Exercises 3.7
Problem 2 - Undetermined Coefficients
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Problem 2a

$$y'(x) - 3\,y(x) = x\,e^{2x} + 6$$

Obtain the homogenous solution:

$$y'(x) - 3\,y(x) = 0$$

$$\textit{Since}, y(x) = e^{\lambda x}, \textit{ thus:}$$

$$\lambda\,e^{\lambda x} - 3\,e^{\lambda x} = 0$$

$$(\lambda - 3)\,e^{\lambda x} = 0$$

$$\textit{Solving for}, \lambda$$

$$\lambda = 3$$

Therefore, the homogenous solution is:

$$y(x) = C_1\,e^{3x}$$

Next, obtain the complementary solution:

$$f(x) = x\,e^{2x} + 6$$

$$\textit{Its first, second, and third derivatives are} = \begin{bmatrix} x\,e^{2x} + 6 \\ e^{2x} + 2\,x\,e^{2x} \\ 4\,e^{2x} + 4\,x\,e^{2x} \end{bmatrix}$$

$$\textit{These are the finite terms} = \begin{bmatrix} 1 \\ e^{2x} \\ x\,e^{2x} \end{bmatrix}$$

Guess the solution:

$$y(x) = A\,e^{2x} + B\,x\,e^{2x} + C$$

And the derivatives:

$$y'(x) = 2\,A\,e^{2x} + B\,e^{2x} + 2\,B\,x\,e^{2x}$$

$$\textit{Plug them back to the orginal problem}, y'(x) - 3\,y(x) = x\,e^{2x} + 6$$

$$-A\,e^{2x} + B\,e^{2x} - B\,x\,e^{2x} - 3\,C = x\,e^{2x} + 6$$

Calculate the constants by matching the coefficients:

$$\{-B = 1, -3\,C = 6, -A + B = 0\}$$

$$\{A = -1, B = -1, C = -2\}$$

Put the constants back to our initial guess:

$$y(x) = -e^{2x} - x\,e^{2x} - 2$$

$$\textit{Therefore, the solution is}, y(x) = C_1\,e^{3x} - e^{2x} - x\,e^{2x} - 2$$

Problem 2b

$$y'(x) + y(x) = x^4 + 2\,x$$

Obtain the homogenous solution:

$$y'(x) + y(x) = 0$$

$$\textit{Since}, y(x) = e^{\lambda x}, \textit{ thus:}$$

$$\lambda\,e^{\lambda x} + e^{\lambda x} = 0$$

$$(\lambda+1)\,e^{\lambda x}=0$$

Solving for, λ

$$\lambda=-1$$

Therefore, the homogenous solution is:

$$y(x)=C_1\,e^{-x}$$

Next, obtain the complementary solution:

$$f(x)=x^4+2x$$

$$\textit{Its first to fifth derivatives are}=\begin{bmatrix} x^4+2x \\ 4x^3+2 \\ 12x^2 \\ 24x \\ 24 \\ 0 \end{bmatrix}$$

$$\textit{These are the finite terms}=\begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{bmatrix}$$

Guess the solution:

$$y(x)=A\,x^4+B\,x^3+Cx^2+\mathrm{D}\,x+E$$

And the derivatives:

$$y'(x)=4\,A\,x^3+3\,B\,x^2+2\,Cx+\mathrm{D}$$

Plug them back to the orginal problem, $y'(x)+y(x)=x^4+2x$

$$A\,x^4+4\,A\,x^3+B\,x^3+3\,B\,x^2+Cx^2+2\,Cx+\mathrm{D}\,x+\mathrm{D}+E=x^4+2x$$

