Exercises 2.5
Problem 5, the exact equations wth integrating factor
Auralius Manurung

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exact_equation:=proc(eqM, eqN)
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exact_equations_integrating_factor:=proc(eqM, eqN)
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Problem 5a

$$3\,y\,dx + dy = 0$$
$$M(x, y) = 3\,y$$
$$N(x, y) = 1$$
$$M_y = 3$$
$$N_x = 0$$
$$M_y \neq N_x$$

Find the integrating factor, σ

$$\frac{M_y - N_x}{M} = \frac{1}{y}$$
$$\frac{M_y - N_x}{N} = 3$$
$$Take\ ,\ \frac{M_y - N_x}{M} = \frac{1}{y},\ function\ of\ y\ alone.$$
$$\sigma(y) = \mathrm{e}^{-\left(\int \frac{M_y - N_x}{M}\,\mathrm{d}y\right)}$$
$$\sigma(y) = \mathrm{e}^{-\left(\int \frac{1}{y}\,\mathrm{d}y\right)}$$
$$\sigma(y) = \frac{1}{y}$$
$$\sigma\,M(x, y)\ dx + \sigma\,N(x, y)\ dy = 0$$
$$3\ dx + \frac{dy}{y} = 0$$

Now, we have a new ODE which is exact.
We then continue with the procedure for the exact equation.

$$3\ dx + \frac{dy}{y} = 0$$
$$M(x, y) = 3$$
$$N(x, y) = \frac{1}{y}$$

Test for exactness:

$$M_y = N_x$$
$$0 = 0$$

We need to find a function F(x,y) such that:

$$M(x, y) = \frac{\partial}{\partial x} F(x, y)$$
$$N(x, y) = \frac{\partial}{\partial y} F(x, y)$$

Therefore:

$$F(x, y) = \int M(x, y) \, dx$$
$$F(x, y) = \int 3 \, dx + A(y)$$
$$F(x, y) = 3x + A(y)$$

Or:

$$F(x, y) = \int N(x, y) \, dy$$
$$F(x, y) = \int \frac{1}{y} \, dy + B(x)$$
$$F(x, y) = \ln(y) + B(x)$$

We have two F(x,y) functions.
Both should be the same
We show this by first calculating the unknown A(y) form the first F(x,y)
and B(x) from the second F(x,y).

Calculating A(y):

$$\frac{\partial}{\partial y} F(x, y) = N(x, y)$$
$$\frac{d}{dy} A(y) = \frac{1}{y}$$
$$\frac{d}{dy} A(y) = \frac{1}{y}$$
$$A(y) = \int \left(\frac{d}{dy} A(y) \right) dy$$
$$A(y) = \int \frac{1}{y} \, dy$$
$$A(y) = \ln(y) + C$$

Thus:

$$F(x, y) = 3x + \ln(y) + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x,y) = M(x,y)$$

$$\frac{d}{dx} B(x) = 3$$

$$\frac{d}{dx} B(x) = 3$$

$$B(x) = \int \left(\frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int 3 \, dx$$

$$B(x) = 3x + D$$

Thus:

$$F(x,y) = 3x + \ln(y) + D$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

Problem 5b

$$y\,dx + x\ln(x)\,dy = 0$$

$$M(x,y) = y$$

$$N(x,y) = x\ln(x)$$

$$M_y = 1$$

$$N_x = \ln(x) + 1$$

$$M_y \neq N_x$$

Find the integrating factor, σ

$$\frac{M_y - N_x}{M} = -\frac{\ln(x)}{y}$$

$$\frac{M_y - N_x}{N} = -\frac{1}{x}$$

$$\text{Take }, \frac{M_y - N_x}{N} = -\frac{1}{x}, \text{ function of } x \text{ alone.}$$

$$\sigma(x) = e^{\int \frac{M_y - N_x}{N} dx}$$

$$\sigma(x) = e^{\int -\frac{1}{x} dx}$$

$$\sigma(x) = \frac{1}{x}$$

$$\sigma M(x,y)\,dx + \sigma N(x,y)\,dy = 0$$

$$\frac{y\,dx}{x} + \ln(x)\,dy = 0$$

Now, we have a new ODE which is exact.
We then continue with the procedure for the exact equation.

$$\frac{y\,dx}{x} + \ln(x)\,dy = 0$$

$$M(x, y) = \frac{y}{x}$$

$$N(x, y) = \ln(x)$$

Test for exactness:

$$M_y = N_x$$

$$\frac{1}{x} = \frac{1}{x}$$

We need to find a function F(x,y) such that:

$$M(x, y) = \frac{\partial}{\partial x} F(x, y)$$

$$N(x, y) = \frac{\partial}{\partial y} F(x, y)$$

Therefore:

$$F(x, y) = \int M(x, y)\,\mathrm{d}x$$

$$F(x, y) = \int \frac{y}{x}\,\mathrm{d}x + A(y)$$

$$F(x, y) = y\ln(x) + A(y)$$

Or:

$$F(x, y) = \int N(x, y)\,\mathrm{d}y$$

$$F(x, y) = \int \ln(x)\,\mathrm{d}y + B(x)$$

$$F(x, y) = y\ln(x) + B(x)$$

We have two F(x,y) functions.
Both should be the same
We show this by first calculating the unknown A(y) form the first F(x,y)
and B(x) from the second F(x,y).

