Exercises 2.5
Problem Sets # 7a, 7b, and 7d
Greenberg's Book

7. Show that the given equation is not exact and that an integrating factor depending on x alone or y alone does not exist. If possible, find an integrating factor in the form $\sigma(x, y) = x^a y^b$, where a and b are suitably chosen constants. If such a σ can be found, then use it to obtain the general solution of the differential equation; if not, state that.

(a) $(3xy - 2y^2)dx + (2x^2 - 3xy)dy = 0$
(b) $(3xy + 2y^2)dx + (3x^2 + 4xy)dy = 0$
(c) $(x + y^2)dx + (x - y)dy = 0$
(d) $ydx - (x^2y - x)dy = 0$

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exact_equation:=proc(eqM, eqN)
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Problem 2a

An ODE:

$$(3\,x\,y - 2\,y^2)\;dx + (2\,x^2 - 3\,x\,y)\;dy = 0$$

Therefore,

$$M(x, y) = 3\,x\,y - 2\,y^2$$
$$N(x, y) = 2\,x^2 - 3\,x\,y$$

Take the partial derivatives:

$$M_y = 3\,x - 4\,y$$
$$N_x = 4\,x - 3\,y$$

Fail the test for exactness.

Show that it does not have any integrating factors in function of x alone or y alone:

$$\frac{M_y - N_x}{M} = \frac{-x - y}{3\,x\,y - 2\,y^2}$$
$$\frac{M_y - N_x}{N} = \frac{-x - y}{2\,x^2 - 3\,x\,y}$$

As an alternative, we will use, $\sigma(x, y) = x^a\,y^b$, *thus:*

$$x^a\,y^b\,M\,dx + x^a\,y^b\,N\,dy = 0$$

$$M(x, y) = x^a\,y^b\,(3\,x\,y - 2\,y^2)$$
$$N(x, y) = x^a\,y^b\,(2\,x^2 - 3\,x\,y)$$

Since, $M_y = N_x$, *we then have:*

$$\frac{x^a y^b b\left(3xy-2y^2\right)}{y}+x^a y^b(3x-4y)=\frac{x^a a y^b\left(2x^2-3xy\right)}{x}+x^a y^b(4x-3y)$$

$$3x^a y^b b x-2x^a y^b b y+3x^a y^b x-4x^a y^b y=2x^a a x y^b-3x^a a y^b y+4x^a y^b x-3x^a y^b y$$

$$3y^b(b+1)x^{a+1}-2x^a y^{b+1}(b+2)=2y^b(a+2)x^{a+1}-3x^a y^{b+1}(a+1)$$

Match the coefficients then find the a and b:

$$-3a-3=-2b-4$$

$$2a+4=3b+3$$

$$\{a=1, b=1\}$$

This is the new ODE that is exact:

$$dx\left(3xy-2y^2\right)xy+dy\left(2x^2-3xy\right)xy=0$$

Let us now start the procedure for the exact-type ODE!

$$dx\left(3xy-2y^2\right)xy+dy\left(2x^2-3xy\right)xy=0$$

$$\left(3x(x,y)\,y(x,y)-2y(x,y)^2\right)dx+\left(2x(x,y)^2-3x(x,y)\,y(x,y)\right)dy=0$$

$$M(x,y)=\left(3xy-2y^2\right)xy$$

$$N(x,y)=\left(2x^2-3xy\right)xy$$

Test for exactness:

$$M_y=N_x$$

$$6xy(x-y)=6xy(x-y)$$

We need to find a function F(x,y) such that:

$$M(x,y)=\frac{\partial}{\partial x}F(x,y)$$

$$N(x,y)=\frac{\partial}{\partial y}F(x,y)$$

Therefore:

$$F(x,y)=\int M(x,y)\,\mathrm{d}x$$

$$F(x,y)=\int\left(3xy-2y^2\right)xy\,\mathrm{d}x+A(y)$$

$$F(x,y)=y^2x^2(x-y)+A(y)$$

Or:

$$F(x,y)=\int N(x,y)\,\mathrm{d}y$$

$$F(x,y)=\int \left(2x^2-3xy\right)xy\,\mathrm{d}y+B(x)$$
$$F(x,y)=y^2x^2(x-y)+B(x)$$

We have two F(x,y) functions.
Both should be the same.
We will show this by first calculating the A(y) form the first F(x,y)
and B(x) from the second F(x,y).

