

## Exercises 2.2

### Problem 3, using the variation of parameters

#### Problem 3a

$$y'(x) - y = 3e^x$$

An ODE:

$$\frac{d}{dx} y(x) - y(x) = 3e^x$$

Thus, we have:

$$p(x) = -1$$

$$q(x) = 3e^x$$

Step 1, find the homogenous solution:

$$y_h = A e^{-\left(\int p(x) dx\right)}$$

$$y_h = A e^x$$

Step 2, find the complementary solution based on the homogenous solution by turning the A into a function A(x):

$$y(x) = A(x) e^{-\left(\int p(x) dx\right)}$$

$$y(x) = A(x) e^x$$

Step 3, find y'(x) from y(x):

$$\frac{d}{dx} y(x) = \left( \frac{d}{dx} A(x) \right) e^x + A(x) e^x$$

Step 4, to find A(x), substitute y(x) and y'(x) back to the ODE:

$$\frac{d}{dx} y(x) - y(x) = 3e^x$$

$$\left( \frac{d}{dx} A(x) \right) e^x = 3e^x$$

$$\frac{d}{dx} A(x) = 3$$

Integrate both sides wrt x:

$$A(x) = 3x + C$$

Step 5: substitute A(x) back to y(x) of the complementary solution:

$$y(x) = A(x) e^x$$

$$y(x) = (3x + C) e^x$$

#### Problem 3b

$$y'(x) + 4y = 8$$

An ODE:

$$\frac{d}{dx} y(x) + 4y(x) = 8$$

Thus, we have:

$$p(x) = 4$$

$$q(x) = 8$$

Step 1, find the homogenous solution:

$$y_h = A e^{-\left(\int p(x) dx\right)}$$

$$y_h = A e^{-4x}$$

Step 2, find the complementary solution based on the homogenous solution by turning the A into a function A(x):

$$y(x) = A(x) e^{-\left(\int p(x) dx\right)}$$

$$y(x) = A(x) e^{-4x}$$

Step 3, find y'(x) from y(x):

$$\frac{d}{dx} y(x) = \left( \frac{d}{dx} A(x) \right) e^{-4x} - 4 A(x) e^{-4x}$$

Step 4, to find A(x), substitute y(x) and y'(x) back to the ODE:

$$\frac{d}{dx} y(x) + 4 y(x) = 8$$

$$\left( \frac{d}{dx} A(x) \right) e^{-4x} = 8$$

$$\frac{d}{dx} A(x) = \frac{8}{e^{-4x}}$$

Integrate both sides wrt x:

$$A(x) = 2 e^{4x} + C$$

Step 5: substitute A(x) back to y(x) of the complementary solution:

$$y(x) = A(x) e^{-4x}$$

$$y(x) = 2 + C e^{-4x}$$

### Problem 3c

$$y'(x) + y = x^2$$

An ODE:

$$\frac{d}{dx} y(x) + y(x) = x^2$$

Thus, we have:

$$p(x) = 1$$

$$q(x) = x^2$$

Step 1, find the homogenous solution:

$$y_h = A e^{-\left(\int p(x) dx\right)}$$

$$y_h = A e^{-x}$$

Step 2, find the complementary solution based on the homogenous solution by turning the A into a function A(x):

$$y(x) = A(x) e^{-\left(\int p(x) dx\right)}$$

$$y(x) = A(x) e^{-x}$$

Step 3, find y'(x) from y(x):

$$\frac{d}{dx} y(x) = \left( \frac{d}{dx} A(x) \right) e^{-x} - A(x) e^{-x}$$

Step 4, to find A(x), substitute y(x) and y'(x) back to the ODE:

$$\frac{d}{dx} y(x) + y(x) = x^2$$

$$\left( \frac{d}{dx} A(x) \right) e^{-x} = x^2$$

$$\frac{d}{dx} A(x) = \frac{x^2}{e^{-x}}$$

Integrate both sides wrt x:

$$A(x) = (x^2 - 2x + 2) e^x + C$$

Step 5: substitute A(x) back to y(x) of the complementary solution:

$$y(x) = A(x) e^{-x}$$

$$y(x) = x^2 - 2x + C e^{-x} + 2$$

### Problem 3d

$$y'(x) = y - \sin(2x)$$

An ODE:

$$\frac{d}{dx} y(x) - y(x) = -\sin(2x)$$

Thus, we have:

$$p(x) = -1$$

$$q(x) = -\sin(2x)$$

Step 1, find the homogenous solution:

$$y_h = A e^{-\left(\int p(x) dx\right)}$$

$$y_h = A e^x$$

Step 2, find the complementary solution based on the homogenous solution by turning the A into a function A(x):

$$y(x) = A(x) e^{-\left(\int p(x) dx\right)}$$

$$y(x) = A(x) e^x$$

Step 3, find y'(x) from y(x):

$$\frac{d}{dx} y(x) = \left( \frac{d}{dx} A(x) \right) e^x + A(x) e^x$$

Step 4, to find A(x), substitute y(x) and y'(x) back to the ODE:

$$\frac{d}{dx} y(x) - y(x) = -\sin(2x)$$

$$\left( \frac{d}{dx} A(x) \right) e^x = -\sin(2x)$$

$$\frac{d}{dx} A(x) = -\frac{\sin(2x)}{e^x}$$

Integrate both sides wrt x:

$$A(x) = \frac{(2 \cos(2x) + \sin(2x)) e^{-x}}{5} + C$$

Step 5: substitute A(x) back to y(x) of the complementary solution:

$$y(x) = A(x) e^x$$

$$y(x) = \frac{2 \cos(2x)}{5} + \frac{\sin(2x)}{5} + C e^x$$

### Problem 3e

$$y'(x) - \tan(x)y = 6$$

An ODE:

$$\frac{d}{dx} y(x) - \tan(x) y(x) = 6$$

Thus, we have:

$$p(x) = -\tan(x)$$

$$q(x) = 6$$

Step 1, find the homogenous solution:

$$y_h = A e^{-\left(\int p(x) dx\right)}$$

$$y_h = \frac{A}{\cos(x)}$$

Step 2, find the complementary solution based on the homogenous solution by turning the A into a function A(x):

$$y(x) = A(x) e^{-\left(\int p(x) dx\right)}$$

$$y(x) = \frac{A(x)}{\cos(x)}$$

Step 3, find  $y'(x)$  from  $y(x)$ :

$$\frac{d}{dx} y(x) = \frac{\frac{d}{dx} A(x)}{\cos(x)} + \frac{A(x) \sin(x)}{\cos(x)^2}$$

Step 4, to find A(x), substitute  $y(x)$  and  $y'(x)$  back to the ODE:

$$\frac{d}{dx} y(x) - \tan(x) y(x) = 6$$

$$\frac{\frac{d}{dx} A(x)}{\cos(x)} + \frac{A(x) \sin(x)}{\cos(x)^2} - \frac{\tan(x) A(x)}{\cos(x)} = 6$$

$$\frac{d}{dx} A(x) = \frac{\tan(x) A(x) \cos(x) + 6 \cos(x)^2 - A(x) \sin(x)}{\cos(x)}$$

Integrate both sides wrt x:

$$A(x) = 6 \sin(x) + C$$

Step 5: substitute A(x) back to  $y(x)$  of the complementary solution:

$$y(x) = \frac{A(x)}{\cos(x)}$$

$$y(x) = \frac{6 \sin(x) + C}{\cos(x)}$$

### Problem 3f

$$xy'(x) + 2y = x^3 \Leftrightarrow y'(x) + \frac{2}{x}y = x^3$$

An ODE:

$$\frac{d}{dx} y(x) + \frac{2y(x)}{x} = x^3$$

Thus, we have:

$$p(x) = \frac{2}{x}$$

$$q(x) = x^3$$

Step 1, find the homogenous solution:

$$y_h = A e^{-\left(\int p(x) dx\right)}$$

$$y_h = \frac{A}{x^2}$$

Step 2, find the complementary solution based on the homogenous solution by turning the A into a

function  $A(x)$ :

$$y(x) = A(x) e^{-\left(\int p(x) dx\right)}$$
$$y(x) = \frac{A(x)}{x^2}$$

Step 3, find  $y'(x)$  from  $y(x)$ :

$$\frac{d}{dx} y(x) = \frac{\frac{d}{dx} A(x)}{x^2} - \frac{2 A(x)}{x^3}$$

Step 4, to find  $A(x)$ , substitute  $y(x)$  and  $y'(x)$  back to the ODE:

$$\frac{d}{dx} y(x) + \frac{2 y(x)}{x} = x^3$$
$$\frac{\frac{d}{dx} A(x)}{x^2} = x^3$$
$$\frac{d}{dx} A(x) = x^5$$

