Problem Set 1.4

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exact_equation:=proc(eqM, eqN)
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exact_equations_integrating_factor:=proc(eqM, eqN)
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Problem 5a

$$2\,x\,y\,dx + x^2\,dy = 0$$
$$M(x, y) = 2\,x\,y$$
$$N(x, y) = x^2$$

Test for exactness:

$$M_y = 2\,x$$
$$N_x = 2\,x$$

The equation is already exact!

We can continue with the procedure for the exact equation.

$$2\,x\,y\,dx + x^2\,dy = 0$$
$$M(x, y) = 2\,x\,y$$
$$N(x, y) = x^2$$

Test for exactness:

$$M_y = N_x$$
$$2\,x = 2\,x$$

We need to find a function F(x,y) such that:

$$M(x, y) = \frac{\partial}{\partial x}\,F(x, y)$$
$$N(x, y) = \frac{\partial}{\partial y}\,F(x, y)$$

Therefore:

$$F(x, y) = \int M(x, y)\,\mathrm{d}x$$
$$F(x, y) = \int 2\,x\,y\,\mathrm{d}x + A(y)$$
$$F(x, y) = x^2\,y + A(y)$$

Or:

$$F(x, y) = \int N(x, y)\,\mathrm{d}y$$
$$F(x, y) = \int x^2\,\mathrm{d}y + B(x)$$

$$F(x, y) = x^2 y + B(x)$$

We have two F(x,y) functions.
Both should be the same
We show this by first calculating the unknown A(y) form the first F(x,y)
and B(x) from the second F(x,y).

Calculating A(y):

$$\frac{\partial}{\partial y} F(x, y) = N(x, y)$$

$$x^2 + \frac{d}{dy} A(y) = x^2$$

$$\frac{d}{dy} A(y) = 0$$

$$A(y) = \int \left(\frac{d}{dy} A(y) \right) dy$$

$$A(y) = \int 0 \, dy$$

$$A(y) = C$$

Thus:

$$F(x, y) = x^2 y + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x, y) = M(x, y)$$

$$2 x y + \frac{d}{dx} B(x) = 2 x y$$

$$\frac{d}{dx} B(x) = 0$$

$$B(x) = \int \left(\frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int 0 \, dx$$

$$B(x) = \mathrm{D}$$

Thus:

$$F(x, y) = x^2 y + \mathrm{D}$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

$$x^3 \, dx + y^3 \, dy = 0$$

$$M(x, y) = x^3$$

$$N(x, y) = y^3$$

Test for exactness:

$$M_y = 0$$

$$N_x = 0$$

The equation is already exact!

We can continue with the procedure for the exact equation.

$$x^3\, dx + y^3\, dy = 0$$

$$M(x, y) = x^3$$

$$N(x, y) = y^3$$

Test for exactness:

$$M_y = N_x$$

$$0 = 0$$

We need to find a function F(x,y) such that:

$$M(x, y) = \frac{\partial}{\partial x} F(x, y)$$

$$N(x, y) = \frac{\partial}{\partial y} F(x, y)$$

Therefore:

$$F(x, y) = \int M(x, y)\, \mathrm{d}x$$

$$F(x, y) = \int x^3\, \mathrm{d}x + A(y)$$

$$F(x, y) = \frac{x^4}{4} + A(y)$$

Or:

$$F(x, y) = \int N(x, y)\, \mathrm{d}y$$

$$F(x, y) = \int y^3\, \mathrm{d}y + B(x)$$

$$F(x, y) = \frac{y^4}{4} + B(x)$$

We have two F(x,y) functions.

Both should be the same

We show this by first calculating the unknown A(y) form the first F(x,y)

and B(x) from the second F(x,y).

