Greenberg 23.2

Problem defintion

Find: $S = \int_C f(z)\, \mathrm{d}z$

where $C : z(r)$, which is the parametrized function of a curve in the complex plane

```
> restart:
```

```
> complex_integral_parameterized:=proc(z_::algebraic,f_::algebraic,
  a_,b_)
    local f, zprime, S;

    print(`Find the integral of:`);
    print('f(z)'=f_);
    print(`along the curve:`);
    print('z'=z_);
    print(`starting from`, r=a_, `moving to`, r=b_);

    f:=eval(f_,z=z_);
    zprime:=diff(z_,r);
    S:=int(f*zprime,r=a_..b_);

    print(`Solution:`);

    print('diff(z(r),r)'=zprime);
    print('Int(f(z)*diff(z(r),r),r=a_..b_)'=Int(f*zprime,r=a_..b_))
  ;
    print(`Hence, the final result is:`);
    print(S);
    print(`------------------------------------------------------------
  -------------------`);
    return(S);
  end proc:
```

```
> z(r):=exp(I*r):
  f(z):=1/z:
  complex_integral_parameterized(z(r),f(z),0,2*Pi):
```

Find the integral of:

$$f(z) = \frac{1}{z}$$

along the curve:

$$z = \mathrm{e}^{\mathrm{I}\, r}$$

starting from, $r = 0$, *moving to*, $r = 2\,\pi$

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}r}\, z(r) = \mathrm{I}\, \mathrm{e}^{\mathrm{I}\, r}$$

$$\int_0^{2\,\pi} f(z) \left(\frac{\mathrm{d}}{\mathrm{d}r}\, z(r)\right) \mathrm{d}r = \int_0^{2\,\pi} \mathrm{I}\, \mathrm{d}r$$

Hence, the final result is:

$$2\,\mathrm{I}\,\pi$$

-- **(1)**

```
> z(r):=r+I*r:
  f(z):=abs(z)^2:
  complex_integral_parameterized(z(r),f(z),0,1):
```

Find the integral of:

$$f(z)=|z|^2$$

along the curve:

$$z=r+\mathrm{I}\,r$$

starting from, $r=0$, *moving to,* $r=1$

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}r}\,z(r)=1+\mathrm{I}$$

$$\int_0^1 f(z)\left(\frac{\mathrm{d}}{\mathrm{d}r}\,z(r)\right)\mathrm{d}r=\int_0^1 (1+\mathrm{I})\,|r+\mathrm{I}\,r|^2\,\mathrm{d}r$$

Hence, the final result is:

$$\frac{2}{3}+\frac{2\,\mathrm{I}}{3}$$

-- **(2)**

```
> z(r):=r+I*r:
  f(z):=conjugate(z):
  complex_integral_parameterized(z(r),f(z),0,1):
```

Find the integral of:

$$f(z)=\overline{z}$$

along the curve:

$$z=r+\mathrm{I}\,r$$

starting from, $r=0$, *moving to,* $r=1$

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}r}\,z(r)=1+\mathrm{I}$$

$$\int_0^1 f(z)\left(\frac{\mathrm{d}}{\mathrm{d}r}\,z(r)\right)\mathrm{d}r=\int_0^1 (1+\mathrm{I})\,\overline{(r+\mathrm{I}\,r)}\,\mathrm{d}r$$

Hence, the final result is:

$$1$$

-- **(3)**

```
> z(r):=2*exp(I*r):
  f(z):=conjugate(z):
  complex_integral_parameterized(z(r),f(z),Pi,0):
```

Find the integral of:

$$f(z) = \overline{z}$$

along the curve:

$$z = 2\,e^{\mathrm{I}\,r}$$

starting from, $r = \pi$, *moving to*, $r = 0$

Solution:

$$\frac{d}{dr}\,z(r) = 2\,\mathrm{I}\,e^{\mathrm{I}\,r}$$

$$\int_{\pi}^{0} f(z)\left(\frac{d}{dr}\,z(r)\right) dr = \int_{\pi}^{0} 4\,\mathrm{I}\,e^{-\mathrm{I}\,\overline{r}}\,e^{\mathrm{I}\,r}\,dr$$

Hence, the final result is:

$$-4\,\mathrm{I}\,\pi$$

-- **(4)**

```
> z(r):=2*exp(I*r):
  f(z):=4*z:
  complex_integral_parameterized(z(r),f(z),0,Pi):
```

