Category Theory in Context by Emily Riehl + Solutions + MUSA 174 + more - Notes

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Contents

§1 Category Theory

§1.1 Abstract and Concrete Categories

Definition 1.1 (isomorphism)**.** An **isomorphism** is a morphism $f : X \to Y$ for which $\exists$ a morphism $g : Y \to X$ so that $gf = 1_X$ and $fg = 1_Y$. The objects X and Y are **isomorphic** whenever there exists an isomorphism between them, expressed X

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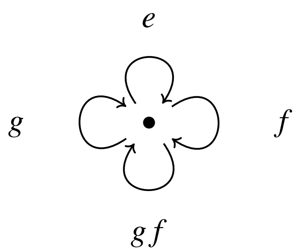
§1.1.1 Class 2: 8/30

Definition 1.2 (discrete category)**.** A category is descrete if all morphisms are identities.

> **Example 1.3**
>
> Given any set X, can make a discrete category:
> $\mathrm{Ob}(X_d) = X$ (every element in the set is an object.
> $\mathrm{Mor}(X_d)$ = identities

Theme is building new categories out of existing mathematical objects.

Definition 1.4 (delooping)**.** Let G be a group. The **delooping** of G, BG
$\mathrm{Ob}(BG) = \cdot$
$\mathrm{Mor}(BG) = \{g \in G\}$



Where $\cdot$ is the set of elements in G, and e, f, g, gf are group actions on $\cdot$

Definition 1.5 (Category of Matrices ($_k$))**.** $\mathrm{Ob}(Mat_k) = (N) \bigcup 0$
$\mathrm{Mor}(_k) = (m, n) = n * m$ matrices with entries in k Composition is matrix multiplication

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Exercise 1.6 (Exercise)**.** Convince yourself that

$$C(a, x) \sim x^a$$

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§1.1.2 Class 3: 9/4

Definition 1.7 (monomorphism)**.** In a category C, a morphism $\alpha : a \longrightarrow b$ is a monomorphism IFF $\alpha f = \alpha g \Rightarrow f = g$ for all f, g.

$$\begin{tikzcd} x \arrow[r, bend left, "g"] \arrow[r, bend right, "f"'] & a \arrow[r, "\alpha"] & b \end{tikzcd}$$

Definition 1.8 (epimorphism)**.** α is an epimorphism IFF $f\alpha = g\alpha \Rightarrow f = g$

$$\begin{tikzcd} a \arrow[r, "\alpha"] & b \arrow[r, bend left, "g"] \arrow[r, bend right, "f"'] & z \end{tikzcd}$$

> **Remark 1.9.** Sometimes, monomorphisms are decorated with a tail $\rightarrowtail$, and epimorphisms are decorated at their head $\twoheadrightarrow$

> **Remark 1.10.** You can think of monomorphisms like injections, and epimorphisms like surjections

Example 1.11

In Set, mono ≡ Left-cancellative functions ≡ post-invertible functions ≡ injections

In Set, epi ≡ right-cancellative functions ≡ pre-invertible functions ≡ surjections

> **Remark 1.12.** This is not just an example; this is a proof

In Grp, mono ≡ Injective Homomorphisms

In Grp, mono ≡ Surjective Homomorphisms

In BG, monomorphisms ≡ epicmorphisms ≡ everything

Example 1.13

In Ring, the inclusion $\iota : \mathbb{Z} \hookrightarrow \mathbb{Q}$, is epic despite not being surjective. ι is epic means that $f\iota$ tells you what f is. Taking $f : \mathbb{Q} \to R$, we know (since ι is an inclusion, that $f\iota$ is restricted to $\mathrm{im}(\iota) = \mathbb{Z}$. Knowing f on $\mathbb{Z}$ is enough to recover all of f because we know that

$$\begin{aligned} & f(a) \\ &= f(p/q) && p, q \in \mathbb{Z} \\ &= f(p)/f(q) \end{aligned}$$

Even just 0, 1 can generate all of $\mathbb{Q}$ using ring operations.
Now we can prove ι is epic:
Let $\iota : \mathbb{Z} \hookrightarrow \mathbb{Q}$ be the inclusion and let $f, g : \mathbb{Q} \to R$ be ring homomorphisms. Assume $f\iota = g\iota$. Then $f = g$ because for each $a \in \mathbb{Q}$ we have

$$\begin{aligned} & f(a) \\ &= f(p/q) && \text{decompose with } p, q \in \mathbb{Z} \\ &= f(p)/f(q) \\ &= (f \circ \iota)(p)/(f \circ \iota)(q) && \iota(p) = p, \iota(q) = q \\ &= (g \circ \iota)(p)/(g \circ \iota)(q) && f\iota = g\iota \\ &= g(p)/g(q) \\ &= g(p/q) \\ &= g(a) \end{aligned}$$

> **Remark 1.14** (equality of morphisms)**.** Two morphisms can have the same domain and codomain, but not be equal

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§1.2 Duality

§1.2.1 Class 4: 9/6

A **dual** takes something in the reverse direction. For example, for every category, we have an associated opposite category by **duality** To a category C, $\exists$ an associated category called C^{op} s.t. $\mathrm{Ob}(C^{\mathrm{op}}) = \mathrm{Ob}(C)$, $\mathrm{Mor}(C^{\mathrm{op}}) = \mathrm{Mor}(C)$ with reversed arrows.

Example 1.15 (A category and it's opposite)

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[rr, bend left=50, "fg"] \arrow[loop below, "1_x"] & y \arrow[r, bend left, "g"] \arrow[loop below, "1_y"] & z \arrow[loop below, "1_z"] \end{tikzcd} \quad vs. \quad \begin{tikzcd} x \arrow[loop below, "1_x^{op}"] & y \arrow[l, bend right, "f"'] \arrow[loop below, "1_y^{op}"] & z \arrow[l, bend right, "g"'] \arrow[ll, bend right=50, "fg"'] \arrow[loop below, "1_z^{op}"] \end{tikzcd}$$

So the the opposite morphism in it's opposite category is $f : c \to d$ is $f^{op} : d^{op} \to c^{op}$.

Theorem 1.17

f is monic if and only if f^{op} is epic

Proof: The commutative diagrams of a monic function α and it's opposite is:

$$\begin{tikzcd} x \arrow[r, bend left, "g"] \arrow[r, bend right, "f"'] & a \arrow[r, "\alpha"] & b \end{tikzcd} \Leftrightarrow \begin{tikzcd} x^{op} & a^{op} \arrow[l, bend right, "g^{op}"'] \arrow[l, bend left, "f^{op}"] & b^{op} \arrow[l, "\alpha^{op}"'] \end{tikzcd}$$

And since the dual diagram is precisely the definition of an epimorphism, we have proved the theorem.

Remark 1.18 (^{op} foundations)**.** A definition of the ^{op} just has to abide by the following 4 rules.

1. $x \mapsto x^{op}$
2. $f : x \to y \mapsto f^{op} : x^{op} \to y^{op}$
3. flips composition
4. $(C^{op})^{op} = C$

So when we are using x^{op} to denote the opposite elements in the opposite category, we are noting that these elements are the same, but their data -that includes the morphisms that go to and from them- may be different. Thus if we are defining categories as: $C := (\mathrm{Ob}(C),\ \mathrm{Mor}(C), \mathrm{Dom}, \mathrm{Cod})$, then $C^{op} := (\mathrm{Ob}(C^{op}),\ \mathrm{Mor}(C^{op}), \mathrm{Cod}, \mathrm{Dom})$

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§1.2.2 Class 5: 9/9

Definition 1.19 (split monic / split epic)**.** Let $c, d \in C$ with

$$c \xrightarrow{f} d \xrightarrow{g} c$$

with $\mathrm{id}_c = fg$.
Then g is split monic and f is split epic.

> **Remark 1.20.** These are just left and right inverses

§1.3 Functoriality

Definition 1.21 (Functor)**.** Let $\mathcal{C}, \mathcal{D}$, be categories. A **covariant functor** (sometimes just a functor) $F : \mathcal{C} \to \mathcal{D}$ is the following data.

1. $c \in \mathrm{Ob}(\mathcal{C}) \mapsto F(c) \equiv Fc \in \mathrm{Ob}(\mathcal{D})$
2. $f \in \mathrm{Mor}(\mathcal{C}) \mapsto F(f) \equiv Ff \in \mathrm{Mor}(\mathcal{D})$
   a) $f : c \to d$ maps to $Ff : Fc \to Fd$ - imposing that source and target are preserved
   b) $F(\mathrm{id}_c) \implies \mathrm{id}_F(c)$ - identity
   c) $f : c \to d$, $g : d \to e$ gives: $gf : c \to e$ and maps to $Fg \circ Ff = F(gf)$ - composition

With b, c, being the 2 **functoriality axioms** that a functor is required to satisfy.

> **Remark 1.22.** A functor is a morphism between categories.

