Exercises 2.4
Problem Set # 10
Greenberg's Book

Problem 10a

$$\frac{d}{dx} y(x) = e^{\frac{y(x)}{x}} + \frac{y(x)}{x}$$

Take:

$$y(x) = u(x)\, x$$

$$\frac{d}{dx} y(x) = \left(\frac{d}{dx} u(x)\right) x + u(x)$$

Substitute them back to the ODE:

$$\left(\frac{d}{dx} u(x)\right) x + u(x) = e^{u(x)} + u(x)$$

$$\frac{d}{dx} u(x) = \frac{e^{u(x)}}{x}$$

The new ODE:

$$\frac{d}{dx} u(x) = \frac{e^{u}}{x}$$

with:

$$X(x) = \frac{1}{x}$$

$$U(u) = e^{u}$$

Therefore,

$$\int \frac{1}{U(u)}\, du = \int X(x)\, dx$$

$$\int \frac{1}{e^{u}}\, du = \int \frac{1}{x}\, dx$$

$$-\frac{1}{e^{u}} = \ln(x) + C$$

Thus, the general solution is:

$$u(x) = \ln\left(-\frac{1}{\ln(x) + C}\right)$$

Since, $u(x) = \frac{y(x)}{x}$, *therefore:*

$$\frac{y(x)}{x} = \ln\left(-\frac{1}{\ln(x) + C}\right)$$

Problem 10b

$$\frac{d}{dx} y(x) = \frac{2\,y(x) - x}{y(x) - 2\,x}$$

Take:

$$y(x) = u(x)\, x$$

$$\frac{d}{dx} y(x) = \left(\frac{d}{dx} u(x)\right) x + u(x)$$

Substitute them back to the ODE:

$$\left(\frac{d}{dx}u(x)\right)x+u(x)=\frac{2\,u(x)\,x-x}{u(x)\,x-2\,x}$$

$$\frac{d}{dx}u(x)=\frac{\frac{2\,u(x)\,x-x}{u(x)\,x-2\,x}-u(x)}{x}$$

$$\frac{d}{dx}u(x)=\frac{-u(x)^2+4\,u(x)-1}{x\,(u(x)-2)}$$

The new ODE:

$$\frac{d}{dx}u(x)=\frac{-u^2+4\,u-1}{x\,(u-2)}$$

with:

$$X(x)=\frac{1}{x}$$

$$U(u)=\frac{-u^2+4\,u-1}{u-2}$$

Therefore,

$$\int\frac{1}{U(u)}\,du=\int X(x)\,dx$$

$$\int\frac{u-2}{-u^2+4\,u-1}\,du=\int\frac{1}{x}\,dx$$

$$-\frac{\ln\left(u^2-4\,u+1\right)}{2}=\ln(x)+C$$

Thus, the general solution is:

$$u(x)=\left(\frac{2\,e^C\,x+\sqrt{3\,\left(e^C\right)^2x^2+1}}{e^C\,x},\frac{2\,e^C\,x-\sqrt{3\,\left(e^C\right)^2x^2+1}}{e^C\,x}\right)$$

Since, $u(x)=\frac{y(x)}{x}$, *therefore:*

$$\frac{y(x)}{x}=\left(\frac{2\,e^C\,x+\sqrt{3\,\left(e^C\right)^2x^2+1}}{e^C\,x},\frac{2\,e^C\,x-\sqrt{3\,\left(e^C\right)^2x^2+1}}{e^C\,x}\right)$$

