MATH 3283W
$\TeX$ Project

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Problem. *(4.3 #3(d)) Prove that the sequence s_n where $s_1 = 1$ and $s_{n+1} = \sqrt{(2s_n) + 2}$ is both monotone and bounded. Then find the limit as n approaches ∞.*

$$s_1 = 1$$
$$s_2 = \sqrt{(2 * 1) + 2} = \sqrt{4} = 2$$
$$s_3 = \sqrt{(2 * 2) + 2} = \sqrt{6} = 2.45$$
$$s_4 = \sqrt{(2 * \sqrt{6}) + 2} = 2.627$$
$$s_4 = \sqrt{(2 * 2.627) + 2} = 2.69$$

First we show that the sequence is monotone by proving that it is increasing for all $n \geq 1$.

Proof: We proceed inductively to show that s_n is increasing for all $n \geq 1$.
$BaseCase$: When $n = 1$ then, $s_1 = 1 < 2 = s_2$.
Suppose that $s_k < s_{k+1}$ for some k.
Show that $s_{k+1} < s_{k+2}$.

$$s_{k+1} = \sqrt{(2 * s_k) + 2} < \sqrt{(2 * s_{k+1}) + 2} = s_{k+2},$$

since $s_k < s_{k+1}$ Thus by induction, we can conclude that s_n is increasing for all $n \geq 1$. $\square$

Now that we have proven that this sequence is increasing, we must prove that it is bounded above so that we can apply the Monotone Convergence Theorem. The Monotone Convergence Theorem states that a monotone sequence converges if and only if it is bounded. Hence, choose an arbitrary upper bound (in this case we will use three) and prove that $s_n \leq 3$ for all $n \geq 1$.

Proof: We proceed inductively to show that $s_n \leq 3$ for all $n \geq 1$.
$BaseCase$: When $n = 1$ then, $s_1 = 1 \leq 3$.
Suppose that $s_k \leq 3$ for some k.

Show that $s_{k+1} \leq 3$. We have,

$$s_{k_1} = \sqrt{(2 * s_k) + 2} \leq \sqrt{2 * 3 + 2} = \sqrt{8} < 3.$$

Thus, we have shown that $s_n \leq 3$ for all $n \geq 1$, which means that the sequence is bounded above. □

Now using the Monotone Convergence Theorem, which states that a monotone sequence s_n converges to some real number s if and only if it is bounded, we know that the sequence above must converge since we have proven that it is both monotone and bounded.

The limit $lims_n = s$ must satisfy the equation $s = \sqrt{(2 * s) + 2}$, because both the limits of s_n and s_{n_1} both converge to the real number s.
Solving for s we have,

$$s^2 - 2s - 2 = 0$$

Using the quadratic formula, we find the answers produced by this equation are $1 \pm \sqrt{3}$, and since all of the terms of the series are positive and increasing, they must converge to the positive number,

$$s = 1 + \sqrt{3}.$$

Therefore, we conclude that the sequence s_n converges to $s = 1 + \sqrt{3}$ as n approaches ∞.