Calculating A(y):
$$\frac{\partial}{\partial y}F(x,y)=N(x,y)$$
$$2x^2y(x-y)-x^2y^2+\frac{\mathrm{d}}{\mathrm{d}y}A(y)=\left(2x^2-3xy\right)xy$$
$$\frac{\mathrm{d}}{\mathrm{d}y}A(y)=0$$
$$A(y)=\int\left(\frac{\mathrm{d}}{\mathrm{d}y}A(y)\right)\mathrm{d}y$$
$$A(y)=\int 0\,\mathrm{d}y$$
$$A(y)=C$$
Thus:
$$F(x,y)=x^3y^2-x^2y^3+C$$

Calculating B(x):
$$\frac{\partial}{\partial x}F(x,y)=M(x,y)$$
$$2xy^2(x-y)+x^2y^2+\frac{\mathrm{d}}{\mathrm{d}x}B(x)=\left(3xy-2y^2\right)xy$$
$$\frac{\mathrm{d}}{\mathrm{d}x}B(x)=0$$
$$B(x)=\int\left(\frac{\mathrm{d}}{\mathrm{d}x}B(x)\right)\mathrm{d}x$$
$$B(x)=\int 0\,\mathrm{d}x$$
$$B(x)=\mathrm{D}$$
Thus:
$$F(x,y)=x^3y^2-x^2y^3+\mathrm{D}$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant.

Problem 2b

An ODE:

$$\left(3xy+2y^2\right)dx+\left(3x^2+4xy\right)dy=0$$

Therefore,

$$M(x,y)=3xy+2y^2$$
$$N(x,y)=3x^2+4xy$$

Take the partial derivatives:

$$M_y=3x+4y$$
$$N_x=6x+4y$$

Fail the test for exactness.

Show that it does not have any integrating factors in function of x alone or y alone:

$$\frac{M_y-N_x}{M}=-\frac{3x}{3xy+2y^2}$$
$$\frac{M_y-N_x}{N}=-\frac{3x}{3x^2+4xy}$$

As an alternative, we will use, $\sigma(x,y)=x^a y^b$, *thus:*

$$x^a y^b M\,dx+x^a y^b N\,dy=0$$

$$M(x,y)=x^a y^b\left(3xy+2y^2\right)$$
$$N(x,y)=x^a y^b\left(3x^2+4xy\right)$$

Since, $M_y=N_x$, *we then have:*

$$\frac{x^a y^b b\left(3xy+2y^2\right)}{y}+x^a y^b(3x+4y)=\frac{x^a a y^b\left(3x^2+4xy\right)}{x}+x^a y^b(6x+4y)$$
$$3x^a y^b bx+2x^a y^b by+3x^a y^b x+4x^a y^b y=3x^a axy^b+4x^a ay^b y+6x^a y^b x+4x^a y^b y$$
$$3y^b(b+1)x^{a+1}+2x^a y^{b+1}(b+2)=3y^b(a+2)x^{a+1}+4x^a y^{b+1}(a+1)$$

Match the coefficients then find the a and b:

$$4a+4=2b+4$$
$$3a+6=3b+3$$
$$\{a=1,b=2\}$$

This is the new ODE that is exact:

$$dx\left(3xy+2y^2\right)xy^2+dy\left(3x^2+4xy\right)xy^2=0$$

Let us now start the procedure for the exact-type ODE!

$$dx\left(3xy+2y^2\right)xy^2+dy\left(3x^2+4xy\right)xy^2=0$$

$$\left(3\,x(x,y)\,y(x,y)+2\,y(x,y)^2\right)\,dx+\left(3\,x(x,y)^2+4\,x(x,y)\,y(x,y)\right)\,dy=0$$
$$M(x,y)=\left(3\,x\,y+2\,y^2\right)\,x\,y^2$$
$$N(x,y)=\left(3\,x^2+4\,x\,y\right)\,x\,y^2$$

Test for exactness:

$$M_y=N_x$$
$$x\,y^2\,(9\,x+8\,y)=x\,y^2\,(9\,x+8\,y)$$

We need to find a function F(x,y) such that:

$$M(x,y)=\frac{\partial}{\partial x}\,F(x,y)$$
$$N(x,y)=\frac{\partial}{\partial y}\,F(x,y)$$

Therefore:

$$F(x,y)=\int M(x,y)\,\mathrm{d}x$$
$$F(x,y)=\int\left(3\,x\,y+2\,y^2\right)\,x\,y^2\,\mathrm{d}x+A(y)$$
$$F(x,y)=y^3\,x^2\,(x+y)+A(y)$$

Or:

$$F(x,y)=\int N(x,y)\,\mathrm{d}y$$
$$F(x,y)=\int\left(3\,x^2+4\,x\,y\right)\,x\,y^2\,\mathrm{d}y+B(x)$$
$$F(x,y)=y^3\,x^2\,(x+y)+B(x)$$

We have two F(x,y) functions.
Both should be the same.
We will show this by first calculating the A(y) form the first F(x,y) and B(x) from the second F(x,y).

