $C = \overline{C}$. If D is some other component and D is not separated from C, then $\overline{C} \cap \overline{D} \neq \emptyset$. Then by theorem 3.42, $C \cup D$ is connected, which means C is not maximal, which is a contradiction. $\square$

We basically took a component, and said that any other component that isn't separated implies a contradiction

Proposition 3.47. Every connected subset of X is in some component.

Proof. Say $A \subseteq X$ is connected. Put C to be the union of all connected subsets containing A. By theorem 3.46, C is connected, so it is a component. $\square$

3.5 Path Connectedness

Recently in complex analysis, we began talking about continuous, piecewise-smooth curves. Curves are important in many fields of mathematics, and paths are just curves that are parameterized.

Definition 3.48. Path A **path** in X is a map $\gamma : [0, 1] \to X$ where $\gamma(0)$ is the **beginning** of γ and $\gamma(1)$ is the **end**.

Definition 3.49. A space is **path connected** if any two points can be joined by a path.

Theorem 3.50. Path-connected implies connected

Proof. Say $A \subset X$ is clopen, and (recalling theorem 3.32) towards contradiction say that $A \neq \emptyset$ and $A \neq X$. Choose $x \in A$ and $y \in X \backslash A$. Join them by a path $\gamma : [0, 1] \to X$. Then $0 \in \gamma^{-1}(A)$ while $1 \notin \gamma^{-1}(A)$, so $\gamma^{-1}(A)$ is neither the empty set nor the whole interval, which by the continuity of γ (a continuous map means the preimage of closed sets are closed, and the preimage of open sets are open) tells us that $\gamma^{-1}(A)$ is clopen in $[0, 1]$. This contradicts that the interval is connected. Then path-connected spaces must be connected. $\square$

Remark 3.51. The converse can fail. A space can be connected but not path connected

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Theorem 3.52 (Paths Compose)**.** Just reparameterize:

$$\gamma(t) = \begin{cases} \alpha(2t) & 0 \le t \le \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

assuming the composability conditions hold.

Theorem 3.53. A connected open subset X of $\mathbb{R}^n$ is path connected.

Proof. Choose $x \in X$. Put $U(x) = \{y \in X \mid y \text{ can be joined to } x \text{ by a path in } X\}$. $x \in U(x)$, so $U(x)$ is nonempty. We'll show that it's clopen.

$U(x)$ **is open)** Say $y \in U(x)$. Choose a ball $B(y)$ such that $y \in B \subseteq X$. If $z \in B$, we can join y to z by a straight line. So $z \in U(x)$. Hence $y \in B \subseteq U(x)$, so $U(x)$ is open.

$U(x)$ **is closed)** Say $y \in X \backslash U(x)$. Take a ball $B(y)$ such that $y \in B \subseteq X$. Choose $z \in B$. If $z \in U(x)$, we can join y to x, which is a contradiction. Then $z \notin U(x)$, i.e., $z \in X \backslash U(x)$. Hence $B \subseteq X \backslash U(x)$. Hence $X \backslash U(x)$ is open.

So $U(x)$ is clopen. Since X is connected, $U(x) = \emptyset$ or $U(x) = X$. $U(x)$ is nonempty, which means X is path connected. ☐

We showed that there is an open ball in $U(x)$ for every point inside of $U(x)$, and an open ball outside of $U(x)$ for every point outside of $U(x)$. Then it's clopen so it's X

Definition 3.54. A **path component** is a maximal path connected subset.

Fact: If A is path connected and nonempty, then A sits inside (is a subset of) a unique path component (take the union of all path connected subsets containing A.)

Theorem 3.55. If $f : X \to Y$ is a map and X is path connected, then $f(X)$ is also path connected.

Proof. Choose $y_1, y_2 \in f(X)$. Then choose $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$. Join x_1 to x_2 by a path γ. Then $f \circ \gamma$ joins y_1 to y_2 by a path in $f(x)$. ☐

Definition 3.56. If X is a topological space then $\pi_0(X)$ is the set of path components in X.

Given a map $f : X \to Y$, we have an induced map $f_* : \pi_0(X) \to \pi_0(Y)$. Given $C \in \pi_0(x)$, $f(C)$ is path connected in Y, so it sits inside a unique path component D of Y. Let $f_*(C) = D$.

Theorem 3.57. If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are maps then $(g \circ f)_* = g_* \circ f_*$ as functions from $\pi_0(X)$ to $\pi_0(Y)$.

Proof. Given $C \in \pi_0(X)$, choose $x \in C$. Then $f_*(C)$ is the path component in Y containing $f(x)$. So $g_*(f_*(C))$ is a path component of z containing $g(f(c)) = g \circ f(x)$, so $(g \circ f)_*(C) = g_*(f_*(C))$. ☐

Corollary 3.58. If $f : X \to Y$ is a homeomorphism, then $f_* : \pi_0(X) \to \pi_0(Y)$ is a bijection.

Proof. Let $g : Y \to X$ such that $g \circ f = \mathrm{Id}_X$ and $f \circ g = \mathrm{Id}_Y$. Then

$$g_* \circ f_* = \mathrm{Id}_{\pi_0(X)}$$

and

$$f_* \circ g_* = \mathrm{Id}_{\pi_0(Y)}$$

Hence f_* is a bijection. ☐

This is starting to look like some real topology!! Two spaces are the same if they have the same amount of pieces? It seems pretty "coffee-cup to donut"-y.

Theorem 3.59. $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for $n > 1$.

Proof. Suppose $F : \mathbb{R} \to \mathbb{R}^n$ is homeomorphic, and say $f = F_{\mathbb{R}\setminus\{0\}}$. Then f is a homeomorphism from $\mathbb{R}\setminus\{0\}$ to $\mathbb{R}^n\setminus\{F(0)\}$. But $\pi_0(\mathbb{R}\setminus\{0\})$ has two elements, whereas $\mathbb{R}_n\setminus\{0\}$ has only 1 element. $\square$

π_0 and π_1 are known as homotopies.
If you recall the motivation for this class from the beginning of the semester, we said that the goal of Algebraic Topology is to assign algebraic objects to topological spaces. Here we have assigned the object of sets to topological spaces.

Corollary 3.60. $[0,1]$ is not homeomorphic to $\triangle$

This is true by the same argument.

Corollary 3.61. The space filling curve is not injective.

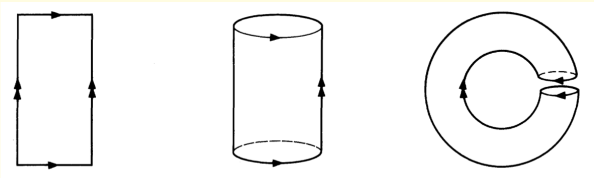
Proof. By Corollary 3.60, our space filling curve is a surjective map that is not homeomorphic. It maps from a compact space to a Hausdorff space, so it would be a homeomorphism if it were a bijective map. This means that the only place it can fail to be Homeomorphic is injectivity, so it is not injective. $\square$

4 Identification Spaces

4.1 The Identification Topology

Example 4.1 (Circle)**.** Say $X = [0,1]$. Identify $\{0\}$ and $\{1\}$. We are basically folding the endpoints onto each other to get a circle

Example 4.2 (Torus)**.** $X = [0,1] \times [0,1]$. Identify the top edge with the bottom edge i.e., $(x,0)$ with $(x,1)$ and the side edges with each other i.e., $(0,y)$ with $(1,y)$.



This picture is taken from the Armstrong book.

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Definition 4.3. A **partition** $\mathcal{P}$ of X is a family of adjoint nonempty subsets whose union is X.

We get a surjective function $\pi : X \to Y$ by saying $\pi(X)$ is the elements of $\mathcal{P}$ containing X. Conversely, given a surjective map $\pi : X \to Y$, we get a partition $\mathcal{P}$ on X whose elements are $\{\pi^{-1}(y)\}_{y\in Y}$. Now suppose we have a topological space X and a partition $\mathcal{P}$ of X. We want to put a topology on Y such that π is continuous.

Definition 4.4. This map $\pi : X \to Y$ induces an **identification topology** aka **quotient topology** where an open subset $\mathcal{O}$ of Y is open if and only if it's preimage is open in X.

Proposition 4.5. This is a topology on Y

Proof.

1. $\pi^{-1}(\emptyset_Y) = \emptyset_X$ is open in X, so $\emptyset_Y$ is open in Y
2. $\pi^{-1}(Y) = X$ is open, so Y is open.
3. If $\{\mathcal{O}_\alpha\}$ are open in Y then $\pi^{-1}(\bigcup_\alpha \mathcal{O}_\alpha) = \bigcup_\alpha(\pi^{-1}(\mathcal{O}_\alpha))$ is open in X. So $\cup_\alpha \mathcal{O}_\alpha$ is open in Y.

4. If $\{\mathcal{O}_i\}_{i=1}^n$ are open in Y, then $\pi^{-1}(\cap_i \mathcal{O}_i) = \cap_i \pi^{-1}(\mathcal{O}_i)$ is open in X, so $\cap_i \mathcal{O}_i$ is open in Y

$\square$

So we have the following definition.

Definition 4.6. Let X be a topological space and $\sim$ an equivalence relation induced by a partition of X. Let Y be the quotient set $X \backslash \sim$, the set of equivalence classes of X. An **identification space** aka **quotient space** is the set Y equipped with the identification (or quotient) topology.

Definition 4.7. Assume X, Y are topological spaces. A map $\pi : X \to Y$ is an identification map if $\mathcal{O} \subset Y$ is open if and only if $\pi^{-1}(\mathcal{O})$ is open in X.

Warning 4.8

If π is an identification map, it does not follow that f of an open set is open.

Example 4.9. Let $X = \{a, b, c, d\}$, and let $\mathcal{P} = \{\{a, b\}, \{c, d\}\}$. Let's say that the open sets of X are

$$\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c, d\}$$

Now call $Y = \{e, f\}$, where $a \mapsto e$, $c \mapsto e$, $b \mapsto f$, and $d \mapsto f$. Then we see that our open sets in Y are

$$\emptyset_Y, \{e, f\}$$

This is because $\pi^{-1}(e) = \{a, c\}$ which is not open in X, and the same follows for $\pi^{-1}(f)$. But $\pi(a) = e$, an open set doesn't necessarily map to an open set.

Theorem 4.10. Say $\pi : X \to Y$ is an identification map. Given a topological space Z, a function $f : Y \to Z$ is continuous if and only if $f \circ \pi : X \to Z$ is continuous

$$\begin{tikzcd} X \arrow[d, "\pi"] & \\ Y \arrow[r, "f"] & Z \end{tikzcd}$$

Proof. $\implies$ Automatic

$\impliedby$ Suppose $f \circ \pi$ is continuous. Then

$$(f \circ \pi)^{-1}(u)$$

is open in X. Then

$$\pi^{-1}(f^{-1}(u))$$

is open in X. Since π is an identification map, $f'^{-1}(u)$ is open in Y. So f is continuous.

$\square$

Theorem 4.11. Suppose $f : X \to Y$ is a surjective map that sends open sets to open sets or closed sets to closed sets. Then f is an identification map.

Proof. Suppose f sends open sets to open sets. We want to show that $\mathcal{O} \subset Y$ is open if and only if $f^{-1}(\mathcal{O})$ is open.
$\mathcal{O} \subset Y$ is open $\implies$ $f^{-1}(\mathcal{O})$ is open, since f is continuous. Suppose $f^{-1}(\mathcal{O})$ is open. Then $f(f^{-1}(\mathcal{O})$ is open, by assumption. The proof is similar if f sends closed sets to closed sets. $\square$

Theorem 4.12. Suppose $f : X \to Y$ is a surjective map. If X is compact and Y is Hausdorff then f is an identification map.

Proof. It's enough to show that if $C \subset X$ is closed, then $f(C) \subset Y$ is closed. Suppose C is closed. Then it is compact, so $f(c)$ is compact, so it is closed (since Y is Hausdorff). $\square$

Why did we only need to show that C is closed implies $f(C)$ is closed? It's because of Theorem 4.11.

Corollary 4.13. If $f : X \to Y$ is a surjective map, C is compact, and Y is Hausdorff, then Y is homeomorphic to the identification space induced by the partition $\mathcal{P}$ of X into $\{f^{-1}(y)\}_{y\in Y}$.

This means we have an induced homeomorphism h such that

$$\begin{tikzcd} X \arrow[d, "\pi"] \arrow[dr, "f"] & \\ Y^* \arrow[r, dashed, "h"] & Y \end{tikzcd}$$

the diagram commutes. Y^* denotes the identification space coming from the partition $\mathcal{P}$

Example 4.14. Consider Example 4.2 where we demonstrated an identification map from $[0,1] \times [0,1]$ to the torus T^2.

$$\mathcal{P} : \begin{cases} \{4 \text{ vertices} \\ \{(x,0),(x,1)\}, 0 < x < 1 \\ \{(0,y),(1,y)\}, 0 < y < 1 \\ \{\text{Each interior point}\} \text{ (singletons)} \end{cases}$$

By definition, T^2 is the identification space

Lets apply Corollary 4.13

Proposition 4.15. T^2 is homeomorphic to $S^1 \times S^1$

Proof. Consider the surjective map $f : [0,1] \times [0,1] \to S^1 \times S^1$ where

$$f(x,y) = (e^{2\pi ix}, e^{2\pi iy})$$

We can check that the preimages of f are exactly the elements of $\mathcal{P}$.
For example, look at $(1,1) \in S^1 \times S^1$. Then $f^{-1}(1,1) = ((0,0),(1,0),(0,1)(1,1))$. By Corollary 4.13, the identification space T^2 is homeomorphic to $S^1 \times S^1$. $\square$

2.21.2025

Since we're ahead of schedule, this class will be mainly examples and review, and next class will be an AMA

Example 4.16 (Cone). Let X be a topological space. Start with $X \times [0,1]$. This space looks like a cylinder. Take a partition

$$\mathcal{P} = \begin{cases} \{(x,1), x \in X\} \\ \text{Each } \{(x,t)\}, \text{ for } x \in X, 0 \leq t \leq 1 \end{cases}$$

This identification space is a **cone** over X

Example 4.17. Let $X = B^n$. Consider the partition

$$\mathcal{P} = \begin{cases} S^{n-1} = \{\vec{x} \in \mathbb{R}^n \mid |\vec{x}| = 1\} \\ \{\vec{x} \text{ if } |\vec{x}| < 1 \end{cases}$$

Call the identification space B^n/S^{n-1}, the quotient of a ball by it's boundary. We claim that B^n/S^{n-1} is homeomorphic to S^n

Proof. Construct $f : B^n \to S^n$ whose preimages are $\mathcal{P}$

$$f(x_1, \ldots, x_n) = (\pi\frac{\sin(r)}{r}x_1, \ldots, \pi\frac{\sin(r)}{r}x_n, \cos(r))$$

where $r = \pi\sqrt{x_1^2 + \cdots + x_n^2}$. This maps points in the ball to points on S^n. $\square$

A **projective space** P^n of some topological space can be imagined as it's shadow. We will give three examples of P^n as an identification space.