Calculating A(y):

$$\frac{\partial}{\partial y} F(x, y) = N(x, y)$$

$$\frac{d}{dy} A(y) = y^3$$

$$\frac{d}{dy} A(y) = y^3$$

$$A(y) = \int \left(\frac{d}{dy} A(y) \right) dy$$

$$A(y) = \int y^3 \, dy$$

$$A(y) = \frac{y^4}{4} + C$$

Thus:

$$F(x, y) = \frac{x^4}{4} + \frac{y^4}{4} + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x, y) = M(x, y)$$

$$\frac{d}{dx} B(x) = x^3$$

$$\frac{d}{dx} B(x) = x^3$$

$$B(x) = \int \left(\frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int x^3 \, dx$$

$$B(x) = \frac{x^4}{4} + \mathrm{D}$$

Thus:

$$F(x, y) = \frac{x^4}{4} + \frac{y^4}{4} + \mathrm{D}$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

$$\sin(x) \cos(y) \, dx + \cos(x) \sin(y) \, dy = 0$$

$$M(x, y) = \sin(x) \cos(y)$$

$$N(x, y) = \cos(x) \sin(y)$$

Test for exactness:

$$M_y = -\sin(x) \sin(y)$$

$$N_x = -\sin(x) \sin(y)$$

The equation is already exact!
We can continue with the procedure for the exact equation.

$$\sin(x)\cos(y)\,dx + \cos(x)\sin(y)\,dy = 0$$
$$M(x, y) = \sin(x)\cos(y)$$
$$N(x, y) = \cos(x)\sin(y)$$

Test for exactness:
$$M_y = N_x$$
$$-\sin(x)\sin(y) = -\sin(x)\sin(y)$$

We need to find a function F(x,y) such that:
$$M(x, y) = \frac{\partial}{\partial x} F(x, y)$$
$$N(x, y) = \frac{\partial}{\partial y} F(x, y)$$

Therefore:
$$F(x, y) = \int M(x, y)\,\mathrm{d}x$$
$$F(x, y) = \int \sin(x)\cos(y)\,\mathrm{d}x + A(y)$$
$$F(x, y) = -\cos(x)\cos(y) + A(y)$$

Or:
$$F(x, y) = \int N(x, y)\,\mathrm{d}y$$
$$F(x, y) = \int \cos(x)\sin(y)\,\mathrm{d}y + B(x)$$
$$F(x, y) = -\cos(x)\cos(y) + B(x)$$

We have two F(x,y) functions.
Both should be the same
We show this by first calculating the unknown A(y) form the first F(x,y)
and B(x) from the second F(x,y).

Calculating A(y):
$$\frac{\partial}{\partial y} F(x, y) = N(x, y)$$
$$\cos(x)\sin(y) + \frac{\mathrm{d}}{\mathrm{d}y} A(y) = \cos(x)\sin(y)$$
$$\frac{\mathrm{d}}{\mathrm{d}y} A(y) = 0$$
$$A(y) = \int \left(\frac{\mathrm{d}}{\mathrm{d}y} A(y)\right) \mathrm{d}y$$
$$A(y) = \int 0\,\mathrm{d}y$$

$$A(y) = C$$

Thus:

$$F(x, y) = -\cos(x)\cos(y) + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x, y) = M(x, y)$$

$$\sin(x)\cos(y) + \frac{d}{dx} B(x) = \sin(x)\cos(y)$$

$$\frac{d}{dx} B(x) = 0$$

$$B(x) = \int \left(\frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int 0 \, dx$$

$$B(x) = \mathrm{D}$$

Thus:

$$F(x, y) = -\cos(x)\cos(y) + \mathrm{D}$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

$$e^{3y}\, dx + 3\, e^{3y}\, x\, dy = 0$$

$$M(x, y) = e^{3y}$$

$$N(x, y) = 3\, e^{3y}\, x$$

Test for exactness:

$$M_y = 3\, e^{3y}$$

$$N_x = 3\, e^{3y}$$

The equation is already exact!
We can continue with the procedure for the exact equation.

$$e^{3y}\, dx + 3\, e^{3y}\, x\, dy = 0$$

$$M(x, y) = e^{3y}$$

$$N(x, y) = 3\, e^{3y}\, x$$

Test for exactness:

$$M_y = N_x$$

$$3\, e^{3y} = 3\, e^{3y}$$

We need to find a function F(x,y) such that:

$$M(x, y) = \frac{\partial}{\partial x} F(x, y)$$
$$N(x, y) = \frac{\partial}{\partial y} F(x, y)$$