Calculate the constants by matching the coefficients:

$$\{A=1,\,4+B=0,\,3\,B+C=0,\,2\,C+\mathrm{D}=2,\,\mathrm{D}+E=0\}$$

$$\{A=1,\,B=-4,\,C=12,\,\mathrm{D}=-22,\,E=22\}$$

Put the constants back to our initial guess:

$$y(x)=x^4-4\,x^3+12\,x^2-22\,x+22$$

Therefore, the solution is, $y(x)=C_1\,e^{-x}+x^4-4\,x^3+12\,x^2-22\,x+22$

Problem 2c

$$y'(x)+2\,y(x)=3\,e^{2x}+4\sin(x)$$

Obtain the homogenous solution:

$$y'(x)+2\,y(x)=0$$

Since , $y(x)=e^{\lambda x}$*, thus:*

$$\lambda\,e^{\lambda x}+2\,e^{\lambda x}=0$$

$$(\lambda+2)\,e^{\lambda x}=0$$

Solving for, λ

$$\lambda=-2$$

Therefore, the homogenous solution is:

$$y(x)=C_1\,e^{-2x}$$

Next, obtain the complementary solution:

$$f(x) = 3\,e^{2x} + 4\sin(x)$$

$$\textit{Its first to fifth derivatives are} = \begin{bmatrix} 3\,e^{2x} + 4\sin(x) \\ 6\,e^{2x} + 4\cos(x) \\ 12\,e^{2x} - 4\sin(x) \\ 24\,e^{2x} - 4\cos(x) \\ 48\,e^{2x} + 4\sin(x) \\ 96\,e^{2x} + 4\cos(x) \end{bmatrix}$$

$$\textit{These are the finite terms} = \begin{bmatrix} e^{2x} \\ \sin(x) \\ \cos(x) \end{bmatrix}$$

Guess the solution:

$$y(x) = A\,e^{2x} + B\sin(x) + C\cos(x)$$

And the derivatives:

$$y'(x) = 2\,A\,e^{2x} + B\cos(x) - C\sin(x)$$

$$\textit{Plug them back to the orginal problem, } y'(x) + 2\,y(x) = 3\,e^{2x} + 4\sin(x)$$

$$4\,A\,e^{2x} + B\cos(x) - C\sin(x) + 2\,B\sin(x) + 2\,C\cos(x) = 3\,e^{2x} + 4\sin(x)$$

Calculate the constants by matching the coefficients:

$$\{4\,A = 3,\ -C + 2\,B = 4,\ 2\,C + B = 0\}$$

$$\left\{A = \frac{3}{4}, B = \frac{8}{5}, C = -\frac{4}{5}\right\}$$

Put the constants back to our initial guess:

$$y(x) = \frac{3\,e^{2x}}{4} + \frac{8\sin(x)}{5} - \frac{4\cos(x)}{5}$$

$$\textit{Therefore, the solution is, } y(x) = C_1\,e^{-2x} + \frac{3\,e^{2x}}{4} + \frac{8\sin(x)}{5} - \frac{4\cos(x)}{5}$$

Problem 2d

$$y'(x) - 3\,y(x) = x\,e^{3x} + 4$$

Obtain the homogenous solution:

$$y'(x) - 3\,y(x) = 0$$

$$\textit{Since}, y(x) = e^{\lambda x}, \textit{ thus:}$$

$$\lambda\,e^{\lambda x} - 3\,e^{\lambda x} = 0$$

$$(\lambda - 3)\,e^{\lambda x} = 0$$

$$\textit{Solving for, } \lambda$$

$$\lambda = 3$$

Therefore, the homogenous solution is:

$$y(x) = C_1\,e^{3x}$$

Next, obtain the complementary solution:

$$f(x) = x\,e^{3x} + 4$$

$$\textit{Its first to fifth derivatives are} = \begin{bmatrix} x e^{3x} + 4 \\ e^{3x} + 3 x e^{3x} \\ 6 e^{3x} + 9 x e^{3x} \\ 27 e^{3x} + 27 x e^{3x} \\ 108 e^{3x} + 81 x e^{3x} \\ 405 e^{3x} + 243 x e^{3x} \end{bmatrix}$$

$$\textit{These are the finite terms} = \begin{bmatrix} 1 \\ e^{3x} \\ x e^{3x} \end{bmatrix}$$

$$\textit{Remove the duplication} = \begin{bmatrix} 1 \\ x e^{3x} \\ x^2 e^{3x} \end{bmatrix}$$

