Support Vector Machines

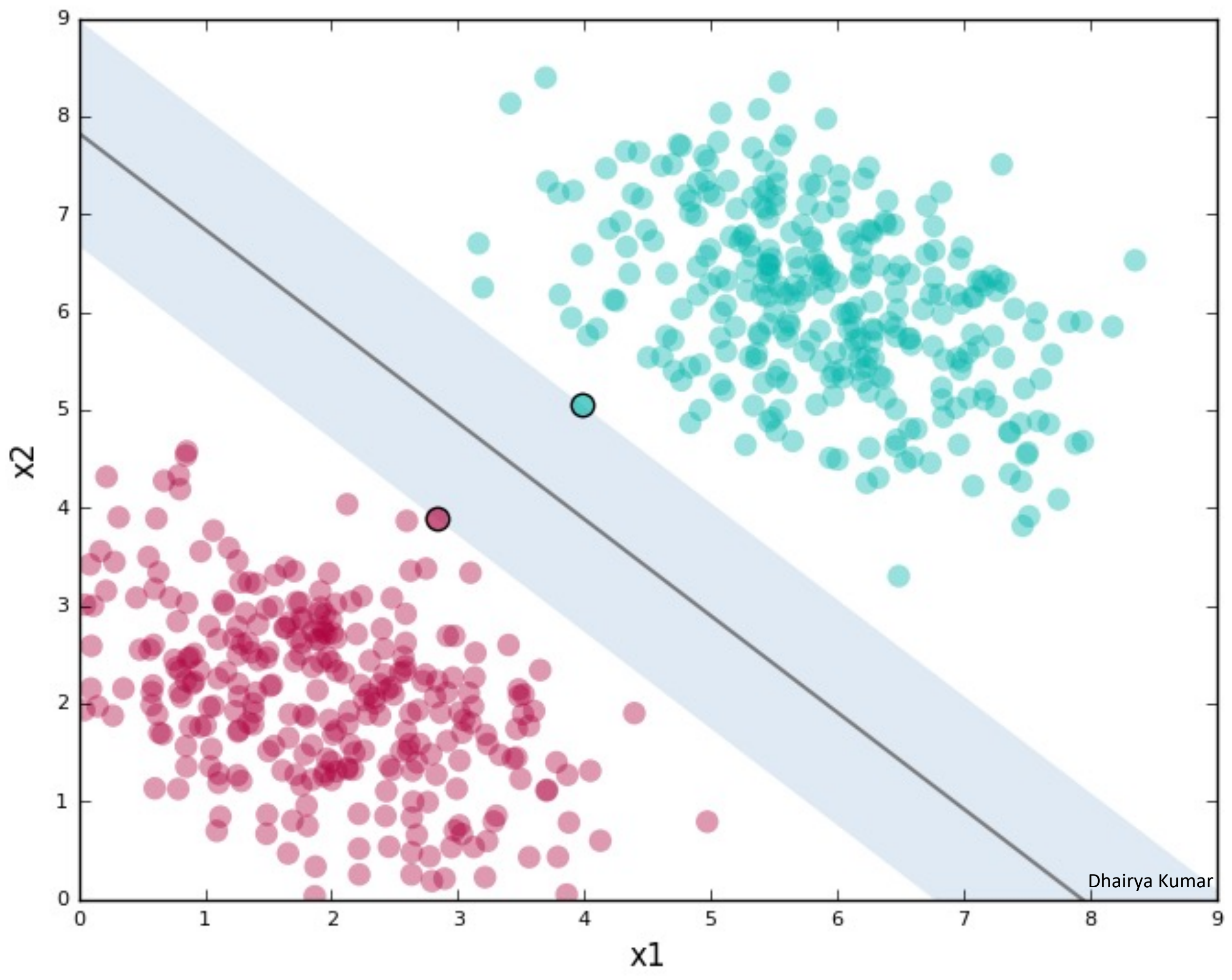
COSC 410: Applied Machine Learning

Spring 2022

Prof. Apthorpe

Outline

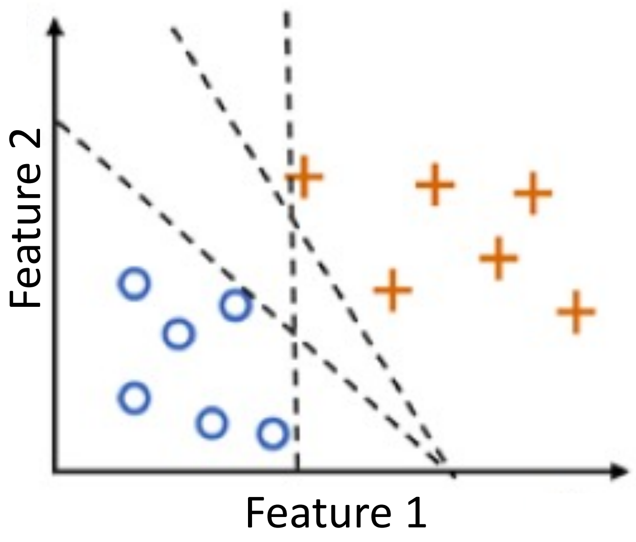
- Linear SVM Intuition
- SVM Model Notation
- Linear SVM
  - Model & Prediction
  - Training
- Non-Linear SVMs
  - Kernel Trick
- Multiclass SVM
- Practical Use of SVMs



Linear SVM Intuition

- Ideally, training data are **linearly separable**
  - An **N-dimensional line** can divide the data by class

Binary classification!
More on multiclass later…



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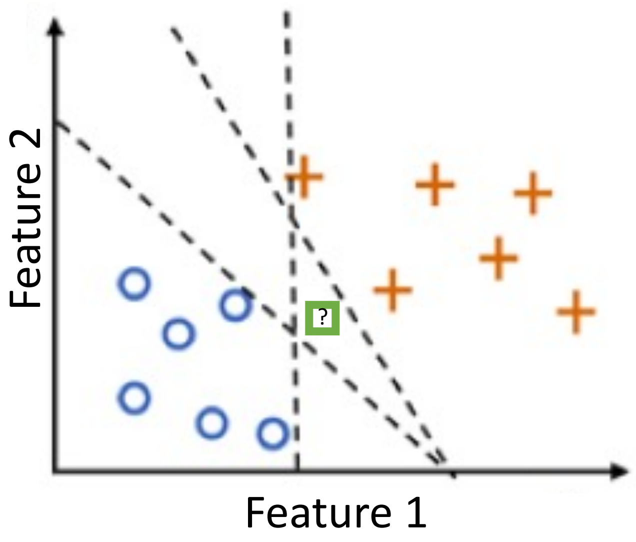
  - Predict new examples based on which side of the line they are located



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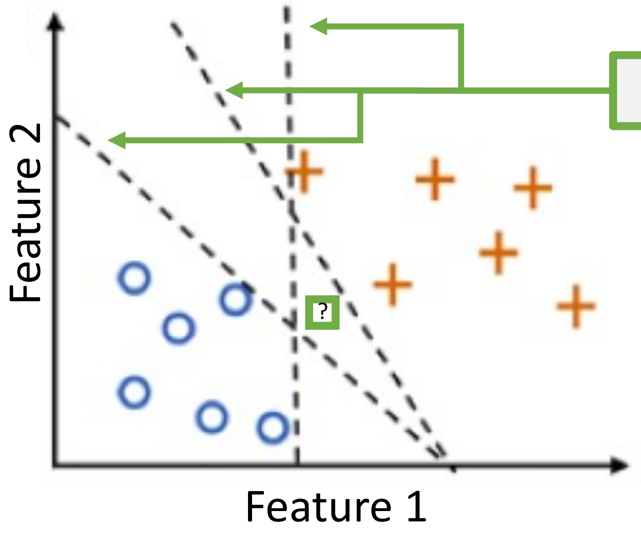
Infinite possible lines separate this data.
Which is the best line to draw?



Why might none of these 3 lines be the best?

Generalizability!
The best line stays as far away from the training examples as possible

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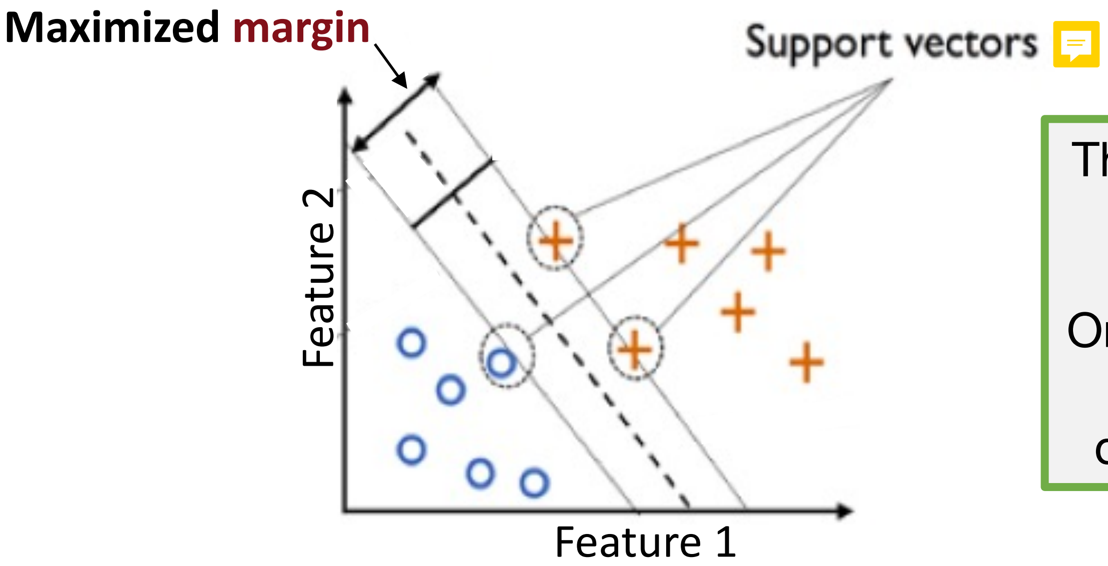
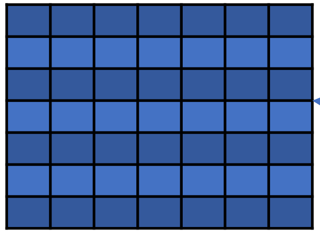


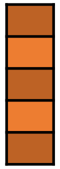
The best line maximizes the distance between it and the **nearest examples**

Only these **support vectors** matter to the line's position, adding more data points further away from the line doesn't change the model → very robust to overfitting

