Exercises 2.2
Problem 2, using the integrating factor
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Problem 2a

$y'(x) - y = 3e^x$

Given an ODE:

$$y'(x) - y(x) = 3\,e^x$$

Thus, we have:

$$p(x) = -1$$

$$q(x) = 3\,e^x$$

Step 1, find the integrating factor:

$$\sigma(x) = e^{\int p(x)\,dx}$$

$$\sigma(x) = e^{-x}$$

Step 2:

$$\frac{d}{dx}\left(\sigma(x)\,y(x)\right) = \sigma(x)\,q(x)$$

$$-e^{-x}y(x) + e^{-x}y'(x) = 3\,e^{-x}e^x$$

Step 3, integrate both sides wrt x:

$$\int\left(\frac{d}{dx}\left(\sigma(x)\,y(x)\right)\right)dx = \int\sigma(x)\,q(x)\,dx$$

$$\int\left(-e^{-x}y(x) + e^{-x}y'(x)\right)dx = \int 3\,e^{-x}e^x\,dx$$

$$e^{-x}y(x) = 3\,x + C$$

$$y(x) = \frac{3\,x + C}{e^{-x}}$$

$$y(x) = (3\,x + C)\,e^x$$

Problem 2b

$y'(x) + 4\,y = 8$

Given an ODE:

$$y'(x) + 4\,y(x) = 8$$

Thus, we have:

$$p(x) = 4$$

$$q(x) = 8$$

Step 1, find the integrating factor:

$$\sigma(x) = e^{\int p(x)\,dx}$$

$$\sigma(x) = e^{4\,x}$$

Step 2:

$$\frac{d}{dx}\left(\sigma(x)\,y(x)\right)=\sigma(x)\,q(x)$$

$$4\,e^{4x}y(x)+e^{4x}y'(x)=8\,e^{4x}$$

Step 3, integrate both sides wrt x:

$$\int\left(\frac{d}{dx}\left(\sigma(x)\,y(x)\right)\right)dx=\int\sigma(x)\,q(x)\,dx$$

$$\int\left(4\,e^{4x}y(x)+e^{4x}y'(x)\right)dx=\int 8\,e^{4x}\,dx$$

$$e^{4x}y(x)=2\,e^{4x}+C$$

$$y(x)=\frac{2\,e^{4x}+C}{e^{4x}}$$

$$y(x)=2+C\,e^{-4x}$$

Problem 2c

$y'(x)+y=x^2$

Given an ODE:

$$y'(x)+y(x)=x^2$$

Thus, we have:

$$p(x)=1$$

$$q(x)=x^2$$

Step 1, find the integrating factor:

$$\sigma(x)=e^{\int p(x)\,dx}$$

$$\sigma(x)=e^{x}$$

Step 2:

$$\frac{d}{dx}\left(\sigma(x)\,y(x)\right)=\sigma(x)\,q(x)$$

$$e^{x}y(x)+e^{x}y'(x)=e^{x}x^2$$

Step 3, integrate both sides wrt x:

$$\int\left(\frac{d}{dx}\left(\sigma(x)\,y(x)\right)\right)dx=\int\sigma(x)\,q(x)\,dx$$

$$\int\left(e^{x}y(x)+e^{x}y'(x)\right)dx=\int e^{x}x^2\,dx$$

$$e^{x}y(x)=\left(x^2-2\,x+2\right)e^{x}+C$$

$$y(x)=\frac{e^{x}x^2-2\,e^{x}x+C+2\,e^{x}}{e^{x}}$$

$$y(x)=x^2-2\,x+C\,e^{-x}+2$$

Problem 2d

$y'(x) = y - \sin(2\,x)$

Given an ODE:

$$y'(x) - y(x) = -\sin(2\,x)$$

Thus, we have:

$$p(x) = -1$$

$$q(x) = -\sin(2\,x)$$

Step 1, find the integrating factor:

$$\sigma(x) = e^{\int p(x)\,dx}$$

$$\sigma(x) = e^{-x}$$

Step 2:

$$\frac{d}{dx}\left(\sigma(x)\,y(x)\right) = \sigma(x)\,q(x)$$

$$-e^{-x}y(x) + e^{-x}y'(x) = -e^{-x}\sin(2\,x)$$

Step 3, integrate both sides wrt x:

$$\int\left(\frac{d}{dx}\left(\sigma(x)\,y(x)\right)\right)dx = \int\sigma(x)\,q(x)\,dx$$

$$\int\left(-e^{-x}y(x) + e^{-x}y'(x)\right)dx = \int -e^{-x}\sin(2\,x)\,dx$$

$$e^{-x}y(x) = \frac{2\,e^{-x}\cos(2\,x)}{5} + \frac{e^{-x}\sin(2\,x)}{5} + C$$

$$y(x) = \frac{2\,e^{-x}\cos(2\,x) + e^{-x}\sin(2\,x) + 5\,C}{5\,e^{-x}}$$

$$y(x) = \frac{2\cos(2\,x)}{5} + \frac{\sin(2\,x)}{5} + C\,e^{x}$$

Problem 2e

$y'(x) - \tan(x)y = 6$

Given an ODE:

$$y'(x) - \tan(x)\,y(x) = 6$$

Thus, we have:

$$p(x) = -\tan(x)$$

$$q(x) = 6$$

Step 1, find the integrating factor:

$$\sigma(x) = e^{\int p(x)\,dx}$$

$$\sigma(x) = \cos(x)$$

Step 2:

$$\frac{d}{dx}\left(\sigma(x)\,y(x)\right)=\sigma(x)\,q(x)$$

$$-\sin(x)\,y(x)+\cos(x)\,y'(x)=6\cos(x)$$

Step 3, integrate both sides wrt x:

$$\int\left(\frac{d}{dx}\left(\sigma(x)\,y(x)\right)\right)dx=\int\sigma(x)\,q(x)\,dx$$

$$\int(-\sin(x)\,y(x)+\cos(x)\,y'(x))\,dx=\int 6\cos(x)\,dx$$

$$\cos(x)\,y(x)=6\sin(x)+C$$

$$y(x)=\frac{6\sin(x)+C}{\cos(x)}$$

Problem 2f

$xy'(x)+2y=x^3 \Leftrightarrow y'(x)+\frac{2}{x}y=x^3$

Given an ODE:

$$y'(x)+\frac{2\,y(x)}{x}=x^3$$

Thus, we have:

$$p(x)=\frac{2}{x}$$

$$q(x)=x^3$$

Step 1, find the integrating factor:

$$\sigma(x)=e^{\int p(x)\,dx}$$

$$\sigma(x)=x^2$$

Step 2:

$$\frac{d}{dx}\left(\sigma(x)\,y(x)\right)=\sigma(x)\,q(x)$$

$$2\,x\,y(x)+x^2\,y'(x)=x^5$$

Step 3, integrate both sides wrt x:

$$\int\left(\frac{d}{dx}\left(\sigma(x)\,y(x)\right)\right)dx=\int\sigma(x)\,q(x)\,dx$$

$$\int\left(2\,x\,y(x)+x^2\,y'(x)\right)dx=\int x^5\,dx$$

$$x^2\,y(x)=\frac{x^6}{6}+C$$

$$y(x)=\frac{x^6+6\,C}{6\,x^2}$$

Problem 2g

$$xy'(x) - 2y = x^3 \Leftrightarrow y'(x) - \frac{2}{x}y = x^3$$

Given an ODE:

$$y'(x) - \frac{2\,y(x)}{x} = x^3$$

Thus, we have:

$$p(x) = -\frac{2}{x}$$

$$q(x) = x^3$$

Step 1, find the integrating factor:

$$\sigma(x) = \mathrm{e}^{\int p(x)\,\mathrm{d}x}$$

$$\sigma(x) = \frac{1}{x^2}$$

Step 2:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sigma(x)\,y(x)\right) = \sigma(x)\,q(x)$$

$$-\frac{2\,y(x)}{x^3} + \frac{y'(x)}{x^2} = x$$

Step 3, integrate both sides wrt x:

$$\int\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(\sigma(x)\,y(x)\right)\right)\mathrm{d}x = \int\sigma(x)\,q(x)\,\mathrm{d}x$$

$$\int\left(-\frac{2\,y(x)}{x^3} + \frac{y'(x)}{x^2}\right)\mathrm{d}x = \int x\,\mathrm{d}x$$

