Exercises 2.5
Problem 1, the exact equations
Auralius Manurung

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exact_equation:=proc(eqM, eqN)
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Example 1

$\sin(y)dx + (x\cos(y) - 2\,y)dy = 0$

$$\sin(y)\,dx + (x\cos(y) - 2\,y)\,dy = 0$$

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$M(x,y) = \sin(y)$$

$$N(x,y) = x\cos(y) - 2\,y$$

Test for exactness:

$$\frac{\partial}{\partial y}\,M(x,y) = \frac{\partial}{\partial x}\,N(x,y)$$

$$\cos(y) = \cos(y)$$

We need to find a function F(x,y) such that:

$$M(x,y) = \frac{\partial}{\partial x}\,F(x,y)$$

$$N(x,y) = \frac{\partial}{\partial y}\,F(x,y)$$

Therefore:

$$F(x,y) = \int M(x,y)\,\mathrm{d}x$$

$$F(x,y) = \int \sin(y)\,\mathrm{d}x + A(y)$$

$$F(x,y) = \sin(y)\,x + A(y)$$

$$F(x,y) = \int N(x,y)\,\mathrm{d}y$$

$$F(x,y) = \int (x\cos(y) - 2\,y)\,\mathrm{d}y + B(x)$$

$$F(x,y) = \sin(y)\,x - y^2 + B(x)$$

However, we ended up with two F(x,y) functions.
We need to make sure they are the same by calculating the unknown A(y) form the first F(x,y) and B(x) from the second F(x,y).
Calculating A(y):

$$\frac{\partial}{\partial y}\,F(x,y) = N(x,y)$$

$$x\cos(y) + \frac{\mathrm{d}}{\mathrm{d}y}\,A(y) = x\cos(y) - 2\,y$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\,A(y) = -2\,y$$

$$A(y) = \int \left(\frac{\mathrm{d}}{\mathrm{d}y}\,A(y)\right)\mathrm{d}y$$

$$A(y) = \int -2\,y\,\mathrm{d}y$$

$$A(y) = -y^2 + C$$

Thus:

$$F(x,y) = -y^2 + \sin(y)\,x + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x,y) = M(x,y)$$

$$\sin(y) + \frac{d}{dx} B(x) = \sin(y)$$

$$\frac{d}{dx} B(x) = 0$$

$$B(x) = \int \left(\frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int 0 \, dx$$

$$B(x) = D$$

Thus:

$$F(x,y) = -y^2 + \sin(y)\, x + D$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

Problem 1a

$$3\, dx - dy = 0$$

$$M(x,y)\, dx + N(x,y)\, dy = 0$$

$$M(x,y) = 3$$

$$N(x,y) = -1$$

Test for exactness:

$$\frac{\partial}{\partial y} M(x,y) = \frac{\partial}{\partial x} N(x,y)$$

$$0 = 0$$

We need to find a function F(x,y) such that:

$$M(x,y) = \frac{\partial}{\partial x} F(x,y)$$

$$N(x,y) = \frac{\partial}{\partial y} F(x,y)$$

Therefore:

$$F(x,y) = \int M(x,y)\, dx$$

$$F(x,y) = \int 3\, dx + A(y)$$

$$F(x,y) = 3\, x + A(y)$$

$$F(x,y) = \int N(x,y)\, dy$$

$$F(x,y) = \int (-1)\, dy + B(x)$$

$$F(x,y) = -y + B(x)$$

However, we ended up with two F(x,y) functions.
We need to make sure they are the same by calculating the unknown A(y) form the first F(x,y) and B(x) from the second F(x,y).
Calculating A(y):

$$\frac{\partial}{\partial y} F(x,y) = N(x,y)$$

$$\frac{d}{dy} A(y) = -1$$

$$\frac{d}{dy} A(y) = -1$$

$$A(y) = \int \left(\frac{d}{dy} A(y) \right) dy$$

$$A(y) = \int (-1)\, dy$$

$$A(y) = -y + C$$

Thus:

$$F(x, y) = 3x - y + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x, y) = M(x, y)$$

$$\frac{d}{dx} B(x) = 3$$

$$\frac{d}{dx} B(x) = 3$$

$$B(x) = \int \left(\frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int 3\, dx$$

$$B(x) = 3x + \mathrm{D}$$

Thus:

$$F(x, y) = 3x - y + \mathrm{D}$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

