Technical aspect for the paper: A new model for control of systems with friction[1]

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Given a LuGre friction model (*see Eq. 1 and 2 of the paper*):

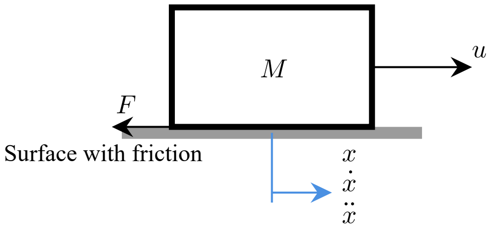
$$\dot{z} = \frac{dz}{dt} = v - \frac{|v|}{g(v)} z \tag{1}$$

and:

$$g(v) = \frac{F_C + (F_S - F_C)e^{-(v/v_s)^2}}{\sigma_0} \tag{2}$$

Substituting Eq. (2) to Eq. (1) gives us:

$$\dot{z} = v - \frac{z|v|\sigma_0}{F_C + (F_S - F_C)e^{-(v/v_s)^2}} \tag{3}$$



The dynamic model of a system with a mass of M, an external force of u, and a friction force of F can be written as:

$$\ddot{x} = \frac{u - F}{M} \tag{4}$$

As explained in the paper, the friction force F here is dynamic and is expressed as follows (*see Eq. 3 of the paper*):

$$F = \sigma_0 z + \sigma_1 \dot{z} + \sigma_2 v \tag{5}$$

where

$$v = \dot{x} \tag{6}$$

Substituting Eq. (5) and Eq. (6) to Eq. (4) gives us:

$$\ddot{x} = \frac{u - \sigma_0 z - \sigma_1 \dot{z} - \sigma_2 v}{M} \tag{7}$$

where:

$$\dot{z} = \dot{x} - \frac{z|\dot{x}|\sigma_0}{F_C + (F_S - F_C)e^{-(\dot{x}/v_s)^2}} \tag{8}$$

Now, let us take:

$$\begin{aligned} q_1 = x &\rightarrow \dot{q}_1 = \dot{x} \\ q_2 = \dot{x} &\rightarrow \dot{q}_2 = \ddot{x} \\ q_3 = z &\rightarrow \dot{q}_3 = \dot{z} \end{aligned} \tag{9}$$

Applying Eq. (9) to Eq. (7) and Eq. (8) then gives us:

$$\begin{aligned} \dot{q}_1 &= q_2 \\ \dot{q}_2 &= \frac{(u - \sigma_0 q_3 - \sigma_1 \dot{z} - \sigma_2 q_2)}{M} \\ \dot{q}_3 &= \dot{z} \end{aligned} \tag{10}$$

[1] C. Canudas de Wit, H. Olsson, K. J. Astrom, and P. Lischinsky, "A new model for control of systems with friction," IEEE Trans. Automat. Contr., vol. 40, no. 3, pp. 419–425, Mar. 1995.

where:

$$\dot{z} = q_2 - \frac{z|q_2|\sigma_0}{F_C + (F_S - F_C)e^{-(q_2/v_s)^2}} \tag{11}$$

We need to be aware that in the right-hand-side of Eq. (10), we do not replace $\dot{z}$ with $\dot{q}_3$ because we will compute $\dot{z}$ independently by using Eq. (11). Eq. (11) merely acts as an additional equation for Eq. (10). Eq. (10) now follows the formal form: $\dot{q} = f(q)$. Thus, we can implement it into MATLAB built-in ODE solver.

Next, we will address $u(t)$. In the selected paper, only two kinds of force input are found: a ramp force input as in the varying break-away force simulation and a piecewise linear force input as in the presliding simulation.