Calculating A(y):

$$\frac{\partial}{\partial y} F(x, y) = N(x, y)$$

$$\ln(x) + \frac{\mathrm{d}}{\mathrm{d}y} A(y) = \ln(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}y} A(y) = 0$$

$$A(y) = \int \left(\frac{\mathrm{d}}{\mathrm{d}y} A(y) \right) \mathrm{d}y$$

$$A(y) = \int 0 \, dy$$

$$A(y) = C$$

Thus:

$$F(x, y) = \ln(x)\, y + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x, y) = M(x, y)$$

$$\frac{y}{x} + \frac{d}{dx} B(x) = \frac{y}{x}$$

$$\frac{d}{dx} B(x) = 0$$

$$B(x) = \int \left(\frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int 0 \, dx$$

$$B(x) = \mathrm{D}$$

Thus:

$$F(x, y) = \ln(x)\, y + \mathrm{D}$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

Problem 5c

$$y\, dx + x \ln(x)\, dy = 0$$

$$M(x, y) = y$$

$$N(x, y) = x \ln(x)$$

$$M_y = 1$$

$$N_x = \ln(x) + 1$$

$$M_y \neq N_x$$

Find the integrating factor, σ

$$\frac{M_y - N_x}{M} = -\frac{\ln(x)}{y}$$

$$\frac{M_y - N_x}{N} = -\frac{1}{x}$$

Take , $\frac{M_y - N_x}{N} = -\frac{1}{x}$, *function of x alone.*

$$\sigma(x) = e^{\int \frac{M_y - N_x}{N} dx}$$

$$\sigma(x) = e^{\int -\frac{1}{x} dx}$$

$$\sigma(x) = \frac{1}{x}$$

$$\sigma\, M(x, y)\ dx + \sigma\, N(x, y)\ dy = 0$$

$$\frac{y\, dx}{x} + \ln(x)\ dy = 0$$

Now, we have a new ODE which is exact.

We then continue with the procedure for the exact equation.

$$\frac{y\, dx}{x} + \ln(x)\ dy = 0$$

$$M(x, y) = \frac{y}{x}$$

$$N(x, y) = \ln(x)$$

Test for exactness:

$$M_y = N_x$$

$$\frac{1}{x} = \frac{1}{x}$$

We need to find a function F(x,y) such that:

$$M(x, y) = \frac{\partial}{\partial x}\ F(x, y)$$

$$N(x, y) = \frac{\partial}{\partial y}\ F(x, y)$$

Therefore:

$$F(x, y) = \int M(x, y)\ \mathrm{d}x$$

$$F(x, y) = \int \frac{y}{x}\ \mathrm{d}x + A(y)$$

$$F(x, y) = \ln(x)\ y + A(y)$$

Or:

$$F(x, y) = \int N(x, y)\ \mathrm{d}y$$

$$F(x, y) = \int \ln(x)\ \mathrm{d}y + B(x)$$

$$F(x, y) = \ln(x)\ y + B(x)$$

We have two F(x,y) functions.

Both should be the same

We show this by first calculating the unknown A(y) form the first F(x,y) and B(x) from the second F(x,y).

Calculating A(y):

$$\frac{\partial}{\partial y} F(x,y) = N(x,y)$$

$$\ln(x) + \frac{d}{dy} A(y) = \ln(x)$$

$$\frac{d}{dy} A(y) = 0$$

$$A(y) = \int \left(\frac{d}{dy} A(y) \right) dy$$

$$A(y) = \int 0 \, dy$$

$$A(y) = C$$

Thus:

$$F(x,y) = \ln(x)\, y + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x,y) = M(x,y)$$

$$\frac{y}{x} + \frac{d}{dx} B(x) = \frac{y}{x}$$

$$\frac{d}{dx} B(x) = 0$$

$$B(x) = \int \left(\frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int 0 \, dx$$

$$B(x) = \mathrm{D}$$

Thus:

$$F(x,y) = \ln(x)\, y + \mathrm{D}$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

Problem 5d

$$dx + \left(x - e^{-y}\right) dy = 0$$

$$M(x,y) = 1$$

$$N(x,y) = x - e^{-y}$$

$$M_y = 0$$

$$N_x = 1$$

$$M_y \neq N_x$$

Find the integrating factor, σ

$$\frac{M_y - N_x}{M} = -1$$

$$\frac{M_y - N_x}{N} = \frac{1}{-x + e^{-y}}$$

$$\textit{Take}\ ,\ \frac{M_y - N_x}{M} = -1,\ \textit{function of } y \textit{ alone.}$$

$$\sigma(y) = e^{-\left(\int \frac{M_y - N_x}{M}\, dy\right)}$$

$$\sigma(y) = e^{-\left(\int (-1)\, dy\right)}$$

$$\sigma(y) = e^y$$

$$\sigma\, M(x, y)\ dx + \sigma\, N(x, y)\ dy = 0$$

$$e^y\, dx + e^y\left(x - e^{-y}\right)\, dy = 0$$

Now, we have a new ODE which is exact.

We then continue with the procedure for the exact equation.

$$e^y\, dx + \left(x\, e^y - 1\right)\, dy = 0$$

$$M(x, y) = e^y$$

$$N(x, y) = x\, e^y - 1$$

Test for exactness:

$$M_y = N_x$$

$$e^y = e^y$$

We need to find a function F(x,y) such that:

$$M(x, y) = \frac{\partial}{\partial x} F(x, y)$$

$$N(x, y) = \frac{\partial}{\partial y} F(x, y)$$

Therefore:

$$F(x, y) = \int M(x, y)\, dx$$

$$F(x, y) = \int e^y\, dx + A(y)$$

$$F(x, y) = x\, e^y + A(y)$$

Or:

$$F(x, y) = \int N(x, y)\, dy$$

$$F(x, y) = \int \left(x\, e^y - 1\right)\, dy + B(x)$$

$$F(x, y) = -y + x\, e^y + B(x)$$

We have two F(x,y) functions.
Both should be the same
We show this by first calculating the unknown A(y) form the first F(x,y)
and B(x) from the second F(x,y).

Calculating A(y):

$$\frac{\partial}{\partial y} F(x,y) = N(x,y)$$

$$x\,e^y + \frac{d}{dy} A(y) = x\,e^y - 1$$

$$\frac{d}{dy} A(y) = -1$$

$$A(y) = \int \left(\frac{d}{dy} A(y) \right) dy$$

$$A(y) = \int (-1)\, dy$$

$$A(y) = -y + C$$

Thus:

$$F(x,y) = e^y x - y + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x,y) = M(x,y)$$

$$e^y + \frac{d}{dx} B(x) = e^y$$

$$\frac{d}{dx} B(x) = 0$$

$$B(x) = \int \left(\frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int 0\, dx$$

$$B(x) = \mathrm{D}$$

Thus:

$$F(x,y) = e^y x - y + \mathrm{D}$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