Training Models Review

Remember to **standardize**



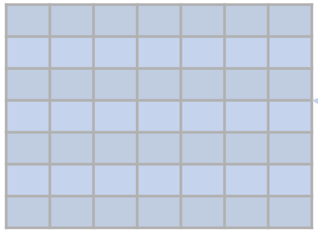
- Model training is a **minimization** process
  - **Training data** $\mathbf{X}$ of individual feature vectors $\mathbf{x}$
  - **Training labels** $\mathbf{y}$ 
  - Chosen **error function** $E()$
  - **Model** $h_{\boldsymbol{\theta}}()$ (features $\mathbf{X}$ → predicted labels $\hat{\mathbf{y}}$)
    - Parameters: $\boldsymbol{\theta} = [\theta_0, \theta_1, \theta_2]$
  - **GOAL:** Find model parameters that minimize $E(h_{\boldsymbol{\theta}}(\mathbf{X}), \mathbf{y})$

predicted labels

actual labels

Training Models: SVM Notation

Remember to **standardize**



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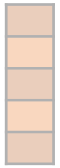
  - **Training labels** $\mathbf{y}$ 
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  - **Model** $h_{\mathbf{w},b}()$ (features $\mathbf{X}$ → predicted labels $\hat{\mathbf{y}}$)
    - Parameters: Weights vector $\mathbf{w} = [w_1, w_2, w_3]$ and bias scalar b

predicted labels

- **GOAL:** Find model parameters that minimize $E(h_{\mathbf{w},b}(\mathbf{X}, \mathbf{y})$

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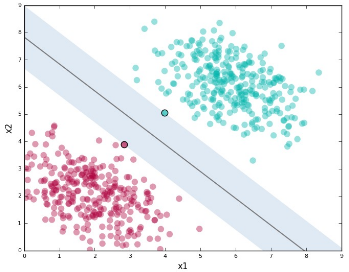
Linear SVM Model



- Predicted labels are a **piecewise linear combination** of features
  - Model parameters: weights $\mathbf{w} = [w_1, w_2, w_3, \ldots]$ and bias b
  - For **each** example $\mathbf{x} = [x_1, x_2, x_3, \ldots]$

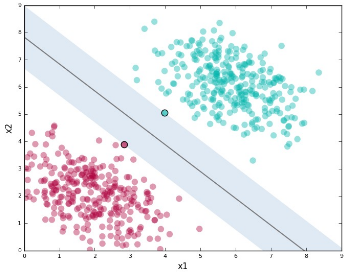
Predicted label

$$\hat{y} = \begin{cases} 0 & \text{if} \quad w_1 x_1 + w_2 x_2 + \cdots + b < 0 \\ 1 & \text{if} \quad w_1 x_1 + w_2 x_2 + \cdots + b \geq 0 \end{cases}$$

Example is “**above**” the decision line

Example is “**below**” the decision line

Linear SVM Model



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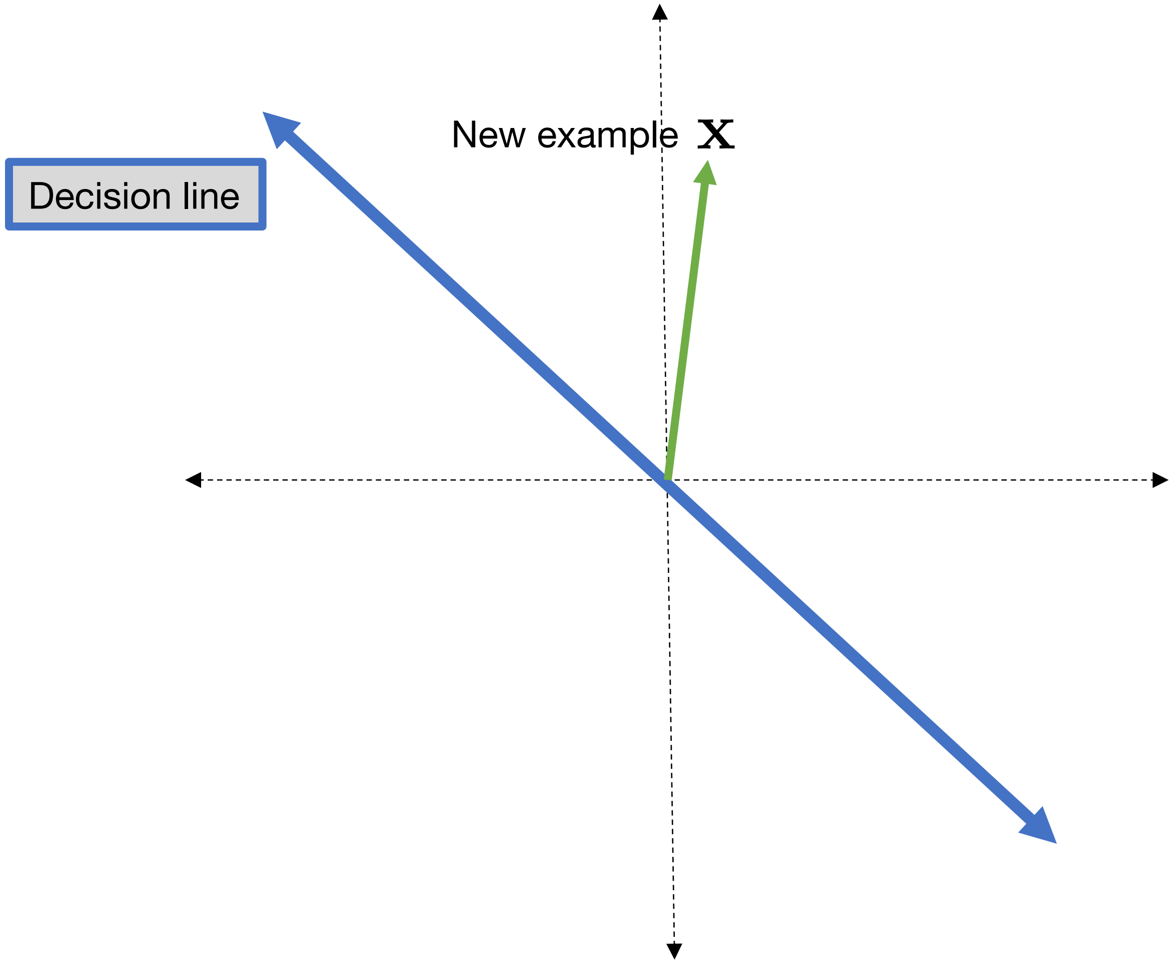
Example is “**below**” the decision line

Example is “**above**” the decision line

$$= \begin{cases} 0 & \text{if} \quad \mathbf{w} \cdot \mathbf{x} + b < 0 \\ 1 & \text{if} \quad \mathbf{w} \cdot \mathbf{x} + b \geq 0 \end{cases} \qquad = \begin{cases} 0 & \text{if} \quad \mathbf{w}^\intercal \mathbf{x} + b < 0 \\ 1 & \text{if} \quad \mathbf{w}^\intercal \mathbf{x} + b \geq 0 \end{cases}$$

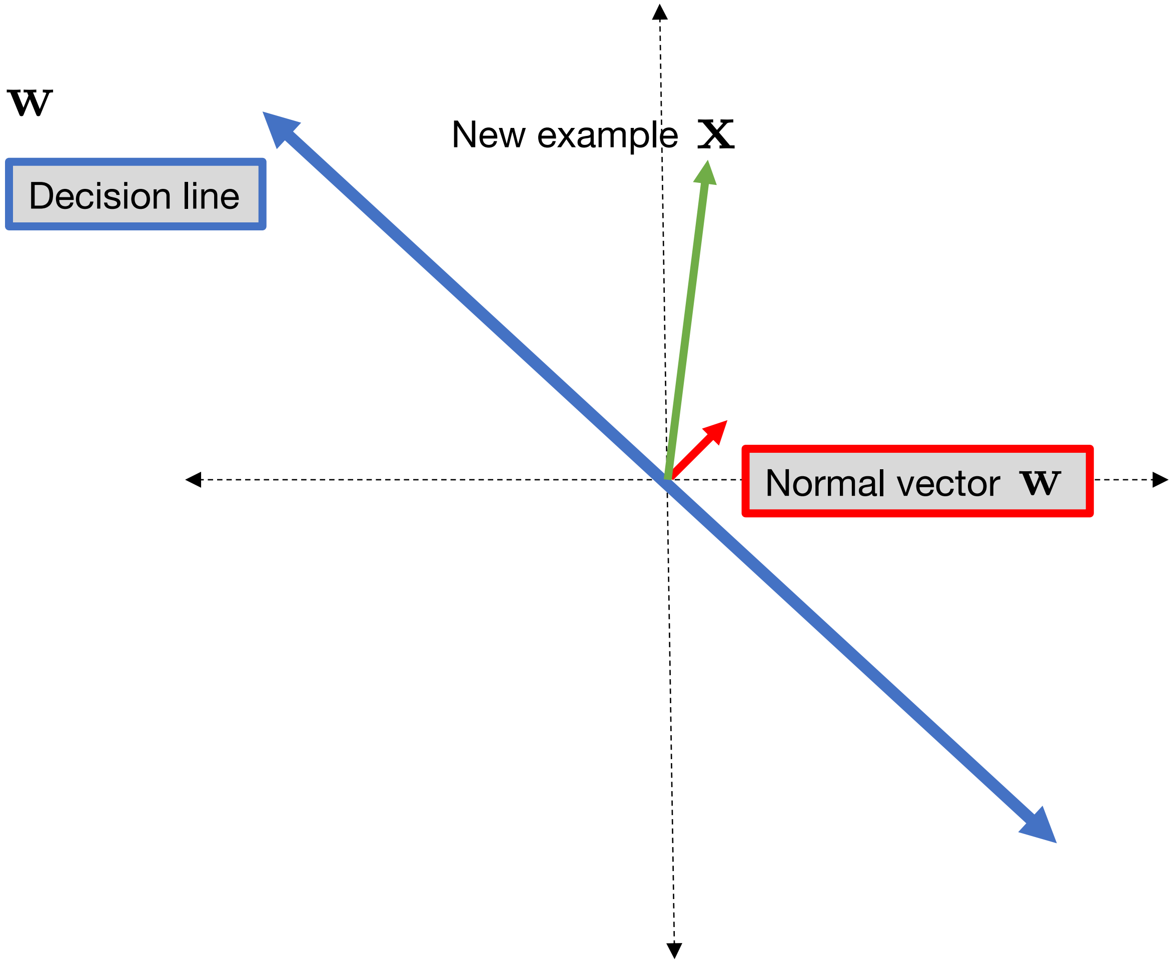
Linear SVM Model

- Is $\mathbf{x}$ "above" or "below" the decision line?



