ANOVA - Analysis of Single Factor Experiments

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Completely Randomized Design

In a completely randomized design all experimental units are randomly assigned to one of k treatment groups

example

Three ways of memorizing words are to be compared. Thirty people are randomly assigned to one of the 3 ways. Each individual is then given 50 words to memorize. The average number of words recalled in each group is used to compare the 3 methods of memorization.

A two independent sample t-test is an example of a completely randomized design.

Randomized Block Design

In a randomized block design, blocks of units that are similar in terms of a blocking factor are used. Treatments are randomly assigned with each block.

example

In the memorization experiment, each person could be asked to evaluation all 3 ways of memorizing words. The order of the 3 types would be randomly assigned. The individuals in this case are the block.

The paired t-test is an example of a randomized block design.

Completely Randomized Design

One-Way Layout

Treatment

1	2	$\cdots$	k
Y_{11}	Y_{21}	$\cdots$	Y_{k1}
Y_{12}	Y_{22}	$\cdots$	Y_{k2}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
Y_{1n_1}	Y_{2n_2}	$\cdots$	Y_{kn_k}
$\bar{Y}_1$	$\bar{Y}_2$	$\cdots$	$\bar{Y}_k$
S_1^2	S_2^2	$\cdots$	S_k^2

- $N = \sum_{i=1}^{k} n_i$
- if $n_1 = n_2 = \cdots = n_k$ we have a balanced design.
- Y_{ij} is the observed value of Y for the jth unit in the ith treatment population $i = 1, \ldots, k$, $j = 1, \ldots, n_i$

Fixed Effects Model

$$Y_{ij} = \mu_i + \epsilon_{ij}$$

- ϵ_{ij} *iid* $N(0, \sigma^2$
- The primary interest is to compare $\mu_1, \mu_2, \ldots, \mu_k$
- We can estimate μ_i by $\bar{Y}_i$, the average of Y in the ith population.
- Define $\bar{Y}_{\bullet\bullet} = \sum_{i=1}^{k} \bar{Y}_i / k$

- The treatment sum of squares measures the amount of variation in the data explained by the treatments.

$$SSA = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (\bar{Y}_i - \bar{Y}_{\bullet\bullet})^2$$

- The treatment mean square is $\dfrac{SSA}{k-1}$

Fixed Efffects Model

Sums of Squares

- The error sum of squares measures the random error in the data

$$SSE = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_i)^2$$

- The mean square error is found as $\dfrac{SSE}{N-k}$
- MSE is used to estimate the variance σ^2

Fixed Effects Model

Sums of Squares

- The total sum of squares measure the total variation in the data

$$SST = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_{\bullet\bullet})^2$$

- We can show that SST=SSE + SSA

Test for Equality of Means

To test $H_0 : \mu_1 = \mu - 2 = \cdots \mu_k$ versus the alternative H_1 : not all μ_i are equal If H_0 is true then $Y_{ij} = \mu + \epsilon_{ij}$ and the only variation in the data is due to the error term. So SSE and SSA are both measure the same variance. Let

$$F = \frac{MSA}{MSE}$$

Under H_0

$$F \sim F_{k-1,N-k}$$

If H_0 is not true F will be too big. Reject H_0 if F is too big.

ANOVA table

ANOVA Table

Source	SS	df	MS	F
Treatment	SSA	k-1	MSA	MSA/MSE
Error	SSE	N-k	MSE	
Total	SST	N-1		——

Example

A manufacturer of plastic bottles has six stations in the plant that produces the bottles independently. The company wants to test whether the stations produce bottles with the same weight. A single factor experiment was set up where 8 bottles were randomly chosen from each station, their weights measured and recorded. The data gave treatment means

Station 1	Station 2	Station 3	Station 4	Station 5	Station 6
$\bar{Y}_1$	$\bar{Y}_2$	$\bar{Y}_3$	$\bar{Y}_4$	$\bar{Y}_5$	$\bar{Y}_6$
51.66	51.335	51.24	51.623	51.699	51.858

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Example

Model

$$Y_{ij} = \mu_i + \epsilon_{ij}$$

Here μ_i is the mean weight of the ith station. A test for an effect of station on the means of Y is

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_k$$

ANOVA table

Source	df	SS	MS	F	p-value
Station	5	2.194	0.439	4.03	0.004
Error	42	4.577	0.109		
Total	47	6.771			

Since the p-value is small, we reject the null hypothesis that the mean weight is the same at each station. At least one station has a different mean weight from the others.

Multiple Comparisons of Means

Once the null hypothesis is rejected we need to find which mean(s) is(are) different. We use pairwise t-tests to see which pairs are different. We have $\binom{k}{2}$ comparisons and we want all comparisons to be at a fixed α level. To do this we must adjust the comparisons to keep overall α level fixed. Some methods for multiple comparisons are:

- Bonferoni Method: When comparing m pairs, divide α by m and each individual t-test has critical level α/m.
- Tukey
- Scheffe

Random Effect Model

Suppose the treatments in a study are selected at random from a larger population of treatments. Inference can then be extended to all treatments in the population.

example

To study the variation between the productive output of workers in a factory, a random sample of workers is selected. Interest is not in company mean production level of the particular workers chosen - they just happened to be selected. Interest is in seeing what this sample says about the population of all workers in the factory.

In a fixed effect model $\mu_1, \mu_2, \ldots, \mu_k$ are numbers. In a random effect model $\mu_1, \mu_2, \ldots, \mu_k$ are random varialbes.

Random Effect Model

$$Y_{ij} = \mu_i + \epsilon_{ij}$$

- μ_i iid $N(\mu, \sigma_a^2)$
- ϵ_{ij} iid $N(0, \sigma_e^2)$
- μ_i and ϵ_{ij} are all independent
- $Var(Y_{ij}) = \sigma_a^2 + \sigma_e^2$
- By estimating the total variance we can determine what proportion of the total variance is attributed to treatment, (σ_a^2).

Random Effect Model

Test for a treatment effect

To test for $H_0 : \sigma_a^2 = 0$ versus $H_1 : \sigma_a^2 > 0$. We use the same ANOVA table to test this. The only difference is in intepretation.

Example

We compare 10 samples from each of 5 different batches of a manufacturing process. The 5 batches are a random sample from the population of all batches. We want to know whether the batch has an effect on the output of the process. The following ANOVA table was generated.

Source	df	SS	MS	F	p-value
Batches	4	30373	7593	3.58	0.013
Error	45	95518	2123		
Total	49	125891			

Reject the null hypothesis that the treatment variance is equal to zero. Conclude there is an batch effect.