Guess the solution:

$$y(x) = A + B x e^{3x} + C x^2 e^{3x}$$

And the derivatives:

$$y'(x) = B e^{3x} + 3 B x e^{3x} + 2 C x e^{3x} + 3 C x^2 e^{3x}$$

$$\textit{Plug them back to the orginal problem, } y'(x) - 3 y(x) = x e^{3x} + 4$$

$$B e^{3x} + 2 C x e^{3x} - 3 A = x e^{3x} + 4$$

Calculate the constants by matching the coefficients:

$$\{B = 0, 3 A = 4, 2 C = 1\}$$

$$\left\{A = \frac{4}{3}, B = 0, C = \frac{1}{2}\right\}$$

Put the constants back to our initial guess:

$$y(x) = \frac{4}{3} + \frac{x^2 e^{3x}}{2}$$

$$\textit{Therefore, the solution is, } y(x) = C_1 e^{3x} + \frac{4}{3} + \frac{x^2 e^{3x}}{2}$$

Problem 2e

$$y'(x) + y(x) = 5 - x e^{-x}$$

Obtain the homogenous solution:

$$y'(x) + y(x) = 0$$

$$\textit{Since , } y(x) = e^{\lambda x}, \textit{ thus:}$$

$$\lambda e^{\lambda x} + e^{\lambda x} = 0$$

$$(\lambda + 1) e^{\lambda x} = 0$$

$$\textit{Solving for, } \lambda$$

$$\lambda = -1$$

Therefore, the homogenous solution is:

$$y(x) = C_1 e^{-x}$$

Next, obtain the complementary solution:

$$f(x) = 5 - x e^{-x}$$

$$\textit{Its first to fifth derivatives are} = \begin{bmatrix} 5 - x e^{-x} \\ -e^{-x} + x e^{-x} \\ 2 e^{-x} - x e^{-x} \\ -3 e^{-x} + x e^{-x} \\ 4 e^{-x} - x e^{-x} \\ -5 e^{-x} + x e^{-x} \end{bmatrix}$$

$$\textit{These are the finite terms} = \begin{bmatrix} 1 \\ e^{-x} \\ x e^{-x} \end{bmatrix}$$

$$\textit{Remove the duplication} = \begin{bmatrix} 1 \\ x^2 e^{-x} \\ x^3 e^{-x} \end{bmatrix}$$

Guess the solution:

$$y(x) = A + B x^2 e^{-x} + C x^3 e^{-x}$$

And the derivatives:

$$y'(x) = 2 B x e^{-x} - B x^2 e^{-x} + 3 C x^2 e^{-x} - C x^3 e^{-x}$$

$$\textit{Plug them back to the orginal problem, } y'(x) + y(x) = 5 - x e^{-x}$$

$$2 B x e^{-x} + 3 C x^2 e^{-x} + A = 5 - x e^{-x}$$

Calculate the constants by matching the coefficients:

$$\{A = 5, 2 B = -1, 3 C = 0\}$$

$$\left\{A = 5, B = -\frac{1}{2}, C = 0\right\}$$

Put the constants back to our initial guess:

$$y(x) = 5 - \frac{x^2 e^{-x}}{2}$$

$$\textit{Therefore, the solution is, } y(x) = C_1 e^{-x} + 5 - \frac{x^2 e^{-x}}{2}$$