1. $X = S^n \subset \mathbb{R}^{n+1}$ Let the partition be
$$\mathcal{P} = \{\vec{x}, -\vec{x}\}, \vec{x} \in S$$
2. $X = \mathbb{R}^{n+1} \backslash \{\vec{0}\}$ Let the partition be
$$\mathcal{P} = \{\text{Lone through } \vec{0}, \text{ minus } \vec{0}\}$$
3. Crosscap model. Let $X = B^n$, and $\mathcal{P} = \begin{cases} \{\vec{x}, -\vec{x}\}, & \text{If } |\vec{x}| = 1 \\ \{\vec{x}\} & \text{If } |\vec{x}| < 1 \end{cases}$

All three of these lead to the same space, as we can see below.

$2 \implies 1$ Given a line ℓ through $\vec{O} \in \mathbb{R}^{n+1}$, see where it intersects S^n

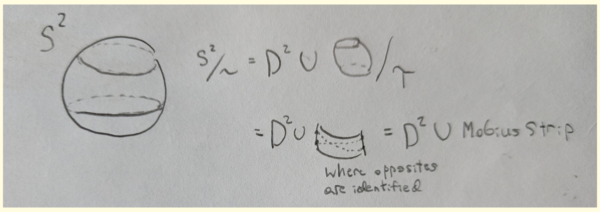
$1 \implies 3$ To see $S^n/\sim$, identify antipodal points. For most pairs of antipodal points, we can keep one in the northern hemisphere.

Another example of identification spaces are **attaching maps**. Let X, Y be topological spaces. Let $A \subset Y$, and $f : A \to X$ is a map. Our goal is to "glue" Y to X. Consider the partition

$$\mathcal{P} = \begin{cases} \{(a, f(a)) \text{ if } a \in A \\ \{x\} \text{ if } x \in X/f(A) \\ \{y\} \text{ if } y \in Y/A \end{cases}$$

The identification space is written $X \bigcup_f Y$, where f is called the attaching map.

Example 4.18. Let X be the mobius strip, and $Y = D^2$, a disc, with $A = S^1$. We claim that $X \cup_f Y = P^2$, because $P^2 = S^2/\sim$, where x is identified with $-\vec{x}$.



The diagram depicts the sphere S^2, where we identify antipodal points. Starting with the northern and southern hemispheres, we get a disc and the middle part of a sphere. Identifying the antipodal points of the rest of the sphere, we see that we get a mobius strip.

4.2 Topological Groups

Definition 4.19. A **group** is a set G with three operations

1. Multiplication $m : G^2 \to G$
2. Inverse $\iota : G^1 \to G$
3. Identity $e : G^0 \to G$

A topological group is a Hausdorff topological space with a group structure, such that m and ι are continuous.

2.26.2025

We actually forgot to talk about Gluings last week.

Definition 4.20 (Gluing of Maps). Say T is a topological space and X, Y are subspaces such that $T = X \cup Y$. Suppose we are given a topological space Z with maps $f : X \to Z$ and $g : Y \to Z$. Suppose $f(t) = g(t)$ for $t \in X \cap Y$. Define $F : T \to Z$ by

$$F(t) = \begin{cases} f(t), & t \in X \\ g(t), & t \in Y \end{cases}$$

Lemma 4.21 (Gluing Lemma). If X, Y are closed, then F is continuous.

Proof. Take $C \subseteq Z$ closed. $F^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$. f is continuous, so $f^{-1}(C)$ is closed in C, so $f^{-1}(c) = X \cap D$ for some D closed in T. X is closed, so $D \cap X = f^{-1}(c)$ is closed in T. And $g^{-1}(c)$ is closed in T by the same logic, so $F^{-1}(c) = f^{-1}(c) \cup g^{-1}(c)$ is closed in T. □

Application:
Consider the two functions $\alpha : [0,1] \to Z$ and $\beta : [0,1] \to Z$, so $\alpha(t) = \beta(0)$ Put

$$\gamma(t) = \begin{cases} \alpha(2t), & 0 < t \leq \frac{1}{2} \\ \beta(2t-1), & \frac{1}{2} \leq t < 1 \end{cases}$$

$T = [0,1]$, $x = [0, \frac{1}{2}]$

Now back to Topological Groups. We defined them above.

Example 4.22. The Circle $G = S^1$ is one. The operations are as follows: $\begin{cases} m(e^{i\theta}, e^{i\phi}) = e^{i(\theta+\phi)} \\ \iota(e^{i\theta}) = e^{-i\theta} \end{cases}$

Example 4.23. Here are some more examples of groups:
$G = (\mathbb{R}, +)$
Any group with the discrete topology
$T^2 = S^1 \times S^1$
$(\mathbb{R}^n, +)$

Example 4.24. **Unit quaternions**: The quaternions are defined as follows $\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$, with the following properties:
$i^2 = k^2 = j^2 = -1$
$ij = -ji = k$
$jk = -kj = i$
$ki = -ik = j$ and the unit quaternions are the set $\{a + bi + cj + dk : a^2 + b^2 + c^2 + d^2 = 1\}$. Topologically, this is $\{(a, b, c, d) \in \mathbb{R}^4 | a^2 + b^2 + c^2 + d^2 = 1\} \cong S^3 \subset \mathbb{R}^4$. The group operations are

$$m : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$$

which restricts to $m : U \times U \to U$, and

$$\iota(a + bi + cj + dk) = -a - bi - cj - dk$$

with $\iota : U \to U$

Example 4.25. The general linear group $GL(n) = \{$Invertible matrices $n \times n$ over $\mathbb{R}$. Write M_n to denote the set of $n \times n$ matrices, and for a matrix $A \in M_n$, we write $A = (a_{ij})$. So for the topology we identify each $A = (a_{ij})$ with the corresponding point

$$(a_{11}, a_{12}, \ldots, a_{21}, a_{22}, \ldots, a_{nn-1}, a_{nn})$$

of $\mathbb{E}^{n^2}$.

Claim: The group operations $m : M_n \times M_n$ and $\iota : M_n \to M_n$ is continuous.

Proof. $(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$ is a polynomial in a_{ij}s and b_{ij}s, respectively. $GL(n) \subset M_n$, so we give it the subspace topology. Then $m : GL(n) \times GL(n) \to GL(n)$ is continuous, since our polynomials are continuous and give us different values continuously when we change A or B.
The inverse is also continuous, see that $(A^{-1})_{ij} = \frac{1}{\det(A)}(\text{ji}^{\text{th}}$ cofactor of $A)$. The point is that this is just a ratio of two polynomials in entries of A, which is continuous in the entries (note that invertible matrices never have determinant 0) □

Definition 4.26. Topological groups are **isomorphic** (denoted $G_1 \cong G_2$) is there exists an isomorphism $\phi : G_1 \to G_2$ which is also a homeomorphism.

Definition 4.27. A **subgroup** of a topological group is a subgroup with the subspace topology.

Definition 4.28. Given $x \in G$, define left multiplication $L_x : G \to G$ by $L_x(g) = xg$, and right multiplication $R_x : G \to G$ be $R_x(g) = gx$.

Claim: L_x and R_x are homeomorphisms of G.

Proof. $L_{x^{-1}}(L_x(g)) = x^{-1}(xg) = g$ and $L_x(L_{x^{-1}}(g)) = x(x^{-1}g) = g$. Then $(L_x)^{-1} = L_{x^{-1}}$. We can write L_x as a map where $\phi(g) = (x, g)$ with $g \mapsto (x, g) \xrightarrow{m} xg$. But m is continuous by assumption, and ϕ is continuous by product topology, so L_x is continuous. Also, $(L_x)^{-1} = L_{x^{-1}}$ is continuous, so L_x is a homeomorphism. The same follows for R_x □

Corollary 4.29. Given $g, g' \in G$, there exists a homeomorphism $h : G \to G$ such that $h(g) = g'$. We call G a **homogeneous space**.

Proof. $h = L_{g'}g^{-1}$. □

For $g \in G$ we have $h(x) = g'g^{-1}g = g'$.

Theorem 4.30. Say G is a topological group and K is the component that contains the identity e. Then K is a closed normal subgroup of G.

Proof. Since K is a component, it is closed. To see K is a subgroup, we know that for any $x \in K$, $Kx^{-1} = R_{x^{-1}}(K)$. But this is continuous, so it's connected (since K is). We also know that $e = xx^{-1} \in Kx^{-1}$, so $Kx^{-1} \in K$, since K is a connected set which contains e. Since $e \in K$, we get $ex^{-1} = x^{-1} \in K$.
If y is in K, $yx' \in K$, so K is preserved by multiplication. Given $g \in G$, $L_g \circ R_{g^{-1}}$ is continuous. So $(L_g \circ R_{g^{-1}})(K)$ is connected. $(L_g \circ R_{g^{-1}})(e) = geg^{-1} = e$, so $e \in (L_g \circ R_{g^{-1}}(K)$, so it contains e and is connected, so $(L_g \circ R_{g^{-1}})(K) \subset K$, i.e.,

$$gkg^{-1} \subset K$$

hence K is a normal subgroup of G. □

2.28.2025

Last week, we saw that if K is a connected component of e then K is a closed normal subgroup.

Corollary 4.31. $\{\text{Components of } G\}$ gets a group structure as $G \backslash K$.

Theorem 4.32. Suppose G is connected. Then any open neighborhood V of e generates G.

Proof. Say $H =< v >$, the group generated by V. i.e., $h \in H$ means we can write $h = v_1^{n_1} v_2^{n_2} \cdots v_k^{n_k}$ where $v_i \in V$, $n_i \in \mathbb{Z}$. $H \neq \emptyset$, since $e \in H$. Since G is connected, it works to show that H is clopen, since the only clopen sets in a connected space are G and the empty set.

We will show that H is open and closed in G. To see that H is open, take $h \in H$. Look at hV. We know that $hV \in H$, since V generated H. We also know that hV is open, since $hV = L_h(V)$, which is a homeomorphism, and the preimage is open.

Now we just need to show that the space $G \backslash H$ is open. Suppose $g \in G \backslash H$. Look at gV. We want to show that it's a neighborhood of g in $G \backslash H$. To see $gV \in G \backslash H$, note that if it were not, $gV \cap H \neq \emptyset$. Take $x \in gV \cap H$. $x = gv$ for some $v \in V$. Then $g = xv^{-1}$. By assumption $x \in H$, so $xv^{-1} \in H$. Hence $g \in H$, which is a contradiction. Therefore we have $gv \subseteq G \backslash H$. $gv = L_g(V)$ is open in G, so we have $g = gV = G \backslash H$. Then $G \backslash H$ is open, and thus H is clopen in G, as desired. □

Nonexample 4.33. Let G be a nontrivial discrete group. Take $V = \{e\}$. Then $H = \{e\}$. Thus we need connectedness

Remark 4.34. Note that connectedness should not be thought of as closure of a group. We should instead think of the fact that G connected $\Leftrightarrow G \backslash K = \{e\}$.

Now we are going to look at more examples of topological groups.

Definition 4.35. The **orthogonal group** is defined as $O(n) = \{M \in M_n \mid MM^T = I\} \subseteq M_n \cong \mathbb{R}^{n^2}$
The **special orthogonal group** $SO(n) = \{M \in O(n) \mid \det M = 1\} \subseteq M_n \cong \mathbb{R}^{n^2}$

Theorem 4.36. $O(n)$ and $SO(n)$ are compact.

Proof. We want to show that they are both closed and bounded in $\mathbb{R}^{n^2}$. We have a map $\phi : M_n \to M_n$ with $\phi(M) = MM^T$, and $O(n) = \phi^{-1}(I_n)$. $\{I_n\}$ is closed in M_n, so $O(n)$ is closed in M_n.
We know that $MM^T = I$ by construction. Then $\mathrm{Tr}(MM^T) = n$[3], since

$$\mathrm{Tr}(MM^T) = \sum_{i,j=1}^{n} m_{ij}^2$$

so $O(n) \subset \{M \in M_n | \sum m_{ij}^2 = n\}$, which means it's bounded in $\mathbb{R}^{n^2}$. $SO_n(n) = \det^{-1}(1)$, a closed subset of $O(n)$, so it too is compact. □

Example 4.37. Let $n = 1$. $SO(1) = \{(1)\}$, and $O(1) = \{(1), (-1)\}$. These are the sets of matrices whose determinants are 1 or roots of 1, respectively

Example 4.38. Let $n = 2$. $SO(2) = \{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\}$ which is a circle. $O(2)$ is two disjoint circles.

