Riemann-sum for complex integral

Given $f(z) = e^z$ and a line made by $z_0 = 0$ to $z_1 = 2 + \frac{I\pi}{4}$

The Riemann-sum of complex integral can be expressed as follows:

$$\int_C f(z)dz = \lim_{n\to\infty} \sum_{k=0}^{n} f(z_k)\Delta z_k$$

Riemann-sum

```
> restart:
  F := z -> exp(z):
  `f(z) ` = F(z);
  a := 0:
  b := 2 + I*Pi/4:
  n := 8:
  `a ` = a, `     b ` = b, `     n ` = n;
  dz := (b-a)/n:
  z := k -> a + k*dz:
  Sum(f(z[k])*Delta*z[k]) = Sum(F(z(k))*dz);

  RS := sum(F(z(k))*dz, k=1..8):
  `The Riemann sum is:`;
  Sum(f(z[k])*Delta*z[k],k=1..8) = RS;
  `The Riemann sum is:`;
  Sum(f(z[k])*Delta*z[k],k=1..8) = evalf(RS);
```

$$f(z) = e^z$$

$$a = 0, \quad b = 2 + \frac{1}{4} I\pi, \quad n = 8$$

$$\sum f(z_k)\,\Delta z_k = \sum e^{k\left(\frac{1}{4} + \frac{1}{32} I\pi\right)} \left(\frac{1}{4} + \frac{1}{32} I\pi\right)$$

The Riemann sum is:

$$\sum_{k=1}^{8} f(z_k)\,\Delta z_k = e^{\frac{1}{4} + \frac{1}{32} I\pi}\left(\frac{1}{4} + \frac{1}{32} I\pi\right) + e^{\frac{1}{2} + \frac{1}{16} I\pi}\left(\frac{1}{4} + \frac{1}{32} I\pi\right) + e^{\frac{3}{4} + \frac{3}{32} I\pi}\left(\frac{1}{4} + \frac{1}{32} I\pi\right) + e^{1 + \frac{1}{8} I\pi}\left(\frac{1}{4} + \frac{1}{32} I\pi\right) + e^{\frac{5}{4} + \frac{5}{32} I\pi}\left(\frac{1}{4} + \frac{1}{32} I\pi\right) + e^{\frac{3}{2} + \frac{3}{16} I\pi}\left(\frac{1}{4} + \frac{1}{32} I\pi\right) + e^{\frac{7}{4} + \frac{7}{32} I\pi}\left(\frac{1}{4} + \frac{1}{32} I\pi\right) + e^{2 + \frac{1}{4} I\pi}\left(\frac{1}{4} + \frac{1}{32} I\pi\right)$$

The Riemann sum is:

$$\sum_{k=1}^{8} f(z_k)\,\Delta z_k = 4.493757363 + 6.125610273\, I$$

Complex integral

```
> restart:
> with(plots,implicitplot):
> z0 := 0:
```

```
x[0] := Re(z0):
y[0] := Im(z0):

z1 := 2 + I*Pi/4:
x[1] := Re(z1):
y[1] := Im(z1):

z := t-> x[0] + (x[1]-x[0])*t + I*(y[0] + (y[1]-y[0])*t):
zd := t-> diff(z(t),t):

f :=z -> exp(z):
F := t->f(z(t)):

`The line can be calculated as follows:`;
'z(t)=x[0] + (x[1]-x[0])*t + I*(y[0] + (y[1]-y[0])*t)';
`where`;
'z[0] = 0';
'x[0] = Re(z[0])', 'y[0] = Im(z[0])';
`and`;
'z[1] = 1 + I';
'x[1] = Re(z[1])', 'y[1] = Im(z[1])';

`\nThus, the line parametric equation is: `;
`C:z(t)` = z(t);

`\nAnd the derivative is:`;
'diff(z(t),t)' = evalc(zd(t));

`\nBecause `, 'f(z) = exp(z)', ` thus `;
'f(z(t))'=F(t);
'f(t)'=evalc(F(t));

` \nTherefore:`;
'int(f(z),z=C..)'='Int(f(z(t)) * diff(z(t),t), t=0..1)';
'int(f(z),z=C..)' = Int(F(t) * zd(t), t=0..1);

`Notice that the anti-derivative is: `;

g:=t->F(t): # Keep the anti-derivative

`g(t)` = g(t);

`Thus: `;
'int(f(z),z=C..)' = 'g(1) - g(0)';
'int(f(z),z=C..)' = g(1) - g(0);
'int(f(z),z=C..)' = evalf(g(1)-g(0));
```

The line can be calculated as follows:

$$z(t) = x_0 + (x_1 - x_0)\, t + \mathrm{I}\, (y_0 + (y_1 - y_0)\, t)$$

where

$$z_0 = 0$$

$$x_0 = \Re(z_0), y_0 = \Im(z_0)$$

and

$$z_1 = 1 + \mathrm{I}$$

$$x_1 = \Re(z_1), y_1 = \Im(z_1)$$

Thus, the line parametric equation is:

$$C{:}z(t) = 2\,t + \frac{\mathrm{I}\,\pi\,t}{4}$$

And the derivative is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,z(t) = 2 + \frac{\mathrm{I}\,\pi}{4}$$

$$\mathit{Because}\,,f(z) = \mathrm{e}^{z},\ \mathit{thus}$$

$$f(z(t)) = \mathrm{e}^{2\,t + \frac{\mathrm{I}\,\pi\,t}{4}}$$

$$f(t) = \mathrm{e}^{2\,t}\cos\left(\frac{\pi\,t}{4}\right) + \mathrm{I}\,\mathrm{e}^{2\,t}\sin\left(\frac{\pi\,t}{4}\right)$$

Therefore:

$$\int_{C}^{(\,)} f(z)\,\mathrm{d}z = \int_{0}^{1} f(z(t))\left(\frac{\mathrm{d}}{\mathrm{d}t}\,z(t)\right)\mathrm{d}t$$

$$\int_{C}^{(\,)} f(z)\,\mathrm{d}z = \int_{0}^{1} \mathrm{e}^{2\,t + \frac{\mathrm{I}\,\pi\,t}{4}}\left(2 + \frac{\mathrm{I}\,\pi}{4}\right)\mathrm{d}t$$

Notice that the anti-derivative is:

$$g(t) = \mathrm{e}^{2\,t + \frac{\mathrm{I}\,\pi\,t}{4}}$$

Thus:

$$\int_{C}^{(\,)} f(z)\,\mathrm{d}z = g(1) - g(0)$$

$$\int_{C}^{(\,)} f(z)\,\mathrm{d}z = \mathrm{e}^{2 + \frac{\mathrm{I}\,\pi}{4}} - 1$$

$$\int_{C}^{(\,)} f(z)\,\mathrm{d}z = 4.224851674 + 5.224851675\,\mathrm{I}$$

Check both results, how different are they?