Exercises 2.2
Problem 3, using the variation of parameters

Problem 3a

$y'(x) - y = 3e^x$

An ODE:

$$\frac{d}{dx} y(x) - y(x) = 3 e^x$$

Thus, we have:

$$p(x) = -1$$

$$q(x) = 3 e^x$$

Step 1, find the homogenous solution:

$$y_h = A e^{-\left(\int p(x)\, dx\right)}$$

$$y_h = A e^x$$

Step 2, find the complementary solution based on the homogenous solution by turning the A into a function A(x):

$$y(x) = A(x) e^{-\left(\int p(x)\, dx\right)}$$

$$y(x) = A(x) e^x$$

Step 3, find y'(x) from y(x):

$$\frac{d}{dx} y(x) = \left(\frac{d}{dx} A(x)\right) e^x + A(x) e^x$$

Step 4, to find A(x), substitute y(x) and y'(x) back to the ODE:

$$\frac{d}{dx} y(x) - y(x) = 3 e^x$$

$$\left(\frac{d}{dx} A(x)\right) e^x = 3 e^x$$

$$\frac{d}{dx} A(x) = 3$$

Integrate both sides wrt x:

$$A(x) = 3x + C$$

Step 5: substitute A(x) back to y(x) of the complementary solution:

$$y(x) = A(x) e^x$$

$$y(x) = (3x + C) e^x$$

Problem 3b

$y'(x) + 4y = 8$

An ODE:

$$\frac{d}{dx} y(x) + 4 y(x) = 8$$

Thus, we have:

$$p(x) = 4$$

$$q(x) = 8$$

Step 1, find the homogenous solution:

$$y_h = A\, e^{-\left(\int p(x)\, dx\right)}$$

$$y_h = A\, e^{-4x}$$

Step 2, find the complementary solution based on the homogenous solution by turning the A into a function A(x):

$$y(x) = A(x)\, e^{-\left(\int p(x)\, dx\right)}$$

$$y(x) = A(x)\, e^{-4x}$$

Step 3, find y'(x) from y(x):

$$\frac{d}{dx}\, y(x) = \left(\frac{d}{dx}\, A(x)\right) e^{-4x} - 4\, A(x)\, e^{-4x}$$

Step 4, to find A(x), substitute y(x) and y'(x) back to the ODE:

$$\frac{d}{dx}\, y(x) + 4\, y(x) = 8$$

$$\left(\frac{d}{dx}\, A(x)\right) e^{-4x} = 8$$

$$\frac{d}{dx}\, A(x) = \frac{8}{e^{-4x}}$$

Integrate both sides wrt x:

$$A(x) = 2\, e^{4x} + C$$

Step 5: substitute A(x) back to y(x) of the complementary solution:

$$y(x) = A(x)\, e^{-4x}$$

$$y(x) = 2 + C\, e^{-4x}$$

Problem 3c

$y'(x) + y = x^2$

An ODE:

$$\frac{d}{dx}\, y(x) + y(x) = x^2$$

Thus, we have:

$$p(x) = 1$$

$$q(x) = x^2$$

Step 1, find the homogenous solution:

$$y_h = A\, e^{-\left(\int p(x)\, dx\right)}$$

$$y_h = A\, e^{-x}$$

Step 2, find the complementary solution based on the homogenous solution by turning the A into a function A(x):

$$y(x) = A(x)\, e^{-\left(\int p(x)\, dx\right)}$$

$$y(x) = A(x)\, e^{-x}$$

Step 3, find y'(x) from y(x):

$$\frac{d}{dx}\, y(x) = \left(\frac{d}{dx}\, A(x)\right) e^{-x} - A(x)\, e^{-x}$$

Step 4, to find A(x), substitute y(x) and y'(x) back to the ODE:

$$\frac{d}{dx} y(x) + y(x) = x^2$$

$$\left(\frac{d}{dx} A(x)\right) e^{-x} = x^2$$

$$\frac{d}{dx} A(x) = \frac{x^2}{e^{-x}}$$

Integrate both sides wrt x:

$$A(x) = \left(x^2 - 2x + 2\right) e^x + C$$

Step 5: substitute A(x) back to y(x) of the complementary solution:

$$y(x) = A(x) e^{-x}$$

$$y(x) = x^2 - 2x + C e^{-x} + 2$$

Problem 3d

$y'(x) = y - \sin(2x)$

An ODE:

$$\frac{d}{dx} y(x) - y(x) = -\sin(2x)$$

Thus, we have:

$$p(x) = -1$$

$$q(x) = -\sin(2x)$$

Step 1, find the homogenous solution:

$$y_h = A e^{-\left(\int p(x)\, dx\right)}$$

$$y_h = A e^x$$

Step 2, find the complementary solution based on the homogenous solution by turning the A into a function A(x):

$$y(x) = A(x) e^{-\left(\int p(x)\, dx\right)}$$

$$y(x) = A(x) e^x$$

Step 3, find y'(x) from y(x):

$$\frac{d}{dx} y(x) = \left(\frac{d}{dx} A(x)\right) e^x + A(x) e^x$$

Step 4, to find A(x), substitute y(x) and y'(x) back to the ODE:

$$\frac{d}{dx} y(x) - y(x) = -\sin(2x)$$

$$\left(\frac{d}{dx} A(x)\right) e^x = -\sin(2x)$$

$$\frac{d}{dx} A(x) = -\frac{\sin(2x)}{e^x}$$

Integrate both sides wrt x:

$$A(x) = \frac{(2\cos(2x) + \sin(2x))\, e^{-x}}{5} + C$$

Step 5: substitute A(x) back to y(x) of the complementary solution:

$$y(x) = A(x) e^x$$

$$y(x) = \frac{2\cos(2x)}{5} + \frac{\sin(2x)}{5} + C\,e^{x}$$

Problem 3e

$y'(x) - \tan(x)y = 6$

An ODE:

$$\frac{d}{dx}\,y(x) - \tan(x)\,y(x) = 6$$

Thus, we have:

$$p(x) = -\tan(x)$$
$$q(x) = 6$$

Step 1, find the homogenous solution:

$$y_h = A\,e^{-\left(\int p(x)\,dx\right)}$$

$$y_h = \frac{A}{\cos(x)}$$

Step 2, find the complementary solution based on the homogenous solution by turning the A into a function A(x):

$$y(x) = A(x)\,e^{-\left(\int p(x)\,dx\right)}$$
$$y(x) = \frac{A(x)}{\cos(x)}$$

Step 3, find y'(x) from y(x):

$$\frac{d}{dx}\,y(x) = \frac{\frac{d}{dx}\,A(x)}{\cos(x)} + \frac{A(x)\,\sin(x)}{\cos(x)^2}$$

Step 4, to find A(x), substitute y(x) and y'(x) back to the ODE:

$$\frac{d}{dx}\,y(x) - \tan(x)\,y(x) = 6$$

$$\frac{\frac{d}{dx}\,A(x)}{\cos(x)} + \frac{A(x)\,\sin(x)}{\cos(x)^2} - \frac{\tan(x)\,A(x)}{\cos(x)} = 6$$

$$\frac{d}{dx}\,A(x) = \frac{\tan(x)\,A(x)\,\cos(x) + 6\cos(x)^2 - A(x)\,\sin(x)}{\cos(x)}$$

Integrate both sides wrt x:

$$A(x) = 6\sin(x) + C$$

Step 5: substitute A(x) back to y(x) of the complementary solution:

$$y(x) = \frac{A(x)}{\cos(x)}$$

$$y(x) = \frac{6\sin(x) + C}{\cos(x)}$$

Problem 3f

$xy'(x) + 2y = x^3 \Leftrightarrow y'(x) + \frac{2}{x}y = x^3$

An ODE:

$$\frac{d}{dx}\,y(x) + \frac{2\,y(x)}{x} = x^3$$

Thus, we have:

$$p(x) = \frac{2}{x}$$

$$q(x) = x^3$$

Step 1, find the homogenous solution:

$$y_h = A\,\mathrm{e}^{-\left(\int p(x)\,\mathrm{d}x\right)}$$

$$y_h = \frac{A}{x^2}$$

Step 2, find the complementary solution based on the homogenous solution by turning the A into a function A(x):

$$y(x) = A(x)\,\mathrm{e}^{-\left(\int p(x)\,\mathrm{d}x\right)}$$

$$y(x) = \frac{A(x)}{x^2}$$

Step 3, find y'(x) from y(x):

$$\frac{\mathrm{d}}{\mathrm{d}x}\,y(x) = \frac{\frac{\mathrm{d}}{\mathrm{d}x}\,A(x)}{x^2} - \frac{2\,A(x)}{x^3}$$