$$\frac{y(x)}{x^2} = \frac{x^2}{2} + C$$

$$y(x) = \frac{(x^2 + 2\,C)\,x^2}{2}$$

$$y(x) = \frac{1}{2}\,x^4 + C\,x^2$$

Problem 2h

$y'(x) + \cot(x)y = 2\cos(x)$

Given an ODE:

$$y'(x) + \cot(x)\,y(x) = 2\cos(x)$$

Thus, we have:

$$p(x) = \cot(x)$$

$$q(x) = 2\cos(x)$$

Step 1, find the integrating factor:

$$\sigma(x) = \mathrm{e}^{\int p(x)\,\mathrm{d}x}$$

$$\sigma(x) = \sin(x)$$

Step 2:

$$\frac{d}{dx}\left(\sigma(x)\, y(x)\right) = \sigma(x)\, q(x)$$

$$\cos(x)\, y(x) + \sin(x)\, y'(x) = 2\sin(x)\cos(x)$$

Step 3, integrate both sides wrt x:

$$\int\left(\frac{d}{dx}\left(\sigma(x)\, y(x)\right)\right) dx = \int \sigma(x)\, q(x)\, dx$$

$$\int (\cos(x)\, y(x) + \sin(x)\, y'(x))\, dx = \int 2\sin(x)\cos(x)\, dx$$

$$\sin(x)\, y(x) = \sin(x)^2 + C$$

$$y(x) = \frac{\sin(x)^2 + C}{\sin(x)}$$

Problem 2i

$$(x-5)(xy' + 3y) = 2$$

$$xy' + 3y = \frac{2}{(x-5)}$$

$$y' + \frac{3}{x}\, y = \frac{2}{x(x-5)}$$

Given an ODE:

$$y'(x) + \frac{3\, y(x)}{x} = \frac{2}{x\,(x-5)}$$

Thus, we have:

$$p(x) = \frac{3}{x}$$

$$q(x) = \frac{2}{x\,(x-5)}$$

Step 1, find the integrating factor:

$$\sigma(x) = e^{\int p(x)\, dx}$$

$$\sigma(x) = x^3$$

Step 2:

$$\frac{d}{dx}\left(\sigma(x)\, y(x)\right) = \sigma(x)\, q(x)$$

$$3x^2\, y(x) + x^3\, y'(x) = \frac{2x^2}{x-5}$$

Step 3, integrate both sides wrt x:

$$\int\left(\frac{d}{dx}\left(\sigma(x)\,y(x)\right)\right)dx=\int\sigma(x)\,q(x)\,dx$$

$$\int\left(3\,x^2\,y(x)+x^3\,y'(x)\right)dx=\int\frac{2\,x^2}{x-5}\,dx$$

$$x^3\,y(x)=x^2+10\,x+50\,\ln(x-5)+C$$

$$y(x)=\frac{x^2+10\,x+50\,\ln(x-5)+C}{x^3}$$

Problem 2j

$x'-6\,x=e^y$

This has a different form since x is the dependent variable and y is the indepenent variable. Thus, we need to change x to y and y to x to make it fit to the Maple procedure.

$y'-6\,y=e^x$

Given an ODE:

$$y'(x)-6\,y(x)=e^x$$

Thus, we have:

$$p(x)=-6$$

$$q(x)=e^x$$

Step 1, find the integrating factor:

$$\sigma(x)=e^{\int p(x)\,dx}$$

$$\sigma(x)=e^{-6\,x}$$

Step 2:

$$\frac{d}{dx}\left(\sigma(x)\,y(x)\right)=\sigma(x)\,q(x)$$

$$-6\,e^{-6\,x}\,y(x)+e^{-6\,x}\,y'(x)=e^{-6\,x}\,e^x$$

Step 3, integrate both sides wrt x:

$$\int\left(\frac{d}{dx}\left(\sigma(x)\,y(x)\right)\right)dx=\int\sigma(x)\,q(x)\,dx$$

$$\int\left(-6\,e^{-6\,x}\,y(x)+e^{-6\,x}\,y'(x)\right)dx=\int e^{-6\,x}\,e^x\,dx$$

$$e^{-6\,x}\,y(x)=-\frac{e^{-5\,x}}{5}+C$$

$$y(x)=\frac{5\,C-e^{-5\,x}}{5\,e^{-6\,x}}$$

$$y(x)=C\,e^x\,e^{5\,x}-\frac{e^x}{5}$$

Problem 2k

$yx'-y^5+3\,x=0$

This has a different form since x is the dependent variable and y is the indepenent variable. Thus, we need to change x to y and y to x to make it fit to the Maple procedure.

