

Math 142: Elementary Algebraic Topology Notes as taught by John Lott

Adrian Fry

2025

Contents

1 Metric Spaces	3
1.1 Introduction to Metric Spaces	3
2 Continuity	5
2.1 Introduction to Topology	5
2.2 Continuous functions	7
2.3 Space Filling Curve	10
2.4 More on Continuity	10
3 Compactness and Completeness	11
3.1 Closed and Bounded Subsets	11
3.2 Properties of Compact Spaces	12
3.3 Product Spaces	13
3.4 Connectedness	16
3.5 Path Connectedness	18
4 Identification Spaces	20
4.1 The Identification Topology	20
4.2 Topological Groups	23
4.3 Group Actions	27
5 The Fundamental Group	30
5.1 Homotopic Maps	30
5.2 The Fundamental Group	32
5.3 The fundamental group of S^1 and Brouwer's fixed-point theorem	35
5.4 Calculations	39
6 Simplicial Complexes	44
6.1 Triangulating Spaces	45
6.2 Simplicial Approximation	48
6.3 Free Groups and Products	49
6.4 The Edge Group of a Complex	51
6.5 Abstract Simplicial Complexes	55
7 Simplicial Homology	56
7.1 Homology Groups	56
7.2 Simplicial Maps	63
7.3 Homological Algebra	64

8 Degree and Lefschetz Number	64
8.1 The Euler-Poincare formula	66
8.1.1 Calculations	68
8.2 The Lefschetz Fixed-Point Theorem	69

Abstract

These are my notes for my math class: Math 142 @UC Berkeley as taught by Dr. John Lott. Textbook used is Basic Topology, By M.A. Armstrong, as well as the first chapter of Mathematical Analysis II, by Vladimir Zorich. Any mistakes are my own.

1 Metric Spaces

1.1 Introduction to Metric Spaces

1.22.2025

Main Idea of This Class

"Assign numbers to spaces"

Of course, by space, we mean a topological space, and by number we mean algebraic objects.

The layout of this course is as follows. We will start out by covering general topology, before moving on to algebraic topology. We will start out with studying the real numbers.

Definition 1.1. Given $x, y \in \mathbb{R}$, let $D(x, y) = |x - y|$. D has the following properties:

1. $D(x, y) \geq 0$
2. $D(x, y) = 0$ if and only if $x = y$
3. D is symmetric
4. Triangle Inequality

Proof. 1. \mathbb{R} is closed, so $x - y \in \mathbb{R}$. And $|z| > 0$ for all $z \in \mathbb{R}$. So 1 holds.¹

2. Suppose $D(x, y) = 0$. Then $|x - y| = 0$, so $x - y = 0$, so $x = y$. The reverse direction is similar

□

Definition 1.2. A **metric** on a set X is a function $d : X \times X \rightarrow \mathbb{R}$, such that

1. $d(x, y)$ is nonnegative
2. $d(x, y) = 0$ if and only if $x = y$
3. d is symmetric
4. Triangle Inequality

Example 1.3 (ℓ_p norm). For a real number p , the ℓ_p norm of a vector $x = \{x_1, \dots, x_n\}$ is

$$\|x\|_p := \left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}}$$

So for $p = 1$, we get the taxicab norm, $p = 2$, we get the euclidean norm, and when $p \rightarrow \infty$ we get the ℓ_∞ norm, where $d(x, y) = \max(|x_i - y_i|)$

Definition 1.4. Let X be a metric space. The ball with center $a \in C$ and radius $\delta > 0$ is $B(a, \delta) = \{x \in X \mid d(a, x) < \delta\}$

Definition 1.5. $G \subset X$ is open if for all $x \in G$, there exists δ such that $B(x, \delta) \subset G$

Definition 1.6. A set is **closed** if its complement is open.

Proposition 1.7. 1. \emptyset and X are always open

2. $B(a, r)$ is open

¹Hopefully this proof was a waste of my time

3. $\{x \in X | d(a, x) > r\}$ is open

1.24.2025

Proposition 1.8 (Properties of Open and Closed Sets).

The union $\bigcup_{\alpha \in A} G_\alpha$ of open sets is open.

The finite intersection $\bigcap_{i=1}^{\infty} G_i$ of open sets is open.

The finite union $\bigcup_{i=1}^{\infty} G_i$ of closed sets is closed.

The intersection $\bigcap_{\alpha \in A} G_\alpha$ of closed sets is closed.

Proof. 1. Let $x \in \bigcup_{\alpha \in A} G_\alpha$. Then there exists α such that $x \in G_\alpha$. And G_α is open, so there exists some $B(x, \delta) \subset G_\alpha \subseteq \bigcup_{\alpha \in A} G_\alpha$, so $B(x, \delta) \subset \bigcup_{\alpha \in A} G_\alpha$, so $\bigcup_{\alpha \in A} G_\alpha$ is open.

2. Let $x \in \bigcap_{i=1}^{\infty} G_i$. For all i , there exists $\delta_i > 0$ such that $B(x, \delta_i) \subset G_i$. Set $\delta = \min_i(\delta_i)$. Then $B(x, \delta) \subseteq B(x, \delta_i) \subset G_i$, therefore $B(x, \delta) \subset \bigcap_{i=1}^n G_i$

3. and 4. follow from DeMorgan's Laws:

- (a) $X - (A \cup B) = (X - A) \cap (X - B)$
- (b) $X - (A \cap B) = (X - A) \cup (X - B)$

$\forall A, B, X$

□

Say (X, d) is a metric space and $X_1 \subset X$. Then there is a canonical metric $d_1(a, b) = d(a, b)$ for $a, b \in X_1$.

$$B_1(a, r) = B(a, r) \cap X_1, \text{ for } a \in X_1$$

where $B_1(a, r)$ is the metric ball with respect to d_1 and $B_1(a, r) \cap X_1$ is the metric ball with respect to d .

Proposition 1.9. If $G_1 \subset X_1$ is open then $G_1 = G \cap X_1$ for some G open in X .

If $F_1 \subset X_1$ is closed then $F_1 = F \cap X$ for some F closed in X

Proof. 1. Say $G_1 \subset C_1$ is open. For all $a \in G_1$, there exists $\delta_a > 0$ such that $B(a, \delta_a) \subset G_1 = \bigcup_{a \in G_1} B_1(a, \delta_a)$. But $G = \bigcup_{a \in G_1} B(a, \delta_a)$ in X . Then G is open in X and $G_1 = G \cap X$.

□

Products in Metric Spaces: Say (X_1, d_1) and (X_2, d_2) are metric spaces.

$$X_1 \times X_2 = \{(x_1, x_2) : x_1 \in X_1, x_2 \in X_2\}$$

Possible metrics on $X_1 \times X_2$ include

1. The euclidean norm = $\sqrt{d_1(x_1, x'_1)^2 + d_2(x_2, x'_2)^2}$
2. The taxicab norm
3. The ℓ_∞ norm

Fact: These 3 metrics all have the same open sets.

Remark 1.10. First, we generalized from \mathbb{R} to metric spaces. Now we generalize to topological spaces. (Using the Armstrong book)

2 Continuity

2.1 Introduction to Topology

Definition 2.1. If X is a set, a **topology** on X is a collection of "open" subsets such that

1. \emptyset and X are open
2. Any union of open sets is open
3. A finite intersection of open sets is open

A set together with a topology is called a **topological space**. So a topological space (X, τ) is a set X , and a topology τ , whose members are open sets (open subsets of X) which satisfy the properties. So we can't define a nontrivial topological space without assigning the property of being open to at least multiple subsets of X .

If our topological space has a subset, we have an **induced topology**:

Definition 2.2 (The subspace topology on Y). $U \subset Y$ is open if $U = \mathcal{O} \cap Y$ for some open set $\mathcal{O} \subseteq X$

1.27.2025

Proposition 2.3. The subspace topology is a topology on Y

Proof. We just need to check that it satisfies the properties of a topology:

1. \emptyset is open because $\emptyset = \emptyset \cap Y$, and $Y = X \cap Y$, therefore Y is open.
2. Say the family of subsets $\{U_\alpha\}_{\alpha \in A}$ are open in Y . We can write $U_\alpha = \mathcal{O}_\alpha \cap Y$, where \mathcal{O}_α is open in X . Then $\bigcup_{\alpha \in A} U_\alpha = (\bigcup_{\alpha \in A} \mathcal{O}_\alpha) \cap Y$, and since the union of open sets is open in X , its intersection with Y is open in Y by definition of the subspace topology, so we have that the union of open sets in Y is open in Y .
3. Say $\{U_i\}_{i \in I}^n$ are open in Y . Like before, we can write $U_i = \mathcal{O}_i \cap Y$ with \mathcal{O}_i open in X . Then $\bigcap_{i=1}^n U_i = (\bigcap_{i=1}^n \mathcal{O}_i) \cap Y$, which is an open set in X intersection with Y , which is open, as desired.

□

Example 2.4. $X = \mathbb{R}^2$, $Y = [-1, 1] \times [-1, 1]$. Let $U = (0, 1] \times [-1, 1]$. As a subset of Y , U is open in Y (but not in X). $U = \mathcal{O} \cap Y$, and \mathcal{O} open in X .

What are some examples of a topology defined on a set X ?

Example 2.5 (Discrete and Indiscrete). In the **Discrete topology**, every set is an open set.

In the **Indiscrete topology**, every set is no sets are open except for X and \emptyset . (Notice that this is also the trivial topology, since we can't define 2 distinct elements of X without them both being closed)

So topological spaces come from the metric space axioms, but we have been abstracting them into some ugly topologies. The question stands:

Question 2.6

Does every topological space come from a metric space?

Let's consider a few examples.

Example 2.7. Say X is a finite set (x_1, \dots, x_n) with some metric d . What is the induced topology? Say $\delta = \frac{1}{2} \min_{i \neq j} \delta(x_i, x_j)$. Then $B(x_i, \delta)$ is open, and is equal to $\{x_i\}$. Every union of these open sets (which are all the subsets of X) are open, and the intersection of any of these singleton sets are \emptyset , which is open. So our finite set induces the discrete topology!

If the discrete topology comes from a metric space, what about the indiscrete topology?

Example 2.8. Let $X = \{x_1, x_2\}$ with the indiscrete topology. This does not come from a metric, since there is no way to define distance between the points. (also, by the example above, every metric induces the discrete topology, which is obviously not the indiscrete topology).

So the answer to our question is no.

In metric spaces we have an idea of distance, so let's define a topological analog of "nearness".

Definition 2.9. A set N is a **neighborhood** of x if there exists an open set O such that $x \in O \subseteq N$. Let \mathring{N} denote the set of points z such that N is a neighborhood of z .

Definition 2.10. $F \subset X$ is **closed** if $F \setminus X$ is open.

Definition 2.11. Let $A \subset X$. Then $p \in X$ is a **limit point** of A if every neighborhood of p intersects A in some point other than p .

Example 2.12. For $X = \mathbb{R}$, $A = \{\frac{1}{n}\}_{n=1}^{\infty}$, the only limit point is 0.

Example 2.13. For $A = [0, 1)$, however, every point in $[0, 1)$ is a limit point. In fact, the set of limit points is larger: $[0, 1]$.

Theorem 2.14. A set $A \subset X$ is closed if and only if it contains all of its limit points.

Proof. For the forward direction, suppose A is closed. Then $X \setminus A$ is open. Suppose $\exists p \notin A$. Then p is not a limit point. $X \setminus A$ is open, so $X \setminus A$ is a neighborhood of p that doesn't intersect A , hence p is not a limit point. For the reverse direction, suppose A contains all its limit points. We want to show that $X \setminus A$ is open. Suppose $p \in X \setminus A$. Then p isn't a limit point, so there exists some open O with $p \in O$ such that $O \cap A = \emptyset$, i.e., $p \in O \subset X \setminus A$, hence $X \setminus A = \bigcup_{p \in X \setminus A} O_p$ is open. \square

Definition 2.15. If $A \subset X$, $\overline{A} = A \cup \{\text{limit points of } A\}$ is denoted the **closure** of A .

Theorem 2.16. \overline{A} is the smallest closed set containing A . i.e., $\overline{A} = \bigcap_{B \supset A} \overline{B}$

Proof. 1. First, we will show that \overline{A} is closed. If $p \in X \setminus \overline{A}$, then there exists some open set U_p such that $p \in U_p \subset X$. But $U_p \subset X \setminus \overline{A}$, since any $q \in U_p$ is not in A and is not a limit point either. Hence the union $\bigcup_{p \in X \setminus \overline{A}} U_p$ is open. Hence intersection of closed sets containing $A \subseteq \overline{A}$

2. Suppose B is a closed set containing A . Any limit point of A is also a limit of B . So $\overline{A} \subset \overline{B} = B$, hence $\overline{A} = \bigcap_{\text{closed } B \supset A} \overline{B}$ \square

In real analysis, we heard about \mathbb{Q} being dense in \mathbb{R} . This meant that for all $x, y \in \mathbb{R}$, there exists $z \in \mathbb{Q}$ such that $x < z < y$. We may also recall the construction of \mathbb{R} from \mathbb{Q} via Dedekind cuts.

Question 2.17

Can we generalize the idea of density to topological spaces in general?

Definition 2.18. $A \subset X$ is **dense** in X if $\overline{A} = X$

Since we defined the closure of a subset, let us define the analogue for open-ness

Definition 2.19. The **interior** of a subset A is the largest open set in A . The interior $\text{int}(A) := \{p \in A \mid \exists \text{ open } \mathcal{O} \text{ such that } p \in \mathcal{O} \subseteq A\}$.

1.29.2025

Definition 2.20. The **frontier** of $A = \overline{A} \cap \overline{X \setminus A}$. You can think of it as the crust.

Example 2.21. For $X = \mathbb{R}^2$, $A = \{x \in \mathbb{R}^2 \mid |x| \leq 1\} = \overline{A}$

$$X \setminus A = \{x \in \mathbb{R}^2 \mid |x| > 1\}$$

$$\overline{X \setminus A} = \{x \in \mathbb{R}^2 \mid |x| \geq 1\}$$

which implies the frontier $\overline{A} \cap \overline{X \setminus A} \subset A$

When we learned about vector spaces, we defined the basis vectors $\{v_1, \dots, v_n\}$ to be the vectors such that any other vector $x \in V$ is a linear combination $x = \sum_{i=1}^n a_i v_i$. This notion of an object having basis elements being those which create all the other elements appears in other mathematical objects.

Definition 2.22 (Basis). Let X be a topological space. A **basis** β is a collection of open sets such that any open set is a union of basis elements

Example 2.23. We could take $\beta = \{\text{open sets}\}$, but this is useless

Example 2.24. We could also take $\beta = \{(a, b) \mid a, b \in \mathbb{Q}\}$ for $X = \mathbb{R}$

Proposition 2.25. A collection of open sets is a basis if and only if given $p \in X$ and an open neighborhood N of p , there exists $B \in \beta$ such that $p \in B \subset N$

Proof. For the forwards direction, suppose β is a basis. Then for all $N \ni p \in X$, N is a union of elements of β , therefore there exists some $B \ni p$, and $B \subset N$.

For the backwards direction, say N is an open set. Then for all $p \in N$, there exists $B_p \subset N$ where $p \in B_p$. So $\bigcup_{p \in N} B_p = N$, so β is a basis. \square

2.2 Continuous functions

In real analysis we defined continuous functions $f : X \rightarrow Y$, where X, Y are subsets of \mathbb{R} . Now, let $f : X \rightarrow Y$, where X, Y are topological spaces. What makes f continuous?

Definition 2.26. f is continuous if for all open sets $\mathcal{O} \subseteq Y$, $f^{-1}(\mathcal{O}) = \{x \in X \mid f(x) \in \mathcal{O}\}$ is open in X

Definition 2.27. A **map** is a continuous function.

Proposition 2.28. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are maps, then so is $g \circ f$

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

Proof. If $\mathcal{O} \subset Z$ is open, then $g^{-1}(\mathcal{O})$ is open, which means $f^{-1}g^{-1}(\mathcal{O})$ is open, so $f \circ g$ is a continuous function. \square

Proposition 2.29. If $f : X \rightarrow Y$ is continuous and $A \subset X$, then $f|_A : A \rightarrow Y$ is continuous, where A gets the subspace topology from X .

Proof. Say $\mathcal{O} \subset Y$ is open. $f|_A^{-1}(\mathcal{O}) = A \cap f^{-1}(\mathcal{O})$ is open in the relative (subspace) topology on A . \square

Example 2.30. The identity map $1_X : X \rightarrow X$ is continuous. If $A \subset X$, then $1_X|_A : A \rightarrow X$ is continuous. Here, $1_X|_A$ is the inclusion map of A into X

Theorem 2.31. The following are equivalent.

1. $f : X \rightarrow Y$ is a map.
2. If β is a basis of Y , and $B \in \beta$, then $f^{-1}(B)$ is open.
3. $f(\overline{A}) \subseteq \overline{f(A)}$
4. $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for all $B \subset Y$
5. The preimage of a closed set is closed.

Proof. 1 \implies 2 Any basis element B is open, therefore $f^{-1}(B)$ is open.

2 \implies 3 We want to use (2) to show that the closure stays in the closure. Suppose $A \subset X$. Then $f(A) \subset \overline{f(A)}$. Let $x \in \overline{A}$. We want to show that $f(x) \in \overline{f(A)}$. There are two cases:

1. If $f(x)$ is in $f(A)$, then we are done
2. Now suppose $f(x)$ is only in the frontier of $f(A)$. Take a neighborhood N of $f(x)$. We can then find an element B of the basis such that $f(x) \in B \subseteq N$. By (2), the preimage is also open, so $f^{-1}(B)$ is a neighborhood of x . But x is a limit point of A , so $f^{-1}(B) \cap A \setminus x \neq \emptyset$, which means there exists some point $a \in f^{-1}(B) \cap A$. Then $f(a) \in B$. So there exists some $f(a) \in B \subseteq N$ such that $f(a) \in A$ for every neighborhood N around x , which is just the definition of a limit point. Then $f(x) \in \overline{f(A)}$, as desired.

3 \implies 4 Suppose we have $B \subset Y$. Put $A = f^{-1}(B)$. By (3), $f(\overline{f^{-1}(B)}) \subset \overline{f(f^{-1}(B))} \subset \overline{B}$. So $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$.

4 \implies 5 First, note that if B is a closed subset, then $B = \overline{B}$. Say $B \subseteq Y$ is closed. $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) = f^{-1}(B)$. But $f^{-1}(B) \subseteq \overline{f^{-1}(B)}$, so $f^{-1}(B) = \overline{f^{-1}(B)}$, so $f^{-1}(B)$ is closed.

5 \implies 1 Say $\mathcal{O} \subseteq Y$ is open. $Y \setminus \mathcal{O}$ is closed. By (5), its inverse $(Y \setminus \mathcal{O})$ is closed in X , but $f^{-1}(\mathcal{O}) = X \setminus f^{-1}(Y \setminus \mathcal{O})$ is open in X . \square

Example 2.32 (The $\epsilon - \delta$ definition). Say X, Y are metric spaces and $f : X \rightarrow Y$ is continuous. Given $a \in X$ and $\epsilon > 0$, look at $B(f(a), \epsilon)$. $f^{-1}(B(f(a), \epsilon))$ is open in X and $a \in f^{-1}(B(f(a), \epsilon))$. We can find some $\delta > 0$ such that

$$a \in B(a, \delta) \subset f^{-1}(B(f(a), \epsilon))$$

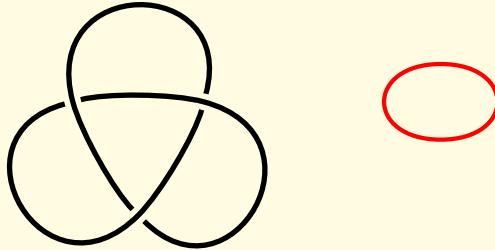
$f(B(a, \delta)) \subseteq B(f(a), \epsilon)$, which is the $\epsilon - \delta$ definition. If $x \in B(a, \delta)$, then $f(x) \in B(f(a), \epsilon)$, that is,

$$d(a, x) < \delta \implies d(f(a), f(x)) < \epsilon$$

Definition 2.33. A **homeomorphism** (coming from the latin "homeo" for same, and "morphism" for form/shape) is a function $h : X \rightarrow Y$ that is injective, surjective, continuous, and has a continuous inverse.

Remark 2.34. Homeomorphisms are isomorphisms in the Category TOP. The morphisms are maps.

Example 2.35. Consider the knots k_1 (the unknot) and k_2



Give k_1 and k_2 the subspace topology from \mathbb{R}^3 . Then they are homeomorphic. (Imagine traveling around the both at a constant speed)

1.31.2025

Recall the definitions of a continuous function and a homeomorphism from last class. How are these different? How are they the same? Let's look at an example of a continuous function that is not a homeomorphism.

Nonexample 2.36. $1_X : (X, \text{Discrete}) \rightarrow (X, \text{Indiscrete})$ does not satisfy all of the properties of a homeomorphism. It is a continuous function, since the two open sets in the indiscrete topology are open in the discrete one, but it is plain to see that many sets in the discrete topology map to closed sets in the indiscrete one.

Intuitively, we can imagine a homeomorphism from a circle excluding a single point to a line, by "unfolding" the circle and just getting a line. But this isn't very "scientific"...

Example 2.37 (S^n is homeomorphic to \mathbb{R}^n). Look at $S^n = \{\bar{x} \in \mathbb{R}^{n+1} \mid |\bar{x}| = 1\}$, with the subspace topology from \mathbb{R}^{n+1} . This is a unit sphere. Consider $p \in S^n$. $S^n \setminus \{p\} \subset S^n$, so give it the subspace topology from S^n . We claim that $S^n \setminus \{p\}$ is a homeomorphism to \mathbb{R}^n .

We will show this using a stereographic projection. (the same one from Complex Analysis). Say p is the north pole. Look at S^n and \mathbb{R}^n in \mathbb{R}^{n+1} . In $n = 1$, S^1 is the unit circle on the horizontal line \mathbb{R} .



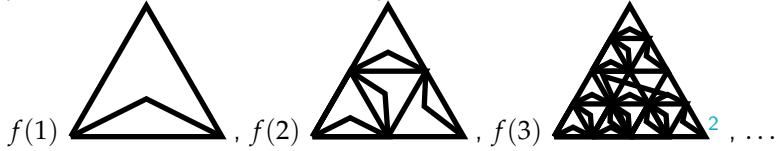
Let

$$\pi : S^n \setminus \{p\} \rightarrow \mathbb{R}^n \quad x \mapsto \text{where } \overline{px} \text{ intersects } \mathbb{R}^n$$

be a projection from the circle to the line, which takes $x \in S^n \setminus \{p\}$ to $\pi(x)$ where the line from the north pole intersects \mathbb{R}^n . Then $S^n \setminus \{p\}$ is homeomorphic to \mathbb{R}^n .

2.3 Space Filling Curve

In this subsection we will explore a curve that can fill up a polygon. Let Δ denote the solid triangle with edge lengths $\frac{1}{2}$. We claim that there exists a continuous map $t : [0,1] \rightarrow \Delta$ which is surjective. Define $f_n : [0,1] \rightarrow \Delta$, where f_n is parameterized by normalized arc length.



Question 2.38

Does this converge? If this does converge, what does this mean for dimensionality?

We will show that it does converge.

Proof. For any two points $x, y \in \mathbb{E}^2$, let $\|x - y\|$ denote the distance between them. Suppose $n \geq m$. Then for any $t \in [0,1]$, $f_n(t)$ and $f_m(t)$ are both in a subtriangle with edge lengths $\frac{1}{2^m}$. Therefore

$$\|f_m(t) - f_n(t)\| \leq \frac{1}{2^m}$$

for every value of t in $[0,1]$, which proves that the sequence $\{f_n\}$ is uniformly convergent. This matters because we know that each f_n is continuous, so, letting $f : [0,1] \rightarrow \Delta$ denote the limit function, f is continuous. Now we must show that f does in fact completely cover Δ .

First, we will show that the image $\text{Im}(f)$ is dense in Δ . Suppose $x \in \Delta$. We want to show that $x \in \overline{\text{Im}(f)}$. If x is in $\text{Im}(f)$, then we are done. If x is not, then it is a limit point of f . Remember that for any n , the image of f_n comes within 2^{-n} of every point of Δ . Say U is a neighborhood of x . Choose N large enough so that the disc $B(x, 2^{-(N-2)})$ lies inside U . Choose some $t_0 \in [0,1]$ such that $\|x - f_N(t_0)\| \leq \frac{1}{2^N} = 2^{-N}$. Since $\|f_N(t) - f(t_0)\| \leq 2^{-N}$ for every $t \in [0,1]$, the triangle inequality gives

$$\begin{aligned} \|x - f(t_0)\| &\leq \|x - f_N(t_0)\| + \|f_N(t_0) - f(t_0)\| \\ &\leq 2^{-N} + 2^{-N} = 2^{-(N-1)} \\ &< 2^{-(N-2)} \end{aligned}$$

so $f(t_0) \in B(x, 2^{-(N-2)})$. Hence x is a limit point of $\text{Im}(f)$.

This means that every neighborhood U_x we can choose a smaller open ball which has an intersection point with the image of f . So we've shown that $\text{Im}(f) = \Delta$. So we know now that there is some curve that is dense in Δ . In the future, we will show that $\text{Im}(f) = \overline{\text{Im}(f)}$, but we will leave it as is \square

2.4 More on Continuity

What makes \mathbb{Q} so nice? It's that there's no limit to how small you can go. In between any two numbers, there is a middle one.

Definition 2.39. A topological space is **Hausdorff** if for all points x, y where $x \neq y$, there exists open sets U, V such that $x \in U$, $y \in V$, and $U \cap V \neq \emptyset$.

Example 2.40 (Metric Space). Let $r = \frac{1}{2}d(x, y)$, $U = B(x, r)$, $V = B(y, r)$. Therefore every metric space is Hausdorff.

Nonexample 2.41. $X = \mathbb{Z}$, equipped with the indiscrete topology. Let $y = x + 1$. The only open set that contains y is the entire space X , which also contains y . This is not Hausdorff.

It's very popular to try and find the distance between a point and a set in \mathbb{E}^n . One may recall finding the perpendicular distance between a point and a plane in multivariable Calculus. That was the shortest distance between the point and any point on the plane. We have something similar when dealing with metric spaces.

²I did not draw this third one correctly but you get the gist

Definition 2.42. If X is a metric space and $A \subset X$, define $d_A : X \rightarrow \mathbb{R}$ by

$$d_A(x) = \inf_{a \in A} d(x, a)$$

Proposition 2.43. $d_A : X \rightarrow \mathbb{R}$ is continuous.

Proof. Let X, \mathbb{R} be metric spaces. Given $x \in X$ and $\epsilon > 0$, we want to define δ such that $d(x, z) < \delta \implies |d_A(x) - d_A(z)| < \epsilon$. Take $\delta = \frac{\epsilon}{2}$. Choose $a \in A$ such that $d(x, a) < d_A(x) + \frac{\epsilon}{2}$. Then

$$\begin{aligned} d_A(z) &\leq d(z, a) \\ &\leq d(z, x) + d(x, a) \\ &< \frac{\epsilon}{2} + d_A(x) + \frac{\epsilon}{2} \\ &< d_A(x) + \epsilon \end{aligned}$$

Also, $d_A(x) < d_A(z) + \epsilon$. So we have $|d_A(x) - d_A(z)| < \epsilon$. \square

2.3.2025

Lemma 2.44. Let X be a metric space, and A a subset. Then $d(x, A) = 0$ if and only if $x \in \overline{A}$

Proof. For the forwards direction, suppose $d(x, A) = 0$. For all $\epsilon > 0$, there exists $a \in A$ such that $d(x, a) < \epsilon$. Hence $B(x, \epsilon) \cap A \neq \emptyset$, so $x \in \overline{A}$.

For the backwards direction, suppose $x \in \overline{A}$, then any $B(x, \epsilon) \cap \overline{A} \neq \emptyset$. Choose $a \in B(x, \epsilon) \cap A$. $d(a, x) < \epsilon$. Hence $d_A(x) = 0$. \square

Proposition 2.45. Say $A, B \subset X$ are closed and $A \cap B \neq \emptyset$. Then there exists a continuous function $f : X \rightarrow [-1, 1]$ such that $f|_A = 1$ and $f|_B = -1$.

Proof. But $f(x) = \frac{d(x, B) - d(x, A)}{d(x, B) + d(x, A)}$. Plug in values and you will see this works. But wait!! What if the denominator is zero? Well, since $d(x, B)$ and $d(x, A)$ are both nonnegative, both of them must be 0. But by the lemma 2.44, $x \in \overline{A}$ and $x \in \overline{B}$. Then $x \in A$ and B since they are both closed, but $A \cap B = \emptyset$, so the denominator can never be 0. \square

Theorem 2.46 (Tietze Extension Theorem). Let X be a metric space. $A \subset X$ is closed if $f : A \rightarrow \mathbb{R}$ is continuous, then there exists a continuous function $F : X \rightarrow \mathbb{R}$ such that $F|_A = f$.

"I just mentioned this so you can impress people at parties, but we won't really be using this."
- John Lott

3 Compactness and Completeness

3.1 Closed and Bounded Subsets

Definition 3.1 (Open Cover). A family of open sets \mathcal{F} is called an **open cover** if $\bigcup \mathcal{F} = X$

Definition 3.2. X is **compact** if every open cover of X has a finite subcover.

Nonexample 3.3. Let $X = \mathbb{R}$, $\mathcal{F} = \{i, i+2 | i \in \mathbb{R}\}$. Since there is no finite subcover, \mathbb{R} is not compact.

The rest of today's notes are just about compactness of subspaces.