Theorem 4.39. $SO(3)$ is homeomorphic to P^3.

Proof. We will construct a map $\phi : S^3 \to SO(3)$.

$$S^3 = \text{ Unit quaternions } = \{a + bi + cj + dk | a^2 + b^2 + c^2 + d^2 = 1\}$$

$$\mathbb{R}^3 = \text{ Pure quaternions } = \{bi + cj + dk | b, c, d \in \mathbb{R}\}$$

[3]The **trace** of a matrix is the sum of its diagonals. It corresponds to the sum of its eigenvalues, among other things.

Given $g \in S^3$, define $\phi(g) \in SO(3)$ by $\phi(g)\overrightarrow{v} = q\overrightarrow{v}q^{-1} \in \mathbb{R}^3$. We want to show that ϕ is a homomorphism, continuous, and onto.

We have $\ker(\phi) = \{\pm 1\}$. Then we can mod the domain by the kernel to get an isomorphism with the image[4]: $S^3/\{\pm 1\} \cong SO(3)$, i.e., quotienting/identifying $q \sim -q$ with respect to the equivalence relation that identifies antipodal points.

Then we get a 1-1, continuous function $\Phi : S^3/\{\pm 1\} \to S)(3)$, so we have a homeomorphism. Note that $S^3/\{\pm 1\} \cong P^3$ are homeomorphic. □

4.3 Group Actions

Definition 4.40 (Group Action)**.** Let G be a group and X a set. An **action** of G on X is a function

$$G \times X \to X$$

$$(g, x) = gx$$

such that $(g_1, g_2)(x) = g_1(g_2 x)$, and $ex = x$. We get a homomorphism $\alpha : G \to \text{Perm}(x)$ with $\alpha(g) = gx$

This was originally how groups were thought of; not as a set with structure, but rather actions on a set.

Example 4.41. $O(n)$ acts on $\mathbb{R}^n$

$$(A, \overrightarrow{v}) \to A\overrightarrow{v}$$

with A an orthogonal matrix, acting on $\overrightarrow{v}$ by left matrix multiplication

Definition 4.42 (Orbit)**.** An **orbit** of $x \in X$ is

$$O(x) = \{gx : g \in G\} \subset X$$

We get a partition of X by orbits, and the orbit space X/G is the set of orbits.

We can think of this as the set of stuff you can get to with a group action.

Example 4.43. $\mathbb{Z}$ (the infinite group of integers under addition) acts on $\mathbb{R}$ with $(n, x) \to x + n$.
$O(\frac{1}{3})$ is the set of numbers $\{x \in \mathbb{R} | x = \frac{1}{3} + k, k \in \mathbb{Z}\}$.
$\mathbb{R}/\mathbb{Z}$ is bijective to S^1.

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Example 4.44. $O(n)$ acts on S^{n-1} by $(M, \overrightarrow{x}) = M\overrightarrow{x}$. We claim this is a transitive action, i.e., there exists only 1 orbit

Proof. Say $\{\overrightarrow{e_1}, \ldots, \overrightarrow{e_n}\}$ is the standard basis of $\mathbb{R}^n$. Given $\overrightarrow{x} \in S^{n-1}$, we can construct a new orthonormal basis with $\overrightarrow{x}$ as its first member:

$$(\overrightarrow{x}, \overrightarrow{f_2}, \ldots, \overrightarrow{f_n})$$

Define $M \in O(n)$ by $M(\overrightarrow{e_1} = \overrightarrow{x})$, and $M(\overrightarrow{e_i} = \overrightarrow{f_i})$ if $i > 1$. Then $M = (\overrightarrow{x}|\overrightarrow{f_2}| \ldots |\overrightarrow{f_n}|)$ is the matrix of the new basis with respect to the standard basis. We see that orbit of e_1 is S^{n-1}, since $M\overrightarrow{e_1} = \overrightarrow{x}$, and we can choose any $\overrightarrow{x}$ and still construct an matrix in $O(n)$. □

Definition 4.45. A topological group G acts on the topological space X if $G \times X \to X$ is continuous. We get a homomorphism $\phi : G \to \text{Homeo}(X)$, where $\text{Homeo}(X)$ is group of homeomorphisms of X

Essentially, each element of G induces a homeomorphism of the space such that

[4] This is known as the **First Isomorphism Theorem**

1. We have associativity: $hg(x) = h(g(x))$
2. The identity $e(x) = x$
3. And continuity: $G \times X \to X$ is continuous

Example 4.46. $\mathbb{Z}^2$ acts on $\mathbb{R}^2$.

$$(n_1, n_2) \cdot (x_1, x_2) = (x_1 + n_1, x_2 + n_2)$$

$$\mathbb{R}^2 / \mathbb{Z}^2 \cong [0,1] \times [0,1] / \sim = T^2$$

where the equivalence relation identifies corners, opposing edges, and we get a torus.

Example 4.47. $G = C_2$, $X = S^n$. The elements of C_2 are $G = \{e, \tau\}$, with τ an involution. We see that $1 \cdot \overrightarrow{x} = \overrightarrow{x}$ and $\tau \cdot \overrightarrow{x} = \pm \overrightarrow{x}$. Then we have

$$S^n / C_2 = \{ \text{ pairs of antipodal points } = P^n \}$$

Example 4.48. G is a topological group, H a subgroup. H acts on G by left multiplication:

$$(h, g) = hg$$

The orbit space is g, g' in some orbit if and only if $g' = hg$ for some $h \in H$, aka $g' \in Hg$. So we notice that the orbits of $g \in G$ are right cosets of H in G, so Orbit space is a set of right cosets $\{Hg\}$.
Somewhat similarly, H acts on G by right multiplication

$$(h, g) \to gh^{-1}$$

The reason it is gh^{-1} should make sense if you consider the action laws.

If G acts on X, we get an identification space X/G coming from a partition of X into the orbits, and we get the identification topology X/G

Theorem 4.49. The identification map $\pi : X \to X/G$ is an open map, i.e., it sends open sets to open sets.

Proof. Say $\mathcal{O}$ is open. We want to show that $\pi(\mathcal{O})$ is open. Say $x \in \pi^{-1}(\pi(\mathcal{O}))$. Then $\pi(x) \in \pi(\mathcal{O})$. Hence $\pi(x) = \pi(y)$ for some $y \in \mathcal{O}$, i.e., x, y are in the same orbit, therefore there exists some $g \in G$ such that $x = gy$. This means that

$$\pi^{-1}(\pi(\mathcal{O})) = \bigcup_{g \in G} g\mathcal{O}$$

where the union is the union of orbits of points in the original open set. We know that L_g is a homeomorphism, so $g\mathcal{O}$ is open in X, hence

$$\bigcup_{g \in G} g\mathcal{O}$$

is open in X, since the union of open sets is open. $\square$

Theorem 4.50. If G acts on X, and G and X/G are connected, then X is connected.

Proof. Suppose $X = U \cup V$, with $U \cap V = \emptyset$ and U, V are nonempty open sets. We want a contradiction. $\pi(U)$ and $\pi(V)$ are open in X/G, and nonempty. Since the quotient space is connected, $\pi(U) \cap \pi(V) \neq \emptyset$. Say $\pi(x) \in \pi(U) \cap \pi(V)$. Then the orbit Gx intersects U and V. But G is connected, so it's orbit is as well. Then $(Gx \cap U) \cup (Gx \cap V) = Gx$ would disconnect Gx (since U and V are disjoint), which is a contradiction, since the orbit Gx is the image of a continuous function $f : G \times X \to X$, which means it is connected, and we just said it wasn't. $\square$

Example 4.51. $O(n)$ acts on S^{n-1} (transitively). Look at the **stabilizer** of $\vec{e}$, $\{g \in O(n) | g\vec{e_1} = \vec{e_1}\}$. If A is in the stabilizer then A has the form:

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & O(n-1) \end{array}\right)$$

where $O(n-1)$ is special because it is orthogonal, which means it fixes $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ (aka e_1). So we get a map $O(n)/O(n-1) \to S^{n-1}$[a] with $[g] \to g\vec{e_1}$, where $[g]$ is the coset $gO(n-1)$. This is a homeomorphism (because it is compact, Hausdorff, and a bijection, which we will not prove). Similarly $SO(n)/SO(n-1)$ is homeomorphic to S^{n-1}

[a] $O(n)/O(n-1)$ is the space of cosets.

Proposition 4.52. $SO(n)$ is connected.

Proof. By induction.

- $n = 1$: $SO(1) = \{(1)\}$, which is connected.
- Suppose $SO(n-1)$ is connected. Then $SO(n)/SO(n-1) \cong S^{n-1}$[5] If $n > 1$, then S^{n-1} is connected. By example 4.51, $SO(n)$ is connected.

□

Note that $O(n)$ is not connected, since it has two components $O(1) = \{(1), (-1)\}$

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Example 4.53. Let p, q be relatively prime integers $0 < p < q$, let $G = C_P$, and let $X = S^3$, which is the set of 4-tuples with magnitude 1 in $\mathbb{R}^4$ or equivalently (and what we will use in this scenario), the set of ordered pairs with magnitude 1 in $\mathbb{C}^2$. Let g be a generator of C_p, and let the action of this group on S^3 be

$$g(z_0, z_1) = (e^{\frac{2\pi i}{p}} z_0, e^{\frac{2\pi i}{p}} z_1)$$

Since we know what g does on elements of S^3, we see that when g acts p times, we get back to (z_0, z_1). The quotient space S^3/C_p is called the **lens space** $L(p, q)$

Example 4.54. Isometries (i.e., distance preserving motions) of $\mathbb{R}^3$ are

$$\{\vec{x} \to A\vec{x} + \vec{b} | A \in O(2), b \in \mathbb{R}^2\}$$

with A a rotation and $\vec{b}$ a translation[a]. We can bestow isometries with group structure, with group left multiplication given by

$$\vec{x} \to A_2\vec{x} + \vec{b_2} \to A_1(A_2\vec{x} + \vec{b}_2) + \vec{b_1}$$

Written as ordered pairs, we have $(\vec{b_1}, A_1) \cdot (A_2, \vec{b_2}) = (A_1A_2, \vec{b_1} + A\vec{b_2})$ We can give this group of isometries the topology of the product space $O(2) \times \mathbb{R}^2$, called the **Euclidean group** $E(2)$. Note that even though $E(2)$ has the product topology, the group structure is not the product structure, but rather the **semidirect product**: Given groups G_1, G_2, and a homomorphism $p : G_2 \to \text{Aut}(G_1)$, multiplication in $G_1 \tilde{\times}_p G_2$ is $(g_1, g_2)(g_1', g_2') = (g_1 p(g_1'), g_2 g_2')$.
In our case, $G_1 = (\mathbb{R}^2, +)$ and $G_2 = O(2)$. Then $\text{Aut}(G_1) = GL_2(\mathbb{R})$, and $p = O(2) \hookrightarrow GL_2(\mathbb{R})$. So

[5] They are homeomorphic. And the n-sphere is connected

$\text{Isom}(\mathbb{R}^2) = \mathbb{R}^2 \tilde{\times}_p O(2)$. We can choose the standard topology on $\mathbb{R}$ so that this is homeomorphic to $\mathbb{R}^2 \times S^1$, so $\text{Isom}(\mathbb{R}^2)$ has two connected components.

[a] The textbook denotes isometries as ordered pairs (θ, v), with $\theta \in O(2)$ and $v \in \mathbb{E}^2$

This leads into the idea of **Wallpaper Groups**:
Say G is a discrete subgroup of $\text{Isom}(\mathbb{R}^2)$. Then G acts on $\mathbb{R}^2$ by isometries

Definition 4.55. G is a **planar crystallographic group** if the orbit space $\mathbb{R}^2/G$ is compact.

Example 4.56. Say $\{\overrightarrow{b_1}, \overrightarrow{b_2}\}$ is a basis of $\mathbb{R}^2$. $G = \{\overrightarrow{x} \to \overrightarrow{x} + n_1\overrightarrow{b_1} + n_2\overrightarrow{b_2}\}$ with $n_1, n_2 \in \mathbb{R}$. Then $\mathbb{R}/G$ is topologically T^2.

There are 17 different isomorphism classes of planar crystallographic groups, and a finite number for any specific higher dimension. The planar crystallographic groups are called wallpaper groups because they classify the set of symmetries of a two-dimensional repetitive pattern, think a tiling of $\mathbb{R}^2$.

5 The Fundamental Group

This is when the algebraic topology really begins.

5.1 Homotopic Maps

In my complex analysis class we already talked a lot about what a homotopy is.

Definition 5.1. A **loop** in a topological space X is a map

$$\alpha : [0,1] \to X$$

such that $\alpha(0) = \alpha(1)$. We say that the loop is **based at** $\alpha(0)$

Idea: Form a group whose multiplication is loop concatenation, and where the inverse of a loop is going around the loop backwards.