$xy' - x^5 + 3y = 0$

$y' + \frac{3}{x}y = x^4$

Given an ODE:

$$y'(x) + \frac{3\,y(x)}{x} = x^4$$

Thus, we have:

$$p(x) = \frac{3}{x}$$

$$q(x) = x^4$$

Step 1, find the integrating factor:

$$\sigma(x) = e^{\int p(x)\,dx}$$

$$\sigma(x) = x^3$$

Step 2:

$$\frac{d}{dx}\left(\sigma(x)\,y(x)\right) = \sigma(x)\,q(x)$$

$$3\,x^2\,y(x) + x^3\,y'(x) = x^7$$

Step 3, integrate both sides wrt x:

$$\int\left(\frac{d}{dx}\left(\sigma(x)\,y(x)\right)\right)dx = \int\sigma(x)\,q(x)\,dx$$

$$\int\left(3\,x^2\,y(x) + x^3\,y'(x)\right)dx = \int x^7\,dx$$

$$x^3\,y(x) = \frac{x^8}{8} + C$$

$$y(x) = \frac{x^8 + 8\,C}{8\,x^3}$$

Problem 2l

$y^2x' + xy - 4\,y^2 = 1$

This has a different form since x is the dependent variable and y is the indepenent variable. Thus, we need to change x to y and y to x to make it fit to the Maple procedure.

$x^2y' + xy - 4\,x^2 = 1$

$y' + \frac{1}{x}y = 4\,x$

Given an ODE:

$$y'(x) + \frac{y(x)}{x} = 4\,x$$

Thus, we have:

$$p(x) = \frac{1}{x}$$

$$q(x) = 4\,x$$

Step 1, find the integrating factor:

$$\sigma(x) = e^{\int p(x)\, dx}$$

$$\sigma(x) = x$$

Step 2:

$$\frac{d}{dx}\left(\sigma(x)\, y(x)\right) = \sigma(x)\, q(x)$$

$$y(x) + x\, y'(x) = 4\, x^2$$

Step 3, integrate both sides wrt x:

$$\int\left(\frac{d}{dx}\left(\sigma(x)\, y(x)\right)\right) dx = \int \sigma(x)\, q(x)\, dx$$

$$\int (y(x) + x\, y'(x))\, dx = \int 4\, x^2\, dx$$

$$x\, y(x) = \frac{4\, x^3}{3} + C$$

$$y(x) = \frac{4\, x^3 + 3\, C}{3\, x}$$

Problem 2m

$tx' - 4\, t^5 = x$

This has a different form since x is the dependent variable and t is the indepenent variable. Thus, we need to change x to y and t to x to make it fit to the Maple procedure.

$xy' - 4\, x^5 = y$

$y' - 4\, x^4 = \frac{1}{x} y$

$y' - \frac{1}{x} y = 4\, x^4$

Given an ODE:

$$y'(x) - \frac{y(x)}{x} = 4\, x^4$$

Thus, we have:

$$p(x) = -\frac{1}{x}$$

$$q(x) = 4\, x^4$$

Step 1, find the integrating factor:

$$\sigma(x) = e^{\int p(x)\, dx}$$

$$\sigma(x) = \frac{1}{x}$$

Step 2:

$$\frac{d}{dx}\left(\sigma(x)\, y(x)\right) = \sigma(x)\, q(x)$$

$$-\frac{y(x)}{x^2} + \frac{y'(x)}{x} = 4x^3$$

Step 3, integrate both sides wrt x:

$$\int \left(\frac{d}{dx} \left(\sigma(x)\, y(x) \right) \right) dx = \int \sigma(x)\, q(x)\, dx$$

$$\int \left(-\frac{y(x)}{x^2} + \frac{y'(x)}{x} \right) dx = \int 4x^3\, dx$$

$$\frac{y(x)}{x} = x^4 + C$$

$$y(x) = \left(x^4 + C\right) x$$