Theorem 3.4 (Heine-Borel). For all $a, b \in \mathbb{R}$, $[a, b]$ is compact.

Proof. Let \mathcal{F} be a open cover of $[a, b]$. Define a subset X of $[a, b]$ as follows:

$$X = \{x \in [a, b] | [a, x] \text{ is contained in the union of some finite subfamily of } \mathcal{F}\}$$

Then X is nonempty and is bounded above by b . So X has a supremum s . We will claim that $s \in X$ and that $s = b$. Let $O \in \mathcal{F}$ contain s . Since O is open we can choose some $\epsilon > 0$ such that $(s - \epsilon, s) \subseteq O$, and if s is less than b we can assume $(s - \epsilon, s + \epsilon) \subseteq O$.

So s is the least upper bound of X and thus there are points of X arbitrarily close to s . And X has the property that $x \in X$ and $a \leq y \leq x$, then $y \in X$. Therefore we can say $s - \frac{\epsilon}{2} \in X$. By the definition of X , the interval $[s - \frac{\epsilon}{2}, s]$ is contained in the union of some finite subfamily \mathcal{F}' of \mathcal{F} . The union $\bigcup \mathcal{F}' \cup O$ is a finite subfamily of \mathcal{F} whose union contains $[a, s]$. Then $s \in X$. If s is less than b then $\bigcup \mathcal{F}' \cup O$ contains $[a, s + \frac{\epsilon}{2}]$, which means $s + \frac{\epsilon}{2} \in X$ and contradicting the fact that s is an upper bound for X . Therefore $s = b$ and all of $[a, b]$ is contained in $\bigcup \mathcal{F}' \cup O$. \square

Intuitively, what we have done is we have taken some "longest" finite subcover, and shown that any epsilon ball around the supremum of that finite subcover must be a finite subcover in the closed interval, which means either the supremum isn't the supremum, or it is just b .

Warning 3.5

If X is a metric space and $A \subseteq X$ is compact, then it will be closed and bounded. The converse, however, does not hold in general. Let $f : X \rightarrow Y$ be a function.

$A \subseteq X \implies f^{-1}(f(A)) \supseteq A$, which has equality if f is injective.

$B \subset Y \text{ implies } f(f^{-1}(B)) \subseteq B$, which has equality if f is surjective.

3.2 Properties of Compact Spaces

Theorem 3.6. If $f : X \rightarrow Y$ is a map and X is compact, then $f(X)$ is compact. (With the subspace topology)

Proof. We can replace Y by $f(X)$, to assume f is surjective. We want to show that Y is compact. Then $\{f^{-1}(\mathcal{O})\}_{\mathcal{O} \in \mathcal{F}}$ is an open cover of X . Since X is compact, we can extract a finite subcover $X = \bigcup_{i=1}^n f^{-1}(\mathcal{O}_i)$. Since f is onto, $Y = f(X) = \bigcup_{i=1}^n f(f^{-1}(\mathcal{O}_i)) = \bigcup_{i=1}^n \mathcal{O}_i$. \square

Definition 3.7. A subset $C \subset X$ is compact if it's compact in the subspace topology.

Theorem 3.8. A closed subset C of a compact space is compact.

Proof. Take a collection \mathcal{F} of open sets in X that cover C . Then $\mathcal{F} = \{\mathcal{O}_\alpha\}_{\alpha \in C}$. Add $X \setminus C$ to the collection. Then we get an open cover of X . Since X is compact, extract a finite subcover $\mathcal{O}_1 \cup \dots \cup \mathcal{O}_n$.

So we get a finite cover $X \setminus C \cup \mathcal{O}_1 \cup \dots \cup \mathcal{O}_n$ of X . Then $C \subseteq \mathcal{O}_1 \cup \dots \cup \mathcal{O}_n$, that is, a finite cover of C , hence it is compact. \square

Today we have shown a few things:

1. Any closed subset of \mathbb{R} is compact
2. The codomain of a map from a compact set is a compact set
3. Any closed subset of a compact space is compact.

2.5.2025

Theorem 3.9. If X is Hausdorff, $A \subset X$ is compact, and $x \in X \setminus A$, then \exists disjoint neighborhoods of $\{x\}$ and A . Particularly, $X \setminus A$ is open, so A is closed.

Proof. Take $z \in A$. X is Hausdorff, so we can find disjoint open sets U_z and V_z such that $x \in U_z$ and $z \in V_z$. $\{V_z\}_{z \in A}$ is an open cover of A , and there exists a finite subcover $\{V_{z_1}, \dots, V_{z_k}\}$ of A .

Put $V = \bigcup_{i=1}^k V_{z_i}$ and $U = \bigcap_{i=1}^k U_{z_i}$. Then U and V are open disjoint sets, $\{x\} \subset U$ and $A \subset V$. In particular, $\{x\} \subset U \subset X \setminus A$, for any $x \notin A$, so $X \setminus A$ is open. \square

Theorem 3.10. Suppose $f : X \rightarrow Y$ is injective, surjective, continuous, X is compact, and Y is Hausdorff. Then f is a homeomorphism.

Proof. Suppose $g = f^{-1}$. We want to show that g is continuous, i.e., If $C \subseteq X$ is closed, then $g^{-1}(C)$ is closed in Y .

Take C closed in X . Since X is compact, C is compact. Then $f(C)$ is compact in Y . Since Y is Hausdorff, $f(C)$ is closed in Y . Thus f is a homeomorphism. \square

Example 3.11. Consider the space filling curve $f : [0, 1] \rightarrow \Delta$. These are not homeomorphic. Since f is continuous, surjective, X is compact, and Y is Hausdorff, we have that f must not be injective

While the curve isn't injective (one-to-one), it can be shown that the space filling curve can actually be four-to-one, i.e., no more than four points in the domain map to a point in the codomain

Theorem 3.12 (Bolzano Weierstrass Theorem). If X is compact then any infinite subset of X has a limit point.

Proof. Suppose S is any subset of X . Suppose S doesn't have any of its limit points. Then for all $x \in S$, x is not a limit point of S . Then there exists some neighborhood U_x such that

$$U_x \cap S = \begin{cases} \{x\} & \text{if } x \in S \\ \emptyset & \text{if } x \notin S \end{cases}$$

$\{U_x\}_{x \in S}$ covers S . Since X is compact, there exists a finite subcover $\{U_{x_i}\}_{i=1}^k$. Each U_{x_i} intersects S in at most one point, so the cardinality of S is at most K . So a lack of limit points implies the subset is finite, therefore the contrapositive holds. \square

Note that in this scenario, we are imagining S as a set of isolated points, rather than a "blob" of points. This makes a lot of sense if you imagine some sequence (s_n) . Bolzano Weierstrass tells us that no matter what subsequence you choose, it will accumulate at some place, and (if you remember from Real Analysis) it has at some convergent subsequence.

Theorem 3.13. A compact subset of \mathbb{R}^n is closed and bounded

Proof. \mathbb{R}^n is Hausdorff, so any compact subset is closed. $\mathbb{R}^n = \bigcup_{\alpha=1}^{\infty} B(0, \alpha)$. Any compact subset C can be written $C = \bigcup_{\alpha=1}^{\infty} B(0, \alpha) \cap C$. Then there exists a finite subcover, so $C \subset B(0, N)$ for some $N \in \mathbb{Z}$. \square

We can imagine just taking bigger and bigger spheres, until one of them encapsulates our compact subset.

Theorem 3.14. If $f : X \rightarrow \mathbb{R}$ is a map and X is compact, then f is bounded above and below, and the infimum and the supremum are realized.

Proof. Since X is compact, $f(X)$ must be compact in \mathbb{R} . From theorem 3.13, $f(X)$ is bounded in \mathbb{R} . It's also closed in \mathbb{R} , so $\sup(f(X)) \in f(X)$. The infimum case follows similarly. \square

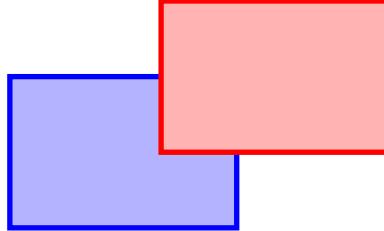
3.3 Product Spaces

Say we have two topological spaces X and Y .

Question 3.15

Can we define topology on the product of their sets $X \times Y$?

Why don't we just try it? Lets say open sets in $X \times Y$ are defined as the product of open sets, i.e., $\{U \times V \mid U \text{ open in } X \text{ and } V \text{ in } Y\}$. But life isn't fair, and this is false.



Imagine the red rectangle being some open set $U \times V$ and the blue rectangle being a different open set $W \times Z$. There are no open sets $A \in X$ and $B \in Y$ where $A \times B = U \times V \cup W \times Z$.

A collection B of subsets of a set X is the basis for some topology on X if and only if whenever $x \in B_1 \cap B_2$, there exists $x \in B_3 \subseteq B_1 \cap B_2$. This allows us to say that open sets are unions of elements of B . We make the following claim:

Theorem 3.16. $\beta = \{U \times V \mid U \text{ is open in } X \wedge V \text{ is open in } Y\}$ forms a basis for a topology.

Proof. Say $B_1 = U_1 \times V_1$, $B_2 = U_2 \times V_2$. Then we can take $B_3 = (U_1 \cap U_2) \times (V_1 \cap V_2)$ □

2.7.2025

Theorem 3.17 (Space Filling Curve Continued). Now we can prove that the space filling is closed, which if you recall before means that the curve is space filling in Δ .

Proof. We know that $f : [0, 1] \rightarrow \Delta$ is continuous and $\text{Im}(f)$ is dense in Δ . $[0, 1]$ is compact, so $\text{Im}(f)$ is compact in Δ . Since Δ is Hausdorff, (since it is a closed subset of \mathbb{R}^2) $\text{Im}(f)$ is closed, so $\text{Im}(f) = \overline{\text{Im}(f)} = \Delta$ □

Recall from last week that we said that $\beta = \{U \times V \mid U \text{ is open in } X \wedge V \text{ is open in } Y\}$ forms a basis for a topology.

Example 3.18. Let $X = Y = \mathbb{R}$. $X \times Y = \mathbb{R}^2 = \mathbb{E}^2$ has the euclidean topology.

Definition 3.19 (Projections). **Projections** $P_1 : X \times Y \rightarrow X$ and $P_2 : X \times Y \rightarrow Y$ are functions where $P_1(x, y) = x$ and $P_2(x, y) = y$

Proposition 3.20. P_1 and P_2 are continuous, and send open sets to open sets.

Proof. Without loss of generality, consider P_1 . Say $U \subseteq X$ is open. Then $P_1^{-1}(U) = U \times Y \in \beta$, which is open. If $U \times V \in \beta$, $P_1(U \times V) = U$. So $P_1(\bigcup_{\alpha} (U_{\alpha} \times V_{\alpha})) = \bigcup_{\alpha} P_1(U_{\alpha} \times V_{\alpha}) = \bigcup_{\alpha} U_{\alpha}$, which is open in X . □

So projections are canonical maps from the product topology. We will now explore the conditions these maps and the product topology preserve.

Theorem 3.21. A function $f : Z \rightarrow X \times Y$ is continuous if and only if $P_1 \circ f$ and $P_2 \circ f$ are continuous.

Proof. The forwards direction is clear because it is a composition of continuous functions. The backwards direction is less so. Suppose $P_1 \circ f$ and $P_2 \circ f$ are continuous. We want to show that $f^{-1}(U \times V)$ is open in Z for every

open set $U \times V$ in the basis. Then

$$\begin{aligned} f^{-1}(U \times V) &= f^{-1}(P_1^{-1}(U) \cap P_2^{-1}(V)) \\ &= f^{-1}(P_1^{-1}(U)) \cap f^{-1}(P_2^{-1}(V)) \\ &= (P_1 \circ f)^{-1}(U) \cap (P_2 \circ f)(V) \end{aligned}$$

which is open in Z , as desired. \square

Theorem 3.22. $X \times Y$ is Hausdorff if and only if X and Y are Hausdorff.

Proof. For the forwards direction, Suppose $X \times Y$ is Hausdorff. Without loss of generality we will show that this means X is Hausdorff. Look at (x_1, y_1) and (x_2, y_2) . We can separate them by basis elements $U_1 \times V_1$ and $U_2 \times V_2$. U_1 and U_2 separate x_1 and x_2 . For the backwards direction, suppose X and Y are Hausdorff. Choose $(x_1, y_1) \neq (x_2, y_2) \in X \times Y$. Without loss of generality, suppose $x_1 \neq x_2$. Choose open sets U_1, U_2 in X such that $U_1 \cap U_2 = \emptyset$ and $x_1 \in U_1$, and $x_2 \in U_2$. Then $U_1 \times Y$ and $U_2 \times Y$ separate (x_1, y_1) and (x_2, y_2) . \square

Lemma 3.23. Say X is a topological space and β is a basis. Then X is compact if and only if every open cover at X by basis elements has a finite subcover.

Proof. The forwards direction holds by definition of compactness. To prove the backwards direction, suppose \mathcal{F} is an open cover of X . Each element $\mathcal{O} \in \mathcal{F}$ is a union of basis elements. Let \mathcal{F}' be that collection of basis elements. \mathcal{F}' is an open cover of X . By assumption, there is a finite subcover \mathcal{F}'' . We will now use this to show that every cover has a finite subcover. For every $B \in \mathcal{F}''$, choose $\mathcal{O}_B \in \mathcal{F}$ so $B \subset \mathcal{O}$. Then $\{\mathcal{O}_B\}_{B \in \mathcal{F}''}$ is a finite subcover if \mathcal{F} , which means it is compact. \square

Theorem 3.24. $X \times Y$ is compact if and only if X and Y are compact.

Proof. In the forwards direction, suppose $X \times Y$ is compact. Without loss of generality, $X = P_1(X \times Y)$ is compact. In the other direction, suppose X, Y are both compact. Say \mathcal{F} is an open cover of $X \times Y$ by basis elements. We will show that \mathcal{F} has a finite subcover, and using lemma 3.23 we will have that $X \times Y$ is compact.

Given $x \in X$, consider $P_1^{-1}(x) = \{x\} \times Y$. This is homeomorphic to Y , so Y is a compact subset of $X \times Y$. We can find a finite subset $\{U_{x,1} \times V_{x,1}, \dots, U_{x,n_x} \times V_{x,n_x}\}$ of \mathcal{F} that covers $\{x\} \times Y$. Their union contains $U_x \times Y$, where $U_x = \bigcap_{i=1}^{n_x} U_{x,i}$. We can do this for all $x \in X$.

To visualize what we have done so far, let $X \subset \mathbb{R} \supset Y$ be compact, and imagine $\{x\} \times Y$ as a vertical line passing through $\{x\}$ that is Y . The finite subset of \mathcal{F} imagine as a bunch of rectangles that cover this rectangle. Then the intersection of their X -components is just a thin strip that covers $\{x\}$.

Now we want to show that we can do this finitely many times to cover $X \times Y$. We have an open cover $\{U_x\}_{x \in X}$ of X . This has a finite subcover $\{U_{x_1}, \dots, U_{x_s}\}$. Since the union of this set is X , we have

$$X \times Y = \bigcup_{j=1}^s (U_{x_j} \times Y) \subseteq \bigcup_{j=1}^s \bigcup_{i=1}^{n_{x_j}} (U_{x_j,i} \times V_{x_j,i})$$

which is a finite subcover of \mathcal{F} , as desired. \square

Now we can say some things about spaces we are more familiar with.

Theorem 3.25. A subset of \mathbb{R}^n is compact if and only if it is closed and bounded.

Proof. The forwards direction we've already done. Suppose $C \subset \mathbb{R}^n$ is closed and bounded. We can find $S > 0$ so that $C \subset [-S, S] \times \dots \times [-S, S]$. We know that $[-S, S]$ is compact in \mathbb{R} , so $[-S, S] \times \dots \times [-S, S]$ is compact in \mathbb{R}^n . C is closed, so it is compact. \square

Definition 3.26. Let X be a metric space, and $C \subset X$. Put the diameter $\text{diam}(C) = \sup_{x,y \in C} d(x, y)$. We say C is **bounded** if $\text{diam}(C) < \infty$

Theorem 3.27. A compact subset of a metric space is closed and bounded.

This holds the same way as \mathbb{R}^n . Note that in general, the converse is not true for metric spaces.

Example 3.28. Let $X = \mathbb{R}$. Define the distance metric

$$d(x, y) = \frac{|x - y|}{1 + |x - y|}$$

This gives the usual topology. Take $X = \mathbb{R}$. C is closed, the diameter is 1, but C is not compact.

2.10.2025

3.4 Connectedness

Definition 3.29. A topological space X is **connected** if for all $X = A \cup B$ such that $A \neq \emptyset, B \neq \emptyset$, we have $\overline{A} \cap B \neq \emptyset$ or $A \cap \overline{B} \neq \emptyset$.

Example 3.30. Imagine $A = (a, b]$ and a different set $B = (b, c)$, where $a < b < c$. Then the topological space $A \cup B$ with the subspace topology is connected.

Nonexample 3.31. But consider $X = [0, 1] \cup [2, 3]$ with the subspace topology. This is disconnected.

In fact, \mathbb{R} is connected, and a subspace with the subspace topology is connected if and only if it's an interval.

Theorem 3.32. The following are equivalent

1. X is connected
2. The only subsets of X that are open and closed (**clopen**) are X and \emptyset
3. X is not the union of 2 nonempty disjoint open subsets.
4. There is no surjective map from X to a discrete space with multiple points.

Proof. 1 \implies 2 Suppose X is connected, and suppose $A \subset X$ is clopen. Then $X \setminus A$ is clopen. $\overline{A} = A$ and $\overline{B} = B$, so $\overline{A} \cap B = A \cap B = \emptyset$ and $A \cap \overline{B} = A \cap B = \emptyset$, which contradicts our definition of connected.

2 \implies 3 Suppose our clopen subsets are \emptyset and X . Suppose $X = U \cup V$ with U, V disjoint open sets. Since $U = X \setminus V$, U is closed, which means by our assumption U or V is empty.

3 \implies 4 Suppose (3) holds, and Y is a discrete space with $|Y| \geq 2$, and some map $f : X \rightarrow Y$. Then write $Y = U \cup V$, with U and V being 2 nonempty disjoint open subsets. Then $X = f^{-1}(U) \cup f^{-1}(V)$. Now we have that $f^{-1}(U)$ and $f^{-1}(V)$ are open, disjoint, and nonempty. This contradicts (3).

4 \implies 1 Suppose (4) holds and X is disconnected. We can write $X = A \cup B$ with A and B nonempty, $\overline{A} \cap B = \emptyset$ and $\overline{B} \cap A = \emptyset$. $B = X \setminus \overline{A}$ is open, as is A . Then define $f : X \rightarrow \{-1, 1\}$ by $f|_A = -1, f|_B = 1$. Then f is continuous and surjective, which is a contradiction to (4)

□

Remark 3.33. This is where the old adage "If a clopen set is can be detected, then your metric space is disconnected" comes from.

Theorem 3.34. If X is connected and f is a map, then $f(x)$ is connected in the subspace topology.

Proof. We can reduce to the case where f is surjective. Suppose we can write $Y = U \cup V$, where U and V are disjoint open sets. Look at $f^{-1}(V)$ and $f^{-1}(U)$, which are open in X and $X = f^{-1}(U) \cup f^{-1}(V)$. Since X is connected, $f^{-1}(U) = \emptyset$ or $f^{-1}(V) = \emptyset$. Then $U = \emptyset$ or $V = \emptyset$, hence Y is connected. \square

Example 3.35 (Intermediate Value Theorem). Consider a function $f : [a, b] \rightarrow \mathbb{R}$. Say y lies between $f(a)$ and $f(b)$. Then there exists some $x \in [a, b]$ such that $f(x) = y$.

Proof. $[a, b]$ is connected, so the image is connected. If $y \notin \text{Im}(f)$, then $f([a, b]) \cap (-\infty, y)$ and $f([a, b]) \cap (y, \infty)$ decomposes $f([a, b])$ into 2 distinct nonempty subsets. This is a contradiction, so $x \in [a, b]$ exists. \square

Definition 3.36. Two subsets are called **separated** if the intersection of their closure is empty.

Theorem 3.37. If X, Y are connected, then $X \times Y$ is connected.

The proof of this is given later.

2.12.2025

Theorem 3.38. Let X be a topological space and Z a dense subset. If Z is connected then so is X .

Example 3.39. Let $X = \{x \in \mathbb{R}^2 \mid |x| \leq 1\}$ and $Z = \{x \in \mathbb{R}^2 \mid |x| < 1\}$. Then X is the closure of Z . Suppose Z is connected and suppose A is a nonempty clopen subset of X . Since Z is dense in X , Z intersects every nonempty open set. In particular, $Z \cap A \neq \emptyset$. $A \cap Z$ is open in Z (under the subspace topology) and $A \cap Z$ is closed in Z . So $A \cap Z$ is a nonempty clopen subset of Z , so it must be Z , i.e., $A \cap Z = Z$, so $Z \subseteq A$. Then $x = \overline{Z} \subseteq \overline{A}$ and $\overline{A} = A$ (since it is clopen), so it is all of X , which means X is connected.

Corollary 3.40. If Z is a connected subset of X , then \overline{Z} is connected.

Proof. Z is dense in \overline{Z} \square

Example 3.41. $X = \mathbb{R}^2$, $z = \{x, \sin(\frac{\pi}{x}) \mid 0 < x \leq 1\}$. Z is connected since it is the image of $(0, 1]$ under a map. But $\overline{Z} = Z \cup \{(0, y) \mid -1 \leq y \leq 1\}$

Theorem 3.42. Say \mathcal{F} is a family of subsets of X , whose union is X . If each $Z \in \mathcal{F}$ is connected and no two $Z, Z' \in \mathcal{F}$ are separated, then X is also connected.

Proof. Suppose $A \subset X$ is clopen. We want to show that $A = \emptyset$ or $A = X$. If $Z \in \mathcal{F}$, then Z is connected and $A \cap Z = \emptyset$ or $A \cap Z = Z$

1. If $A \cap Z = \emptyset$ for all $Z \in \mathcal{F}$, then $A = \emptyset$, since the union of Z 's is X
2. Suppose $A \cap Z = Z$ for some $Z \in \mathcal{F}$, i.e., $Z \subseteq A$, then suppose $W \in \mathcal{F}$. If $W \cap A = \emptyset$, then Z and W are separated. This is impossible, so $W \cap A = W$, so $W \subseteq A$

Hence $\cup_{W \in \mathcal{F}} W \subseteq A$, so $A = X$ \square

Theorem 3.43. If X and Y are connected, then $X \times Y$ is also connected.

Proof. If $x \in X$, then $\{x\} \times Y$ is connected, and if $y \in Y$, then $X \times \{y\}$ is connected. Then

$$(\{x\} \times Y) \cap (X \times \{y\}) = \{(x, y)\}$$

so $(\{x\} \times Y)$ and $(X \times \{y\})$ are not separated, so

$$Z(x, y) := (\{x\} \times Y) \cup (X \times \{y\}) \text{ is connected}$$

Then $X \times Y = \bigcup_{(x,y) \in X \times Y} Z(x, y)$, and $(x, y') \in Z(x, y) \cap Z(x', y')$. So by theorem 3.42, $X \times Y$ is connected. \square

Definition 3.44 (Component). A **component** of X is a maximal connected subset

This means that any two components are separated from each other. A lot of this terminology is similar to that of graph theory.

Example 3.45. $X = \mathbb{Q}$ with the subspace topology from \mathbb{R} . The components are just singleton sets. (yet notice this isn't the discrete topology)

Theorem 3.46. Each component of a topological space is closed and distinct components are separated from one another.

Proof. Suppose C is a component. Then C is connected. By corollary 3.40, \overline{C} is connected. But C is maximal, so we must have $C = \overline{C}$. If D is some other component and D is not separated from C , then $\overline{C} \cap \overline{D} \neq \emptyset$. Then by theorem 3.42, $C \cup D$ is connected, which means C is not maximal, which is a contradiction. \square

We basically took a component, and said that any other component that isn't separated implies a contradiction

Proposition 3.47. Every connected subset of X is in some component.

Proof. Say $A \subseteq X$ is connected. Put C to be the union of all connected subsets containing A . By theorem 3.46, C is connected, so it is a component. \square

3.5 Path Connectedness

Recently in complex analysis, we began talking about continuous, piecewise-smooth curves. Curves are important in many fields of mathematics, and paths are just curves that are parameterized.

Definition 3.48. Path A **path** in X is a map $\gamma : [0, 1] \rightarrow X$ where $\gamma(0)$ is the **beginning** of γ and $\gamma(1)$ is the **end**.

Definition 3.49. A space is **path connected** if any two points can be joined by a path.

Theorem 3.50. Path-connected implies connected

Proof. Say $A \subset X$ is clopen, and (recalling theorem 3.32) towards contradiction say that $A \neq \emptyset$ and $A \neq X$. Choose $x \in A$ and $y \in X \setminus A$. Join them by a path $\gamma : [0, 1] \rightarrow X$. Then $0 \in \gamma^{-1}(A)$ while $1 \notin \gamma^{-1}(A)$, so $\gamma^{-1}(A)$ is neither the empty set nor the whole interval, which by the continuity of γ (a continuous map means the preimage of closed sets are closed, and the preimage of open sets are open) tells us that $\gamma^{-1}(A)$ is clopen in $[0, 1]$. This contradicts that the interval is connected. Then path-connected spaces must be connected. \square

Remark 3.51. The converse can fail. A space can be connected but not path connected

Theorem 3.52 (Paths Compose). Just reparameterize:

$$\gamma(t) = \begin{cases} \alpha(2t) & 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

assuming the compositability conditions hold.

Theorem 3.53. A connected open subset X of \mathbb{R}^n is path connected.

Proof. Choose $x \in X$. Put $U(x) = \{y \in X \mid y \text{ can be joined to } x \text{ by a path in } X\}$. $x \in U(x)$, so $U(x)$ is nonempty. We'll show that it's clopen.

$U(x)$ is open) Say $y \in U(x)$. Choose a ball $B(y)$ such that $y \in B \subseteq X$. If $z \in B$, we can join y to z by a straight line. So $z \in U(x)$. Hence $y \in B \subseteq U(x)$, so $U(x)$ is open.

$U(x)$ is closed) Say $y \in X \setminus U(x)$. Take a ball $B(y)$ such that $y \in B \subseteq X$. Choose $z \in B$. If $z \in U(x)$, we can join y to x , which is a contradiction. Then $z \notin U(x)$, i.e., $z \in X \setminus U(x)$. Hence $B \subseteq X \setminus U(x)$. Hence $X \setminus U(x)$ is open.

So $U(x)$ is clopen. Since X is connected, $U(x) = \emptyset$ or $U(x) = X$. $U(x)$ is nonempty, which means X is path connected. \square

We showed that there is an open ball in $U(x)$ for every point inside of $U(x)$, and an open ball outside of $U(x)$ for every point outside of $U(x)$. Then it's clopen so it's X

Definition 3.54. A **path component** is a maximal path connected subset.

Fact: If A is path connected and nonempty, then A sits inside (is a subset of) a unique path component (take the union of all path connected subsets containing A .)

Theorem 3.55. If $f : X \rightarrow Y$ is a map and X is path connected, then $f(X)$ is also path connected.

Proof. Choose $y_1, y_2 \in f(X)$. Then choose $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$. Join x_1 to x_2 by a path γ . Then $f \circ \gamma$ joins y_1 to y_2 by a path in $f(X)$. \square

Definition 3.56. If X is a topological space then $\pi_0(X)$ is the set of path components in X .

Given a map $f : X \rightarrow Y$, we have an induced map $f_* : \pi_0(X) \rightarrow \pi_0(Y)$. Given $C \in \pi_0(X)$, $f(C)$ is path connected in Y , so it sits inside a unique path component D of Y . Let $f_*(C) = D$.

Theorem 3.57. If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are maps then $(g \circ f)_* = g_* \circ f_*$ as functions from $\pi_0(X)$ to $\pi_0(Y)$.

Proof. Given $C \in \pi_0(X)$, choose $x \in C$. Then $f_*(C)$ is the path component in Y containing $f(x)$. So $g_*(f_*(C))$ is a path component of z containing $g(f(x)) = g \circ f(x)$, so $(g \circ f)_*(C) = g_*(f_*(C))$. \square

Corollary 3.58. If $f : X \rightarrow Y$ is a homeomorphism, then $f_* : \pi_0(X) \rightarrow \pi_0(Y)$ is a bijection.

Proof. Let $g : Y \rightarrow X$ such that $g \circ f = \text{Id}_X$ and $f \circ g = \text{Id}_Y$. Then

$$g_* \circ f_* = \text{Id}_{\pi_0(X)}$$

and

$$f_* \circ g_* = \text{Id}_{\pi_0(Y)}$$

Hence f_* is a bijection. \square

This is starting to look like some real topology!! Two spaces are the same if they have the same amount of pieces? It seems pretty "coffee-cup to donut"-y.

Theorem 3.59. \mathbb{R} is not homeomorphic to \mathbb{R}^n for $n > 1$.

Proof. Suppose $F : \mathbb{R} \rightarrow \mathbb{R}^n$ is homeomorphic, and say $f = F_{\mathbb{R} \setminus \{0\}}$. Then f is a homeomorphism from $\mathbb{R} \setminus \{0\}$ to $\mathbb{R}^n \setminus \{F(0)\}$. But $\pi_0(\mathbb{R} \setminus \{0\})$ has two elements, whereas $\mathbb{R}_n \setminus \{0\}$ has only 1 element. \square

π_0 and π_1 are known as homotopies.

If you recall the motivation for this class from the beginning of the semester, we said that the goal of Algebraic Topology is to assign algebraic objects to topological spaces. Here we have assigned the object of sets to topological spaces.