Problem 1b

$$x^2\, dx + y^2\, dy = 0$$

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

$$M(x, y) = x^2$$

$$N(x, y) = y^2$$

Test for exactness:

$$\frac{\partial}{\partial y} M(x, y) = \frac{\partial}{\partial x} N(x, y)$$

$$0 = 0$$

We need to find a function F(x,y) such that:

$$M(x, y) = \frac{\partial}{\partial x} F(x, y)$$

$$N(x, y) = \frac{\partial}{\partial y} F(x, y)$$

Therefore:

$$F(x, y) = \int M(x, y)\, dx$$

$$F(x, y) = \int x^2\, dx + A(y)$$

$$F(x, y) = \frac{x^3}{3} + A(y)$$

$$F(x,y) = \int N(x,y)\, \mathrm{d}y$$

$$F(x,y) = \int y^2\, \mathrm{d}y + B(x)$$

$$F(x,y) = \frac{y^3}{3} + B(x)$$

However, we ended up with two F(x,y) functions.
We need to make sure they are the same by calculating the unknown A(y) form the first F(x,y) and B(x) from the second F(x,y).
Calculating A(y):

$$\frac{\partial}{\partial y} F(x,y) = N(x,y)$$

$$\frac{\mathrm{d}}{\mathrm{d}y} A(y) = y^2$$

$$\frac{\mathrm{d}}{\mathrm{d}y} A(y) = y^2$$

$$A(y) = \int \left(\frac{\mathrm{d}}{\mathrm{d}y} A(y) \right) \mathrm{d}y$$

$$A(y) = \int y^2\, \mathrm{d}y$$

$$A(y) = \frac{y^3}{3} + C$$

Thus:

$$F(x,y) = \frac{x^3}{3} + \frac{y^3}{3} + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x,y) = M(x,y)$$

$$\frac{\mathrm{d}}{\mathrm{d}x} B(x) = x^2$$

$$\frac{\mathrm{d}}{\mathrm{d}x} B(x) = x^2$$

$$B(x) = \int \left(\frac{\mathrm{d}}{\mathrm{d}x} B(x) \right) \mathrm{d}x$$

$$B(x) = \int x^2\, \mathrm{d}x$$

$$B(x) = \frac{x^3}{3} + \mathrm{D}$$

Thus:

$$F(x,y) = \frac{x^3}{3} + \frac{y^3}{3} + \mathrm{D}$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

Problem 1c

$$x\, dx + 2\, y\, dy = 0$$

$$M(x,y)\, dx + N(x,y)\, dy = 0$$

$$M(x,y) = x$$

$$N(x,y) = 2\, y$$

Test for exactness:

$$\frac{\partial}{\partial y} M(x, y) = \frac{\partial}{\partial x} N(x, y)$$

$$0 = 0$$

We need to find a function F(x,y) such that:

$$M(x, y) = \frac{\partial}{\partial x} F(x, y)$$

$$N(x, y) = \frac{\partial}{\partial y} F(x, y)$$

Therefore:

$$F(x, y) = \int M(x, y) \, dx$$

$$F(x, y) = \int x \, dx + A(y)$$

$$F(x, y) = \frac{x^2}{2} + A(y)$$

$$F(x, y) = \int N(x, y) \, dy$$

$$F(x, y) = \int 2 y \, dy + B(x)$$

$$F(x, y) = y^2 + B(x)$$

However, we ended up with two F(x,y) functions.
We need to make sure they are the same by calculating the unknown A(y) form the first F(x,y) and B(x) from the second F(x,y).
Calculating A(y):

$$\frac{\partial}{\partial y} F(x, y) = N(x, y)$$

$$\frac{d}{dy} A(y) = 2 y$$

$$\frac{d}{dy} A(y) = 2 y$$

$$A(y) = \int \left(\frac{d}{dy} A(y) \right) dy$$

$$A(y) = \int 2 y \, dy$$

$$A(y) = y^2 + C$$

Thus:

$$F(x, y) = \frac{x^2}{2} + y^2 + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x, y) = M(x, y)$$

$$\frac{d}{dx} B(x) = x$$

$$\frac{d}{dx} B(x) = x$$

$$B(x) = \int \left(\frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int x \, dx$$

$$B(x) = \frac{x^2}{2} + D$$

Thus:

$$F(x,y) = \frac{x^2}{2} + y^2 + D$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