Step 4, to find A(x), substitute y(x) and y'(x) back to the ODE:

$$\frac{\mathrm{d}}{\mathrm{d}x}\,y(x) + \frac{2\,y(x)}{x} = x^3$$

$$\frac{\frac{\mathrm{d}}{\mathrm{d}x}\,A(x)}{x^2} = x^3$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\,A(x) = x^5$$

Integrate both sides wrt x:

$$A(x) = \frac{x^6}{6} + C$$

Step 5: substitute A(x) back to y(x) of the complementary solution:

$$y(x) = \frac{A(x)}{x^2}$$

$$y(x) = \frac{x^6 + 6\,C}{6\,x^2}$$

Problem 3g

$xy'(x) - 2\,y = x^3 \Leftrightarrow y'(x) - \frac{2}{x}y = x^3$

An ODE:

$$\frac{\mathrm{d}}{\mathrm{d}x}\,y(x) - \frac{2\,y(x)}{x} = x^3$$

Thus, we have:

$$p(x) = -\frac{2}{x}$$

$$q(x) = x^3$$

Step 1, find the homogenous solution:

$$y_h = A\,\mathrm{e}^{-\left(\int p(x)\,\mathrm{d}x\right)}$$

$$y_h = A\,x^2$$

Step 2, find the complementary solution based on the homogenous solution by turning the A into a function A(x):

$$y(x) = A(x)\,\mathrm{e}^{-\left(\int p(x)\,\mathrm{d}x\right)}$$

$$y(x) = A(x)\,x^2$$

Step 3, find y'(x) from y(x):

$$\frac{\mathrm{d}}{\mathrm{d}x}\,y(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\,A(x)\right)x^2 + 2\,A(x)\,x$$

Step 4, to find A(x), substitute y(x) and y'(x) back to the ODE:

$$\frac{\mathrm{d}}{\mathrm{d}x}\,y(x) - \frac{2\,y(x)}{x} = x^3$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\,A(x)\right)x^2 = x^3$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\,A(x) = x$$

Integrate both sides wrt x:

$$A(x) = \frac{x^2}{2} + C$$

Step 5: substitute A(x) back to y(x) of the complementary solution:

$$y(x) = A(x)\,x^2$$

$$y(x) = \frac{1}{2}\,x^4 + C\,x^2$$

Problem 3h

$y'(x) + \cot(x)y = 2\cos(x)$

An ODE:

$$\frac{\mathrm{d}}{\mathrm{d}x}\,y(x) + \cot(x)\,y(x) = 2\cos(x)$$

Thus, we have:

$$p(x) = \cot(x)$$

$$q(x) = 2\cos(x)$$

Step 1, find the homogenous solution:

$$y_h = A\,\mathrm{e}^{-\left(\int p(x)\,\mathrm{d}x\right)}$$

$$y_h = \frac{A}{\sin(x)}$$

Step 2, find the complementary solution based on the homogenous solution by turning the A into a function A(x):

$$y(x) = A(x)\,\mathrm{e}^{-\left(\int p(x)\,\mathrm{d}x\right)}$$

$$y(x) = \frac{A(x)}{\sin(x)}$$

Step 3, find y'(x) from y(x):

$$\frac{d}{dx}y(x) = \frac{\frac{d}{dx}A(x)}{\sin(x)} - \frac{A(x)\cos(x)}{\sin(x)^2}$$

Step 4, to find A(x), substitute y(x) and y'(x) back to the ODE:

$$\frac{d}{dx}y(x) + \cot(x)\,y(x) = 2\cos(x)$$

$$\frac{\frac{d}{dx}A(x)}{\sin(x)} - \frac{A(x)\cos(x)}{\sin(x)^2} + \frac{\cot(x)\,A(x)}{\sin(x)} = 2\cos(x)$$

$$\frac{d}{dx}A(x) = \frac{2\cos(x)\sin(x)^2 - \cot(x)\,A(x)\sin(x) + A(x)\cos(x)}{\sin(x)}$$

