Exercises 2.4
Problem 1 & 2, using separation of variables
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Problem 1a

An ODE:

$$\frac{d}{dx} y(x) = 3x^2 e^{-y}$$

with:

$$X(x) = 3x^2$$
$$Y(y) = e^{-y}$$

Therefore

$$\int \frac{1}{Y(y)}\, dy = \int X(x)\, dx$$
$$\int \frac{1}{e^{-y}}\, dy = \int 3x^2\, dx$$
$$\frac{1}{e^{-y}} = x^3 + C$$

General solution:

$$y(x) = \ln(x^3 + C)$$

Particular solution for the given IC:

$$y(0) = 0$$

Thus:

$$y(0) = \ln(C)$$
$$0 = \ln(C)$$
$$C = 1$$

Therefore, the particular solution is:

$$y(x) = \ln((x+1)(x^2 - x + 1))$$

Verify the IC:

$$y(0) = 0$$

Problem 1b

An ODE:

$$\frac{d}{dx} y(x) = 6x^2 + 5$$

with:

$$X(x) = 6x^2 + 5$$
$$Y(y) = 1$$

Therefore

$$\int \frac{1}{Y(y)}\, dy = \int X(x)\, dx$$
$$\int 1\, dy = \int (6x^2 + 5)\, dx$$
$$y = 2x^3 + C + 5x$$

General solution:

$$y(x) = 2x^3 + 5x + C$$

Particular solution for the given IC:

$$y(0) = 0$$

Thus:

$$y(0) = C$$

$$0 = C$$
$$C = 0$$

Therefore, the particular solution is:

$$y(x) = 2x^3 + 5x$$

Verify the IC:

$$y(0) = 0$$

Problem 1c

An ODE:

$$\frac{d}{dx} y(x) = -4y$$

with:

$$X(x) = -4$$
$$Y(y) = y$$

Therefore

$$\int \frac{1}{Y(y)}\, dy = \int X(x)\, dx$$
$$\int \frac{1}{y}\, dy = \int (-4)\, dx$$
$$\ln(y) = -4x + C$$

General solution:

$$y(x) = e^{-4x + C}$$

Particular solution for the given IC:

$$y(-1) = 0$$

Thus:

$$y(-1) = e^{4 + C}$$
$$0 = e^{4 + C}$$
$$C = (\ )$$

Fail to calculate C, need to be creative here.

Problem 1d

An ODE:

$$\frac{d}{dx} y(x) = 6x^2 + 5$$

with:

$$X(x) = 6x^2 + 5$$
$$Y(y) = 1$$

Therefore

$$\int \frac{1}{Y(y)}\, dy = \int X(x)\, dx$$
$$\int 1\, dy = \int \left(6x^2 + 5\right) dx$$
$$y = 2x^3 + 5x + C$$

General solution:

$$y(x) = 2x^3 + 5x + C$$

Particular solution for the given IC:

$$y(0) = 0$$

Thus:

$$y(0) = C$$
$$0 = C$$

$$C=0$$

Therefore, the particular solution is:

$$y(x)=2x^3+5x$$

Verify the IC:

$$y(0)=0$$

Problem 1e

An ODE:

$$\frac{d}{dx}\,y(x)=e^x\left(y^2-y\right)$$

with:

$$X(x)=e^x$$

$$Y(y)=y^2-y$$

Therefore

$$\int\frac{1}{Y(y)}\,dy=\int X(x)\,dx$$

$$\int\frac{1}{y^2-y}\,dy=\int e^x\,dx$$

$$\ln(y-1)-\ln(y)=e^x+C$$

General solution:

$$y(x)=-\frac{1}{e^{e^x+C}-1}$$

Particular solution for the given IC:

$$y(0)=2$$

Thus:

$$y(0)=-\frac{1}{e^{1+C}-1}$$

$$2=-\frac{1}{e^{1+C}-1}$$

$$C=-\ln(2)-1$$

Therefore, the particular solution is:

$$y(x)=-\frac{2}{e^{e^x-1}-2}$$

Verify the IC:

$$y(0)=2$$

Problem 1f

An ODE:

$$\frac{d}{dx}\,y(x)=y^2+y-6$$

with:

$$X(x)=1$$

$$Y(y)=y^2+y-6$$

Therefore

$$\int\frac{1}{Y(y)}\,dy=\int X(x)\,dx$$

$$\int\frac{1}{y^2+y-6}\,dy=\int 1\,dx$$

$$-\frac{\ln(y+3)}{5}+\frac{\ln(y-2)}{5}=x+C$$

General solution:

$$y(x)=-\frac{3\,e^{5x+5C}+2}{-1+e^{5x+5C}}$$

Particular solution for the given IC:

$$y(5)=10$$

Thus:

$$y(5)=-\frac{3\,e^{25+5C}+2}{-1+e^{25+5C}}$$

$$10=-\frac{3\,e^{25+5C}+2}{-1+e^{25+5C}}$$

$$C=-5+\frac{\ln\left(\frac{8}{13}\right)}{5}$$

Therefore, the particular solution is:

$$y(x)=\frac{-24\,e^{5x-25}-26}{-13+8\,e^{5x-25}}$$

Verify the IC:

$$y(5)=10$$

Problem 1g

An ODE:

$$\frac{d}{dx}\,y(x)=y\,(y+3)$$

with:

$$X(x)=1$$

$$Y(y)=y\,(y+3)$$

Therefore

$$\int\frac{1}{Y(y)}\,dy=\int X(x)\,dx$$

$$\int\frac{1}{y\,(y+3)}\,dy=\int 1\,dx$$

$$-\frac{\ln(y+3)}{3}+\frac{\ln(y)}{3}=x+C$$

General solution:

$$y(x)=-\frac{3\,e^{3x+3C}}{-1+e^{3x+3C}}$$

Particular solution for the given IC:

$$y(0)=-4$$

Thus:

$$y(0)=-\frac{3\,e^{3C}}{-1+e^{3C}}$$

$$-4=-\frac{3\,e^{3C}}{-1+e^{3C}}$$

$$C=\frac{2\ln(2)}{3}$$

Therefore, the particular solution is:

$$y(x) = -\frac{12\,e^{3x}}{-1+4\,e^{3x}}$$

Verify the IC:

$$y(0) = -4$$

Problem 1h

An ODE:

$$\frac{d}{dx}\,y(x) = \frac{6\,y\ln(y)}{x}$$

with:

$$X(x) = \frac{6}{x}$$

$$Y(y) = y\ln(y)$$

Therefore

$$\int \frac{1}{Y(y)}\,dy = \int X(x)\,dx$$

$$\int \frac{1}{y\ln(y)}\,dy = \int \frac{6}{x}\,dx$$

$$\ln(\ln(y)) = 6\ln(x) + C$$

General solution:

$$y(x) = e^{e^{C}x^{6}}$$

Particular solution for the given IC:

$$y(1) = e$$

Thus:

$$y(1) = e^{e^{C}}$$

$$e = e^{e^{C}}$$

$$C = 0$$

Therefore, the particular solution is:

$$y(x) = e^{x^{6}}$$

Verify the IC:

$$y(1) = e$$

Problem 1i

An ODE:

$$\frac{d}{dx}\,y(x) = e^{x}\,e^{2y}$$

with:

$$X(x) = e^{x}$$

$$Y(y) = e^{2y}$$

Therefore

$$\int \frac{1}{Y(y)}\,dy = \int X(x)\,dx$$

$$\int \frac{1}{e^{2y}}\,dy = \int e^{x}\,dx$$

$$-\frac{1}{2\,e^{2y}} = e^{x} + C$$

General solution:

$$y(x) = \frac{\ln\left(-\frac{1}{2\,(e^x + C)}\right)}{2}$$

Particular solution for the given IC:

$$y(0) = 1$$

Thus:

$$y(0) = \frac{\ln\left(-\frac{1}{2\,(1 + C)}\right)}{2}$$

$$1 = \frac{\ln\left(-\frac{1}{2\,(1 + C)}\right)}{2}$$

$$C = -\frac{2\,e^2 + 1}{2\,e^2}$$

Therefore, the particular solution is:

$$y(x) = 1 + \frac{\ln\left(\frac{1}{-2\,e^{x+2} + 2\,e^2 + 1}\right)}{2}$$

Verify the IC:

$$y(0) = 1$$

Problem 1j

An ODE:

$$\frac{d}{dx}\,y(x) = \frac{y}{2\,x}$$

with:

$$X(x) = \frac{1}{2\,x}$$

$$Y(y) = y$$

Therefore

$$\int \frac{1}{Y(y)}\,dy = \int X(x)\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{2\,x}\,dx$$

$$\ln(y) = \frac{\ln(x)}{2} + C$$

General solution:

$$y(x) = e^{\frac{\ln(x)}{2} + C}$$

Particular solution for the given IC:

$$y(3) = -1$$

Thus:

$$y(3) = e^{\frac{\ln(3)}{2} + C}$$

$$-1 = e^{\frac{\ln(3)}{2} + C}$$

$$C = I\,\pi - \frac{\ln(3)}{2}$$

Therefore, the particular solution is:

$$y(x) = -\frac{\sqrt{x}\,\sqrt{3}}{3}$$

Verify the IC:

$$y(3) = -1$$

Problem 1k

An ODE:

$$\frac{d}{dx}\, y(x) = -3\sin(2x)\, y\,(y+1)$$

with:

$$X(x) = -\sin(2x)$$
$$Y(y) = 3\,y\,(y+1)$$

Therefore

$$\int \frac{1}{Y(y)}\, dy = \int X(x)\, dx$$
$$\int \frac{1}{3\,y\,(y+1)}\, dy = \int -\sin(2x)\, dx$$
$$-\frac{\ln(y+1)}{3} + \frac{\ln(y)}{3} = \frac{\cos(2x)}{2} + C$$

General solution:

$$y(x) = -\frac{e^{\frac{3\cos(2x)}{2} + 3C}}{-1 + e^{\frac{3\cos(2x)}{2} + 3C}}$$

Particular solution for the given IC:

$$y(0) = 1$$

Thus:

$$y(0) = -\frac{e^{\frac{3}{2} + 3C}}{-1 + e^{\frac{3}{2} + 3C}}$$
$$1 = -\frac{e^{\frac{3}{2} + 3C}}{-1 + e^{\frac{3}{2} + 3C}}$$
$$C = -\frac{1}{2} - \frac{\ln(2)}{3}$$

Therefore, the particular solution is:

$$y(x) = -\frac{e^{\frac{3\cos(2x)}{2} - \frac{3}{2}}}{-2 + e^{\frac{3\cos(2x)}{2} - \frac{3}{2}}}$$

Verify the IC:

$$y(0) = 1$$

Problem 1l

An ODE:

$$\frac{d}{dx}\, y(x) = e^{y}$$

with:

$$X(x) = 1$$

$$Y(y) = e^{y}$$

Therefore

$$\int \frac{1}{Y(y)}\, dy = \int X(x)\, dx$$

$$\int \frac{1}{e^{y}}\, dy = \int 1\, dx$$

$$-\frac{1}{e^{y}} = x + C$$

General solution:

$$y(x) = \ln\left(-\frac{1}{x + C}\right)$$

Particular solution for the given IC:

$$y(0) = 5$$

Thus:

$$y(0) = \ln\left(-\frac{1}{C}\right)$$

$$5 = \ln\left(-\frac{1}{C}\right)$$

$$C = -\frac{1}{e^{5}}$$

Therefore, the particular solution is:

$$y(x) = \ln\left(\frac{1}{-x + e^{-5}}\right)$$

Verify the IC:

$$y(0) = 5$$

Problem 1m

An ODE:

$$\frac{d}{dx}\, y(x) = y \ln(y)$$

with:

$$X(x) = 1$$

$$Y(y) = y \ln(y)$$

Therefore

$$\int \frac{1}{Y(y)}\, dy = \int X(x)\, dx$$

$$\int \frac{1}{y \ln(y)}\, dy = \int 1\, dx$$

$$\ln(\ln(y)) = x + C$$

General solution:

$$y(x) = e^{e^{x + C}}$$

Particular solution for the given IC:

$$y(0) = 5$$

Thus:

$$y(0) = e^{e^{C}}$$

$$5 = e^{e^{C}}$$

$$C = \ln(\ln(5))$$

Therefore, the particular solution is:

$$y(x) = 5^{e^x}$$

Verify the IC:

$$y(0) = 5$$

Problem 1n

An ODE:

$$\boxed{\frac{d}{dx}\, y(x) = y^2 - 2\,y + 1}$$

with:

$$X(x) = 1$$

$$Y(y) = y^2 - 2\,y + 1$$

Therefore

$$\int \frac{1}{Y(y)}\, dy = \int X(x)\, dx$$

$$\int \frac{1}{y^2 - 2\,y + 1}\, dy = \int 1\, dx$$

$$-\frac{1}{y - 1} = x + C$$

General solution:

$$y(x) = \frac{x + C - 1}{x + C}$$

Particular solution for the given IC:

$$y(-3) = 0$$

Thus:

$$y(-3) = \frac{-4 + C}{-3 + C}$$

$$0 = \frac{-4 + C}{-3 + C}$$

$$C = 4$$

Therefore, the particular solution is:

$$y(x) = \frac{x + 3}{x + 4}$$

Verify the IC:

$$y(-3) = 0$$