Definition 1.23 (Forgetful Functor)**.** A functor that forgets structure is called a forgetful functor like in the following examples:

- U : Grp $\to$ Set - forgets its group structure
- U : Ab $\to$ Grp - forgets it's Abelian
- U : Top $\to$ Set - forgets it's a topological space

Like morphisms, functors compose. That is, given categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$, and $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{E}$, $GF : \mathcal{C} \to \mathcal{E}$ is a functor, that takes the objects in $\mathcal{C}$ $c \mapsto GF_c$, and the morphisms $f \in \mathcal{C}$ have $f \mapsto GF_f$

Lemma 1.24

Let $\mathcal{C}, \mathcal{D}$ be categories, $F : \mathcal{C} \to \mathcal{D}$ be a functor, $f : c \to c'$ be split monic. Then $Ff = Fc \to Fc'$ is also split monic.
Proof: $f : c \to c'$ is split monic, then there exists some $g : c' \to c$ s.t. $gf = id_c$. Then $F(gf) = F(id_c) = id_{Fc} = FgFf$. □
So Fg acts as a left inverse. Dually, if f is split epic, then Ff is split epic.

Example 1.25 (Constant functor)

The constant functor $\Delta(c) : \mathcal{J} \to C$, takes

- Objects to c
- Morphisms to id_c

In other words, the constant functor maps each object of the category $\mathcal{J}$ to a fixed object $c \in C$ and each morphism of $\mathcal{J}$ to the identity morphism of that fixed object.

Definition 1.26 (Mor$(x, -)$ or "Represented Functor")**.** This definition will come in handy when talking about representability and the Yoneda lemma; try to understand it.

Let $x, y \in C$, $\mathrm{Mor}(c, y) = \begin{cases} \text{morphisms} \\ \text{f:x} \to y \end{cases}$

The functor $\mathrm{Mor}(x, -) : C \to \textsc{Set}$ has

1. For $y \in C$, $y \mapsto \mathrm{Mor}(x, y)$[1]
2. For $f : y \to y$, maps to the **post-composition function** $f_* : \mathrm{Mor}(x, y) \to \mathrm{Mor}(x, y')$

$$\begin{tikzcd} x \arrow[d, "g"'] \arrow[dr, dashed] & \\ y \arrow[r, "f"'] & y' \end{tikzcd} \qquad \text{ex. } f \mapsto fg$$

Looking at the description in Riehl, we have:

$$\begin{tikzcd} C \arrow[rr, "\mathrm{Mor}(c,-)"] & & \textsc{Set} \end{tikzcd}$$

$$\begin{tikzcd} x \arrow[d, "f"] & \longmapsto & \mathrm{Mor}(c, x) \arrow[d, "\mathrm{Mor}(c,-)(f)"] \\ y & \longmapsto & \mathrm{Mor}(c, y) \end{tikzcd}$$

With f mapping to $\mathrm{Mor}(c, -)(f)$, the post-composition function $\mathrm{Mor}(c, -)(f) : \mathrm{Mor}(c, x) \to \mathrm{Mor}(c, y)$. That is, f_* takes in morphisms g and post-composes with them as follows: $f \circ -(g) = fg$

Definition 1.27 (Contravariant Functor)**.** Let $C, \mathcal{D}$ be categories. A **contravariant functor** $F : C \to D$ has the following data:

1. A object $Fc \in \mathcal{D}$ for each object $c \in C$

[1] $\mathrm{Mor}(x, y)$ represents the set of morphisms from x to y. It is sometimes denoted $\mathrm{Hom}(x, y), C(x, y)$

2. The mapping of morphisms: $[f : c \to c'] \mapsto [Ff : Fc' \to Fc \in \mathcal{D}]$
    a) And the following two functoriality axioms
    b) $Ff \circ Fg = F(g \circ f)$ - Composability
    c) $F(1_c) = 1_{Fc}$ - Identity

> **Remark 1.28.** A contravariant Functor $F : C \to \mathcal{D}$ is the same as a covariant functor $F' : C^{\text{op}} \to \mathcal{D}^{\text{op}}$

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§1.3.1 Class 6: 9/11

Example 1.29 (Dual functor)

Set example: Let $P : \textsc{Set} \to \textsc{Set}$ take $x \mapsto \mathcal{P}(x) = \{\text{the set of subsets of x}\}$. $[f : xy] \mapsto [f^{-1} : \mathcal{P}(y) \to \mathcal{P}(x)]$

§1.4 Naturality

Definition 1.30 (Natural Transformation)**.** A **natural transformation** $\eta : F \to G$ is a collection of morphisms $\eta_c : Fc \to Gc$ s.t. the diagram

$$\begin{tikzcd} F_c \arrow[r, "Ff"] \arrow[d, "\eta_c"'] & F_{c'} \arrow[d, "\eta_{c'}"] \\ G_c \arrow[r, "Gf"'] & G_{c'} \end{tikzcd}$$

commutes.
Note that F, G are functors, c, c' are objects. So a natural transformation just shows that two functors preserve the same structure. They are a family of morphisms that take objects in FC (which are in D) to GC (also in D), and maintain the morphisms between them.

> **Remark 1.31.** This globular diagram adopted from Reihl's book helped me understand it better
>
> $$\begin{tikzcd} C \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{D} \end{tikzcd}$$
>
> Note that our natural transformation is denoted α here.

Definition 1.32 (Natural Isomorphism)**.** A natural transformation is a natural isomorphism if $\eta_c : F_c \to G_c$ is an isomorphism $\forall\, c \in C$. (If all ηs are isomorphisms). A natural isomorphism

$\eta : F \cong G$ is denoted $\eta : F \cong G$. (Think: the arrow in the globular diagram above goes both ways)

> **Remark 1.33.** It's important to notice the similarities between a functor and a function, they both have a domain and codomain, and both have some idea of an image and preimage for their inputs. Both times, the image isn't necessarily equal to the codomain, nor is it even a significant portion.

A natural transformation is an is an assignment to every object $c \in C$ a morphism α_c.

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§1.4.1 Class 7: 9/13

This is going to be an attempt to give a better conceptual understanding of *naturality*.

Example 1.34 (The natural transformation (first))

Let $(X)*: \textsc{Set} \to \textsc{Set}$ for any set X, that denotes the set of finite sequences on X, and Let $(-) + \{\star\} : \textsc{Set} \to \textsc{Set}$, where $(-)$ means you are plugging in a set into the dash, $(-) + \{\star\}$ is the name of the functor, and $\{\star\}$ is a singleton set with some $\star$ disjoint with the set.

Now, Let first : $(-)* \implies (-) + \{\star\}$, be a natural transformation where we essentially have:

$$\text{first}_X : \begin{cases} () \mapsto & \star_X \text{ (if no first element give star)} \\ (x_1, x_2, \dots, x_n) \mapsto & x_1 \text{ (otherwise give the first element)} \end{cases}$$

Then we are trying to show that for the two functors $(-)*, (-) + \{\star\} : \textsc{Set} \to \textsc{Set}$, and any two objects A, B, with the corresponding morphism f, the natural transformations

$$\text{first} : (A)* \to A + \{\star_A\} \text{ and, first} : (B)* \to B + \{\star_B\}$$

are able to commute with morphisms through the natural transformation square and the globular diagram in 1.28 and 1.29, respectively.

Observe: Basically, the transformation

$$\text{first} : (-) * [(-) + \{\star\}]$$

works the same no matter our choice of $(-)$.

> It is not too misleading, at least historically, to say that categories are what one must define in order to define functors, and that functors are what one must define in order to define natural transformations.

- Peter Freyd, *Abelian Categories*

Just to be clear, there are usually very many natural transformations between two functors, since each natural transformation is a set of morphisms between objects through F and objects through G.

> **Remark 1.35.** Remember that an isomorphism is a morphism that has an inverse morphism, and a natural isomorphism is a collection of isomorphisms between the images of two functors with the same domain and codomain.[a]
>
> [a] this is just an attempt at being succinct.

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§1.5 Equivalence

§1.5.1 Class 8: 9/16

Definition 1.36. For $C, \mathcal{D}$, an **equivalence** $F : C \simeq \mathcal{D} : G$ consists of functors $\begin{tikzcd} C \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$ s.t. we have two natural isomorphisms $\eta : GF \cong 1_C$ and $\epsilon : FG \cong 1_{\mathcal{D}}$. We say C and $\mathcal{D}$ are equivalent.

> **Remark 1.37.** After going through the functor and back, the object may be different, but it still is in the same isomorphism class, since there are natural isomorphisms to the identity functors.