Problem 2f

$$y'(x) - y(x) = x^2 e^{x}$$

Obtain the homogenous solution:

$$y'(x) - y(x) = 0$$

$$\textit{Since }, y(x) = e^{\lambda x}, \textit{ thus:}$$

$$\lambda e^{\lambda x} - e^{\lambda x} = 0$$

$$(\lambda - 1) e^{\lambda x} = 0$$

$$\textit{Solving for}, \lambda$$

$$\lambda = 1$$

Therefore, the homogenous solution is:

$$y(x) = C_1 e^{x}$$

Next, obtain the complementary solution:

$$f(x) = x^2 e^{x}$$

$$\textit{Its first to fifth derivatives are} = \begin{bmatrix} x^2 e^x \\ 2 x e^x + x^2 e^x \\ 2 e^x + 4 x e^x + x^2 e^x \\ 6 e^x + 6 x e^x + x^2 e^x \\ 12 e^x + 8 x e^x + x^2 e^x \\ 20 e^x + 10 x e^x + x^2 e^x \end{bmatrix}$$

$$\textit{These are the finite terms} = \begin{bmatrix} e^x \\ x e^x \\ x^2 e^x \end{bmatrix}$$

$$\textit{Remove the duplication} = \begin{bmatrix} x e^x \\ x^2 e^x \\ x^3 e^x \end{bmatrix}$$

Guess the solution:

$$y(x) = A x e^x + B x^2 e^x + C x^3 e^x$$

And the derivatives:

$$y'(x) = A e^x + A x e^x + 2 B x e^x + B x^2 e^x + 3 C x^2 e^x + C x^3 e^x$$

$$\textit{Plug them back to the orginal problem, } y'(x) - y(x) = x^2 e^x$$

$$A e^x + 2 B x e^x + 3 C x^2 e^x = x^2 e^x$$

Calculate the constants by matching the coefficients:

$$\{A = 0, 2 B = 0, 3 C = 1\}$$

$$\left\{A = 0, B = 0, C = \frac{1}{3}\right\}$$

Put the constants back to our initial guess:

$$y(x) = \frac{x^3 e^x}{3}$$

$$\textit{Therefore, the solution is, } y(x) = C_1 e^x + \frac{x^3 e^x}{3}$$

Problem 2g

$$y''(x) - y'(x) = 5 \sin(2 x)$$

Obtain the homogenous solution:

$$y''(x) - y'(x) = 0$$

Since:

$$y(x) = e^{\lambda x}$$

$$y'(x) = \lambda e^{\lambda x}$$

$$y''(x) = \lambda^2 e^{\lambda x}$$

Thus:

$$\lambda^2 e^{\lambda x} - \lambda e^{\lambda x} = 0$$

$$(\lambda^2 - \lambda) e^{\lambda x} = 0$$

$$\textit{Solving for, } \lambda$$

$$\lambda_1 = 0$$

$$\lambda_2 = 1$$

Therefore, the homogenous solution is:

$$\boxed{y(x) = C_1 + C_2 e^x}$$

Next, obtain the complementary solution:

$$f(x) = 5\sin(2x)$$

$$\textit{Its first to fifth derivatives are} = \begin{bmatrix} 5\sin(2x) \\ 10\cos(2x) \\ -20\sin(2x) \\ -40\cos(2x) \\ 80\sin(2x) \\ 160\cos(2x) \end{bmatrix}$$

$$\textit{These are the finite terms} = \begin{bmatrix} \sin(2x) \\ \cos(2x) \end{bmatrix}$$

Guess the solution:

$$y(x) = A\sin(2x) + B\cos(2x)$$

And the derivatives:

$$y'(x) = 2A\cos(2x) - 2B\sin(2x)$$

$$y''(x) = -4A\sin(2x) - 4B\cos(2x)$$

$$\textit{Plug them back to the orginal problem, } y''(x) - y'(x) = 5\sin(2x)$$

$$-4A\sin(2x) - 4B\cos(2x) - 2A\cos(2x) + 2B\sin(2x) = 5\sin(2x)$$

Calculate the constants by matching the coefficients:

$$\{-4A + 2B = 5, -4B - 2A = 0\}$$

$$\left\{A = -1, B = \frac{1}{2}\right\}$$

Put the constants back to our initial guess:

$$\boxed{y(x) = -\sin(2x) + \frac{\cos(2x)}{2}}$$

$$\boxed{\textit{Therefore, the solution is, } y(x) = C_1 + C_2 e^x - \sin(2x) + \frac{\cos(2x)}{2}}$$