Therefore:
$$F(x, y) = \int M(x, y) \, \mathrm{d}x$$
$$F(x, y) = \int \mathrm{e}^{3y} \, \mathrm{d}x + A(y)$$
$$F(x, y) = \mathrm{e}^{3y} x + A(y)$$

Or:
$$F(x, y) = \int N(x, y) \, \mathrm{d}y$$
$$F(x, y) = \int 3 \, \mathrm{e}^{3y} x \, \mathrm{d}y + B(x)$$
$$F(x, y) = \mathrm{e}^{3y} x + B(x)$$

We have two F(x,y) functions.
Both should be the same
We show this by first calculating the unknown A(y) form the first F(x,y)
and B(x) from the second F(x,y).

Calculating A(y):
$$\frac{\partial}{\partial y} F(x, y) = N(x, y)$$
$$3 \, \mathrm{e}^{3y} x + \frac{\mathrm{d}}{\mathrm{d}y} A(y) = 3 \, \mathrm{e}^{3y} x$$
$$\frac{\mathrm{d}}{\mathrm{d}y} A(y) = 0$$
$$A(y) = \int \left(\frac{\mathrm{d}}{\mathrm{d}y} A(y) \right) \mathrm{d}y$$
$$A(y) = \int 0 \, \mathrm{d}y$$
$$A(y) = C$$
Thus:
$$F(x, y) = \mathrm{e}^{3y} x + C$$

Calculating B(x):
$$\frac{\partial}{\partial x} F(x, y) = M(x, y)$$
$$\mathrm{e}^{3y} + \frac{\mathrm{d}}{\mathrm{d}x} B(x) = \mathrm{e}^{3y}$$

$$\frac{d}{dx} B(x) = 0$$

$$B(x) = \int \left(\frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int 0 \, dx$$

$$B(x) = D$$

Thus:

$$F(x, y) = e^{3y} x + D$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

$$(x^2 + y^2)\, dx - 2\, x\, y\, dy = 0$$

$$M(x, y) = x^2 + y^2$$

$$N(x, y) = -2\, x\, y$$

Test for exactness:

$$M_y = 2\, y$$

$$N_x = -2\, y$$

$$M_y \neq N_x$$

Not an exact equation!

Find the integrating factor, σ

$$\frac{M_y - N_x}{M} = \frac{4\, y}{x^2 + y^2}$$

$$\frac{M_y - N_x}{N} = -\frac{2}{x}$$

Take , $\frac{M_y - N_x}{N} = -\frac{2}{x}$, *function of x alone.*

$$\sigma(x) = e^{\int \frac{M_y - N_x}{N} dx}$$

$$\sigma(x) = e^{\int -\frac{2}{x} dx}$$

$$\sigma(x) = \frac{1}{x^2}$$

$$\sigma\, M(x, y)\, dx + \sigma\, N(x, y)\, dy = 0$$

$$\frac{(x^2 + y^2)\, dx}{x^2} - \frac{2\, y\, dy}{x} = 0$$

Now, we have a new ODE which is exact.
We then continue with the procedure for the exact equation.

$$\frac{(x^2 + y^2)\ dx}{x^2} - \frac{2\ y\ dy}{x} = 0$$

$$M(x, y) = \frac{x^2 + y^2}{x^2}$$

$$N(x, y) = -\frac{2\ y}{x}$$

Test for exactness:

$$M_y = N_x$$

$$\frac{2\ y}{x^2} = \frac{2\ y}{x^2}$$

We need to find a function F(x,y) such that:

$$M(x, y) = \frac{\partial}{\partial x}\ F(x, y)$$

$$N(x, y) = \frac{\partial}{\partial y}\ F(x, y)$$

Therefore:

$$F(x, y) = \int M(x, y)\ \mathrm{d}x$$

$$F(x, y) = \int \frac{x^2 + y^2}{x^2}\ \mathrm{d}x + A(y)$$

$$F(x, y) = \frac{x^2 - y^2}{x} + A(y)$$

Or:

$$F(x, y) = \int N(x, y)\ \mathrm{d}y$$

$$F(x, y) = \int -\frac{2\ y}{x}\ \mathrm{d}y + B(x)$$

$$F(x, y) = -\frac{y^2}{x} + B(x)$$

We have two F(x,y) functions.
Both should be the same
We show this by first calculating the unknown A(y) form the first F(x,y)
and B(x) from the second F(x,y).