Corollary 3.60. $[0, 1]$ is not homeomorphic to Δ

This is true by the same argument.

Corollary 3.61. The space filling curve is not injective.

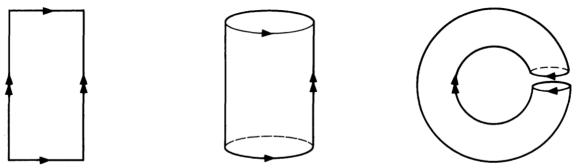
Proof. By Corollary 3.60, our space filling curve is a surjective map that is not homeomorphic. It maps from a compact space to a Hausdorff space, so it would be a homeomorphism if it were a bijective map. This means that the only place it can fail to be Homeomorphic is injectivity, so it is not injective. \square

4 Identification Spaces

4.1 The Identification Topology

Example 4.1 (Circle). Say $X = [0, 1]$. Identify $\{0\}$ and $\{1\}$. We are basically folding the endpoints onto each other to get a circle

Example 4.2 (Torus). $X = [0, 1] \times [0, 1]$. Identify the top edge with the bottom edge i.e., $(x, 0)$ with $(x, 1)$ and the side edges with each other i.e., $(0, y)$ with $(1, y)$.



This picture is taken from the Armstrong book.

2.19.2025

Definition 4.3. A **partition** \mathcal{P} of X is a family of disjoint nonempty subsets whose union is X .

We get a surjective function $\pi : X \rightarrow Y$ by saying $\pi(X)$ is the elements of \mathcal{P} containing X . Conversely, given a surjective map $\pi : X \rightarrow Y$, we get a partition \mathcal{P} on X whose elements are $\{\pi^{-1}(y)\}_{y \in Y}$. Now suppose we have a topological space X and a partition \mathcal{P} of X . We want to put a topology on Y such that π is continuous.

Definition 4.4. This map $\pi : X \rightarrow Y$ induces an **identification topology** aka **quotient topology** where an open subset \mathcal{O} of Y is open if and only if its preimage is open in X .

Proposition 4.5. This is a topology on Y

Proof. 1. $\pi^{-1}(\emptyset_Y) = \emptyset_X$ is open in X , so \emptyset_Y is open in Y

2. $\pi^{-1}(Y) = X$ is open, so Y is open.

3. If $\{\mathcal{O}_\alpha\}$ are open in Y then $\pi^{-1}(\bigcup_\alpha \mathcal{O}_\alpha) = \bigcup_\alpha (\pi^{-1}(\mathcal{O}_\alpha))$ is open in X . So $\bigcup_\alpha \mathcal{O}_\alpha$ is open in Y .

4. If $\{\mathcal{O}_i\}_{i=1}^n$ are open in Y , then $\pi^{-1}(\cap_i \mathcal{O}_i) = \cap_i \pi^{-1}(\mathcal{O}_i)$ is open in X , so $\cap_i \mathcal{O}_i$ is open in Y

□

So we have the following definition.

Definition 4.6. Let X be a topological space and \sim an equivalence relation induced by a partition of X . Let Y be the quotient set $X \setminus \sim$, the set of equivalence classes of X . An **identification space** aka **quotient space** is the set Y equipped with the identification (or quotient) topology.

Definition 4.7. Assume X, Y are topological spaces. A map $\pi : X \rightarrow Y$ is an identification map if $\mathcal{O} \subset Y$ is open if and only if $\pi^{-1}(\mathcal{O})$ is open in X .

Warning 4.8

If π is an identification map, it does not follow that f of an open set is open.

Example 4.9. Let $X = \{a, b, c, d\}$, and let $\mathcal{P} = \{\{a, b\}, \{c, d\}\}$. Let's say that the open sets of X are

$$\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c, d\}$$

Now call $Y = \{e, f\}$, where $a \mapsto e$, $c \mapsto e$, $b \mapsto f$, and $d \mapsto f$. Then we see that our open sets in Y are

$$\emptyset_Y, \{e, f\}$$

This is because $\pi^{-1}(e) = \{a, c\}$ which is not open in X , and the same follows for $\pi^{-1}(f)$. But $\pi(a) = e$, an open set doesn't necessarily map to an open set.

Theorem 4.10. Say $\pi : X \rightarrow Y$ is an identification map. Given a topological space Z , a function $f : Y \rightarrow Z$ is continuous if and only if $f \circ \pi : X \rightarrow Z$ is continuous

$$\begin{array}{ccc} X & & \\ \downarrow \pi & & \\ Y & \xrightarrow{f} & Z \end{array}$$

Proof. \implies Automatic

\Leftarrow Suppose $f \circ \pi$ is continuous. Then

$$(f \circ \pi)^{-1}(u)$$

is open in X . Then

$$\pi^{-1}(f^{-1}(u))$$

is open in X . Since π is an identification map, $f^{-1}(u)$ is open in Y . So f is continuous.

□

Theorem 4.11. Suppose $f : X \rightarrow Y$ is a surjective map that sends open sets to open sets or closed sets to closed sets. Then f is an identification map.

Proof. Suppose f sends open sets to open sets. We want to show that $\mathcal{O} \subset Y$ is open if and only if $f^{-1}(\mathcal{O})$ is open.

$\mathcal{O} \subset Y$ is open $\implies f^{-1}(\mathcal{O})$ is open, since f is continuous. Suppose $f^{-1}(\mathcal{O})$ is open. Then $f(f^{-1}(\mathcal{O}))$ is open, by assumption. The proof is similar if f sends closed sets to closed sets.

□

Theorem 4.12. Suppose $f : X \rightarrow Y$ is a surjective map. If X is compact and Y is Hausdorff then f is an identification map.

Proof. It's enough to show that if $C \subset X$ is closed, then $f(C) \subset Y$ is closed. Suppose C is closed. Then it is compact, so $f(C)$ is compact, so it is closed (since Y is Hausdorff).

□

Why did we only need to show that C is closed implies $f(C)$ is closed? It's because of Theorem 4.11.

Corollary 4.13. If $f : X \rightarrow Y$ is a surjective map, C is compact, and Y is Hausdorff, then Y is homeomorphic to the identification space induced by the partition \mathcal{P} of X into $\{f^{-1}(y)\}_{y \in Y}$.

This means we have an induced homeomorphism h such that

$$\begin{array}{ccc} X & & \\ \downarrow \pi & \searrow f & \\ Y^* & \dashrightarrow & Y \end{array}$$

the diagram commutes. Y^* denotes the identification space coming from the partition \mathcal{P}

Example 4.14. Consider Example 4.2 where we demonstrated an identification map from $[0, 1] \times [0, 1]$ to the torus T^2 .

$$\mathcal{P} : \begin{cases} \{4 \text{ vertices}\} \\ \{(x, 0), (x, 1)\}, 0 < x < 1 \\ \{(0, y), (1, y)\}, 0 < y < 1 \\ \{\text{Each interior point}\} \text{ (singletons)} \end{cases} \quad \text{By definition, } T^2 \text{ is the identification space}$$

Lets apply Corollary 4.13

Proposition 4.15. T^2 is homeomorphic to $S^1 \times S^1$

Proof. Consider the surjective map $f : [0, 1] \times [0, 1] \rightarrow S^1 \times S^1$ where

$$f(x, y) = (e^{2\pi i x}, e^{2\pi i y})$$

We can check that the preimages of f are exactly the elements of \mathcal{P} .

For example, look at $(1, 1) \in S^1 \times S^1$. Then $f^{-1}(1, 1) = ((0, 0), (1, 0), (0, 1), (1, 1))$. By Corollary 4.13, the identification space T^2 is homeomorphic to $S^1 \times S^1$. \square

2.21.2025

Since we're ahead of schedule, this class will be mainly examples and review, and next class will be an AMA

Example 4.16 (Cone). Let X be a topological space. Start with $X \times [0, 1]$. This space looks like a cylinder. Take a partition

$$\mathcal{P} = \begin{cases} \{(x, 1), x \in X\} \\ \text{Each } \{(x, t)\}, \text{ for } x \in X, 0 \leq t \leq 1 \end{cases}$$

This identification space CX is a **cone** over X

Example 4.17. Let $X = B^n$. Consider the partition

$$\mathcal{P} = \begin{cases} S^{n-1} = \{\vec{x} \in \mathbb{R}^n \mid |\vec{x}| = 1\} \\ \{\vec{x} \text{ if } |\vec{x}| < 1 \} \end{cases}$$

Call the identification space B^n/S^{n-1} , the quotient of a ball by its boundary. We claim that B^n/S^{n-1} is homeomorphic to S^n

Proof. Construct $f : B^n \rightarrow S^n$ whose preimages are \mathcal{P}

$$f(x_1, \dots, x_n) = (\pi \frac{\sin(r)}{r} x_1, \dots, \pi \frac{\sin(r)}{r} x_n, \cos(r))$$

where $r = \sqrt{x_1^2 + \dots + x_n^2}$. This maps points in the ball to points on S^n . \square

A **projective space** P^n of some topological space can be imagined as its shadow. We will give three examples of P^n as an identification space.

1. $X = S^n \subset \mathbb{R}^{n+1}$ Let the partition be

$$\mathcal{P} = \{\vec{x}, -\vec{x}\}, \vec{x} \in S$$

2. $X = \mathbb{R}^{n+1} \setminus \{\vec{0}\}$ Let the partition be

$$\mathcal{P} = \{\text{Line through } \vec{0}, \text{ minus } \vec{0}\}$$

3. Crosscap model. Let $X = B^n$, and $\mathcal{P} = \begin{cases} \{\vec{x}, -\vec{x}\}, & \text{If } |\vec{x}| = 1 \\ \{\vec{x}\} & \text{If } |\vec{x}| < 1 \end{cases}$

All three of these lead to the same space, as we can see below.

2 \implies 1 Given a line ℓ through $\vec{O} \in \mathbb{R}^{n+1}$, see where it intersects S^n

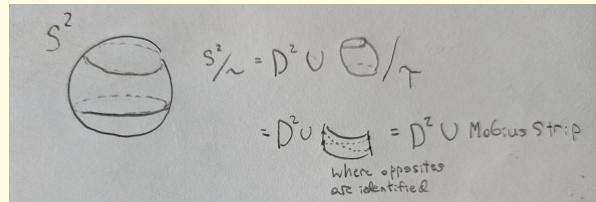
1 \implies 3 To see S^n / \sim , identify antipodal points. For most pairs of antipodal points, we can keep one in the northern hemisphere.

Another example of identification spaces are **attaching maps**. Let X, Y be topological spaces. Let $A \subset Y$, and $f : A \rightarrow X$ is a map. Our goal is to "glue" Y to X . Consider the partition

$$\mathcal{P} = \begin{cases} \{(a, f(a)) \text{ if } a \in A \\ \{x\} \text{ if } x \in X/f(A) \\ \{y\} \text{ if } y \in Y/A \end{cases}$$

The identification space is written $X \cup_f Y$, where f is called the attaching map.

Example 4.18. Let X be the mobius strip, and $Y = D^2$, a disc, with $A = S^1$. We claim that $X \cup_f Y = P^2$, because $P^2 = S^2 / \sim$, where x is identified with $-\vec{x}$.



The diagram depicts the sphere S^2 , where we identify antipodal points. Starting with the northern and southern hemispheres, we get a disc and the middle part of a sphere. Identifying the antipodal points of the rest of the sphere, we see that we get a mobius strip.

4.2 Topological Groups

Definition 4.19. A **group** is a set G with three operations

1. Multiplication $m : G^2 \rightarrow G$
2. Inverse $\iota : G^1 \rightarrow G$
3. Identity $e : G^0 \rightarrow G$

A topological group is a Hausdorff topological space with a group structure, such that m and ι are continuous.

2.26.2025

We actually forgot to talk about Gluings last week.

Definition 4.20 (Gluing of Maps). Say T is a topological space and X, Y are subspaces such that $T = X \cup Y$. Suppose we are given a topological space Z with maps $f : X \rightarrow Z$ and $g : Y \rightarrow Z$. Suppose $f(t) = g(t)$ for $t \in X \cap Y$. Define $F : T \rightarrow Z$ by

$$F(t) = \begin{cases} f(t), & t \in X \\ g(t), & t \in Y \end{cases}$$

Lemma 4.21 (Gluing Lemma). If X, Y are closed, then F is continuous.

Proof. Take $C \subseteq Z$ closed. $F^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$. f is continuous, so $f^{-1}(C)$ is closed in C , so $f^{-1}(c) = X \cap D$ for some D closed in T . X is closed, so $D \cap X = f^{-1}(c)$ is closed in T . And $g^{-1}(c)$ is closed in T by the same logic, so $F^{-1}(c) = f^{-1}(c) \cup g^{-1}(c)$ is closed in T . \square

Application:

Consider the two functions $\alpha : [0, 1] \rightarrow Z$ and $\beta : [0, 1] \rightarrow Z$, so $\alpha(t) = \beta(0)$. Put

$$\gamma(t) = \begin{cases} \alpha(2t), & 0 < t \leq \frac{1}{2} \\ \beta(2t - 1), & \frac{1}{2} \leq t < 1 \end{cases}$$

$T = [0, 1]$, $x = [0, \frac{1}{2}]$

Now back to Topological Groups. We defined them above.

Example 4.22. The Circle $G = S^1$ is one. The operations are as follows: $\begin{cases} m(e^{i\theta}, e^{i\phi}) = e^{i(\theta+\phi)} \\ \iota(e^{i\theta}) = e^{-i\theta} \end{cases}$

Example 4.23. Here are some more examples of groups:

$$G = (\mathbb{R}, +)$$

Any group with the discrete topology

$$T^2 = S^1 \times S^1$$

$$(\mathbb{R}^n, +)$$

Example 4.24. Unit quaternions: The quaternions are defined as follows $\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$, with the following properties:

$$i^2 = k^2 = j^2 = -1$$

$$ij = -ji = k$$

$$jk = -kj = i$$

$ki = -ik = j$ and the unit quaternions are the set $\{a + bi + cj + dk : a^2 + b^2 + c^2 + d^2 = 1\}$. Topologically, this is $\{(a, b, c, d) \in \mathbb{R}^4 | a^2 + b^2 + c^2 + d^2 = 1\} \cong S^3 \subset \mathbb{R}^4$. The group operations are

$$m : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H}$$

which restricts to $m : U \times U \rightarrow U$, and

$$\iota(a + bi + cj + dk) = -a - bi - cj - dk$$

$$\text{with } \iota : U \rightarrow U$$

Example 4.25. The general linear group $GL(n) = \{\text{Invertible matrices } n \times n \text{ over } \mathbb{R}\}$. Write M_n to denote the set of $n \times n$ matrices, and for a matrix $A \in M_n$, we write $A = (a_{ij})$. So for the topology we identify each $A = (a_{ij})$ with the corresponding point

$$(a_{11}, a_{12}, \dots, a_{21}, a_{22}, \dots, a_{nn-1}, a_{nn})$$

of \mathbb{E}^{n^2} .

Claim: The group operations $m : M_n \times M_n$ and $\iota : M_n \rightarrow M_n$ is continuous.

Proof. $(AB)_{ij} = \sum_{k=1}^n a_{ik}b_{kj}$ is a polynomial in a_{ij} s and b_{ij} s, respectively. $GL(n) \subset M_n$, so we give it the subspace topology. Then $m : GL(n) \times GL(n) \rightarrow GL(n)$ is continuous, since our polynomials are continuous and give us different values continuously when we change A or B .

The inverse is also continuous, see that $(A^{-1})_{ij} = \frac{1}{\det(A)}(j\text{th cofactor of } A)$. The point is that this is just a ratio of two polynomials in entries of A , which is continuous in the entries (note that invertible matrices never have determinant 0) \square

Definition 4.26. Topological groups are **isomorphic** (denoted $G_1 \cong G_2$) if there exists an isomorphism $\phi : G_1 \rightarrow G_2$ which is also a homeomorphism.

Definition 4.27. A **subgroup** of a topological group is a subgroup with the subspace topology.

Definition 4.28. Given $x \in G$, define left multiplication $L_x : G \rightarrow G$ by $L_x(g) = xg$, and right multiplication $R_x : G \rightarrow G$ by $R_x(g) = gx$.

Claim: L_x and R_x are homeomorphisms of G .

Proof. $L_{x^{-1}}(L_x(g)) = x^{-1}(xg) = g$ and $L_x(L_{x^{-1}}(g)) = x(x^{-1}g) = g$. Then $(L_x)^{-1} = L_{x^{-1}}$. We can write L_x as a map where $\phi(g) = (x, g)$ with $g \mapsto (x, g) \xrightarrow{m} xg$. But m is continuous by assumption, and ϕ is continuous by product topology, so L_x is continuous. Also, $(L_x)^{-1} = L_{x^{-1}}$ is continuous, so L_x is a homeomorphism. The same follows for R_x \square

Corollary 4.29. Given $g, g' \in G$, there exists a homeomorphism $h : G \rightarrow G$ such that $h(g) = g'$. We call G a **homogeneous space**.

Proof. $h = L_{g'}g^{-1}$. \square

For $g \in G$ we have $h(x) = g'g^{-1}g = g'$.

Theorem 4.30. Say G is a topological group and K is the component that contains the identity e . Then K is a closed normal subgroup of G .

Proof. Since K is a component, it is closed. To see K is a subgroup, we know that for any $x \in K$, $Kx^{-1} = R_{x^{-1}}(K)$. But this is continuous, so it's connected (since K is). We also know that $e = xx^{-1} \in Kx^{-1}$, so $Kx^{-1} \in K$, since K is a connected set which contains e . Since $e \in K$, we get $ex^{-1} = x^{-1} \in K$.

If y is in K , $yx' \in K$, so K is preserved by multiplication. Given $g \in G$, $L_g \circ R_{g^{-1}}$ is continuous. So $(L_g \circ R_{g^{-1}})(K)$ is connected. $(L_g \circ R_{g^{-1}})(e) = geg^{-1} = e$, so $e \in (L_g \circ R_{g^{-1}})(K)$, so it contains e and is connected, so $(L_g \circ R_{g^{-1}})(K) \subset K$, i.e.,

$$gkg^{-1} \subset K$$

hence K is a normal subgroup of G . \square

2.28.2025

Last week, we saw that if K is a connected component of e then K is a closed normal subgroup.

Corollary 4.31. $\{ \text{Components of } G \}$ gets a group structure as $G \setminus K$.

Theorem 4.32. Suppose G is connected. Then any open neighborhood V of e generates G .

Proof. Say $H = \langle v \rangle$, the group generated by V . i.e., $h \in H$ means we can write $h = v_1^{n_1} v_2^{n_2} \cdots v_k^{n_k}$ where $v_i \in V$, $n_i \in \mathbb{Z}$. $H \neq \emptyset$, since $e \in H$. Since G is connected, it works to show that H is clopen, since the only clopen sets in a connected space are G and the empty set.

We will show that H is open and closed in G . To see that H is open, take $h \in H$. Look at hV . We know that $hV \in H$, since V generated H . We also know that hV is open, since $hV = L_h(V)$, which is a homeomorphism, and the preimage is open.

Now we just need to show that the space $G \setminus H$ is open. Suppose $g \in G \setminus H$. Look at gV . We want to show that it's a neighborhood of g in $G \setminus H$. To see $gV \in G \setminus H$, note that if it were not, $gV \cap H \neq \emptyset$. Take $x \in gV \cap H$. $x = gv$ for some $v \in V$. Then $g = xv^{-1}$. By assumption $x \in H$, so $xv^{-1} \in H$. Hence $g \in H$, which is a contradiction. Therefore we have $gV \subseteq G \setminus H$. $gV = L_g(V)$ is open in G , so we have $g = gV = G \setminus H$. Then $G \setminus H$ is open, and thus H is clopen in G , as desired. \square

Nonexample 4.33. Let G be a nontrivial discrete group. Take $V = \{e\}$. Then $H = \{e\}$. Thus we need connectedness

Remark 4.34. Note that connectedness should not be thought of as closure of a group. We should instead think of the fact that G connected $\Leftrightarrow G \setminus K = \{e\}$.

Now we are going to look at more examples of topological groups.

Definition 4.35. The **orthogonal group** is defined as $O(n) = \{M \in M_n \mid MM^T = I\} \subseteq M_n \cong \mathbb{R}^{n^2}$. The **special orthogonal group** $SO(n) = \{M \in O(n) \mid \det M = 1\} \subseteq M_n \cong \mathbb{R}^{n^2}$

Theorem 4.36. $O(n)$ and $SO(n)$ are compact.

Proof. We want to show that they are both closed and bounded in \mathbb{R}^{n^2} . We have a map $\phi : M_n \rightarrow M_n$ with $\phi(M) = MM^T$, and $O(n) = \phi^{-1}(I_n)$. $\{I_n\}$ is closed in M_n , so $O(n)$ is closed in M_n . We know that $MM^T = I$ by construction. Then $\text{Tr}(MM^T) = n^3$, since

$$\text{Tr}(MM^T) = \sum_{i,j=1}^n m_{ij}^2$$

so $O(n) \subset \{M \in M_n \mid \sum m_{ij}^2 = n\}$, which means it's bounded in \mathbb{R}^{n^2} . $SO(n) = \det^{-1}(1)$, a closed subset of $O(n)$, so it too is compact. \square

Example 4.37. Let $n = 1$. $SO(1) = \{(1)\}$, and $O(1) = \{(1), (-1)\}$. These are the sets of matrices whose determinants are 1 or roots of 1, respectively

Example 4.38. Let $n = 2$. $SO(2) = \left\{ \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \right\}$ which is a circle. $O(2)$ is two disjoint circles.

Theorem 4.39. $SO(3)$ is homeomorphic to P^3 .

Proof. We will construct a map $\phi : S^3 \rightarrow SO(3)$.

$$S^3 = \text{Unit quaternions} = \{a + bi + cj + dk \mid a^2 + b^2 + c^2 + d^2 = 1\}$$

$$\mathbb{R}^3 = \text{Pure quaternions} = \{bi + cj + dk \mid b, c, d \in \mathbb{R}\}$$

³The **trace** of a matrix is the sum of its diagonals. It corresponds to the sum of its eigenvalues, among other things.

Given $g \in S^3$, define $\phi(g) \in SO(3)$ by $\phi(g)\vec{v} = q\vec{v}q^{-1} \in \mathbb{R}^3$. We want to show that ϕ is a homomorphism, continuous, and surjective.

We have $\ker(\phi) = \{\pm 1\}$. Then we can mod the domain by the kernel to get an isomorphism with the image⁴: $S^3/\{\pm 1\} \cong SO(3)$, i.e., quotienting/identifying $q \sim -q$ with respect to the equivalence relation that identifies antipodal points.

Then we get a 1-1, continuous function $\Phi : S^3/\{\pm 1\} \rightarrow SO(3)$, so we have a homeomorphism. Note that $S^3/\{\pm 1\} \cong P^3$ are homeomorphic. \square

4.3 Group Actions

Definition 4.40 (Group Action). Let G be a group and X a set. An **action** of G on X is a function

$$G \times X \rightarrow X$$

$$(g, x) = gx$$

such that $(g_1 g_2)(x) = g_1(g_2 x)$, and $ex = x$. We get a homomorphism $\alpha : G \rightarrow \text{Perm}(X)$ with $\alpha(g) = gx$

This was originally how groups were thought of; not as a set with structure, but rather actions on a set.

Example 4.41. $O(n)$ acts on \mathbb{R}^n

$$(A, \vec{v}) \rightarrow A\vec{v}$$

with A an orthogonal matrix, acting on \vec{v} by left matrix multiplication

Definition 4.42 (Orbit). An **orbit** of $x \in X$ is

$$O(x) = \{gx | g \in G\} \subset X$$

We get a partition of X by orbits, and the orbit space X/G is the set of orbits.

We can think of this as the set of stuff you can get to with a group action.

Example 4.43. \mathbb{Z} (the infinite group of integers under addition) acts on \mathbb{R} with $(n, x) \rightarrow x + n$.

$O(\frac{1}{3})$ is the set of numbers $\{x \in \mathbb{R} | x = \frac{1}{3} + k, k \in \mathbb{Z}\}$.

\mathbb{R}/\mathbb{Z} is bijective to S^1 .

3.3.2025

Example 4.44. $O(n)$ acts on S^{n-1} by $(M, \vec{x}) = M\vec{x}$. We claim this is a transitive action, i.e., there exists only 1 orbit

Proof. Say $\{\vec{e}_1, \dots, \vec{e}_n\}$ is the standard basis of \mathbb{R}^n . Given $\vec{x} \in S^{n-1}$, we can construct a new orthonormal basis with \vec{x} as its first member:

$$(\vec{x}, \vec{f}_2, \dots, \vec{f}_n)$$

Define $M \in O(n)$ by $M(\vec{e}_1 = \vec{x})$, and $M(\vec{e}_i = \vec{f}_i)$ if $i > 1$. Then $M = (\vec{x} | \vec{f}_2 | \dots | \vec{f}_n|)$ is the matrix of the new basis with respect to the standard basis. We see that orbit of e_1 is S^{n-1} , since $M\vec{e}_1 = \vec{x}$, and we can choose any \vec{x} and still construct an matrix in $O(n)$. \square

Definition 4.45. A topological group G acts on the topological space X if $G \times X \rightarrow X$ is continuous. We get a homomorphism $\phi : G \rightarrow \text{Homeo}(X)$, where $\text{Homeo}(X)$ is group of homeomorphisms of X

Essentially, each element of G induces a homeomorphism of the space such that

⁴This is known as the **First Isomorphism Theorem**

1. We have associativity: $hg(x) = h(g(x))$
2. The identity $e(x) = x$
3. And continuity: $G \times X \rightarrow X$ is continuous

Example 4.46. \mathbb{Z}^2 acts on \mathbb{R}^2 .

$$(n_1, n_2) \cdot (x_1, x_2) = (x_1 + n_1, x_2 + n_2)$$

$$\mathbb{R}^2 / \mathbb{Z}^2 \cong [0, 1] \times [0, 1] / \sim = T^2$$

where the equivalence relation identifies corners, opposing edges, and we get a torus.

Example 4.47. $G = C_2$, $X = S^n$. The elements of C_2 are $G = \{e, \tau\}$, with τ an involution. We see that $1 \cdot \vec{x} = \vec{x}$ and $\tau \cdot \vec{x} = \pm \vec{x}$. Then we have

$$S^n / C_2 = \{ \text{pairs of antipodal points} = P^n \}$$

Example 4.48. G is a topological group, H a subgroup. H acts on G by left multiplication:

$$(h, g) = hg$$

The orbit space is g, g' in some orbit if and only if $g' = hg$ for some $h \in H$, aka $g' \in Hg$. So we notice that the orbits of $g \in G$ are right cosets of H in G , so Orbit space is a set of right cosets $\{Hg\}$. Somewhat similarly, H acts on G by right multiplication

$$(h, g) \rightarrow gh^{-1}$$

The reason it is gh^{-1} should make sense if you consider the action laws.

If G acts on X , we get an identification space X/G coming from a partition of X into the orbits, and we get the identification topology X/G

Theorem 4.49. The identification map $\pi : X \rightarrow X/G$ is an open map, i.e., it sends open sets to open sets.

Proof. Say \mathcal{O} is open. We want to show that $\pi(\mathcal{O})$ is open. Say $x \in \pi^{-1}(\pi(\mathcal{O}))$. Then $\pi(x) \in \pi(\mathcal{O})$. Hence $\pi(x) = \pi(y)$ for some $y \in \mathcal{O}$, i.e., x, y are in the same orbit, therefore there exists some $g \in G$ such that $x = gy$. This means that

$$\pi^{-1}(\pi(\mathcal{O})) = \bigcup_{g \in G} g\mathcal{O}$$

where the union is the union of orbits of points in the original open set. We know that L_g is a homeomorphism, so $g\mathcal{O}$ is open in X , hence

$$\bigcup_{g \in G} g\mathcal{O}$$

is open in X , since the union of open sets is open. □

Theorem 4.50. If G acts on X , and G and X/G are connected, then X is connected.

Proof. Suppose $X = U \cup V$, with $U \cap V = \emptyset$ and U, V are nonempty open sets. We want a contradiction. $\pi(U)$ and $\pi(V)$ are open in X/G , and nonempty. Since the quotient space is connected, $\pi(U) \cap \pi(V) \neq \emptyset$. Say $\pi(x) \in \pi(U) \cap \pi(V)$. Then the orbit Gx intersects U and V . But G is connected, so its orbit is as well. Then $(Gx \cap U) \cup (Gx \cap V) = Gx$ would disconnect Gx (since U and V are disjoint), which is a contradiction, since the orbit Gx is the image of a continuous function $f : G \times X \rightarrow X$, which means it is connected, and we just said it wasn't. □

Example 4.51. $O(n)$ acts on S^{n-1} (transitively). Look at the **stabilizer** of \vec{e}_1 , $\{g \in O(n) | g\vec{e}_1 = \vec{e}_1\}$. If A is in the stabilizer then A has the form:

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & O(n-1) \end{array} \right)$$

where $O(n-1)$ is special because it is orthogonal, which means it fixes $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ (aka e_1). So we get a map

$O(n)/O(n-1) \rightarrow S^{n-1}$ ^a with $[g] \mapsto g\vec{e}_1$, where $[g]$ is the coset $gO(n-1)$. This is a homeomorphism (because it is compact, Hausdorff, and a bijection, which we will not prove). Similarly $SO(n)/SO(n-1)$ is homeomorphic to S^{n-1}

^a $O(n)/O(n-1)$ is the space of cosets.

Proposition 4.52. $SO(n)$ is connected.

Proof. By induction.