Linear SVM Model

- Is $\mathbf{x}$ "above" or "below" the decision line?
- To find out, *project* $\mathbf{x}$ onto the line's normal vector $\mathbf{w}$



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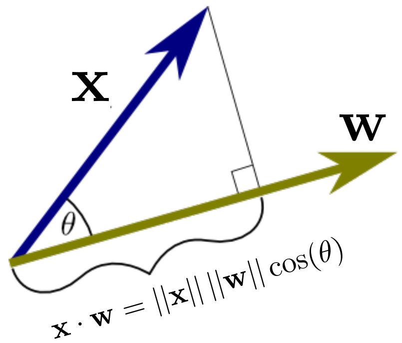

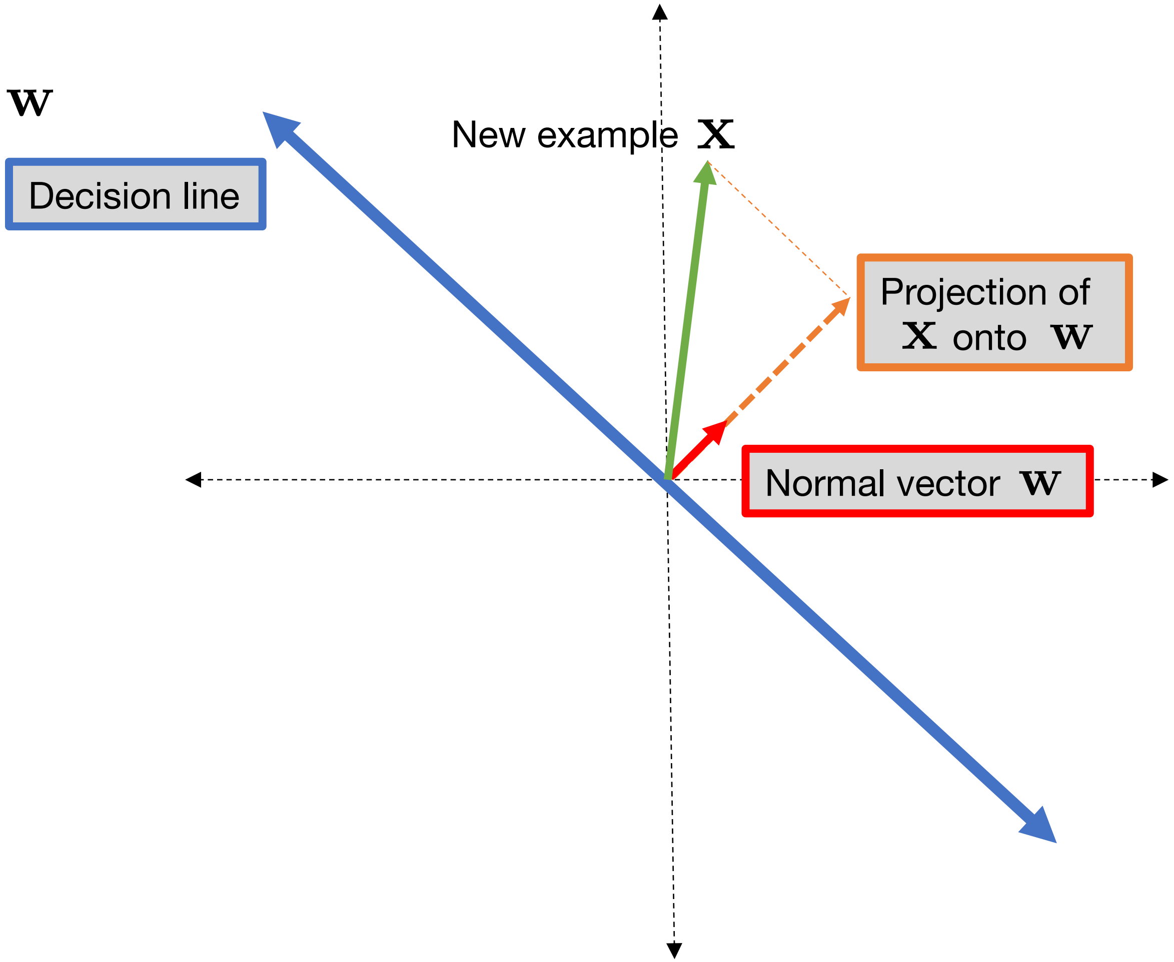

- This is the magnitude of $\mathbf{x}$ *in the direction of* $\mathbf{w}$

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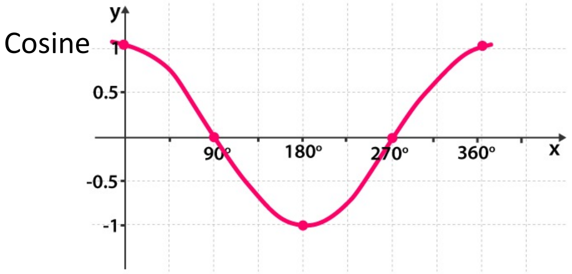

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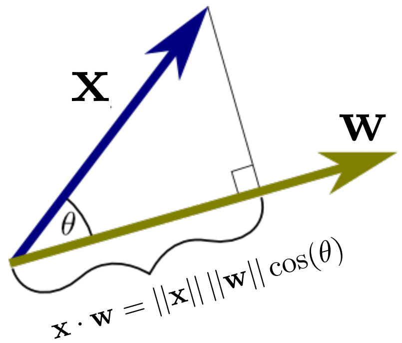

- This is the magnitude of $\mathbf{x}$ *in the direction of* $\mathbf{w}$
- If **positive,** $\mathbf{x}$ is **above** the decision line
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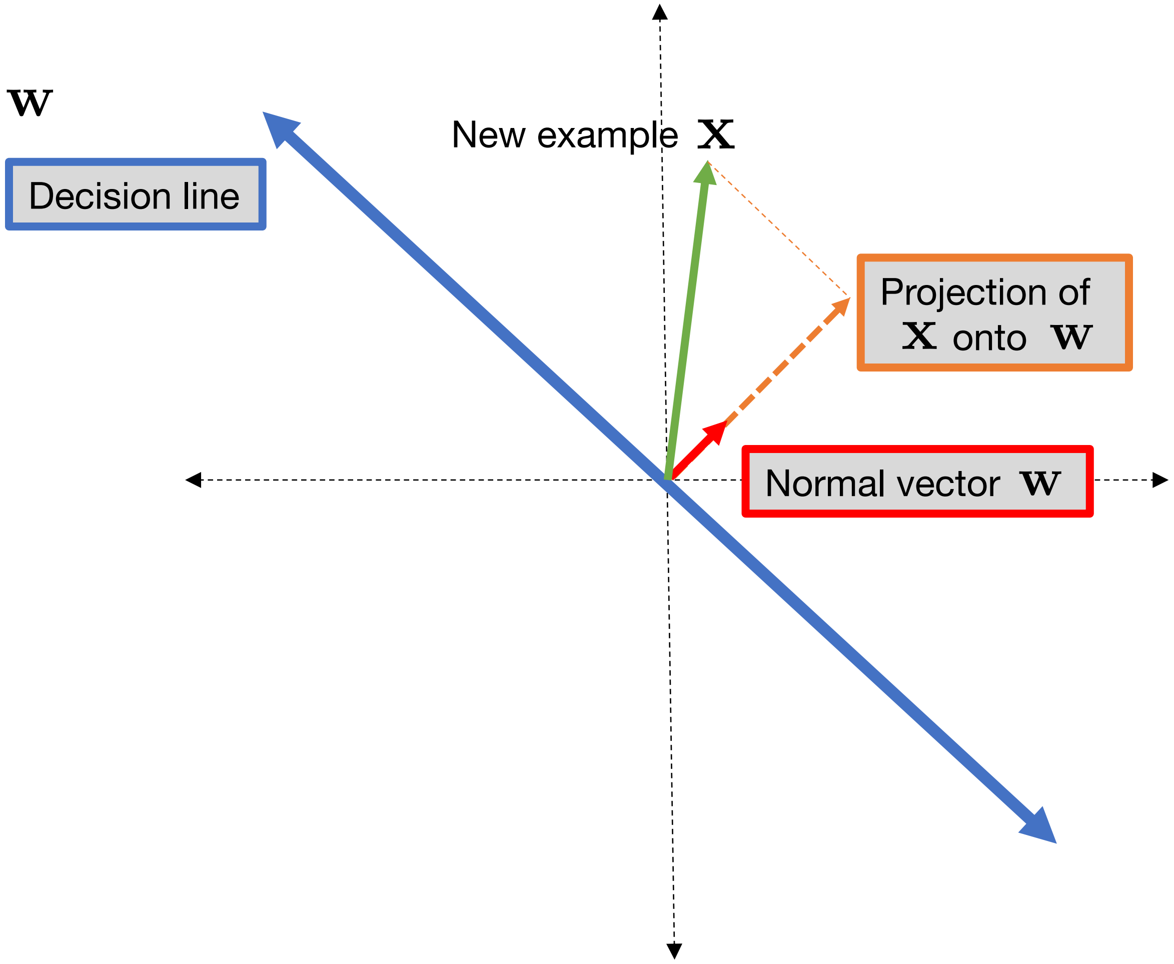

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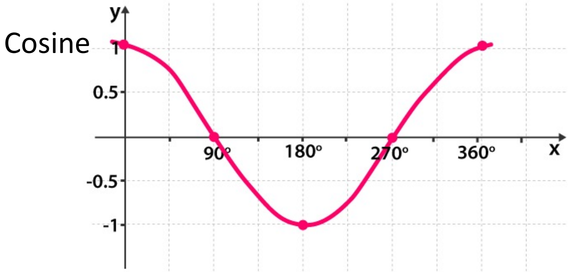

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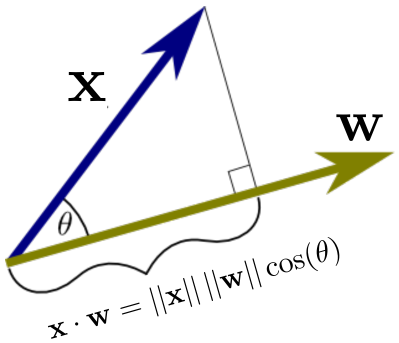