Calculating A(y):

$$\frac{\partial}{\partial y}\,F(x,y)=N(x,y)$$
$$3\,y^2\,x^2\,(x+y)+x^2\,y^3+\frac{\mathrm{d}}{\mathrm{d}y}\,A(y)=\left(3\,x^2+4\,x\,y\right)\,x\,y^2$$
$$\frac{\mathrm{d}}{\mathrm{d}y}\,A(y)=0$$
$$A(y)=\int\left(\frac{\mathrm{d}}{\mathrm{d}y}\,A(y)\right)\,\mathrm{d}y$$

$$A(y) = \int 0 \, dy$$

$$A(y) = C$$

Thus:

$$F(x, y) = x^3 y^3 + x^2 y^4 + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x, y) = M(x, y)$$

$$2 x y^3 (x + y) + x^2 y^3 + \frac{d}{dx} B(x) = (3 x y + 2 y^2) x y^2$$

$$\frac{d}{dx} B(x) = 0$$

$$B(x) = \int \left(\frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int 0 \, dx$$

$$B(x) = \mathrm{D}$$

Thus:

$$F(x, y) = x^3 y^3 + x^2 y^4 + \mathrm{D}$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant.

Problem 2d

An ODE:

$$y \, dx + (-x^2 y + x) \, dy = 0$$

Therefore,

$$M(x, y) = y$$

$$N(x, y) = -x^2 y + x$$

Take the partial derivatives:

$$M_y = 1$$

$$N_x = -2 x y + 1$$

Fail the test for exactness.

Show that it does not have any integrating factors in function of x alone or y alone:

$$\frac{M_y - N_x}{M} = 2 x$$

$$\frac{M_y - N_x}{N} = \frac{2 x y}{-x^2 y + x}$$

As an alternative, we will use, $\sigma(x, y) = x^a y^b$, *thus:*

$$x^a y^b M\,dx + x^a y^b N\,dy = 0$$

$$M(x, y) = x^a y^b y$$

$$N(x, y) = x^a y^b \left(-x^2 y + x\right)$$

Since, $M_y = N_x$, *we then have:*

$$x^a y^b b + x^a y^b = \frac{x^a a y^b \left(-x^2 y + x\right)}{x} + x^a y^b (-2xy + 1)$$

$$x^a y^b (b + 1) = -y^{b+1} (a + 2) x^{a+1} + x^a y^b (a + 1)$$

Match the coefficients then find the a and b:

$$b + 1 = a + 1$$

$$a + 2 = 0$$

$$\{a = -2, b = -2\}$$

This is the new ODE that is exact:

$$\frac{dx}{y x^2} + \frac{dy\left(-x^2 y + x\right)}{x^2 y^2} = 0$$

Let us now start the procedure for the exact-type ODE!

$$\frac{dx}{y x^2} + \frac{dy\left(-x^2 y + x\right)}{x^2 y^2} = 0$$

$$y(x, y)\,dx + \left(-x(x, y)^2 y(x, y) + x(x, y)\right) dy = 0$$

$$M(x, y) = \frac{1}{y x^2}$$

$$N(x, y) = \frac{-x^2 y + x}{x^2 y^2}$$

Test for exactness:

$$M_y = N_x$$

$$-\frac{1}{x^2 y^2} = -\frac{1}{x^2 y^2}$$

We need to find a function F(x,y) such that:

$$M(x, y) = \frac{\partial}{\partial x} F(x, y)$$

$$N(x, y) = \frac{\partial}{\partial y} F(x, y)$$

Therefore:

$$F(x,y) = \int M(x,y)\, dx$$

$$F(x,y) = \int \frac{1}{y\,x^2}\, dx + A(y)$$

$$F(x,y) = -\frac{1}{y\,x} + A(y)$$

Or:

$$F(x,y) = \int N(x,y)\, dy$$

$$F(x,y) = \int \frac{-y\,x^2 + x}{x^2\,y^2}\, dy + B(x)$$

$$F(x,y) = -\ln(y) - \frac{1}{y\,x} + B(x)$$

We have two F(x,y) functions.
Both should be the same.
We will show this by first calculating the A(y) form the first F(x,y)
and B(x) from the second F(x,y).

Calculating A(y):

$$\frac{\partial}{\partial y} F(x,y) = N(x,y)$$

$$\frac{1}{y^2\,x} + \frac{d}{dy} A(y) = \frac{-y\,x^2 + x}{x^2\,y^2}$$

$$\frac{d}{dy} A(y) = -\frac{1}{y}$$

$$A(y) = \int \left(\frac{d}{dy} A(y) \right) dy$$

$$A(y) = \int -\frac{1}{y}\, dy$$

$$A(y) = -\ln(y) + C$$

Thus:

$$F(x,y) = -\frac{1}{x\,y} - \ln(y) + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x,y) = M(x,y)$$

$$\frac{1}{y\,x^2} + \frac{d}{dx} B(x) = \frac{1}{y\,x^2}$$

$$\frac{d}{dx} B(x) = 0$$

$$B(x) = \int \left(\frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int 0 \, dx$$

$$B(x) = D$$

Thus:

$$F(x, y) = -\frac{1}{x\,y} - \ln(y) + D$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant.