Problem 1d

$$4\cos(2x)\,dx - e^{-5y}\,dy = 0$$
$$M(x,y)\,dx + N(x,y)\,dy = 0$$
$$M(x,y) = 4\cos(2x)$$
$$N(x,y) = -e^{-5y}$$

Test for exactness:

$$\frac{\partial}{\partial y} M(x,y) = \frac{\partial}{\partial x} N(x,y)$$
$$0 = 0$$

We need to find a function F(x,y) such that:

$$M(x,y) = \frac{\partial}{\partial x} F(x,y)$$
$$N(x,y) = \frac{\partial}{\partial y} F(x,y)$$

Therefore:

$$F(x,y) = \int M(x,y) \, dx$$
$$F(x,y) = \int 4\cos(2x) \, dx + A(y)$$
$$F(x,y) = 2\sin(2x) + A(y)$$

$$F(x,y) = \int N(x,y) \, dy$$
$$F(x,y) = \int -e^{-5y} \, dy + B(x)$$
$$F(x,y) = \frac{e^{-5y}}{5} + B(x)$$

However, we ended up with two F(x,y) functions.
We need to make sure they are the same by calculating the unknown A(y) form the first F(x,y) and B(x) from the second F(x,y).
Calculating A(y):

$$\frac{\partial}{\partial y} F(x,y) = N(x,y)$$
$$\frac{d}{dy} A(y) = -e^{-5y}$$
$$\frac{d}{dy} A(y) = -e^{-5y}$$
$$A(y) = \int \left(\frac{d}{dy} A(y) \right) dy$$

$$A(y) = \int -e^{-5y}\, dy$$

$$A(y) = \frac{e^{-5y}}{5} + C$$

Thus:

$$F(x, y) = 2\sin(2x) + \frac{e^{-5y}}{5} + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x, y) = M(x, y)$$

$$\frac{d}{dx} B(x) = 4\cos(2x)$$

$$\frac{d}{dx} B(x) = 4\cos(2x)$$

$$B(x) = \int \left(\frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int 4\cos(2x)\, dx$$

$$B(x) = 2\sin(2x) + D$$

Thus:

$$F(x, y) = \frac{e^{-5y}}{5} + 2\sin(2x) + D$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

Problem 1e

$$e^{y}\, dx + \left(x\, e^{y} - 1 \right)\, dy = 0$$

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

$$M(x, y) = e^{y}$$

$$N(x, y) = x\, e^{y} - 1$$

Test for exactness:

$$\frac{\partial}{\partial y} M(x, y) = \frac{\partial}{\partial x} N(x, y)$$

$$e^{y} = e^{y}$$

We need to find a function F(x,y) such that:

$$M(x, y) = \frac{\partial}{\partial x} F(x, y)$$

$$N(x, y) = \frac{\partial}{\partial y} F(x, y)$$

Therefore:

$$F(x, y) = \int M(x, y)\, dx$$

$$F(x, y) = \int e^{y}\, dx + A(y)$$

$$F(x, y) = x\, e^{y} + A(y)$$

$$F(x, y) = \int N(x, y)\, dy$$

$$F(x,y) = \int (x\,e^{y} - 1)\,dy + B(x)$$

$$F(x,y) = -y + x\,e^{y} + B(x)$$

However, we ended up with two F(x,y) functions.
We need to make sure they are the same by calculating the unknown A(y) form the first F(x,y) and B(x) from the second F(x,y).
Calculating A(y):

$$\frac{\partial}{\partial y} F(x,y) = N(x,y)$$

$$x\,e^{y} + \frac{d}{dy} A(y) = x\,e^{y} - 1$$

$$\frac{d}{dy} A(y) = -1$$

$$A(y) = \int \left(\frac{d}{dy} A(y) \right) dy$$

$$A(y) = \int (-1)\,dy$$

$$A(y) = -y + C$$

Thus:

$$F(x,y) = e^{y} x - y + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x,y) = M(x,y)$$

$$e^{y} + \frac{d}{dx} B(x) = e^{y}$$

$$\frac{d}{dx} B(x) = 0$$

$$B(x) = \int \left(\frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int 0\,dx$$

$$B(x) = \mathrm{D}$$

Thus:

$$F(x,y) = e^{y} x - y + \mathrm{D}$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

