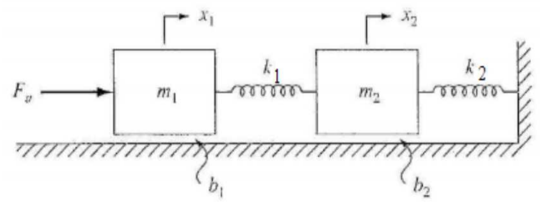
```
> restart;

  with(LinearAlgebra):
  with(Physics):
  with(DEtools):

  with(plots):
  with(plottools):

  with(Typesetting):
  interface(typesetting=extended):
  interface(showassumed=0):
  Settings(typesetdot=true):

  read "derive_msd_model.maple";
```

Example 1




```
> #setup matrix K and B
  K:=<0,0;k__1,k__2>:
  B:=<b__1,0;0,b__2>:
  M=<m__1, m__2>:
  F:=<F_a(t),0>:

  #write down the equations
  eq:=mass_spring_damper(2, M, K, B, F):
  in_matrix(2,eq);
```

The equations of motion:

$$M_1\ddot{x}_1(t)+b_1\dot{x}_1(t)-k_1\left(x_2(t)-x_1(t)\right)=F\_a(t)$$

$$M_2\ddot{x}_2(t)+b_2\dot{x}_2(t)+k_1\left(x_2(t)-x_1(t)\right)+k_2x_2(t)=0$$

when simplified:

$$\ddot{x}_1(t)=-\frac{b_1\dot{x}_1(t)}{M_1}-\frac{k_1x_1(t)}{M_1}+\frac{k_1x_2(t)}{M_1}+\frac{F\_a(t)}{M_1}$$

$$\ddot{x}_2(t)=-\frac{b_2\dot{x}_2(t)}{M_2}+\frac{k_1x_1(t)}{M_2}-\frac{(k_1+k_2)\,x_2(t)}{M_2}$$

in matrix form:

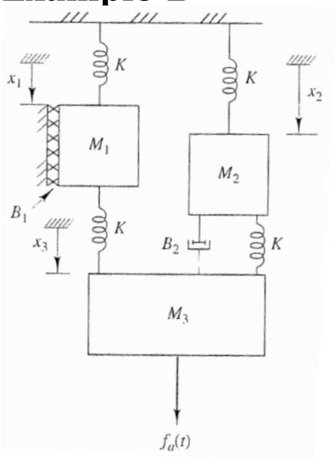
$$\dot{x}(t)=A\,x(t)+f(t)$$

where:

$$\dot{x}(t)=\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \ddot{x}_1(t)\\ \ddot{x}_2(t)\end{bmatrix},A=\begin{bmatrix}0&0&1&0\\ 0&0&0&1\\ -\frac{k_1}{M_1}&\frac{k_1}{M_1}&-\frac{b_1}{M_1}&0\\ \frac{k_1}{M_2}&-\frac{k_1+k_2}{M_2}&0&-\frac{b_2}{M_2}\end{bmatrix}$$

$$x(t)=\begin{bmatrix}x_1(t)\\ x_2(t)\\ \dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix},f(t)=\begin{bmatrix}0\\ 0\\ \frac{F\_a(t)}{M_1}\\ 0\end{bmatrix}$$

Example 2




```
> #clear used variables
  K:='K':
  M:='M':

  #setup matrix K and B
  matK:=<K,0,0;0,K,0;K,K,0>:
  matB:=<B__1,0,0;0,0,0;0,B__2,0>:
  matM:=<M__1,M__2,M__3>:
  matF:=<M__1*g,M__2*g,M__3*g+f__a(t)>:

  #write down the equations
  eq:=mass_spring_damper(3, matM, matK, matB, matF):
  in_matrix(3,eq);
```

The equations of motion:

$$M_1 \ddot{x}_1(t) + B_1 \dot{x}_1(t) + K x_1(t) - K\left(x_3(t) - x_1(t)\right) = M_1 g$$

$$M_2 \ddot{x}_2(t) - B_2\left(\dot{x}_3(t) - \dot{x}_2(t)\right) + K x_2(t) - K\left(x_3(t) - x_2(t)\right) = M_2 g$$

$$M_3 \ddot{x}_3(t) + B_2\left(\dot{x}_3(t) - \dot{x}_2(t)\right) + K\left(x_3(t) - x_1(t)\right) + K\left(x_3(t) - x_2(t)\right) = M_3 g + f_a(t)$$

when simplified:

$$\ddot{x}_1(t) = -\frac{B_1 \dot{x}_1(t)}{M_1} - \frac{2 K x_1(t)}{M_1} + \frac{K x_3(t)}{M_1} + g$$

$$\ddot{x}_2(t) = -\frac{B_2 \dot{x}_2(t)}{M_2} + \frac{B_2 \dot{x}_3(t)}{M_2} - \frac{2 K x_2(t)}{M_2} + \frac{K x_3(t)}{M_2} + g$$

$$\ddot{x}_3(t) = \frac{B_2 \dot{x}_2(t)}{M_3} - \frac{B_2 \dot{x}_3(t)}{M_3} + \frac{K x_1(t)}{M_3} + \frac{K x_2(t)}{M_3} - \frac{2 K x_3(t)}{M_3} + \frac{M_3 g + f_a(t)}{M_3}$$

in matrix form:

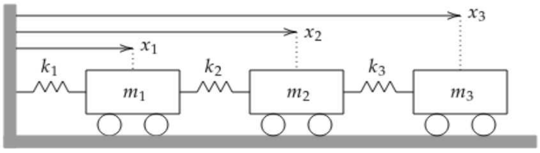
$$\dot{x}(t) = A\,x(t) + f(t)$$

where:

$$\dot{x}(t)=\left[\begin{array}{c}\dot{x}_1(t)\\\dot{x}_2(t)\\\dot{x}_3(t)\\\ddot{x}_1(t)\\\ddot{x}_2(t)\\\ddot{x}_3(t)\end{array}\right], A=\left[\begin{array}{cccccc}0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\-\frac{2K}{M_1}&0&\frac{K}{M_1}&-\frac{B_1}{M_1}&0&0\\0&-\frac{2K}{M_2}&\frac{K}{M_2}&0&-\frac{B_2}{M_2}&\frac{B_2}{M_2}\\\frac{K}{M_3}&\frac{K}{M_3}&-\frac{2K}{M_3}&0&\frac{B_2}{M_3}&-\frac{B_2}{M_3}\end{array}\right]$$

$$x(t)=\left[\begin{array}{c}x_1(t)\\x_2(t)\\x_3(t)\\\dot{x}_1(t)\\\dot{x}_2(t)\\\dot{x}_3(t)\end{array}\right], f(t)=\left[\begin{array}{c}0\\0\\0\\g\\g\\\frac{M_3\,g+f_a(t)}{M_3}\end{array}\right]$$

Example 3




```
> #clear used variables
  K:='K':
  M:='M':

  #setup matrix K and B
  matK:=<k__1,0,0;k__2,0,0;0,k__3,0>:
  matB:=<0,0,0;0,0,0;0,0,0>:
  matM:=<m__1,m__2,m__3>:
  matF:=<0,0,0>:

  #write down the equations
  eq:=mass_spring_damper(3, matM, matK, matB, matF):
  in_matrix(3,eq);
```

The equations of motion:

$$m_1 \ddot{x}_1(t) + k_1 x_1(t) - k_2 \left(x_2(t) - x_1(t)\right) = 0$$

$$m_2 \ddot{x}_2(t) + k_2 \left(x_2(t) - x_1(t)\right) - k_3 \left(x_3(t) - x_2(t)\right) = 0$$

$$m_3 \ddot{x}_3(t) + k_3 \left(x_3(t) - x_2(t)\right) = 0$$

when simplified:

$$\ddot{x}_1(t) = \frac{\left(-k_1 - k_2\right) x_1(t)}{m_1} + \frac{k_2 x_2(t)}{m_1}$$

$$\ddot{x}_2(t) = \frac{k_2 x_1(t)}{m_2} - \frac{\left(k_2 + k_3\right) x_2(t)}{m_2} + \frac{k_3 x_3(t)}{m_2}$$

$$\ddot{x}_3(t) = \frac{k_3 x_2(t)}{m_3} - \frac{k_3 x_3(t)}{m_3}$$

in matrix form:

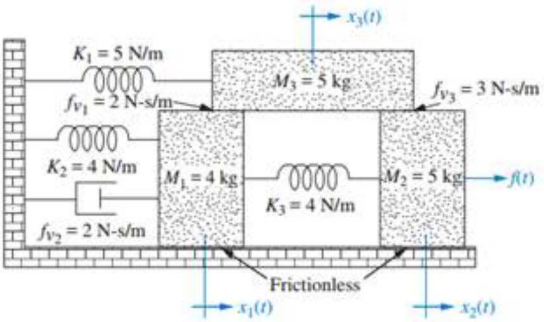
$$\dot{x}(t) = A\, x(t) + f(t)$$

where:

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \ddot{x}_3(t) \end{bmatrix}, A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{-k_1 - k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0 & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_2 + k_3}{m_2} & \frac{k_3}{m_2} & 0 & 0 & 0 \\ 0 & \frac{k_3}{m_3} & -\frac{k_3}{m_3} & 0 & 0 & 0 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix}, f(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 4




```
> #clear used variables
  K:='K':
  M:='M':

  #setup matrix K and B
  matK:=<4,0,0;4,0,0;0,0,5>:
  matB:=<2,0,0;0,0,0;2,3,0>:
  matM:=<4,5,5>:
  matF:=<0,f(t),0>:

  #write down the equations
  eq:=mass_spring_damper(3, matM, matK, matB, matF):
  in_matrix(3,eq);
```