- $n=1$: $SO(1) = \{(1)\}$, which is connected.
- Suppose $SO(n-1)$ is connected. Then $SO(n)/SO(n-1) \cong S^{n-1}$ ⁵. If $n > 1$, then S^{n-1} is connected. By example 4.51, $SO(n)$ is connected.

□

Note that $O(n)$ is not connected, since it has two components $O(1) = \{(1), (-1)\}$

3.5.2025

Example 4.53. Let p, q be relatively prime integers $0 < p < q$, let $G = C_p$, and let $X = S^3$, which is the set of 4-tuples with magnitude 1 in \mathbb{R}^4 or equivalently (and what we will use in this scenario), the set of ordered pairs with magnitude 1 in \mathbb{C}^2 . Let g be a generator of C_p , and let the action of this group on S^3 be

$$g(z_0, z_1) = (e^{\frac{2\pi i}{p}} z_0, e^{\frac{2\pi i}{p}} z_1)$$

Since we know what g does on elements of S^3 , we see that when g acts p times, we get back to (z_0, z_1) . The quotient space S^3/C_p is called the **lens space** $L(p, q)$

Example 4.54. Isometries (i.e., distance preserving motions) of \mathbb{R}^3 are

$$\{ \vec{x} \rightarrow A\vec{x} + \vec{b} \mid A \in O(2), b \in \mathbb{R}^2 \}$$

with A a rotation and \vec{b} a translation^a. We can bestow isometries with group structure, with group left multiplication given by

$$\vec{x} \rightarrow A_2 \vec{x} + \vec{b}_2 \rightarrow A_1(A_2 \vec{x} + \vec{b}_2) + \vec{b}_1$$

Written as ordered pairs, we have $(\vec{b}_1, A_1) \cdot (A_2, \vec{b}_2) = (A_1 A_2, \vec{b}_1 + A_2 \vec{b}_2)$. We can give this group of isometries the topology of the product space $O(2) \times \mathbb{R}^2$, called the **Euclidean group** $E(2)$. Note that even though $E(2)$ has the product topology, the group structure is not the product structure, but rather the **semidirect product**: Given groups G_1, G_2 , and a homomorphism $p : G_2 \rightarrow \text{Aut}(G_1)$, multiplication in $G_1 \tilde{\times}_p G_2$ is $(g_1, g_2)(g'_1, g'_2) = (g_1 p(g'_1), g_2 g'_2)$.

In our case, $G_1 = (\mathbb{R}^2, +)$ and $G_2 = O(2)$. Then $\text{Aut}(G_1) = GL_2(\mathbb{R})$, and $p = O(2) \hookrightarrow GL_2(\mathbb{R})$. So

^aThey are homeomorphic. And the n -sphere is connected

$\text{Isom}(\mathbb{R}^2) = \mathbb{R}^2 \tilde{\times}_p O(2)$. We can choose the standard topology on \mathbb{R} so that this is homeomorphic to $\mathbb{R}^2 \times S^1$, so $\text{Isom}(\mathbb{R}^2)$ has two connected components.

^aThe textbook denotes isometries as ordered pairs (θ, v) , with $\theta \in O(2)$ and $v \in \mathbb{E}^2$

This leads into the idea of **Wallpaper Groups**:

Say G is a discrete subgroup of $\text{Isom}(\mathbb{R}^2)$. Then G acts on \mathbb{R}^2 by isometries

Definition 4.55. G is a **planar crystallographic group** if the orbit space \mathbb{R}^2/G is compact.

Example 4.56. Say $\{\vec{b}_1, \vec{b}_2\}$ is a basis of \mathbb{R}^2 . $G = \{\vec{x} \rightarrow \vec{x} + n_1 \vec{b}_1 + n_2 \vec{b}_2\}$ with $n_1, n_2 \in \mathbb{R}$. Then \mathbb{R}/G is topologically T^2 .

There are 17 different isomorphism classes of planar crystallographic groups, and a finite number for any specific higher dimension. The planar crystallographic groups are called wallpaper groups because they classify the set of symmetries of a two-dimensional repetitive pattern, think a tiling of \mathbb{R}^2 .

5 The Fundamental Group

This is when the algebraic topology really begins.

5.1 Homotopic Maps

In my complex analysis class we already talked a lot about what a homotopy is.

Definition 5.1. A **loop** in a topological space X is a map

$$\alpha : [0, 1] \rightarrow X$$

such that $\alpha(0) = \alpha(1)$. We say that the loop is **based at** $\alpha(0)$

Idea: Form a group whose multiplication is loop concatenation, and where the inverse of a loop is going around the loop backwards.

Definition 5.2 (loop concatenation). If $\alpha, \beta : [0, 1] \rightarrow X$ are loops based at p , but $\gamma : [0, 1] \rightarrow X$ to be

$$\gamma(t) = \begin{cases} \alpha(2t) : 0 \leq t \leq \frac{1}{2} \\ \beta(2t - 1) : \frac{1}{2} \leq t \leq 1 \end{cases}$$

We have a few issues with this idea:

1. Too many loops
2. Multiplication is not associative. This is because $((\alpha\beta)\gamma)(\frac{2}{3}) = \gamma(\frac{1}{3})$, but $(\alpha(\beta\gamma))(\frac{2}{3}) = \beta(\frac{2}{3})$

We want some idea of a continuous deformation so that we can solve problem number one above. Say $f, g : X \rightarrow Y$. We want some family of maps

$$\{F_t\}_{t \in [0, 1]}$$

such that $F_0 = f$, $F_1 = g$. To make this rigorous, look at the product space $X \times I$. Given a map $F : X \times I \rightarrow Y$, we get a family of maps $\{f_t\}_{t \in [0, 1]}$ with $f_t : X \rightarrow Y$ by

$$F_t(x) = F(x, t)$$

Definition 5.3. Suppose $f, g : X \rightarrow Y$ are maps. We'll say f is **homotopic** to g if we can find $F : X \times I \rightarrow Y$ a map such that

$$F(x, 0) = f, F(x, 1) = g \quad \forall x \in X$$

We write $f \underset{F}{\sim} g$

Variation: If $A \subset X$ and $H_A = g|_A$, we may want to preserve this in the homotopy, i.e., $F(a, t) = f(a)$ for all $t \in [0, 1]$. Then we say f and g are homotopic relative to A , and write $f \underset{F}{\sim} g$ (rel A)

Example 5.4. Say $\alpha, \beta : I \rightarrow X$ are two loops based at p . We can ask if α, β are homotopic relative to $A = \{0, 1\}$. If so, we get $F : I \times I \rightarrow X$ with $F(s, 0) = \alpha(s), F(s, 1) = \beta(s)$, and $F(0, t) = p = F(1, t)$ for all $t \in [0, 1]$. So our homotopy is a family of curves centered at p .

Suppose $X \subset \mathbb{R}^n$ is **convex**, i.e., (any two points in C can be joined by a straight line in C)⁶. Given $\vec{f}, \vec{g} : X \rightarrow C$, put

$$F(x, t) = t\vec{g}(x) + (1 - t)\vec{f}(x)$$

a linear homotopy. Then $f \underset{F}{\sim} g$

Example 5.5. Say $f, g : X \rightarrow S^n$. Suppose $\vec{f}(x) + \vec{g}(x) \neq 0$ for all $x \in X$. We can homotope from f to g in \mathbb{R}^{n+1} , avoiding 0 ,

$$(1 - t)\vec{f}(x) + t\vec{g}(x)$$

And our homotopy is

$$F(x, t) = \frac{(1 - t)\vec{f}(x) + t\vec{g}(x)}{|(1 - t)\vec{f}(x) + t\vec{g}(x)|}$$

Then $f \underset{F}{\sim} g$ as maps from X to S^n .

3.7.2025

Today we will cover the fundamental group

Lemma 5.6. Homotopy is an equivalence relation on the set of maps from X to Y

Proof. 1. $f \underset{F}{\sim} f$ by $F(x, t) = f(x)$

2. $f \underset{F}{\sim} g$ and $g \underset{F}{\sim} f$ if $G(x, t) = F(x, 1 - t)$.

3. Suppose $f \underset{F}{\sim} g$ and $g \underset{G}{\sim} h$. Then put

$$H(x, t) = \begin{cases} F(x, 2t), & \text{If } 0 \leq t \leq \frac{1}{2} \\ G(x, 2t - 1), & \text{If } \frac{1}{2} \leq t \leq 1 \end{cases}$$

so $f \underset{F}{\sim} h$

□

Lemma 5.7. Homotopy (rel A) is an equivalence relation on maps $X \rightarrow Y$ with given restriction to $A \subset X$

Consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y & \xrightarrow{\quad h \quad} & Z \\ & \underbrace{\hspace{1cm}}_{g} & & & \end{array}$$

with X, Y, Z topological spaces and f, g, h maps. We ask if the diagram commutes.

Below are some definitions from my category theory class last semester:

⁶This is the definition of convex

Definition 5.8 (monomorphism). In a category \mathcal{C} , a morphism $\alpha : a \rightarrow b$ is a **monomorphism** if and only if $\alpha f = \alpha g \Rightarrow f = g$ for all f, g .

$$\begin{array}{ccccc} & & g & & \\ & x & \swarrow \curvearrowright & a & \xrightarrow{\alpha} b \\ & & f & & \end{array}$$

Definition 5.9 (epimorphism). α is an **epimorphism** if and only if $f\alpha = g\alpha \Rightarrow f = g$

$$\begin{array}{ccccc} & & g & & \\ a & \xrightarrow{\alpha} & b & \swarrow \curvearrowright & z \\ & & f & & \end{array}$$

Back to the topic at hand; the question we asked is actually not asking if homotopies are monomorphisms and epimorphisms⁷, but it's interesting that algebraic topology and category theory are so connected.

Lemma 5.10. If $f, g : X \rightarrow Y$ are homotopic and $h : Y \rightarrow Z$ is a map, then $h \circ f$ is homotopic to $h \circ g$.

Proof. Say $f \underset{F}{\simeq} g$. Then $h \circ f \underset{F}{\simeq} h \circ g$. □

We have commutativity for

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \xrightarrow{\text{f}} Z \\ & & \swarrow \curvearrowright \\ & & Z \xrightarrow{g} \end{array}$$

as well, i.e., $f \underset{F}{\simeq} g \implies h \circ f \underset{F}{\simeq} h \circ g$.

Remark 5.11. If $g \simeq h$ (rel B), then $h \circ f \simeq h \circ g$ (rel $f^{-1}(B)$)

5.2 The Fundamental Group

We know that homotopy is an equivalence relation of loops based at p . The equivalence classes are called **homotopy classes** of loops. For any loop α , we will let $\langle \alpha \rangle$ denote its homotopy class.

Example 5.12. Let the topological space $X = S^1$ and consider point $p = 1 \in S^1$. Consider the two curves $\alpha : [0, 1] \rightarrow X$, $\alpha(s) = p$ for all s and $\beta : [0, 1] \rightarrow X$ with $\beta(s) = e^{2\pi i s}$. α, β are not homotopic, and so are in two different equivalence classes.

Definition 5.13 (Multiplication). $\langle \alpha \rangle \cdot \langle \beta \rangle = \langle \alpha \cdot \beta \rangle$, where

$$\langle \alpha \cdot \beta \rangle(s) = \begin{cases} \alpha(2s), & 0 \leq s \leq \frac{1}{2} \\ \beta(2s - 1), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

Lemma 5.14. This multiplication is well-defined, i.e., for $\alpha, \alpha' \in \langle \alpha \rangle$, $\beta, \beta' \in \langle \beta \rangle$, $\langle \alpha \rangle \cdot \langle \beta \rangle = \langle \alpha' \rangle \cdot \langle \beta' \rangle$

Proof. Let $\alpha \underset{F}{\simeq} \alpha'$ and $\beta \underset{G}{\simeq} \beta'$. We know $\alpha \cdot \beta \underset{H}{\simeq} \alpha' \cdot \beta'$ with

$$H(s, t) = \begin{cases} F(2s, t), & 0 \leq s \leq \frac{1}{2} \\ G(2s - 1, t), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

Hence $\langle \alpha \cdot \beta \rangle = \langle \alpha' \cdot \beta' \rangle$ □

Theorem 5.15. These homotopy classes of loops form a group under multiplication.

⁷They would be monomorphisms if $f \underset{F}{\simeq} g \Leftrightarrow h \circ f \underset{F}{\simeq} h \circ g$, whereas we only care that $f \underset{F}{\simeq} g \implies h \circ f \underset{F}{\simeq} h \circ g$.

Proof. 1. Associativity: We want to show that $\langle \alpha \rangle \cdot \langle \beta \rangle \cdot \langle \gamma \rangle = \langle \alpha \rangle \cdot (\langle \beta \rangle \cdot \langle \gamma \rangle)$. This is equivalent to showing $\langle \alpha \cdot \beta \rangle \cdot \langle \gamma \rangle = \langle \alpha \rangle \cdot \langle \beta \cdot \gamma \rangle$ which is

$$\langle (\alpha \cdot \beta) \cdot \gamma \rangle = \langle \alpha \cdot (\beta \cdot \gamma) \rangle$$

Claim: $((\alpha \cdot \beta) \cdot \gamma) = (\alpha \cdot (\beta \cdot \gamma)) \circ f$, with $f : I \rightarrow I$ and

$$f(s) = \begin{cases} 2s, & 0 \leq s \leq \frac{1}{4} \\ s + \frac{1}{4}, & \frac{1}{4} \leq s \leq \frac{1}{2} \\ \frac{s+1}{2}, & \frac{1}{2} \leq s \leq 1 \end{cases}$$

Since I is convex and $f(0) = 0$, $f(1) = 1$, there exists a homotopy of f to 1_I (rel $\{0, 1\}$)⁸. Then

$$\langle \alpha \cdot \beta \rangle \cdot \gamma = \langle \alpha \cdot (\beta \cdot \gamma) \rangle \circ f \simeq \langle \alpha \cdot (\beta \cdot \gamma) \rangle \circ 1_I = \langle \alpha \cdot (\beta \cdot \gamma) \rangle$$

2. The identity element is the constant loop $e(s) = p$, $0 \leq s \leq 1$. Then

$$\langle \alpha \rangle \cdot \langle e \rangle = \langle e \rangle \cdot \langle \alpha \rangle = \langle \alpha \rangle$$

3. The inverse of $\langle \alpha \rangle$ is $\langle \alpha^{-1} \rangle$, with $\alpha^{-1}(s) = \alpha(1-s)$. Then $\alpha \cdot \alpha^{-1} = \alpha \circ f$, where

$$f(s) = \begin{cases} 2s, & 0 \leq s \leq \frac{1}{2} \\ 2 - 2s, & \frac{1}{2} \leq s \leq 1 \end{cases}$$

$f : I \rightarrow I$ is convex, so we can homotope $f \rightarrow g$ (rel $\{0, 1\}$), where $g(s) = 0$ for all s . Then

$$\alpha \cdot \alpha^{-1} = \alpha \circ f \simeq \alpha \circ g = e(\text{rel}\{0, 1\})$$

□

Definition 5.16 (Fundamental Group). A group with elements $\langle \alpha \rangle$ is called the **Fundamental Group** of X based at p . It is denoted $\pi_1(X, p)$

3.10.2025

Remark 5.17. Note that $\pi_1(X, p)$ depends only on the path component containing p .

Theorem 5.18. If X is path connected then for all $p, q \in X$, $\pi_1(X, p)$ is isomorphic to $\pi_1(X, q)$.

Proof. Choose a path $\gamma : [0, 1] \rightarrow X$. Given a loop α at p , we get a loop $\gamma^{-1} \cdot \alpha \cdot \gamma$ based at q . Define a function $\pi_1(X, p) \rightarrow \pi_1(X, q)$ be

$$\langle \alpha \rangle \rightarrow \langle \gamma^{-1} \cdot \alpha \cdot \gamma \rangle$$

- This is well defined: A homotopy of α gives rise to a homotopy of $\gamma^{-1} \alpha \gamma$, i.e., a homotopy H from α to α' induces a new one by keeping γ, γ^{-1} but homotoping $\alpha \rightarrow \alpha'$.
- It's a homomorphism because $\langle \alpha \rangle \cdot \langle \beta \rangle = \langle \alpha \cdot \beta \rangle$ maps to

$$\langle \gamma^{-1} \cdot \alpha \cdot \beta \cdot \gamma \rangle = \langle \gamma^{-1} \cdot \alpha \cdot \gamma \cdot \gamma^{-1} \cdot \beta \cdot \gamma \rangle$$

and of course $\gamma^{-1} \cdot \gamma$ is homotopic to the point map, so we have a homomorphism

- It's an isomorphism (inverse). Consider the map $\pi_1(X, q) \rightarrow \pi_1(X, p)$, and a loop $\langle \beta \rangle$ based at q . We have a map

$$\langle \beta \rangle \rightarrow \langle \gamma \cdot \beta \cdot \gamma^{-1} \rangle$$

based at p . So we have an isomorphism between $\pi_1(X, p)$ and $\pi_1(X, q)$.

□

⁸This keeps our composite loop centered at p

Theorem 5.19. Given a map $f : X \rightarrow Y$, we get a group homomorphism $f_* : \pi_1(X, p) \rightarrow \pi_1(Y, f(p))$ by

$$f_*(\langle \alpha \rangle) = \langle f \circ \alpha \rangle$$

with the corresponding diagram $[0, 1] \xrightarrow{\alpha} X \xrightarrow{f} Y$

Proof.

$$\begin{aligned} f_*(\langle \alpha \rangle \cdot \langle \beta \rangle) &= f_*(\langle \alpha \cdot \beta \rangle) \\ &= \langle f \circ (\alpha \cdot \beta) \rangle \\ &= \langle (f \circ \alpha) \cdot (f \circ \beta) \rangle \\ &= \langle f \circ \alpha \rangle \cdot \langle f \circ \beta \rangle \\ &= f_*(\langle \alpha \rangle) \cdot f_*(\langle \beta \rangle) \end{aligned}$$

□

Theorem 5.20. Given maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, we get $(g \circ f)_* : \pi_1(X, q) \rightarrow \pi_1(Z, r)$ where $(g \circ f)_* = g_* \circ f_*$,

$$f_* : \pi_1(X, p) \rightarrow \pi_1(Y, q), \quad q = f(p),$$

$$g_* : \pi_1(Y, q) \rightarrow \pi_1(Z, r), \quad r = g(q)$$

Corollary 5.21. If we have a homeomorphism $h : X \rightarrow Y$, then $h_* : \pi_1(X, p) \rightarrow \pi_1(Y, h(p))$ is an isomorphism of groups.

Proof. We have

$$\begin{array}{ccccc} X & \xrightarrow{h} & Y & \xrightarrow{h^{-1}} & X \\ & \searrow^{\text{Id}_X} & & & \swarrow^{\text{Id}_Y} \\ & & Y & \xrightarrow{h^{-1}} & X \xrightarrow{h} Y \end{array}$$

These induce the following homomorphisms

$$\begin{array}{ccccc} \pi_1(X, p) & \xrightarrow{h_*} & \pi_1(Y, h(p)) & \xrightarrow{h_*^{-1}} & \pi_1(X, p) \\ & \searrow^{\text{Id}_{\pi_1(X, p)}} & & & \swarrow^{\text{Id}_{\pi_1(Y, h(p))}} \\ & & \pi_1(Y, h(p)) & \xrightarrow{h_*} & \pi_1(X, p) \\ & & & \searrow^{\text{Id}_{\pi_1(Y, h(p))}} & \\ & & & & \pi_1(Y, h(p)) \end{array}$$

So $h_* : \pi_1(X, p) \rightarrow \pi_1(Y, h(p))$ has an inverse homomorphism $(h^{-1})_* : \pi_1(Y, h(p)) \rightarrow \pi_1(X, p)$. □

Up until now, all our examples of fundamental groups have been trivial. Now we will do nontrivial ones. Towards this goal, we introduce a lemma about metric spaces.

Definition 5.22. Let X be a metric space, $S \subseteq X$. The diameter of S is

$$\text{diam}(S) := \sup_{s_1, s_2 \in S} d(s_1, s_2)$$

Lemma 5.23 (Lebesgue Lemma). Say X is a compact metric space and \mathcal{F} an open cover of X . Then there exists $\delta > 0$ (called the lebesgue number of the covering) such that for all $A \subseteq X$ with $\text{diam}(A) < \delta$, there exists some $U \in \mathcal{F}$ such that $A \subseteq U$

Basically, we can always take a speck of dust of diameter δ and it is covered by a single member of the open covering.

Proof. Suppose no such δ exists. Then for all n , there exists a subset $A_n \subseteq X$ such that $\text{diam}(A_n) < \frac{1}{n}$ but A_n is not contained in a member of \mathcal{F} . For each n , choose $x_n \in A_n$. We can find a subsequence such that $\lim_{n \rightarrow \infty} x_n = p$ (Since X is compact, there exists a convergent subsequence). Choose $U \in \mathcal{F}$ such that $p \in U$. U is open, so we can find $\epsilon > 0$ such that $B(p, \epsilon) \subset U$. Choose N such that

1. $\text{diam}(A_n) < \frac{1}{2}\epsilon$
2. $d(x_n, p) < \frac{1}{2}\epsilon$.

Look at $x \in A_n$. Then $d(x, x_n) < \frac{1}{2}\epsilon$ (from 1), and $d(x_n, p) < \frac{1}{2}\epsilon$ (from 2) So

$$d(x, p) < d(x, x_n) + d(x_n, p) < \epsilon$$

, Hence $x \in B(p, \epsilon) \subset U$. Since x was any point in A_n , we have $A_n \subseteq U$, which is a contradiction. \square

Example 5.24. Identify the circle with the unit circle in \mathbb{C} , and let $\pi : \mathbb{R} \rightarrow S^1$, denote the mapping $\pi(s) = e^{2\pi i s}$. Take $1 \in S^1$ as the basepoint. Given $n \in \mathbb{Z}$, put $\gamma_n(s) = ns$, with $0 \leq s \leq 1$. Then $\pi \circ \gamma_n$ is a loop in S^1 based at 1.

This motivates the following theorem, which we will prove next class.

Theorem 5.25. A function $\phi : \mathbb{Z} \rightarrow \pi_1(S^1, 1)$ given by $\phi(n) = \langle \pi \circ \gamma_n \rangle$ is an isomorphism, i.e., the circle has a nontrivial fundamental group.

3.12.2025

5.3 The fundamental group of S^1 and Brouwer's fixed-point theorem

Today we will compute the fundamental group of S^1 , that is, $\pi_1(S^1)$. Following example 5.24, let $\pi : \mathbb{R} \rightarrow S^1$, denote the mapping $\pi(x) = e^{2\pi i x}$. Take $1 \in S^1$ as the basepoint.

Construct a map $\phi : \mathbb{Z} \rightarrow \pi_1(S^1)$ as follows: let $\gamma_n(s) = ns$ for $0 \leq s \leq 1$, and $\gamma_n : [0, 1] \rightarrow \mathbb{R}$. Then $\pi \circ \gamma_n$ is a loop in S^1 based at $1 \in S^1$ that winds around S^1 n times⁹. Put $\phi(n) = \langle \pi \circ \gamma_n \rangle$. We want to show that it is a group isomorphism.

Remark 5.26. If γ is any path in \mathbb{R} from 0 to n , then $\gamma \simeq \gamma_n(\text{rel}\{0, 1\})$, since \mathbb{R} is convex. So

$$\langle \pi \circ \gamma \rangle \simeq \langle \pi \circ \gamma_n \rangle$$

Lemma 5.27. $\langle \pi \circ \gamma \rangle$ is a homomorphism.

Proof. We want to show that $\phi(m+n) = \phi(m) + \phi(n)$. This is true if $\langle \gamma_{m+n} \rangle = \langle \gamma_m \rangle \cdot \langle \gamma_n \rangle$, which needs,

$$\gamma_{m+n} \simeq \gamma_m \cdot \gamma_n$$

These are both paths in \mathbb{R} from 0 to $m+n$, so they are both homotopic, so we have a homomorphism. \square

Next, we claim that ϕ is onto (surjective), i.e., any $\langle \alpha \rangle \in \pi_1(S^1, 1)$ is such that $\alpha \simeq \pi \circ \gamma_n$ ¹⁰. In order to do this, we try to 'lift' the loop α to a path γ in \mathbb{R} such that $\pi \circ \gamma = \alpha$, where $\gamma(0) = 0$. If we can do this, then $\pi(\gamma(1)) = 1$, so $\gamma(1) \in \mathbb{Z}$ (by construction). Denote this integer as n , and call it the **degree** of α . γ is homotopic to γ_n , and so $\alpha = \pi \circ \gamma \simeq \pi \circ \gamma_n$ (rel $\{0, 1\}$), i.e., $\langle \alpha \rangle = \langle \pi \circ \gamma \rangle = \langle \pi \circ \gamma_n \rangle = \phi(n)$.

In order to do the lifting process, we must understand $\pi : \mathbb{R} \rightarrow S^1$ in more detail. Let U be the open set $S^1 \setminus \{-1\}$, and $V = S^1 \setminus \{1\}$ open as well. Then $\{U, V\}$ is an open cover of S^1 . Then the inverses are as follows:

$$\pi^{-1}(V) = \mathbb{R} \setminus \mathbb{Z} = \bigcup_{n \in \mathbb{Z}} (n, n+1)$$

⁹Winding number reference

¹⁰Take the time to understand this: we can deform α (any loop) into the path looping around the circle n times.

$$\pi^{-1}(U) = \mathbb{R} \setminus \left\{ \frac{1}{2} + \mathbb{Z} \right\} = \bigcup_{n \in \mathbb{Z}} \left(n - \frac{1}{2}, n + \frac{1}{2} \right)$$

We see that the inverses map each open set to open sets such that restricting to any one of the open intervals gives a homeomorphism. So the idea is that if we have a loop in S^1 , we can break it up into segments such that each segment lies in U or V , and then lift each one back into \mathbb{R} individually.

Lemma 5.28 (Path-lifting lemma). If σ is a path in S^1 beginning at 1, there exists a unique path $\tilde{\sigma}$ in \mathbb{R} such that $\sigma(0) = 0$ and $\pi \circ \tilde{\sigma} = \sigma$, i.e., the diagram commutes:

$$\begin{array}{ccc} & \mathbb{R} & \\ \tilde{\sigma} \nearrow & \downarrow \pi & \\ [0, 1] & \xrightarrow{\sigma} & S^1 \end{array}$$

Proof. The pair $\{\sigma^{-1}(U), \sigma^{-1}(V)\}$ is an open cover of $[0, 1]$. By the Lebesgue lemma, there exists a $d > 0$ such that any subset A of $[0, 1]$ with $\text{diam}(A) < d$ lies in $\sigma^{-1}(U)$ or $\sigma^{-1}(V)$, i.e., $\sigma(A) \subset U$ or $\sigma(A) \subset V$. Divide $[0, 1]$ as $0 = T_0 < T_1 < \dots < T_n = 1$, such that $|T_{i+1} - T_i| < d$. Since $\sigma(0) = 1 \in U$, we know that $\sigma([T_0, T_1]) \subset U$. Remember that $\pi|_{(-\frac{1}{2}, \frac{1}{2})} : (-\frac{1}{2}, \frac{1}{2}) \rightarrow U$ is a homeomorphism, so let f denote its inverse.

Now, put $\tilde{\sigma}(s) = f(\sigma(s))$ for $0 \leq s \leq t$. Inductively, suppose $\tilde{\sigma}$ is defined on $[0, T_k]$. We want to define it (i.e., $\tilde{\sigma}$) on $[T_k, T_{k+1}]$. We know that $\sigma([T_k, T_{k+1}])$ is entirely in U or V . If $\sigma([T_k, T_{k+1}]) \subset U$ and $\tilde{\sigma}(T_k) \in (n - \frac{1}{2}, n + \frac{1}{2})$. Let g be the inverse of $\pi|_{(n - \frac{1}{2}, n + \frac{1}{2})} : (n - \frac{1}{2}, n + \frac{1}{2}) \rightarrow U$, and set $\tilde{\sigma}(s) = g(\sigma(s))$ for $T_k \leq s \leq T_{k+1}$.

Considering the other case, if $\sigma([T_k, T_{k+1}]) \subset V$, then $\tilde{\sigma}(T_k) \in (n, n+1)$ for some n . The restriction $\pi|_{(n, n+1)} : (n, n+1) \rightarrow V$ is a homeomorphism with an inverse, say, h . Then we can define $\tilde{\sigma}(s) := h(\sigma(s))$ for $T_k \leq s \leq T_{k+1}$. This completes our inductive definition of the lifted path. Also, see that since after we define $\tilde{\sigma}$ on $[0, T_k]$, there is only one way to extend it over $[T_k, T_{k+1}]$, so $\tilde{\sigma}$ is unique. \square

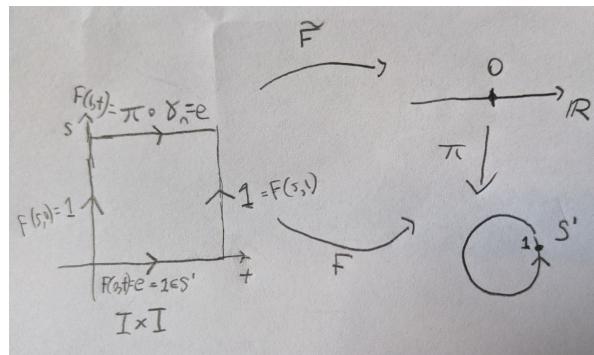
So we know that ϕ is surjective. In order to prove that ϕ is injective, we will need to 'lift' homotopies from the circle to \mathbb{R} .