Problem 1f

$$(e^{x} + y)\,dx + (-\sin(y) + x)\,dy = 0$$

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$M(x,y) = e^{x} + y$$

$$N(x,y) = -\sin(y) + x$$

Test for exactness:

$$\frac{\partial}{\partial y} M(x,y) = \frac{\partial}{\partial x} N(x,y)$$

$$1 = 1$$

We need to find a function F(x,y) such that:

$$M(x,y) = \frac{\partial}{\partial x} F(x,y)$$

$$N(x,y) = \frac{\partial}{\partial y} F(x,y)$$

Therefore:

$$F(x,y) = \int M(x,y)\, \mathrm{d}x$$

$$F(x,y) = \int \left(\mathrm{e}^{x} + y\right) \mathrm{d}x + A(y)$$

$$F(x,y) = y\,x + \mathrm{e}^{x} + A(y)$$

$$F(x,y) = \int N(x,y)\, \mathrm{d}y$$

$$F(x,y) = \int (-\sin(y) + x)\, \mathrm{d}y + B(x)$$

$$F(x,y) = y\,x + \cos(y) + B(x)$$

However, we ended up with two F(x,y) functions.
We need to make sure they are the same by calculating the unknown A(y) form the first F(x,y) and B(x) from the second F(x,y).
Calculating A(y):

$$\frac{\partial}{\partial y} F(x,y) = N(x,y)$$

$$x + \frac{\mathrm{d}}{\mathrm{d}y} A(y) = -\sin(y) + x$$

$$\frac{\mathrm{d}}{\mathrm{d}y} A(y) = -\sin(y)$$

$$A(y) = \int \left(\frac{\mathrm{d}}{\mathrm{d}y} A(y)\right) \mathrm{d}y$$

$$A(y) = \int -\sin(y)\, \mathrm{d}y$$

$$A(y) = \cos(y) + C$$

Thus:

$$F(x,y) = x\,y + \mathrm{e}^{x} + \cos(y) + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x,y) = M(x,y)$$

$$y + \frac{\mathrm{d}}{\mathrm{d}x} B(x) = \mathrm{e}^{x} + y$$

$$\frac{\mathrm{d}}{\mathrm{d}x} B(x) = \mathrm{e}^{x}$$

$$B(x) = \int \left(\frac{\mathrm{d}}{\mathrm{d}x} B(x)\right) \mathrm{d}x$$

$$B(x) = \int \mathrm{e}^{x}\, \mathrm{d}x$$

$$B(x) = \mathrm{e}^{x} + \mathrm{D}$$

Thus:

$$F(x,y) = x\,y + \cos(y) + \mathrm{e}^{x} + \mathrm{D}$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

Problem 1g

$$(x - 2y)\ dx + (-2x + y)\ dy = 0$$
$$M(x, y)\ dx + N(x, y)\ dy = 0$$
$$M(x, y) = x - 2y$$
$$N(x, y) = -2x + y$$

Test for exactness:

$$\frac{\partial}{\partial y} M(x, y) = \frac{\partial}{\partial x} N(x, y)$$
$$-2 = -2$$

We need to find a function F(x,y) such that:

$$M(x, y) = \frac{\partial}{\partial x} F(x, y)$$
$$N(x, y) = \frac{\partial}{\partial y} F(x, y)$$

Therefore:

$$F(x, y) = \int M(x, y)\, \mathrm{d}x$$
$$F(x, y) = \int (x - 2y)\, \mathrm{d}x + A(y)$$
$$F(x, y) = \frac{x\,(x - 4y)}{2} + A(y)$$

$$F(x, y) = \int N(x, y)\, \mathrm{d}y$$
$$F(x, y) = \int (-2x + y)\, \mathrm{d}y + B(x)$$
$$F(x, y) = -2xy + \frac{y^2}{2} + B(x)$$

However, we ended up with two F(x,y) functions.
We need to make sure they are the same by calculating the unknown A(y) form the first F(x,y) and B(x) from the second F(x,y).
Calculating A(y):

$$\frac{\partial}{\partial y} F(x, y) = N(x, y)$$
$$-2x + \frac{\mathrm{d}}{\mathrm{d}y} A(y) = -2x + y$$
$$\frac{\mathrm{d}}{\mathrm{d}y} A(y) = y$$
$$A(y) = \int \left(\frac{\mathrm{d}}{\mathrm{d}y} A(y)\right) \mathrm{d}y$$
$$A(y) = \int y\, \mathrm{d}y$$
$$A(y) = \frac{y^2}{2} + C$$