Definition 5.2 (loop concatenation)**.** If $\alpha, \beta : [0,1] \to X$ are loops based at p, but $\gamma : [0,1] \to X$ to be

$$\gamma(t) = \begin{cases} \alpha(2t) : 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) : \frac{1}{2} \leq t \leq 1 \end{cases}$$

We have a few issues with this idea:

1. Too many loops
2. Multiplication is not associative. This is because $((\alpha\beta)\gamma))(\frac{2}{3}) = \gamma(\frac{1}{3})$, but $(\alpha(\beta\gamma))(\frac{2}{3}) = \beta(\frac{2}{3})$

We want some idea of a continuous deformation so that we can solve problem number one above. Say $f, g : X \to Y$. We want some family of maps

$$\{F_t\}_{t\in[0,1]}$$

such that $F_0 = f$, $F_1 = g$. To make this rigorous, look at the product space $X \times I$. Given a map $F : X \times I \to Y$, we get a family of maps $\{f_t\}_{t\in[9,1]}$ with $F_t : X \to Y$ by

$$F_t(x) = F(x,t)$$

Definition 5.3. Suppose $f, g : X \to Y$ are maps. We'll say f is **homotopic** to g if we can find $F : X \times I \to Y$ a map such that

$$F(x,0) = f, F(x,1) = g \ \forall x \in X$$

We write $f \underset{F}{\simeq} g$

Variation: If $A \subset X$ and $H_A = g|_A$, we may want to preserve this in the homotopy, i.e., $F(a,t) = f(a)$ for all $t \in [0,1]$. Then we say f and g are homotopic relative to A, and write $f \underset{F}{\simeq} g$ (rel$A$)

Example 5.4. Say $\alpha, \beta : I \to X$ are two loops based at p. We can ask if α, β are homotopic relative to $A = \{0,1\}$. If so, we get $F : I \times I \to X$ with $F(s,0) = \alpha(s), F(s,1) = \beta(s)$, and $F(0,t) = p = F(0,1)$ for all $t \in [0,1]$. So our homotopy is a family of curves centered at p.

Suppose $X \subset \mathbb{R}^n$ is **convex**, i.e., (any two points in C can be joined by a straight line in C)[6]. Given $\vec{f}, \vec{g} : X \to C$, put

$$F(x,t) = t\vec{g}(x) + (1-t)\vec{f}(x)$$

a linear homotopy. Then $f \underset{F}{\simeq} g$

Example 5.5. Say $f, g : X \to S^n$. Suppose $\vec{f}(x) + \vec{g}(x) \neq 0$ for all $x \in X$. We can homotope from f to g in $\mathbb{R}^{n+1}$, avoiding $\vec{0}$,

$$(1-t)\vec{f}(x) + t\vec{g}(x)$$

And our homotopy is

$$F(x,t) = \frac{(1-t)\vec{f}(x) + t\vec{g}(x)}{|(1-t)\vec{f}(x) + t\vec{g}(x)|}$$

Then $f \underset{F}{\simeq} g$ as maps from X to S^n.

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Today we will cover the fundamental group

Lemma 5.6. Homotopy is an equivalence relation on the set of maps from X to Y

Proof.

1. $f \underset{F}{\simeq} f$ by $F(x,t) = f(x)$
2. $f \underset{F}{\simeq} g$ and $g \underset{F}{\simeq} f$ if $G(x,t) = F(x, 1-t)$.
3. Suppose $f \underset{F}{\simeq} g$ and $g \underset{G}{\simeq} h$. Then put

$$H(x,t) = \begin{cases} F(x,2t), & \text{If } 0 \leq t \leq \frac{1}{2} \\ G(x,2t-1), & \text{If } \frac{1}{2} \leq t \leq 1 \end{cases}$$

so $f \underset{F}{\simeq} h$

□

Lemma 5.7. Homotopy (rel A) is an equivalence relation on maps $X \to Y$ with given restriction to $A \subset X$

Consider the diagram

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \arrow[r, "h"] & Z \end{tikzcd}$$

with X, Y, Z topological spaces and f, g, h maps. We ask if the diagram commutes.
Below are some definitions from my category theory class last semester:

[6] This is the definition of convex

Definition 5.8 (monomorphism)**.** In a category $\mathcal{C}$, a morphism $\alpha : a \to b$ is a **monomorphism** if and only if $\alpha f = \alpha g \Rightarrow f = g$ for all f, g.

$$\begin{tikzcd} x \arrow[r, bend left, "g"] \arrow[r, bend right, "f"'] & a \arrow[r, "\alpha"] & b \end{tikzcd}$$

Definition 5.9 (epimorphism)**.** α is an **epimorphism** if and only if $f\alpha = g\alpha \Rightarrow f = g$

$$\begin{tikzcd} a \arrow[r, "\alpha"] & b \arrow[r, bend left, "g"] \arrow[r, bend right, "f"'] & z \end{tikzcd}$$

Back to the topic at hand; the question we asked is actually not asking if homotopies are monomorphisms and epimorphisms[7], but it's interesting that algebraic topology and category theory are so connected.

Lemma 5.10. If $f, g : X \to Y$ are homotopic and $h : Y \to Z$ is a map, then $h \circ f$ is homotopic to $h \circ g$.

Proof. Say $f \underset{F}{\simeq} g$. Then $h \circ f \underset{F}{\simeq} h \circ g$. □

We have commutativity for

$$\begin{tikzcd} X \arrow[r, "h"] & Y \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Z \end{tikzcd}$$

as well, i.e., $f \underset{F}{\simeq} g \implies h \circ f \underset{F}{\simeq} h \circ g$.

Remark 5.11. If $g \simeq h$ (rel B), then $h \circ f \simeq h \circ g$ (rel $f^{-1}(B)$)

5.2 The Fundamental Group

We know that homotopy is an equivalence relation of loops based at p. The equivalence classes are called **homotopy classes** of loops. For any loop α, we will let $< \alpha >$ denote it's homotopy class.

Example 5.12. Let the topological space $X = S^1$ and consider point $p = 1 \in S^1$. Consider the two curves $\alpha : [0,1] \to X$, $\alpha(s) = p$ for all S and $\beta : [0,1] \to X$ with $\beta(s) = e^{2\pi i s}$. α, β are not homotopic, and so are in two different equivalence classes.

Definition 5.13 (Multiplication)**.** $< \alpha > \cdot < \beta > = < \alpha \cdot \beta >$, where

$$< \alpha \cdot \beta > (s) = \begin{cases} \alpha(2s), & 0 \leq s \leq \frac{1}{2} \\ \beta(2s-1), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

Lemma 5.14. This multiplication is well-defined, i.e., for $\alpha, \alpha' \in < \alpha >$, $\beta, \beta' \in < \beta >$, $< \alpha > \cdot < \beta > = < \alpha' > \cdot < \beta' >$

Proof. Let $\alpha \underset{F}{\simeq} \alpha'$ and $\beta \underset{G}{\simeq} \beta'$ We know $\alpha \cdot \beta \underset{H}{\simeq} \alpha' \cdot \beta'$ with

$$H(s,t) = \begin{cases} F(2s,t), & 0 \leq s \leq \frac{1}{2} \\ G(2s-1,t), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

Hence $< \alpha \cdot \beta > = < \alpha' \cdot \beta' >$ □

Theorem 5.15. These homotopy classes of loops form a group under multiplication.

[7] They would be monomorphisms if $f \underset{F}{\simeq} g \Leftrightarrow h \circ f \underset{F}{\simeq} h \circ g$, whereas we only care that $f \underset{F}{\simeq} g \implies h \circ f \underset{F}{\simeq} h \circ g$.

Proof. 1. Associativity: We want to show that $(<\alpha>\cdot<\beta>)\cdot<\gamma>=<\alpha>\cdot(<\beta>\cdot\gamma>)$. This is equivalent to showing $<\alpha\cdot\beta>\cdot<\gamma>=<\alpha>\cdot<\beta\cdot\gamma>$ which is

$$<(\alpha\cdot\beta)\cdot\gamma>=<\alpha\cdot(\beta\cdot\gamma)>$$

Claim: $((\alpha\cdot\beta)\cdot\gamma)=(\alpha\cdot(\beta\cdot\gamma))\circ f$, with $f: I\to I$ and

$$f(s)=\begin{cases}2s, & 0\le s\le \frac{1}{4}\\ s+\frac{1}{4}, & \frac{1}{4}\le s\le\frac{1}{2}\\ \frac{s+1}{2}, & \frac{1}{2}\le s\le 1\end{cases}$$

Since I is convex and $f(0)=0$, $f(1)=1$, there exists a homotopy of f to 1_I (rel $\{0,1\}$)[8]. Then

$$(\alpha\cdot\beta)\cdot\gamma=(\alpha\cdot(\beta\cdot\gamma))\circ f\simeq(\alpha\cdot(\beta\cdot\gamma))\circ 1_I=\alpha\cdot(\beta\cdot\gamma)$$

.

2. The identity element is the constant loop $e(s)=p$, $0\le s\le 1$. Then

$$<\alpha>\cdot<e>=<e>\cdot<\alpha>=<\alpha>$$

3. The inverse of $<\alpha>$ is $<\alpha^{-1}>$, with $\alpha^{-1}(s)=\alpha(1-s)$. Then $\alpha\cdot\alpha^{-1}=\alpha\circ f$, where

$$f(s)=\begin{cases}2s, & 0\le s\le\frac{1}{2}\\ 2-2s, & \frac{1}{2}\le s\le 1\end{cases}$$

$f: I\to I$ is convex, so we can homotope $f\to g(\text{rel }\{0,1\})$, where $g(s)=0$ for all s. Then

$$\alpha\cdot\alpha^{-1}=\alpha\circ f\simeq\alpha\circ g=e(\text{rel}\{0,1\})$$

□

Definition 5.16 (Fundamental Group)**.** A group with elements $<\alpha>$ is called the **Fundamental Group** of X based at p. It is denoted $\pi_1(X,p)$

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Remark 5.17. Note that $\pi_1(X,p)$ depends only on the path component containing p.

Theorem 5.18. If X is path connected then for all $p,q\in X$, $\pi_1(X,p)$ is isomorphic to $\pi_1(X,q)$.

Proof. Choose a path $\gamma:[0,1]\to X$. Given a loop α at p, we get a loop $\gamma^{-1}\cdot\alpha\cdot\gamma$ based at q. Define a function $\pi_1(X,p)\to\pi_1(X,q)$ be

$$<\alpha>\to<\gamma^{-1}\cdot\alpha\cdot\gamma>$$

- This is well defined: A homotopy of α gives rise to a homotopy of $\gamma^{-1}\alpha\gamma$, i.e., a homotopy H from α to α' induces a new one by keeping γ,γ^{-1} but homotoping $\alpha\to\alpha'$.
- It's a homomorphism because $<\alpha>\cdot<\beta>=<\alpha\cdot\beta>$ maps to

$$<\gamma^{-1}\cdot\alpha\cdot\beta\cdot\gamma>=<\gamma^{-1}\cdot\alpha\cdot\gamma\cdot\gamma^{-1}\cdot\beta\cdot\gamma>$$

  and of course $\gamma^{-1}\cdot\gamma$ is homotopic to the point map, so we have a homomorphism
- It's an isomorphism (inverse). Consider the map $\pi_1(X,q)\to\pi_1(x,p)$, and a loop $<\beta>$ based at q. We have a map

$$<\beta>\to<\gamma\cdot\beta\cdot\gamma^{-1}>$$

  based at p. So we have an isomorphism between $\pi_1(X,p)$ and $\pi_1(X,q)$.

□

[8]This keeps our composite loop centered at p

Theorem 5.19. Given a map $f : X \to Y$, we get a group homomorphism $f_* : \pi_1(X,p) \to \pi_1(Y, f(p))$ by

$$f_*(<\alpha>) = <f \circ \alpha>$$

with the corresponding diagram $[0,1] \xrightarrow{\alpha} X \xrightarrow{f} Y$

Proof.

$$\begin{aligned} f_*(<\alpha> \cdot <\beta>) &= f_*(<\alpha \cdot \beta>) \\ &= < f \circ (\alpha \cdot \beta) > \\ &= < (f \circ \alpha) \cdot (f \circ \beta) > \\ &= < f \circ \alpha > \cdot < f \circ \beta > \\ &= f_*(<\alpha>) \cdot f_*(<\beta>) \end{aligned}$$

□

Theorem 5.20. Given maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, we get $(g \circ f)_* : \pi_1(X,q) \to \pi_1(Z,r)$ where $(g \circ f)_* = g_* \circ f_*$,

$$f_* : \pi(X,p) \to \pi_1(Y,q), \quad q = f(p),$$

$$f_* : \pi(Y,q) \to \pi_1(Z,r), \quad r = g(q)$$

Corollary 5.21. If we have a homeomorphism $h : X \to Y$, then $h_* : \pi_1(X,p) \to \pi_1(Y, h(p))$ is an isomorphism of groups.

Proof. We have

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[rr, bend right, "\mathsf{Id}_X"'] & Y \arrow[r, "h^{-1}"] & X \end{tikzcd} \quad \begin{tikzcd} Y \arrow[r, "h^{-1}"] \arrow[rr, bend right, "\mathsf{Id}_Y"'] & X \arrow[r, "h"] & y \end{tikzcd}$$

These induce the following homomorphisms

$$\begin{tikzcd} \pi_1(X,p) \arrow[r, "h_*"] \arrow[rr, bend right, "\mathsf{Id}_{\pi_1(X,p)}"'] & \pi_1(Y,h(p)) \arrow[r, "h_*^{-1}"] & \pi_1(X,p) \end{tikzcd}$$

$$\begin{tikzcd} \pi_1(Y,h(p)) \arrow[r, "h_*^{-1}"] \arrow[rr, bend right, "\mathsf{Id}_{\pi_1(Y,h(p))}"'] & \pi_1(X,p) \arrow[r, "h_*"] & \pi_1(Y,h(p)) \end{tikzcd}$$

So $h_* : \pi_1(X,p) \to \pi_1(X, h(p))$ has an inverse homomorphism $(h^{-1})_* : \pi_1(Y, h(p)) \to \pi_1(X,p)$. □

Up until now, all our examples of fundamental groups have been trivial. Now we will do nontrivial ones. Towards this goal, we introduce a lemma about metric spaces.

Definition 5.22. Let X be a metric space, $S \subseteq X$. The diameter of S is

$$\operatorname{diam}(S) := \sup_{s_1, s_2 \in S} d(s_1, s_2)$$

Lemma 5.23 (Lebesgue Lemma)**.** Say X is a compact metric space and $\mathcal{F}$ an open cover of X. Then there exists $\delta > 0$ (called the lebesgue number of the covering) such that for all $A \subseteq X$ with $\operatorname{diam}(A) < \delta$, there exists some $U \in \mathcal{F}$ such that $A \subseteq U$

Basically, we can always take a speck of dust of diameter δ and it is covered by a single member of the open covering.