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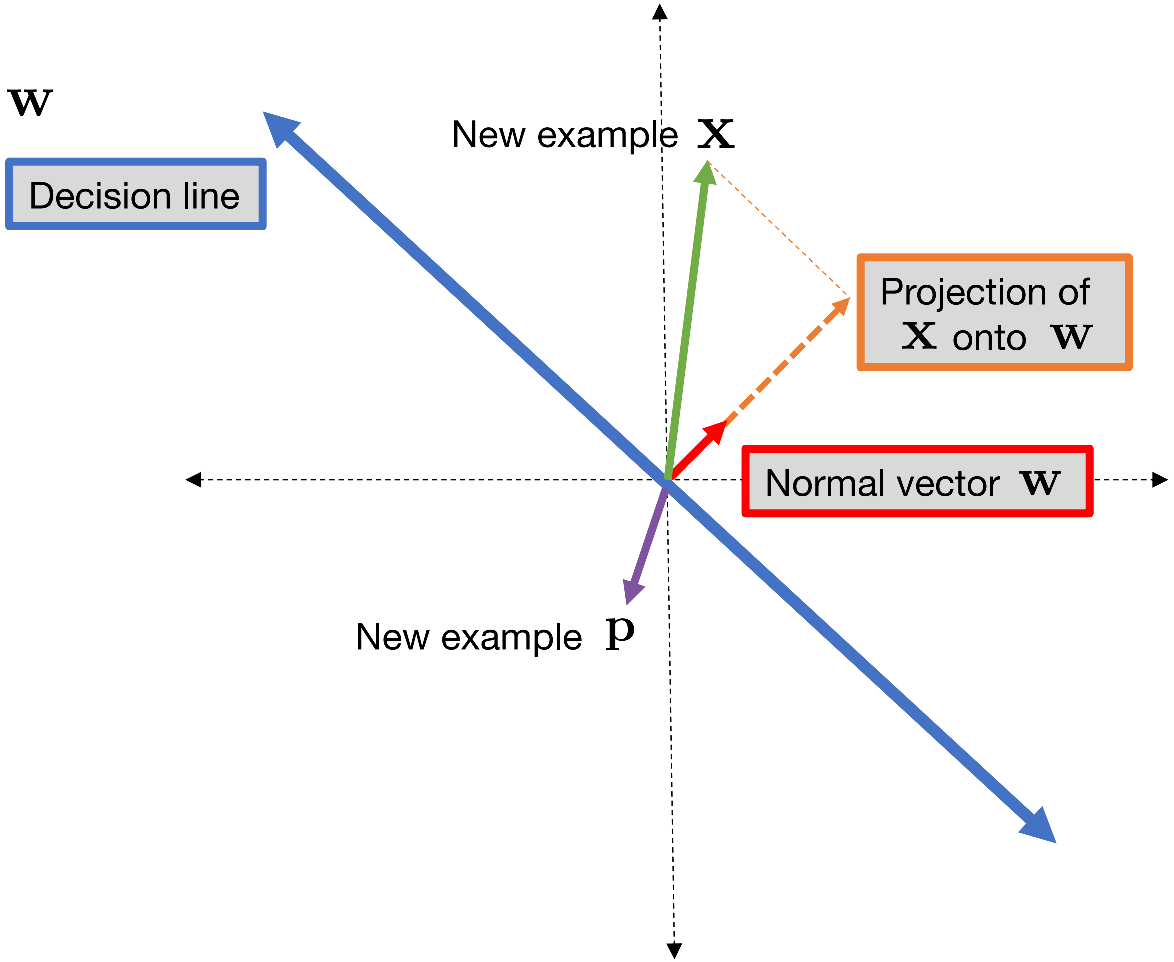

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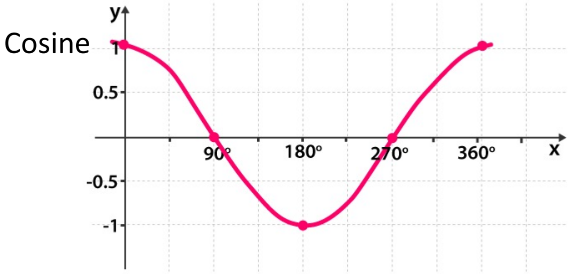

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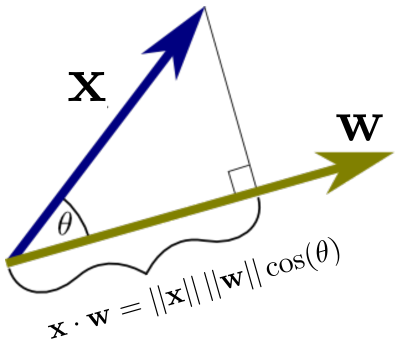

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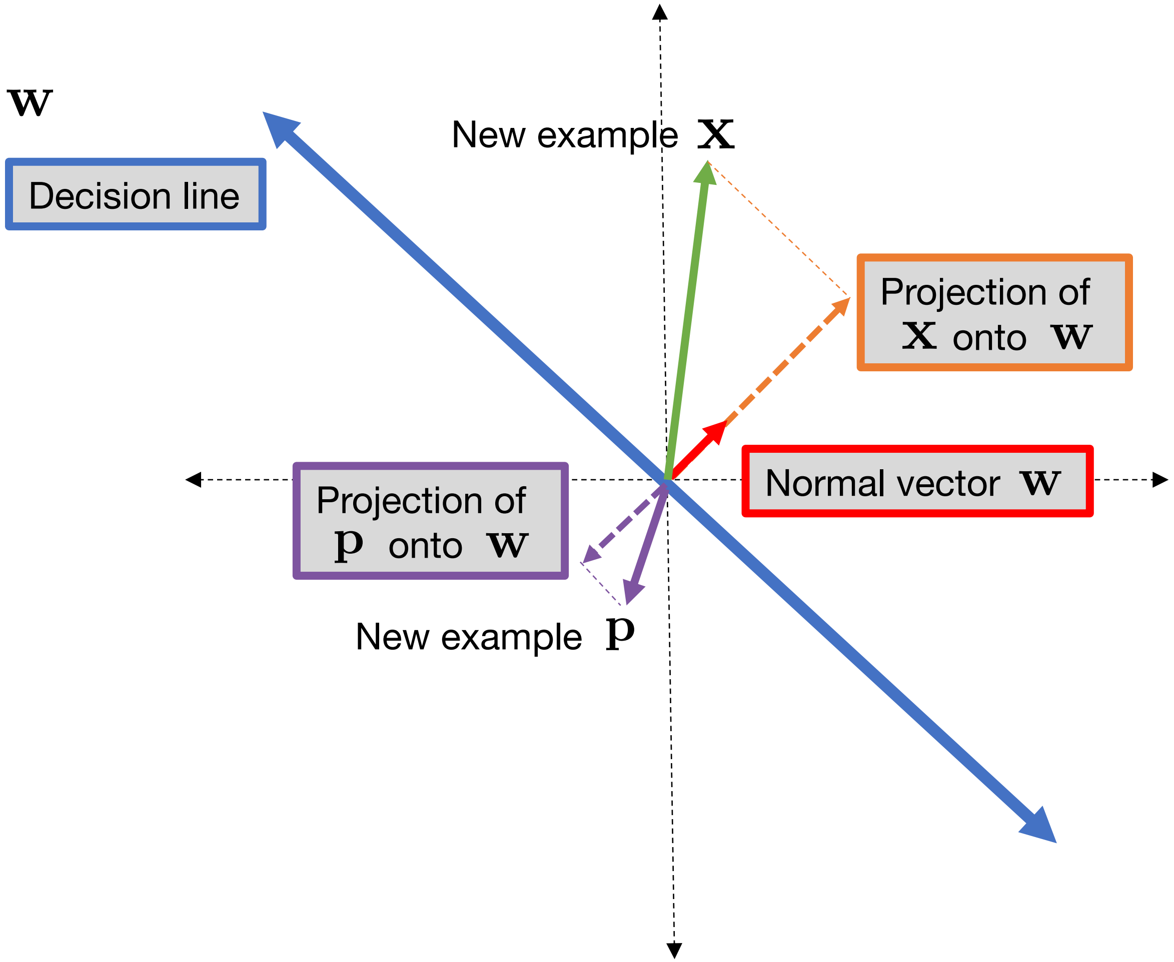

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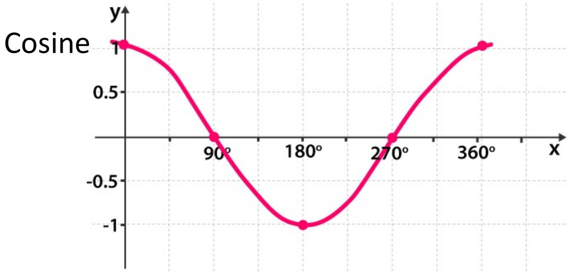

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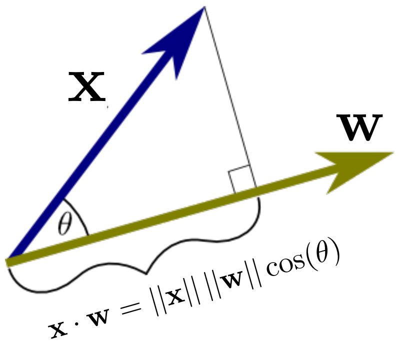