The equations of motion:

$$4\,\ddot{x}_1(t) + 4\,\dot{x}_1(t) - 2\,\dot{x}_3(t) + 8\,x_1(t) - 4\,x_2(t) = 0$$

$$5\,\ddot{x}_2(t) - 3\,\dot{x}_3(t) + 3\,\dot{x}_2(t) + 4\,x_2(t) - 4\,x_1(t) = f(t)$$

$$5\,\ddot{x}_3(t) + 5\,\dot{x}_3(t) - 2\,\dot{x}_1(t) - 3\,\dot{x}_2(t) + 5\,x_3(t) = 0$$

when simplified:

$$\ddot{x}_1(t) = -\dot{x}_1(t) + \frac{\dot{x}_3(t)}{2} - 2\,x_1(t) + x_2(t)$$

$$\ddot{x}_2(t) = \frac{3\,\dot{x}_3(t)}{5} - \frac{3\,\dot{x}_2(t)}{5} - \frac{4\,x_2(t)}{5} + \frac{4\,x_1(t)}{5} + \frac{f(t)}{5}$$

$$\ddot{x}_3(t) = -\dot{x}_3(t) + \frac{2\,\dot{x}_1(t)}{5} + \frac{3\,\dot{x}_2(t)}{5} - x_3(t)$$

in matrix form:

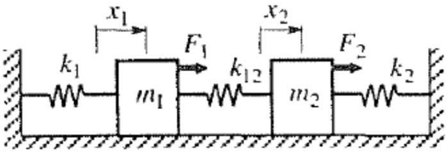
$$\dot{x}(t) = A\,x(t) + f(t)$$

where:

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \ddot{x}_3(t) \end{bmatrix}, A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & -1 & 0 & \frac{1}{2} \\ \frac{4}{5} & -\frac{4}{5} & 0 & 0 & -\frac{3}{5} & \frac{3}{5} \\ 0 & 0 & -1 & \frac{2}{5} & \frac{3}{5} & -1 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix}, f(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{f(t)}{5} \\ 0 \end{bmatrix}$$

Example 5




```
> #setup matrix K and B
  K:=<k__1,0;k__12,k__2>:
  B:=<0,0;0,0>:
  M=<m__1, m__2>:
  F:=<F__1,F__2>:

  #write down the equations
  eq:=mass_spring_damper(2, M, K, B, F):
  in_matrix(2,eq);
```

The equations of motion:

$$M_1\,\ddot{x}_1(t) + k_1\,x_1(t) - k_{12}\,\big(x_2(t) - x_1(t)\big) = F_1$$

$$M_2\,\ddot{x}_2(t) + k_{12}\,\big(x_2(t) - x_1(t)\big) + k_2\,x_2(t) = F_2$$

when simplified:

$$\ddot{x}_1(t) = \frac{(-k_1 - k_{12})\,x_1(t)}{M_1} + \frac{k_{12}\,x_2(t)}{M_1} + \frac{F_1}{M_1}$$

$$\ddot{x}_2(t) = \frac{k_{12}\,x_1(t)}{M_2} + \frac{(-k_{12} - k_2)\,x_2(t)}{M_2} + \frac{F_2}{M_2}$$

in matrix form:

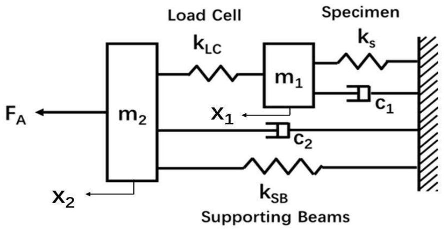
$$\dot{x}(t) = A\,x(t) + f(t)$$

where:

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix},\ A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k_1 - k_{12}}{M_1} & \frac{k_{12}}{M_1} & 0 & 0 \\ \frac{k_{12}}{M_2} & \frac{-k_{12} - k_2}{M_2} & 0 & 0 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix},\ f(t) = \begin{bmatrix} 0 \\ 0 \\ \frac{F_1}{M_1} \\ \frac{F_2}{M_2} \end{bmatrix}$$

Example 6




```
> #clear used variables
  K:='K':
  M:='M':

  #setup matrix K and B
  matB:=<c__1,0;0,c__2>:
  matK:=<k__S,k__LC;0, k__SB>:
  matM:=<M__1,M__2>:
  matF:=<0, F__A>:

  #write down the equations
  eq:=mass_spring_damper(2, matM, matK, matB, matF):
  in_matrix(2,eq);
```