Calculating A(y):

$$\frac{\partial}{\partial y}\ F(x, y) = N(x, y)$$

$$-\frac{2y}{x} + \frac{d}{dy}A(y) = -\frac{2y}{x}$$

$$\frac{d}{dy}A(y) = 0$$

$$A(y) = \int \left(\frac{d}{dy}A(y)\right) dy$$

$$A(y) = \int 0\, dy$$

$$A(y) = C$$

Thus:

$$F(x, y) = \frac{x^2 - y^2 + C x}{x}$$

Calculating B(x):

$$\frac{\partial}{\partial x}F(x, y) = M(x, y)$$

$$\frac{y^2}{x^2} + \frac{d}{dx}B(x) = \frac{x^2 + y^2}{x^2}$$

$$\frac{d}{dx}B(x) = 1$$

$$B(x) = \int \left(\frac{d}{dx}B(x)\right) dx$$

$$B(x) = \int 1\, dx$$

$$B(x) = x + \mathrm{D}$$

Thus:

$$F(x, y) = \frac{x^2 - y^2 + \mathrm{D}\, x}{x}$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

$$2\, x \tan(y)\; dx + \sec(y)^2\, dy = 0$$

$$M(x, y) = 2\, x \tan(y)$$

$$N(x, y) = \sec(y)^2$$

Test for exactness:

$$M_y = 2\, x\, (1 + \tan(y)^2)$$

$$N_x = 0$$

$$M_y \neq N_x$$

Not an exact equation!

Find the integrating factor, σ

$$\frac{M_y - N_x}{M} = \frac{1}{\cos(y)\sin(y)}$$

$$\frac{M_y - N_x}{N} = 2x$$

Take , $\frac{M_y - N_x}{N} = 2x$, *function of x alone.*

$$\sigma(x) = e^{\int \frac{M_y - N_x}{N}\,dx}$$

$$\sigma(x) = e^{\int 2x\,dx}$$

$$\sigma(x) = e^{x^2}$$

$$\sigma\, M(x, y)\, dx + \sigma\, N(x, y)\, dy = 0$$

$$2\, e^{x^2}\, x \tan(y)\, dx + e^{x^2} \sec(y)^2\, dy = 0$$

Now, we have a new ODE which is exact.

We then continue with the procedure for the exact equation.

$$2\, e^{x^2}\, x \tan(y)\, dx + \frac{e^{x^2}\, dy}{\cos(y)^2} = 0$$

$$M(x, y) = 2\, e^{x^2}\, x \tan(y)$$

$$N(x, y) = \frac{e^{x^2}}{\cos(y)^2}$$

Test for exactness:

$$M_y = N_x$$

$$\frac{2\, e^{x^2}\, x}{\cos(y)^2} = \frac{2\, e^{x^2}\, x}{\cos(y)^2}$$

We need to find a function F(x,y) such that:

$$M(x, y) = \frac{\partial}{\partial x} F(x, y)$$

$$N(x, y) = \frac{\partial}{\partial y} F(x, y)$$

Therefore:

$$F(x, y) = \int M(x, y)\, dx$$

$$F(x, y) = \int 2\, e^{x^2}\, x \tan(y)\, dx + A(y)$$

$$F(x, y) = e^{x^2} \tan(y) + A(y)$$

Or:

$$F(x, y) = \int N(x, y)\, \mathrm{d}y$$

$$F(x, y) = \int \frac{e^{x^2}}{\cos(y)^2}\, \mathrm{d}y + B(x)$$

$$F(x, y) = e^{x^2} \tan(y) + B(x)$$

We have two F(x,y) functions.
Both should be the same
We show this by first calculating the unknown A(y) form the first F(x,y)
and B(x) from the second F(x,y).