The important part of this definition is a functor (remember that functors compose so GF and FG are like "unit" functors that "pass through" an equivalent category. We can imagine natural isomorphism $\alpha : GF \to 1_C$ with the following naturality square, where $\sim$ denotes isomorphisms (or rather, elements in the same isomorphism class).

$$\begin{tikzcd} GF_c \arrow[r, "\sim", "\alpha_c"'] \arrow[d, "GF(f)"'] & c \arrow[d, "f"] \\ GF_{c'} \arrow[r, "\sim", "\alpha_{c'}"'] & c' \end{tikzcd}$$

Then we also notice that $f = \alpha_{c'} \circ GF(f) \circ \alpha_c^{-1}$ (Remember that we can do α_c^{-1} since it is a natural isomorphism)

Lemma 1.38 (equivalence is an equivalence relation)

$\simeq$ is an equivalence relation if it obeys the following 3 properties:

1. Reflexive ' $C \cong \mathcal{D} \to C \simeq \mathcal{D}$
2. Symmetric The definition of an equivalence relation is symmetric with respect to C and $\mathcal{D}$
3. Transitive Say we have $\begin{tikzcd} C \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "I"] \arrow[r, bend left, "G"] & \mathcal{E} \arrow[l, bend left, "H"] \end{tikzcd}$ with $C \simeq \mathcal{D} \simeq \mathcal{E}$, then (GF, IH) witness $C \simeq \mathcal{E}$. Then we have

$$IGHF = I(HG)F \qquad = I1_d F = IF \qquad = 1_c$$

Lemma 1.39 (Box Lemma)

Given natural isomorphisms $\alpha : a \cong a'$ and $\beta : b \cong b'$, each of the following diagrams induce

$$\begin{tikzcd} a \arrow[d, "f"] & a' \arrow[l, "\cong"'] \arrow[d, dotted, "f'"] \\ b \arrow[r, "\cong"'] & b' \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "\cong"] \arrow[d, "f"'] & a' \arrow[d, dotted, "f'"] \\ b \arrow[r, "\cong"'] & b' \end{tikzcd} \qquad \begin{tikzcd} a \arrow[d, "f"'] & a' \arrow[l, "\cong"'] \arrow[d, dotted, "f'"] \\ b & b' \arrow[l, "\cong"] \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "\cong"] \arrow[d, "f"'] & a' \arrow[d, dotted, "f'"] \\ b & b' \arrow[l, "\cong"] \end{tikzcd}$$

Thus follows because every natural isomorphism has a two-sided inverse, and because diagram 1 uniquely defines f' through composition.

Definition 1.40 (Full, Faithful, and Essentially Surjective)**.** A functor $f : C \to \mathcal{D}$ is:
full IFF it is surjective for all $c, c' \in C$
faithful IFF it is injective on morphisms
Essentially Surjective IFF it hits every isomorphism class i.e., $\forall d \in \mathcal{D}$, there exists $c \in C$ where $Fc \cong d$

§1.5.2 Class 9: 9/18

Definition 1.41. If C is a category, $\mathrm{Sk}(C)$ is "the" subcategory of C consisting of one object from each isomorphism class of C, plus as many morphisms as possible. $\mathrm{Sc}(C)$ is called **skeletal**, if it contains just one object in each isomorphism class.

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§2 Yoneda

§2.1 Representable Functors

Universal Properties

Definition 2.1. An object is $c \in C$ is **initial**[2] if for all other objects x, the set $C(c, x)$ (also written $\mathrm{Mor}(c, x)$) has one element.

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§2.1.1 Class 10: 9/20

Example 2.2 (Products)

An important thing about products is that maps $f : C \to C \times Y$ are actually a pair of maps
$f_1 : C \to C$
$f_2 : C \to Y$

Example 2.3 (Quotients: $f : \frac{x}{\sim} \to y$)

A **Quotient** is a map out of x that respects $\sim$ (equivalence classes).

$$\frac{x}{\sim} = \{[x]_\sim\} \longleftarrow \text{ the set of equivalence classes}$$

If $f : x \to y$ that satisfies $x \sim y \implies f(x) = f(y)$ induces a map $f : \frac{x}{\sim} \to y$

We will talk about the universal property of being a representing object for a functor.
Recall definition 1.25

Definition 2.4 (Representable functors)**.** Let C be a locally small category, and $F : C \to \textsc{Set}$ is a functor. We say F is **representable** if there exists an object $x \in C$ such that there exists a natural isomorphism between F and the functor $\mathrm{Mor}(x, -)$, which takes:

1. For $y \in C$, $y \mapsto \mathrm{Mor}(x, y)$
2. For $f : y \to y$, maps to $f_* : \mathrm{Mor}(x, y) \to \mathrm{Mor}(x, y')$

$$\begin{tikzcd} x \arrow[d, "g"'] \arrow[dr, dashed] & \\ y \arrow[r, "f"'] & y' \end{tikzcd} \qquad \text{ex. } g \mapsto fg$$

We say c **represents** F if there exists $\alpha_c : \mathrm{Mor}(c, -) \cong F$. (If there exists a natural isomorphism between F and it's Mor functor).

So what does this mean? Say we have some natural isomorphism $\alpha : F \cong \mathrm{Mor}(c, -)$. Then for every object $d \in C$, we have $\alpha_d : F \cong \mathrm{Mor}(c, d)$, which means every morphism from $\mathrm{Mor}(c, d) \in \textsc{Set}$ to $F(d) \in \textsc{Set}$, there exists an inverse morphism from $F(d) \to \mathrm{Mor}(c, d)$.

[2] We will discuss initial and terminal objects later but we will use a different(?) context.

A representable functor $F : C \to \textsc{Set}$ "represent" morphisms out of a representing object.

Example 2.5

$1_{\textsc{Set}} : \textsc{Set} \to \textsc{Set}$ is represented by $\{\star\}$ because an object in $F(\textsc{Set}) = \textsc{Set}$ has elements, which can be thought of as morphisms $x : \{\star\} \to s$

Theorem 2.6

An object c in a category C is initial IFF $C(c, -) : C \to \textsc{Set}$ is naturally isomorphic to the constant functor $\Delta : C \to \textsc{Set}$ that sends every object to the singleton set. Recall that an initial functor has a singlton set of morphisms between it and any other object.

Proof. To show that this is a natural isomorphism, we can construct the following commutative diagrams

$$\begin{tikzcd} \Delta(x) = \{\star\} \arrow[r, "\Delta f = \mathrm{Id}_{\{\star\}}"] \arrow[d, "\eta_x"'] & \Delta(y) = \{\star\} \arrow[d, "\eta_y"] \\ \mathrm{Mor}(c, x) \arrow[r, "\mathrm{Mor}(c,-)f = f_*"'] & \mathrm{Mor}(c, y) \end{tikzcd}$$

□

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§2.1.2 Class 11: 9/23

More examples of representing objects

Example 2.7

The forgetful functor $\mathcal{U} : \textsc{Grp} \to \textsc{Set}$ is represented by $\mathbb{Z}$. That is, $\textsc{Grp}(\mathbb{Z}, -) \cong \mathcal{U}$.
Intuition: A group homomorphism Φ out of $\mathbb{Z}$ tells us $\Phi(1)$, and then we get $\Phi(1)^2 = \Phi(2)$, $\Phi(1)^3 = \Phi(3), \ldots$.

Proof. For every group H, there exists some $\alpha_G : \textsc{Grp}(\mathbb{Z}, G) \to \mathcal{U}(G)$ that takes, for each $g \in \mathcal{U}G$, the unique homomorphism $\mathbb{Z} \to G$ that maps $1 \to g$. This is a bijection because every homomorphism is determined by what 1 maps to (since 1 is the generator). This bijection is natural because every composite group homomorphism $\mathbb{Z} \xrightarrow{g} G \xrightarrow{\psi} H$ has $1 \mapsto \psi(g)$ □

Remark 2.8. The proof of this is short, but its super nutty to visualize. Try drawing a diagram, and realize that composing the natural transformations (which are from objects to morphisms) means composing morphisms.

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§2.1.3 Class 12: 9/25

§2.2 The Yoneda Lemma

This class is a proof of the **Yoneda Lemma**

Lemma 2.9

Let C be a category, and c, d are isomorphic objects. Then for the represented functors:

$$\mathrm{Mor}(c, -) \cong \mathrm{Mor}(d, -)$$
$$\mathrm{Mor}(-, c) \cong \mathrm{Mor}(-, d)$$

(are also isomorphic).

Definition 2.10 (Presheaf)**.** Let C be a category. A **Presheaf** in C is a contravariant functor $F : C \to \textsc{Set}$. Alternatively, a presheaf is a covariant functor $F : C^{op} \to \textsc{Set}$. That is: $\mathrm{Fun}(C^{op}, \textsc{Set}) := \mathrm{PSh}(C)$

Note that sometimes we use terminology like $\mathrm{Nat}(F, G) = \{\text{natural transformations } \alpha : F \to G\}$ or $Fun(C, \mathcal{D}) = F : C \to \mathcal{D}$

Theorem 2.11 (Yoneda Lemma)

Let C be a locally small category, $F : C \to \textsc{Set}$, and $c \in C$. Then $\mathrm{Nat}(\mathrm{Mor}(c, -), F) \cong F(c)$. Note that the natural transformation and $F(c)$ are both in $\textsc{Set}$.