Integrate both sides wrt x:

$$A(x) = -\cos(x)^2 + C$$

Step 5: substitute A(x) back to y(x) of the complementary solution:

$$y(x) = \frac{A(x)}{\sin(x)}$$

$$y(x) = \frac{-\cos(x)^2 + C}{\sin(x)}$$

Problem 3i

$(x-5)(xy' + 3y) = 2$

$xy' + 3y = \frac{2}{(x-5)}$

$y' + \frac{3}{x}y = \frac{2}{x(x-5)}$

An ODE:

$$\frac{d}{dx}y(x) + \frac{3\,y(x)}{x} = \frac{2}{x(x-5)}$$

Thus, we have:

$$p(x) = \frac{3}{x}$$

$$q(x) = \frac{2}{x(x-5)}$$

Step 1, find the homogenous solution:

$$y_h = A\,e^{-\left(\int p(x)\,dx\right)}$$

$$y_h = \frac{A}{x^3}$$

Step 2, find the complementary solution based on the homogenous solution by turning the A into a function A(x):

$$y(x) = A(x)\,e^{-\left(\int p(x)\,dx\right)}$$

$$y(x) = \frac{A(x)}{x^3}$$

Step 3, find y'(x) from y(x):

$$\frac{d}{dx}\,y(x) = \frac{\frac{d}{dx}\,A(x)}{x^3} - \frac{3\,A(x)}{x^4}$$

Step 4, to find A(x), substitute y(x) and y'(x) back to the ODE:

$$\frac{d}{dx}\,y(x) + \frac{3\,y(x)}{x} = \frac{2}{x\,(x-5)}$$

$$\frac{\frac{d}{dx}\,A(x)}{x^3} = \frac{2}{x\,(x-5)}$$

$$\frac{d}{dx}\,A(x) = \frac{2\,x^2}{x-5}$$

Integrate both sides wrt x:

$$A(x) = x^2 + 10\,x + 50\,\ln(x-5) + C$$

Step 5: substitute A(x) back to y(x) of the complementary solution:

$$y(x) = \frac{A(x)}{x^3}$$

$$y(x) = \frac{x^2 + 10\,x + 50\,\ln(x-5) + C}{x^3}$$

Problem 3j

$x' - 6\,x = e^y$

This has a different form since x is the dependent variable and y is the indepenent variable. Thus, we need to change x to y and y to x to make it fit to the Maple procedure.

$y' - 6\,y = e^x$

An ODE:

$$\frac{d}{dx}\,y(x) - 6\,y(x) = e^x$$

Thus, we have:

$$p(x) = -6$$

$$q(x) = e^x$$

Step 1, find the homogenous solution:

$$y_h = A\,e^{-\left(\int p(x)\,dx\right)}$$

$$y_h = A\,e^{6\,x}$$

Step 2, find the complementary solution based on the homogenous solution by turning the A into a function A(x):

$$y(x) = A(x)\,e^{-\left(\int p(x)\,dx\right)}$$

$$y(x) = A(x)\,e^{6\,x}$$

Step 3, find y'(x) from y(x):

$$\frac{d}{dx}\,y(x) = \left(\frac{d}{dx}\,A(x)\right)e^{6\,x} + 6\,A(x)\,e^{6\,x}$$

Step 4, to find A(x), substitute y(x) and y'(x) back to the ODE:

$$\frac{d}{dx}\,y(x) - 6\,y(x) = e^x$$

$$\left(\frac{d}{dx} A(x)\right) e^{6x} = e^{x}$$

$$\frac{d}{dx} A(x) = \frac{e^{x}}{e^{6x}}$$

Integrate both sides wrt x:

$$A(x) = -\frac{e^{-5x}}{5} + C$$

Step 5: substitute A(x) back to y(x) of the complementary solution:

$$y(x) = A(x) e^{6x}$$

$$y(x) = C e^{x} e^{5x} - \frac{e^{x}}{5}$$

Problem 3k

$yx' - y^5 + 3x = 0$

This has a different form since x is the dependent variable and y is the indepenent variable. Thus, we need to change x to y and y to x to make it fit to the Maple procedure.