Proof. Suppose no such δ exists. Then for all n, there exists a subset $A_n \subseteq X$ such that $\text{diam}(A_n) < \frac{1}{n}$ but A_n is not contained in a member of $\mathcal{F}$. For each n, choose $x_n \in A_n$. We can find a subsequence such that $\lim_{n\to\infty} x_n = p$ (Since X is compact, there exists a convergent subsequence). Choose $U \in \mathcal{F}$ such that $p \in U$. U is open, so we can find $\epsilon > 0$ such that $B(p, \epsilon) \subset U$. Choose N such that

1. $\text{diam}(A_n) < \frac{1}{2}\epsilon$
2. $d(x_n, p) < \frac{1}{2}\epsilon$.

Look at $x \in A_n$. Then $d(x, x_n) < \frac{1}{2}\epsilon$ (from 1), and $d(x_n, p) < \frac{1}{2}\epsilon$ (from 2) So

$$d(x, p) < d(x, x_n) + d(x_n, p) < \epsilon$$

, Hence $x \in B(p, \epsilon) \subset U$. Since x was any point in A_n, we have $A_n \subseteq U$, which is a contradiction. □

Example 5.24. Identify the circle with the unit circle in $\mathbb{C}$, and let $\pi : \mathbb{R} \to S^1$, denote the mapping $\pi(s) = e^{2\pi i s}$. Take $1 \in S^1$ as the basepoint. Given $n \in \mathbb{Z}$, put $\gamma_n(s) = ns$, with $0 \le s \le 1$. Then $\pi \circ \gamma_n$ is a loop in S^1 based at 1.

This motivates the following theorem, which we will prove next class.

Theorem 5.25. A function $\phi : \mathbb{Z} \to \pi_1(S^1, 1)$ given by $\phi(n) =< \pi \circ \gamma_n >$ is an isomorphism, i.e., the circle has a nontrivial fundamental group.

3.12.2025

5.3 The fundamental group of S^1 and Brouwer's fixed-point theorem

Today we will compute the fundamental group of S^1, that is, $\pi_1(S^1)$. Following example 5.24, let $\pi : \mathbb{R} \to S^1$, denote the mapping $\pi(x) = e^{2\pi i x}$. Take $1 \in S^1$ as the basepoint.

Construct a map $\phi : \mathbb{Z} \to \pi_1(S^1)$ as follows: let $\gamma_n(s) = ns$ for $0 \le s \le 1$, and $\gamma_n : [0, 1] \to \mathbb{R}$. Then $\pi \circ \gamma_n$ is a loop in S^1 based at $1 \in S^1$ that winds around S^1 n times[9]. Put $\phi(n) =< \pi \circ \gamma_n >$. We want to show that it is a group isomorphism.

Remark 5.26. If γ is any path in $\mathbb{R}$ from 0 to n, then $\gamma \simeq \gamma_n(\text{Rel}\{0, 1\})$, since $\mathbb{R}$ is convex. So

$$< \pi \circ \gamma > \simeq < \pi \circ \gamma_n >$$

Lemma 5.27. $< \pi \circ \gamma >$ is a homomorphism.

Proof. We want to show that $\phi(m + n) = \phi(m) + \phi(n)$. This is true if $< \gamma_{m+n} >=< \gamma_m > \cdot < \gamma_n >$, which needs,

$$\gamma_{m+n} \simeq \gamma_m \cdot \gamma_n$$

These are both paths in $\mathbb{R}$ from 0 to $m + n$, so they are both homotopic, so we have a homomorphism. □

Next, we claim that ϕ is onto (surjective), i.e., any $< \alpha > \in \pi_1(S^1, 1)$ is such that $\alpha \simeq \pi \circ \gamma_n$[10]. In order to do this, we try to 'lift' the loop α to a path γ in $\mathbb{R}$ such that $\pi \circ \gamma = \alpha$, where $\gamma(0) = 0$. If we can do this, then $\pi(\gamma(1)) = 1$, so $\gamma(1) \in \mathbb{Z}$ (by construction). Denote this integer as n, and call it the **degree** of α. γ is homotopic to γ_n, and so $\alpha = \pi \circ \gamma \simeq \pi \circ \gamma_n$ (rel$\{0, 1\}$), i.e., $< \alpha >=< \pi \circ \gamma >=< \pi \circ \gamma_n >= \phi(n)$.

In order to do the lifting process, we must understand $\pi : \mathbb{R} \to \mathbb{S}^1$ in more detail. Let U be the open set $S^1 \backslash \{-1\}$, and $V = S^1 \backslash \{1\}$ open as well. Then $\{U, V\}$ is an open cover of S^1. Then the inverses are as follows:

$$\pi^{-1}(V) = \mathbb{R} \backslash \mathbb{Z} = \bigcup_{n \in \mathbb{Z}} (n, n + 1)$$

[9]Winding number reference

[10]Take the time to understand this: we can deform α (any loop) into the path looping around the circle n times.

$$\pi^{-1}(U) = \mathbb{R} \backslash \{\frac{1}{2} + \mathbb{Z}\} = \bigcup_{n \in \mathbb{Z}} (n - \frac{1}{2}, n + \frac{1}{2})$$

We see that the inverses map each open set to open sets such that restricting to any one of the open intervals gives a homeomorphism. So the idea is that if we have a loop in S^1, we can break it up into segments such that eat segment lies in U or V, and then lift each one back into $\mathbb{R}$ individually.

Lemma 5.28 (Path-lifting lemma). If σ is a path in S^1 beginning at 1, there exists a unique path $\tilde{\sigma}$ in $\mathbb{R}$ such that $\sigma(0) = 0$ and $\pi \circ \tilde{\sigma} = \sigma$, i.e., the diagram commutes:

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "\pi"] \\ {[0,1]} \arrow[ur, "\tilde{\sigma}"] \arrow[r, "\sigma"'] & S^1 \end{tikzcd}$$

Proof. The pair $\{\sigma^{-1}(U), \sigma^{-1}(V)\}$ is an open cover of $[0,1]$. By the lebesgue lemma, there exists a $d > 0$ such that any subset A of $[0,1]$ with $\text{diam}(A) < d$ lies in $\sigma^{-1}(U)$ or $\sigma^{-1}(V)$, i.e., $\sigma(A) \subset U$ or $\sigma(A) \subset V$. Divide $[0,1]$ as $0 = T_0 < T_1 < \cdots < T_n = 1$, such that $|T_{i+1} - T_i| < d$. Since $\sigma(0) = 1 \in U$, we know that $\sigma([T_0, T_1]) \subset U$. Remember that $\pi|_{(-\frac{1}{2},\frac{1}{2})} : (-\frac{1}{2}, \frac{1}{2}) \to U$ is a homeomorphism, so let f denote it's inverse.

Now, put $\tilde{\sigma}(s) = f(\sigma(s))$ for $0 \leq s \leq t$. Inductively, suppose $\tilde{\sigma}$ is defined on $[0, T_k]$. We want to define it (i.e., $\tilde{\sigma}$) on $[T_k, T_{k+1}]$. We know that $\sigma([T_k, T_{k+1}])$ is entirely in U or V. If $\sigma([T_k, T_{k+1}]) \subseteq U$ and $\tilde{\sigma}(T_k) \in (n - \frac{1}{2}, n + \frac{1}{2})$. Let g be the inverse of $\pi|_{(n-\frac{1}{2},n+\frac{1}{2})} : (n - \frac{1}{2}, n + \frac{1}{2}) \to U$, and set $\tilde{\sigma}(s) = g(\sigma(s))$ for $T_k \leq s \leq T_{k+1}$.

Considering the other case, if $\sigma([T_k, T_{k+1}]) \subseteq V$, then $\tilde{\sigma}(T_k) \in (n, n+1)$ for some n. The restriction $\pi|_{(n,n+1)} : (n, n+1) \to V$ is a homeomorphism with an inverse, say, h. Then we can define $\tilde{\sigma}(s) := h(\sigma(s))$ for $T_k \leq s \leq T_{k+1}$. This completes our inductive definition of the lifted path. Also, see that since after we define $\tilde{\sigma}$ on $[0, T_k]$, there is only one way to extend it over $[T_k, T_k + 1]$, so $\tilde{\sigma}$ is unique. $\square$

So we know that ϕ is surjective. In order to prove that ϕ is injective, we will need to 'lift' homotopies from the circle to $\mathbb{R}$.

Lemma 5.29 (Homotopy-lifting lemma). If $F : I \times I \to S^1$ is a map with $F(0,t) = F(1,t) = 1 \in S^1$ for all $0 \leq t \leq 1$, then there exists a unique map $\tilde{F} : I \times I \to \mathbb{R}$ such that $\tilde{F}(0,t) = 0$ for all $0 \leq t \leq 1$ and $\pi \circ \tilde{F} = F$.

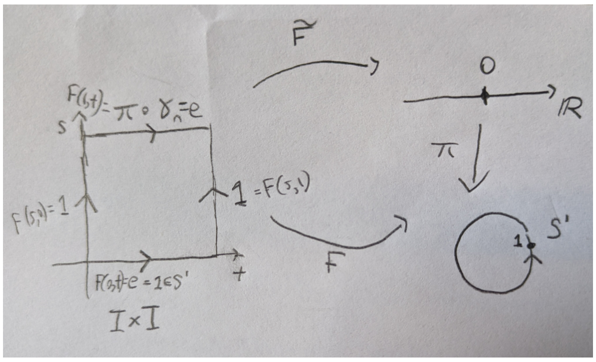
Proof. The proof for this follows similarly to the Path-lifting lemma. $\{F^{-1}(U), F^{-1}(V)\}$ is an open cover of $[0,1] \times [0,1]$. The Lebesgue lemma gives $\delta > 0$, for which $A \subseteq I \times I$, $\text{diam}(A) < \delta \implies F(A) \subset U$ or $F(A) \subset V$. Subdivide the unit square so that each square has diam$< \delta$. The idea is we lift F on squares iteratively, starting from the lower left, to get $\tilde{F}$. $\square$

So now we can prove Theorem 5.25, that $\phi(n) =< \pi \circ \gamma_n >$ is a group isomorphism.

1. We've shown that ϕ is a homomorphism.

2. We have that ϕ is onto by the path lifting lemma, as stated earlier.

3. Finally, we show that ϕ is injective, or one-to-one. Suppose $n \in \ker(\phi)$, i.e., $\phi(n) =< e >\in \pi_1(S^1, 1)$, where $e(s) = 1$ for all $0 \leq s \leq 1$. Then $\pi \circ \gamma_n = e$. We call $\pi \circ \gamma_n$ a **null-homotopic loop**. Say $F : I \times I \to S^1$ is a homotopy from e (the constant loop at 1) to $\pi \circ \gamma_n$. By the Homotopy-lifting lemma, we can lift F to some $\tilde{F} : I \times I \to \mathbb{R}$ with $\tilde{F}(0,t) = 0$ for all $t \in [0,1]$.



Say $p = (\text{Left edge}) \cup (\text{Bottom edge}) \cup (\text{Right edge})$. Then p is connected and $F(p) = \pi(\tilde{F}(p)) = 1 \in S^1$. So $\tilde{F}(p) = m$ for some $m \in \mathbb{Z}$, i.e., all of p is sent to some single integer. But $\tilde{F}(0,0) = 0$, so $N = 0$, and all of p is sent there. The path in $\mathbb{R}$ defined by $\{F(\tilde{S}, 1)$ is a lift of $\pi \circ \gamma_n$ which begins at 0, and by uniqueness, it must be γ_n. Since $\tilde{F}(1,1) = 0$, we get $\gamma_n(1) = n = 0$, so the kernel $\ker(\phi) = \{0\}$. This implies that ϕ is injective.

So we've shown that there is a homomorphism between the fundamental group of S^1 and the group $\mathbb{Z}$, i.e., $\pi_1(S^1) \cong \mathbb{Z}$.

3.14.2025

Definition 5.30. Say X is a set, $f : X \to X$ a function. We say $x \in X$ is a **fixed point** for f if $f(x) = x$.

Proposition 5.31. If $f : [0,1] \to [0,1]$ is a map then f has a fixed point.

Proof. Define $g(x) = f(x) - x$, $g(0) = g(0) - 0 \geq 0$. Then

$$g(1) = f(1) - 1 \leq 0$$

So by the intermediate value theorem, there exists an x such that it is a fixed point for f. □

Remark 5.32. Not all maps $f : S^1 \to S^1$ have a fixed point

Theorem 5.33 (Brouwer's fixed-point Theorem)**.** Any $f : D^2 \to D^2$ has a fixed point, where D^2 is the closed unit disc in $\mathbb{R}^2$

Proof. Towards contradiction, suppose f has no fixed points, i.e., $f(x) \neq x$ for all $x \in D^2$. Draw a line segment from $f(x)$ to x and extend the line until it hits the circle. Call this point $g(x)$. We see that

$$g : D^2 \to S^1$$

If $x \in S^1$, then $g(x) = x$, i.e., g fixes the points on the circle.