Integrate both sides wrt  $x$ :

$$A(x) = \frac{x^6}{6} + C$$

Step 5: substitute  $A(x)$  back to  $y(x)$  of the complementary solution:

$$y(x) = \frac{A(x)}{x^2}$$
$$y(x) = \frac{x^6 + 6 C}{6 x^2}$$

### Problem 3g

$$xy'(x) - 2y = x^3 \Leftrightarrow y'(x) - \frac{2}{x}y = x^3$$

An ODE:

$$\frac{d}{dx} y(x) - \frac{2 y(x)}{x} = x^3$$

Thus, we have:

$$p(x) = -\frac{2}{x}$$
$$q(x) = x^3$$

Step 1, find the homogenous solution:

$$y_h = A e^{-\left(\int p(x) dx\right)}$$
$$y_h = A x^2$$

Step 2, find the complementary solution based on the homogenous solution by turning the  $A$  into a function  $A(x)$ :

$$y(x) = A(x) e^{-\left(\int p(x) dx\right)}$$
$$y(x) = A(x) x^2$$

Step 3, find  $y'(x)$  from  $y(x)$ :

$$\frac{d}{dx} y(x) = \left( \frac{d}{dx} A(x) \right) x^2 + 2 A(x) x$$

Step 4, to find A(x), substitute y(x) and y'(x) back to the ODE:

$$\begin{aligned}\frac{d}{dx} y(x) - \frac{2 y(x)}{x} &= x^3 \\ \left( \frac{d}{dx} A(x) \right) x^2 &= x^3 \\ \frac{d}{dx} A(x) &= x\end{aligned}$$

Integrate both sides wrt x:

$$A(x) = \frac{x^2}{2} + C$$

Step 5: substitute A(x) back to y(x) of the complementary solution:

$$\begin{aligned}y(x) &= A(x) x^2 \\ y(x) &= \frac{1}{2} x^4 + C x^2\end{aligned}$$

### Problem 3h

$$y'(x) + \cot(x)y = 2 \cos(x)$$

An ODE:

$$\frac{d}{dx} y(x) + \cot(x) y(x) = 2 \cos(x)$$

Thus, we have:

$$\begin{aligned}p(x) &= \cot(x) \\ q(x) &= 2 \cos(x)\end{aligned}$$

Step 1, find the homogenous solution:

$$\begin{aligned}y_h &= A e^{-\left(\int p(x) dx\right)} \\ y_h &= \frac{A}{\sin(x)}\end{aligned}$$

Step 2, find the complementary solution based on the homogenous solution by turning the A into a function A(x):

$$\begin{aligned}y(x) &= A(x) e^{-\left(\int p(x) dx\right)} \\ y(x) &= \frac{A(x)}{\sin(x)}\end{aligned}$$

Step 3, find y'(x) from y(x):

$$\frac{d}{dx} y(x) = \frac{\frac{d}{dx} A(x)}{\sin(x)} - \frac{A(x) \cos(x)}{\sin(x)^2}$$

Step 4, to find A(x), substitute y(x) and y'(x) back to the ODE:

$$\begin{aligned}\frac{d}{dx} y(x) + \cot(x) y(x) &= 2 \cos(x) \\ \frac{\frac{d}{dx} A(x)}{\sin(x)} - \frac{A(x) \cos(x)}{\sin(x)^2} + \frac{\cot(x) A(x)}{\sin(x)} &= 2 \cos(x) \\ \frac{d}{dx} A(x) &= \frac{2 \cos(x) \sin(x)^2 - \cot(x) A(x) \sin(x) + A(x) \cos(x)}{\sin(x)}\end{aligned}$$