Thus:

$$F(x, y) = \frac{1}{2} x^2 - 2xy + \frac{1}{2} y^2 + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x, y) = M(x, y)$$

$$-2\,y+\frac{d}{dx}\,B(x)=x-2\,y$$

$$\frac{d}{dx}\,B(x)=x$$

$$B(x)=\int\left(\frac{d}{dx}\,B(x)\right)dx$$

$$B(x)=\int x\,dx$$

$$B(x)=\frac{x^2}{2}+D$$

Thus:

$$F(x,y)=\frac{1}{2}\,x^2-2\,x\,y+\frac{1}{2}\,y^2+D$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

Problem 1h

$$(\sin(y)+y\cos(x))\,dx+(\sin(x)+x\cos(y))\,dy=0$$
$$M(x,y)\,dx+N(x,y)\,dy=0$$
$$M(x,y)=\sin(y)+y\cos(x)$$
$$N(x,y)=\sin(x)+x\cos(y)$$

Test for exactness:

$$\frac{\partial}{\partial y}\,M(x,y)=\frac{\partial}{\partial x}\,N(x,y)$$

$$\cos(y)+\cos(x)=\cos(y)+\cos(x)$$

We need to find a function F(x,y) such that:

$$M(x,y)=\frac{\partial}{\partial x}\,F(x,y)$$

$$N(x,y)=\frac{\partial}{\partial y}\,F(x,y)$$

Therefore:

$$F(x,y)=\int M(x,y)\,dx$$

$$F(x,y)=\int(\sin(y)+y\cos(x))\,dx+A(y)$$

$$F(x,y)=y\sin(x)+\sin(y)\,x+A(y)$$

$$F(x,y)=\int N(x,y)\,dy$$

$$F(x,y)=\int(\sin(x)+x\cos(y))\,dy+B(x)$$

$$F(x,y)=y\sin(x)+\sin(y)\,x+B(x)$$

However, we ended up with two F(x,y) functions.
We need to make sure they are the same by calculating the unknown A(y) form the first F(x,y) and B(x) from the second F(x,y).
Calculating A(y):

$$\frac{\partial}{\partial y}\,F(x,y)=N(x,y)$$

$$\sin(x)+x\cos(y)+\frac{d}{dy}\,A(y)=\sin(x)+x\cos(y)$$

$$\frac{d}{dy} A(y) = 0$$
$$A(y) = \int \left(\frac{d}{dy} A(y) \right) dy$$
$$A(y) = \int 0 \, dy$$
$$A(y) = C$$

Thus:

$$F(x, y) = \sin(y)\, x + \sin(x)\, y + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x, y) = M(x, y)$$
$$y \cos(x) + \sin(y) + \frac{d}{dx} B(x) = \sin(y) + y \cos(x)$$
$$\frac{d}{dx} B(x) = 0$$
$$B(x) = \int \left(\frac{d}{dx} B(x) \right) dx$$
$$B(x) = \int 0 \, dx$$
$$B(x) = \mathrm{D}$$

Thus:

$$F(x, y) = \sin(y)\, x + \sin(x)\, y + \mathrm{D}$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

Problem 1i

$$(\sin(x\,y) + x\,y \cos(x\,y))\, dx + x^2 \cos(x\,y)\, dy = 0$$
$$M(x, y)\, dx + N(x, y)\, dy = 0$$
$$M(x, y) = \sin(x\,y) + x\,y \cos(x\,y)$$
$$N(x, y) = x^2 \cos(x\,y)$$

Test for exactness:

$$\frac{\partial}{\partial y} M(x, y) = \frac{\partial}{\partial x} N(x, y)$$
$$2\,x \cos(x\,y) - x^2\, y \sin(x\,y) = 2\,x \cos(x\,y) - x^2\, y \sin(x\,y)$$

We need to find a function F(x,y) such that:

$$M(x, y) = \frac{\partial}{\partial x} F(x, y)$$
$$N(x, y) = \frac{\partial}{\partial y} F(x, y)$$