Problem 5e

$$x\,dy + dx = 0$$

$$M(x,y) = 1$$

$$N(x,y) = x$$

$$M_y = 0$$

$$N_x = 1$$

$$M_y \neq N_x$$

Find the integrating factor, σ

$$\frac{M_y - N_x}{M} = -1$$

$$\frac{M_y - N_x}{N} = -\frac{1}{x}$$

Take , $\dfrac{M_y - N_x}{N} = -\dfrac{1}{x}$, *function of x alone.*

$$\sigma(x) = e^{\int \frac{M_y - N_x}{N}\, dx}$$

$$\sigma(x) = e^{\int -\frac{1}{x}\, dx}$$

$$\sigma(x) = \frac{1}{x}$$

$$\sigma\, M(x, y)\ dx + \sigma\, N(x, y)\ dy = 0$$

$$\frac{dx}{x} + dy = 0$$

Now, we have a new ODE which is exact.
We then continue with the procedure for the exact equation.

$$\frac{dx}{x} + dy = 0$$

$$M(x, y) = \frac{1}{x}$$

$$N(x, y) = 1$$

Test for exactness:

$$M_y = N_x$$

$$0 = 0$$

We need to find a function F(x,y) such that:

$$M(x, y) = \frac{\partial}{\partial x} F(x, y)$$

$$N(x, y) = \frac{\partial}{\partial y} F(x, y)$$

Therefore:

$$F(x, y) = \int M(x, y)\, dx$$

$$F(x, y) = \int \frac{1}{x}\, dx + A(y)$$

$$F(x,y) = \ln(x) + A(y)$$

Or:

$$F(x,y) = \int N(x,y)\,\mathrm{d}y$$

$$F(x,y) = \int 1\,\mathrm{d}y + B(x)$$

$$F(x,y) = y + B(x)$$

We have two F(x,y) functions.

Both should be the same

We show this by first calculating the unknown A(y) form the first F(x,y) and B(x) from the second F(x,y).

Calculating A(y):

$$\frac{\partial}{\partial y} F(x,y) = N(x,y)$$

$$\frac{\mathrm{d}}{\mathrm{d}y} A(y) = 1$$

$$\frac{\mathrm{d}}{\mathrm{d}y} A(y) = 1$$

$$A(y) = \int \left(\frac{\mathrm{d}}{\mathrm{d}y} A(y)\right)\mathrm{d}y$$

$$A(y) = \int 1\,\mathrm{d}y$$

$$A(y) = y + C$$

Thus:

$$F(x,y) = y + \ln(x) + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x,y) = M(x,y)$$

$$\frac{\mathrm{d}}{\mathrm{d}x} B(x) = \frac{1}{x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} B(x) = \frac{1}{x}$$

$$B(x) = \int \left(\frac{\mathrm{d}}{\mathrm{d}x} B(x)\right)\mathrm{d}x$$

$$B(x) = \int \frac{1}{x}\,\mathrm{d}x$$

$$B(x) = \ln(x) + \mathrm{D}$$

Thus:

$$F(x,y) = y + \ln(x) + \mathrm{D}$$

Pay attention, the two F(x,y) functions MUST be the same.

The solution is the F(x,y) = constant and it is in an implicit form.

Problem 5f

$$\left(y\,e^{-x}+1\right)\,dx+x\,e^{-x}\,dy=0$$

$$M(x,y)=y\,e^{-x}+1$$

$$N(x,y)=x\,e^{-x}$$

$$M_y=e^{-x}$$

$$N_x=e^{-x}-x\,e^{-x}$$

$$M_y\neq N_x$$

$$\textit{Find the integrating factor, }\sigma$$

$$\frac{M_y-N_x}{M}=\frac{x\,e^{-x}}{y\,e^{-x}+1}$$

$$\frac{M_y-N_x}{N}=1$$

$$\textit{Take },\ \frac{M_y-N_x}{N}=1,\ \textit{function of x alone.}$$

$$\sigma(x)=e^{\int\frac{M_y-N_x}{N}\,dx}$$

$$\sigma(x)=e^{\int 1\,dx}$$

$$\sigma(x)=e^{x}$$

$$\sigma\,M(x,y)\,dx+\sigma\,N(x,y)\,dy=0$$

$$e^{x}\left(y\,e^{-x}+1\right)\,dx+e^{x}\,x\,e^{-x}\,dy=0$$

Now, we have a new ODE which is exact.

We then continue with the procedure for the exact equation.

$$\left(y+e^{x}\right)\,dx+x\,dy=0$$

$$M(x,y)=y+e^{x}$$

$$N(x,y)=x$$

Test for exactness:

$$M_y=N_x$$

$$1=1$$

We need to find a function F(x,y) such that:

$$M(x,y)=\frac{\partial}{\partial x}\,F(x,y)$$

$$N(x,y)=\frac{\partial}{\partial y}\,F(x,y)$$

Therefore:

$$F(x,y) = \int M(x,y)\,\mathrm{d}x$$

$$F(x,y) = \int \left(y + \mathrm{e}^{x}\right)\,\mathrm{d}x + A(y)$$

$$F(x,y) = y\,x + \mathrm{e}^{x} + A(y)$$

Or:

$$F(x,y) = \int N(x,y)\,\mathrm{d}y$$

$$F(x,y) = \int x\,\mathrm{d}y + B(x)$$

$$F(x,y) = y\,x + B(x)$$

We have two F(x,y) functions.
Both should be the same
We show this by first calculating the unknown A(y) form the first F(x,y)
and B(x) from the second F(x,y).