The equations of motion:

$$M_1 \ddot{x}_1(t) + c_1 \dot{x}_1(t) + k_S x_1(t) + k_{LC}\left(x_1(t) - x_2(t)\right) = 0$$

$$M_2 \ddot{x}_2(t) + c_2 \dot{x}_2(t) - k_{LC}\left(x_1(t) - x_2(t)\right) + k_{SB} x_2(t) = F_A$$

when simplified:

$$\ddot{x}_1(t) = -\frac{c_1 \dot{x}_1(t)}{M_1} - \frac{\left(k_{LC} + k_S\right) x_1(t)}{M_1} + \frac{k_{LC} x_2(t)}{M_1}$$

$$\ddot{x}_2(t) = -\frac{c_2 \dot{x}_2(t)}{M_2} + \frac{k_{LC} x_1(t)}{M_2} + \frac{\left(-k_{LC} - k_{SB}\right) x_2(t)}{M_2} + \frac{F_A}{M_2}$$

in matrix form:

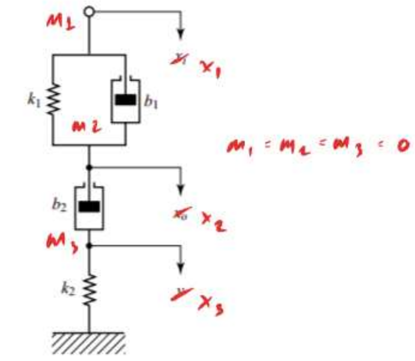
$$\dot{x}(t) = A\, x(t) + f(t)$$

where:

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix}, A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_{LC}+k_S}{M_1} & \frac{k_{LC}}{M_1} & -\frac{c_1}{M_1} & 0 \\ \frac{k_{LC}}{M_2} & \frac{-k_{LC}-k_{SB}}{M_2} & 0 & -\frac{c_2}{M_2} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}, f(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{F_A}{M_2} \end{bmatrix}$$

Example 7




```
> #clear used variables
  K:='K':
  M:='M':

  #setup matrix K and B
  matK := <0,    0, 0;
           k__1, 0, 0;
           0,    0, k__2>:

  matB := <0,    0,    0;
           b__1, 0,    0;
           0,    b__2, 0>:

  matM := <M__1, M__2, M__3>:

  matF := <0, 0, 0>:

  #write down the equations
  eq:=mass_spring_damper(3, matM, matK, matB, matF):
  in_matrix(3,eq);
```

The equations of motion:

$$M_1\,\ddot{x}_1(t) - b_1\left(\dot{x}_2(t) - \dot{x}_1(t)\right) - k_1\left(x_2(t) - x_1(t)\right) = 0$$

$$M_2\,\ddot{x}_2(t) + b_1\left(\dot{x}_2(t) - \dot{x}_1(t)\right) - b_2\left(\dot{x}_3(t) - \dot{x}_2(t)\right) + k_1\left(x_2(t) - x_1(t)\right) = 0$$

$$M_3\,\ddot{x}_3(t) + b_2\left(\dot{x}_3(t) - \dot{x}_2(t)\right) + k_2\,x_3(t) = 0$$

when simplified:

$$\ddot{x}_1(t) = -\frac{b_1\,\dot{x}_1(t)}{M_1} + \frac{b_1\,\dot{x}_2(t)}{M_1} - \frac{k_1\,x_1(t)}{M_1} + \frac{k_1\,x_2(t)}{M_1}$$

$$\ddot{x}_2(t) = \frac{b_1\,\dot{x}_1(t)}{M_2} - \frac{(b_1 + b_2)\,\dot{x}_2(t)}{M_2} + \frac{b_2\,\dot{x}_3(t)}{M_2} + \frac{k_1\,x_1(t)}{M_2} - \frac{k_1\,x_2(t)}{M_2}$$

$$\ddot{x}_3(t) = \frac{b_2\,\dot{x}_2(t)}{M_3} - \frac{b_2\,\dot{x}_3(t)}{M_3} - \frac{k_2\,x_3(t)}{M_3}$$

in matrix form:

$$\dot{x}(t) = A\,x(t) + f(t)$$

where:

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \ddot{x}_3(t) \end{bmatrix}, A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{k_1}{M_1} & \frac{k_1}{M_1} & 0 & -\frac{b_1}{M_1} & \frac{b_1}{M_1} & 0 \\ \frac{k_1}{M_2} & -\frac{k_1}{M_2} & 0 & \frac{b_1}{M_2} & -\frac{b_1+b_2}{M_2} & \frac{b_2}{M_2} \\ 0 & 0 & -\frac{k_2}{M_3} & 0 & \frac{b_2}{M_3} & -\frac{b_2}{M_3} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix}, f(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$