Calculating A(y):

$$\frac{\partial}{\partial y} F(x, y) = N(x, y)$$

$$e^{x^2} \left(1 + \tan(y)^2\right) + \frac{d}{dy} A(y) = \frac{e^{x^2}}{\cos(y)^2}$$

$$\frac{d}{dy} A(y) = 0$$

$$A(y) = \int \left(\frac{d}{dy} A(y)\right) \mathrm{d}y$$

$$A(y) = \int 0\, \mathrm{d}y$$

$$A(y) = C$$

Thus:

$$F(x, y) = e^{x^2} \tan(y) + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x, y) = M(x, y)$$

$$2\, e^{x^2} x \tan(y) + \frac{d}{dx} B(x) = 2\, e^{x^2} x \tan(y)$$

$$\frac{d}{dx} B(x) = 0$$

$$B(x) = \int \left(\frac{d}{dx} B(x)\right) \mathrm{d}x$$

$$B(x) = \int 0\, \mathrm{d}x$$

$$B(x) = \mathrm{D}$$

Thus:

$$F(x, y) = e^{x^2} \tan(y) + \mathrm{D}$$

Pay attention, the two F(x,y) functions MUST be the same.

The solution is the F(x,y) = constant and it is in an implicit form.

$$e^{x}\cos(y)\,dx - e^{x}\sin(y)\,dy = 0$$

$$M(x, y) = e^{x}\cos(y)$$

$$N(x, y) = -e^{x}\sin(y)$$

Test for exactness:

$$M_{y} = -e^{x}\sin(y)$$

$$N_{x} = -e^{x}\sin(y)$$

The equation is already exact!

We can continue with the procedure for the exact equation.

$$e^{x}\cos(y)\,dx - e^{x}\sin(y)\,dy = 0$$

$$M(x, y) = e^{x}\cos(y)$$

$$N(x, y) = -e^{x}\sin(y)$$

Test for exactness:

$$M_{y} = N_{x}$$

$$-e^{x}\sin(y) = -e^{x}\sin(y)$$

We need to find a function F(x,y) such that:

$$M(x, y) = \frac{\partial}{\partial x} F(x, y)$$

$$N(x, y) = \frac{\partial}{\partial y} F(x, y)$$

Therefore:

$$F(x, y) = \int M(x, y)\,dx$$

$$F(x, y) = \int e^{x}\cos(y)\,dx + A(y)$$

$$F(x, y) = e^{x}\cos(y) + A(y)$$

Or:

$$F(x, y) = \int N(x, y)\,dy$$

$$F(x, y) = \int -e^{x}\sin(y)\,dy + B(x)$$

$$F(x, y) = e^{x}\cos(y) + B(x)$$

We have two F(x,y) functions.

Both should be the same

We show this by first calculating the unknown A(y) form the first F(x,y) and B(x) from the second F(x,y).

Calculating A(y):

$$\frac{\partial}{\partial y} F(x, y) = N(x, y)$$

$$-e^{x} \sin(y) + \frac{d}{dy} A(y) = -e^{x} \sin(y)$$

$$\frac{d}{dy} A(y) = 0$$

$$A(y) = \int \left(\frac{d}{dy} A(y) \right) dy$$

$$A(y) = \int 0 \, dy$$

$$A(y) = C$$

Thus:

$$F(x, y) = e^{x} \cos(y) + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x, y) = M(x, y)$$

$$e^{x} \cos(y) + \frac{d}{dx} B(x) = e^{x} \cos(y)$$

$$\frac{d}{dx} B(x) = 0$$

$$B(x) = \int \left(\frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int 0 \, dx$$

$$B(x) = \mathrm{D}$$

Thus:

$$F(x, y) = e^{x} \cos(y) + \mathrm{D}$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

$$2 e^{2x} \cos(y) \, dx - e^{2x} \sin(y) \, dy = 0$$

$$M(x, y) = 2 e^{2x} \cos(y)$$

$$N(x, y) = -e^{2x} \sin(y)$$

Test for exactness:

$$M_y = -2 e^{2x} \sin(y)$$

$$N_x = -2 e^{2x} \sin(y)$$

The equation is already exact!