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§2.3 Universal Properties and universal elements

A representable functor $F : C \to \textsc{Set}$ or $F : C^{\mathrm{op}} \to \textsc{Set}$ encodes a universal property of its representing object.

Definition 2.12. A **Universal Property** of an object $c \in C$ is expressed by a representable functor F together with a **Universal Element** $x \in Fc$ that defines a natural isomorphism $\mathrm{Mor}\, c, - \cong F$, via the Yoneda Lemma.

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§2.3.1 Class 13: 9/27

よ $F : \mathcal{C} \to \textsc{Set}, c \in \mathcal{C}$

[Yoneda Embedding よ] よ $: \mathcal{C} \to \mathrm{PS}\textsc{h}(\mathcal{C})$

with $c \mapsto \mathrm{Mor}(-, c)$

Proposition 2.13

よ $: \mathcal{C} \to \textsc{Psh}(\mathcal{C})$ is full and faithful

§3 Limits and Colimits

§3.1 Limits and Colimits as universal cones

Definition 3.1 (Initial and Terminal)**.** Let $\mathcal{C}$ be a category. An object $c \in \mathcal{C}$ is

- **initial** if for all other objects, there exists a unique morphism $f : c \to d$
- **final** or **terminal** if for all $b \in \mathcal{C}$, there exists a unique map $g : b \to c$

Example 3.2 1. $\emptyset$ is initial in $\textsc{Set}$

2. $\{\star\}$ is final in $\textsc{Set}$

3. In $\textsc{Grp}$, the group $\{e\}$ is initial and final

Proposition 3.3

Initial and final objects are unique up to unique isomorphism

Proof. Let c, c' be initial. Then there exists a unique $f : c \to c'$

□

Definition 3.4. Let $\mathcal{I}, \mathcal{C}$ be categories. A **diagram** is a functor $F : \mathcal{I} \to \mathcal{C}$, where $\mathcal{I}$ is some indexing category

Example 3.5

$\mathcal{I}$ is a directed graph

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet & \bullet \arrow[l] \end{tikzcd}$$

then $F : \mathcal{I} \to$ Set is

$$\begin{tikzcd} & B \arrow[d] \\ A & C \arrow[l] \end{tikzcd}$$

Definition 3.6. Let $F : I \to C$ be a diagram. A **cone** over F is

1. An object $z \in C$ called the **apex**
2. For every $i \in \mathcal{I}$, a map $\lambda_i = z \to F(i)$ called the legs, all compatibly with each other i.e., the diagram commutes

$$\begin{tikzcd} & z \arrow[dl, "\lambda_1"'] \arrow[d, "\lambda_2"'] \arrow[dr, "\lambda_3"] & \cdots & \\ A_1 \arrow[r] & A_2 \arrow[r] & A_3 \arrow[r] & \cdots \end{tikzcd}$$

3. A natural transformation $\lambda : c \implies F$, with the domain being the constant functor at c, so that you can just think of the family of morphisms coming out of c. Note that the domain is a functor but the cone diagram shows c as an object.

Definition 3.7. A **morphism of cones** Z, W is a map $f : Z \to W$ compatible with legs

Ex.

$$\begin{tikzcd} & z \arrow[d, dashed, "f"] \arrow[ddl, bend right, "\eta_1"'] \arrow[ddr, bend left, "\eta_2"] & \\ & w \arrow[dl, "\lambda_1"'] \arrow[dr, "\lambda_2"] & \\ A & & B \end{tikzcd}$$

Definition 3.8. Cone(F) is the category of cones over $F : \mathcal{I} \to C$

Definition 3.9. A limit over F is a final object in Cone(F)

---

§3.1.1 Class 14: 9/30

Definition 3.10. A **limit** is a terminal object in Cone(F).
A **colimit** is an initial object in Cocone(F)

$$\begin{tikzcd}
 & c \arrow[d, dashed, "\exists!"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & \lim F \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_{j'}"] & \\
F_j \arrow[rr, "Ff"] & & F_{j'}
\end{tikzcd}$$

So $(\lim F, \lambda_j)$ is a limit of $F : \mathcal{J} \to C$, which implies $(c^{\mathrm{op}}, \lambda_j^{\mathrm{op}})$ is a colimit of $F^{\mathrm{op}} : \mathcal{J}^{\mathrm{op}} \to C^{\mathrm{op}}$

Problem 3.11. What is the limit over $F : \emptyset \to C$?[3]

Drawing our limit diagram, we have

$$c \xrightarrow{\exists!} \lim F$$

Where $\lim(F)$ is the terminal object, since Cone($\emptyset$) is just the apex and nothing else

Problem 3.12. Preliminary Understanding: Let $F : C \to \mathcal{D}$ be a diagram. What is the limit of F?[4]
Then, write out the limit using logical connectives, and try to prove that the limit in a one object diagram an inverse (split epimorphism or split monomorphism or full isomorphism).

Solution: $\forall\ c \in$ Cone(F), $\exists! f : c \to \lim(F) \in \mathrm{Mor}(F)$, such that the diagram
$$\begin{tikzcd}
c \arrow[d, "\exists! f"] \arrow[dd, bend right=60, "\lambda"'] \\
\lim(F) \arrow[d, "g"] \\
\bullet
\end{tikzcd}$$
commutes. We know that $gf_i = \lambda_i$ and that g must be a monomorphism, since every morphism $f_i : c_i \to \lim(F)$ is unique. $\square$

And thus we can construct the following table of limits:

diagram\category	Set	Grp	Vect
$\emptyset$	$\{\star\}$	$\{e\}$	$\{\vec{0}\}$
c	Isomorphic to c$\longrightarrow$		
$x\ y$	$X \times Y \longrightarrow$		

---

§3.1.2 Class 15: 10/2

Definition 3.13. An **equalizer** is a limit of a diagram indexed by the **parallel pair**, the category $\bullet \rightrightarrows \bullet$. A diagram of this shape is $f, g : A \rightrightarrows B$ in our target category C (remember

[3] $\emptyset$ represents the empty diagram
[4] Answer: A terminal object in the category of cones over F

that a diagram is a functor from $\mathcal{J} \to C$). A cone over this diagram with summit C consists of a pair of morphisms in the commutative diagram below:

$$\begin{tikzcd} & C \arrow[dl, "a"'] \arrow[dr, "b"] & \\ A \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & B \end{tikzcd}$$

where $ga = fa = b$. The equalizer $h : E \to A$ is the morphism h and object E satisfying

$$\begin{tikzcd} C \arrow[d, "k"', "\exists!"] \arrow[dr, "a"] & & \\ E \arrow[r, "h"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

with the morphisms from C to B and from E to B implied because the diagram commutes.

Definition 3.14. A **pullback** is a limit of a diagram indexed by the poset category

$$\bullet \to \bullet \leftarrow \bullet$$

comprised of two non-identity morphisms with common codomain. If we write f, g as the morphisms, a cone with summit D is three morphisms so that the triangles in this diagram commute:

$$\begin{tikzcd} D \arrow[r, "c"] \arrow[d, "b"'] \arrow[dr, "a"] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

The pullback, also known as the **fiber product** denoted $B \times_A C$, is a commutative square $fh = gk$ with the following universal property: $\forall$ commutative squares if the form above, the following commutative diagram holds:

$$\begin{tikzcd} D \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "c"] \arrow[ddr, bend right, "b"'] & & \\ & P \arrow[r, "k"] \arrow[d, "h"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\ & B \arrow[r, "f"'] & A \end{tikzcd}$$

Definition 3.15. [5] A **product** is a limit of a diagram indexed by a discrete category $\mathcal{J}$ (recall that a discrete category is a category with only identity morphisms. A diagram in C indexed by $\mathcal{J}$ is a collection of objects $F_j \in C$ indexed by $j \in \mathcal{J}$. A cone over this diagram is also

[5] This is taken from Riehl's book essentially verbatim, I will write more to understand it better, but like a lot of parts of this document, my notes are just attempts to understand the textbook as best I can, which may involve copying down some stuff.

indexed by $\mathcal{J}$ and is a family of morphisms $(\lambda_j : c \to F_j)_{j\in\mathcal{J}}$. The limit is typically denoted by $\Pi_{j\in\mathcal{J}} F_j$ and the legs of the limit cones are maps

$$(\pi_k : \Pi_{j\in\mathcal{J}} F_j \to F_k)_{k\in\mathcal{J}}$$

called **(product) projections**. The universal property asserts that composition with the product projections defines a natural isomorphism

$$C(c, \Pi_{j\in\mathcal{J}} F_j) \xrightarrow[\cong]{(\pi_k)_*} \Pi_{k\in\mathcal{J}} C(c, F_k) \cong \mathrm{Cone}(c, F)$$

What does this mean!?!?!?
To understand what $C(c, \Pi_{j\in\mathcal{J}} F_j) \xrightarrow[\cong]{(\pi_k)_*} \Pi_{k\in\mathcal{J}} C(c, F_k) \cong \mathrm{Cone}(c, F)$ means, this is a breakdown of every component:

1. $C(c, \Pi_{j\in\mathcal{J}} F_j)$ refers to the mapping of objects through the represented functor in C, which is the set of morphisms, or **Hom-set**, taking the object c to the product $\Pi_{j\in\mathcal{J}} F_j$ of all the objects F_j for $j \in \mathcal{J}$

2. $(\pi_k)_*$ is just the name of natural isomorphism map induced by projection morphisms π_k for each k

3. $\Pi_{k\in\mathcal{J}} C(c, F_k)$ is the product of all hom-sets, which means it consists of n-tuples of morphisms $\{f_k : c \to F_k\}_{k\in\mathcal{J}}$

4. $\mathrm{Cone}(c, F)$ is the category of cones from the object c. Note that this is a category which is isomorphic to $\Pi_{k\in\mathcal{J}} C(c, F_k)$
   This whole statement tells you that

   i. (The natural isomorphism tells us that): The set of morphisms from c to the product of all F_js (which are also objects) is "the same" as the set of morphisms f_k from c to F_k for each f_k. (Its just moving around where the product is!)

   ii. (The isomorphism between the two categories tells us that): The family of morphisms from c to objects in the diagram F is "the same as" (isomorphic to) a cone with apex c in the diagram F

   Since $\mathcal{J}$ is a discrete category in this situation, we can easily intuit the idea of indexing over its objects

> **Remark 3.16.** After finally understanding what this all meant, it seems a lot easier[a] to understand than it looked. This complicated notation may look like a lot, but really just look at what changes from the first part to the second part of the natural isomorphism.
>
> [a]This took me over an hour to understand

We can dualize all of these definitions above to define the **coproduct**, **coequalizer**, **initial object**, and a **pushout**

Definition 3.17. A **bifunctor** is a functor whose domain is the product category:

$$\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$$

Example 3.18

A well known bifunctor is the represented functor, or **Hom-Functor**, or mor-functor, denoted in so many different ways but this time denoted

$$\mathrm{Hom}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{SET}$$

---

§3.2 Limits in the Category of Sets

§3.2.1 Class 16: 10/4

What is the limit of $n \preccurlyeq n+1 \preccurlyeq \cdots \preccurlyeq m$ in the poset category $(\mathbb{N}, \preccurlyeq)$? [6]

Definition 3.19. A category C is **complete** if for all $F : \mathcal{J} \to \mathcal{C}$ such that $\mathcal{J}$ is not a proper class (is a set), $\lim_{\mathcal{J}} F$ exists.

A category is **cocomplete** if the above holds for colimits.

Theorem 3.20

A cone over F with apex x is a natural transformation $\Delta x : \mathcal{J} \to \mathcal{C} \implies (F : \mathcal{J} \to \mathcal{C})$

Proof. We know that $\Delta(-)$ is functorial (is a functor) and maps each object in its domain to $(-)$ in its codomain. The morphisms map to identity morphisms. Then $\Delta : \mathcal{C} \to (\mathcal{J} \to \mathcal{C})$, where $\mathcal{J} \to \mathcal{C}$ is the category of functors from $\mathcal{J}$ to $\mathcal{C}$ with the objects being functors and the morphisms being natural transformations. Then, for $(f : x \to y) \in \mathcal{C}$ we have $(\Delta f) : \Delta x \implies \Delta y$. □

Example 3.21

The product of sets A_j indexed by elements of $j \in \mathcal{J}$, where $\mathcal{J}$ is discrete, is the limit of the diagram. The legs of the limit cones are maps

$$(\pi_k : \Pi_{j \in \mathcal{J}} A_j \to A_k)_{k \in \mathcal{J}}$$

So for our legs we have $\pi_1 : \Pi A_j \to A_1$ and $\pi_2 : \Pi A_j \to A_2$. So our product is just the cartesian product.

[6] **Solution**: n

§3.2.2 Class 17: 10/7

We have learned all these definitions and notations, but what do they really mean? To gain a better understanding, we construct these in the category Set

Example 3.22

Consider the diagram $A \xrightarrow{f} B \xleftarrow{g} C$

The limit of this diagram is the pullback, denoted $A \times_B C$.

Proof.

$$\begin{aligned}\lim &= \{\mu : \{\star\} \implies A \xrightarrow{f} B \xleftarrow{g} C\} \\ &= \{\{\mu_a : \{\star\} \to A,\ \mu_b : \{\star\} \to B,\ \mu_c : \{\star\} \to C\} |\ \mu \text{ is a cone } \} \\ &= \{(a, b, c) \in A \times B \times C |\ f(a) = b = g(c)\} \\ &= \{(a, c) \in A \times C |\ fa = gc\} \\ &= A \times_B C\end{aligned}$$

□

The pullback may be constructed as an equalizer of a diagram involving a product.
Given a pair of functions $f : A \to C$ and $g : B \to C$, the pullback $A \times_C B$ is the same as the equalizer of the following diagram:

$$B \times_A C \rightarrowtail B \times C \underset{(b,c)\mapsto gc}{\overset{(b,c)\mapsto fb}{\rightrightarrows}} A$$

Recall that the equalizer is the limit of a diagram indexed by the parallel pair. From our construction of the pullback in Set, we see that $fb = gc = a$. In fact, the equalizer in Set, of a pair of functions $f, g : X \rightrightarrows Y$, is the set of maps $1 \xrightarrow{x} X$ so that $fx = gx$. So $E := \{x \in X |\ fx = gx\}$.

---

§3.2.3 Class 18: 10/9

Theorem 3.23

A category C is complete IFF it has products and equalizers.

Proof. Let $F : \mathcal{J} \to C$ be a diagram, and let x be the equalizer of the following diagram.

$$\begin{tikzcd} x \arrow[r] & \Pi_{j\in \mathrm{ob}(\mathcal{J})} F_j \arrow[r, shift left, "c"] \arrow[r, shift right, "d"'] & \Pi_{j\in \mathrm{ob}(\mathcal{J})} F(\mathrm{cod} f) \end{tikzcd}$$

It is necessary and sufficient to define each component morphisms, i.e., the composites of c and d with the projection π_f in order to define c and d.
The components of c are projections, as shown in the top half of the diagram below, that map –at the indexing element $f \in \mathrm{mor}(\mathcal{J})$– the product of the objects in $\mathcal{J}$ to the individual component of $\mathcal{J}$ which is the codomain of morphism f.

$$\begin{tikzcd} & & F(\mathrm{cod} f) \\ x \arrow[r] & \Pi_{j\in \mathrm{ob}(\mathcal{J})} F_j \arrow[ur, "\pi_{\mathrm{cod} f}"] \arrow[r, shift left, "c"] \arrow[r, shift right, "d"'] \arrow[d, "\pi_{\mathrm{dom} f}"'] & \Pi_{f\in \mathrm{Mor}(\mathcal{J})} F(\mathrm{cod} f) \arrow[u, "\pi_f"'] \arrow[d, "\pi_f"] \\ & F(\mathrm{dom} f) \arrow[r, "Ff"] & F(\mathrm{cod} f) \end{tikzcd}$$

The component at f of the map d, on the bottom half of the diagram, is the composition of a projections, to the individual component of $\mathcal{J}$ which is the domain of morphism f, composed with the morphism f composed with F. □

---

§3.2.4 Class 21: 10/16

Lemma 3.24

Let C be a category. $f : x \to y$ a morphism. f is monic IFF

$$\begin{tikzcd} x \arrow[r, "\mathrm{id}_x"] \arrow[d, "\mathrm{id}_x"'] \arrow[dr, phantom, "\lrcorner", very near start] & x \arrow[d, "f"] \\ x \arrow[r, "f"'] & y \end{tikzcd}$$

is a pullback.