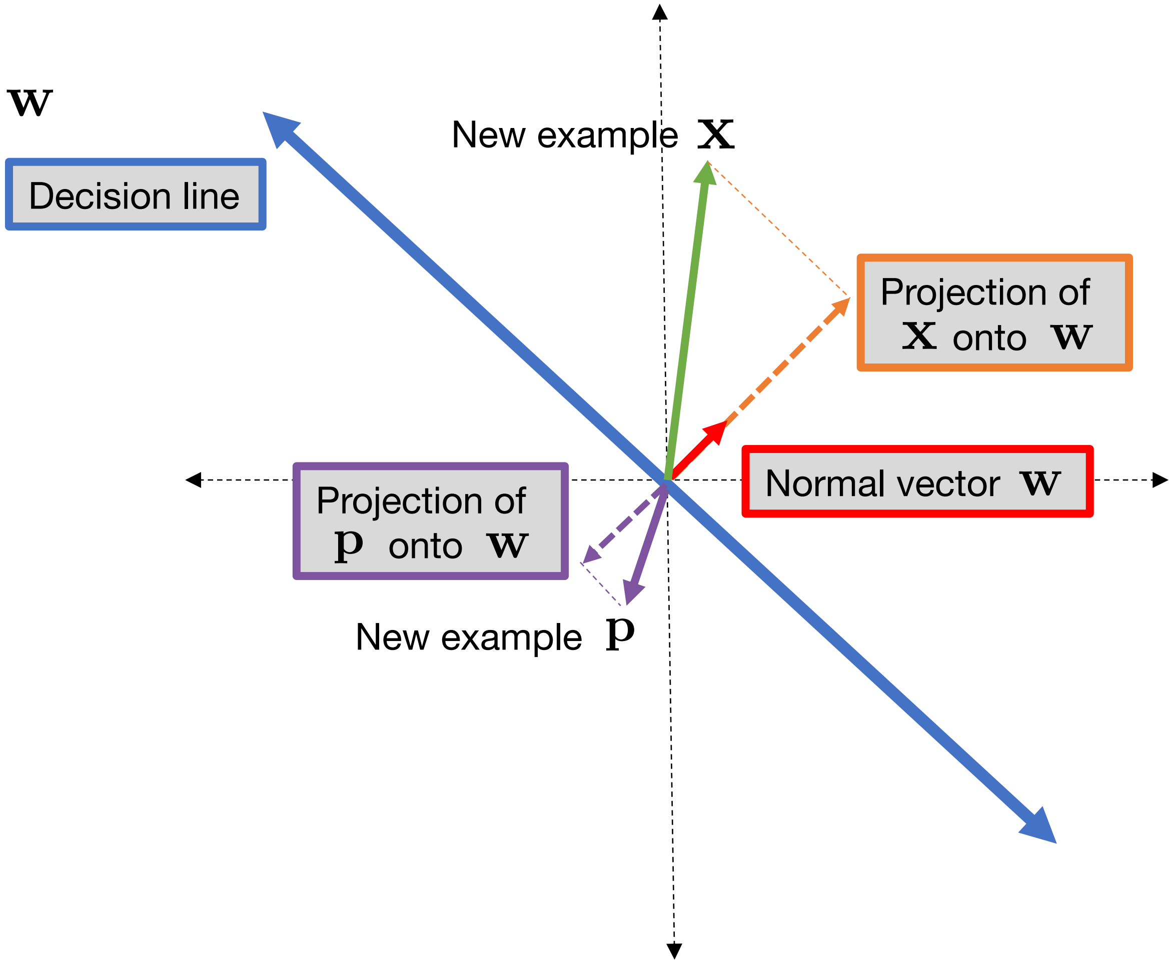

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- If best decision line isn’t through origin, add bias b

Linear SVM Model

- Predicted labels

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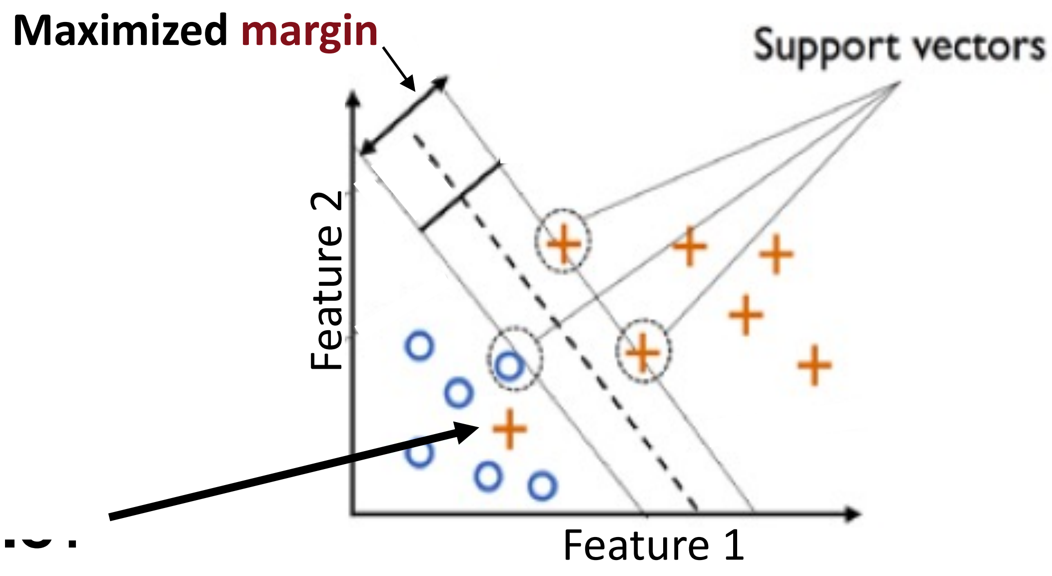
Example is "**below**" the decision line