We can continue with the procedure for the exact equation.

$$2\,e^{2x}\cos(y)\,dx - e^{2x}\sin(y)\,dy = 0$$
$$M(x, y) = 2\,e^{2x}\cos(y)$$
$$N(x, y) = -e^{2x}\sin(y)$$

Test for exactness:

$$M_y = N_x$$
$$-2\,e^{2x}\sin(y) = -2\,e^{2x}\sin(y)$$

We need to find a function F(x,y) such that:

$$M(x, y) = \frac{\partial}{\partial x}\,F(x, y)$$
$$N(x, y) = \frac{\partial}{\partial y}\,F(x, y)$$

Therefore:

$$F(x, y) = \int M(x, y)\,dx$$
$$F(x, y) = \int 2\,e^{2x}\cos(y)\,dx + A(y)$$
$$F(x, y) = e^{2x}\cos(y) + A(y)$$

Or:

$$F(x, y) = \int N(x, y)\,dy$$
$$F(x, y) = \int -e^{2x}\sin(y)\,dy + B(x)$$
$$F(x, y) = e^{2x}\cos(y) + B(x)$$

We have two F(x,y) functions.
Both should be the same
We show this by first calculating the unknown A(y) form the first F(x,y)
and B(x) from the second F(x,y).

Calculating A(y):

$$\frac{\partial}{\partial y}\,F(x, y) = N(x, y)$$
$$-e^{2x}\sin(y) + \frac{d}{dy}\,A(y) = -e^{2x}\sin(y)$$
$$\frac{d}{dy}\,A(y) = 0$$

$$A(y) = \int \left(\frac{d}{dy} A(y) \right) dy$$

$$A(y) = \int 0 \, dy$$

$$A(y) = C$$

Thus:

$$F(x, y) = e^{2x} \cos(y) + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x, y) = M(x, y)$$

$$2\, e^{2x} \cos(y) + \frac{d}{dx} B(x) = 2\, e^{2x} \cos(y)$$

$$\frac{d}{dx} B(x) = 0$$

$$B(x) = \int \left(\frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int 0 \, dx$$

$$B(x) = \mathrm{D}$$

Thus:

$$F(x, y) = e^{2x} \cos(y) + \mathrm{D}$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

$$2 \cosh(x) \cos(y)\, dx - \sinh(x) \sin(y)\, dy = 0$$

$$M(x, y) = 2 \cosh(x) \cos(y)$$

$$N(x, y) = -\sinh(x) \sin(y)$$

Test for exactness:

$$M_y = -2 \cosh(x) \sin(y)$$

$$N_x = -\cosh(x) \sin(y)$$

$$M_y \neq N_x$$

Not an exact equation!

Find the integrating factor, σ

$$\frac{M_y - N_x}{M} = -\frac{\sin(y)}{2 \cos(y)}$$

$$\frac{M_y - N_x}{N} = \frac{\cosh(x)}{\sinh(x)}$$

Take , $\dfrac{M_y - N_x}{N} = \dfrac{\cosh(x)}{\sinh(x)}$, *function of x alone.*

$$\sigma(x) = e^{\int \frac{M_y - N_x}{N} \, dx}$$

$$\sigma(x) = e^{\int \frac{\cosh(x)}{\sinh(x)} \, dx}$$

$$\sigma(x) = \sinh(x)$$

$$\sigma M(x, y)\, dx + \sigma N(x, y)\, dy = 0$$

$$2 \sinh(x) \cosh(x) \cos(y)\, dx - \sinh(x)^2 \sin(y)\, dy = 0$$

Now, we have a new ODE which is exact.

We then continue with the procedure for the exact equation.

$$2 \sinh(x) \cosh(x) \cos(y)\, dx - \sinh(x)^2 \sin(y)\, dy = 0$$

$$M(x, y) = 2 \sinh(x) \cosh(x) \cos(y)$$

$$N(x, y) = -\sinh(x)^2 \sin(y)$$

Test for exactness:

$$M_y = N_x$$

$$-2 \sinh(x) \cosh(x) \sin(y) = -2 \sinh(x) \cosh(x) \sin(y)$$

We need to find a function F(x,y) such that:

$$M(x, y) = \frac{\partial}{\partial x} F(x, y)$$

$$N(x, y) = \frac{\partial}{\partial y} F(x, y)$$

Therefore:

$$F(x, y) = \int M(x, y) \, dx$$

$$F(x, y) = \int 2 \sinh(x) \cosh(x) \cos(y) \, dx + A(y)$$

$$F(x, y) = \cosh(x)^2 \cos(y) + A(y)$$

Or:

$$F(x, y) = \int N(x, y) \, dy$$

$$F(x, y) = \int -\sinh(x)^2 \sin(y) \, dy + B(x)$$

$$F(x, y) = \sinh(x)^2 \cos(y) + B(x)$$

We have two F(x,y) functions.