Lemma 5.29 (Homotopy-lifting lemma). If $F : I \times I \rightarrow S^1$ is a map with $F(0, t) = F(1, t) = 1 \in S^1$ for all $0 \leq t \leq 1$, then there exists a unique map $\tilde{F} : I \times I \rightarrow \mathbb{R}$ such that $\tilde{F}(0, t) = 0$ for all $0 \leq t \leq 1$ and $\pi \circ \tilde{F} = F$.

Proof. The proof for this follows similarly to the Path-lifting lemma. $\{F^{-1}(U), F^{-1}(V)\}$ is an open cover of $[0, 1] \times [0, 1]$. The Lebesgue lemma gives $\delta > 0$, for which $A \subseteq I \times I$, $\text{diam}(A) < \delta \implies F(A) \subset U$ or $F(A) \subset V$. Subdivide the unit square so that each square has $\text{diam} < \delta$. The idea is we lift F on squares iteratively, starting from the lower left, to get \tilde{F} . \square

So now we can prove Theorem 5.25, that $\phi(n) = \langle \pi \circ \gamma_n \rangle$ is a group isomorphism.

1. We've shown that ϕ is a homomorphism.
2. We have that ϕ is surjective by the path lifting lemma, as stated earlier.
3. Finally, we show that ϕ is injective, or one-to-one. Suppose $n \in \ker(\phi)$, i.e., $\phi(n) = \langle e \rangle \in \pi_1(S^1, 1)$, where $e(s) = 1$ for all $0 \leq s \leq 1$. Then $\pi \circ \gamma_n = e$. We call $\pi \circ \gamma_n$ a **null-homotopic loop**. Say $F : I \times I \rightarrow S^1$ is a homotopy from e (the constant loop at 1) to $\pi \circ \gamma_n$. By the Homotopy-lifting lemma, we can lift F to some $\tilde{F} : I \times I \rightarrow \mathbb{R}$ with $\tilde{F}(0, t) = 0$ for all $t \in [0, 1]$.



Say $p = (\text{Left edge}) \cup (\text{Bottom edge}) \cup (\text{Right edge})$. Then p is connected and $F(p) = \pi(\tilde{F}(p)) = 1 \in S^1$. So $\tilde{F}(p) = m$ for some $m \in \mathbb{Z}$, i.e., all of p is sent to some single integer. But $\tilde{F}(0,0) = 0$, so $N = 0$, and all of p is sent there. The path in \mathbb{R} defined by $\{F(\tilde{S}, 1)\}$ is a lift of $\pi \circ \gamma_n$ which begins at 0, and by uniqueness, it must be γ_n . Since $\tilde{F}(1,1) = 0$, we get $\gamma_n(1) = n = 0$, so the kernel $\ker(\phi) = \{0\}$. This implies that ϕ is injective.

So we've shown that there is a homomorphism between the fundamental group of S^1 and the group \mathbb{Z} , i.e., $\pi_1(S^1) \cong \mathbb{Z}$.

3.14.2025

Definition 5.30. Say X is a set, $f : X \rightarrow X$ a function. We say $x \in X$ is a **fixed point** for f if $f(x) = x$.

Proposition 5.31. If $f : [0,1] \rightarrow [0,1]$ is a map then f has a fixed point.

Proof. Define $g(x) = f(x) - x$, $g(0) = g(0) - 0 \geq 0$. Then

$$g(1) = f(1) - 1 \leq 0$$

So by the intermediate value theorem, there exists an x such that it is a fixed point for f . \square

Remark 5.32. Not all maps $f : S^1 \rightarrow S^1$ have a fixed point

Theorem 5.33 (Brouwer's fixed-point Theorem). Any $f : D^2 \rightarrow D^2$ has a fixed point, where D^2 is the closed unit disc in \mathbb{R}^2

Proof. Towards contradiction, suppose f has no fixed points, i.e., $f(x) \neq x$ for all $x \in D^2$. Draw a line segment from $f(x)$ to x and extend the line until it hits the circle. Call this point $g(x)$. We see that

$$g : D^2 \rightarrow S^1$$

If $x \in S^1$, then $g(x) = x$, i.e., g fixes the points on the circle.

Say i is the inclusion map $i : S^1 \rightarrow D^2$. Then we have

$$S^1 \xrightarrow{i} D^2 \xrightarrow{g} S^1$$

with $g \circ i = \text{Id}_{S^1}$. Say $p = 1$. Then we get

$$\begin{array}{ccccc} & & & & \text{Id}_{S^1} \\ & & & & \downarrow \\ \pi_1(S^1, p) & \xrightarrow{i_*} & \pi_1(D^2, p) & \xrightarrow{g_*} & \pi_1(S^1, p) \\ & & & & \downarrow \\ & & & & \text{Id}_{\pi_1(S^1, p)} \end{array}$$

With the fundamental groups of $\pi_1(S^1, p)$ and $\pi_1(D^2, p)$ being \mathbb{Z} and 0^{11} respectively. We see that $g_* \circ i_* = (\text{Id}_{S^1})_* = \text{Id}_{\pi_1(S^1, p)}$. But $g_* \circ i_* = 0 : \mathbb{Z} \rightarrow \mathbb{Z}$, since $\pi_1(D^2, p) = 0$, i.e., it factors through the trivial group. This is a contradiction, so there must be a fixed point. \square

Corollary 5.34. If X is homeomorphic to D^2 , then any map $f : X \rightarrow X$ has a fixed point.

Proof. Say $h : D^2 \rightarrow X$ is a homeomorphism. $h^{-1} \circ f \circ h : D^2 \rightarrow D^2$ is a map. By Brouwer's fixed-point theorem, there exists a $z \in D^2$ such that $h^{-1} \circ f \circ h = z$. Then $(f \circ h)(z) = h(z)$, i.e., $f(h(z)) = h(z)$, so $h(z)$ is a fixed point for f . \square

¹¹ $\pi_1(D^2, p)$ is trivial since D^2 is convex in \mathbb{R}^2

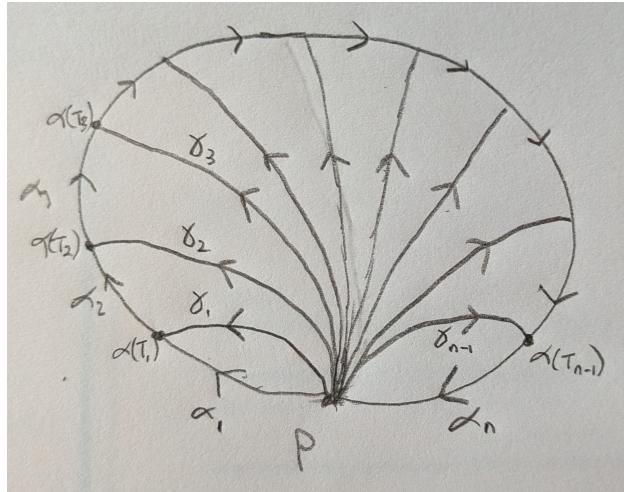
Definition 5.35. X is **simply connected** if it is path connected and its fundamental group is trivial, i.e., $\pi_1(X) \cong \{e\}$

Theorem 5.36. If $X = A \cup B$, where A, B are open, simply connected, and their intersection path connected, then X is simply connected.

Proof. It's easy to see that X is path connected. Take $p \in A \cup B$ to be the base point, and consider a loop $\alpha : [0, 1] \rightarrow X$ based at p , i.e., $\alpha(0) = \alpha(1) = p$. Then

$$\{\alpha^{-1}(A), \alpha^{-1}(B)\}$$

(that is, the inverse of the subset is the set of points in the interval which map into said subset through the path) is an open cover of $[0, 1]$. By the Lebesgue lemma, we can split the interval into $0 = T_0 < T_1 < \dots < T_n = 1$, such that $[T_i, T_{i+1}]$ is entirely contained in $\alpha^{-1}(A)$ or $\alpha^{-1}(B)$.



- If $\alpha(T_k) \in A$, choose a path γ_k from p to $\alpha(T_k)$ which lies entirely within A .
- If $\alpha(T_k) \in B$, choose γ_k from p to $\alpha(T_k)$ that lies entirely in B .

Note: If $\alpha(T_k) \in A \cap B$, γ_k lies in $A \cap B$ because their intersection is path connected. We can see from the diagram that

$$\begin{aligned} <\alpha> &= <\alpha_1 \gamma_1^{-1} \gamma_1 \alpha_2 \gamma_2^{-1} \gamma_2 \cdots \alpha_{n-1} \gamma_{n-1}^{-1} \gamma_{n-1} \alpha_n> \\ &= <\alpha_1 \gamma_1^{-1}> <\gamma_1 \alpha_2 \gamma_2^{-1}> \cdots <\gamma_{n-1} \alpha_n> \end{aligned}$$

which is a product of homotopy classes of loops based at p , each lying entirely in A or B . But these are all homotopic to a point map at p , hence $<\alpha> = <e_p>$. \square

Example 5.37. Let $X = S^n$, with $n > 1$, and $A = S^n \setminus \{\text{North Pole}\}$, and $B = S^n \setminus \{\text{South Pole}\}$. A, B are homeomorphic to \mathbb{R}^n , so they are simply connected. $A \cap B = S^1 \setminus \{\text{North, South poles}\}$ is path connected, since $n > 1$. Then we see that S^n must be simply connected.

Nonexample 5.38. Let $X = S^1$, and A, B the same sets as before. Their intersection is not path connected, so we cannot apply the result.

Nonexample 5.39. Let $X = S^1$. Suppose A is a clopen, simply connected subset, and all of the other assumptions hold (so B and one side of A share a boundary). Obviously this doesn't suddenly make S^1 path connected. Let this be a lesson not to forget these conditions, however. The Mayor of Evanston Illinois forgot to use the fact that A, B had to be open, and now he's the Mayor of Evanston.

3.17.2025

5.4 Calculations

Definition 5.40. A group acts **freely** if for all x , $\text{Stab}(x) = e$.

Example 5.41. $(\mathbb{Z}, +)$ acts freely on \mathbb{R}

Theorem 5.42. Suppose G is a discrete group acting on a simply connected topological space X such that for all $x \in X$, there exists an open neighborhood U such that $U \cap g(U) = \emptyset$ for all $g \neq e$. Then $\pi_1(X/G) \cong G$

Example 5.43. $G = \mathbb{Z}$, $X = \mathbb{R}$, $U = (x - \frac{1}{6}, x + \frac{1}{6})$.

$$X/G = S^1, \text{ so } \pi_1(X/G) \cong G, \text{ i.e., } \pi_1(S^1) \cong \mathbb{Z}$$

We will model our proof based on the above example, so keep it in mind when understanding the proof.

Proof. (of the theorem). Fix $x_0 \in X$. Given $g \in G$, join x_0 to $g(x_0)$ by a path γ_g . If $\pi : X \rightarrow X/G$ denotes the quotient map, $\pi \circ \gamma_g$ is a loop of $\pi(x_0)$ (since $(\pi \circ \gamma_g)(1) = \pi(\gamma_g(1)) = \pi(g(x_0)) = \pi(x_0)$).

Define $\phi : G \rightarrow \pi_1(X, \pi(x_0))$ by

$$\phi(g) = \langle \pi \circ \gamma_g \rangle$$

We claim this is independent of our choice of path γ_g . If γ'_g is another choice, then $\gamma'_g \circ \gamma_g^{-1}$ is a loop at x_0 . Since X is simply connected, this is homotopic to $e_{x_0}(\text{rel}\{0,1\})$. Say $\{\eta_T\}_{T \in [0,1]}$ is a 1-parameter family of loops from $\gamma'_g \circ \gamma_g^{-1}$ to e_{x_0} based at x_0 . Then γ'_g is homotopic to $\gamma_g(\text{rel}\{0,1\})$ by $\{\eta_T \circ \gamma_g\}_{T \in [0,1]}$, so $\langle \pi \circ \gamma'_g \rangle = \langle \pi \circ \gamma_g \rangle$. So ϕ is well defined.

Now, we want to show that this is an isomorphism (similar to what we did for S^1 , using the path and homotopy-lifting lemmas).

- To show ϕ is surjective, consider a path σ in X/G that begins at $\pi(x_0)$. Then there exists a unique $\tilde{\sigma}$, a path in X , such that $\tilde{\sigma}(0) = x_0$ and $\sigma = \pi \circ \tilde{\sigma}$.
- To show that it is injective, we use the homotopy-lifting lemma. If a map $F : I \times I \rightarrow X/G$ with $F(0, t) = F(1, t) = \pi(x_0)$, then there is a unique $\tilde{F} : I \times I \rightarrow X$ such that $\tilde{F}(0, t) = X$ for all $t \in [0, 1]$ and $F = \pi \circ \tilde{F}$.

Of course, one might wonder if these lemmas truly hold on this space. These lemmas hold for any map $\pi : X \rightarrow Y$ with the following property. For any $y \in Y$, there is an open neighborhood V such that $\pi^{-1}(V)$ can be decomposed into a family $\{U_\alpha\}$ of pairwise disjoint sets such that the restriction of π to each U_α is a homeomorphism from U_α to V . This map π is called a **covering map**, and X is called a **covering space** of Y . \square

So in the example above, \mathbb{R} is a covering space of S^1

Example 5.44. $G = C^n$, $X = \mathbb{R}^n$, acting by integer translation. By definition, $X/G = T^n$. So $\pi_1(T^n) \cong \mathbb{Z}^n$

Example 5.45. C_2 on $X = S^n$. $X/G = P^n$, and so $\pi_1(P^n) = C_2$

Example 5.46. Let $G = \mathbb{Z}/p^a$ on S^3 . $S^3/\mathbb{Z}/p = L(p, q)$, the lens space. So $\pi_1(L(p, q)) \cong \mathbb{Z}/p$.

\mathbb{Z}/p is the cyclic group with p elements C_p also written \mathbb{Z}_p .

Remark 5.47. $\pi_1(X)$ is not necessarily abelian.

Theorem 5.48. Let X, Y be connected. Then $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.

Proof. Choose base points $x_0 \in X, y_0 \in Y$. Use (x_0, y_0) as a basepoint in $X \times Y$. We have canonical projections $p_1 : X \times Y \rightarrow X, p_2 : X \times Y \rightarrow Y$. Then these projections induce homomorphisms

$$p_{1*} : \pi_1(X \times Y) \rightarrow \pi_1(X)$$

and the same for p_{2*}

Define $\psi : \pi_1(X \times Y, (x_0, y_0)) \rightarrow \pi_1(X, x_0) \times \pi_1(Y, y_0)$ by $\psi(<\gamma>) = (<p_1 \circ \gamma>, <p_2 \circ \gamma>)$. This is a homomorphism, and we want to show that it is an isomorphism.

Claim: ψ is injective: Suppose $\alpha \in \ker(\psi)$, i.e.,

$$\psi(\alpha) = (<e_{x_0}>, <e_{y_0}>)$$

i.e.,

$$\begin{aligned} p_1 \circ \alpha &\xrightarrow{F} e_{x_0} \\ p_2 \circ \alpha &\xrightarrow{G} e_{y_0} \end{aligned}$$

Then $\alpha \xrightarrow{H} (e_{x_0}, e_{y_0})$ by $H(s, t) = (F(s, t), G(s, t))$.

We also claim ψ is surjective: Given $<\alpha> \in \pi_1(X, x_0)$ and $<\beta> \in \pi_1(Y, y_0)$, put $\gamma(s) = (\alpha(s), \beta(s))$, a loop in $X \times Y$. Then $\psi(<\gamma>) = (<\alpha>, <\beta>)$ So we are done. \square

Example 5.49. Let $X, Y = S^1$. Then $X \times Y = T^2$. So this means $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$. We can realize the trefoil knot as a nontrivial loop on T^2

Definition 5.50. Topological spaces X, Y are **homotopy equivalent** (aka have the same **homotopy type**) if there exist maps

$$F : X \rightarrow Y$$

and

$$G : Y \rightarrow X$$

such that $G \circ F \simeq \text{Id}_X$ and $F \circ G \simeq \text{Id}_Y$. We say G is a homotopy inverse to F and the pair F, G is a homotopy equivalence of X, Y

Theorem 5.51. Homotopy equivalence is an equivalence relation.

Proof. 1.

$$\begin{array}{ccc} & \text{Id}_X & \\ X & \xrightarrow{\quad} & X \\ & \text{Id}_X & \end{array}$$

implies $X \simeq X$

2.

$$\begin{array}{ccc} & f & \\ X & \xrightarrow{\quad} & Y \\ & g & \end{array}$$

implies

$$\begin{array}{ccc} & g & \\ Y & \xrightarrow{\quad} & X \\ & f & \end{array}$$

therefore, $X \simeq Y \implies Y \simeq X$

3.

$$X \xrightarrow{f} Y, \quad Y \xrightarrow{u} Z$$

$$X \xleftarrow{g} Y, \quad Y \xleftarrow{v} Z$$

implies

$$X \xrightarrow{u \circ f} Z$$

$$X \xleftarrow{g \circ v} Z$$

$$(g \circ v) \circ (u \circ f) = g \circ (v \circ u) \circ f$$

$$\simeq g \circ \text{Id}_Y \circ f$$

$$= g \circ f$$

$$\simeq \text{Id}_X$$

$$(u \circ f) \circ (g \circ v) = u \circ (f \circ g) \circ v$$

$$\simeq u \circ \text{Id}_X \circ v$$

$$= u \circ v$$

$$\simeq \text{Id}_Y$$

therefore we have transitivity

□

Definition 5.52 (Algebraic Topology). So now we have a new definition of the field which we are aiming to study: **Algebraic Topology** is the study of topological spaces up to homotopy equivalence.

3.19.2025

Just to recap, we have two types of homotopy equivalence

- Of maps
- Of spaces

Example 5.53. Homeomorphic spaces are homotopy equivalent. Given two homeomorphic spaces as follows,

$$X \xrightarrow{f} Y$$

$$X \xleftarrow{g} Y$$

we have:

$$f \circ g = \text{Id}_X, \quad g \circ f = \text{Id}_Y$$

Example 5.54. A convex subset C of \mathbb{R}^n is homotopy equivalent to a point. Pick $p \in C$. Take $X = \{p\}$, $Y = C$. We have two maps:

$$f : p \hookrightarrow C \text{ the inclusion map}$$

$$g : C \rightarrow p$$

Then $g \circ f = \text{Id}_{\{p\}}$. $g \circ f : C \rightarrow C$ sends C to p .

Put $F : I \times C \rightarrow C$ with $F(t, \vec{x}) = (1-t)\vec{x} + t\vec{p}$. Then $F(0, \vec{x}) = \vec{x} = \text{Id}_X$ and $F(1, \vec{x}) = \vec{p} = f \circ g$. Then $f \circ g \underset{F}{\sim} \text{Id}_C$.

The idea is we have a linear homotopy between the identity and a point in C because it is convex.

Example 5.55. $\mathbb{R}^n \setminus \{0\}$ is homotopy equivalent to S^{n-1} .

Let $f : S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$ be the inclusion map, and $g : \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}$ be such that $g(\vec{x}) = \frac{\vec{x}}{|\vec{x}|}$ such that it normalizes the vector.

Then $g \circ f = \text{Id}_{S^{n-1}}$, and $f \circ g : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^n \setminus \{0\}$ takes a vector \vec{x} and normalizes it. Put $F(t, \vec{x}) = (1-t)\vec{x} + t\frac{\vec{x}}{|\vec{x}|}$. So we have a homotopy from any vector to the unit vector with same direction.

Then $f \circ g \xrightarrow{F} \text{Id}_{\mathbb{R}^n \setminus \{0\}}$. Similar to above, the homotopy is linear.

You'll notice that the last two homotopy equivalence examples seem pretty similar. This leads us to the following.

Definition 5.56. Let $A \subseteq X$. A **Deformation Retract** of X to A is a homotopy $G : X \times I \rightarrow X$ (rel A) such that

$$G(X, 0) = X \text{ and } G(X, 1) \in A$$

Remark 5.57. Then X and A are homotopy equivalent. Take $f : A \hookrightarrow X$ to be the inclusion map and $g : X \rightarrow A$ to be $g(x) = G(x, 1) \in A$. Then $f \circ g = \text{Id}_A$ and $g \circ f \xrightarrow{G} \text{Id}_X$.

Recall that if $f : X \rightarrow Y$, we get the induced group homomorphism $f_* : \pi_1(X, p) \rightarrow \pi_1(Y, f(p))$ by

$$f_*(\langle \alpha \rangle) = \langle f \circ \alpha \rangle$$

Theorem 5.58. If $f \xrightarrow{F} g : X \rightarrow Y$, then

$$g_* : \pi_1(X, p) \rightarrow \pi_1(Y, g(p))$$

is equal to the composition

$$\pi_1(X, p) \xrightarrow{f_*} \pi_1(Y, f(p)) \xrightarrow{\gamma_*} \pi_1(Y, g(p))$$

with γ the path from $f(p)$ to $g(p)$ given by $\gamma(s) = F(p, s)$ and γ_* the corresponding isomorphism

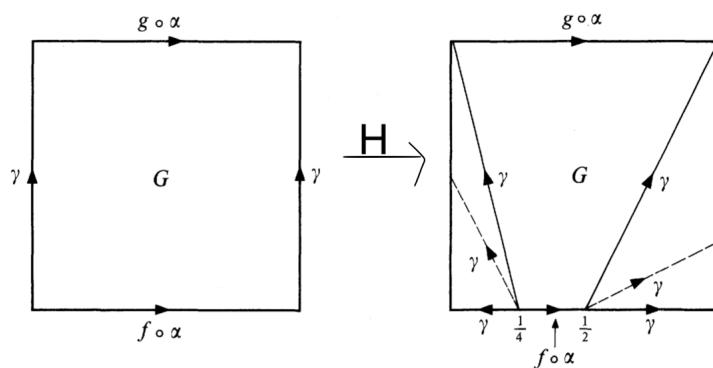
$$\gamma_* : \pi_1(Y, f(p)) \rightarrow \pi_1(Y, g(p))$$

Proof. Say α is a loop in X based at p . Then $g_*(\langle \alpha \rangle) = \langle g \circ \alpha \rangle$ and

$$\gamma_*(f_*(\langle \alpha \rangle)) = \gamma_*(\langle f \circ \alpha \rangle) = \langle \gamma^{-1} \circ (f \circ \alpha) \circ \gamma \rangle$$

We want to show that $g \circ \alpha \simeq \gamma^{-1} \circ (f \circ \alpha) \circ \gamma$ (rel $\{0, 1\}$)

Define $G : I \times I \rightarrow Y$ by $G(s, t) = F(\alpha(s), t)$. Using G , we construct a homotopy between our two loops, the effect illustrated here:



□

Theorem 5.59. If two path connected spaces are homotopy equivalent, they have the same π_1 up to isomorphism.

Proof. Suppose

$$\begin{array}{ccc} & f & \\ X & \xleftarrow{\quad g \quad} & Y \end{array}$$

with $g \circ f \xrightarrow{F} \text{Id}_X$ and $f \circ g \xrightarrow{G} \text{Id}_Y$. Choose $q \in Y$, and put $p = g(q)$ (Then p is in the image of g). We'll show $f_* : \pi_1(X, p) \rightarrow \pi_1(Y, f(p))$ is an isomorphism.

Join $p = \text{Id}_X(p)$ to $(g \circ f)(p)$ by $\gamma(s) = F(p, s)$. Applying theorem 5.58¹², we have

$$g_* \circ f_* = (g \circ f)_* = \text{Id}_X \circ \gamma_* : \pi_1(X, p) \rightarrow \pi_1(X, (g \circ f)(p))$$

which means $g_* \circ f_*$ is an isomorphism. This implies that f_* is injective¹³.

Now we just want to show that f_* is also surjective. $f_* \circ g_*$ is an isomorphism as well, by the same logic. Then f_* is surjective, so it is an isomorphism. \square

Corollary 5.60. If $n > 2$ then \mathbb{R}^n is not homeomorphic to \mathbb{R}^2

Proof. Suppose $f : \mathbb{R}^2 \rightarrow \mathbb{R}^n$ is a homeomorphism. Then $f_{|\mathbb{R}^2 \setminus \{0\}} : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}^n \setminus \{f(0)\}$ is a homeomorphism, hence $\pi_1(\mathbb{R}^2 \setminus \{0\})$ is isomorphic to $\pi_1(\mathbb{R}^n \setminus \{f(0)\})$.

$\mathbb{R}^2 \setminus \{0\}$ deformation retracts to S^1 . So

$$\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \pi_1(S^1) \cong \mathbb{Z}$$

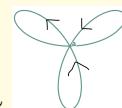
. But $\mathbb{R}^n \setminus \{0\}$ deformation retracts to S^{n-1} , so

$$\pi_1(\mathbb{R}^n \setminus \{f(0)\}) \cong \pi_1(S^{n-1}) = \{e\}$$

which is a contradiction \square

3.21.2025

Last time we talked about deformation retractions.



Example 5.61. Let X be the unit ball in $\mathbb{R}^2 - 3$ balls. Then $X \simeq$

Definition 5.62. X is **contractible** if Id_X is homotopic to a constant map. (i.e., A map that sends X to a point $p \in X$)

Theorem 5.63. 1. X is contractible if and only if X is homotopy equivalent to a point

2. Contractible implies simply connected
3. If X is contractible, then any two $f, g : Z \rightarrow X$ are homotopic.
4. If X is contractible, then $\text{Id}_X \simeq C_x$ for all $x \in X$, where C_x is the constant map $C_x : X \rightarrow \{x\}$

¹²Theorem 5.58 has two functions going from X to Y . IN this case, those two functions are $g_* \circ f$ and Id_X , both going from X to X .

¹³and g_* is surjective!

Warning 5.64

2) is not an if and only if.

Proof. 1. For the forward direction, suppose X is contractible. Say $\alpha_p : X \rightarrow \{p\}$ is a constant map, and $\iota : \{p\} \rightarrow X$ is the inclusion map. We claim this is homotopy equivalence. We see that

$$\alpha_p \circ \iota = \text{Id}_{\{p\}}$$

and by contractibility assumption,

$$\iota \circ \alpha_p \simeq \text{Id}_{\{p\}}$$

So X and $\{p\}$ are homotopy equivalent.

For the backwards direction, Say X is homotopic to $\{p\}$. Then we have

$$\begin{array}{ccc} & f & \\ X & \xrightarrow{\quad} & \{p\} \\ & g & \end{array}$$

Put $q = g(p) \in X$. We have $f \circ g = \text{Id}_{\{p\}}$. Then we see that $\text{Id}_X \simeq g \circ f \simeq \alpha_q$.

2. Suppose X is contractible. We want to show that it is path connected, and that the fundamental group $\pi_1(X)$ is trivial ($= \{e\}$). If $\text{Id}_X \xrightarrow{F} \alpha_p$, then $\gamma(s) = \{F(x, s)\}_{s \in [0,1]}$ is a path from X to p . Then

$$\pi_1(X) \cong \pi_1(\{p\}) = \{e\}$$

by theorem 5.59.

3. Given $\text{Id}_X \simeq \alpha_p$, we have

$$f = \text{Id}_X \circ f \simeq \alpha_p \circ f = \alpha_p \circ g \simeq \text{Id}_X \circ g = g$$

4. We want to show that X being contractible implies $\text{Id}_X \simeq C_x$ for all $x \in X$. We know that $\text{Id}_X \simeq \alpha_p$. From 2), any two maps into X are homotopic, so $\alpha_p \simeq C_x : X \rightarrow X$. Then $\text{Id}_X \simeq C_x$. □

Remark 5.65. If X deformation retracts onto a point in X , then X is contractible. (The converse isn't true)

Definition 5.66 (Retract). If $A \subseteq X$, a **retraction** of X to A is a map $g : X \rightarrow A$ such that $g|_A = \text{Id}_A$

Example 5.67. $\{1\} \in S^1$ is a retract of S^1 , but is not a deformation retract.

Proposition 5.68. If $g : X \rightarrow A$ is a retraction, then $g_* : \pi_1(X, a) \rightarrow \pi_1(A, a)$ is a surjection.

Proof. Say $\iota : A \rightarrow X$ is the inclusion map. then $g \circ \iota = \text{Id}_A$. So $g_* \circ \iota_* = \text{Id}_{\pi_1(A, a)}$, and

$$\begin{array}{ccccc} \pi_1(A, a) & \xrightarrow{\iota_*} & \pi_1(X, p) & \xrightarrow{g_*} & \pi_1(A, p) \\ & & \searrow & & \\ & & & & \text{Id}_{\pi_1(A, a)} \end{array}$$

Given $\eta \in \pi_1(A, a)$, then $g_*(\iota_*(\eta)) = \eta$, so $\eta \in \text{Im}(g_*)$. So g_* is a surjection. □

Think of retraction as something that loses information.

6 Simplicial Complexes

There are so many topological spaces, and so it's nice to classify them. Algebraic invariants such as the fundamental group allow us retrieve information about spaces, but they are only of use if we can calculate them. Simplicial complexes let us compute these things with only a finite amount of data.

6.1 Triangulating Spaces

We begin with a string of definitions and examples.

Definition 6.1. Say $\vec{v}_0, \dots, \vec{v}_k \in \mathbb{R}^n$. The hyperplane H spanned by these points is

$$\begin{aligned} H &= \{\lambda_0 \vec{v}_0 + \dots + \lambda_k \vec{v}_k \mid \sum \lambda_i = 1\} \\ &= \{\vec{v}_0 + r_1(\vec{v}_1 - \vec{v}_0) + \dots + r_k(\vec{v}_k - \vec{v}_0) \mid r_i \in \mathbb{R}\} \end{aligned}$$

These points are in **General Position** if $\{\vec{v}_1 - \vec{v}_0, \dots, \vec{v}_k - \vec{v}_0\}$ are linearly independent. (Then $\dim(H) = k$)

Example 6.2. Let $\vec{v}_0, \vec{v}_1 \in \mathbb{R}^2$ and let $H = \{\lambda_0 \vec{v}_0 + \lambda_1 \vec{v}_1 \mid \lambda_1 + \lambda_0 = 1\}$. Assuming \vec{v}_0 and \vec{v}_1 are linearly independent, this is a line in \mathbb{R}^2 .

