**Problem 1.4 No. 1,2, and 3**

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exact_equation:=proc(eqM, eqN)
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Problem 1

$2\,xy\,dx + x^2 dy = 0$

$$2\,x\,y\,dx + x^2\,dy = 0$$
$$M(x, y)\,dx + N(x, y)\,dy = 0$$
$$M(x, y) = 2\,x\,y$$
$$N(x, y) = x^2$$

Test for exactness:

$$\frac{\partial}{\partial y}\,M(x, y) = \frac{\partial}{\partial x}\,N(x, y)$$
$$2\,x = 2\,x$$

So it is true that this in an exact equation!

We need to find a function F(x,y) such that:

$$M(x, y) = \frac{\partial}{\partial x}\,F(x, y)$$
$$N(x, y) = \frac{\partial}{\partial y}\,F(x, y)$$

Therefore:

$$F(x, y) = \int M(x, y)\,\mathrm{d}x$$
$$F(x, y) = \int 2\,x\,y\,\mathrm{d}x + A(y)$$
$$F(x, y) = x^2\,y + A(y)$$

$$F(x, y) = \int N(x, y)\,\mathrm{d}y$$
$$F(x, y) = \int x^2\,\mathrm{d}y + B(x)$$
$$F(x, y) = x^2\,y + B(x)$$

However, we ended up with two F(x,y) functions.
We need to make sure they are the same by calculating the unknown A(y) form the first F(x,y) and B(x) from the second F(x,y).
Calculating A(y):

$$\frac{\partial}{\partial y}\,F(x, y) = N(x, y)$$
$$x^2 + \frac{\mathrm{d}}{\mathrm{d}y}\,A(y) = x^2$$
$$\frac{\mathrm{d}}{\mathrm{d}y}\,A(y) = 0$$
$$A(y) = \int \left(\frac{\mathrm{d}}{\mathrm{d}y}\,A(y)\right)\mathrm{d}y$$
$$A(y) = \int 0\,\mathrm{d}y$$
$$A(y) = C$$

Thus:

$$F(x, y) = x^2 y + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x, y) = M(x, y)$$

$$2 x y + \frac{d}{dx} B(x) = 2 x y$$

$$\frac{d}{dx} B(x) = 0$$

$$B(x) = \int \left( \frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int 0 \, dx$$

$$B(x) = \mathrm{D}$$

Thus:

$$F(x, y) = x^2 y + \mathrm{D}$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

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Problem 2
$x^3 dx + y^3 dy = 0$

$$x^3 dx + y^3 dy = 0$$

$$M(x, y) dx + N(x, y) dy = 0$$

$$M(x, y) = x^3$$

$$N(x, y) = y^3$$

Test for exactness:

$$\frac{\partial}{\partial y} M(x, y) = \frac{\partial}{\partial x} N(x, y)$$

$$0 = 0$$

So it is true that this in an exact equation!

We need to find a function F(x,y) such that:

$$M(x, y) = \frac{\partial}{\partial x} F(x, y)$$

$$N(x, y) = \frac{\partial}{\partial y} F(x, y)$$

Therefore:

$$F(x, y) = \int M(x, y) \, dx$$

$$F(x, y) = \int x^3 \, dx + A(y)$$

$$F(x, y) = \frac{x^4}{4} + A(y)$$

$$F(x, y) = \int N(x, y) \, dy$$

$$F(x, y) = \int y^3 \, dy + B(x)$$

$$F(x, y) = \frac{y^4}{4} + B(x)$$

However, we ended up with two F(x,y) functions.
We need to make sure they are the same by calculating the unknown A(y) form the first F(x,y) and B(x) from the second F(x,y).
Calculating A(y):

$$\frac{\partial}{\partial y} F(x, y) = N(x, y)$$

$$\frac{d}{dy} A(y) = y^3$$

$$\frac{d}{dy} A(y) = y^3$$

$$A(y) = \int \left( \frac{d}{dy} A(y) \right) dy$$

$$A(y) = \int y^3 \, dy$$

$$A(y) = \frac{y^4}{4} + C$$

Thus:

$$F(x, y) = \frac{x^4}{4} + \frac{y^4}{4} + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x, y) = M(x, y)$$

$$\frac{d}{dx} B(x) = x^3$$

$$\frac{d}{dx} B(x) = x^3$$

$$B(x) = \int \left( \frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int x^3 \, dx$$

$$B(x) = \frac{x^4}{4} + \mathrm{D}$$

Thus:

$$F(x, y) = \frac{x^4}{4} + \frac{y^4}{4} + \mathrm{D}$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

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Problem 3
$\sin(x) \cos(y) \, dx + \cos(x) \sin(y) \, dy = 0$

$$\sin(x) \cos(y) \, dx + \cos(x) \sin(y) \, dy = 0$$

$$M(x, y) \, dx + N(x, y) \, dy = 0$$

$$M(x, y) = \sin(x) \cos(y)$$

$$N(x, y) = \cos(x)\ \sin(y)$$

Test for exactness:

$$\frac{\partial}{\partial y}\ M(x, y) = \frac{\partial}{\partial x}\ N(x, y)$$

$$-\sin(x)\ \sin(y) = -\sin(x)\ \sin(y)$$

So it is true that this in an exact equation!

We need to find a function F(x,y) such that:

$$M(x, y) = \frac{\partial}{\partial x}\ F(x, y)$$

$$N(x, y) = \frac{\partial}{\partial y}\ F(x, y)$$

Therefore:

$$F(x, y) = \int M(x, y)\ dx$$

$$F(x, y) = \int \sin(x)\ \cos(y)\ dx + A(y)$$

$$F(x, y) = -\cos(x)\ \cos(y) + A(y)$$

$$F(x, y) = \int N(x, y)\ dy$$

$$F(x, y) = \int \cos(x)\ \sin(y)\ dy + B(x)$$

$$F(x, y) = -\cos(x)\ \cos(y) + B(x)$$

However, we ended up with two F(x,y) functions.
We need to make sure they are the same by calculating the unknown A(y) form the first F(x,y) and B(x) from the second F(x,y).
Calculating A(y):

$$\frac{\partial}{\partial y}\ F(x, y) = N(x, y)$$

$$\cos(x)\ \sin(y) + \frac{d}{dy}\ A(y) = \cos(x)\ \sin(y)$$

$$\frac{d}{dy}\ A(y) = 0$$

$$A(y) = \int \left(\frac{d}{dy}\ A(y)\right) dy$$

$$A(y) = \int 0\ dy$$

$$A(y) = C$$

Thus:

$$F(x, y) = -\cos(x)\ \cos(y) + C$$

Calculating B(x):

$$\frac{\partial}{\partial x}\ F(x, y) = M(x, y)$$

$$\sin(x)\ \cos(y) + \frac{d}{dx}\ B(x) = \sin(x)\ \cos(y)$$

$$\frac{d}{dx}\ B(x) = 0$$

$$B(x) = \int \left( \frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int 0 \, dx$$

$$B(x) = D$$

Thus:

$$F(x, y) = -\cos(x)\cos(y) + D$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.