Problem 10c

$$\frac{d}{dx}y(x)=\frac{x\,y(x)+2\,y(x)^2}{x^2}$$

Take:

$$y(x)=u(x)\,x$$

$$\frac{d}{dx}y(x)=\left(\frac{d}{dx}u(x)\right)x+u(x)$$

Substitute them back to the ODE:

$$\left(\frac{d}{dx}u(x)\right)x+u(x)=\frac{x^2\,u(x)+2\,u(x)^2x^2}{x^2}$$

$$\frac{d}{dx}\, u(x) = \frac{\frac{x^2\, u(x) + 2\, u(x)^2\, x^2}{x^2} - u(x)}{x}$$

$$\frac{d}{dx}\, u(x) = \frac{2\, u(x)^2}{x}$$

The new ODE:

$$\frac{d}{dx}\, u(x) = \frac{2\, u^2}{x}$$

with:

$$X(x) = \frac{1}{x}$$

$$U(u) = 2\, u^2$$

Therefore,

$$\int \frac{1}{U(u)}\, du = \int X(x)\, dx$$

$$\int \frac{1}{2\, u^2}\, du = \int \frac{1}{x}\, dx$$

$$-\frac{1}{2\, u} = \ln(x) + C$$

Thus, the general solution is:

$$u(x) = -\frac{1}{2\,(\ln(x) + C)}$$

$$\mathit{Since},\ u(x) = \frac{y(x)}{x},\ \mathit{therefore:}$$

$$\frac{y(x)}{x} = -\frac{1}{2\,(\ln(x) + C)}$$

Problem 10d

$$\frac{d}{dx}\, y(x) = -\frac{2\, x + y(x)}{x}$$

Take:

$$y(x) = u(x)\, x$$

$$\frac{d}{dx}\, y(x) = \left(\frac{d}{dx}\, u(x)\right) x + u(x)$$

Substitute them back to the ODE:

$$\left(\frac{d}{dx}\, u(x)\right) x + u(x) = -\frac{2\, x + u(x)\, x}{x}$$

$$\frac{d}{dx}\, u(x) = \frac{-\frac{2\, x + u(x)\, x}{x} - u(x)}{x}$$

$$\frac{d}{dx}\, u(x) = \frac{-2\, u(x) - 2}{x}$$

The new ODE:

$$\frac{d}{dx}\, u(x) = \frac{-2\,u-2}{x}$$

with:

$$X(x) = \frac{1}{x}$$

$$U(u) = -2\,u - 2$$

Therefore,

$$\int \frac{1}{U(u)}\, du = \int X(x)\, dx$$

$$\int \frac{1}{-2\,u-2}\, du = \int \frac{1}{x}\, dx$$

$$-\frac{\ln(-2\,u-2)}{2} = \ln(x) + C$$

Thus, the general solution is:

$$u(x) = -\frac{2\,x^2 + e^{-2\,C}}{2\,x^2}$$

Since, $u(x) = \frac{y(x)}{x}$, *therefore:*

$$\frac{y(x)}{x} = -\frac{2\,x^2 + e^{-2\,C}}{2\,x^2}$$

Problem 10e

$$\frac{d}{dx}\, y(x) = e^{\frac{y(x)}{x}} + \frac{y(x)}{x}$$

Take:

$$y(x) = u(x)\, x$$

$$\frac{d}{dx}\, y(x) = \left(\frac{d}{dx}\, u(x)\right) x + u(x)$$

Substitute them back to the ODE:

$$\left(\frac{d}{dx}\, u(x)\right) x + u(x) = e^{u(x)} + u(x)$$

$$\frac{d}{dx}\, u(x) = \frac{e^{u(x)}}{x}$$

The new ODE:

$$\frac{d}{dx}\, u(x) = \frac{1}{x\, e^{u}}$$

with:

$$X(x) = \frac{1}{x}$$

$$U(u) = \frac{1}{e^{u}}$$

Therefore,

$$\int \frac{1}{U(u)}\, du = \int X(x)\, dx$$

$$\int e^u \, du = \int \frac{1}{x} \, dx$$

$$e^u = \ln(x) + C$$

Thus, the general solution is:

$$u(x) = \ln(\ln(x) + C)$$

Since, $u(x) = \frac{y(x)}{x}$, *therefore:*

$$\frac{y(x)}{x} = \ln(\ln(x) + C)$$