Definition 6.3. If $(\vec{v}_0, \dots, \vec{v}_k)$ are in general position, then the smallest convex set in \mathbb{R}^n containing them is

$$\{\lambda_0 \vec{v}_0 + \dots + \lambda_k \vec{v}_k \mid \sum \lambda_i = 1 \wedge \lambda_i \geq 0\}$$

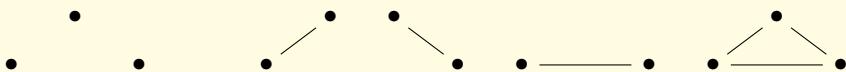
This is a **simplex** with vertices $\vec{v}_0, \dots, \vec{v}_k$

Example 6.4. For

- $k = 0$ •
- $k = 1$ • ————— •
- $k = 2$ • ————— •
- $k = 3$

Definition 6.5. If A, B are simplices, and $\{\text{Vertices of } B\} \subseteq \{\text{Vertices of } A\}$, then B is a **face** of A .

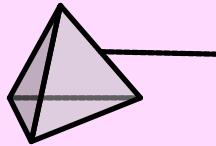
Example 6.6. A 2-simplex has 7 faces



Definition 6.7. A finite collection of simplices in some \mathbb{R}^n is a **simplicial complex** if

1. Whenever a simplex is in the collection, its faces are as well.
2. Whenever two simplices in the collection meet, they intersect in a common face.

Example 6.8.



Nonexample 6.9.

3.31.2025

If K is a simplicial complex (a union of simplices), it can be regarded as a subset of Euclidean space. $|K|$ denotes a complex when regarded as a topological space by giving it the subspace topology.

Definition 6.10. A **triangulation** of a topological space X is a simplicial complex K and a homeomorphism between $|K|$ and X .

Example 6.11. Let $X = S^2$, and K the union of simplices that make up the surface of a tetrahedron. A triangulation is a homeomorphism between $|K|$ and X

Definition 6.12. A simplicial complex K in \mathbb{R}^n can be thought of as $K \subset \mathbb{R}^n \subset \mathbb{R}^{n+1}$. If A is a simplex (v_0, \dots, v_k) in K , put $v = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$ and $CA = (v_0, \dots, v_k, v)$. Let

$$CK = K \cup \{v\} \cup \{CA \mid A \in K\}$$

CK is known as the **cone over K** .

Notice that $|CK| = C|K|$ (see Example 4.16)

Definition 6.13 (Isomorphism of simplicial complexes). Say K, L are simplicial complexes. An isomorphism $K \rightarrow L$ is a bijection

$$\phi : \{\text{vertices of } K\} \rightarrow \{\text{vertices of } L\}$$

such that

$$\{v_0, \dots, v_k\}$$

is a simplex of K if and only if

$$\{\phi(v_0), \dots, \phi(v_k)\}$$

is a simplex of L . Then $|L|$ and $|K|$ are homeomorphic.

The idea is that we want to map simplices of K to simplices of L piecewise linearly.

Definition 6.14. Consider a simplex A with vertices (v_0, \dots, v_k) . Then the **interior** is

$$\text{int}(A) = \{\sum \lambda_i v_i \mid \sum \lambda_i = 1, \lambda_i > 0\}$$

Lemma 6.15. Say K is a simplicial complex in \mathbb{R}^n

1. $|K|$ is compact
2. Each point of $|K|$ lies in the interior of exactly 1 simplex.
3. The closed subsets of K are the subsets whose intersection with each simplex is closed.
4. If the underlying topological space ($|K|$) is connected then K is path connected.

Proof. 1. Each simplex is closed and bounded in \mathbb{R}^n . The same is true for the finite union, therefore they are compact.

2. Each point of $|K|$ lies in a simplex, so it lies in the interior of some simplex in K . (A point on the edge is in the edge, a vertex is in itself) Suppose $x \in |K|$ and $x \in \text{int}(A) \cap \text{int}(B)$ for simplices $A, B \in K$. Two simplices can only intersect in a common face, and the only face which contains interior points is the whole simplex, so $A = B$
3. If $C \subseteq |K|$ is closed, then $C \cap |A|$ is closed for each simplex A , since $|A|$ is closed in \mathbb{R}^n . Then $C \cap |A|$ is closed in $|K|$, since $|A|$ is closed in $|K|$. Since the $C = \bigcup_A C \cap |A|$, that is, C is the finite union of the intersection of C with each $|A|$, we get that C is closed in $|K|$.
4. If $x \in |K|$, there exists a $\delta > 0$ such that $|K| \cap B(x, \delta)$ is path connected because every $y \in |K| \cap B(x, \delta)$ can be joined to x by a line segment in $|K| \cap B(x, \delta)$. Hence $|K|$ is locally path connected, and so its path connected.

□

Definition 6.16. Let A be simplex with vertices (v_0, \dots, v_k) . Its **barycenter** is defined as

$$\hat{A} := \frac{1}{k+1} \sum_{i=0}^k v_i$$

Barycentric subdivision K^1 of a simplicial complex K divides K into smaller triangles by drawing a line from each vertex to the barycenter and extending it until it hits the center of the other edge.

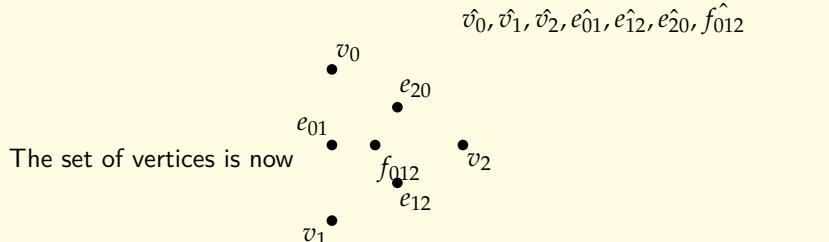
For notation, write K^k for the k th barycentric subdivision of K . Also for notation, if A is a proper face of B , write $A < B$.

Definition 6.17. If $A_0 < A_1 < \dots < A_k$ is an increasing chain of simplices in K , then

$$(\hat{A}_0, \hat{A}_1, \dots, \hat{A}_k)$$

is a simplex in K^1

Example 6.18. Given $K = \begin{array}{c} v_0 \\ \backslash \\ v_1 \end{array}$ with edges e_{01}, e_{12}, e_{20} and face f_{012} , the vertices of K^1 are



The set of vertices is now $\{v_0, v_1, v_2, \hat{f}_{012}\}$



with the barycenter at \hat{f}_{012} , and edge set all 9 edges in this figure:

$$\begin{aligned} & (\hat{v}_0, e_{01}), (\hat{v}_0, e_{12}), (\hat{v}_1, e_{01}), (\hat{v}_1, e_{20}), (\hat{v}_2, e_{02}), (\hat{v}_2, e_{12}), \\ & (\hat{v}_0, f_{012}), (\hat{v}_1, f_{012}), (\hat{v}_2, f_{012}), (e_{01}, f_{012}), (e_{12}, f_{012}), (e_{20}, f_{012}) \end{aligned}$$

Finally, there are 6 faces of K^1 , of the form

$$(\hat{v}_0, e_{01}, f_{012})$$

4.2.2025

6.2 Simplicial Approximation

Nonexample 6.19. A simplex is not a simplicial complex (unless it is a point).

If (v_0, \dots, v_k) are vertices of a simplex, we get a simplicial complex whose vertices are the nonempty subsets of $\{v_0, \dots, v_k\}$. There are $2^{k+1} - 1$ faces. Let K be the two simplex, let Y be disjoint of simplices in K . We have a function $\pi : Y \rightarrow |K|$. Y has a the disjoint topology. Then Lemma 6.15 (3) tells us that $|K|$ has the identification topology from Y and π .

Definition 6.20. The **dimension** of a simplicial complex is the maximum dimension of a simplex in K .

Example 6.21. $\dim(\text{triangle}) = 2$

Definition 6.22. The **mesh** $\mu(K)$ is the longest diameter of a simplex in K , that is, the length of the longest edge in K .

Lemma 6.23. Let K^1 be a simplicial complex.

1. Each simplex of K^1 is contained in a simplex of K .
2. $|K| = |K^1|$
3. If $\dim(K) = m$, then $\mu(K^1) \leq \frac{m}{m+1}\mu(K)$

Right now, simplices are pretty useless. We have only defined maps between topological spaces, and right now these simplices don't really provide us with any invariants. We want a more combinatorial object to describe maps between simplices.

Definition 6.24. Say K, L are simplicial complexes. A **simplicial map** $s : |K| \rightarrow |L|$ is a map that takes vertices of K to vertices of L , and is linear on every simplex of K , that is, if (v_0, \dots, v_k) are vertices of a simplex in K , then $s(\sum \lambda_i v_i) = \sum \lambda_i s(v_i)$, where $\sum \lambda_i = 1$ and $\lambda_i \geq 0$.

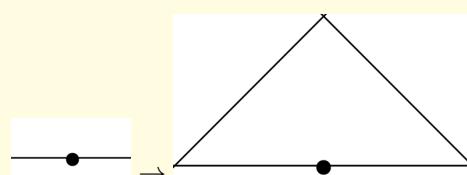
Remark 6.25. Note that if A is a simplex in K , $\dim(s(A))$ can be less than $\dim(A)$.

Example 6.26. $|K| \rightarrow \text{pt}$ is a simplicial map.

We want to "approximate" maps from K to L by simplicial maps.

Definition 6.27. Suppose $f : |K| \rightarrow |L|$ is a map. Given $x \in |K|$, $f(x)$ is in the interior of a unique simplex of L . Call this simplex the **carrier** of $f(x)$.

Example 6.28. Suppose we have a map $|K| \rightarrow |L|$ and consider a point going through this map as follows.



The carrier of $f(\bullet)$ is the bottom face of L .

Definition 6.29. Given a map $f : |K| \rightarrow |L|$, a simplicial map $s : |K| \rightarrow |L|$ is a **simplicial approximation** to f if for all $x \in |K|$, $s(x)$ is in the carrier of $f(x)$.

Note that if $f(x)$ is a vertex, $f(x) = s(x)$.

Lemma 6.30. s and f are homotopic maps

Proof. Say $|L| \subset \mathbb{R}^n$. Put $F(x, t) = (1 - t)\vec{s}(x) + t\vec{f}(x)$. Since $s(x)$ and $f(x)$ are in the same simplex, so is $F(x, t)$. \square

Theorem 6.31 (Simplicial Approximation Theorem). Say $f : |K| \rightarrow |L|$ is a map. Then if m is large enough, there exists a simplicial approximation $s : |K^m| \rightarrow |L|$ of $f : |K^m| \rightarrow |L|$

Idea of Proof. Let v be a vertex of a simplicial complex K . Define $\text{Star}(v, K)$ to be the union of interiors of the simplices containing V . This is an open neighborhood of V .

$$|L| = \bigcup_{v \in L} \text{star}(v, L)$$

is an open covering. $\{f^{-1}(\text{star}(v, L))\}_{v \in L}$ is an open cover of $|K|$.

$|K|$ is compact, so say δ is its Lebesgue number. Choose m such that $\mu(k^m) < \frac{1}{2}\delta$. For each vertex u of k^m , $\text{diam}(\text{star}(u, k^m)) < \delta$, so $\text{star}(u, k^m) \subset f^{-1}(\text{star}(v, L))$ for some v , i.e., $f(\text{star}(v, k^m)) \subset \text{star}(v, L)$. Then for all vertices u of k^m , choose a vertex v of L such that $f(\text{star}(u, k^m)) \subset \text{star}(v, L)$. Put $S(u) = v$. s sends simplices of k^m to simplices of L . Extend s linearly to $s : |k^m| \rightarrow |L|$. This map simplicially approximates f . \square

Also there is a midterm on Friday.

4.7.2025

6.3 Free Groups and Products

Today we will discuss some important group theory, specifically material on generators, relations, free groups and free products

Definition 6.32. Given a set X , a **word** is a finite product $x_1^{n_1} \dots x_k^{n_k}$ with $x_i \in X$ and $n \in \mathbb{Z}$. It's **reduced** if $n_i \neq 0$ and $x_i \neq x_{i+1}$

Definition 6.33. A subset X of a group G is a **generating set** if each $g \neq e$ can be written

$$g = x_1^{n_1} \dots x_k^{n_k}$$

where $x_i \in X$ and $n_k \in \mathbb{Z} \setminus \{0\}$. That is, each element of the group can be written as a word of X

Definition 6.34. A subset X of a group G is a **free set of generators** if each $g \neq e$ can be written in a **unique** way as a product

$$g = x_1^{n_1} \dots x_k^{n_k}$$

where $x_i \in X$ and $n_k \in \mathbb{Z} \setminus \{0\}$. If a group G has a free set of generators then it is a **free group**.

The idea behind a free group is unlike a regular group, where one might have $x_1^2 x_2 = x_3$, in a free group each element has a unique representation as a product of the generating set, and so $x_1^2 x_2 \neq x_3$, since these are two different product expressions.

Reduced words form a group $F(X)$, the free group generated by X . Let F_k denote the free group on k letters.

Example 6.35. We are given a word in F_2 and reducing it:

$$(x_1^3 x_2^2)(x_2^{-2} x_1) = x_1^3 x_2^2 x_2^{-2} x_1 = x_1^3 x_2^0 x_1 = x_1^3 x_1 = x_1^4$$

Example 6.36. $F_1 = \mathbb{Z} = \{x^n\}_{n \in \mathbb{Z}}$

You may have heard about some group being generated by a set of generators and a set of relations. For example, D_4^{14} is determined by two elements x, y , and the relations $x^4 = e$, $y^2 = e$, and $xy = yx^{-1}$. We will make this notion precise using the concept of a free group.

Proposition 6.37. Say G is a group and X a set of generators. There exists a homomorphism

$$\phi : F(X) \rightarrow G$$

$$\phi(x_1^{n_1} \dots x_k^{n_k}) = x_1^{n_1} \dots x_k^{n_k} \in G$$

ϕ is surjective since X generates G . Put $N = \ker(\phi)$, a normal subgroup of $F(X)$. Then $F(X)/N \cong G$ so N "determines" G .

Say R is a collection of words in N such that N is the smallest normal subgroup containing them. (Normal subgroup means N is closed under multiplication and if $y \in N$ and $g \in F(X)$, then $gyg^{-1} \in N$). These words determine which words in $F(X)$ become the identity, that is, which products in G are the identity in G . We'll say the pair $\langle X, R \rangle$ is a **presentation** of G .

If X is finite with elements $\{x_1, \dots, x_n\}$ and R a finite set of words with elements $\{r_1, \dots, r_m\}$, then we say G is **finitely presented**, and write

$$G = \langle x_1, \dots, x_n | r_1, \dots, r_m \rangle$$

Idea: Given $G = \langle x_1, \dots, x_n | r_1, \dots, r_m \rangle$, form a group by taking words in r and imposing relations $r_1 = e, \dots, r_m = e$.

Example 6.38. $\mathbb{Z} = \langle x | \emptyset \rangle$

Example 6.39. $C_n = \langle x | x^n \rangle$

Example 6.40. $\mathbb{Z} \times \mathbb{Z} = \langle x, y \mid xyx^{-1}y^{-1} \rangle$

Say G, H are groups. We form words $x_1 \dots x_m$ with $x_i \in G \cup H$. This time, say a word is **reduced** if x_i and x_{i+1} are never in the same group, and x_i is never the identity in G or H . Add the empty word, and get a **free product** $G * H$.

Example 6.41. $\mathbb{Z} * \mathbb{Z}$. They have generators x, y respectively, and so we have words

$$x^{n_1} y^{n_2} x^{n_3} y^{n_4} \dots$$

$$y_{n_1} x^{n_2} y^{n_3} x^{n_4} \dots$$

So $\mathbb{Z} * \mathbb{Z} = F_2$, the free group on two letters.

If $G = \langle x_1, \dots, x_m | r_1, \dots, r_n \rangle$, and $H = \langle y_1, \dots, y_k | s_1, \dots, s_r \rangle$, then

$$G * H = \langle x_1, \dots, x_m, y_1 \dots y_k | r_1 \dots r_n, s_1 \dots s_r, x_i y_j x_i^{-1} y_j^{-1} \rangle$$

Not only this, but the direct product

$$G \times H = \langle x_1, \dots, x_m, y_1 \dots y_k | r_1 \dots r_n, s_1 \dots s_r, x_i y_j x_i^{-1} y_j^{-1} \rangle$$

Notice that $x_i y_j x_i^{-1} y_j^{-1}$ means every element of G commutes with every element of H , since it implies $x_i y_j = y_j x_i$.

¹⁴The dihedral group of the square

Example 6.42. Say $G = H = \mathbb{Z}_2$, with $G = \langle x|x^2 \rangle$, $H = \langle y|y^2 \rangle$, Then

$$G * H = \langle x, y|x^2, y^2 \rangle = \{e, x, y, xy, yx, xyx, yxy, \dots\}$$

So it's an infinite group. There are no x^{-1} elements or x^2 elements since $x^{-1} = x$ and $x^2 = e$. This should show why reduced words are defined differently here.

Consider the homomorphism $\alpha : \mathbb{Z}_2 * \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$ with $\alpha(\text{word}) = \text{the number of letters } (\mod 2)$. We can say that $\mathbb{Z}_2 * \mathbb{Z}_2$ acts on \mathbb{R} , with x sending U to $-U$, and y sending $U \mapsto 2 - U$. In this sense, x reflects across 0 and y across 1. So α just tells us if the word did an even or odd number of "reflections". α is surjective.

The kernel $\ker(\alpha) = \{(xy)^n\}_{n \in \mathbb{Z}} \cong \mathbb{Z}$. Note that this includes that words of the form $yxyx\dots$, since $(yx)^n = (xy)^{-n}$. Thinking of how it acts on \mathbb{R} , the kernel is the set of translations. We have the short exact sequence^a

$$1 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 * \mathbb{Z}_2 \rightarrow \mathbb{Z} \rightarrow 1$$

and the free product is isomorphic to the infinite dihedral group D_∞

^aI don't know what this means

6.4 The Edge Group of a Complex

Say K is a connected simplicial complex. Recall that if $f : |K| \rightarrow |L|$ is a map, s is a simplicial approximation to f , and $f(x)$ is a vertex of L , then $s(x) = f(x)$

Definition 6.43 (Tree). A **tree** in K is a one-dimensional subcomplex L such that $|L|$ is path connected and simply connected.

From graph theory, we know that

Lemma 6.44. A maximal tree in K contains all the vertices.

Proof. Say T is a maximal tree. Suppose $v \notin T$ for some vertex v . Take a vertex u in T , and join u to v by a path γ in $|K|$. By simplicial approximation, we can assume γ goes along edges. Say $\gamma = uv_1 \cdots v_kv$, and that v_i is the "last" vertex in T . Add v_iv_{i+1} to T . This is still a tree, which contradicts the assumption that T was maximal. \square

We want an idea for how to find generators and relations for $\pi_1(|K|, v)$.

Idea: Define a maximal tree. It's contractible. Imagine contracting it to a point. We get a loop for each edge in $K \setminus T$. These are the generators for π . The triangles in K give relations for π_1 .

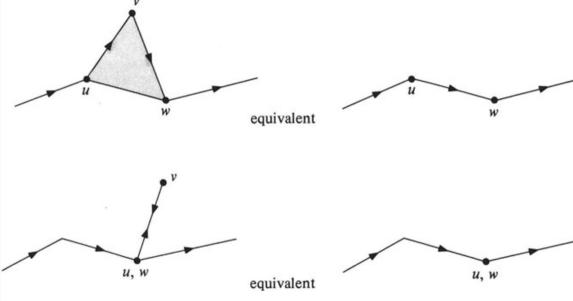
4.9.2025

Let K be a simplicial complex. We want to compute $\pi_1(|K|, v)$. By last time's idea, first, we reduce to edge paths. Then, we simplify using a maximal tree.

Definition 6.45. An edge path is a sequence of vertices v_0, v_1, \dots, v_k in K such that each $v_i v_{i+1}$ is a simplex (i.e., it is an edge or vertex). We allow $v_i = v_{i+1}$. If $v_0 = v_k = v$, we have an **edge loop** based at v .

Definition 6.46. Two edge paths are **equivalent** if one comes from the other by a sequence of the following moves:

- If three vertices uvw span a simplex, they can be replaced with uw in any edge path where they appear in order.
- Any repeated vertex uu can be replaced with u



This picture is taken from the Armstrong book.

Write the equivalence class of an edge class $v_0v_1 \dots v_k$ as $\{v_0v_1 \dots v_k\}$. The edge loops (modulo the equivalence relation) form a group $E(K, v)$

Theorem 6.47.

$$E(K, v) \cong \pi_1(|K|, v)$$

Proof. Define $\phi : E(K, v) \rightarrow \pi_1(|K|, v)$. Given an edge loop $vv_1 \dots v_{k-1}v$, divide $[0, 1]$ into k segments. Let $\alpha : I \rightarrow |K|$ be a linear extension of $\begin{cases} \alpha(0), \alpha(1) = v \\ \alpha(\frac{i}{k}) = v_i \text{ for } 1 \leq i \leq k-1 \end{cases}$. Then α is a loop in $|K|$, based at v .

If E_1, E_2 are equivalent edge loops, then α_{E_1} and α_{E_2} are homotopic in $|K|$, so we get

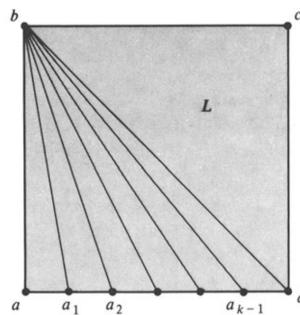
$$\phi(\{vv_1 \dots v_{k-1}v\}) = <\alpha>$$

ϕ is a homomorphism, and we want to show it is an isomorphism. To show it is surjective, say $\alpha : I \rightarrow |K|$ is any loop based at v_0 . Think of $I = [0, 1]$ as the underlying space of a simplicial complex L . By the simplicial approximation theorem, there exists a simplicial map $s : |L^m| \rightarrow |K|$ homotopic to α . Since $\alpha(0) = \alpha(1) = v$, we have $s(0) = s(1) = v$, so s gives an edge loop. $|L^m|$ has $2^m + 1$ vertices, specifically $\{\frac{1}{2^m}\}_{i=0}^{2^m}$. Put $v_i = s_i(\frac{i}{2^m})$. Then by construction,

$$\phi(\{vv_1 \dots v_{k-1}v\}) = <s> = <\alpha>$$

Hence ϕ is surjective.

To show that it's injective, we want to show that its kernel is trivial. Suppose $v_1 \dots v_{k-1}v$ is an edge loop which is null homotopic as a loop in $|K|$. We have a map $F : I \times I \rightarrow |K|$ such that $F(s, 0) = \alpha(s)$ and $F(s, 1) = F(0, t) = F(1, t) = v$. Think of $I \times I$ as $|L|$ shown in the figure



with a_i the point $(\frac{i}{k}, 0)$ and a, b, c, d the four corners, and we note that $F(a_i) = v_i$. The edge paths $E_1 =$ Bottom edge, and $E_2 = abcd$ (the rest of the edges) are equivalent in L (the simplicial complex of $I \times I$, as edge paths). By the simplicial approximation theorem, there exists an m such that we have a simplicial approximation $s : |L^m| \rightarrow |K|$ of $F : |L| \rightarrow |K|$. $|L^m|$ is a barycentric subdivision of the figure.

In $|L^m|$, E_1 and E_2 are still equivalent edge paths in $|L^m|$, so $s(E_1)$ and $s(E_2)$ are equivalent edge paths in K . Also, s sends E_2 to v_1 since F did. See that

$$F([a_i, a_{i+1}]) = [v_i, v_{i+1}]$$

When we subdivide, any new vertex between a_i and a_{i+1} goes to v_i or v_{i+1} (by the carrier property in simplicial approximation). So $s(E_1)$ is equivalent to $\{vv_1 \dots v_{k-1}v\}$ in $E(|K|, v)$. The upshot is that $vv_1 \dots v_{k-1}v$ can be transformed to an edge loop $s(E_2)$ using moves. \square

Say K is a simplicial complex and L is a path connected simply connected subcomplex which contains all the vertices of K , such as a maximal tree. Label the vertices of K as $v = v_0, v_1, \dots, v_s$.

Definition 6.48. The group $G(K, L)$ has

Generators g_{ij} for every ordered pair of vertices v_i, v_j that span a simplex (edge or vertex) in K

Relations 1. $g_{ij} = e$ if $\overline{v_i v_j}$ is a simplex in L

2. $g_{ij}g_{jk} = g_{ik}$ if $v_i v_j v_k$ spans a simplex.

$G(K, L)$ has a finite presentation.

Theorem 6.49. $G(K, L) \cong E(K, v)$

Proof. Next time. □

4.11.2025

$G(K, L) = \{\text{the free group generated by } g_{ij}\text{'s}\} / \{\text{the normal subgroup generated by relations}\}$. The number of g_{ij} 's = 2(number of edges) + (number of vertices).

$G(K, L) \cong E(K, v)$. We want to show

$$\begin{array}{ccc} & \phi & \\ G(K, L) & \xrightarrow{\quad} & E(K, v) \\ & \theta \curvearrowleft & \end{array}$$

To construct $\phi : G(K, L) \rightarrow E(K, v)$, join v to each v_i by a path E_i in L . Put $\phi(g_{ij}) = E_i \overline{v_i v_j} E_j^{-1}$. We need to know ϕ preserves relations.

a) If $\overline{v_i v_j} \subset L$, then $E_i \overline{v_i v_j} E_j^{-1} \subset L$. So it's a loop in L . But since L is simply connected, it vanishes in $\pi_1(|K|, v) \cong E(K, v)$

b) If $v_i v_j v_k$ span a simplex in K , then $\phi(g_{ij})\phi(g_{jk}) = (E_i \overline{v_i v_j} E_j^{-1}) \cdot (E_j \overline{v_j v_k} E_k^{-1}) \simeq E_i \overline{v_i v_j} \overline{v_j v_k} E_k^{-1} \simeq E_i \overline{v_j v_k} E_k^{-1} = \phi(g_{ik})$

To define $\theta : E(K, v) \rightarrow G(K, L)$, say $\Theta(vv_k v_l \cdots v_n v) = g_{0k} g_{kl} \cdots g_{n0}$. This respects the relations in $E(K, v)$, so it is well defined.

Look at $\theta\phi : G(K, L) \rightarrow G(K, L)$:

$$\theta\phi(g_{ij}) = \theta(E_i v_{ij} E_j^{-1}) = e \cdot e \cdot g_{ij} \cdot e \cdots \cdots \cdot e = g_{ij}$$

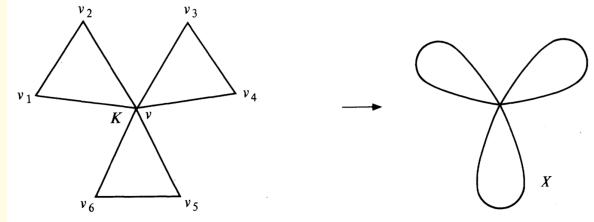
We want to show that $\phi\theta : E(K, v) \rightarrow E(K, v)$ is the identity. If $vv_k \dots v_n v$ is an edge loop, we can write it as $(E_0 \overline{v v_k} E_k^{-1})(E_k \overline{v_k v_l} E_l^{-1}) \cdots (E_n \overline{v_n v} E_0^{-1})$ in $E(K, v)$. It is enough to show that $\phi\theta(E_k \overline{v_k v_l} E_l^{-1}) = E_k \overline{v_k v_l} E_l^{-1}$.

$$\phi\theta(E_k \overline{v_k v_l} E_l^{-1}) = \phi(g_{kl}) = E_k \overline{v_k v_l} E_l$$

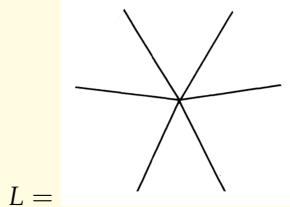
So $\phi\theta = \text{Id}$ on $E(K, v)$ □

We now have a simplified presentation of $\pi_1(|K|, v)$. Order vertices v_0, v_1, \dots, v_s . It is generated by the generators g_{ij} for an edge $\overline{v_i v_j}$ not in L with $i < j$. And we have the relation $g_{ij}g_{jk} = g_{ik}$ whenever $v_i v_j v_k$ is a triangle with $i < j < k$.

Example 6.50. Given the **bouquet** of 3 circles X , we have the triangulation K as follows



with maximal tree



Then $E(K, v) \cong G(K, L)$ is generated by g_{12}, g_{34}, g_{56} and it has no relations. Then $\pi_1(K, v) = F_3$ the free group on three generators.

Example 6.51. Consider the same bouquet of circles as last time, but with one of the circles filled in. That is, we have a 2-simplex with vertices v_1, v_2, v .

It has the same generators, but with the relations $g_{01}g_{12} = g_{02}$, and since g_{01} and g_{02} are in L , we have $g_{12} = e$, which implies $g_{12} = e$. Then

$$G(K, L) = F(g_{12}, g_{34}, g_{56}) / \langle g_{12} \rangle \cong F(g_{34}g_{56}) = F_2$$

Corollary 6.52. If K is a finite simplicial complex then $\pi_1(|K|, v)$ is finitely presented

Corollary 6.53. If $K(2)$ is the subcomplex of K consisting of all vertices, edges, and triangles (the **2-skeleton** of K), then $\pi_1(|K|, v) = \pi_1(|K(2)|, v)$.