Proof. (*of the forward direction*). Suppose f is monic. Given $h : z \to x$ with $gf = hf$, then

$g = h$. Drawing the fiber product diagram gives us

$$
\begin{tikzcd}
z \arrow[drr, bend left, "\lambda_x = h"] \arrow[ddr, bend right, "\lambda_x = g"'] \arrow[dr, shift left, "h(\exists!)"] \arrow[dr, shift right, "g"'] & & \\
& x \arrow[r, "\mathrm{id}_x"] \arrow[d, "\mathrm{id}_x"'] \arrow[dr, phantom, "\lrcorner", very near start] & x \arrow[d, "f"] \\
& x \arrow[r, "f"'] & y
\end{tikzcd}
$$

But $g = h$, so our morphism from z to x is unique. Then our pullback diagram holds and we have a pullback. $\square$

---

§4 Exercises and MUSA 174 HW

§4.1 Exercise 1.1.iii

Exercise 1.1.iii (ii) There is a category C/c whose objects are morphisms $f : x \to c$ with codomain c and in which a morphism from $f : x \to c$ to $g : y \to c$ is a map $h : x \to y$ between the domains so that the triangle commutes.

$$\begin{tikzcd} x \arrow[rr, "h"] \arrow[dr, "f"'] & & y \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

ie., so that $f = gh$

Proof: Here we want to define a category C/c, which is a collection of objects and morphisms, where every object has a designated identity morphism and so that for any pair of objects g, f where the codomain of f is equal to the domain of g $\exists$ a composite morphism gf.
By assumption, our objects are defined as morphisms $f : x \to c$ in C. A morphism $F : f \to g$ in our new category C/c is between two morphisms $f, g \in C$, and is a map $h : x \to y$ between the domains so that the triangle commutes.
Now let's cover what it means for two morphisms to compose in C/c. Looking at the following diagram

$$\begin{tikzcd} x \arrow[r, "h"] \arrow[dr, "f"'] & y \arrow[r, "i"] \arrow[d, "g"] & z \arrow[dl, "j"] \\ & c & \end{tikzcd}$$

, we can define $G : g \to j \circ F : f \to g \in C/c$ as a map $i : y \to z \circ h : x \to y \in C$, so we have to show, that this implies that the diagram commutes. Since $j, i, h \in C$, we know that $(ji)h = j(ih)$, and that implies that our morphism composition GF is a morphism.
Our other criteria for a Category is that every object has an identity morphism. Take some element $f : x \to c \in C/c$. Our identity morphism 1_f must map $1x : x \to x$, so that the triangle

$$\begin{tikzcd} x \arrow[rr, "1_x"] \arrow[dr, "f"'] & & x \arrow[dl, "f"] \\ & c & \end{tikzcd}$$

commutes. Then 1_f (and 1_g) can compose with $F : f \to g$ by the following diagram.

$$\begin{tikzcd} x, \arrow[r, "1_x"] \arrow[drr, "f"'] & x \arrow[rr, "h"] \arrow[dr, "f"] & & y \arrow[r, "1_y"] \arrow[dl, "g"'] & y \arrow[dll, "g"] \\ & & c & & \end{tikzcd}$$

, where $F \circ 1_f = F : f \to g = 1_g \circ F$

So every category C has a slice category C/c of C over c $\square$

§4.2 HW1

§4.2.1 Problem 1

Let C be a category, and $f : x \to y$ be a morphism. Show that if f is an isomorphism, its inverse is unique.

§4.2.2 Solution

Let $g : y \to x$ and $h : y \to x$ be two morphisms such that $gf = hf = 1_x$ and $fg = fh = 1_y$.

> **Remark 4.1.** We can create a commutative diagram:
>
> $$\begin{tikzcd} x \arrow[r, "f"] & y \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] & x \end{tikzcd}$$
>
> and notice that $g = h$ implies f is an epimorphism. By dual logic, $g = h$ also implies f is monic.

Since we are assuming both g and h hold the properties of the isomorphism, we know that $gf = hf = 1_x$. Then $gfh = hfh = gfg = hfg = 1_x g = 1_x h = g = h$. Thus $g = h$ and thus the inverse of an isomorphism is unique. □

§4.2.3 Problem 2

Let C be a category, and $f : x \to y$ be a morphism. Suppose we have morphisms $g, h : y \to x$ such that $gf = 1_x$ and $fh = 1_y$. Show that $g = h$ and conclude that f is an isomorphism.

§4.2.4 Solution

By assumption, we have $gf = 1_x$ and $fh = 1_y$. This implies $fgf = f1_x = f$, and $fhf = 1_y f = f$, which gives $fgf = fhf = f$, or $(fg)f = (fh)f = f$, since morphism composition is associative. Then $fh = fg = 1_y$.
Since we know by assumption that $gf = 1_x$, we have $gfg = gfh = 1_x g = 1_x h = g = h$. Thus $g = h$ and so by the definition of an isomorphism, $f : x \to y$ is an isomorphism because $\exists$ a morphism $g : y \to x$ s.t. $gf = 1_x$ and $fh = fg = 1_y$. □

---

§4.3 HW2

§4.3.1 Problem

Construct a category C where every morphism is monic but there exists a non-epic morphism.

§4.3.2 Solution

A subset of Set where the only morphisms are injective (but not necessarily surjective) functions, and the objects are the same. Since every injective function is monic, and injective functions can be not surjective and thus not epic, this fulfills the criteria. Since this still has identity morphisms, associativity, and every morphism still has a domain and codomain, it is still a category.

---

§4.3.3 Instructor Notes

yeah, nice. good for verifying it's a category

you're missing justification that in your subcategory, "monic = injective, epi = surjective" holds. maybe this automatically holds for subcategories of Set, maybe not

---

§4.4 HW3

§4.4.1 Problem

Prove $1, 2$ by arguing a or b and then using duality. Conclude that the monomorphisms in a category define a subcategory and that the same is true for the epimorphisms by duality.

1. a) If $f : x \rightarrowtail y$ and $g : y \rightarrowtail z$ are monomorphisms, then so is $gf : x \rightarrowtail z$.

   b) if $f : x \twoheadrightarrow y$ and $g : y \twoheadrightarrow z$ are epimorphisms, then so is $gf : x \twoheadrightarrow z$

2. a) If $f : x \to y$ and $g : y \to z$ are morphisms so that gf is monic, then f is monic.

   b) If $f : x \to y$ and $g : y \to z$ are morphisms so that gf is epic, then g is epic

§4.4.2 solution

1. b) We are given f, g are both epimorphisms, which imply that for the following commutative diagrams,

$$\begin{tikzcd} x \arrow[r, "f"] & y \arrow[r, bend left, "h"] \arrow[r, bend right, "i"'] & a \end{tikzcd}$$

we have, $hf = if$ implies $h = i$
AND

$$\begin{tikzcd} y \arrow[r, "g"] & z \arrow[r, bend left, "j"] \arrow[r, bend right, "k"'] & b \end{tikzcd}$$

$jg = kg$ implies $j = k$
Where a, b are arbitrary objects and h, i, j, k arbitrary morphisms.
Looking at the composite morphism $gf : x \to z$, we just have to show that for

$$\begin{tikzcd} x \arrow[r, "gf"] & z \arrow[r, bend left, "j"] \arrow[r, bend right, "k"'] & b \end{tikzcd}$$

$jgf = kgf$ implies $j = k$.
We know by assumption that $(jg)f = (kg)f$ implies $(jg = kg)$. But $jg = kg$ implies $j = k$. So epimorphisms compose.
By duality, the same is true for monomorphisms

2. a) Suppose fg is monic, and Suppose f is not monic. Then for the following diagram,

$$\begin{tikzcd} b \arrow[r, bend left, "k"] \arrow[r, bend right, "h"'] & x \arrow[r, "f"] & y \end{tikzcd}$$

$fg = fk$ does not imply $i = j$. Then for any morphism $g : y \to z$, $gf : z \to z$ has

$$\begin{tikzcd} b \arrow[r, bend left, "k"] \arrow[r, bend right, "h"'] & x \arrow[r, "gf"] & z \end{tikzcd}$$

$gfh = gfk = g(fh) = g(fk)$. But $(fk) = (fh)$ do not imply that $k = h$, so

$$\begin{tikzcd} b \arrow[r, bend left, "fk"] \arrow[r, bend right, "fh"'] & y \arrow[r, "g"] & z \end{tikzcd}$$

gives $gfk = gfh$ may or not imply $fk = fh$, which does not imply $k = h$.

But by assumption, fg is monic, which is a contradiction.
By duality, the same is true for epimorphisms

Then the monomorphisms of a category form a subcategory, as monomorphisms have a specified domain and codomain, identity morphisms are monic (and epic), and monomorphisms compose. The axioms still hold because it is in the larger category. The same is true for the epimorphisms of a category by duality. □

§4.5 HW 4

§4.5.1 Problem

Fixing a parallel pair of functors $F, G : C \rightrightarrows \mathcal{D}$, natural transformations $\alpha : F \Rightarrow G$ correspond bijectively to functors $H : C \times 2 \to D$ such that H restricts along i_0 and i_1 to the functors F and G, i.e., so that

$$\begin{tikzcd} C \arrow[r, "i_0"] \arrow[dr, "F"'] & C \times 2 \arrow[d, "H"] & C \arrow[l, "i_1"'] \arrow[dl, "G"] \\ & \mathcal{D} & \end{tikzcd}$$

commutes

§4.5.2 Solution

$i_0 : C \to C \times 2$ just turns all elements $c \in C$ into ordered pairs $(C, 0)$. Since α is a natural transformation, the diagram

$$\begin{tikzcd} F_c \arrow[r, "Ff"] \arrow[d, "\alpha_c"'] & F_{c'} \arrow[d, "\alpha_{c'}"] \\ G_c \arrow[r, "Gf"'] & G_{c'} \end{tikzcd}$$

commutes.
Then $(\alpha_c : F_c \to G_c) \circ (Gf : Gc \to Gc')$ has $Gf\alpha c = \alpha c' Ff$.
We want to show that for F, G, the natural transformations correspond bijectively to functors $H : C \times 2 \to \mathcal{D}$ because every collection of morphisms from $Fc \to Gc$ has a functor which defines morphisms from $Fc \to Gc$ through $Hi_0 \to Hi_1$. That is, morphisms in $C \times 2$ which go between elements $(c, 0)$ to $(c', 1)$, obey the natural transformation square so that $Hf : (c, 0) \to (c', 1) = Gf\alpha_c = a_{c'}Ff$.
We can see that for each natural transformation α which is a collection of morphisms, there is some H_α, in which:

1. H_α takes objects $(c, 0) \mapsto c$, and $(d, 1) \mapsto d$
2. H_α takes morphisms $f_n : (c, n) \to (c', n)$ to morphisms $Ff : Fc \to Fc'$ if $n = 0$ and $Gf : Gc \to Gc'$ if $n = 1$.
3. H_α takes morphisms $f_n : (c, 0) \to (c', 1)$ to $Gf\alpha_c = a_{c'}Ff$ for the corresponding natural transformation α

There is a bijection from α to $H_\alpha f$, where f is a morphism from $(c, 0) \to (c, 1)$, if we can show that the function $K : \alpha \to H_\alpha f$ is injective and surjective.
K is injective, since our definition of H_α requires that for each $H_{\alpha 1} f_n$, there is some unique α_{1c} where $Gf\alpha_c = a_{c'}Ff$. Then $H_{\alpha 1}$ is defined in terms of α_1, and thus $K(\alpha_1) \neq K(\alpha_2)$ implies

$\alpha_1 = \alpha_2$
Suppose K is not surjective. Then there exists some $H_\alpha f$ where for all α, $K(\alpha) \neq H_\alpha f$. But then $H_\alpha f$ is undefined. Thus K is surjective.
Then natural transformations do bijectively correspond to functors $H_\alpha : C \times 2 \to \mathcal{D}$ such that the diagram commutes. $\square$

---

§4.6 HW 5

§4.6.1 Problem

Prove that Set^{∂} is equivalent Set_* using one of the three formulations of equivalence

1. $GF = 1_C$ and $FG = 1_{\mathcal{D}}$
2. F that is full, faithful, and essentially surjective
3. Skeletons

As a reminder, Set^{∂} is the category whose objects are sets and partial functions between sets. Set_* is the category whose objects are pairs (X, x_0) of a set and an element of that set, and morphisms $(X, x_0) \to (Y, y_0)$ are functions $f : X \to Y$ abiding by $f(x_0) = y_0$

§4.6.2 Solution

We want to prove that for our two categories, there exists some functors F, G such that $F : \text{Set}^{\partial} \to \text{Set}_*$ and $G : \text{Set}_* \to \text{Set}^{\partial}$, and there exists natural transformations $\alpha : \text{id}_{\text{Set}^{\partial}} \cong GF$ and $\varepsilon : FG \cong \text{id}_{\text{Set}_*}$

First we should find such functors:

Let $F : \text{Set}^{\partial} \to \text{Set}_*$ take objects $C \in \text{Set}^{\partial}$ to objects $(C', \perp_{C'}) \in \text{Set}_*$. These objects are pairs of a set C' and an element $\perp_C \in C'$ but $\perp_C \notin C$. But we can construct $(C', \perp_{C'})$ as $(C \cup \perp_C, \perp_C)$, where $\perp_{C'} = \{C\}$ (the singleton set of the whole set C).
Let F take morphisms (which are partial functions between sets) $f : C \to D$ to $Ff : FC \to FD$ where $c \in C$ (remember that C and D are sets) goes to $f(c)$ if $c \in \text{dom}(f)$, $\perp_D$ if $c \notin \text{dom}(f)$.

Let G take objects; pairs of the form $(D, \perp_D) \in \text{Set}_*$ where $\perp_D \in D$, to objects without the basepoint; $(D, \perp_D) \mapsto D/\{\perp_D\} \in \text{Set}^{\partial}$.
Let G take morphisms $g : (C, \perp_C) \to (D, \perp_D)$ to $Gg : G(C, \perp_C) \to G(D, \perp_D)$ where for every element $a \mapsto \begin{cases} f(a) & \text{if } a \neq \perp_C \\ \emptyset & \text{if } a = \perp_C \end{cases}$ $(AnelementainCmapstof(a) \in \text{Set}_*$ unless it is the basepoint). Note that in Set_*, our objects $(D, \perp_D)$ are pairs of a set D and a basepoint in that set. In Set^{∂}, however, our counterpart objects are sets which should not contain that basepoint element.

Next we want to see to where our composite functors GF and FG map our categories.

Our mapping $GF : \text{Set}^{\partial} \to \text{Set}_* \to \text{Set}^{\partial}$ takes a morphism $f : X \to Y$ to $f : (X \cup \perp_X, \perp_X) \to (Y \cup \perp_Y, \perp_Y)$ to $f : X \cup \perp_X /\{\perp_X\} \to Y \cup \perp_Y /\{\perp_Y\} = f : X \to Y$ (By letting $\perp_X = \{X\}$ as mentioned above, we have a basepoint that is not in X). So GF is the identity mapping on Set^{∂}

Our mapping $FG : \text{Set}_* \to \text{Set}^{\partial} \to \text{Set}_*$ takes a morphism $f : (X, \perp_X) \to (Y, \perp_Y)$ to $f : X/\{\perp_X\} \to Y/\{\perp_Y\}$ to $f : (X/\{\perp_X\} \cup \{X/\{\perp_X\}\}, \{X/\{\perp_X\}\}) \to f : (Y/\{\perp_Y\} \cup \{Y/\{\perp_Y\}$

$\}\}, \{Y/\{\perp_Y\}\})$. Note that this last term is not the same as $f : (X, \perp_X) \to (Y, \perp_Y)$.

Next, we want to show that there exists a natural isomorphism between GF and $\mathrm{id}_{\mathrm{SET}^\partial}$ and between FG and $\mathrm{id}_{\mathrm{SET}_*}$. Since $GF = \mathrm{id}_{\mathrm{SET}^\partial}$, they are isomorphic and so a natural isomorphism exists between them. We just have to show that $FG \cong \mathrm{id}_{\mathrm{SET}_*}$
Looking at the two naturality squares for an arbitrary morphism $f : (C, \perp_C) \to (D, \perp_D)$ and the natural transformation ε and it's inverse ε^{-1}, we have:

$$\begin{tikzcd}
FG((C,\perp_C)) \arrow[r, "\varepsilon_{(C,\perp_C)}"] \arrow[d, "FGf"'] & (C,\perp_C) \arrow[d, "f"] \\
FG((D,\perp_D)) \arrow[r, "\varepsilon_{(D,\perp_D)}"] & (D,\perp_D)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
(C,\perp_C) \arrow[r, "\varepsilon^{-1}_{(D,\perp_D)}"] \arrow[d, "g"'] & FG((C,\perp_C)) \arrow[d, "FGg"] \\
(D,\perp_D) \arrow[r, "\varepsilon^{-1}_{(C,\perp_C)}"'] & FG((D,\perp_D))
\end{tikzcd}$$

These squares commute, because our natural transformation $\varepsilon : FG((C, \perp_C)) \to (C, \perp_C)$ takes in objects (defined when we defined the action of FG on morphisms) of the form $(X/\{\perp_X\} \cup \{X/\{\perp_X\}\}, \{X/\{\perp_X\}\})$. So let

$$\varepsilon_C : (C, \perp_C) \to (D, \perp_D) = \left(x \mapsto \begin{cases} x & \text{if} x \in X/\{\perp_X\} \\ \perp_x & \text{if} x = \{X/\{\perp_X\}\} \end{cases} \right) : (C, \perp_C) \to (D, \perp_D)$$

and it follows that $\varepsilon^{-1} : (C, \perp_C) \to FG((C, \perp_C))$ takes in objects in SET_* and returns objects of the form $(X/\{\perp_X\} \cup \{X/\{\perp_X\}\}, \{X/\{\perp_X\}\})$, so

$$\varepsilon_C^{-1} : (C, \perp_C) \to (D, \perp_D) = \left(x \mapsto \begin{cases} x & \text{if} x \neq \perp_X \\ \{X/\{\perp_X\}\} & \text{if} x = \perp_x \end{cases} \right) : (C, \perp_C) \to (D, \perp_D)$$

Then our naturality squares commute for all f, so $FG \cong \mathrm{id}_{\mathrm{SET}_*}$, which along with $GF \cong \mathrm{id}_{\mathrm{SET}^\partial}$ which implies $\mathrm{SET}^\partial \simeq \mathrm{SET}_*$

Some of the first half I looked at the example on Maynard's Site, which has this as an unfinished example, but I tried to use it as sparsely as possible.

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§4.7 HW 7

§4.7.1 Problem

2.1.ii) Prove that if $F : C \longrightarrow \textsc{Set}$ is representable, then F preserves monomorphisms i.e., sends every monomorphism in C to an injective function. Use the contrapositive to find a covariant set-valued functor defined on your favorite category that is not representative. Hint: Prove $C(x, -)$ (aka $\mathrm{Mor}(x, -)$) preserves monomorphisms first.