Calculating A(y):

$$\frac{\partial}{\partial y}\,F(x,y) = N(x,y)$$

$$x + \frac{\mathrm{d}}{\mathrm{d}y}\,A(y) = x$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\,A(y) = 0$$

$$A(y) = \int \left(\frac{\mathrm{d}}{\mathrm{d}y}\,A(y)\right)\mathrm{d}y$$

$$A(y) = \int 0\,\mathrm{d}y$$

$$A(y) = C$$

Thus:

$$F(x,y) = x\,y + \mathrm{e}^{x} + C$$

Calculating B(x):

$$\frac{\partial}{\partial x}\,F(x,y) = M(x,y)$$

$$y + \frac{\mathrm{d}}{\mathrm{d}x}\,B(x) = y + \mathrm{e}^{x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\,B(x) = \mathrm{e}^{x}$$

$$B(x) = \int \left(\frac{\mathrm{d}}{\mathrm{d}x}\,B(x)\right)\mathrm{d}x$$

$$B(x) = \int \mathrm{e}^{x}\,\mathrm{d}x$$

$$B(x) = \mathrm{e}^{x} + \mathrm{D}$$

Thus:

$$F(x, y) = x\,y + e^{x} + D$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

Problem 5g

$$\cos(y)\,dx + (2\,(x-y)\sin(y) + \cos(y))\,dy = 0$$

$$M(x, y) = \cos(y)$$

$$N(x, y) = 2\,(x-y)\sin(y) + \cos(y)$$

$$M_y = -\sin(y)$$

$$N_x = 2\sin(y)$$

$$M_y \neq N_x$$

Find the integrating factor, σ

$$\frac{M_y - N_x}{M} = -\frac{3\sin(y)}{\cos(y)}$$

$$\frac{M_y - N_x}{N} = -\frac{3\sin(y)}{(2\,x - 2\,y)\sin(y) + \cos(y)}$$

$$\text{Take}\,,\ \frac{M_y - N_x}{M} = -\frac{3\sin(y)}{\cos(y)},\ \text{function of } y \text{ alone.}$$

$$\sigma(y) = e^{-\left(\int \frac{M_y - N_x}{M}\,dy\right)}$$

$$\sigma(y) = e^{-\left(\int -\frac{3\sin(y)}{\cos(y)}\,dy\right)}$$

$$\sigma(y) = \frac{1}{\cos(y)^3}$$

$$\sigma\,M(x, y)\,dx + \sigma\,N(x, y)\,dy = 0$$

$$\frac{dx}{\cos(y)^2} + \frac{(2\,(x-y)\sin(y) + \cos(y))\,dy}{\cos(y)^3} = 0$$

Now, we have a new ODE which is exact.
We then continue with the procedure for the exact equation.

$$\frac{dx}{\cos(y)^2} + \frac{((2\,x - 2\,y)\sin(y) + \cos(y))\,dy}{\cos(y)^3} = 0$$

$$M(x, y) = \frac{1}{\cos(y)^2}$$

$$N(x, y) = \frac{(2\,x - 2\,y)\sin(y) + \cos(y)}{\cos(y)^3}$$

Test for exactness:

$$M_y = N_x$$

$$\frac{2\sin(y)}{\cos(y)^3} = \frac{2\sin(y)}{\cos(y)^3}$$

We need to find a function F(x,y) such that:

$$M(x,y) = \frac{\partial}{\partial x} F(x,y)$$

$$N(x,y) = \frac{\partial}{\partial y} F(x,y)$$

Therefore:

$$F(x,y) = \int M(x,y)\,dx$$

$$F(x,y) = \int \frac{1}{\cos(y)^2}\,dx + A(y)$$

$$F(x,y) = \frac{x}{\cos(y)^2} + A(y)$$

Or:

$$F(x,y) = \int N(x,y)\,dy$$

$$F(x,y) = \int \frac{(2x - 2y)\sin(y) + \cos(y)}{\cos(y)^3}\,dy + B(x)$$

$$F(x,y) = \frac{2\sin(y)\cos(y) + x - y}{\cos(y)^2} + B(x)$$

We have two F(x,y) functions.
Both should be the same
We show this by first calculating the unknown A(y) form the first F(x,y) and B(x) from the second F(x,y).

Calculating A(y):

$$\frac{\partial}{\partial y} F(x,y) = N(x,y)$$

$$\frac{2x\sin(y)}{\cos(y)^3} + \frac{d}{dy} A(y) = \frac{(2x - 2y)\sin(y) + \cos(y)}{\cos(y)^3}$$

$$\frac{d}{dy} A(y) = \frac{-2\sin(y)\,y + \cos(y)}{\cos(y)^3}$$

$$A(y) = \int \left(\frac{d}{dy} A(y)\right) dy$$

$$A(y) = \int \frac{-2\sin(y)\,y + \cos(y)}{\cos(y)^3}\,dy$$

$$A(y) = \frac{2\sin(y)\cos(y) - y}{\cos(y)^2} + C$$

Thus:

$$F(x, y) = \frac{x - y + C\cos(y)^2 + 2\sin(y)\cos(y)}{\cos(y)^2}$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x, y) = M(x, y)$$

$$\frac{1}{\cos(y)^2} + \frac{d}{dx} B(x) = \frac{1}{\cos(y)^2}$$

$$\frac{d}{dx} B(x) = 0$$

$$B(x) = \int \left(\frac{d}{dx} B(x)\right) dx$$

$$B(x) = \int 0 \, dx$$

$$B(x) = D$$

Thus:

$$F(x, y) = \frac{x - y + 2\sin(y)\cos(y)}{\cos(y)^2} + D$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

Problem 5h

$$(1 - x - y)\, dx + dy = 0$$

$$M(x, y) = 1 - x - y$$

$$N(x, y) = 1$$

$$M_y = -1$$

$$N_x = 0$$

$$M_y \neq N_x$$

Find the integrating factor, σ

$$\frac{M_y - N_x}{M} = \frac{1}{-1 + x + y}$$

$$\frac{M_y - N_x}{N} = -1$$

$$\textit{Take}, \ \frac{M_y - N_x}{N} = -1, \ \textit{function of } x \textit{ alone.}$$

$$\sigma(x) = e^{\int \frac{M_y - N_x}{N} dx}$$

$$\sigma(x) = e^{\int (-1)\, dx}$$

$$\sigma(x) = e^{-x}$$

$$\sigma\, M(x, y)\, dx + \sigma\, N(x, y)\, dy = 0$$

$$e^{-x}\,(1 - x - y)\, dx + e^{-x}\, dy = 0$$

Now, we have a new ODE which is exact.
We then continue with the procedure for the exact equation.

$$-e^{-x}\,(-1 + x + y)\, dx + e^{-x}\, dy = 0$$

$$M(x, y) = -e^{-x}\,(-1 + x + y)$$

$$N(x, y) = e^{-x}$$

Test for exactness:

$$M_y = N_x$$

$$-e^{-x} = -e^{-x}$$

We need to find a function F(x,y) such that:

$$M(x, y) = \frac{\partial}{\partial x}\, F(x, y)$$

$$N(x, y) = \frac{\partial}{\partial y}\, F(x, y)$$

Therefore:

$$F(x, y) = \int M(x, y)\, dx$$

$$F(x, y) = \int -e^{-x}\,(-1 + x + y)\, dx + A(y)$$

$$F(x, y) = (x + y)\, e^{-x} + A(y)$$

Or:

$$F(x, y) = \int N(x, y)\, dy$$

$$F(x, y) = \int e^{-x}\, dy + B(x)$$

$$F(x, y) = y\, e^{-x} + B(x)$$

We have two F(x,y) functions.
Both should be the same
We show this by first calculating the unknown A(y) form the first F(x,y)
and B(x) from the second F(x,y).