We are motivated to find a better way of calculating these groups, since even for simple spaces we can have too many generators and relations such that calculation can be annoying. Let J, K be simplicial complexes with $J \cap K = v$ a single vertex. Then

$$\pi_1(J \cup K, v) = \pi_1(J, v) * \pi_1(K, v)$$

since any loop in this union is clearly just a free product of loops in either complex. This is a much quicker way to find the free group for something like the bouquet of n circles, but what if there is more in the intersection?

Theorem 6.54 (Von Kampen's Theorem). Suppose J, K are simplicial complexes in \mathbb{R}^n , and $J \cap K$ is a subcomplex of each. Suppose $J, K, J \cap K$ are path connected. Choose a vertex v in $J \cap K$. Put $j : J \cap K \hookrightarrow J$ and $k : J \cap K \hookrightarrow K$ inclusion maps. Then $\pi_1(|J \cup K|, v)$ is obtained from $\pi_1(|J|, v) * \pi_1(|K|, v)$ by adding relations $j * (z) = k * (z)$ for a generating set of z s that generate $\pi_1(|J \cap K|, v)$.

4.14.2025

Proof. Take a maximal tree T_0 in $J \cap K$. Extend to maximal trees T_1, T_2 for J, K respectively. Then $T_1 \cup T_2$ is a maximal tree for $J \cup K$. $\pi_1(|J \cup K|, v)$ is generated by edges in $(J \cup K)/(T_1 \cup T_2)$. These are edges in J/T_1 and K/T_2 .

But what about overcounting of edges in $(J \cap K)/T_0$?! We will add new relations to get around this: $a_{ij} = b_{ij}$ with $a_{ij} \in G(K, T_1)$ and $b_{ij} \in G(K, T_2)$ for all edges $\overline{v_i v_j} \in (J \cap K)/T_0$. \square

We have the **amalgamated free product**

$$\pi_1(|J \cup K|) = \pi_1(|J|) *_{\pi_1(|J \cap K|)} \pi_1(|K|)$$

Example 6.55. We showed before that $P^2 = (\text{mobius}) \cup D^2$. Then the intersection is the circle S^1 , and so for a generator z of $\pi_1(S^1)$, we have

$$\begin{aligned} \pi_1(P^2) &= \pi_1(\text{mobius}) * \pi_1(D^2) / \langle j_*(z) = k*(z) \rangle \\ &= (\mathbb{Z} * e) / \langle j_*(z) = k*(z) \rangle \\ &= \mathbb{Z} / \langle 2z = 0 \rangle \\ &= \mathbb{Z}/2\mathbb{Z} \\ &= \mathbb{Z}_2 \end{aligned}$$

6.5 Abstract Simplicial Complexes

All our simplices are subsets of \mathbb{R}^n , but we don't care about this fact. How can we make these objects purely combinatorial?

Definition 6.56. A finite abstract simplicial complex is a pair (V, S) with V its vertex set, and S a collection of nonempty subsets of V such that

1. Any $v \in V \implies \{v\} \in S$
2. If $x \in S$, then any nonempty subset of X is in S

Theorem 6.57. Any finite abstract simplicial complex can be realized as a simplicial complex in \mathbb{R}^n , for some n .

Proof. Say V has k elements. Say Δ is the standard $(k-1)$ -simplex in \mathbb{R}^k , i.e., with vertices

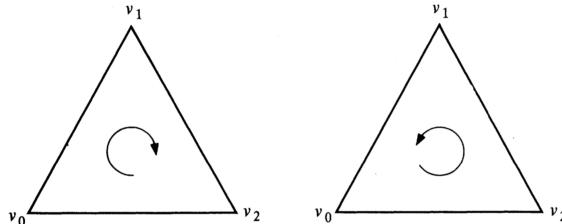
$$v_1 = (1, 0, \dots, 0), v_2 = (0, 1, 0, \dots, 0), \dots, v_k = (0, \dots, 0, 1)$$

Every nonempty subset of $\{v_1, \dots, v_k\}$ forms a simplex. Keep some vertices, and just keep the faces of Δ specified by S . \square

Example 6.58. Let $V = \{a, b, c\}$, and $S = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$. This is an abstract simplicial complex.

Since we're skipping ahead to chapter 8 of Armstrong, here is some material that we will have skipped from chapter 7:

Orientation of a simplex. We want to distinguish between the following:



First case : Keep the information of $(v_0 v_1 v_2)$ or $(v_1 v_2 v_0)$ or $(v_2 v_0 v_1)$

Second case : Keep the information of $(v_0 v_2 v_1)$ or $(v_2 v_1 v_0)$ or $(v_1 v_0 v_2)$

Definition 6.59. If A is a simplex other than a point, consider the ordering of its vertices. Say two orderings are equivalent if they differ by an even permutation. There are two equivalence classes, each called an **orientation** of A .

Definition 6.60. Say vertices of a simplex A are ordered as v_0, v_1, \dots, v_k . Say B_i is the face obtained by removing v_i . We get an ordering $(v_0, \dots, \hat{v}_i, \dots, v_k)$ ^a. If i is even, call the orientation the **right orientation** for B_i . If i is odd, the right orientation is the opposite one.

^aFor a sequence of elements (k_i) , \hat{k}_j means remove that element.

Example 6.61. Considering the first triangle we looked at in the diagram above. Choose the orientation $(v_0 v_1 v_2)$. If we remove v_i when $i = 0, 2$, then B_i has the right orientation. However, if we remove v_i , then $B_1 = (v_0, v_2)$ has the wrong orientation.

4.16.2025

7 Simplicial Homology

Idea: We have $\pi_1(X, p)$. Can we extend this to a higher degree?

- Homotopy groups $\pi_j(X, p)$, with $j \geq 0$ look at maps $S^j \rightarrow X$ up to homotopy, since π_1 is a loop S^1 . The problem with this is it's hard to compute.
- Homology groups $H_j(X)$, $j \geq 0$. The idea is a loop is trivial if it's the boundary of something. More specifically, a loop in H_1 is trivial if it's the boundary of a 2-dimensional something. That's the idea: while π_1 asks if a loop bounds a disc, H_j asks if you bound something.

Here are some revelatory remarks from the beginning of the chapter in Armstrong

- S_k has a $k + 1$ -dimensional hole. We only can recognize this fact for S^1 using the fundamental group, but this is true for larger k as well
- A loop has no boundary, but a disc does
-

7.1 Homology Groups

Rehashing our discussion about free groups from 6.3, we will consider how it's different if our group is abelian, since $H_j(x)$ will be an abelian group.

Say $(G, +)$ is an abelian group. We will assume this for the rest of the section.

Definition 7.1. A subset $X \subseteq G$ is a **generating set** if every nonzero element can be written

$$g = n_1x_1 + \cdots + n_kx_k$$

with $x_i \in X$, $n_i \in \mathbb{Z} \setminus \{0\}$, and $x_i \neq x_j$ if $i \neq j$.

Definition 7.2. A generating set is a **free set of generators** if every $g \neq 0$ can be written **uniquely** as above.

Then G is a **free abelian group**. Given a set X , an **abelian word** is a finite sum $\sum n_i x_i$ with $x_i \in X$, $n_i \in \mathbb{Z}$. It's **reduced** if $n_i \neq 0$ and $x_i \neq x_j$ if $i \neq j$. Reduced words form an abelian group, the free abelian group $FA(x)$ generated by X .

Say G is an abelian group, X a generating set. We get a homomorphism

$$\phi : FA(X) \rightarrow G \text{ by } \phi(\sum n_i x_i) = \sum n_i x_i$$

, with the sum $\sum n_i x_i$ in $\phi(\sum n_i x_i)$ a formal word, the $\sum n_i x_i$ on the right side of the equation an element of G .¹⁵ ϕ is onto.

Put $N = \ker(\phi)$. Then $G = FA(X)/N^{16}$. Put $T = \{g \in G \mid ng = 0 \text{ for some } n \neq 0\}$. This is called the **torsion subgroup**. G/T is **torsion-free**.

Suppose G is finitely generated. So is G/T . $G/T \cong \mathbb{Z}^N$ for some N , called the **rank** of G . There's an isomorphism $G \cong T \oplus \mathbb{Z}^N$, but it is not unique if $T \neq 0$.

Nonexample 7.3. Consider the group

$$G = \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \dots$$

G is infinitely generated, and then the torsion subgroup is infinite.

Let K be a finite simplicial complex.

Definition 7.4. $C_q(K)$ with $q \geq 0$ is the free abelian group generated by the oriented q -simplices in K modulo the relation $\sigma + \tau = 0$ if σ and τ are opposite orientations of the same simplex.

An element of $C_q(K)$ is a finite sum¹⁷

$$\sum \lambda_i \sigma_i$$

where $\lambda_i \in \mathbb{Z}$ and σ_i is an oriented q -simplex. We call each element a **q -dimensional chain** or **q -chain**

Remark 7.5. Note:

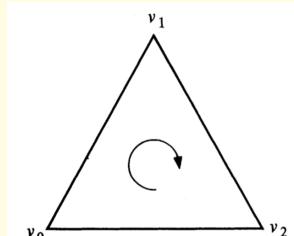
$$\lambda(-\sigma) = (-\lambda)(\sigma)$$

where $-\sigma$ denotes σ with opposite orientation.

So $C_q(K) \cong \mathbb{Z}^{n_q(K)}$ where $n_q(K)$ is the number of q -simplices.

Definition 7.6. Given an oriented q -simplex σ with vertices $(v_0 v_1 \dots v_q)$, its **boundary** is defined

$$\partial\sigma = \sum_{i=0}^q (-1)^i (v_0 \dots \hat{v}_i \dots v_q) \in C_{q-1}(K)$$



Example 7.7. Consider the simplex

Its boundary is

$$\partial(v_0 v_1 v_2) = (v_1 v_2) - (v_0 v_2) + (v_0 v_1)$$

So we get a homomorphism

$$\partial : C_q(K) \rightarrow C_{q-1}(K)$$

by $\partial(\sum \lambda_i \sigma_i) = \sum \lambda_i \partial\sigma_i$. If $q = 0$, $C_{-1}(K) = 0$, so $\partial : C_0(K) \rightarrow 0$ is a zero map.

Definition 7.8. We call

$$Z_q(K) = \ker(\partial : C_q(K) \rightarrow C_{q-1}(K))$$

the **group of q -cycles** of K , and

$$B_q(K) = \text{Im}(\partial : C_{q+1}(K) \rightarrow C_q)$$

¹⁵See how these are different?

¹⁶First isomorphism theorem

¹⁷This is a formal sum. We are treating simplices as objects which we can perform the addition and subtraction operations on, but we aren't thinking about the resultant sum as some new thing, just an abstract object.

the **group of q -boundaries** of K . These are both abelian subgroups of $C_q(K)$

Example 7.9. Let $q = 1$, so consider the 1-simplices in some simplicial complex K . Say $v_1v_1 \dots v_k$ is an edge path in K . We get a 1-chain

$$\alpha = (v_0v_1) + (v_1v_2) + \dots + (v_{k-1}v_k) \in C_1(K)$$

with

$$\begin{aligned}\partial\alpha &= ((v_1) - (v_0)) + ((v_2) - (v_1)) + \dots + ((v_k) - (v_{k-1})) \\ &= (v_k) - (v_0) \in C_0(K)\end{aligned}$$

If α is an edge loop, i.e., $v_0 = v_k$, then $\partial\alpha = 0$ and $\alpha \in Z_1(K)$ is a 1-cycle.

Lemma 7.10. $\partial^2 = 0$, that is, the composite map

$$C_{q+1}(K) \xrightarrow{\partial} C_q(K) \xrightarrow{\partial} C_{q-1}(K)$$

is the zero map.

Proof. Compute $\partial^2(\sigma)$ for a oriented $(q+1)$ -simplex $(v_0v_1 \dots v_{q+1})$.

$$\begin{aligned}\partial^2(v_0 \dots v_{q+1}) &= \partial(\partial(v_0 \dots v_{q+1})) \\ &= \partial\left(\sum_{i=0}^{q+1} (-1)^i (v_0 \dots \hat{v}_i \dots v_{q+1})\right) \\ &= \sum_{i=0}^{q+1} \sum_{j=0}^{q+1} (-1)^{j-1} (-1)^i (v_0 \dots \hat{v}_i \dots \hat{v}_j \dots v_{q+1}) + \sum_{i=0}^{q+1} \sum_{j=0}^{i-1} (-1)^j (-1)^i (v_0 \dots \hat{v}_j \dots \hat{v}_i \dots v_{q+1}) \\ &= 0\end{aligned}$$

On the second to last line, the first term is the $i < j$ scenario, and the second term is that $j < i$ scenario. \square

Corollary 7.11.

$$B_q(K) \subset Z_q(K)$$

Proof. Say $b \in B_q(K)$. Then $b = \partial c$ for some $c \in C_{q+1}(K)$. $\partial b = \partial(\partial c) = 0$, so $b \in Z_q(K)$ \square

Definition 7.12. For $q \geq 0$,

$$H_q(K) = Z_q(K) / B_q(K)$$

is the **q th Homology group**

4.18.2025

The final is in Physics 2

Last time, we saw that for K a finite simplicial complex, we have the group

$$C_q(K) = \{\sum n_i \sigma_i \mid n_i \in \mathbb{Z}, \sigma_i \text{ an oriented } q\text{-simplex}\} / ((-\sigma) = (\sigma))$$

with each element a q -chain. We also have the boundary map $\partial : C_q(K) \rightarrow C_{q-1}(K)$ and saw that $\partial^2 = 0$. Then we have an **exact sequence**¹⁸

$$0 \rightarrow C_{\dim(K)}(K) \xrightarrow{\partial} \dots \xrightarrow{\partial} C_1(K) \xrightarrow{\partial} C_0(K) \rightarrow 0$$

¹⁸more on this later

From Corollary 7.11 and Definition 7.8, we have that

$$\begin{array}{ccc} B_q(K) & \subset & Z_q(K) \subset C_q(K) \\ \text{boundaries} & & \text{cycles} & \text{chains} \end{array}$$

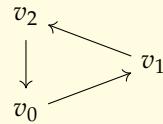
Noether proved that $H_q(K) = Z_q(K)/B_q(K)$ is a finitely generated abelian group.

Put $T_q(K) = \{\alpha \in H_q(K) \mid n\alpha = 0 \text{ for some } n \neq 0\}$. This is the torsion subgroup, a finite abelian group. It follows that $H_q(K)/T_q(K) \cong \mathbb{Z}^{\beta_q(K)}$, for some $\beta_q \geq 0$.

Definition 7.13. $\beta_q(K)$ is the q th **Betti number** of K . Note that $H_q(K) \cong T_q(K) \oplus \mathbb{Z}^{\beta_q}$ not canonically.

Definition 7.14. The element of $H_q(z)$ determined by a q -cycle z will be called the **homology class** of z denoted $[z]$. If 2 q -cycles α_1, α_2 differ by a bounding q -cycle in $B_q(K)$, we say they are **homologous**. Equivalently, $[\alpha_1] = [\alpha_2]$ in $H_q(K)$.

Example 7.15. Consider the one-dimensional simplicial complex k



The group of 0-cycles $C_0 = \text{span}((v_0), (v_1), (v_2))$.

The group of 1-cycles $C_1 = \text{span}((v_0v_1), (v_1v_2), (v_2v_0))$.

The boundary map

$$\partial : C_1(K) \rightarrow C_0(K)$$

With

$$\begin{aligned} \partial(n_1(v_0v_1) + n_2(v_1v_2) + n_3(v_2v_0)) &= n_1((v_1) - (v_0)) + n_2((v_2) - (v_1)) + n_3((v_0) - (v_2)) \\ &= (n_3 - n_1)(v_0) + (n_1 - n_2)(v_1) + (n_2 - n_3)(v_2) \end{aligned}$$

And we have

$$0 \rightarrow C_1(K) \xrightarrow{\partial} C_0(K) \rightarrow 0$$

Finally, the kernel is

$$\begin{aligned} Z_1(K) &= \ker(\partial) = \{(n_1(v_0v_1) + n_2(v_1v_2) + n_3(v_2v_0)) \mid n_1 = n_2 = n_3\} \\ &= \mathbb{Z}((v_0v_1) + (v_1v_2) + (v_2v_0)) \end{aligned}$$

$B_1(K) = 0$, so the first homology group $H_1(K) = Z_1(K)/B_1(K) \cong \mathbb{Z}$.

The zeroth homology group $H_0(K)$ is confusing to determine. We know $Z_0(K) = \ker(\partial : C_0(K) \rightarrow C_{-1}(K)) = C_0(K)$. The image of the boundary map $B_0(K) = \{m_1(v_0) + m_2(v_1) + m_3(v_2) \mid m_1 + m_2 + m_3 = 0\}$. We want to know what $H_0(K) = Z_0(K)/B_0(K)$ is.

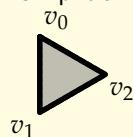
Define $\phi : Z_0(K) \rightarrow \mathbb{Z}$ by

$$\phi(m_1(v_0) + m_2(v_1) + m_3(v_2)) = m_1 + m_2 + m_3$$

ϕ is surjective, and $\ker(\phi) = B_0(K)$. Then

$$H_0(K) = Z_0(K)/B_0(K) = Z_0(K)/\ker(\phi) \cong \mathbb{Z}$$

Example 7.16. Now consider the 2-dimensional simplicial complex:



$(v_0v_1) + (v_1v_2) + (v_2v_0)$ is a 1-cycle, and it's a boundary since $\partial(v_0v_1v_2) = (v_1v_2) - (v_0v_2) + (v_1v_2)$.

So the cycle is **null-homologous**, meaning it gives you nothing in homology. We will see that $H_0(K) = \mathbb{Z}$ and that $H_1(K) = H_2(K) = 0$.

Remark 7.17. If $q > \dim(K)$, then $H_q(K) = 0$.

Theorem 7.18. $H_0(K) \cong \mathbb{Z}^r$, where r is the number of connected components.

Proof.

$$C_1(K) \xrightarrow{\partial} C_0(K) \rightarrow 0$$

$$Z_0(K) = C_0(K) = \sum n_i(v_i)$$

Say the connected components of K are labeled c_1, c_2, \dots, c_r . Put

$$\phi : C_0(K) \rightarrow \mathbb{Z}^r$$

by $\phi(\sum n_i(v_i)) = \{\text{sum of } n_i\text{s with } v_i \text{ in } c_1, \text{sum of } n_i\text{s with } v_i \text{ in } c_2, \dots, \text{sum of } n_i\text{s with } v_i \text{ in } c_n\}$.

ϕ is **surjective**: Say $z \in \ker(\phi)$. In any component c_j , the sum of n_i s for $v_i \in c_j$ is zero. Choose $v_j \in c_j$. For each $v_i \in c_j$, choose a simplicial (edge) path p_i from v_j to v_i . Put

$$\alpha = \sum_j \sum_{v_i \in c_j} n_i p_i \in C_1(K)$$

$$\partial\alpha = \sum_j \sum_{v_i \in c_j} n_i((v_i) - (v_j)) = \sum_j \sum_{v_i \in c_j} n_i(v_i) = \sum n_i v_i$$

So the kernel $\ker(\phi) \subseteq B_0(K)$. Also, $B_0(K) \subseteq \ker(\phi)$, since if e is an edge then it lives in one component, so $\phi(\partial e) = \vec{0}$. So $H_0(K) = Z_0(K)/B_0(K) = C_0(K)/\ker(\phi) \cong \mathbb{Z}^r$. \square

4.21.2025

Theorem 7.19 (Homology of cones). Let L be a simplicial complex. Then $H_q(CL) = \begin{cases} \mathbb{Z} & \text{if } q = 0 \\ 0 & \text{if } q > 0 \end{cases}$

Proof. Idea: CL deformation retracts to v . Also, if X is a cycle in L , then X is the boundary of CX , the cone over X .

If $q = 0$, we're done, since CL is connected and so $H_0(CL) \cong \mathbb{Z}$. Suppose $q > 0$. Define $d : C_q(K) \rightarrow C_{q+1}(K)$ by

$$d(\sigma) = \begin{cases} (vv_0 \dots v_q) & \text{if } \sigma = (v_0, \dots, v_q) \text{ is an oriented } q\text{-simplex in } K = CL \\ 0 & \text{otherwise} \end{cases}$$

Lemma 7.20. $\partial d(\sigma) = \sigma - d(\partial\sigma)$ for any oriented q -simplex in $K = CL$.

Proof. If $\sigma = (vv_0 \dots v_q)$ with $(v_0 \dots v_q)$ a simplex in L , then $d(\sigma) = 0$.

$$\begin{aligned} d(\partial\sigma) &= d((v_0 \dots v_q) - \sum_{i=0}^q (-1)^i (vv_0 \dots \hat{v}_i \dots v_q)) \\ &= d(v_0 \dots v_q) \quad (\text{all the other terms go to zero}) \\ &= (vv_0 \dots v_q) \end{aligned}$$

So in this case $\partial d(\sigma) = 0 \stackrel{?}{=} \partial - d(\partial\sigma) = \sigma - \sigma = 0$, so it holds. Now for the $\sigma \in L$ case, let $\sigma = (v_0 \dots v_q)$, get

$$\begin{aligned} \partial d(\sigma) &= \partial(vv_0 \dots v_q) \\ &= (v_0 \dots v_q) - \sum (-1)^i (vv_0 \dots \hat{v}_i \dots v_q) \\ &= \sigma - d(\partial\sigma) \end{aligned}$$

So the lemma holds. \square

We have that $\sigma d(c) = c - d(\partial c)$ for any $c \in C_q(k)$. Suppose $z \in Z_q(K)$ is a q -cycle. Then

$$\partial d(z) = z - d(\partial z) = z - d(0) = z$$

that is, $z = \partial(d(z))$. Hence $z \in B_q(K)$, so $H_q(K) = Z_q(K)/B_q(K) = 0$. \square

Definition 7.21. The k -skeleton of a simplicial complex K is the union of the simplices with dimension $\leq k$.

Observe that $H_q(K)$ only depends on the $(q+1)$ -skeleton of K , since $H_q(K)$ depends on C_{q+1}, C_q, C_{q-1} , and ∂ . Put

$$\begin{aligned} \Delta^n &= \text{standard } n\text{-simplex in } \mathbb{R}^{n+1} \\ &= \{(v_0, \dots, v_n) \mid v_i \geq 0, \sum_{i=0}^n v_i = 1\} \end{aligned}$$

Definition 7.22. Let Σ^n denote the simplices in boundary of Δ^{n+1} , an $n+1$ -simplex. Then $|\Sigma^n| = S^n$.

Theorem 7.23. $H_q(\Sigma^n) = \begin{cases} \mathbb{Z} & \text{if } q = 0, n \\ 0 & \text{otherwise} \end{cases}$

Proof. Note that $\Delta^{n+1} = C\Delta^n$, so

$$H_q(\Delta^{n+1}) = \begin{cases} \mathbb{Z} & \text{if } q = 0 \\ 0 & \text{if } q > 0 \end{cases}$$

The n -skeleton of Δ^{n+1} is Σ^n . So $H_q(\Sigma^n) \cong H_q(\Delta^{n+1})$ if $q < n-1$. Also, $H_q(\Sigma^n) = 0$ if $q > n$. So we need to find $H_n(\Sigma^n)$. We have

$$0 \rightarrow C_n(\Sigma) \xrightarrow{\partial} C_{n-1}(\Sigma) \rightarrow \dots$$

where the first term is 0 because there are no $(n+1)$ -simplices in Σ^n . So $B_n(\Sigma) = 0$ and $H_n(\Sigma) = Z_n(\Sigma)/0 = Z_n(\Sigma)$. Note that $Z_n(\Sigma) \cong Z_n(\Delta^{n+1}) = B_n(\Delta^{n+1})$. Since

$$0 \rightarrow C_{n+1}(\Delta^{n+1}) \xrightarrow{\partial} C_n(\Delta^{n+1}) \rightarrow \dots$$

$$\begin{matrix} \mathbb{Z} \\ \parallel \\ \mathbb{Z} \end{matrix}$$

$C_{n+1}(\Delta^{n+1})$ is generated by a single $(n+1)$ -simplex $\sigma = (v_0 \dots v_n)$, and $B_n(\Delta^{n+1}) = \mathbb{Z}\partial(\sigma) = \sum (-1)^i (v_0 \dots \hat{v}_i \dots v_n)$, the sum of the faces of Σ^n . So $H_n(\Sigma^n) \cong \mathbb{Z}$, generated by

$$\sum_{i=0}^n (-1)^i (v_0 \dots \hat{v}_i \dots v_n)$$

\square

$H_0(|K|) = \mathbb{Z}^r$, where r is the number of connected components. $H_1(|K|)$ is an abelian group, while π_1 can be non abelian.

Definition 7.24 (The commutator subgroup). If G is a group, the **commutator subgroup** consists of products

$$(g_1\hat{g}_1g_1^{-1}\hat{g}_1^{-1})(g_2\hat{g}_2g_2^{-1}\hat{g}_2^{-1}) \cdots (g_n\hat{g}_n g_n^{-1}\hat{g}_n^{-1})$$

for $g_1, \dots, g_n, \hat{g}_1, \dots, \hat{g}_n \in G$. This is a normal subgroup of G .

$$g(g_1\hat{g}_1g_1^{-1}\hat{g}_1^{-1} = (gg_1g^{-1})(g\hat{g}_1g^{-1})(gg_1g^{-1})(g\hat{g}_1g^{-1})^{-1}$$

Note that the **commutator** $[g, \hat{g}] = g\hat{g}g^{-1}\hat{g}^{-1} = e \Leftrightarrow g\hat{g} = \hat{g}g$. The quotient $G/[G, G]$ is abelian, and is called the **abelianization** of G .

Example 7.25. Let $G = F_n$. $G/[G, G] \cong \mathbb{Z}^n$

Definition 7.26. Let K be a simplicial complex. An **elementary 1-cycle** is the cycle coming from an oriented simple polygonal loop in K .

4.23.2025

Assume K is connected.

Theorem 7.27. The abelianization of $\pi_1(|K|, v)$ is $H_1(K)$

Proof. Given an edge loop $\alpha = vv_1 \dots v_kv$, we get a 1-cycle

$$\begin{aligned} Z(\alpha) &= (vv_1) + (v_1v_2) + \cdots + (v_kv) \in Z_1(K) \\ &\quad (\text{omitting } (v_iv_{i+1}) \text{ if } v_i = v_{i+1}) \end{aligned}$$

Recall that $E(K, v)$ is generated by edge loops, quotient by moves that are equivalent (see definition 6.46).

Use $\pi_1(|K|, v) \cong E(K, v)$. We want to show that

$$E(K, v)/[E(K, v), E(K, v)] \cong H_1(K)$$

Consider the map

$$z : \{\text{Edge loops at } v\} \rightarrow Z_1(K) = 1\text{-cycles}$$

by $z(vv_1 \dots v_kv) = (vv_1) + (v_1v_2) + \cdots + (v_kv)$. If α, β are edge loops based at v , and $\alpha \sim \beta$, then $z(\alpha) - z(\beta) \in B_1(K)$.

We get a homomorphism $\phi : E(K, v) \rightarrow H_1(K)$ by saying that if $\langle \alpha \rangle \in E(K, v)$, then $\phi(\langle \alpha \rangle) = z(\alpha) \bmod B_1(K)$. We want to show that ϕ is surjective and that $\ker(\phi) = [E(K, v), E(K, v)]$.

ϕ is **surjective**: $Z_1(K)$ is generated by elementary 1-cycles α (Homework 12). We get an edge loop such that $z(\alpha) = x$. So $\phi(\langle \alpha \rangle) = \langle x \rangle \in H_1(K)$. $\ker(\phi) = [E(K, v), E(K, v)]$. Suppose α is an edge loop at v such that $\phi(\langle \alpha \rangle) = 0$ in $H_1(K)$. Then $z(\alpha) = \partial(\xi)$ for some 2-chain ξ .

Since $H_1(K)$ is abelian, the kernel of ϕ contains the commutator subgroup. We want to show that if α is an edge loop for which $z(\alpha)$ is a bounding cycle, then $\{\alpha\}$ lies in the commutator subgroup of $E(K, v)$. Say $\alpha = vv_1 \dots v_kv$, and $\xi = \sum \lambda_i \sigma_i$ with $\lambda_i \in \mathbb{Z}$, σ_i oriented 2-simplices. $(vv_1) + \cdots + (v_kv) = \partial(\sum \lambda_i \sigma_i)$. Say $\sigma_i = (a_i b_i c_i)$, and for each i choose an edge path from v to a_i . The edge loop $\gamma a_i b_i c_i \gamma^{-1}$ is equivalent to the trivial edge loop v , and therefore so is the product

$$\beta = \prod_i (\gamma_i a_i b_i c_i \gamma_i^{-1}) \lambda_i$$

Then $\langle \alpha \rangle = \langle \alpha \beta^{-1} \rangle$ in $E(K, v)$. $z(\gamma_i a_i b_i c_i \gamma_i^{-1}) = \partial(a_i b_i c_i) = \partial \sigma_i$. Get $z(\alpha \beta^{-1}) = 0 \in Z_1(K)$.

Now the only way an edge loop can map to the zero 1-cycle under $\alpha \mapsto z(\alpha)$ is if, whenever an oriented edge (a, b) occurs n times in it, then (b, a) also occurs n times. We had an isomorphism $\Theta : E(K, v) \rightarrow G(K, L)$ with L a maximal tree in K . The generators of $G(K, L)$ are the edges in K/L . Under Θ the equivalence class of a loop like $\alpha \beta^{-1}$ will map to a product of group elements. Therefore if we apply Θ and then we abelianize $G(K, L)$, it vanishes

"After we abelianize...

$$\cdots g_{ij} \cdots g_{ij}^{-1} \cdots$$

These two will kill each other. That is the devious trick!"