Therefore:

$$F(x, y) = \int M(x, y)\, dx$$
$$F(x, y) = \int (\sin(x\,y) + x\,y \cos(x\,y))\, dx + A(y)$$
$$F(x, y) = x \sin(x\,y) + A(y)$$

$$F(x, y) = \int N(x, y)\, dy$$

$$F(x, y) = \int x^2 \cos(x\,y)\,dy + B(x)$$

$$F(x, y) = x \sin(x\,y) + B(x)$$

However, we ended up with two F(x,y) functions.
We need to make sure they are the same by calculating the unknown A(y) form the first F(x,y) and B(x) from the second F(x,y).
Calculating A(y):

$$\frac{\partial}{\partial y} F(x, y) = N(x, y)$$

$$x^2 \cos(x\,y) + \frac{d}{dy} A(y) = x^2 \cos(x\,y)$$

$$\frac{d}{dy} A(y) = 0$$

$$A(y) = \int \left(\frac{d}{dy} A(y) \right) dy$$

$$A(y) = \int 0\,dy$$

$$A(y) = C$$

Thus:

$$F(x, y) = \sin(x\,y)\,x + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x, y) = M(x, y)$$

$$\sin(x\,y) + x\,y \cos(x\,y) + \frac{d}{dx} B(x) = \sin(x\,y) + x\,y \cos(x\,y)$$

$$\frac{d}{dx} B(x) = 0$$

$$B(x) = \int \left(\frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int 0\,dx$$

$$B(x) = \mathrm{D}$$

Thus:

$$F(x, y) = \sin(x\,y)\,x + \mathrm{D}$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

Problem 1j

$$\left(3\,x^2 \sin(2\,y) - 2\,x\,y\right) dx + \left(2\,x^3 \cos(2\,y) - x^2\right) dy = 0$$

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

$$M(x, y) = 3\,x^2 \sin(2\,y) - 2\,x\,y$$

$$N(x, y) = 2\,x^3 \cos(2\,y) - x^2$$

Test for exactness:

$$\frac{\partial}{\partial y} M(x, y) = \frac{\partial}{\partial x} N(x, y)$$

$$6\,x^2 \cos(2\,y) - 2\,x = 6\,x^2 \cos(2\,y) - 2\,x$$

We need to find a function F(x,y) such that:

$$M(x, y) = \frac{\partial}{\partial x} F(x, y)$$

$$N(x,y) = \frac{\partial}{\partial y} F(x,y)$$

Therefore:

$$F(x,y) = \int M(x,y)\, dx$$

$$F(x,y) = \int \left(3x^2 \sin(2y) - 2xy\right) dx + A(y)$$

$$F(x,y) = x^2 (\sin(2y)\, x - y) + A(y)$$

$$F(x,y) = \int N(x,y)\, dy$$

$$F(x,y) = \int \left(2x^3 \cos(2y) - x^2\right) dy + B(x)$$

$$F(x,y) = x^2 (\sin(2y)\, x - y) + B(x)$$

However, we ended up with two F(x,y) functions.
We need to make sure they are the same by calculating the unknown A(y) form the first F(x,y) and B(x) from the second F(x,y).
Calculating A(y):

$$\frac{\partial}{\partial y} F(x,y) = N(x,y)$$

$$x^2 (2\cos(2y)\, x - 1) + \frac{d}{dy} A(y) = 2x^3 \cos(2y) - x^2$$

$$\frac{d}{dy} A(y) = 0$$

$$A(y) = \int \left(\frac{d}{dy} A(y)\right) dy$$

$$A(y) = \int 0\, dy$$

$$A(y) = C$$

Thus:

$$F(x,y) = \sin(2y)\, x^3 - x^2 y + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x,y) = M(x,y)$$

$$2x (\sin(2y)\, x - y) + x^2 \sin(2y) + \frac{d}{dx} B(x) = 3x^2 \sin(2y) - 2xy$$

$$\frac{d}{dx} B(x) = 0$$

$$B(x) = \int \left(\frac{d}{dx} B(x)\right) dx$$

$$B(x) = \int 0\, dx$$

$$B(x) = \mathrm{D}$$

Thus:

$$F(x,y) = \sin(2y)\, x^3 - x^2 y + \mathrm{D}$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

Problem 1k

$$\left(4x^3y^5\sin(3x)+3x^4y^5\cos(3x)\right)dx+5x^4y^4\sin(3x)\,dy=0$$
$$M(x,y)\,dx+N(x,y)\,dy=0$$
$$M(x,y)=4x^3y^5\sin(3x)+3x^4y^5\cos(3x)$$
$$N(x,y)=5x^4y^4\sin(3x)$$