Calculating A(y):

$$\frac{\partial}{\partial y} F(x, y) = N(x, y)$$

$$e^{-x} + \frac{d}{dy} A(y) = e^{-x}$$

$$\frac{d}{dy} A(y) = 0$$

$$A(y) = \int \left(\frac{d}{dy} A(y) \right) dy$$

$$A(y) = \int 0 \, dy$$

$$A(y) = C$$

Thus:

$$F(x, y) = (x + y) e^{-x} + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x, y) = M(x, y)$$

$$-y e^{-x} + \frac{d}{dx} B(x) = -e^{-x} (-1 + x + y)$$

$$\frac{d}{dx} B(x) = -e^{-x} (x - 1)$$

$$B(x) = \int \left(\frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int -e^{-x} (x - 1) \, dx$$

$$B(x) = x e^{-x} + D$$

Thus:

$$F(x, y) = (x + y) e^{-x} + D$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

Problem 5i

$$\left(2 + \tan(x)^2\right) \left(1 + e^{-y}\right) dx + e^{-y} \tan(x) \, dy = 0$$

$$M(x, y) = \left(2 + \tan(x)^2\right) \left(1 + e^{-y}\right)$$

$$N(x, y) = e^{-y} \tan(x)$$

$$M_y = -\left(2 + \tan(x)^2\right) e^{-y}$$

$$N_x = e^{-y} \left(1 + \tan(x)^2\right)$$

$$M_y \neq N_x$$

Find the integrating factor, σ

$$\frac{M_y - N_x}{M} = -\frac{e^{-y} \left(2 + \cos(x)^2\right)}{\left(1 + e^{-y}\right) \left(\cos(x)^2 + 1\right)}$$

$$\frac{M_y - N_x}{N} = \frac{-2 - \cos(x)^2}{\sin(x)\cos(x)}$$

Take , $\frac{M_y - N_x}{N} = \frac{-2 - \cos(x)^2}{\sin(x)\cos(x)}$, *function of x alone.*

$$\sigma(x) = e^{\int \frac{M_y - N_x}{N}\, dx}$$

$$\sigma(x) = e^{\int \frac{-2 - \cos(x)^2}{\sin(x)\cos(x)}\, dx}$$

$$\sigma(x) = \frac{\cos(x)^2}{\sin(x)^3}$$

$$\sigma M(x, y)\, dx + \sigma N(x, y)\, dy = 0$$

$$\frac{\cos(x)^2 \left(2 + \tan(x)^2\right) \left(1 + e^{-y}\right) dx}{\sin(x)^3} + \frac{\cos(x)^2 e^{-y} \tan(x)\, dy}{\sin(x)^3} = 0$$

Now, we have a new ODE which is exact.

We then continue with the procedure for the exact equation.

$$-\frac{\left(1 + e^{-y}\right) \left(\cos(x)^2 + 1\right) dx}{\sin(x) \left(\cos(x)^2 - 1\right)} + \frac{e^{-y} \cos(x)\, dy}{\sin(x)^2} = 0$$

$$M(x, y) = -\frac{\left(1 + e^{-y}\right) \left(\cos(x)^2 + 1\right)}{\sin(x) \left(\cos(x)^2 - 1\right)}$$

$$N(x, y) = \frac{e^{-y} \cos(x)}{\sin(x)^2}$$

Test for exactness:

$$M_y = N_x$$

$$\frac{e^{-y} \left(\cos(x)^2 + 1\right)}{\sin(x) \left(\cos(x)^2 - 1\right)} = \frac{e^{-y} \left(\cos(x)^2 + 1\right)}{\sin(x) \left(\cos(x)^2 - 1\right)}$$

We need to find a function F(x,y) such that:

$$M(x, y) = \frac{\partial}{\partial x} F(x, y)$$

$$N(x, y) = \frac{\partial}{\partial y} F(x, y)$$

Therefore:

$$F(x, y) = \int M(x, y)\, dx$$

$$F(x, y) = \int -\frac{\left(1 + e^{-y}\right) \left(\cos(x)^2 + 1\right)}{\sin(x) \left(\cos(x)^2 - 1\right)}\, dx + A(y)$$

$$F(x,y) = -\frac{\left(1+e^{-y}\right)\cos(x)}{\sin(x)^2} + A(y)$$

Or:

$$F(x,y) = \int N(x,y)\,dy$$

$$F(x,y) = \int \frac{e^{-y}\cos(x)}{\sin(x)^2}\,dy + B(x)$$

$$F(x,y) = -\frac{e^{-y}\cos(x)}{\sin(x)^2} + B(x)$$

We have two F(x,y) functions.
Both should be the same
We show this by first calculating the unknown A(y) form the first F(x,y)
and B(x) from the second F(x,y).