Problem 2h

$$y''(x) + y'(x) = 4x e^x + 3\sin(x)$$

Obtain the homogenous solution:

$$y''(x) + y'(x) = 0$$

Since:

$$y(x) = e^{\lambda x}$$

$$y'(x) = \lambda e^{\lambda x}$$

$$y''(x) = \lambda^2 e^{\lambda x}$$

Thus:

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} = 0$$

$$(\lambda^2 + \lambda) e^{\lambda x} = 0$$

$$\textit{Solving for, } \lambda$$

$$\lambda_1 = -1$$

$$\lambda_2 = 0$$

Therefore, the homogenous solution is:

$$y(x) = C_1 e^{-x} + C_2$$

Next, obtain the complementary solution:

$$f(x) = 4 x e^{x} + 3 \sin(x)$$

$$\textit{Its first to fifth derivatives are} = \begin{bmatrix} 4 x e^{x} + 3 \sin(x) \\ 4 e^{x} + 4 x e^{x} + 3 \cos(x) \\ 8 e^{x} + 4 x e^{x} - 3 \sin(x) \\ 12 e^{x} + 4 x e^{x} - 3 \cos(x) \\ 16 e^{x} + 4 x e^{x} + 3 \sin(x) \\ 20 e^{x} + 4 x e^{x} + 3 \cos(x) \end{bmatrix}$$

$$\textit{These are the finite terms} = \begin{bmatrix} \sin(x) \\ \cos(x) \\ e^{x} \\ x e^{x} \end{bmatrix}$$

Guess the solution:

$$y(x) = A \sin(x) + B \cos(x) + C e^{x} + D x e^{x}$$

And the derivatives:

$$y'(x) = A \cos(x) - B \sin(x) + C e^{x} + D e^{x} + D x e^{x}$$

$$y''(x) = -A \sin(x) - B \cos(x) + C e^{x} + 2 D e^{x} + D x e^{x}$$

$$\textit{Plug them back to the orginal problem, } y''(x) + y'(x) = 4 x e^{x} + 3 \sin(x)$$

$$-A \sin(x) - B \cos(x) + 2 C e^{x} + 3 D e^{x} + 2 D x e^{x} + A \cos(x) - B \sin(x) = 4 x e^{x} + 3 \sin(x)$$

Calculate the constants by matching the coefficients:

$$\{2 D = 4, -A - B = 3, -B + A = 0, 2 C + 3 D = 0\}$$

$$\left\{A = -\frac{3}{2}, B = -\frac{3}{2}, C = -3, D = 2\right\}$$

Put the constants back to our initial guess:

$$y(x) = -\frac{3 \sin(x)}{2} - \frac{3 \cos(x)}{2} - 3 e^{x} + 2 x e^{x}$$

$$\textit{Therefore, the solution is, } y(x) = C_1 e^{-x} + C_2 - \frac{3 \sin(x)}{2} - \frac{3 \cos(x)}{2} - 3 e^{x} + 2 x e^{x}$$

Problem 2i

$$y''(x) + y'(x) = 3 \sin(2 x) - 5 + 2 x^2$$

Obtain the homogenous solution:

$$y''(x) + y'(x) = 0$$

Since:

$$y(x) = e^{\lambda x}$$

$$y'(x) = \lambda e^{\lambda x}$$

$$y''(x) = \lambda^2 e^{\lambda x}$$

Thus:

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} = 0$$

$$\left(\lambda^2 + \lambda\right) e^{\lambda x} = 0$$

Solving for, λ

$$\lambda_1 = -1$$

$$\lambda_2 = 0$$

Therefore, the homogenous solution is:

$$y(x) = C_1 e^{-x} + C_2$$

Next, obtain the complementary solution:

$$f(x) = 3\sin(2x) - 5 + 2x^2$$

$$\textit{Its first to fifth derivatives are} = \begin{bmatrix} 3\sin(2x) - 5 + 2x^2 \\ 6\cos(2x) + 4x \\ -12\sin(2x) + 4 \\ -24\cos(2x) \\ 48\sin(2x) \\ 96\cos(2x) \end{bmatrix}$$

$$\textit{These are the finite terms} = \begin{bmatrix} \sin(2x) \\ \cos(2x) \\ 1 \\ x \\ x^2 \end{bmatrix}$$

$$\textit{Remove the duplicate terms} = \begin{bmatrix} \sin(2x) \\ \cos(2x) \\ x \\ x^2 \\ x^3 \end{bmatrix}$$