Example is "**above**" the decision line

- What about the **margin?**
  - Important for **training**, not prediction

- What if data are **not linearly separable?**



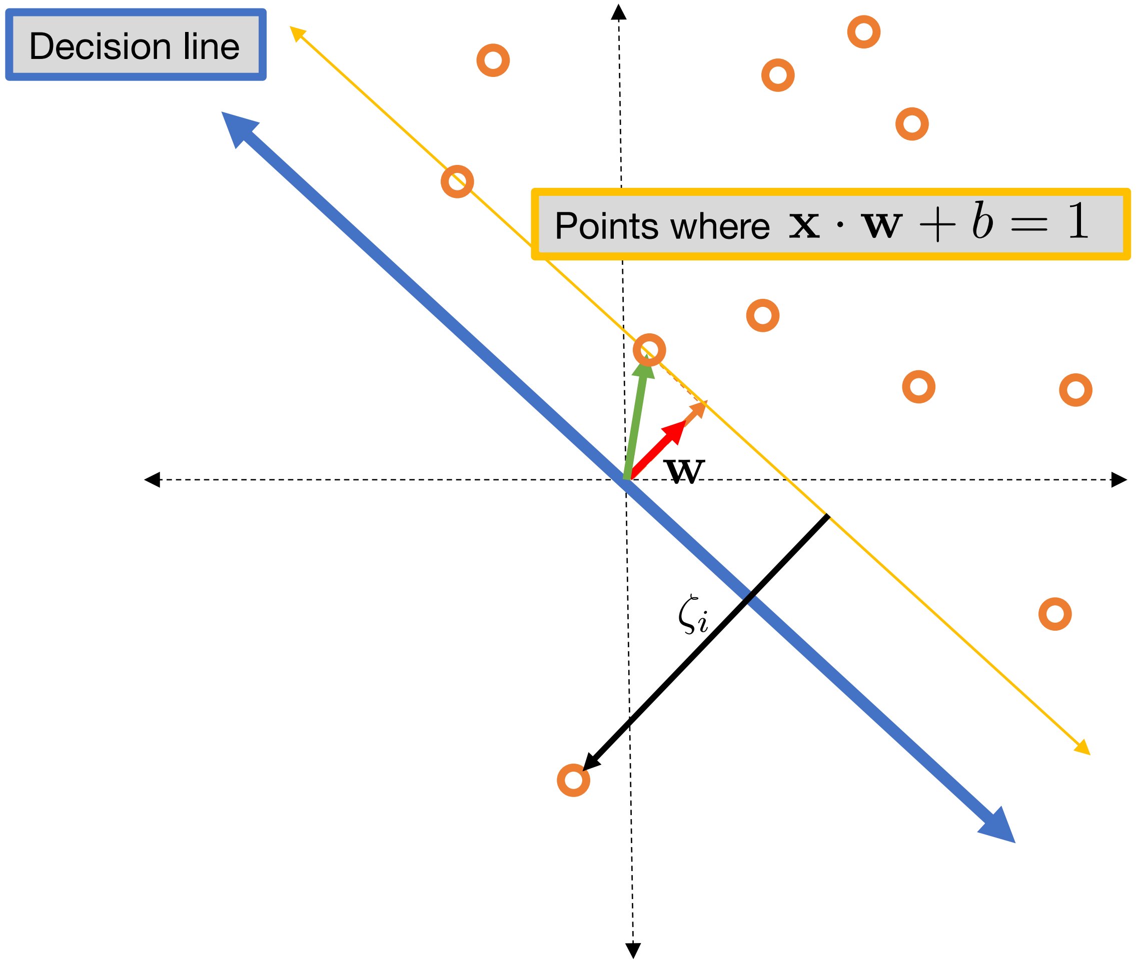
Linear SVM Training

Slack variable ζ accounts for non-separable data in a **soft-margin SVM**

- **Goal pt. 1:** Find $\mathbf{w}$ and b such that

$$\mathbf{w}^{\intercal}\mathbf{x} + b \geq 1 - \zeta$$

for all training examples in “1” class AND



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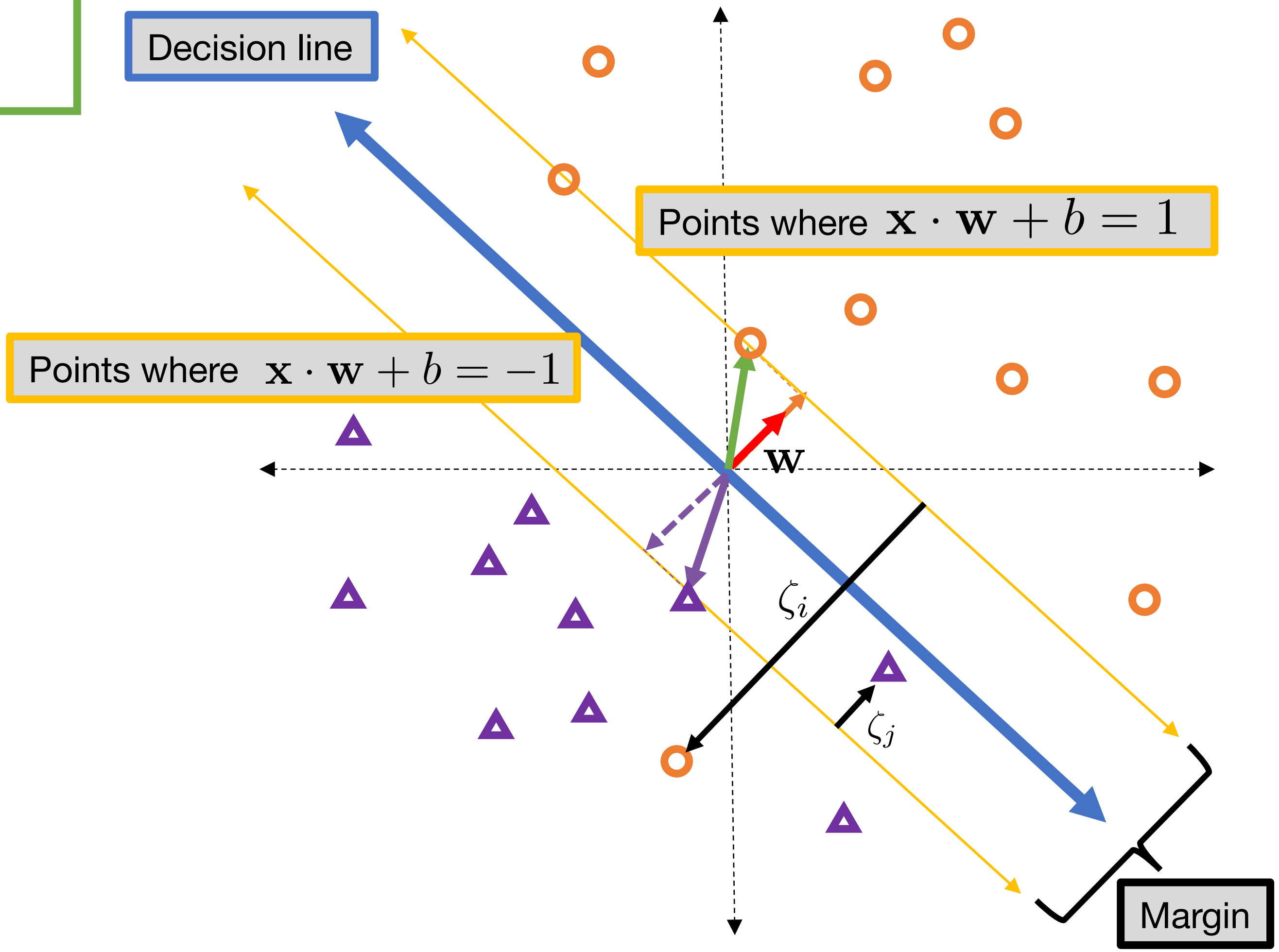

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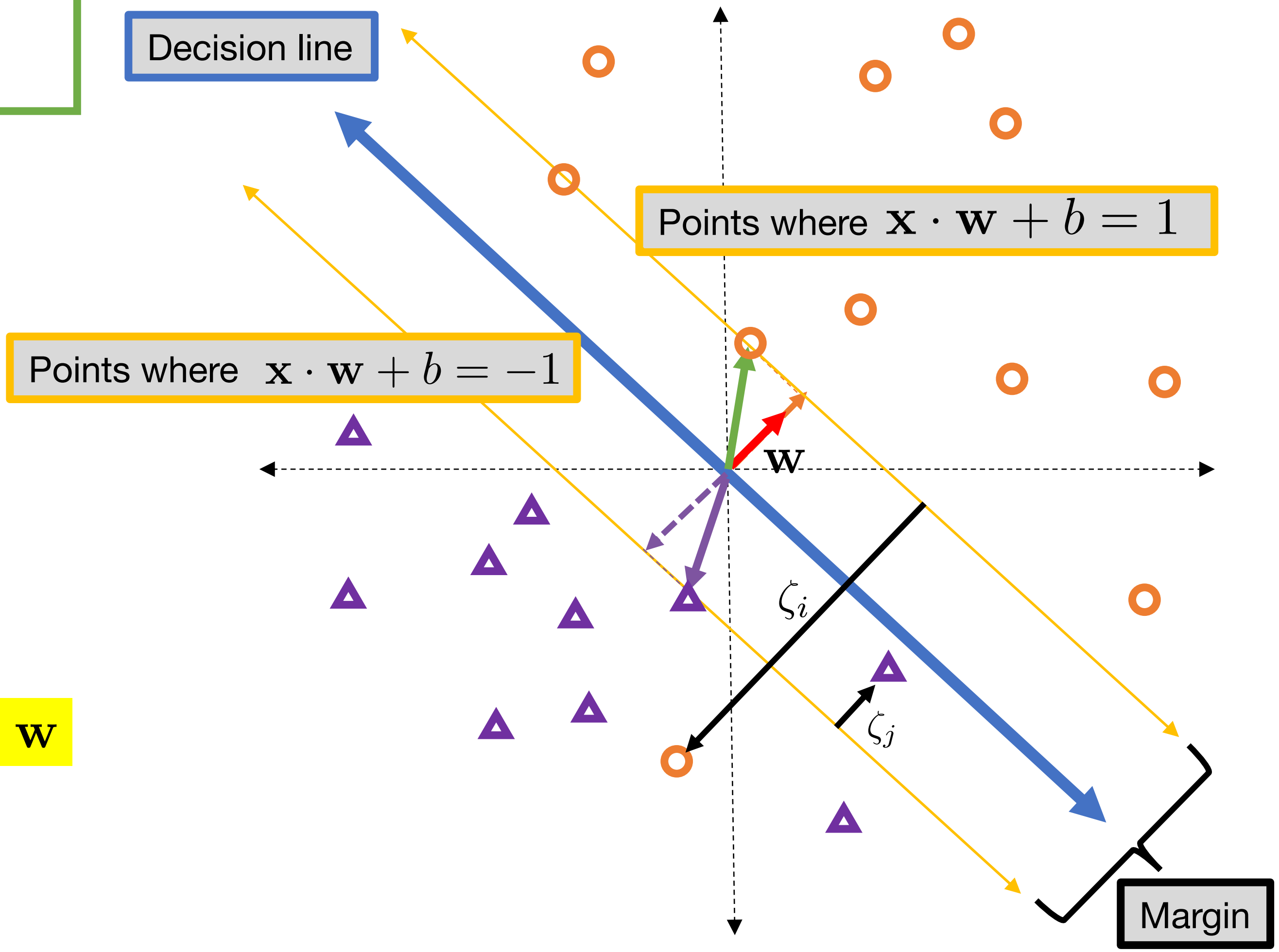
  - Equivalent to minimizing the magnitude of $\mathbf{w}$



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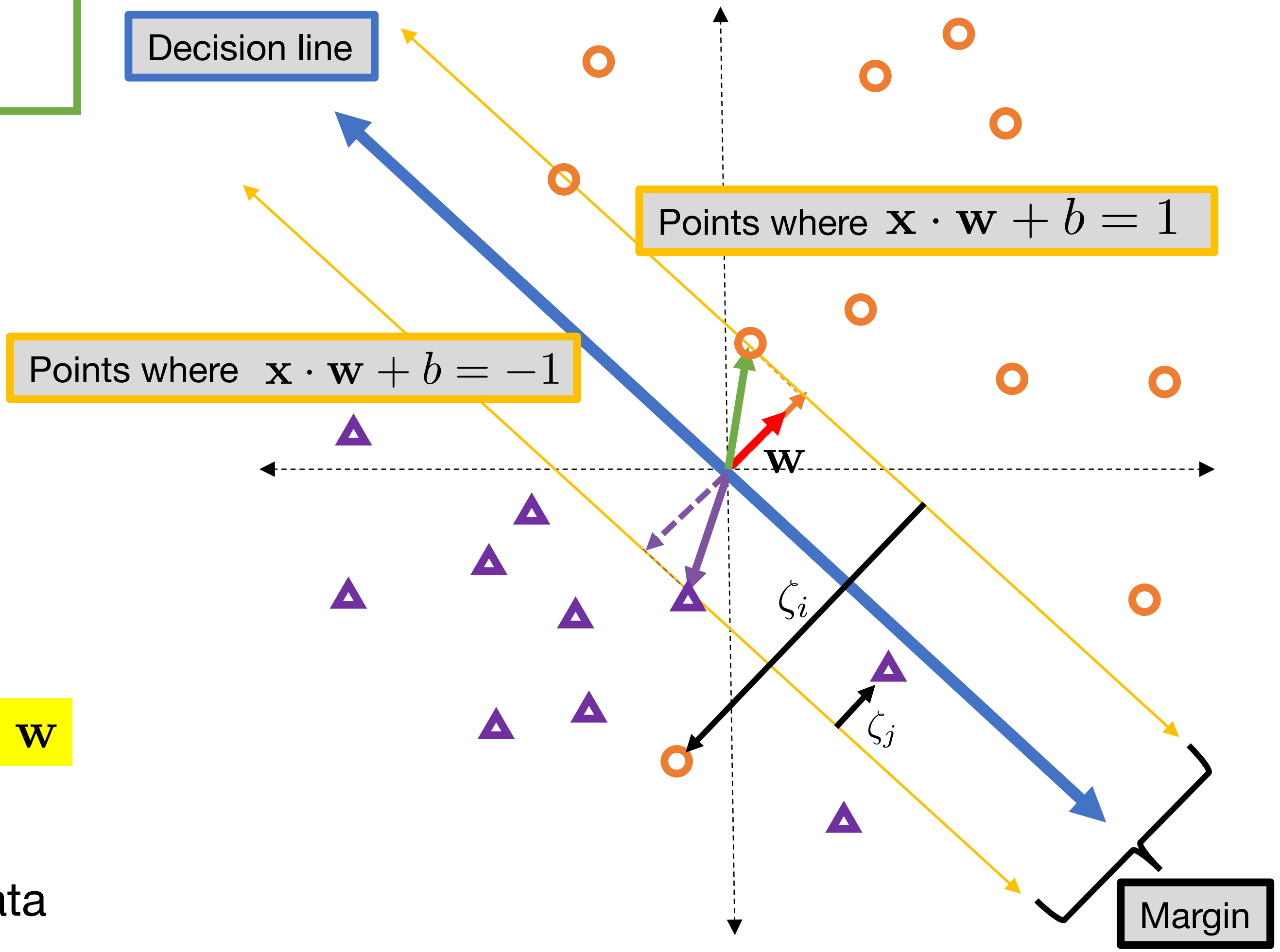

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- **Option 2:** Minimize error function with gradient descent

Error function to minimize

Hyperparameter (weight placed on examples on “wrong side” of margin)

$$E(\mathbf{w}, b) = \frac{1}{2}\mathbf{w}^\intercal \mathbf{w} + C \sum_{i=1}^{m} hinge(\mathbf{w}^\intercal \mathbf{x_i} + b)$$