Calculating A(y):

$$\frac{\partial}{\partial y} F(x,y) = N(x,y)$$

$$\frac{e^{-y}\cos(x)}{\sin(x)^2} + \frac{d}{dy} A(y) = \frac{e^{-y}\cos(x)}{\sin(x)^2}$$

$$\frac{d}{dy} A(y) = 0$$

$$A(y) = \int \left(\frac{d}{dy} A(y)\right) dy$$

$$A(y) = \int 0\,dy$$

$$A(y) = C$$

Thus:

$$F(x,y) = \frac{-C\cos(x)^2 - e^{-y}\cos(x) + C - \cos(x)}{\sin(x)^2}$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x,y) = M(x,y)$$

$$\frac{2\,e^{-y}\cos(x)^2}{\sin(x)^3} + \frac{e^{-y}}{\sin(x)} + \frac{d}{dx} B(x) = -\frac{\left(1+e^{-y}\right)\left(\cos(x)^2+1\right)}{\sin(x)\left(\cos(x)^2-1\right)}$$

$$\frac{d}{dx} B(x) = \frac{-\cos(x)^2 - 1}{\sin(x)\left(\cos(x)^2-1\right)}$$

$$B(x) = \int \left(\frac{d}{dx} B(x)\right) dx$$

$$B(x) = \int \frac{-\cos(x)^2 - 1}{\sin(x)\,\left(\cos(x)^2 - 1\right)}\,dx$$

$$B(x) = -\frac{\cos(x)}{\sin(x)^2} + D$$

Thus:

$$F(x, y) = \frac{-D\cos(x)^2 - e^{-y}\cos(x) + D - \cos(x)}{\sin(x)^2}$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

Problem 5j

$$\left(3x^2\sinh(3y) - 2x\right)dx + 3x^3\cosh(3y)\,dy = 0$$

$$M(x, y) = 3x^2\sinh(3y) - 2x$$

$$N(x, y) = 3x^3\cosh(3y)$$

$$M_y = 9x^2\cosh(3y)$$

$$N_x = 9x^2\cosh(3y)$$

The equation is already exact!
We can continue with the procedure for the exact equation.

$$\left(3x^2\sinh(3y) - 2x\right)dx + 3x^3\cosh(3y)\,dy = 0$$

$$M(x, y) = 3x^2\sinh(3y) - 2x$$

$$N(x, y) = 3x^3\cosh(3y)$$

Test for exactness:

$$M_y = N_x$$

$$9x^2\cosh(3y) = 9x^2\cosh(3y)$$

We need to find a function F(x,y) such that:

$$M(x, y) = \frac{\partial}{\partial x}F(x, y)$$

$$N(x, y) = \frac{\partial}{\partial y}F(x, y)$$

Therefore:

$$F(x, y) = \int M(x, y)\,dx$$

$$F(x, y) = \int \left(3x^2\sinh(3y) - 2x\right)dx + A(y)$$

$$F(x, y) = x^2\left(\sinh(3y)\,x - 1\right) + A(y)$$

Or:

$$F(x,y) = \int N(x,y)\, dy$$

$$F(x,y) = \int 3\,x^3 \cosh(3\,y)\, dy + B(x)$$

$$F(x,y) = x^3 \sinh(3\,y) + B(x)$$

We have two F(x,y) functions.
Both should be the same
We show this by first calculating the unknown A(y) form the first F(x,y)
and B(x) from the second F(x,y).

Calculating A(y):

$$\frac{\partial}{\partial y} F(x,y) = N(x,y)$$

$$3\,x^3 \cosh(3\,y) + \frac{d}{dy} A(y) = 3\,x^3 \cosh(3\,y)$$

$$\frac{d}{dy} A(y) = 0$$

$$A(y) = \int \left(\frac{d}{dy} A(y) \right) dy$$

$$A(y) = \int 0\, dy$$

$$A(y) = C$$

Thus:

$$F(x,y) = \sinh(3\,y)\, x^3 - x^2 + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x,y) = M(x,y)$$

$$3\,x^2 \sinh(3\,y) + \frac{d}{dx} B(x) = 3\,x^2 \sinh(3\,y) - 2\,x$$

$$\frac{d}{dx} B(x) = -2\,x$$

$$B(x) = \int \left(\frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int -2\,x\, dx$$

$$B(x) = -x^2 + \mathrm{D}$$

Thus:

$$F(x,y) = \sinh(3\,y)\, x^3 - x^2 + \mathrm{D}$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

Problem 5k

$$\cos(x)\, dx + (3\sin(x) + 3\cos(y) - \sin(y))\, dy = 0$$

$$M(x,y) = \cos(x)$$

$$N(x,y) = 3\sin(x) + 3\cos(y) - \sin(y)$$

$$M_y = 0$$

$$N_x = 3\cos(x)$$

$$M_y \neq N_x$$

Find the integrating factor, σ

$$\frac{M_y - N_x}{M} = -3$$

$$\frac{M_y - N_x}{N} = \frac{3\cos(x)}{-3\sin(x) - 3\cos(y) + \sin(y)}$$

$$\textit{Take}\ ,\ \frac{M_y - N_x}{M} = -3,\ \textit{function of } y \textit{ alone.}$$

$$\sigma(y) = e^{-\left(\int \frac{M_y - N_x}{M}\, dy\right)}$$

$$\sigma(y) = e^{-\left(\int (-3)\, dy\right)}$$

$$\sigma(y) = e^{3y}$$

$$\sigma M(x,y)\, dx + \sigma N(x,y)\, dy = 0$$

$$e^{3y}\cos(x)\, dx + e^{3y}(3\sin(x) + 3\cos(y) - \sin(y))\, dy = 0$$

Now, we have a new ODE which is exact.

We then continue with the procedure for the exact equation.

$$e^{3y}\cos(x)\, dx + e^{3y}(3\sin(x) + 3\cos(y) - \sin(y))\, dy = 0$$

$$M(x,y) = e^{3y}\cos(x)$$

$$N(x,y) = e^{3y}(3\sin(x) + 3\cos(y) - \sin(y))$$

Test for exactness:

$$M_y = N_x$$

$$3\, e^{3y}\cos(x) = 3\, e^{3y}\cos(x)$$

We need to find a function F(x,y) such that:

$$M(x,y) = \frac{\partial}{\partial x} F(x,y)$$

$$N(x,y) = \frac{\partial}{\partial y} F(x,y)$$

Therefore:

$$F(x,y) = \int M(x,y)\, dx$$

$$F(x,y) = \int e^{3y} \cos(x)\, dx + A(y)$$

$$F(x,y) = e^{3y} \sin(x) + A(y)$$

Or:

$$F(x,y) = \int N(x,y)\, dy$$

$$F(x,y) = \int e^{3y} (3 \sin(x) + 3 \cos(y) - \sin(y))\, dy + B(x)$$

$$F(x,y) = e^{3y} (\cos(y) + \sin(x)) + B(x)$$

We have two F(x,y) functions.
Both should be the same
We show this by first calculating the unknown A(y) form the first F(x,y) and B(x) from the second F(x,y).