Guess the solution:

$$y(x) = A\sin(2x) + B\cos(2x) + Cx + \mathrm{D}x^2 + Ex^3$$

And the derivatives:

$$y'(x) = 2A\cos(2x) - 2B\sin(2x) + C + 2\mathrm{D}x + 3Ex^2$$

$$y''(x) = -4A\sin(2x) - 4B\cos(2x) + 2\mathrm{D} + 6Ex$$

Plug them back to the orginal problem, $y''(x) + y'(x) = 3\sin(2x) - 5 + 2x^2$

$$-4A\sin(2x) - 4B\cos(2x) + 2\mathrm{D} + 6Ex + 2A\cos(2x) - 2B\sin(2x) + C + 2\mathrm{D}x + 3Ex^2 = 3\sin(2x) - 5 + 2x^2$$

Calculate the constants by matching the coefficients:

$$\{3E = 2,\ -4A - 2B = 3,\ -4B + 2A = 0,\ 2\mathrm{D} + C = -5,\ 6E + 2\mathrm{D} = 0\}$$

$$\left\{A = -\frac{3}{5}, B = -\frac{3}{10}, C = -1, \mathrm{D} = -2, E = \frac{2}{3}\right\}$$

Put the constants back to our initial guess:

$$y(x) = -\frac{3\sin(2x)}{5} - \frac{3\cos(2x)}{10} - x - 2x^2 + \frac{2x^3}{3}$$

Therefore, the solution is, $y(x) = C_1 e^{-x} + C_2 - \frac{3\sin(2x)}{5} - \frac{3\cos(2x)}{10} - x - 2x^2 + \frac{2x^3}{3}$

Problem 2j

$$y''(x) + y'(x) - 2y(x) = x^3 - e^{-x}$$

Obtain the homogenous solution:

$$y''(x) + y'(x) - 2y(x) = 0$$

Since:

$$y(x) = e^{\lambda x}$$

$$y'(x) = \lambda e^{\lambda x}$$

$$y''(x) = \lambda^2 e^{\lambda x}$$

Thus:

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} - 2 e^{\lambda x} = 0$$
$$(\lambda^2 + \lambda - 2)\, e^{\lambda x} = 0$$
Solving for, λ
$$\lambda_1 = 1$$
$$\lambda_2 = -2$$
Therefore, the homogenous solution is:
$$y(x) = C_1 e^x + C_2 e^{-2x}$$

Next, obtain the complementary solution:
$$f(x) = x^3 - e^{-x}$$
$$\textit{Its first to fifth derivatives are} = \begin{bmatrix} x^3 - e^{-x} \\ 3x^2 + e^{-x} \\ 6x - e^{-x} \\ 6 + e^{-x} \\ -e^{-x} \\ e^{-x} \end{bmatrix}$$
$$\textit{These are the finite terms} = \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ e^{-x} \end{bmatrix}$$
Guess the solution:
$$y(x) = A + Bx + Cx^2 + Dx^3 + E e^{-x}$$
And the derivatives:
$$y'(x) = B + 2Cx + 3Dx^2 - E e^{-x}$$
$$y''(x) = 2C + 6Dx + E e^{-x}$$
Plug them back to the orginal problem, $y''(x) + y'(x) - 2y(x) = x^3 - e^{-x}$
$$2C + 6Dx - 2E e^{-x} + B + 2Cx + 3Dx^2 - 2A - 2Bx - 2Cx^2 - 2Dx^3 = x^3 - e^{-x}$$
Calculate the constants by matching the coefficients:
$$\{-2D = 1,\ -2E = -1,\ 3D - 2C = 0,\ 2C + B - 2A = 0,\ 6D + 2C - 2B = 0\}$$
$$\left\{A = -\frac{15}{8}, B = -\frac{9}{4}, C = -\frac{3}{4}, D = -\frac{1}{2}, E = \frac{1}{2}\right\}$$
Put the constants back to our initial guess:
$$y(x) = -\frac{15}{8} - \frac{9x}{4} - \frac{3x^2}{4} - \frac{x^3}{2} + \frac{e^{-x}}{2}$$