Integrate both sides wrt x:

$$A(x) = -\cos(x)^2 + C$$

Step 5: substitute A(x) back to y(x) of the complementary solution:

$$y(x) = \frac{A(x)}{\sin(x)}$$

$$y(x) = \frac{-\cos(x)^2 + C}{\sin(x)}$$

### Problem 3i

$$(x-5)(xy' + 3y) = 2$$

$$xy' + 3y = \frac{2}{(x-5)}$$

$$y' + \frac{3}{x}y = \frac{2}{x(x-5)}$$

An ODE:

$$\frac{d}{dx} y(x) + \frac{3y(x)}{x} = \frac{2}{x(x-5)}$$

Thus, we have:

$$p(x) = \frac{3}{x}$$

$$q(x) = \frac{2}{x(x-5)}$$

Step 1, find the homogenous solution:

$$y_h = A e^{-\left(\int p(x) dx\right)}$$

$$y_h = \frac{A}{x^3}$$

Step 2, find the complementary solution based on the homogenous solution by turning the A into a function A(x):

$$y(x) = A(x) e^{-\left(\int p(x) dx\right)}$$

$$y(x) = \frac{A(x)}{x^3}$$

Step 3, find y'(x) from y(x):

$$\frac{d}{dx} y(x) = \frac{\frac{d}{dx} A(x)}{x^3} - \frac{3A(x)}{x^4}$$

Step 4, to find A(x), substitute y(x) and y'(x) back to the ODE:

$$\frac{d}{dx} y(x) + \frac{3y(x)}{x} = \frac{2}{x(x-5)}$$

$$\frac{\frac{d}{dx} A(x)}{x^3} = \frac{2}{x(x-5)}$$

$$\frac{d}{dx} A(x) = \frac{2x^2}{x-5}$$

Integrate both sides wrt x:

$$A(x) = x^2 + 10x + 50 \ln(x-5) + C$$

Step 5: substitute A(x) back to y(x) of the complementary solution:

$$y(x) = \frac{A(x)}{x^3}$$

$$y(x) = \frac{x^2 + 10x + 50 \ln(x-5) + C}{x^3}$$

### Problem 3j

$$x' - 6x = e^y$$

This has a different form since  $x$  is the dependent variable and  $y$  is the independent variable. Thus, we need to change  $x$  to  $y$  and  $y$  to  $x$  to make it fit to the Maple procedure.

$$y' - 6y = e^x$$

An ODE:

$$\frac{d}{dx} y(x) - 6y(x) = e^x$$

Thus, we have:

$$p(x) = -6$$

$$q(x) = e^x$$

Step 1, find the homogenous solution:

$$y_h = A e^{-\left(\int p(x) dx\right)}$$

$$y_h = A e^{6x}$$

Step 2, find the complementary solution based on the homogenous solution by turning the  $A$  into a function  $A(x)$ :

$$y(x) = A(x) e^{-\left(\int p(x) dx\right)}$$

$$y(x) = A(x) e^{6x}$$

Step 3, find  $y'(x)$  from  $y(x)$ :

$$\frac{d}{dx} y(x) = \left( \frac{d}{dx} A(x) \right) e^{6x} + 6A(x) e^{6x}$$

Step 4, to find  $A(x)$ , substitute  $y(x)$  and  $y'(x)$  back to the ODE:

$$\frac{d}{dx} y(x) - 6y(x) = e^x$$

$$\left( \frac{d}{dx} A(x) \right) e^{6x} = e^x$$

$$\frac{d}{dx} A(x) = \frac{e^x}{e^{6x}}$$

Integrate both sides wrt  $x$ :

$$A(x) = -\frac{e^{-5x}}{5} + C$$

Step 5: substitute  $A(x)$  back to  $y(x)$  of the complementary solution:

$$y(x) = A(x) e^{6x}$$

$$y(x) = C e^x e^{5x} - \frac{e^x}{5}$$

### Problem 3k

$$yx' - y^5 + 3x = 0$$

This has a different form since  $x$  is the dependent variable and  $y$  is the independent variable. Thus, we need to change  $x$  to  $y$  and  $y$  to  $x$  to make it fit to the Maple procedure.

$$xy' - x^5 + 3y = 0$$

$$y' + \frac{3}{x}y = x^4$$

An ODE:



$$\frac{d}{dx} y(x) + \frac{3 y(x)}{x} = x^4$$

Thus, we have:

$$p(x) = \frac{3}{x}$$

$$q(x) = x^4$$

Step 1, find the homogenous solution:

$$y_h = A e^{-\left(\int p(x) dx\right)}$$

$$y_h = \frac{A}{x^3}$$

Step 2, find the complementary solution based on the homogenous solution by turning the A into a function A(x):

$$y(x) = A(x) e^{-\left(\int p(x) dx\right)}$$

$$y(x) = \frac{A(x)}{x^3}$$

Step 3, find y'(x) from y(x):