Calculating A(y):

$$\frac{\partial}{\partial y} F(x,y) = N(x,y)$$

$$3\, e^{3y} \sin(x) + \frac{d}{dy} A(y) = e^{3y} (3 \sin(x) + 3 \cos(y) - \sin(y))$$

$$\frac{d}{dy} A(y) = (3 \cos(y) - \sin(y))\, e^{3y}$$

$$A(y) = \int \left(\frac{d}{dy} A(y) \right) dy$$

$$A(y) = \int (3 \cos(y) - \sin(y))\, e^{3y}\, dy$$

$$A(y) = e^{3y} \cos(y) + C$$

Thus:

$$F(x,y) = e^{3y} (\cos(y) + \sin(x)) + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x,y) = M(x,y)$$

$$e^{3y} \cos(x) + \frac{d}{dx} B(x) = e^{3y} \cos(x)$$

$$\frac{d}{dx} B(x) = 0$$

$$B(x) = \int \left(\frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int 0\, dx$$

$$B(x) = \mathrm{D}$$

Thus:

$$F(x,y) = e^{3y}(\cos(y) + \sin(x)) + D$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

Problem 5l

$$\left(y\ln(y) + 2xy^2\right) dx + \left(x^2 y + x\right) dy = 0$$
$$M(x,y) = y\ln(y) + 2xy^2$$
$$N(x,y) = x^2 y + x$$
$$M_y = \ln(y) + 1 + 4xy$$
$$N_x = 2xy + 1$$
$$M_y \neq N_x$$

Find the integrating factor, σ

$$\frac{M_y - N_x}{M} = \frac{1}{y}$$
$$\frac{M_y - N_x}{N} = \frac{\ln(y) + 2xy}{x^2 y + x}$$
$$Take\ ,\ \frac{M_y - N_x}{M} = \frac{1}{y},\ function\ of\ y\ alone.$$
$$\sigma(y) = e^{-\left(\int \frac{M_y - N_x}{M}\, dy\right)}$$
$$\sigma(y) = e^{-\left(\int \frac{1}{y}\, dy\right)}$$
$$\sigma(y) = \frac{1}{y}$$
$$\sigma M(x,y)\, dx + \sigma N(x,y)\, dy = 0$$
$$\frac{\left(y\ln(y) + 2xy^2\right) dx}{y} + \frac{\left(x^2 y + x\right) dy}{y} = 0$$

Now, we have a new ODE which is exact.
We then continue with the procedure for the exact equation.

$$(\ln(y) + 2xy)\, dx + \frac{x(xy+1)\, dy}{y} = 0$$
$$M(x,y) = \ln(y) + 2xy$$
$$N(x,y) = \frac{x(xy+1)}{y}$$

Test for exactness:

$$M_y = N_x$$

$$\frac{2xy+1}{y} = \frac{2xy+1}{y}$$

We need to find a function F(x,y) such that:

$$M(x,y) = \frac{\partial}{\partial x} F(x,y)$$

$$N(x,y) = \frac{\partial}{\partial y} F(x,y)$$

Therefore:

$$F(x,y) = \int M(x,y)\,dx$$

$$F(x,y) = \int (\ln(y) + 2xy)\,dx + A(y)$$

$$F(x,y) = x\,(xy + \ln(y)) + A(y)$$

Or:

$$F(x,y) = \int N(x,y)\,dy$$

$$F(x,y) = \int \frac{x\,(xy+1)}{y}\,dy + B(x)$$

$$F(x,y) = x\,(xy + \ln(y)) + B(x)$$

We have two F(x,y) functions.
Both should be the same
We show this by first calculating the unknown A(y) form the first F(x,y)
and B(x) from the second F(x,y).

Calculating A(y):

$$\frac{\partial}{\partial y} F(x,y) = N(x,y)$$

$$x\left(x + \frac{1}{y}\right) + \frac{d}{dy} A(y) = \frac{x\,(xy+1)}{y}$$

$$\frac{d}{dy} A(y) = 0$$

$$A(y) = \int \left(\frac{d}{dy} A(y)\right) dy$$

$$A(y) = \int 0\,dy$$

$$A(y) = C$$

Thus:

$$F(x,y) = x^2 y + \ln(y)\,x + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x, y) = M(x, y)$$

$$\ln(y) + 2xy + \frac{d}{dx} B(x) = \ln(y) + 2xy$$

$$\frac{d}{dx} B(x) = 0$$

$$B(x) = \int \left(\frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int 0 \, dx$$

$$B(x) = D$$

Thus:

$$F(x, y) = x^2 y + \ln(y) x + D$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

Problem 5m

$$\boxed{(3x - 2y)\, dx - x\, dy = 0}$$

$$M(x, y) = 3x - 2y$$

$$N(x, y) = -x$$

$$M_y = -2$$

$$N_x = -1$$

$$M_y \neq N_x$$

Find the integrating factor, σ

$$\frac{M_y - N_x}{M} = -\frac{1}{3x - 2y}$$

$$\frac{M_y - N_x}{N} = \frac{1}{x}$$

$$\text{Take}, \ \frac{M_y - N_x}{N} = \frac{1}{x}, \ \text{function of } x \text{ alone.}$$

$$\sigma(x) = e^{\int \frac{M_y - N_x}{N} dx}$$

$$\sigma(x) = e^{\int \frac{1}{x} dx}$$

$$\sigma(x) = x$$

$$\sigma M(x, y)\, dx + \sigma N(x, y)\, dy = 0$$

$$x (3x - 2y)\, dx - x^2\, dy = 0$$

Now, we have a new ODE which is exact.
We then continue with the procedure for the exact equation.

$$x\,(3\,x-2\,y)\,dx-x^2\,dy=0$$
$$M(x,y)=x\,(3\,x-2\,y)$$
$$N(x,y)=-x^2$$

Test for exactness:

$$M_y=N_x$$
$$-2\,x=-2\,x$$

We need to find a function F(x,y) such that:

$$M(x,y)=\frac{\partial}{\partial x}\,F(x,y)$$
$$N(x,y)=\frac{\partial}{\partial y}\,F(x,y)$$

Therefore:

$$F(x,y)=\int M(x,y)\,\mathrm{d}x$$
$$F(x,y)=\int x\,(3\,x-2\,y)\,\mathrm{d}x+A(y)$$
$$F(x,y)=x^2\,(x-y)+A(y)$$

Or:

$$F(x,y)=\int N(x,y)\,\mathrm{d}y$$
$$F(x,y)=\int -x^2\,\mathrm{d}y+B(x)$$
$$F(x,y)=-x^2\,y+B(x)$$

We have two F(x,y) functions.
Both should be the same
We show this by first calculating the unknown A(y) form the first F(x,y) and B(x) from the second F(x,y).

