Banach Algebras

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Abstract

This is a description of what a Banach Algebra is without assuming very much knowledge of algebra. It is the first part of my journey in learning about the Corona Problem. This is a draft and may have mistakes; it should not be taken as something particularly useful.

1 What is a Banach Algebra?

Definition 1.1 (Normed Vector Space)**.** A vector space V is a set of vectors which can be added together and multiplied by scalars. We can define a norm over V, which is a map

$$||\cdot|| : V \to \mathbb{R}$$

satisfying the following properties:

1. For all $v \in V$, $||v|| \geq 0$
2. $||v|| = 0$ if and only if $v = 0$.
3. For every vector V and every scalar λ, $||\lambda v|| = |\lambda| ||v||$
4. Triangle inequality: for all $w, v \in V$, $||v + w|| \leq ||v|| + ||w||$

If V is a real or complex vector space (i.e., the scalars are in $\mathbb{R}$ or $\mathbb{C}$) then the pair $(V, ||\cdot||)$ is called a **normed vector space**.

Definition 1.2 (Banach Space)**.** A norm induces a metric $d : V \times V \to \mathbb{R}$ (concept of distance) by

$$d(v, w) := ||w - v||$$

This metric space is called a **Banach Space** if it is **complete**. (that is, every Cauchy sequence of points in V has a limit that is also in V)

Example 1.3 (L^p space)**.** An L^p space is the space of measurable functions for which the pth power of the absolute value is Lebesgue integrable. For a function f, the L^p norm is defined as

$$||f||_p := \left(\int_{\mathbb{R}} |f(x)|^p dx \right)^{\frac{1}{p}}$$

We can then say that the L^p space is the space of functions for which $||f||_p < \infty$.

The normed vector space

$$(L^p, ||\cdot||_p)$$

is a Banach space for every $1 \leq p \leq \infty$[1]

Definition 1.4 (Commutative ring). A **ring** is a set R with two binary operations $\cdot : R \times R \to R$ and $+ : R \times R \to R$ where $(R, \cdot)$ is a monoid, $(R, +)$ is an abelian group, and multiplication **distributes** over addition, i.e.,

$$a(b + c) = ab + ac$$

A ring is **commutative** if $(R, \cdot)$ is a commutative monoid (multiplication commutes).

Definition 1.5 (Ring things). The **center** $Z(R)$ of a ring R is the **subring** (a subset of R which is closed under $+$ and $\cdot$) consisting of all the elements $x \in R$ such that $xy = yx$ for all elements $y \in R$.

A **ring homomorphism** is a map $\phi : R \to S$ with

1. $\phi(r_1 + r_2) = \phi(r_1) + \phi(r_2)$
2. $\phi(r_1 r_2) = \phi(r_1)\phi(r_2)$
3. $\phi(1_R) = 1_S$

Example 1.6. The inclusion map $\iota : Z(R) \hookrightarrow R$ is an example of a ring homomorphism. If we have a commutative ring, there is a homomorphism from $R \to Z(R)$, specifically the identity map.

Definition 1.7 (Associative Algebra). An **Associative algebra** over a commutative ring K is a ring A with a ring homomorphism from K into the center of A.

This is also a vector space, with vectors elements of a, and with scalar multiplication induced by the homomorphism $\phi : k \to a$. in particular, for $x \in a$, $c \in K$, define scalar multiplication $c * x$ by $\phi(c) \cdot x$ (multiplication in the ring A).

Example 1.8 ($n \times n$ Matrices over $\mathbb{R}$). The set of $n \times n$ matrices with elements in $\mathbb{R}$ forms a ring $M_n(\mathbb{R})$ since you can do addition and multiplication of matrices. It is also a vector space since you can multiply matrices by scalars in $\mathbb{R}$. Thus it forms an associative algebra.

Finally, we can introduce the setting of the theorem:

Definition 1.9 (Banach Algebra). A Banach Algebra is an Associative Algebra A over the real or complex numbers that is also a Banach space, with the norm satisfying

$$||xy|| = ||x||\ ||y||,\ \forall\ x, y \in A$$

[1]This property is required for the metric space to be complete, by the Riesz–Fischer theorem.

> This is required for multiplication to be continuous with respect to the metric

Some examples of Banach Algebras are:

- The set of real (or complex) numbers with the norm given by absolute value
- $M_n(\mathbb{R})$ with a matrix norm which is **sub-multiplicative**, i.e. $||AB|| \leq ||A||\ ||B||$.
- The algebra of all bounded real or complex valued functions defined on some set (with pointwise multiplication and the supremum norm).

This last one makes sense because $(L^\infty, ||\cdot||_\infty)$ is a Banach space, and because the space of real or complex valued functions is a vector space, with a **bilinear product** given by pointwise multiplication[2].

To be more succinct, a Banach Algebra is a vector space V, equipped with a norm such that it is also a complete metric space. It also has an associative binary operation on the vectors which distributes over addition (just like in a ring), which behaves well with scalar multiplication (because it's bilinear), and which works nicely with the norm, i.e., for all $x, y \in V$,

$$||xy|| = ||x||\ ||y||$$

It should be very clear that this is an extremely appropriate setting in which to study functions and spaces of functions, since it is, in short, a vector space with vector multiplication and an ability to measure distances and take limits of the vectors, where everything fits together nicely.

[2] This is the multiplication in the commutative ring.