§4.7.2 Solution

A morphism from x to any other object d in C we will call f_d. A morphism $g : c \longrightarrow c'$ after it passes through a covariant hom-functor $C(x, -)$ will be denoted f_g.

First, we show that functor $C(x, -)$ preserves monomorphisms. $C(x, -)$ takes objects c in C to the set of morphisms $\{f_c : x \longrightarrow c\}$, and morphisms $g : c \longrightarrow c'$ to the $\textsc{Set}$-valued morphism (aka. post-composition function) f_g which takes in a C-valued morphism f_c and gives us a C-valued compound morphism gf_c [7]. Then suppose we have some morphism $\alpha : z \longrightarrow a \in C$, such that for all $g, f : y \rightrightarrows z$, the diagram:

$$\begin{tikzcd} y \arrow[r, bend left, "h"] \arrow[r, bend right, "g"'] & z \arrow[r, "\alpha"] & a \end{tikzcd}$$

commutes.
Then we are showing that f_α is a monomorphism. We know f_α takes the morphism $f_z : x \longrightarrow z$ to the morphism αf_z. Then we need to show that $f_\alpha f_g = f_\alpha f_h$ implies $f_g = f_h$. We know that $f_\alpha f_h = \alpha \circ -(h \circ -(f_y)) = \alpha h f_y = \alpha g f_y = f_\alpha f_g$. Thus f_α be a monomorphism.

Now we know that $C(x, -)$ preserves monomorphisms. Then we want to show that if there exists a natural isomorphism $\eta : F \cong C(x, -)$, then F preserves monomorphisms; i.e., if $C(x, -)(\alpha)$ is a monomorphism then $F\alpha$ is a monomorphism. Looking at the naturality square, we have:

$$\begin{tikzcd} F_c \arrow[r, "Fg"] \arrow[d, "\eta_c"'] & F_d \arrow[d, "\eta_d"] \\ f_c : x \longrightarrow c \arrow[r, "f_g = g\circ -"'] & f_d : x \longrightarrow d \end{tikzcd}$$

Recall that both of our functors go to $\textsc{Set}$. Then we just have to prove that our functions are injective. We know that for our monomorphism α, $F\alpha = \eta_z^{-1} f_\alpha \eta_a$. f_α is injective, and η_a and η_z^{-1} are bijective, which means the composite function $\eta_z^{-1} f_\alpha \eta_a$ is injective. Then $F\alpha$ is injective, which means it is monic, as desired. □

[7] f_g is a **post-composition function** which takes in some object k in the Hom-set $x \longrightarrow c$ and returns $g \circ -(k)$

§4.7.3 Problem

2.2.ii) Explain why the Yoneda lemma does not dualize to classify natural transformations from an arbitrary set-valued functor to a represented functor.

§4.7.4 Solution

We are trying to show that there is some functor F out of some category C and some object c where $\alpha : C(c, -) \implies F$ does not map in bijection to $\alpha_c(1_c)$. Taking C to be $\textsc{Set}$, and letting $c = \emptyset$ be the empty set, we note that the functor $\textsc{Set}(\emptyset, -)$, which maps objects $-$ to the morphisms $f_- : \emptyset \to -$. So every morphism just maps to the identity.
Then the natural transformation $\alpha : (F : \textsc{Set} \to \textsc{Set}) \implies \textsc{Set}(\emptyset, -)$ is not in bijection to $\alpha_\emptyset(1_\emptyset) = \emptyset$.

I definitely looked a lot of stuff up in order to wrap my head around this one, I hope this at least shows understanding.

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§4.8 HW 8

§4.8.1 Problem

Construct colimits for each of the following diagrams in both Set and Grp:
The empty diagram:
The one-object diagram:$\bullet$
The coproduct diagram: $\bullet\bullet$
The coequalizer diagram: $\bullet \rightrightarrows \bullet$

§4.8.2 Solution

1. The colimit is the initial object of the category of cocones over $F : \mathcal{J} \to C$. In Set, a cocone under the empty diagram is an object in Set. Then we just need the initial object of Set. For every set, there exists a unique function from the initial object to the set, which is true for the empty set.

2. The colimit of the one-object diagram is an object such that for all $c_i \in \text{cocone}(F)$, there exists a unique morphism $f_i : \text{colim} \to c_i$ such that the diagram
$$\begin{tikzcd} \bullet \arrow[d, "g"] \arrow[dd, bend right=60, "\lambda_i"'] \\ \text{colim} \arrow[d, "\exists! f"] \\ c_i \end{tikzcd}$$
commutes.

   We know that $f_i g = \lambda_i$ and that g must be an epimorphism, since every morphism $f_i : c_i \to \lim(F)$ is unique. But also notice that

3. The colimit of the coproduct diagram $\bullet\bullet isthedisjointunionofsets.Thisisbecauseitsatisfiestheuniversalpr$

1. The colimit in the empty diagram for Grp is the trivial group.

2.

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§4.9 HW 9

§4.9.1 Problem

Let F be the following set valued diagram:

$A \qquad B$

Prove that the functor $\text{Cone}(-, F) : \textsc{Set} \to \textsc{Set}$, sending a set x to the set of cones over F with apex x and who's action on morphisms was described in lecture is a representable functor that is represented by the limit of F (namely, the cartesian product $A \times B$)

Added comment: note that the cone functor $\text{Cone}(-, F)$ is a contravariant functor. Thus, a representation of $\text{Cone}(-, F)$ by the object $\lim F$ in $\textsc{Set}$ consists of a natural isomorphism $\textsc{Set}(-, \lim F) \leftrightarrow \text{Cone}(-, F)$.

§4.9.2 Solution

The functor $\text{Cone}(-, F) : \textsc{Set} \to \textsc{Set}$ acts the following:

1. On objects $x \mapsto \{\text{Cone}(x, F)\}$
2. On morphisms $f : x \to y$, $f \mapsto f_0 : \text{Cone}(x, F) \to \text{Cone}(y, F)$

And we just need to show that there exists a natural isomorphism $\alpha : \text{Cone}(-, F) \cong \textsc{Set}(-, A \times B)$. Then for all $a \in \textsc{Set}$, we have $\alpha_a : \text{Cone}(a, F) \cong \textsc{Set}(a, A \times B)$, so for every morphism $\text{Cone}(a, F) \cong \textsc{Set}(a, A \times B)$

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§5 bibliography

https://qchu.wordpress.com/2012/10/01/split-epimorphisms-and-split-monomorphisms/ https://qchu.wordpress. and-epimorphisms/: :text=Somehttps://ncatlab.org/nlab/show/bifunctor

https://ncatlab.org/nlab/show/pullback https://ncatlab.org/nlab/show/limit https://math.uchicago.edu/ may/VI-GRE/VIGRE2007/REUPapers/FINALAPP/Messick.pdf https://ncatlab.org/nlab/show/cone https://cronokirby.com/posts/2020/06/monomorphisms-vs-epimorphisms/

https://ncatlab.org/nlab/show/folklore

https://math.ucr.edu/home/baez/act$_{c}ourse/https://ncatlab.org/nlab/show/over+category$

https://www2.math.ou.edu/ mmolander/Equivalence$_{o}f_{C}ategories.pdf$

https://en.wikipedia.org/wiki/Equaliser$_{(}mathematics)In_{c}ategory_{t}heoryhttps://en.wikipedia.org/wik$
$text = A$

References

[1] **Monomorphisms vs Epimorphisms**, by f. https://cronokirby.com/posts/2020/06/monomorphisms-vs-epimorphisms/.

[2] **Over Category**, by f. https://ncatlab.org/nlab/show/over+category

[3] **Wikipedia**, https://en.wikipedia.org/wiki/Main_Page

[4] **Math Stack Exchange**, https://math.stackexchange.com/ https://math.stackexchange.com/questions/3889401/what-are-post-composition-and-pre-composition-in-category-theory https://math.stackexchange.com/questions/1948753/good-notation-for-hom-functors https://math.stackexchange.com/questions/3375799/monomorphisms-are-preserved-by-representable-functors https://math.stackexchange.com/questions/3165094/why-does-mathbbz-represent-the-forgetful-functor-u-mathbfgrp-to-mathbf https://math.stackexchange.com/questions/1948753/good-notation-for-hom-functors

[5] **Maynards Site**, https://cats.z.maynards.site/

[6] **Quiver**, https://q.uiver.app/

[7] **ncatlab**, https://ncatlab.org/nlab/show/HomePage

[8] **math3ma**, https://www.math3ma.com/

[9] **just categories**, https://justcategories.wordpress.com/2012/11/26/universal-arrows/

[10] **ncatlab**, https://ncatlab.org/nlab/show/HomePage

[11] **ncatlab**, https://ncatlab.org/nlab/show/HomePage