$xy' - x^5 + 3y = 0$

$y' + \frac{3}{x} y = x^4$

An ODE:

$$\frac{d}{dx} y(x) + \frac{3\, y(x)}{x} = x^4$$

Thus, we have:

$$p(x) = \frac{3}{x}$$

$$q(x) = x^4$$

Step 1, find the homogenous solution:

$$y_h = A\, e^{-\left(\int p(x)\, dx\right)}$$

$$y_h = \frac{A}{x^3}$$

Step 2, find the complementary solution based on the homogenous solution by turning the A into a function A(x):

$$y(x) = A(x)\, e^{-\left(\int p(x)\, dx\right)}$$

$$y(x) = \frac{A(x)}{x^3}$$

Step 3, find y'(x) from y(x):

$$\frac{d}{dx} y(x) = \frac{\frac{d}{dx} A(x)}{x^3} - \frac{3\, A(x)}{x^4}$$

Step 4, to find A(x), substitute y(x) and y'(x) back to the ODE:

$$\frac{d}{dx} y(x) + \frac{3\, y(x)}{x} = x^4$$

$$\frac{\frac{d}{dx} A(x)}{x^3} = x^4$$

$$\frac{d}{dx} A(x) = x^7$$

Integrate both sides wrt x:

$$A(x) = \frac{x^8}{8} + C$$

Step 5: substitute A(x) back to y(x) of the complementary solution:

$$y(x) = \frac{A(x)}{x^3}$$

$$y(x) = \frac{x^8 + 8\,C}{8\,x^3}$$

Problem 3l

$y^2 x' + xy - 4\,y^2 = 1$

This has a different form since x is the dependent variable and y is the indepenent variable. Thus, we need to change x to y and y to x to make it fit to the Maple procedure.

$x^2 y' + xy - 4\,x^2 = 1$

$y' + \frac{1}{x} y = 4\,x$

An ODE:

$$\frac{d}{dx} y(x) + \frac{y(x)}{x} = 4\,x$$

Thus, we have:

$$p(x) = \frac{1}{x}$$

$$q(x) = 4\,x$$

Step 1, find the homogenous solution:

$$y_h = A\,e^{-\left(\int p(x)\,dx\right)}$$

$$y_h = \frac{A}{x}$$

Step 2, find the complementary solution based on the homogenous solution by turning the A into a function A(x):

$$y(x) = A(x)\,e^{-\left(\int p(x)\,dx\right)}$$

$$y(x) = \frac{A(x)}{x}$$

Step 3, find y'(x) from y(x):

$$\frac{d}{dx} y(x) = \frac{\frac{d}{dx} A(x)}{x} - \frac{A(x)}{x^2}$$

Step 4, to find A(x), substitute y(x) and y'(x) back to the ODE:

$$\frac{d}{dx} y(x) + \frac{y(x)}{x} = 4\,x$$

$$\frac{\frac{d}{dx} A(x)}{x} = 4x$$

$$\frac{d}{dx} A(x) = 4x^2$$

Integrate both sides wrt x:

$$A(x) = \frac{4x^3}{3} + C$$

Step 5: substitute A(x) back to y(x) of the complementary solution:

$$y(x) = \frac{A(x)}{x}$$

$$y(x) = \frac{4x^3 + 3C}{3x}$$

Problem 3m

$tx' - 4t^5 = x$

This has a different form since x is the dependent variable and t is the indepenent variable. Thus, we need to change x to y and t to x to make it fit to the Maple procedure.

$xy' - 4x^5 = y$

$y' - 4x^4 = \frac{1}{x} y$

$y' - \frac{1}{x} y = 4x^4$

An ODE:

$$\frac{d}{dx} y(x) - \frac{y(x)}{x} = 4x^4$$

Thus, we have:

$$p(x) = -\frac{1}{x}$$

$$q(x) = 4x^4$$

Step 1, find the homogenous solution:

$$y_h = A\,e^{-\left(\int p(x)\,dx\right)}$$

$$y_h = A\,x$$

Step 2, find the complementary solution based on the homogenous solution by turning the A into a function A(x):

$$y(x) = A(x)\,e^{-\left(\int p(x)\,dx\right)}$$

$$y(x) = A(x)\,x$$

Step 3, find y'(x) from y(x):

$$\frac{d}{dx} y(x) = \left(\frac{d}{dx} A(x)\right) x + A(x)$$

Step 4, to find A(x), substitute y(x) and y'(x) back to the ODE:

$$\frac{d}{dx} y(x) - \frac{y(x)}{x} = 4x^4$$

$$\left(\frac{d}{dx} A(x)\right) x = 4x^4$$

$$\frac{d}{dx} A(x) = 4x^3$$

Integrate both sides wrt x:

$$A(x) = x^4 + C$$

Step 5: substitute A(x) back to y(x) of the complementary solution:

$$y(x) = A(x)\, x$$

$$y(x) = \left(x^4 + C\right) x$$