Therefore, the solution is, $y(x) = C_1 e^x + C_2 e^{-2x} - \frac{15}{8} - \frac{9x}{4} - \frac{3x^2}{4} - \frac{x^3}{2} + \frac{e^{-x}}{2}$

Problem 2k

$$y''(x) + y'(x) = 6\cos(x) + 2$$

Obtain the homogenous solution:
$$y''(x) + y'(x) = 0$$
Since:
$$y(x) = e^{\lambda x}$$

$$y'(x) = \lambda e^{\lambda x}$$

$$y''(x) = \lambda^2 e^{\lambda x}$$

Thus:

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} = 0$$

$$\left(\lambda^2 + \lambda\right) e^{\lambda x} = 0$$

Solving for, λ

$$\lambda_1 = -1$$

$$\lambda_2 = 0$$

Therefore, the homogenous solution is:

$$y(x) = C_1 e^{-x} + C_2$$

Next, obtain the complementary solution:

$$f(x) = 6\cos(x) + 2$$

$$\textit{Its first to fifth derivatives are} = \begin{bmatrix} 6\cos(x) + 2 \\ -6\sin(x) \\ -6\cos(x) \\ 6\sin(x) \\ 6\cos(x) \\ -6\sin(x) \end{bmatrix}$$

$$\textit{These are the finite terms} = \begin{bmatrix} 1 \\ \cos(x) \\ \sin(x) \end{bmatrix}$$

$$\textit{Remove the duplicates} = \begin{bmatrix} x \\ \cos(x) \\ \sin(x) \end{bmatrix}$$

Guess the solution:

$$y(x) = A x + B\cos(x) + C\sin(x)$$

And the derivatives:

$$y'(x) = A - B\sin(x) + C\cos(x)$$

$$y''(x) = -B\cos(x) - C\sin(x)$$

Plug them back to the orginal problem, $y''(x) + y'(x) = 6\cos(x) + 2$

$$-B\cos(x) - C\sin(x) + A - B\sin(x) + C\cos(x) = 6\cos(x) + 2$$

Calculate the constants by matching the coefficients:

$$\{A = 2,\ -B + C = 6,\ -C - B = 0\}$$

$$\{A = 2,\ B = -3,\ C = 3\}$$

Put the constants back to our initial guess:

$$y(x) = 2x - 3\cos(x) + 3\sin(x)$$

Therefore, the solution is, $y(x) = C_1 e^{-x} + C_2 + 2x - 3\cos(x) + 3\sin(x)$

Problem 2l (repeated roots)

$$y''(x) + 2y'(x) + y(x) = x^2 e^x$$

Obtain the homogenous solution:

$$y''(x) + 2y'(x) + y(x) = 0$$

Since:

$$y(x) = e^{\lambda x}$$
$$y'(x) = \lambda e^{\lambda x}$$
$$y''(x) = \lambda^2 e^{\lambda x}$$
Thus:
$$\lambda^2 e^{\lambda x} + 2\lambda e^{\lambda x} + e^{\lambda x} = 0$$
$$(\lambda^2 + 2\lambda + 1)\, e^{\lambda x} = 0$$
Solving for, λ
$$\lambda = -1$$
Therefore, the homogenous solution is:
$$y(x) = (x C_2 + C_1)\, e^{-x}$$