Say i is the inclusion map $i : S^1 \to D^2$. Then we have

$$\begin{tikzcd} S^1 \arrow[r, "i"] \arrow[rr, bend right, "Id_{S^1}"'] & D^2 \arrow[r, "g"] & S^1 \end{tikzcd}$$

with $g \circ i = Id_{S^1}$. Say $p = 1$. Then we get

$$\begin{tikzcd} \pi_1(S^1, p) \arrow[r, "i_*"] \arrow[rr, bend right, "Id_{\pi_1(S^1,p)}"'] & \pi_1(D^2, p) \arrow[r, "g_*"] & \pi_1(S^1, p) \end{tikzcd}$$

With the fundamental groups of $\pi_1(S^1, p)$ and $\pi_1(D^2, p)$ being $\mathbb{Z}$ and 0[11] respectively. We see that $g_* \circ i_* = (\mathrm{Id}_{S^1})_* = \mathrm{Id}_{\pi_1(S^1,p)}$. But $g_* \circ i_* = 0 : \mathbb{Z} \to \mathbb{Z}$, since $\pi_1(D^2, p) = 0$, i.e., it factors through the trivial group. This is a contradiction, so there must be a fixed point. □

Corollary 5.34. If X is homeomorphic to D^2, then any map $f : X \to X$ has a fixed point.

Proof. Say $h : D^2 \to X$ is a homeomorphism. $h^{-1} \circ f \circ h : D^2 \to D^2$ is a map. By Brouwer's fixed-point theorem, there exists a $z \in D^2$ such that $h^{-1} \circ f \circ h = z$. Then $(f \circ h)(z) = h(z)$, i.e., $f(h(z)) = h(z)$, so $h(z)$ is a fixed point for f. □

[11] $\pi_1(D^2, p)$ is trivial since D^2 is convex in $\mathbb{R}^2$

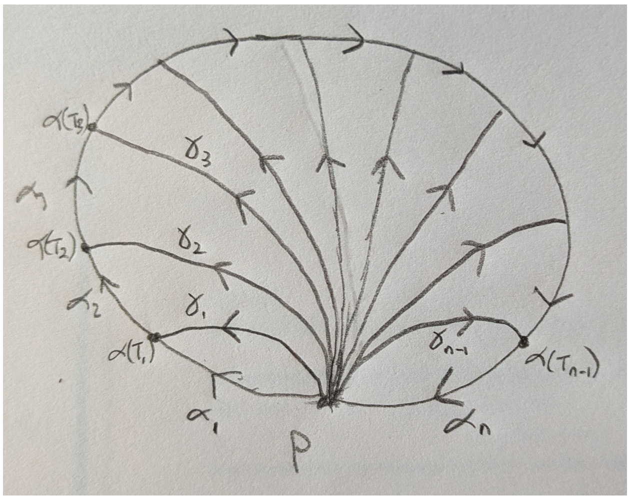
Definition 5.35. X is **simply connected** if it is path connected and its fundamental group is trivial, i.e., $\pi_1(X) \cong \{e\}$

Theorem 5.36. If $X = A \cup B$, where A, B are open, simply connected, and their intersection path connected, then X is simply connected.

Proof. It's easy to see that X is path connected. Take $p \in A \cup B$ to be the base point, and consider a loop $\alpha : [0,1] \to X$ based at p, i.e., $\alpha(0) = \alpha(1) = p$. Then

$$\{\alpha^{-1}(A), \alpha^{-1}(B)\}$$

(that is, the inverse of the subset is the set of points in the interval which map into said subset through the path) is an open cover of $[0,1]$. By the Lebesgue lemma, we can split the interval into $0 = T_0 < T_1 < \cdots < T_n = 1$, such that $[T_i.T_{i+1}]$ is entirely contained in $\alpha^{-1}(A)$ or $\alpha^{-1}(B)$.



- If $\alpha(T_k) \in A$, choose a path γ_k from p to $\alpha(T_k)$ which lies entirely within A.
- If $\alpha(T_k) \in B$, choose γ_k from p to $\alpha(T_k)$ that lies entirely in B.

Note: If $\alpha(T_k) \in A \cap B$, γ_k lies in $A \cap B$ because their intersection is path connected.
We can see from the diagram that

$$\begin{aligned} <\alpha> &=< \alpha_1\gamma_1^{-1}\gamma_1\alpha_2\gamma_2^{-1}\gamma_2 \cdots \alpha_{n-1}\gamma_{n-1}^{-1}\gamma_{n-1}\alpha_n > \\ &=< \alpha_1\gamma_1^{-1} >< \gamma_1\alpha_2\gamma_2^{-1} > \cdots < \gamma_{n-1}\alpha_n > \end{aligned}$$

which is a product of homotopy classes of loops based at p, each lying entirely in A or B. But these are all homotopic to a point map at p, hence $<\alpha>=< e_p >$. ☐

Example 5.37. Let $X = S^n$, with $n > 1$, and $A = S^n \backslash \{\text{North Pole}\}$, and $B = S^n \backslash \{\text{South Pole}\}$. A, B are homeomorphic to $\mathbb{R}^n$, so they are simply connected. $A \cap B = S^1 \backslash \{\text{North, South poles}\}$ is path connected, since $n > 1$. Then we see that S^n must be simply connected.

Nonexample 5.38. Let $X = S^1$, and A, B the same sets as before. Their intersection is not path connected, so we cannot apply the result.

Nonexample 5.39. Let $X = S^1$. Suppose A is a clopen, simply connected subset, and all of the other assumptions hold (so B and one side of A share a boundary). Obviously this doesn't suddenly make S^1 path connected. Let this be a lesson not to forget these conditions, however. The Mayor of Evanston Illinois forgot to use the fact that A, B had to be open, and now he's the Mayor of Evanston.

3.17.2025

5.4 Calculations

Definition 5.40. A group acts **freely** if for all x, $\mathrm{Stab}(x) = e$.

Example 5.41. $(\mathbb{Z}, +)$ acts freely on $\mathbb{R}$

Theorem 5.42. Suppose G is a discrete group acting on a simply connected topological space X such that for all $x \in X$, there exists an open neighborhood U such that $U \cap g(U) = \emptyset$ for all $g \neq e$. Then $\pi_1(X/G) \cong G$

Example 5.43. $G = \mathbb{Z}$, $X = \mathbb{R}$, $U = (x - \frac{1}{6}, x + \frac{1}{6})$.

$$X/G = S^1, \text{ so } \pi_1(X/G) \cong G, \text{i.e., } \pi_1(S^1) \cong \mathbb{Z}$$

We will model our proof based on the above example, so keep it in mind when understanding the proof.

Proof. (of the theorem). Fix $x_0 \in X$. Given $g \in G$, join x_0 to $g(x_0)$ by a path γ_g. If $\pi : X \to X/G$ denotes the quotient map, $\pi \circ \gamma_g$ is a loop of $\pi(x_0)$ (since $(\pi \circ \gamma_g)(1) = \pi(\gamma_g(1)) = \pi(g(x_0)) = \pi(x_0)$).

Define $\phi : G \to \pi_1(X, \pi(x_0))$ by

$$\phi(g) = < \pi \circ \gamma_g >$$

We claim this is independent of our choice of path γ_g. If γ_g' is another choice, then $\gamma_g' \circ \gamma_g^{-1}$ is a loop at x_0. Since X is simply connected, this is homotopic to $e_{x_0}(\mathrm{rel}\{0,1\})$. Say $\{\eta_T\}_{T \in [0,1]}$ is a 1-parameter family of loops from $\gamma_g' \circ \gamma_g^{-1}$ to e_{x_0} based at x_0. Then γ_g' is homotopic to $\gamma_g(\mathrm{rel}\{0,1\})$ by $\{\eta_T \circ \gamma_g\}_{T \in [0,1]}$, so $< \pi \circ \gamma_g' > = < \pi \circ \gamma_g >$. So ϕ is well defined.

Now, we want to show that this is an isomorphism (similar to what we did for S^1, using the path and homotopy-lifting lemmas).

- To show ϕ is onto, consider a path σ in X/G that begins at $\pi(x_0)$. Then there exists a unique $\tilde{\sigma}$, a path in X, such that $\tilde{\sigma}(0) = x_0$ and $\sigma = \pi \circ \tilde{\sigma}$.
- To show that it is injective, we use the homotopy-lifting lemma. If a map $F : I \times I \to X/G$ with $F(0,t) = F(1,t) = \pi(x_0)$, then there is a unique $\tilde{F} : I \times I \to X$ such that $\tilde{F}(0,t) = X$ for all $t \in [0,1]$ and $F = \pi \circ \tilde{F}$.

Of course, one might wonder if these lemmas truly hold on this space. These lemmas hold for any map $\pi : X \to Y$ with the following property. For any $y \in Y$, there is an open neighborhood V such that $\pi^{-1}(V)$ can be decomposed into a family $\{U_\alpha\}$ of pairwise disjoint sets such that the restriction of π to each U_α is a homeomorphism from U_α to V. This map π is a called a **covering map**, and X is called a **covering space** of Y. ◻

So in the example above, $\mathbb{R}$ is a covering space of S^1

Example 5.44. $G = C^n$, $X = \mathbb{R}^n$, acting by integer translation. By definition, $X/G = T^n$. So $\pi_1(T^n) \cong \mathbb{Z}^n$

Example 5.45. C_2 on $X = S^n$. $X/G = P^n$, and so $\pi_1(P^n) = C_2$

Example 5.46. Let $G = \mathbb{Z}/p$ on S^3. $S^3/\mathbb{Z}/p = L(p,q)$, the lens space. So $\pi_1(L(p,q)) \cong \mathbb{Z}/p$.

Remark 5.47. $\pi_1(X)$ is not necessarily abelian.

Theorem 5.48. Let X, Y be connected. Then $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.

Proof. Choose base points $x_0 \in X$, $y_0 \in Y$. Use (x_0, y_0) as a basepoint in $X \times Y$. We have canonical projections $p_1 : X \times Y \to X$, $p_2 : X \times Y \to Y$. Then these projections induce homomorphisms

$$p_{1_*} : \pi_1(X \times Y) \to \pi_1(X)$$

and the same for p_{2_*}

Define $\psi : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ by $\psi(<\gamma>) = (<p_1 \circ \gamma>, <p_2 \circ \gamma>)$. This is a homomorphism, and we want to show that it is an isomorphism.

Claim: ψ is injective: Suppose $\alpha \in \ker(\psi)$, i.e.,

$$\psi(\alpha) = (<e_{x_0}>, <e_{y_0}>)$$

i.e.,

$$p_1 \circ \alpha \underset{F}{\simeq} e_{x_0}$$
$$p_1 \circ \alpha \underset{G}{\simeq} e_{y_0}$$

Then $\alpha \underset{H}{\simeq} (e_{x_0}, e_{y_0})$ by $H(s,t) = (F(s,t), G(s,t))$.

We also claim ψ is surjective: Given $<\alpha> \in \pi_1(X, x_0)$ and $<p> \in \pi_1(Y, y_0)$, put $\gamma(s) = (\alpha(s), \beta(s))$, a loop in $X \times Y$. Then $\psi(<\gamma>) = (<\alpha>, <\beta>)$ So we are done. ☐

Example 5.49. Let $X, Y = S^1$. Then $X \times Y = T^2$. So this means $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$. We can realize the trefoil knot as a nontrivial loop on T^2

Definition 5.50. Topological spaces X, Y are **homotopy equivalent** if there exist maps

$$F : X \to Y$$

and

$$G : Y \to X$$

such that $G \circ F \simeq \mathrm{Id}_X$ and $F \circ G \simeq \mathrm{Id}_Y$. We say G is a homotopy inverse to F and the pair F, G is a homotopy equivalence of X, Y

Theorem 5.51. Homotopy equivalence is an equivalence relation.

Proof.

1.
$$\begin{tikzcd} X \arrow[r, bend left, "\mathrm{Id}_X"] \arrow[r, bend right, "\mathrm{Id}_X"'] & X \end{tikzcd}$$
implies $X \simeq X$

2.
$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \end{tikzcd}$$
implies
$$\begin{tikzcd} Y \arrow[r, bend left, "g"] \arrow[r, bend right, "f"'] & X \end{tikzcd}$$
therefore, $X \simeq Y \implies Y \simeq X$

3.
$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \end{tikzcd}, \quad \begin{tikzcd} Y \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] & Z \end{tikzcd}$$

implies

$$\begin{tikzcd} X \arrow[r, bend left, "u \circ f"] \arrow[r, bend right, "g \circ v"'] & Z \end{tikzcd}$$

$$\begin{aligned} (g \circ v) \circ (u \circ f) &= g \circ (v \circ u) \circ f && (1)\\ &\simeq g \circ \mathrm{Id}_Y \circ f && (2)\\ &= g \circ f && (3)\\ &\simeq \mathrm{Id}_X && (4) \end{aligned}$$

$$\begin{aligned} (u \circ f) \circ (g \circ v) &= u \circ (f \circ g) \circ v && (5)\\ &\simeq u \circ \mathrm{Id}_X \circ v && (6)\\ &= u \circ v && (7)\\ &\simeq \mathrm{Id}_Y && (8) \end{aligned}$$

therefore we have transitivity

□

Definition 5.52 (Algebraic Topology)**.** So now we have a new definition of the field which we are aiming to study: **Algebraic Topology** is the study of topological spaces up to homotopy equivalence.