$$\frac{d}{dx} y(x) = \frac{\frac{d}{dx} A(x)}{x^3} - \frac{3 A(x)}{x^4}$$

Step 4, to find A(x), substitute y(x) and y'(x) back to the ODE:

$$\frac{d}{dx} y(x) + \frac{3 y(x)}{x} = x^4$$

$$\frac{\frac{d}{dx} A(x)}{x^3} = x^4$$

$$\frac{d}{dx} A(x) = x^7$$

Integrate both sides wrt x:

$$A(x) = \frac{x^8}{8} + C$$

Step 5: substitute A(x) back to y(x) of the complementary solution:

$$y(x) = \frac{A(x)}{x^3}$$

$$y(x) = \frac{x^8 + 8 C}{8 x^3}$$

### Problem 31

$$y^2 x' + xy - 4 y^2 = 1$$

This has a different form since  $x$  is the dependent variable and  $y$  is the independent variable. Thus, we need to change  $x$  to  $y$  and  $y$  to  $x$  to make it fit to the Maple procedure.

$$x^2 y' + xy - 4 x^2 = 1$$

$$y' + \frac{1}{x} y = 4 x$$

An ODE:

$$\frac{d}{dx} y(x) + \frac{y(x)}{x} = 4 x$$

Thus, we have:

$$p(x) = \frac{1}{x}$$

$$q(x) = 4x$$

Step 1, find the homogenous solution:

$$y_h = A e^{-\left(\int p(x) dx\right)}$$

$$y_h = \frac{A}{x}$$

Step 2, find the complementary solution based on the homogenous solution by turning the A into a function A(x):

$$y(x) = A(x) e^{-\left(\int p(x) dx\right)}$$

$$y(x) = \frac{A(x)}{x}$$

Step 3, find  $y'(x)$  from  $y(x)$ :

$$\frac{d}{dx} y(x) = \frac{\frac{d}{dx} A(x)}{x} - \frac{A(x)}{x^2}$$

Step 4, to find A(x), substitute  $y(x)$  and  $y'(x)$  back to the ODE:

$$\frac{d}{dx} y(x) + \frac{y(x)}{x} = 4x$$

$$\frac{\frac{d}{dx} A(x)}{x} = 4x$$

$$\frac{d}{dx} A(x) = 4x^2$$

Integrate both sides wrt x:

$$A(x) = \frac{4x^3}{3} + C$$

Step 5: substitute A(x) back to  $y(x)$  of the complementary solution:

$$y(x) = \frac{A(x)}{x}$$

$$y(x) = \frac{4x^3 + 3C}{3x}$$

### Problem 3m

$$tx' - 4t^5 = x$$

This has a different form since  $x$  is the dependent variable and  $t$  is the independent variable. Thus, we need to change  $x$  to  $y$  and  $t$  to  $x$  to make it fit to the Maple procedure.

$$xy' - 4x^5 = y$$

$$y' - 4x^4 = \frac{1}{x}y$$

$$y' - \frac{1}{x}y = 4x^4$$

An ODE:

$$\frac{d}{dx} y(x) - \frac{y(x)}{x} = 4x^4$$

Thus, we have:

$$p(x) = -\frac{1}{x}$$

$$q(x) = 4x^4$$

Step 1, find the homogenous solution:

$$y_h = A e^{-\left(\int p(x) dx\right)}$$

$$y_h = A x$$

Step 2, find the complementary solution based on the homogenous solution by turning the A into a function A(x):

$$y(x) = A(x) e^{-\left(\int p(x) dx\right)}$$

$$y(x) = A(x) x$$

Step 3, find y'(x) from y(x):

$$\frac{d}{dx} y(x) = \left( \frac{d}{dx} A(x) \right) x + A(x)$$

Step 4, to find A(x), substitute y(x) and y'(x) back to the ODE:

$$\frac{d}{dx} y(x) - \frac{y(x)}{x} = 4x^4$$

$$\left( \frac{d}{dx} A(x) \right) x = 4x^4$$

$$\frac{d}{dx} A(x) = 4x^3$$

Integrate both sides wrt x:

$$A(x) = x^4 + C$$

Step 5: substitute A(x) back to y(x) of the complementary solution:

$$y(x) = A(x) x$$

$$y(x) = (x^4 + C) x$$