Find the integral of:

$$f(z) = 4\,z$$

along the curve:

$$z = 2\,e^{\mathrm{I}\,r}$$

starting from, $r = 0$, *moving to*, $r = \pi$

Solution:

$$\frac{d}{dr}\,z(r) = 2\,\mathrm{I}\,e^{\mathrm{I}\,r}$$

$$\int_{0}^{\pi} f(z)\left(\frac{d}{dr}\,z(r)\right) dr = \int_{0}^{\pi} 16\,\mathrm{I}\left(e^{\mathrm{I}\,r}\right)^{2} dr$$

Hence, the final result is:

$$0$$

-- **(5)**

```
> z(r):=4*exp(I*r):
  f(z):=1/((z-3*I)*(z+5)):
  complex_integral_parameterized(z(r),f(z),0,2*Pi):
```

Find the integral of:

$$f(z) = \frac{1}{(z-3\,\mathrm{I})\,(z+5)}$$

along the curve:

$$z = 4\,e^{\mathrm{I}\,r}$$

starting from, $r = 0$, *moving to*, $r = 2\,\pi$

Solution:

$$\frac{d}{dr} z(r) = 4\,I\,e^{I r}$$

$$\int_0^{2\pi} f(z)\left(\frac{d}{dr} z(r)\right) dr = \int_0^{2\pi} \frac{4\,I\,e^{I r}}{\left(4\,e^{I r} - 3\,I\right)\left(4\,e^{I r} + 5\right)}\, dr$$

Hence, the final result is:

$$\frac{3\,\pi}{17} + \frac{5\,I\,\pi}{17}$$

-- **(6)**

```
> z(r):=6*exp(I*r):
f(z):=1/((z-3*I)*(z+5)):
complex_integral_parameterized(z(r),f(z),0,2*Pi):
```

Find the integral of:

$$f(z) = \frac{1}{(z - 3\,I)\,(z + 5)}$$

along the curve:

$$z = 6\,e^{I r}$$

starting from, $r = 0$, *moving to,* $r = 2\,\pi$

Solution:

$$\frac{d}{dr} z(r) = 6\,I\,e^{I r}$$

$$\int_0^{2\pi} f(z)\left(\frac{d}{dr} z(r)\right) dr = \int_0^{2\pi} \frac{6\,I\,e^{I r}}{\left(6\,e^{I r} - 3\,I\right)\left(6\,e^{I r} + 5\right)}\, dr$$

Hence, the final result is:

$$0$$

-- **(7)**

```
> z(r):=exp(I*r):
f(z):=z^2*sin(1/z):
complex_integral_parameterized(z(r),f(z),2*Pi,0):
```

Find the integral of:

$$f(z) = z^2 \sin\left(\frac{1}{z}\right)$$

along the curve:

$$z = e^{I r}$$

starting from, $r = 2\,\pi$, *moving to,* $r = 0$

Solution:

$$\frac{d}{dr}\, z(r) = \mathrm{I}\, e^{\mathrm{I}\, r}$$

$$\int_{2\pi}^{0} f(z) \left(\frac{d}{dr}\, z(r)\right) dr = \int_{2\pi}^{0} \mathrm{I} \left(e^{\mathrm{I}\, r}\right)^3 \sin\left(\frac{1}{e^{\mathrm{I}\, r}}\right) dr$$

Hence, the final result is:

$$\frac{\mathrm{I}}{3}\, \pi$$

-- **(8)**

```
> z(r):=exp(I*r):
  f(z):=1/(z*(z-2)):
  complex_integral_parameterized(z(r),f(z),0,2*Pi):
```

Find the integral of:

$$f(z) = \frac{1}{z\,(z-2)}$$

along the curve:

$$z = e^{\mathrm{I}\, r}$$

starting from, $r = 0$, *moving to,* $r = 2\,\pi$

Solution:

$$\frac{d}{dr}\, z(r) = \mathrm{I}\, e^{\mathrm{I}\, r}$$

$$\int_{0}^{2\pi} f(z) \left(\frac{d}{dr}\, z(r)\right) dr = \int_{0}^{2\pi} \frac{\mathrm{I}}{e^{\mathrm{I}\, r} - 2}\, dr$$

Hence, the final result is:

$$-\mathrm{I}\,\pi$$

-- **(9)**