3.19.2025

Definition 5.53. Homotopy

3.21.2025

6 Homework

6.1 Homework 1, due 2/2/2025

p. 31: 1(a,c,d), 3, 12

Homework 6.1.1 (1 (a, c, d)). Verify each of the following for arbitrary subsets A, B of a space X:

a) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

c) $\overline{\overline{A}} = \overline{A}$

d) $(A \cup B)^{\circ} \supseteq \mathring{A} \cup \mathring{B}$

a)
- Suppose $x \in \overline{A}$. If x is not a limit point, then it is in A and thus in both $\overline{A \cup B}$ and $\overline{A} \cup \overline{B}$
- Then suppose x is a limit point of A. Then for every neighborhood U containing x, $U \cap A \backslash \{x\} \neq \emptyset$. Then $U \cap (A \cup B) \backslash \{x\} \neq \emptyset$, since $U \cap A \backslash \{x\} \subseteq U \cap (A \cup B) \backslash \{x\}$.
- The same holds for $x \in \overline{B}$, so every element $\in \overline{A}$ is also in $\overline{A \cup B}$, so $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$
- Now suppose $y \in \overline{A \cup B}$ is a limit point. This means that for every neighborhood N of y, $N \cap A \cup B \backslash \{y\} \neq 0$. Then $N \cap \overline{A} \cup \overline{B} \backslash \{y\} \neq 0$ as well, since $A \cup B \subseteq \overline{A} \cup \overline{B}$. But then y is in $\overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$, since the intersection of closed sets is closed, and the closure of a closed set is just the set itself. (which we will show in (c)).

c)
- We know that $\overline{A} \subseteq \overline{\overline{A}}$, since the closure of a set contains that set. Now we just need to show that $\overline{\overline{A}} \subseteq \overline{A}$
- By definition, $\overline{\overline{A}}$ is the intersection of all closed sets which contain $\overline{A}$. But $\overline{A}$ is itself a closed set, so $\overline{\overline{A}} = \overline{A}$ which is the smallest closed set containing itself.

d) Without loss of generality, suppose $z \in \mathring{A}$. Then there exists some open set $\mathcal{O}$ such that $x \in \mathcal{O} \subseteq A$. Then $\mathcal{O} \subseteq A \cup B$, so $z \in (A \mathring{\cup} B)$. The case for $z \in \mathring{B}$ is the same.

Homework 6.1.2 (3). Specify the interior, closure, and frontier of the following subsets of the plane:

a) $\{(x, y) | 1 < x^2 + y^2 \leq 2\}$

b) $\mathbb{E}^2$ with both axes removed

c) $\mathbb{E}^2 \backslash \{(x, \sin(\frac{1}{x})) | x > 0\}$

a)
- i. The interior is the set $\{(x, y) | 1 < x^2 + y^2 < 2\}$
- ii. The closure is the set $\{(x, y) | 1 \leq x^2 + y^2 \leq 2\}$
- iii. The frontier is the set $\{(x, y) | x^2 + y^2 = 1\} \cup \{(x, y) | x^2 + y^2 = 2\}$

b)
- i. The interior is the set $\{(x, y) | x \neq 0 \text{ and } y \neq 0\}$
- ii. The closure is $\mathbb{E}^2$
- iii. The frontier is the set $\{(x, y) | x = 0 \text{ or } y = 0\}$

c)
- i. The interior is the set $\{(x, y) | y \neq \sin(\frac{1}{x}) \text{ and } x > 0\}$
- ii. The closure is the $\mathbb{E}^2$
- iii. The frontier is the curve $y = \sin(\frac{1}{x})$ where $x > 0$

Homework 6.1.3 (12). Show that if X has a countable basis for its topology, then X contains a countable dense subset. A space whose topology has a countable base is called a **second countable space**. A space which contains a countable dense subset is said to be **separable**.

Suppose X has a countable basis β for its topology. Then for every open subset, there is a sequence of basis elements $\{B_n\}_{n\in N}$ such that their union is the subset. Then for every neighborhood N around every point p, there exists a basis element B_n around it such that $p \in B_n \subseteq N$. Consider the set of points $Q = \{p_n\}_{B_n \in \beta}$ where $p_n \in B_n$. Take any point $x \in X$. If $x = p_n$ for some $p_n \in Q$, then $x \in P$. If x is not in P, then for any open neighborhood N, there exists a basis element B_j such that $x \in B_j \subseteq N$. And $B_j \cap Q \backslash \{x\} = p_j$, which means x is a limit point of P. Then every point $x \in X$ is either in P or is a limit point of P, so $x \in \overline{P}$. But if there are only countably many basis elements, and each $p_n \in P$ corresponds to a single B_n, then there is a bijection from $\beta \to P$, which means P is countable, which means X is separable.

6.2 Homework 2, due 2/9/2025

p. 35: 17,21
p. 41: 28

Homework 6.2.1 (17). Let X denote the set of real numbers with the finite-complement topology, and define $f : \mathbb{E} \to X$ by $f(x) = x$. Show that f is continuous, but not a homeomorphism.

Suppose $\mathcal{O} \in X$ is some open set. Then the set $X \backslash \mathcal{O}$ has a finite number of elements. For every $x \in \mathcal{O}$, consider $f(x) = x \in \mathbb{E}$, and let N be some open neighborhood around it. If N does not intersect $f(X \backslash \mathcal{O})$, then $f(\mathcal{O})$ is open. But for every $p \in N$, there exists some open ball $B(p, \epsilon)$ with $\epsilon = \frac{d(\mathcal{O},p)}{2}$ such that $B(p,\epsilon) \cap \mathcal{O} = \emptyset$, so $N \cap \mathcal{O} = \emptyset$, so $\mathcal{O}$ is open, as desired.
f is not a homeomorphism because there exists open sets in $\mathbb{E}$ whose complements are infinite and thus are closed under the finite-complement topology.

Homework 6.2.2 (21). Show that the unit ball in $\mathbb{E}^n$ and the unit cube (points whose coordinates satisfy $|x_i| \leq 1 \; \forall \; 0 < i \leq n$) are homeomorphic if they are both given the subspace topology from $\mathbb{E}^n$.

There is a function $f : B(0,1) \to \square^n$, where $\square^n$ represents the unit cube. We recognize that the square is a unit ball under the ℓ^∞ maximum metric, whereas the regular unit ball in $\mathbb{E}^n$ is a unit ball under the ℓ^2 euclidean metric.
We define f where for each $(x_1, \ldots, x_n) \in B(0,1)$, $(x_1, \ldots, x_n) \neq (0, \ldots, 0) \mapsto \frac{\max(|x_1|,\ldots,|x_n|)}{\sqrt{x_1^2 + \cdots + x_n^2}}$ but when $x = (0, \ldots, 0)$ it maps to $(0, \ldots, 0)$. The inverse map is defined similarly , but when $x \neq (0, \ldots, 0)$, we have $(x_1, \ldots, x_n) \mapsto \frac{\sqrt{x_1^2+\cdots+x_n^2}}{\max(|x_1|,\ldots,|x_n|)}(x_1, \ldots, x_n)$.
This function is continuous, since for every open ball $\mathcal{O} \in \square^n$, $f^{-1}(\mathcal{O})$ is open in $B(0,1)$, which means for every point $p \in \mathcal{O}$, for every open ball, $B(p, \epsilon) \in \mathcal{O}$, there exists some ball $B(f^{-1}(p), \frac{\epsilon}{2}) \subset f^{-1}(\mathcal{O})$. So for every point in an open set $\mathcal{O}$, there exists some open ball such that it is contained in $f(\mathcal{O})$, and since its defined on every point in $\mathcal{O}$, $f(\mathcal{O}) = \bigcup_{p \in \mathcal{O}} B(f^{-1}(p), \frac{\epsilon}{2})$. The same logic holds for continuous inverse, since for every open ball $\mathcal{G}$ around some point y, we can take some other point q and some δ such that $B(q, \delta) \subset \mathcal{G}$, so there exists some ball $B(f(q), \frac{\delta}{2}) \subset f(\mathcal{G})$.
The map is a bijection because every element maps to and from a unique element. Thus it is a homeomorphism.

Homework 6.2.3. If A, B are disjoint closed subsets of a metric space X, find disjoint open sets U, V such that $A \subseteq U$ and $B \subseteq V$.

Without loss of generality, consider the frontier of A. We know that for every $p \in \text{Frontier}(A)$, for every $\epsilon > 0$, the ball $B(p, \epsilon) \cap X \backslash A \neq \emptyset$. For every point x on the frontier of A, consider the distance $d(x, B)$. We know that for any point y on the frontier of B, $d(y, A) \cap d(x, B) = \emptyset$. Then let $\epsilon_x = \frac{d(x,B)}{2}$, and consider the open set

$$\bigcup_{x \in \text{Frontier}(A)} B(x, \epsilon_x) \cup A$$

and the open set around B defined similarly:

$$\bigcup_{y \in \text{Frontier}(B)} B(y, \frac{d(y,A)}{2}) \cup B$$

The intersection of these sets is empty, and each set is open because the set $A \backslash \text{Frontier}(A)$ is open and the union of open sets is open.

6.3 Homework 3, due 2/16/2025

p. 47, 3
p. 50, 14
p. 55, 21

> **Homework 6.3.1 (3).** Use Heine-Borel to show that an infinite subset of a closed interval must have a limit point.

By Heine-Borel, every closed interval is compact. Then every closed interval has a finite subcover. So for every open cover, there exists a finite subcover of open sets of the interval. Then for every infinite subset, there exists some open set in our open cover which contains infinite elements of our infinite subset. So no matter how small of an epsilon neighborhood we choose, there will always be an infinite number of elements in that neighborhood, which means we have a limit point.

> **Homework 6.3.2 (14).** If $f : X \to Y$ is a one-to-one map, and if $f : X \to f(X)$ is a homeomorphism when we give $f(X)$ the induced topology from Y, we call f an **embedding** of X in Y. Show that a one-to-one map from a compact space to a Hausdorff space must be an embedding.

Suppose $f : X \to Y$ is a one-to-one map from a compact space to a Hausdorff space. Then we just need to show that $f : X \to f(X)$ is surjective. A function is surjective if every element in $f(X)$ has a preimage. But this is true by construction, so we have a bijective, continuous map from a compact space to a Hausdorff space, which is therefore a homeomorphism and so we have an embedding.

> **Homework 6.3.3 (21).** If A and B are compact, and if W is a neighborhood of $A \times B$ in $X \times Y$, find a neighborhood U of A in X and a neighborhood V of B in Y such that $U \times V \subseteq W$.

If W is a neighborhood of $A \times B$, then it is the union of a collection of basis vectors

$$\bigcup_{i \in I, j \in J} X_i \times Y_j = W$$

where I, J are indexing sets. Let U be the collection of basis elements such that $X_i \times B \subseteq W$ and $i \in I$, and let V be the collection of basis elements such that $A \times Y_j \subseteq W$ and $j \in J$. Then we have

$$A \times B \subseteq U \times V \subseteq W$$

where U and V are both unions of open sets, and therefore open. And WLOG, U is a neighborhood of A since for every $p \in A$, we have some basis element X_i such that $p \times B \subseteq X_i \times B \subseteq W$. The same holds for B, which means we have found a neighborhood V of B in Y and a neighborhood U of A in X such that $U \times V \subseteq W$.

6.4 Homework 4, due 2/23/2025

p. 60, 31, 34

Homework 6.4.1 (31). Give the set of real numbers the finite-complement topology. What are the components of the resulting space? Answer the same question for the half-open interval topology.

A space has the finite-complement topology when a set is open if its complement is finite or all of X.

- Consider a finite subset C of $\mathbb{R}$. Any two subsets of C are finite, and thus are closed. But we can partition C into two disjoint finite subsets A, B, which are closed, so $\overline{A} \cap \overline{B} = \emptyset$. So a finite set is not connected.

- Now consider a subset C which is infinite. Then when we have A, B such that $A \cup B = C$, then at least one of A or B is infinite. But the closure of any infinite set (WLOG assume it is A) is just $\mathbb{R}$, since there is no finite subset S such that $A \subseteq S$. This means C is connected.

So any infinite subset of $\mathbb{R}$ is connected. Then the maximal connected subset is just $\mathbb{R}$ itself.

Definition 6.1. Let $X = \mathbb{R}$ and β the family of subsets of the form $\{x \mid a \leq x < b,$ s.t. $a < b$. This forms the basis for a topology, known as the **half-open interval topology**. Each member of β is both open and closed.

The half-open interval topology is totally disconnected. Consider some nonempty subset C of $\mathbb{R}$. We can always find some element $a \in C$ and then take the sets $A = \{x \mid x \in C \wedge x < a\}$ and $B = \{y \mid y \in C \wedge a \leq y\}$. The closure of these sets do not intersect. So our only connected sets are singleton sets, since we cannot choose two disjoint sets A, B as defined above. Thus the components are just each point.

Homework 6.4.2 (34). A space X is locally connected if for each $x \in X$, and each neighborhood U of x, there is a connected neighborhood V of x which is contained in U. Show that any euclidean space, and therefore any space which is locally euclidean (like a surface), is locally connected. If $X = \{0\} \cup \{\frac{1}{n} \mid n = 1, 2, \dots\}$ with the subspace topology from the real line, show that X is not locally connected.

We know that any interval in $\mathbb{R}$ is connected. For any neighborhood U around any point $(r_1, \dots, r_n) \in \mathbb{R}^n$, we can find an open ball B such that $x \in B \subset U$, since $\mathbb{R}^n$ is Hausdorff. And any open ball is path connected, since for any two points $a, b \in B_\epsilon$, there exists a straight line $\gamma(t) = ta + (1-t)b$ connecting them. This straight line segment is contained in B_ϵ since it is convex. So we have that any open ball in $\mathbb{R}^n$ is path connected, which means it is connected, which means $\mathbb{R}^n$ is locally connected.
Consider $X = \{0\} \cup \{\frac{1}{n} \mid n = 1, 2, \dots\}$ with the subspace topology from $\mathbb{R}$. Consider an open neighborhood U around $x = 0$. We know that there exists some epsilon ball B_ϵ contained in any open neighborhood such that $x \in B_\epsilon$. But for every $\epsilon > 0$, we can find some $n > \frac{1}{\epsilon}$ such that $S_\epsilon = \{x \mid x < n, x \in X\} \subset B_\epsilon$. So for any $V \subset U$ such that $x \in V$, we have some ϵ such that $S_\epsilon \subset V$. But S_ϵ is not an interval, so V is not connected. Then X is not connected locally.

6.5 Homework 5, due 3/2/2025

p. 63: 39, 40
p. 72: 7 (Only describing the space with some explanation is enough, no rigorous proof is required.)