Test for exactness:

$$\frac{\partial}{\partial y}M(x,y)=\frac{\partial}{\partial x}N(x,y)$$
$$20x^3y^4\sin(3x)+15x^4y^4\cos(3x)=20x^3y^4\sin(3x)+15x^4y^4\cos(3x)$$

We need to find a function F(x,y) such that:

$$M(x,y)=\frac{\partial}{\partial x}F(x,y)$$
$$N(x,y)=\frac{\partial}{\partial y}F(x,y)$$

Therefore:

$$F(x,y)=\int M(x,y)\,dx$$
$$F(x,y)=\int\left(4x^3y^5\sin(3x)+3x^4y^5\cos(3x)\right)dx+A(y)$$
$$F(x,y)=x^4y^5\sin(3x)+A(y)$$

$$F(x,y)=\int N(x,y)\,dy$$
$$F(x,y)=\int 5x^4y^4\sin(3x)\,dy+B(x)$$
$$F(x,y)=x^4y^5\sin(3x)+B(x)$$

However, we ended up with two F(x,y) functions.
We need to make sure they are the same by calculating the unknown A(y) form the first F(x,y) and B(x) from the second F(x,y).
Calculating A(y):

$$\frac{\partial}{\partial y}F(x,y)=N(x,y)$$
$$5x^4y^4\sin(3x)+\frac{d}{dy}A(y)=5x^4y^4\sin(3x)$$
$$\frac{d}{dy}A(y)=0$$
$$A(y)=\int\left(\frac{d}{dy}A(y)\right)dy$$
$$A(y)=\int 0\,dy$$
$$A(y)=C$$

Thus:

$$F(x,y)=\sin(3x)\,x^4y^5+C$$

Calculating B(x):

$$\frac{\partial}{\partial x}F(x,y)=M(x,y)$$
$$4x^3y^5\sin(3x)+3x^4y^5\cos(3x)+\frac{d}{dx}B(x)=4x^3y^5\sin(3x)+3x^4y^5\cos(3x)$$
$$\frac{d}{dx}B(x)=0$$

$$B(x) = \int \left(\frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int 0 \, dx$$

$$B(x) = D$$

Thus:

$$F(x, y) = \sin(3\,x)\, x^4 y^5 + D$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

Problem 1l

$$3\,x^2\,y \ln(y)\; dx + \left(x^3 \ln(y) + x^3 - 2\,y\right) dy = 0$$

$$M(x, y)\; dx + N(x, y)\; dy = 0$$

$$M(x, y) = 3\,x^2\,y \ln(y)$$

$$N(x, y) = x^3 \ln(y) + x^3 - 2\,y$$

Test for exactness:

$$\frac{\partial}{\partial y} M(x, y) = \frac{\partial}{\partial x} N(x, y)$$

$$3\,x^2 \ln(y) + 3\,x^2 = 3\,x^2 \ln(y) + 3\,x^2$$

We need to find a function F(x,y) such that:

$$M(x, y) = \frac{\partial}{\partial x} F(x, y)$$

$$N(x, y) = \frac{\partial}{\partial y} F(x, y)$$

Therefore:

$$F(x, y) = \int M(x, y) \, dx$$

$$F(x, y) = \int 3\,x^2\,y \ln(y) \, dx + A(y)$$

$$F(x, y) = x^3\,y \ln(y) + A(y)$$

$$F(x, y) = \int N(x, y) \, dy$$

$$F(x, y) = \int \left(x^3 \ln(y) + x^3 - 2\,y\right) dy + B(x)$$

$$F(x, y) = y\left(x^3 \ln(y) - y\right) + B(x)$$

However, we ended up with two F(x,y) functions.
We need to make sure they are the same by calculating the unknown A(y) form the first F(x,y) and B(x) from the second F(x,y).
Calculating A(y):

$$\frac{\partial}{\partial y} F(x, y) = N(x, y)$$

$$x^3 \ln(y) + x^3 + \frac{d}{dy} A(y) = x^3 \ln(y) + x^3 - 2\,y$$

$$\frac{d}{dy} A(y) = -2\,y$$

$$A(y) = \int \left(\frac{d}{dy} A(y) \right) dy$$

$$A(y) = \int -2\,y\,\mathrm{d}y$$

$$A(y) = -y^2 + C$$

Thus:

$$F(x, y) = \ln(y)\,x^3\,y - y^2 + C$$

Calculating B(x):

$$\frac{\partial}{\partial x}\,F(x, y) = M(x, y)$$

$$3\,x^2\,y\ln(y) + \frac{\mathrm{d}}{\mathrm{d}x}\,B(x) = 3\,x^2\,y\ln(y)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\,B(x) = 0$$

$$B(x) = \int \left(\frac{\mathrm{d}}{\mathrm{d}x}\,B(x)\right)\mathrm{d}x$$

$$B(x) = \int 0\,\mathrm{d}x$$

$$B(x) = \mathrm{D}$$

Thus:

$$F(x, y) = \ln(y)\,x^3\,y - y^2 + \mathrm{D}$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

Problem 1m

$$\left(2\,y\,\mathrm{e}^{2\,x\,y}\sin(x) + \mathrm{e}^{2\,x\,y}\cos(x) + 1\right)dx + 2\,x\,\mathrm{e}^{2\,x\,y}\sin(x)\,dy = 0$$

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

$$M(x, y) = 2\,y\,\mathrm{e}^{2\,x\,y}\sin(x) + \mathrm{e}^{2\,x\,y}\cos(x) + 1$$

$$N(x, y) = 2\,x\,\mathrm{e}^{2\,x\,y}\sin(x)$$

Test for exactness:

$$\frac{\partial}{\partial y}\,M(x, y) = \frac{\partial}{\partial x}\,N(x, y)$$

$$2\,\mathrm{e}^{2\,x\,y}\sin(x) + 4\,y\,x\,\mathrm{e}^{2\,x\,y}\sin(x) + 2\,x\,\mathrm{e}^{2\,x\,y}\cos(x) = 2\,\mathrm{e}^{2\,x\,y}\sin(x) + 4\,y\,x\,\mathrm{e}^{2\,x\,y}\sin(x) + 2\,x\,\mathrm{e}^{2\,x\,y}\cos(x)$$

We need to find a function F(x,y) such that:

$$M(x, y) = \frac{\partial}{\partial x}\,F(x, y)$$

$$N(x, y) = \frac{\partial}{\partial y}\,F(x, y)$$

Therefore:

$$F(x, y) = \int M(x, y)\,\mathrm{d}x$$

$$F(x, y) = \int \left(2\,y\,\mathrm{e}^{2\,x\,y}\sin(x) + \mathrm{e}^{2\,x\,y}\cos(x) + 1\right)\mathrm{d}x + A(y)$$

$$F(x, y) = \mathrm{e}^{2\,x\,y}\sin(x) + x + A(y)$$

$$F(x, y) = \int N(x, y)\,\mathrm{d}y$$

$$F(x, y) = \int 2\,x\,\mathrm{e}^{2\,x\,y}\sin(x)\,\mathrm{d}y + B(x)$$

$$F(x, y) = e^{2xy}\sin(x) + B(x)$$

However, we ended up with two F(x,y) functions.
We need to make sure they are the same by calculating the unknown A(y) form the first F(x,y) and B(x) from the second F(x,y).
Calculating A(y):

$$\frac{\partial}{\partial y} F(x, y) = N(x, y)$$

$$2\,x\,e^{2xy}\sin(x) + \frac{d}{dy} A(y) = 2\,x\,e^{2xy}\sin(x)$$

$$\frac{d}{dy} A(y) = 0$$

$$A(y) = \int \left(\frac{d}{dy} A(y)\right) dy$$

$$A(y) = \int 0\, dy$$

$$A(y) = C$$

Thus:

$$F(x, y) = x + e^{2xy}\sin(x) + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x, y) = M(x, y)$$

$$2\,y\,e^{2xy}\sin(x) + e^{2xy}\cos(x) + \frac{d}{dx} B(x) = 2\,y\,e^{2xy}\sin(x) + e^{2xy}\cos(x) + 1$$

$$\frac{d}{dx} B(x) = 1$$

$$B(x) = \int \left(\frac{d}{dx} B(x)\right) dx$$

$$B(x) = \int 1\, dx$$

$$B(x) = x + \mathrm{D}$$

Thus:

$$F(x, y) = x + e^{2xy}\sin(x) + \mathrm{D}$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.