- John Lott

Then $\Theta(<\alpha\beta^{-1}>) \subset [G(K,L), G(K,L)]$. And since Θ is a group isomorphism, $<\alpha> = <\alpha\beta^{-1}> \in [E(K,v), E(K,v)]$, so $\ker(\phi) \subset [E(K,v), E(K,v)]$.

ϕ is onto, so $E(K,v)/\ker(\phi) = E(K,v)/[E(K,v), E(K,v)] \cong H_1(K)$

□

7.2 Simplicial Maps

Let K, L be simplicial complexes and $s : K \rightarrow L$ a simplicial map (it sends simplices to simplices linearly). We want to define $s_{q*} : H_q(K) \rightarrow H_q(L)$.

First, define $s_q : C_q(K) \rightarrow C_q(L)$. Say $(v_0 \dots v_q)$ is an oriented q -simplex of K . Note that $\dim(s(v_0 \dots v_q)) \leq q$. Define

$$s_q(\sigma) = \begin{cases} (s(v_0), \dots, s(v_q)) & \text{in } s(v_0), \dots, s(v_q) \text{ all distinct} \\ 0 & \text{otherwise} \end{cases}$$

Extend it linearly to $C_q(K)$.

Lemma 7.28. $\partial s_q = s_{q-1} \partial : C_q(K) \rightarrow C_q(L)$, that is,

$$C_q(K) \xrightarrow{\partial} C_{q-1}(K)$$

$$\begin{array}{ccc} & & \\ s_q \downarrow & & \downarrow s_{q-1} \\ & & \end{array}$$

$$C_q(L) \xrightarrow[\partial]{} C_{q-1}(L)$$

commutes.

Proof. It's enough to show

$$\partial s_q(\sigma) = s_{q-1}(\partial \sigma)$$

This is true if $s(v_0), \dots, s(v_q)$ are all distinct. Suppose $s(v_j) = s(v_k)$ for some $j < k$. By definition, $s_q(\sigma) = 0$, which means $\partial s_q(\sigma) = 0$

$$\begin{aligned} s_{q-1}(\partial \sigma) &= s_{q-1}(\sum (-1)^i (v_0 \dots \hat{v}_i \dots v_q)) \\ &= \sum (-1)^i s_{q-1}(v_0 \dots \hat{v}_i \dots v_q) = 0 \end{aligned}$$

If $i \notin \{j, k\}$, then $s_{q-1}(v_0 \dots \hat{v}_i \dots v_q) = 0$ since we still have v_j and v_k , and $s(v_j) = s(v_k)$. So

$$s_{q-1}(\partial \sigma) = (-1)^j (v_0 \dots \hat{v}_j \dots v_q) + (-1)^k (v_0 \dots \hat{v}_k \dots v_q)$$

But

$$\begin{aligned} s_{q-1}(v_0 \dots \hat{v}_j \dots v_q) &= (s(v_0) \dots s(\hat{v}_j) \dots s(v_q)) \\ &= (s(v_0) \dots s(\hat{v}_j) \dots s(v_k) \dots s(v_q)) \\ &= (s(v_0) \dots s(\hat{v}_j) \dots s(v_j) \dots s(v_q)) \\ &= (-1)^{k-j-1} (s(v_0) \dots s(v_j) \dots s(\hat{v}_k) \dots s(v_q)) \end{aligned}$$

So $s_{q-1}(\partial \sigma) = [(-1)^j + (-1)^k (-1)^{k-j-1}] s_{q-1}(v_0 \dots \hat{v}_j \dots v_q) = 0$

□

Corollary 7.29.

$$s_q(Z_q(K)) \subset Z_q(L) \text{ and } s_q(B_q(K)) = B_q(L)$$

Proof. ▀ Say $z \in Z_q(K)$, so $\partial z = 0$. Then $\partial s_q(z) = s_{q-1}(\partial z) = s_{q-1}(0) = 0$, so $s_q(z) \in Z_q(L)$

▀ Say $b \in B_q(K)$. Then $b = \partial c$ for some $c \in C_{q-1}(K)$. Then $\partial s_{q+1}(c) = s_q(\partial c) = s_q(b)$, so $s_q(b) \in B_q(L)$. □

Corollary 7.30. $s_{q*} : H_q(K) \rightarrow H_q(L)$ is well-defined.

4.25.2025

Let $s : K \rightarrow L$. We defined $s_q : C_q(K) \rightarrow C_q(L)$, and showed $\partial(S_q C) = S_{q-1}(\partial C)$. s_q sends $Z_q(K)$ and $B_q(K)$ to $Z_q(L)$ and $B_q(L)$ respectively. We get an induced homomorphism

$$s_* : H_q(K) \rightarrow H_q(L)$$

Explicitly, say $\alpha \subset H_q(K)$. Choose $z \in Z_q(K)$ such that (the homology class of z) $[z] = \alpha$. Choose $z \in Z_q(K)$ such that $[z] = \alpha$. Look at $s_q(z) \in Z_q(L)$. Then $s_*(\alpha) = [s_q(z)] \in H_q(L)$. Suppose z' is another choice such that $[z'] = \alpha$. Then $[z' - z] = 0$ in $H_q(K)$, so $z' - z \in B_q(K)$.

$$s_q(z' - z) \in B_q(L)$$

so $[s_q(z')] - [s_q(z)] = [s_q(z' - z)] = 0$ in $H_q(L)$.

7.3 Homological Algebra

$$\cdots \xrightarrow{\partial} C_q(K) \xrightarrow{\partial} C_{q-1}(K) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0(K) \rightarrow 0$$

This is called the **Chain Complex** of K , written $C(K)$. "Complex" means $\partial \circ \partial = 0$.

Definition 7.31. A chain map $\phi : C(K) \rightarrow C(L)$ is a collection of homomorphisms $\phi : C_q(K) \rightarrow C_q(L)$ such that

$$\partial \circ \phi_q = \phi_{q-1} \circ \partial$$

And so we have the following commutative diagram.

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\partial} & C_q(K) & \xrightarrow{\partial} & C_{q-1}(K) & \xrightarrow{\partial} & \cdots \xrightarrow{\partial} C_0(K) \longrightarrow 0 \\ & & \downarrow \phi_q & & \downarrow \phi_{q-1} & & \downarrow \phi_0 \\ \cdots & \xrightarrow{\partial} & C_q(L) & \xrightarrow{\partial} & C_{q-1}(L) & \xrightarrow{\partial} & \cdots \xrightarrow{\partial} C_0(L) \longrightarrow 0 \end{array}$$

we get a homomorphism $\phi_* : H(K) \rightarrow H(L)$, that is,

$$\phi_{*q} : H_q(K) \rightarrow H_q(L)$$

for all degrees q .

Given $\phi : C(K) \rightarrow C(L)$ and $\psi : C(L) \rightarrow C(M)$, the composition $\psi \circ \phi$ is also a chain map and

$$(\psi \circ \phi)_* = \psi_* \circ \phi_* : H(K) \rightarrow H(M)$$

See (Homework 13).

8 Degree and Lefschetz Number

This section is about some applications of Homology theory.

Question 8.1 (1920s)

If $|K|$ and $|L|$ are homeomorphic, is $H_q(K)$ isomorphic to $H_q(L)$?

Let K be a simplicial complex, and K^1 its barycentric subdivision.

Theorem 8.2. There's a chain map $\lambda : C(K) \rightarrow C(K^1)$ such that

$$\lambda_* : H(K) \rightarrow H(K^1)$$

is an isomorphism.

Example 8.3. If K is the oriented 2-simplex, λ maps it to K^1 with the induced orientation.

Conjecture 8.4 (Hauptvermutung)

If $|K|$ is homeomorphic to $|L|$, then some subdivision of K is simplicially isomorphic to some subdivision of L .

But this was disproved by a counterexample in the 1960s. Before that and after Hauptvermutung, came the simplicial approximation theorem, and the field went in a different direction.

Theorem 8.5. Any map $f : |K| \rightarrow |L|$ induces a map

$$f_* : H(K) \rightarrow H(L)$$

Proof. Say $s : K^m \rightarrow L$ is a simplicial approximation to f . We get $H(K) \cong H(K^m) \rightarrow H(L)$. Call this f_* , and show this is independent of the choice of simplicial approximation s . This proof is difficult and we will skip it. \square

Theorem 8.6. If $f, g : |K| \rightarrow |L|$ are homotopic, then

$$f_* = g_* : H(K) \rightarrow H(L)$$

Corollary 8.7. If $|K|$ and $|L|$ are homotopy equivalent, then $H(K) \cong H(L)$. In particular, if they are homeomorphic then they have the same homology, that is,

$$H(K) \cong H(L)$$

Proof. Say $f : |K| \rightarrow |L|$ is a homotopy equivalence. Say $g : |L| \rightarrow |K|$ is a homotopy inverse, that is, $g \circ f \simeq \text{Id}_{|K|}$ and $f \circ g \simeq \text{Id}_{|L|}$. Then we get $g_* \circ f_* = \text{Id}_{H(K)}$ and $f_* \circ g_* = \text{Id}_{H(L)}$. Hence $f_* : H(K) \rightarrow H(L)$ is an isomorphism. \square

What is this proof really saying? It's telling us that homology is another one of our invariants under a homeomorphism, along with the fundamental group and Hausdorff and more.

Corollary 8.8. If $m \neq n$, then S^n is not homotopy equivalent to S^m .

Proof. Suppose these were homeomorphic. Then $\mathbb{R}^n \setminus \{0\}$ is homeomorphic to $\mathbb{R}^n \setminus f(0)$. But $\mathbb{R}^m \setminus 0 \simeq S^{m-1}$, and $\mathbb{R}^n \setminus f(0) \simeq S^{n-1}$. But S^{m-1} is not homotopy equivalent to S^{n-1} , so we have a contradiction. \square

Now we can actually prove the general case of the Brouwer fixed point theorem

Theorem 8.9 (Brouwer Fixed Point Theorem). Any $f : B^n \rightarrow B^n$ has a fixed point.

Proof. The proof is the same as before, but with homology instead of π_1 . Suppose f has no fixed points. Assume we have no fixed points, and say $g(x)$ is where the line segment from $f(x)$ to x intersects the boundary S^{n-1} . Then $g|_{S^{n-1}} = \text{Id}|_{S^{n-1}}$. Then we have the composition

$$S^{n-1} \xrightarrow{i} B^n \xrightarrow{g} S^{n-1}$$

where $g \circ i = \text{Id}_{S^{n-1}}$. Then $g_* \circ i_* = \text{Id}_{H(S^{n-1})}$, which means the maps

$$H_{n-1}(S^{n-1}) \xrightarrow{i_*} H_{n-1}(B^n) \xrightarrow{g_*} H_{n-1}(S^{n-1})$$

Where $H_{n-1}(S^{n-1}) = \mathbb{Z}$, and $H_{n-1}(B^n) = 0$. So composition $g_* \circ i_* = 0$, which is a contradiction. \square

It turns out that the Nash Equilibrium is a special case of this.

8.1 The Euler-Poincare formula

Say K is a simplicial complex. α_q = number of q -simplices.

Definition 8.10. The **Euler characteristic** of K is $\chi(K) = \sum_{q=0}^n (-1)^q \alpha_q = \alpha_0 - \alpha_1 + \alpha_2 - \cdots$

Example 8.11. Let $X = S^2$. The triangulation of S^2 is just a hollow tetrahedron. $\chi(K) = 4 - 6 + 4 = 2$.

Example 8.12. $\chi(T^2) = 0$, $\chi(2\text{-torus}) = -2$, $\chi(\text{pretzel aka 3-torus}) = -4$.

Recall that $H_q(K)$ has a torsion subgroup $T_q(K)$ and that $H_q(K)/T_q(K) \cong \mathbb{Z}^{\beta_q}$, for some β_q the q th Betti number of K .

Claim: $\chi(K) = \sum_{q=0}^n (-1)^q \beta_q$.

Corollary 8.13. $\chi(K)$ is a homotopy invariant of $|K|$.

Definition 8.14. Let V be the set

$$V = \left\{ \sum r_i \sigma_i \mid r_i \in \mathbb{Q}, \sigma_i \text{ oriented } q\text{-simplex} \right\}$$

This set forms a vector space in a natural way. Consider the subspace W where its span: $\text{span}_{\mathbb{Q}}\{\sigma + \tau \mid \sigma, \tau \text{ are the same simplex but differ by orientation}\}$. Call the quotient space $V/W = C_q(K, \mathbb{Q})$.

Then $C_q(K, \mathbb{Q})$ is a vector space over \mathbb{Q} with dimension α_q , the number of q -simplices in K . We get $\partial : C_q(K, \mathbb{Q}) \rightarrow C_{q-1}(K, \mathbb{Q})$ with

$$Z_q(K, \mathbb{Q}) = \ker(\partial)$$

$$B_q(K, \mathbb{Q}) = \text{Im}(\partial)$$

$$H_q(K, \mathbb{Q}) = Z_q(K, \mathbb{Q}) / B_q(K, \mathbb{Q})$$

The homology group is called the **q th homology group of K with rational coefficients**

Theorem 8.15. $\dim_{\mathbb{Q}} H_q(K, \mathbb{Q}) = \beta_q$

4.28.2025

We will prove that the q th betti number β_q is the dimension of $H_q(K, \mathbb{Q})$ as a vector space over \mathbb{Q} .

Proof.

$$H_q(K) \cong T_q \oplus \mathbb{Z}^{\beta_q}$$

Choose a minimal set of generators $[z_1], \dots, [z_{\beta_q}], [w_1], \dots, [w_{\gamma_q}]$, where the $[z_k]$ s generate the free part and the $[w_i]$ s have finite order so are part of the torsion subgroup. Given a q -cycle $z \in Z_q(K)$, we get $[z] \in H_q(K)$ and $\{z\} \in H_q(K, \mathbb{Q})$.

Claim 1: $\{z_1\}, \dots, \{z_{\beta_q}\}, \{w_i\}, \dots, \{w_{\gamma_q}\}$ span $H_q(K, \mathbb{Q})$.

A q -cycle z with integer coefficients can be thought of as having rational coefficients, and therefore determines an element of $H_q(K, \mathbb{Q})$.

Suppose

$$\frac{a_1}{b_1} \sigma_1 + \dots + \frac{a_s}{b_s} \sigma_s \in Z_q(K, \mathbb{Q})$$

where σ_i s are oriented q -simplices with $a_i, b_i \in \mathbb{Z}$. Then

$$\begin{aligned} z &= \frac{1}{b_1 \dots b_s} \times (\text{a cycle with integer coefficients}) \\ &= \frac{1}{b_1 \dots b_s} \times (\text{a linear combination of } z_i \text{s, } w_i \text{s modulo the boundary } B_q(K, \mathbb{Q})) \end{aligned}$$

Claim 2: $\{w_i\} = 0$ in $H_q(K, \mathbb{Q})$.

For some $m \in \mathbb{Z}$, $m[w_i] = 0$ in $H_q(K)$, that is, $mw_i \in B_q(K)$, so $w_i \in B_q(K, \mathbb{Q})$, so $\{w_i\} = 0$ in $H_q(K, \mathbb{Q})$.

Claim 3: $\{z_1\}, \dots, \{z_{\beta_q}\}$ are linearly independent.

Suppose $\sum \frac{a_i}{b_i} \{z_i\} = 0$ in $H_q(K, \mathbb{Q})$. Then $\sum \frac{a_i}{b_i} z_i \in B_q(K, \mathbb{Q})$, so there exists N such that $\sum N \frac{a_i}{b_i} \in B_q(K)$ with $N \frac{a_i}{b_i} \in \mathbb{Z}$ for all i ,¹⁹ so $\sum N \frac{a_i}{b_i} [z_i] = 0$ in $H_q(K)$. Hence, $N \frac{a_i}{b_i} = 0$ for all i , so $\frac{a_i}{b_i} = 0$.

□

Theorem 8.16 (Euler-Poincare Formula).

$$\chi(K) = \sum (-1)^q \beta_q$$

Proof. We will work over \mathbb{Q} , with $B_q \subseteq Z_q \subseteq C_q$ rational vector spaces.

$$\begin{aligned} C_q &\simeq Z_q \oplus C_q / Z_q \\ &\simeq B_q \oplus Z_q / B_q \oplus C_q / Z_q \end{aligned}$$

with $C_{q+1} \cong B_{q+1} \oplus Z_{q+1} / B_{q+1} \oplus C_{q+1} / Z_{q+1}$

Lemma 8.17. $\partial : C_{q+1} / Z_{q+1} \rightarrow B_q$ is an isomorphism.

The map is surjective because it is a boundary map by definition. We just need to show injectivity. Suppose $\partial(C_{q+1} \bmod Z_{q+1}) = 0$. Then $\partial C_{q+1} = 0$, so C_{q+1} is a cycle, that is, $C_{q+1} \in Z_{q+1}$, so

$$C_{q+1} (\bmod Z_{q+1}) = 0 (\bmod Z_{q+1})$$

Hence $\dim_{\mathbb{Q}} C_q / Z_q = \dim_{\mathbb{Q}} B_{q-1}$, hence

$$\begin{aligned} \chi(K) &= \sum (-1)^q \dim_{\mathbb{Q}} C_q \\ &= \sum (-1)^q [\dim_{\mathbb{Q}} B_q + \dim_{\mathbb{Q}} Z_q / B_q + \dim_{\mathbb{Q}} C_q / Z_q] \\ &= \sum (-1)^q [\dim_{\mathbb{Q}} B_q + \dim_{\mathbb{Q}} H_q(K, \mathbb{Q}) + \dim_{\mathbb{Q}} B_{q-1}] \\ &= \sum (-1)^q \dim H_q(K, \mathbb{Z}) \\ &= \sum (-1)^q \beta_q \end{aligned}$$

□

Example 8.18. Consider the space S^2 . Its triangulation K is the tetrahedron, and $\chi = 2$.

$$H_q(S^2) = \begin{cases} \mathbb{Q} & \text{if } q = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

¹⁹This is just some really big N

$\beta_0 = 1, \beta_1 = 0, \beta_2 = 1$, so $\chi(K) = 2$.

Lemma 8.19. If $\dim(K) = n$, then $H_n(K)$ is a free abelian group, isomorphic to \mathbb{Z} .

Proof.

$$H_n(K) = \ker(\partial : C_n(K) \rightarrow C_{n-1}(K)) / \text{Im}(\partial : C_{n+1}(K) \rightarrow C_n(K)) = Z_n(K) \subseteq C_n(K)$$

□

Example 8.20. Consider the torus T^2 .

$$\begin{aligned} M(T^2) &= \alpha_1 - \alpha_2 + \alpha_3 = 0 \\ &= \beta_0 - \beta_1 - \beta_2 \end{aligned}$$

1. β_0 is the number of connected components = 1.
2. $H_1(K)$ is the abelianization of $\pi_1(T^2) \cong \mathbb{Z}^2$, so $H_q(T^2) = \mathbb{Z}^2$, and $\beta(T^2) = 2$.
3. $0 = \chi(K) = 1 - 2 + \beta_2(T^2)$, so $\beta_2 = 1$.

$$\text{Then } H_q(K) = \begin{cases} \mathbb{Z} & \text{if } q = 0, 2 \\ 0 & \text{if } q = 1 \end{cases}$$

4.30.2025

Example 8.21. Consider the genus two surface X . It's connected, so $H_0(X) = \mathbb{Z}$.

$$\pi_1(X) = \langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle$$

$H_1(X) = \mathbb{Z}^4$, since the relation goes away once you abelianize.

$H_2(X) = \mathbb{Z}$, and $\chi(X) = 1 - 4 + 1 = -2$

8.1.1 Calculations

Here are some calculations of the Euler characteristics of common spaces.

We have the following table

Genus	χ
0	2
1	0
2	-2
\vdots	\vdots
g	$2 - 2g$

$$\begin{aligned} H_q(S^n) &= \begin{cases} \mathbb{Z} & \text{if } q = 0, n \\ 0 & \text{otherwise} \end{cases} \\ \chi(S^n) &= 1 + (-1)^n = \begin{cases} 2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

$$P^2 = S^2 / \mathbb{Z}_2$$

If $\mathbb{Z}_2 = \{1, \tau\}$, then $\tau : S^2 \rightarrow S^2$ is the antipodal map. Take the icosahedron (a triangulation of S^2) and quotient it by \mathbb{Z}_2 -action, to get a triangulation of P^2 . Then we have six vertices, fifteen edges, and ten faces, so $\chi = 6 - 15 + 10 = 1$. This corresponds to:

1. $H_0(P^2) = \mathbb{Z}$
2. $\pi_1(P^2) = \mathbb{Z}_2 = H_1$
3. $H_2(P^2) = 0$, since $\mathbb{Z}_2(P^2) = 0$

so $\chi = 2 - 1 = 1$.

8.2 The Lefschetz Fixed-Point Theorem

Say X is a triangulable space, and $f : X \rightarrow X$ is a map. This induces a map $f_{q*} : H_q(X, \mathbb{Z}) \rightarrow H_q(X, \mathbb{Q})$

Definition 8.22 (Lefschetz Number).

$$\Lambda f = \sum (-1)^q \text{Tr}(f_{q*})$$

We call Λ the **Lefschetz Number** of f .

Example 8.23. If $f : X \rightarrow X$ is the identity, then

$$f_{q*} = \text{Id}_{H_q(X, \mathbb{Q})}$$

so $\text{Tr}(f_{q*}) = \beta_q$ and $\Lambda f = \sum (-1)^q \beta_q = \chi(X)$

Lemma 8.24. If $f, g : X \rightarrow X$ are homotopic, then $\Lambda f = \Lambda g$.

Proof. $f_{q*} = g_{q*}$, so Λ is the same □

Corollary 8.25. If $f : X \rightarrow X$ is homotopic to the identity map, then $\Lambda f = \chi(X)$.

We'll show that if $\Lambda f \neq 0$, then f has no fixed point.

Idea: Suppose f is a simplicial map and $f(\sigma) \neq \sigma$ for all simplices σ . Then we have

$$f_q : C_q(K, \mathbb{Q}) \rightarrow C_q(K, \mathbb{Q})$$

on the chains. Then $\text{Tr}(f_q) = 0$, since q -simplices form a basis of $C_q(K, \mathbb{Q})$ so f_q has zeros on the diagonal. We'll show that

$$\sum (-1)^q \text{Tr}(f_q) = \sum (-1)^q \text{Tr}(f_{q*})$$

so $\Lambda f = 0$.

Theorem 8.26 (Hopf Trace Formula). If K is a simplicial complex and $\phi : C(K, \mathbb{Q}) \rightarrow C(K, \mathbb{Q})$ is a chain map, then

$$\sum (-1)^q \text{Tr}\phi_q = \sum (-1)^q \text{Tr}\phi_{q*}$$

where $\phi_{q*} : H_q(K, \mathbb{Q}) \rightarrow H_q(K, \mathbb{Q})$

Proof. We had over the rationals

$$\begin{aligned} C_q &\simeq Z_q \oplus C_q/Z_q \\ &\simeq B_q \oplus Z_q/B_q \oplus C_q/Z_q \end{aligned}$$

with $C_{q+1} \cong B_{q+1} \oplus Z_{q+1}/B_{q+1} \oplus C_{q+1}/Z_{q+1}$ and

$$\partial : C_{q+1}/Z_{q+1} \rightarrow B_q$$

an isomorphism. Since ϕ is a chain map, $\phi_q(Z_q) \subset Z_q$ and $\phi_q(B_q) \subset B_q$.

$$\begin{aligned} \sum (-1)^q \text{Tr}\phi_q &= \sum (-1)^q [\text{Tr}\phi_q|_{B_q} + \text{Tr}\phi_q|_{Z_q/B_q} + \text{Tr}\phi_q|_{C_q/Z_q}] \\ &= \sum (-1)^q [\text{Tr}\phi_q|_{B_q} + \text{Tr}\phi_q|_{H_q} + \text{Tr}\phi_q|_{B_{q-1}}] \\ &= \sum (-1)^q \text{Tr}\phi_q|_{H_q} \\ &= \sum (-1)^q \text{Tr}(\phi_{q-1}) \end{aligned}$$

□

5.2.2025

Theorem 8.27. If $f : |K| \rightarrow |K|$ is a map with no fixed points, then $\Lambda f = 0$.

Proof. Say d is the metric on $|K|$ coming from embedding in \mathbb{R}^N . Put $H(x) = d(x, f(x))$, for $x \in |K|$. Then $H(x) > 0$ and $H : |K| \rightarrow \mathbb{R}$ is continuous, so there's some $\delta > 0$ such that $H(x) > \delta$ for all $x \in |K|$.

Subdivide so $\text{mesh}(k) < \frac{\delta}{3}$. Rename K as this. Choose a simplicial approximation $s : K^m \rightarrow K$ of $f : |K^m| \rightarrow |K|$. Then we have a chain map $\chi : C(K, \mathbb{Q}) \rightarrow C(K^m, \mathbb{Q})$ such that the induced map $\chi_* : H(K, \mathbb{Q}) \rightarrow H(K^m, \mathbb{Q})$ is an isomorphism.

If σ is an oriented q -simplex of K , then $\chi(\sigma)$ is the sum of oriented simplices τ of K^m inside of σ . Then f_{q*} is the composition

$$H_q(K, \mathbb{Q}) \xrightarrow{\chi_{q*}} H_q(K^m, \mathbb{Q}) \xrightarrow{s_{q*}} H_q(K, \mathbb{Q})$$

so $f_{q*} = s_{q*} \circ \chi_{q*} = (s_q \circ \chi_q)$ by the homework.

$s_q \circ \chi_q : C_q(K, \mathbb{Q}) \rightarrow C_q(K, \mathbb{Q})$. By the Hopf Trace Formula, we see that $\Lambda f = 0$, which is enough to show that $\text{Tr}(s_q \circ \chi_q) = 0$ for all q . Then, we show that the "diagonal entries" of $s_q \circ \chi_q$ vanish, and use oriented q -simplices as a basis for $C_q(K, \mathbb{Q})$.

Suppose σ is an oriented q -simplex, and that γ is in the support of $\chi_q(\sigma)$. We want to show that $s_\tau \cap \sigma = \emptyset$. Suppose $\chi \in \tau$ and $s(\chi) \in \sigma$. Then $d(f(s(\chi)), s(\chi)) \leq d(s(\chi), s(\chi)) + d(s(\chi), f(s(\chi))) < \frac{\delta}{3} + \frac{\delta}{3} < \delta$. □

We showed that if $f : |K| \rightarrow |K|$ has no fixed points, then $\Lambda f = 0$.

Corollary 8.28. If X is a compact, contractible, triangulable space, then any $f : X \rightarrow X$ has a fixed point.

$$\text{Proof. } H_q(X, \mathbb{Q}) = H_q(\text{pt}, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } q = 0 \\ 0 & \text{if } q > 0 \end{cases}$$

$\Lambda f = \text{Tr}(f_{0*})$, with $f_{0*} : H_0(X, \mathbb{Q}) \rightarrow H_0(X, \mathbb{Q})$. $C_0(X, \mathbb{Q}) = Z_0(X, \mathbb{Q}) \rightarrow H_0(X, \mathbb{Q}) \cong \mathbb{Q}$, sends $\sum \lambda_i v_i \rightarrow \sum \lambda_i$, and commutes with our simplicial map $s_0 : C_0(X, \mathbb{Q}) \rightarrow C_0(X, \mathbb{Q})$, a simplicial approximation, so $f_{0*} = 1_* : \mathbb{Q} \rightarrow \mathbb{Q}$ (i.e., $\{1\}$) and $\Lambda f = 1$. □

Clearly, letting $X = B^n$ is Brouwer's fixed point theorem.

Corollary 8.29. If $\chi(X) \neq 0$ and $f : X \rightarrow X$ is homotopic to the identity map, then f has a fixed point.

Proof. $\Lambda f = \Lambda_{Id} = \chi(X) \neq 0$. □

Corollary 8.30 (Hairy Ball Theorem). If n is even then any vector field v on S^n has a zero

Proof. Here $S^n \subset \mathbb{R}^{n+1}$ and $V(\vec{x})$ is a vector so $\overrightarrow{V(x)} \perp \vec{x}$:

Suppose V is nowhere vanishing. Then replace it by $\frac{\overrightarrow{V(x)}}{|V(x)|}$, the unit vector field. Look at $f_T : S^n \rightarrow S^n$ given by

$$f_T(\vec{x}) = (\cos(T))\vec{x} + (\sin(T))\overrightarrow{V(x)}$$

which is continuous in T . $f_\pi(\vec{x}) = -\vec{x}$, so f_π is the antipodal map. $\chi(S^n) = 2$, and f_π is homotopic to $\text{Id} = f_0$. But f_π does not fix any point, which is a contradiction. \square

Definition 8.31. If f is a map $f : S^n \rightarrow S^n$, then the **degree** of f is the number $\deg(f)$ so

$$f_{n*} : H_n(S^n) \rightarrow H_n(S^n)$$

is multiplication by $\deg(f)$. ($H_n(S^n) = \mathbb{Z}$)

Example 8.32. If $f(e^{i\theta}) = e^{im\theta}$, then $\deg(f) = m$.

Then

$$\begin{aligned}\Lambda_f &= \text{Tr}(f_{0*}) + (-1)^n \text{Tr}(f_{n*}) \\ &= 1 + (-1)^n \deg(f)\end{aligned}$$

Corollary 8.33. 1. If $|\deg(f)| > 1$, then f has a fixed point.

2. If $\deg(f) = 0$, then f has a fixed point.
3. If $\deg(f) = 1$ and n is even, then f has a fixed point.
4. If $\deg(f) = -1$ and n is odd, then f has a fixed point.