Next, obtain the complementary solution:
$$f(x) = x^2 e^x$$
$$\textit{Its first to fifth derivatives are} = \begin{bmatrix} x^2 e^x \\ 2x e^x + x^2 e^x \\ 2e^x + 4x e^x + x^2 e^x \\ 6e^x + 6x e^x + x^2 e^x \\ 12 e^x + 8x e^x + x^2 e^x \\ 20 e^x + 10 x e^x + x^2 e^x \end{bmatrix}$$
$$\textit{These are the finite terms} = \begin{bmatrix} e^x \\ x e^x \\ x^2 e^x \end{bmatrix}$$
Guess the solution:
$$y(x) = A e^x + B x e^x + C x^2 e^x$$
And the derivatives:
$$y'(x) = A e^x + B e^x + B x e^x + 2 C x e^x + C x^2 e^x$$
$$y''(x) = A e^x + 2 B e^x + B x e^x + 2 C e^x + 4 C x e^x + C x^2 e^x$$
$$\textit{Plug them back to the orginal problem, } y''(x) + 2y'(x) + y(x) = x^2 e^x$$
$$4 A e^x + 4 B e^x + 4 B x e^x + 2 C e^x + 8 C x e^x + 4 C x^2 e^x = x^2 e^x$$
Calculate the constants by matching the coefficients:
$$\{4C = 1, 4B + 8C = 0, 4A + 4B + 2C = 0\}$$
$$\left\{A = \frac{3}{8}, B = -\frac{1}{2}, C = \frac{1}{4}\right\}$$
Put the constants back to our initial guess:
$$y(x) = \frac{3e^x}{8} - \frac{x e^x}{2} + \frac{x^2 e^x}{4}$$

$$\textit{Therefore, the solution is, } y(x) = (x C_2 + C_1)\, e^{-x} + \frac{3e^x}{8} - \frac{x e^x}{2} + \frac{x^2 e^x}{4}$$

Problem 2l ~~(repeated roots)~~

$$y''(x) - y'(x) = 2 x e^x$$

Obtain the homogenous solution:
$$y''(x) - y'(x) = 0$$
Since:

$$y(x) = e^{\lambda x}$$
$$y'(x) = \lambda e^{\lambda x}$$
$$y''(x) = \lambda^2 e^{\lambda x}$$
Thus:
$$\lambda^2 e^{\lambda x} - \lambda e^{\lambda x} = 0$$
$$(\lambda^2 - \lambda) e^{\lambda x} = 0$$
Solving for, λ
$$\lambda_1 = 0$$
$$\lambda_2 = 1$$
Therefore, the homogenous solution is:
$$y(x) = C_1 + C_2 e^x$$

Next, obtain the complementary solution:
$$f(x) = 2 x e^x$$
$$\textit{Its first to fifth derivatives are} = \begin{bmatrix} 2 x e^x \\ 2 e^x + 2 x e^x \\ 4 e^x + 2 x e^x \\ 6 e^x + 2 x e^x \\ 8 e^x + 2 x e^x \\ 10 e^x + 2 x e^x \end{bmatrix}$$
$$\textit{These are the finite terms} = \begin{bmatrix} e^x \\ x e^x \end{bmatrix}$$
$$\textit{Remove the duplicates} = \begin{bmatrix} x e^x \\ x^2 e^x \end{bmatrix}$$
Guess the solution:
$$y(x) = A x e^x + B x^2 e^x$$
And the derivatives:
$$y'(x) = A e^x + A x e^x + 2 B x e^x + B x^2 e^x$$
$$y''(x) = 2 A e^x + A x e^x + 2 B e^x + 4 B x e^x + B x^2 e^x$$
Plug them back to the orginal problem, $y''(x) - y'(x) = 2 x e^x$
$$A e^x + 2 B e^x + 2 B x e^x = 2 x e^x$$
Calculate the constants by matching the coefficients:
$$\{2 B = 2, A + 2 B = 0\}$$
$$\{A = -2, B = 1\}$$
Put the constants back to our initial guess:
$$y(x) = -2 x e^x + x^2 e^x$$

Therefore, the solution is, $y(x) = C_1 + C_2 e^x - 2 x e^x + x^2 e^x$