Both should be the same

We show this by first calculating the unknown A(y) form the first F(x,y)

and B(x) from the second F(x,y).

Calculating A(y):

$$\frac{\partial}{\partial y} F(x, y) = N(x, y)$$

$$-\cosh(x)^2 \sin(y) + \frac{d}{dy} A(y) = -\sinh(x)^2 \sin(y)$$

$$\frac{d}{dy} A(y) = \sin(y)$$

$$A(y) = \int \left(\frac{d}{dy} A(y) \right) dy$$

$$A(y) = \int \sin(y)\, dy$$

$$A(y) = -\cos(y) + C$$

Thus:

$$F(x, y) = \cosh(x)^2 \cos(y) - \cos(y) + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x, y) = M(x, y)$$

$$2 \sinh(x) \cosh(x) \cos(y) + \frac{d}{dx} B(x) = 2 \sinh(x) \cosh(x) \cos(y)$$

$$\frac{d}{dx} B(x) = 0$$

$$B(x) = \int \left(\frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int 0\, dx$$

$$B(x) = \mathrm{D}$$

Thus:

$$F(x, y) = \cosh(x)^2 \cos(y) + \mathrm{D} - \cos(y)$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

$$2\, x\, y\, e^{x^2}\, dx + e^{x^2}\, dy = 0$$

$$M(x, y) = 2\, x\, y\, e^{x^2}$$

$$N(x, y) = e^{x^2}$$

Test for exactness:

$$M_y = 2\, x\, e^{x^2}$$

$$N_x = 2\, x\, e^{x^2}$$

The equation is already exact!
We can continue with the procedure for the exact equation.

$$2\,x\,y\,e^{x^2}\,dx + e^{x^2}\,dy = 0$$
$$M(x, y) = 2\,x\,y\,e^{x^2}$$
$$N(x, y) = e^{x^2}$$

Test for exactness:
$$M_y = N_x$$
$$2\,x\,e^{x^2} = 2\,x\,e^{x^2}$$

We need to find a function F(x,y) such that:
$$M(x, y) = \frac{\partial}{\partial x} F(x, y)$$
$$N(x, y) = \frac{\partial}{\partial y} F(x, y)$$

Therefore:
$$F(x, y) = \int M(x, y)\, dx$$
$$F(x, y) = \int 2\,x\,y\,e^{x^2}\, dx + A(y)$$
$$F(x, y) = y\,e^{x^2} + A(y)$$

Or:
$$F(x, y) = \int N(x, y)\, dy$$
$$F(x, y) = \int e^{x^2}\, dy + B(x)$$
$$F(x, y) = y\,e^{x^2} + B(x)$$

We have two F(x,y) functions.
Both should be the same
We show this by first calculating the unknown A(y) form the first F(x,y)
and B(x) from the second F(x,y).

Calculating A(y):
$$\frac{\partial}{\partial y} F(x, y) = N(x, y)$$
$$e^{x^2} + \frac{d}{dy} A(y) = e^{x^2}$$
$$\frac{d}{dy} A(y) = 0$$

$$A(y) = \int \left(\frac{d}{dy} A(y) \right) dy$$

$$A(y) = \int 0 \, dy$$

$$A(y) = C$$

Thus:

$$F(x, y) = e^{x^2} y + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x, y) = M(x, y)$$

$$2\, x\, y\, e^{x^2} + \frac{d}{dx} B(x) = 2\, x\, y\, e^{x^2}$$

$$\frac{d}{dx} B(x) = 0$$

$$B(x) = \int \left(\frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int 0 \, dx$$

$$B(x) = \mathrm{D}$$

Thus:

$$F(x, y) = e^{x^2} y + \mathrm{D}$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.