Maximize margin

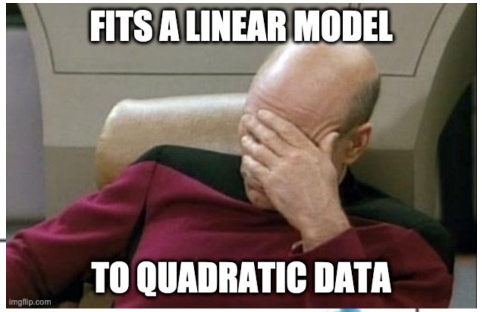
Penalty for examples on “wrong side” of margin

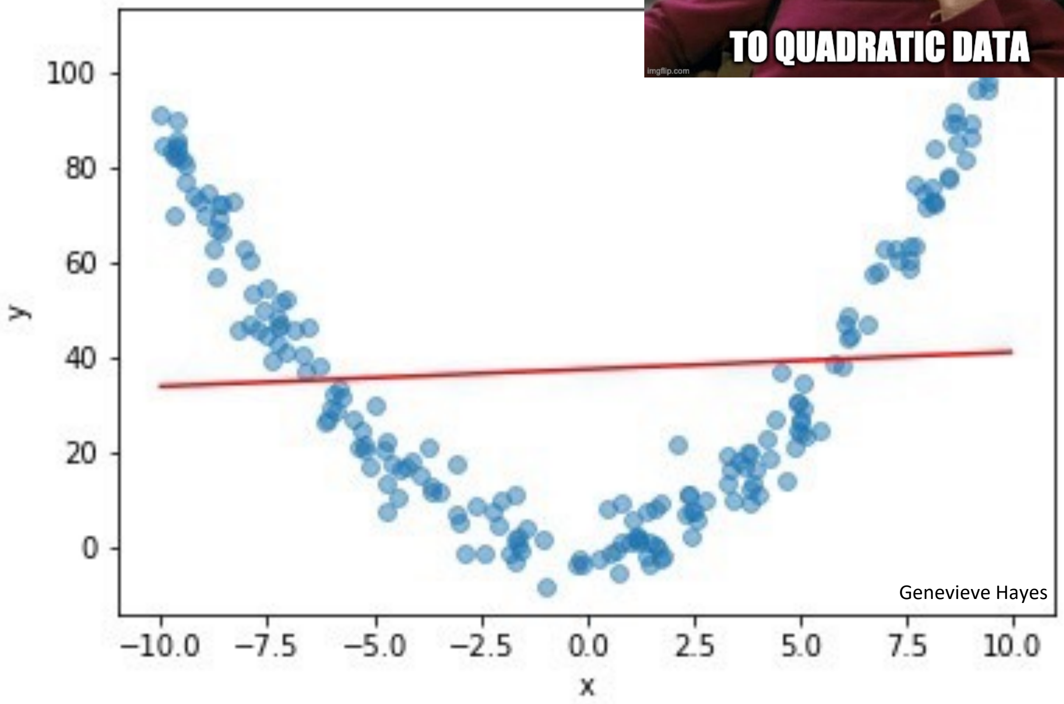
0 if on “right side” of margin, ζ_i otherwise

Non-Linear SVM

Non-Linear SVM

- Many datasets derive from nonlinear phenomena
  - Poor match for linear models

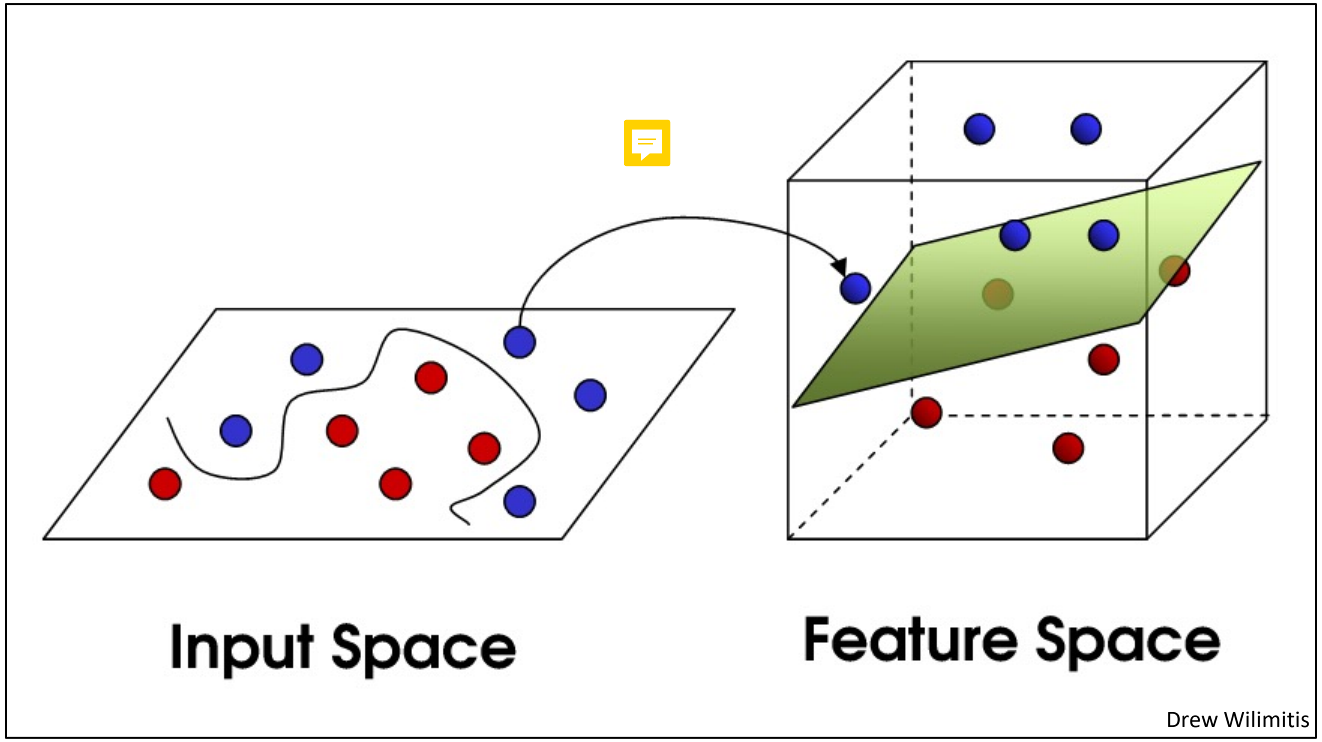




- Naïve option:
  - Add polynomial features to data
  - BUT produces too many features for high-degree polynomials

Non-Linear SVM: Kernel Trick

- Many non-linear datasets become linearly separable if projected to higher dimensional spaces



Non-Linear SVM: Kernel Trick

- Many non-linear datasets become linearly separable if projected to higher dimensional spaces

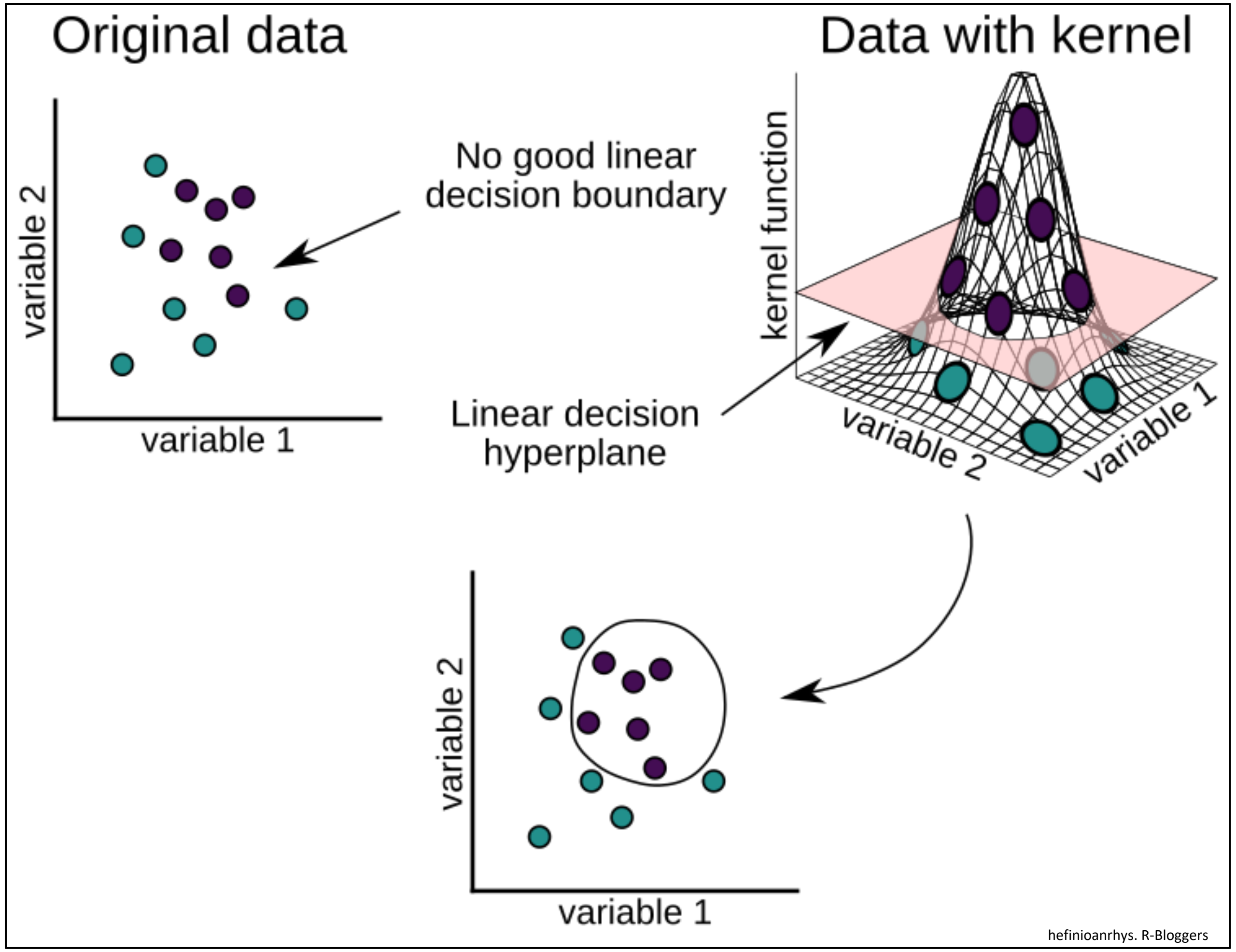
- BUT resulting projections often have too many features
    - Computationally infeasible training

Non-Linear SVM: **Kernel Trick**

- Re-derive SVM training minimization problem into **dual** format
  - See textbook for overview of math
  - ★ Only requires a **similarity metric** between **pairs of features,** not the features themselves