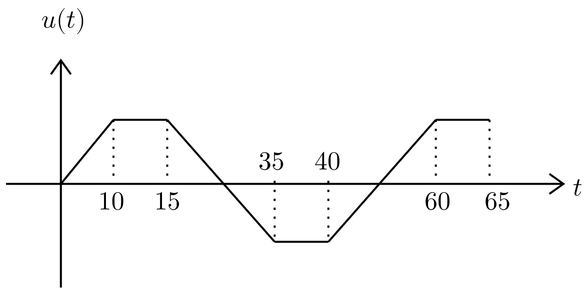
Ramp Force Input

```
function xdot = sim_mass_with_ramp_force_input(t, q, M, Fs, Fc, sigma_0, sigma_1, sigma_2, vs, force_rate)
    u = force_rate*t;      % ramped-up force input

    zdot = q(2) - ( (q(3)*abs(q(2))*sigma_0) / (Fc+(Fs-Fc)*exp(-(q(2)/vs)^2)) );
    F = sigma_0*q(3) + sigma_1 * zdot + sigma_2*q(2);

    qdot_1 = q(2);
    qdot_2 = (u - F) / M;
    qdot_3 = zdot;
    xdot = [qdot_1 ; qdot_2; qdot_3 ];
end
```

Piecewise Linear Force Input




```
function xdot = sim_presliding(t, q, M, Fs, Fc, sigma_0, sigma_1, sigma_2, vs)
    rate = 0.1425;

    if t <= 10
        u = rate*t;
    elseif t > 10 && t <= 15
        u = 1.425;
    elseif t > 15 && t <= 35
        u = 1.425 - rate*(t-15);
    elseif t > 35 && t <= 40
        u = -1.425;
    elseif t > 40 && t <= 60
        u = -1.425 + rate*(t-40);
    elseif t > 60
```

```
        u = 1.425;
    end

    zdot = q(2) - ( (q(3)*abs(q(2))*sigma_0) / (Fc+(Fs-Fc)*exp(-(q(2)/vs)^2)) );
    F = sigma_0*q(3) + sigma_1 * zdot + sigma_2*q(2);

    qdot_1 = q(2);
    qdot_2 = (u - F) / M;
    qdot_3 = zdot;
    xdot = [qdot_1 ; qdot_2; qdot_3 ];
end
```

A System with PID

Given a system:

$$\ddot{x} = \frac{u - F}{M} \tag{12}$$

For a system with a PID control, we simply set u with the following PID control law (*see Eq. 11 of the paper*):

$$u = -K_v v - K_p e - K_i \int e \; dt \tag{13}$$

where:

$$e = x - x_d \tag{14}$$

By substituting Eq. (13) to Eq. (12), we get the following equation:

$$\ddot{x} = \frac{\overbrace{-K_v v - K_p e - K_i \int e \; dt}^{u} - \overbrace{(\sigma_0 z + \sigma_1 \dot{z} + \sigma_2 v)}^{F}}{M} \tag{15}$$

Next, we introduce a new variable (q_4) to accommodate the integral term:

$$q_4 = \int e \; dt \rightarrow \dot{q}_4 = e \tag{16}$$

Substituting Eq. (16) to Eq. (15) and replacing v with q_2 and $\ddot{x}$ with $\dot{q}_2$ give us:

$$\dot{q}_2 = \frac{\overbrace{-K_v q_2 - K_p e - K_i \int e \; dt}^{u} - \overbrace{(\sigma_0 z + \sigma_1 \dot{z} + \sigma_2 q_2)}^{F}}{M} \tag{17}$$

Hence, we now have a new system as follows:

$$\begin{gathered} \dot{q}_1 = q_2 \\ \dot{q}_2 = \frac{\left(-K_v q_2 - K_p q_4 - K_i \int e \; dt - \sigma_0 q_3 - \sigma_1 \dot{z} - \sigma_2 q_2\right)}{M} \\ \dot{q}_3 = \dot{z} \\ \dot{q}_4 = e \end{gathered} \tag{18}$$

The MATLAB implementation is as follows:

```
function xdot = sim_pid(t, q, M, Fs, Fc, sigma_0, sigma_1, sigma_2, vs, xd)
    Kp = 3;
    Ki = 4;
    Kv = 6;

    e = q(1) - xd;

    u = -Kv*q(2)-Kp*e-Ki*q(4);

    zdot = q(2) - ( (q(3)*abs(q(2))*sigma_0) / (Fc+(Fs-Fc)*exp(-(q(2)/vs)^2)) );
    F = sigma_0*q(3) + sigma_1 * zdot + sigma_2*q(2);

    qdot_1 = q(2);
    qdot_2 = (u - F) / M;
    qdot_3 = zdot;
    qdot_4 = e;
    xdot = [qdot_1 ; qdot_2; qdot_3; qdot_4];
end
```