> **Homework 6.5.1 (39).** Prove that the product of two path-connected spaces is path connected.

For any two points (x, y) and (a, b) in our new space $A \times B$, we have two paths $\gamma_1 : [0, 1] \to A$ and $\gamma_2 : [0, 1] \to B$, with $\gamma_1(0) = x$, $\gamma_1(1) = a$, $\gamma_2(0) = y$, and $\gamma_2(1) = b$. Then we have a map $\gamma = (\gamma_1, \gamma_2)$, such that $\gamma(0) = (x, y)$ and $\gamma(1) = (a, b)$. We know that this path is continuous because the projections of the resultant map pre-composed with γ are γ_1 and γ_2, which are both maps and therefore continuous, which means γ is continuous.

> **Homework 6.5.2 (40).** If A and B are path-connected subsets of a space, and if $A \cap B$ is nonempty, prove that $A \cup B$ is path-connected.

We want to show that A and B are contained in the same path component, since that would imply $A \cup B$ is path connected If $A \cap B$ is nonempty, then there exists some point $c \in A \cap B$. For any two points a, b in $A \cup B$, we have a path between each of them and c. To show this is true, we know that $a \in A$ or $a \in B$. Without loss of generality, assume $a \in A$. So is c, so there exists some path γ_1 such that $\gamma_1(0) = a$, and connecting them in A. Then γ_1 connects them in $A \cup B$ as well. The same holds for b, whether it is in A or B, and so we see that there is some other path γ_2 such that $\gamma_2(0) = c$, $\gamma_2(1) = b$. And since paths compose, there exists some path γ such that $\gamma(0) = a$and $\gamma(1) = b$.

> **Homework 6.5.3 (7).** Describe each of the following spaces
>
> **a)** The cylinder with each of its boundary circles identified to a point
>
> **b)** he torus with the subset consisting of a meridional and a longitudinal circle identified to a point
>
> **c)** S^2 with the equator identified to a point
>
> **d)** $\mathbb{E}^2$ with each of the circles center the origin and of integer radius identified to a point.

a) The resultant space is shaped like a donut with a meridian shrunk down to a point, sort of like a croissant with both ends touching.

b) The resultant space looks similar to the one above, except there is no hole in the middle, but rather a point. If we inflate it it becomes a sphere.

c) The space is two spheres touching each other next to each other.

d) This space looks like an infinite sequence of beads on a string, each one next to the following one.

6.6 Homework 6, due 3/9/2025

p. 78: 16
p. 85, 27, 31

Homework 6.6.1 (16)**.** Prove that $O(n)$ is homeomorphic to $SO(n) \times Z_2$. Are these two isomorphic as topological groups?

We have some intuition for these since we know that $SO(n)$ is connected, whereas $O(n)$ has two connected components. We want to show that $SO(n) \times Z_2$ is compact and $O(n)$ is Hausdorff, and there is a bijection between them, since this would imply they are homeomorphic.

We know that they are compact as shown in class, or rather, we know that $SO(n)$ and $O(n)$ are compact. $SO(n) \times Z_2$ is compact since Z_2 is a finite set and thus is compact, and the product of two compact sets is compact. $O(n)$ is Hausdorff by definition. So we just need to show that the two topological groups have a bijection between them.

There is a bijection between $SO(n) \times Z_2$ and $O(n)$ as follows. Z_2 has two elements, $1, -1$, where -1 is an involution. $SO(n)$ is the number of matrices in $O(n)$ with $\det(1)$. The remaining matrices are the ones in $O(n)$ with $\det(-1)$. There is a bijection $\phi : O(n) \to SO(n) \times Z_2$ with $\phi(M) = (M \begin{pmatrix} \det(M) & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ \cdots & 0 & 0 & 1 \end{pmatrix}, \det(M))$, since changing the sign of one row of a matrix changes the sign of the determinant.

The two groups are isomorphic if there is a group isomorphism between them. There is no bijection between $O(n)$ and $SO(n) \times Z_2$, so they are not isomorphic as groups. Consider our bijection above as an example. $\phi : O(n) \to SO(n) \times Z_2$ with $\phi(M) = (M \begin{pmatrix} \det(M) & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ \cdots & 0 & 0 & 1 \end{pmatrix}, \det(M))$ is not a homomorphism, since if we have two matrices M, N with determinant -1, then we have as follows: Put $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $N = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. Then $\phi(MN) = (\begin{pmatrix} ab + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}, 1)$. But $\phi(M)\phi(N) = (\begin{pmatrix} ae - bg & af - bh \\ -ce + dg & -cf + dh \end{pmatrix}, 1)$. This is true for all other bijections.

Homework 6.6.2 (27)**.** Find an action of $\mathbb{Z}_2$ on the torus with orbit space the cylinder.

$\mathbb{Z}_2$ acts on the torus with orbit space the cylinder. So we have $T^2/\mathbb{Z}_2 \cong$ Cylinder. $\mathbb{Z}_2 = \{e, \tau\}$, with τ an involution. If we plot T^2 in $\mathbb{R}^3$, we the operations act on T^2 as follows:

$$1 \cdot (x, y, z) = (x, y, z) \in T^2$$

$$\tau \cdot (x, y, z) = (-x, y, z,) \in T^2$$

So when quotienting we are identifying opposite points across the y-z place. This is a cylinder since it looks like we are cutting the torus in half. □

Homework 6.6.3 (31)**.** The stabilizer of a point $x \in X$ consists of those elements g in G for which $g(x) = x$. Show that the stabilizer of any point is a closed subgroup of G when X is Hausdorff, and that points in the same orbit have conjugate stabilizers for any x.

The stabilizer obeys the properties of a subgroup:

- Closure: for all $g, h \in \text{Stab}(x)$, $gh(x) = g(h(x)) = g(x) = x$
- Associativity: $ex = x$, so $e \in \text{Stab}(x)$
- Inverses: for all $g \in \text{Stab}(x)$, $x = ex = g^{-1}gx = g^{-1}(gx) = g^{-1}(x)$, so g^{-1} is in $\text{Stab}(x)$

The stabilizer subgroup is closed when X is Hausdorff because the group action is a continuous function $f : G \to X$. Then $\text{Stab}(x) = f^{-1}(x)$, and x is closed, so by continuity the preimage must be closed as well.

Any point y in the same orbit has a conjugate stabilizer if $h\text{Stab}(x)h^{-1} = \text{Stab}(y)$ for some $h \in G$. y is in the same orbit of x if there exists h such that $hx = y$.

Let $g \in \mathsf{Stab}(y)$. We see that $h^{-1}gh(x) = h^{-1}gy = h^{-1}y = x$. Then $h^{-1}\mathsf{Stab}(y)h = \mathsf{Stab}(x)$, which implies $h\mathsf{Stab}(x)h^{-1} = \mathsf{Stab}(y)$, as desired. $\square$

6.7 Homework 7, due 3/16/2025

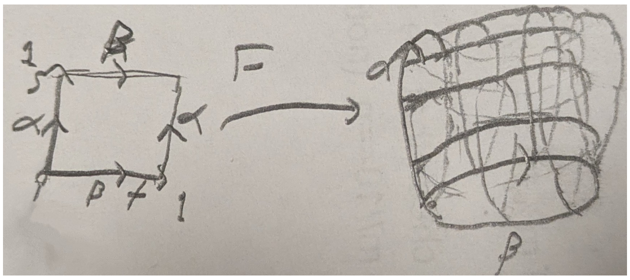
p. 91: 1
p. 95: 13

Homework 6.7.1. Let C denote the unit circle in the plane. Suppose $f : C \to C$ is a map which is not homotopic to the identity. Prove that $f(x) = -x$ for some point x of C.

By the example 5.1.2, we have that if we have two functions which are never antipodal, then they have a homotopy between them. We can let $g(x) = x$, and since $C = S^1$, we see that $f(x)$, which is never antipodal to $g(x)$, must not be homotopic to g.

Homework 6.7.2. Let G be a path-connected topological group. Given two loops α, β based at e in G, define a map $F : [0,1] \times [0,1] \to G$ by $F(s,t) = \alpha(s) \cdot \beta(t)$, where the dot denotes multiplication in G. Draw a diagram to show the effect of this map on the square, and prove that the fundamental group of G is abelian.

We want to show that $< \alpha > \cdot < \beta > = < \beta > \cdot < \alpha >$. Notice that $\alpha(0) = e$, $\beta(0) = e$. This implies $F(0,t) = F(1,t) = \beta(t)$, and $F(s,0) = F(s,1) = \alpha(s)$. So $\alpha \underset{F}{\simeq} \alpha(s)$ and $\beta \underset{F}{\simeq} \beta(t)$. We know that we can concatenate loops based at the same point. But F is a continuous deformation of loops, which means these loops are simply connected, so we have the trivial group, which is abelian.



6.8 Homework 8, due 3/23/2025

p. 95: 10
p. 102: 18, 21

Homework 6.8.1. Let γ, σ be two paths in the space X which begin at the point p and end at q. As in the proof of theorem (5.6), these paths induce isomorphisms γ_*, σ_* of $\pi_1(X,q)$ with $\pi_1(X,q)$. Show that σ_* is the composition of γ_* and the inner automorphism of $\pi_1(X,q)$ induced by the element $< \sigma^{-1}\gamma >$.

Let $< \sigma^{-1}\gamma > \_ < \sigma^{-1}\gamma >^{-1}$ denote the inner automorphism of $\pi_1(X,q)$ induced by the elements $< \sigma^{-1}\gamma >$. We want to show that the diagram

$$\begin{tikzcd}
\pi_1(X,p) \arrow[r, "\sigma_*"] \arrow[dr, "\gamma_*"'] & \pi_1(X,q) \\
 & \pi_1(X,q) \arrow[u, "<\sigma^{-1}\gamma>_<\sigma^{-1}\gamma>^{-1}"']
\end{tikzcd}$$

commutes.

Consider some loop α based at p and its equivalence class $< \alpha > \in \pi_1(X,p)$. Passing it through the diagram we see that

$$\begin{aligned}
(< \sigma^{-1}\gamma > _ < \sigma^{-1}\gamma >^{-1} \circ \gamma_*)(< \alpha >) &= (< \sigma^{-1}\gamma > _ < \sigma^{-1}\gamma >^{-1})(\gamma_*(< \alpha >)) \\
&= (< \sigma^{-1}\gamma > _ < \sigma^{-1}\gamma >^{-1})(< \gamma^{-1} \cdot \alpha \cdot \gamma >) \\
&= < \sigma^{-1}\gamma >< \gamma^{-1} \cdot \alpha \cdot \gamma >< \sigma^{-1}\gamma >^{-1} \\
&= < \sigma^{-1}\gamma\gamma^{-1} \cdot \alpha \cdot \gamma\gamma^{-1}\sigma > \\
&= < \sigma^{-1} \cdot \alpha \cdot \sigma > \\
&= \sigma_*(< \alpha >)
\end{aligned}$$

Homework 6.8.2. Let $\pi : X \to Y$ be a covering map. So each point $y \in Y$ has a neighborhood V for which $\pi^{-1}(V)$ breaks up as a union of disjoint open sets, each of which maps homeomorphically onto V under π. Call such a neighborhood 'canonical'. If α is a path in Y, show how to find points $0 = t_0 < t_1 < \cdots < t_m = 1$ such that $\alpha([t_i, t_{i+1}])$ lies in a canonical neighborhood for $0 \le i \le m-1$. Hence lift α piece by piece to a (unique) path in X which begins at any preassigned point of $\pi^{-1}(\alpha(0))$.

For each point $y \in Y$, consider the canonical neighborhood V. We have an open cover of canonical neighborhoods $\mathcal{V} = \{V_y\}_{y \in Y}$. Then we see that $\{\alpha^{-1}(V_y) : V_y \in \mathcal{V}\}$ is an open cover of the interval $[0,1]$. Since the interval is compact, we can apply Lebesgue lemma, and see that there exists some $\delta > 0$ such that we can divide $[0,1]$ into intervals with diameter $< \delta$ and have that each interval $[t_i, t_{i+1}]$ is completely contained in one of the V_y. So each $\alpha([t_i, t_{i+1}])$ is completely contained in a canonical neighborhood.

Let $\tilde{\alpha} : [0,1] \to X$ be a path in X. We can fix $\tilde{\alpha}(0) = p$ where p is some preassigned point in $\pi^{-1}(\alpha(0))$. Then for each i indexing α, put $\tilde{\alpha}(t_i) \in (\pi|_{A_i})^{-1}(\alpha(t_i))$, where $A_i = \pi^{-1}(\alpha(t_i))$. This lifts α in Y to some path $\tilde{\alpha}$ in X.

Homework 6.8.3. Describe the homomorphism $f_* : \pi_1(S^1, 1) \to \pi_1(S^1, f(1))$ induced by each of the following maps:

a) The antipodal map $f(e^{i\theta}) = e^{i(\theta+\pi)}$, $0 \le \theta \le 2\pi$

b) $f(e^{i\theta}) = e^{in\theta}$, $0 \le \theta \le 2\pi$, where $n \in \mathbb{Z}$

c) $f(e^{i\theta}) = \begin{cases} e^{i\theta} & 0 \le \theta \le \pi \\ e^{i(2\pi-\theta)} & \pi \le \theta \le 2\pi \end{cases}$

a) This homomorphism is the identity map

b) By De Moivre's formula, this moves each point around the circle n times it's current angle. Since $\pi_1(S^1) \cong \mathbb{Z}$, we can use that terminology. f_* maps 0 to itself, and then any map $x \ne 0$ has $x \mapsto nx$

c) This takes any curve and flips it around when it gets to $\theta = \pi$. So everything maps to the identity 0.

6.9 Homework 9, due 4/6/2025

p. 91: 1
p. 95: 13