- A **kernel function** computes the similarity between vectors in a higher dimensional space…*without needing to project the vectors into that space!!!*
  - Plug-and-play kernel functions into linear SVM to train a non-linear model

Non-Linear SVM: **Kernel Trick**



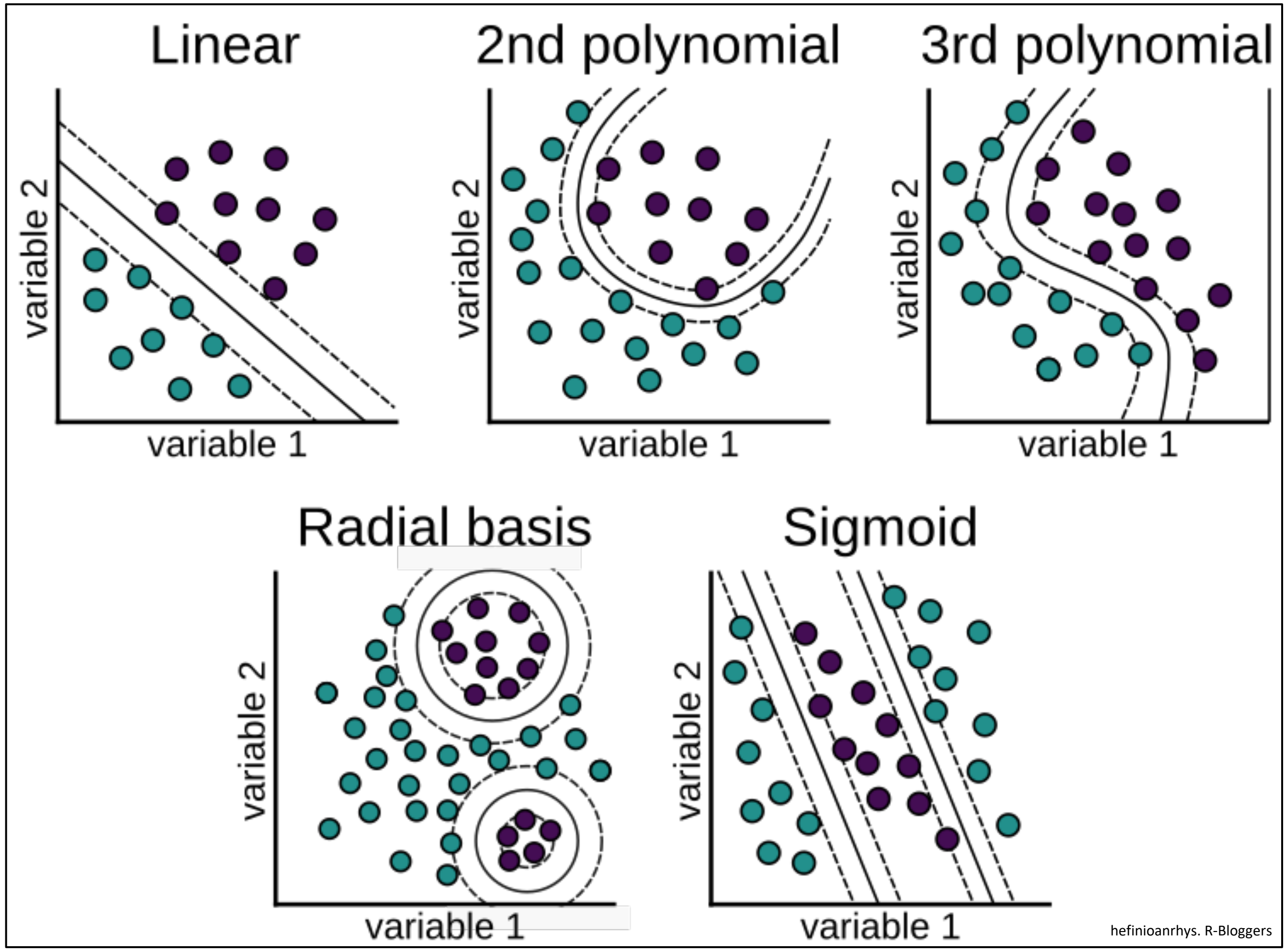
- d-degree polynomial kernel

$$K(\mathbf{x}_1, \mathbf{x}_2) = (\mathbf{x_1}^\intercal \mathbf{x_2})^d$$

- Similarity metric between vectors with d-degree polynomial features

- No need to compute the features themselves!

Non-Linear SVM: Kernel Trick



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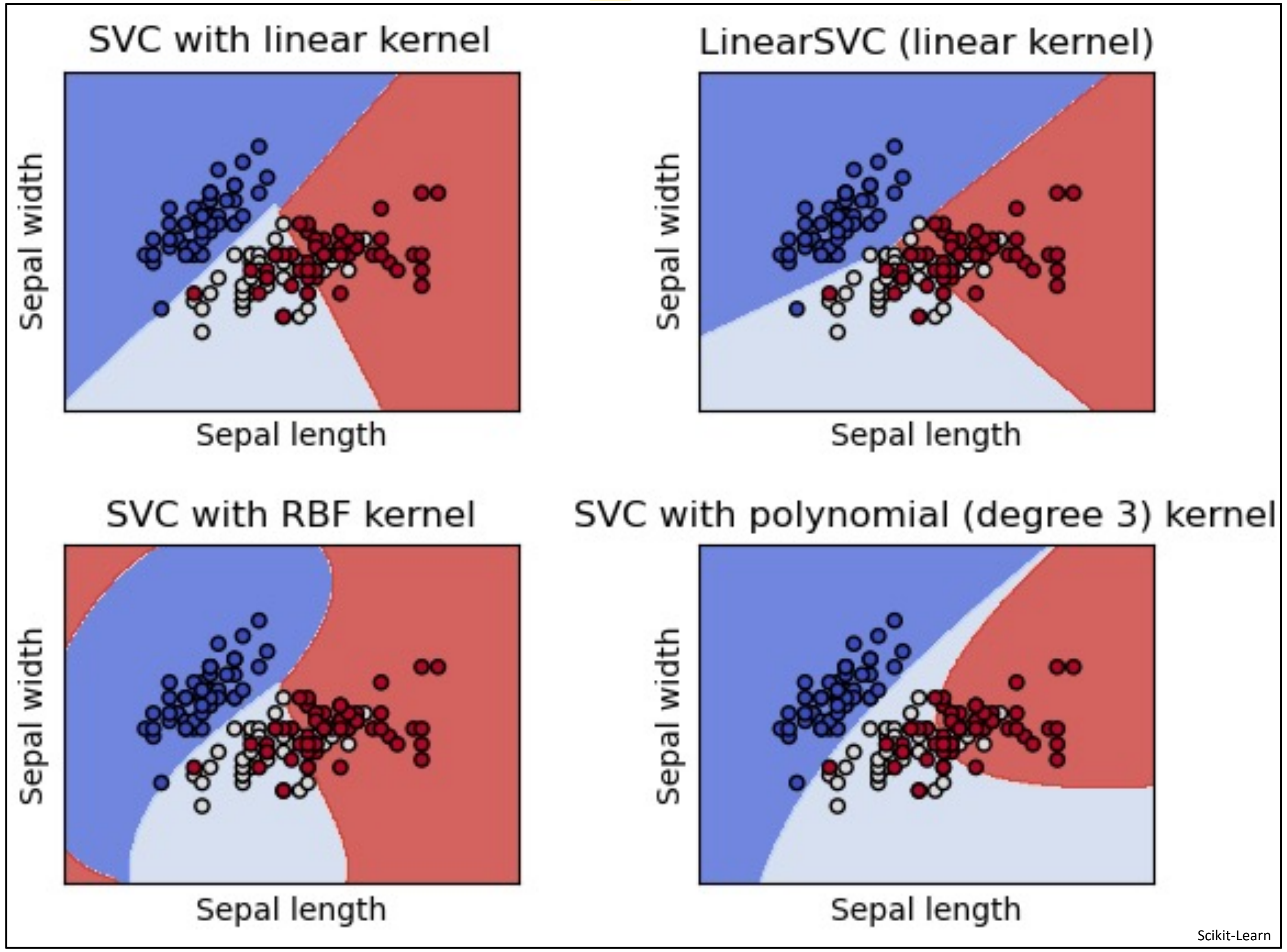
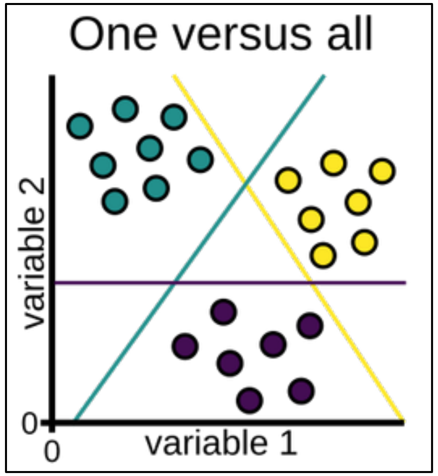
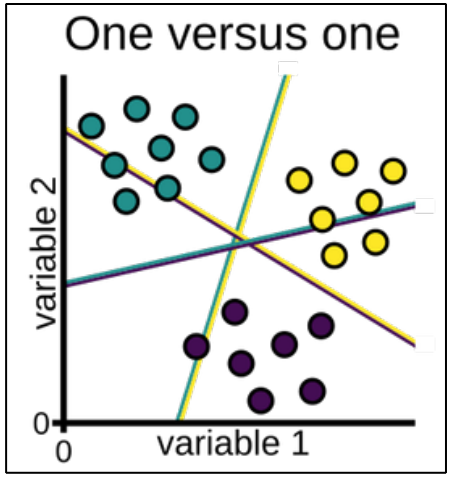


Multiclass SVM

Multiclass SVM

- One-versus-rest
  - For N classes, train N SVMs (each binary…one class versus all others)
- One-versus-one
  - For N classes, train $\binom{N}{2}$ classifiers (all pairwise combinations)
- Both
  - Predict new data using all models
  - Keep prediction that is *furthest from its decision line*





hefinioanrhys, R-Bloggers

Practical Use of SVMs

★ "Small" datasets

- Not enough data to train a neural network
- 1000s to 100,000s of examples
  - Very approximate limit → depends heavily on learning task

- Support vector **regression**
  - Fit data inside margin instead of outside margin
  - See textbook for more

Programming Practice

SVM.ipynb

Questions?