Calculating A(y):

$$\frac{\partial}{\partial y}\,F(x,y)=N(x,y)$$
$$-x^2+\frac{\mathrm{d}}{\mathrm{d}y}\,A(y)=-x^2$$
$$\frac{\mathrm{d}}{\mathrm{d}y}\,A(y)=0$$
$$A(y)=\int\left(\frac{\mathrm{d}}{\mathrm{d}y}\,A(y)\right)\mathrm{d}y$$
$$A(y)=\int 0\,\mathrm{d}y$$

$$A(y) = C$$

Thus:

$$F(x, y) = x^3 - x^2 y + C$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x, y) = M(x, y)$$

$$-2 x y + \frac{d}{dx} B(x) = x (3 x - 2 y)$$

$$\frac{d}{dx} B(x) = 3 x^2$$

$$B(x) = \int \left(\frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int 3 x^2 \, dx$$

$$B(x) = x^3 + \mathrm{D}$$

Thus:

$$F(x, y) = x^3 - x^2 y + \mathrm{D}$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

Problem 5n

$$y\, dx + (x^2 - x)\, dy = 0$$

$$M(x, y) = y$$

$$N(x, y) = x^2 - x$$

$$M_y = 1$$

$$N_x = 2 x - 1$$

$$M_y \neq N_x$$

Find the integrating factor, σ

$$\frac{M_y - N_x}{M} = \frac{2 - 2 x}{y}$$

$$\frac{M_y - N_x}{N} = -\frac{2}{x}$$

$$\text{Take },\ \frac{M_y - N_x}{N} = -\frac{2}{x},\ \text{function of } x \text{ alone.}$$

$$\sigma(x) = e^{\int \frac{M_y - N_x}{N} dx}$$

$$\sigma(x) = e^{\int -\frac{2}{x} dx}$$

$$\sigma(x) = \frac{1}{x^2}$$

$$\sigma M(x,y)\ dx + \sigma N(x,y)\ dy = 0$$

$$\frac{y\,dx}{x^2} + \frac{(x^2 - x)\ dy}{x^2} = 0$$

Now, we have a new ODE which is exact.
We then continue with the procedure for the exact equation.

$$\frac{y\,dx}{x^2} + \frac{(x-1)\ dy}{x} = 0$$

$$M(x,y) = \frac{y}{x^2}$$

$$N(x,y) = \frac{x-1}{x}$$

Test for exactness:

$$M_y = N_x$$

$$\frac{1}{x^2} = \frac{1}{x^2}$$

We need to find a function F(x,y) such that:

$$M(x,y) = \frac{\partial}{\partial x} F(x,y)$$

$$N(x,y) = \frac{\partial}{\partial y} F(x,y)$$

Therefore:

$$F(x,y) = \int M(x,y)\ \mathrm{d}x$$

$$F(x,y) = \int \frac{y}{x^2}\ \mathrm{d}x + A(y)$$

$$F(x,y) = -\frac{y}{x} + A(y)$$

Or:

$$F(x,y) = \int N(x,y)\ \mathrm{d}y$$

$$F(x,y) = \int \frac{x-1}{x}\ \mathrm{d}y + B(x)$$

$$F(x,y) = \frac{(x-1)\ y}{x} + B(x)$$

We have two F(x,y) functions.

Both should be the same
We show this by first calculating the unknown A(y) form the first F(x,y)
and B(x) from the second F(x,y).

Calculating A(y):

$$\frac{\partial}{\partial y} F(x,y) = N(x,y)$$

$$-\frac{1}{x} + \frac{d}{dy} A(y) = \frac{x-1}{x}$$

$$\frac{d}{dy} A(y) = 1$$

$$A(y) = \int \left(\frac{d}{dy} A(y) \right) dy$$

$$A(y) = \int 1 \, dy$$

$$A(y) = y + C$$

Thus:

$$F(x,y) = \frac{(y+C)\,x - y}{x}$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x,y) = M(x,y)$$

$$-\frac{(x-1)\,y}{x^2} + \frac{y}{x} + \frac{d}{dx} B(x) = \frac{y}{x^2}$$

$$\frac{d}{dx} B(x) = 0$$

$$B(x) = \int \left(\frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int 0 \, dx$$

$$B(x) = \mathrm{D}$$

Thus:

$$F(x,y) = \frac{(y+\mathrm{D})\,x - y}{x}$$

Pay attention, the two F(x,y) functions MUST be the same.
The solution is the F(x,y) = constant and it is in an implicit form.

Problem 5o

$$2\,x\,y\,dx + \left(-x^2 + y^2\right)\,dy = 0$$

$$M(x,y) = 2\,x\,y$$

$$N(x,y) = -x^2 + y^2$$

$$M_y = 2\,x$$

$$N_x = -2x$$
$$M_y \neq N_x$$

Find the integrating factor, σ

$$\frac{M_y - N_x}{M} = \frac{2}{y}$$
$$\frac{M_y - N_x}{N} = -\frac{4x}{x^2 - y^2}$$
$$\textit{Take },\ \frac{M_y - N_x}{M} = \frac{2}{y},\ \textit{function of } y \textit{ alone.}$$
$$\sigma(y) = e^{-\left(\int \frac{M_y - N_x}{M}\, dy\right)}$$
$$\sigma(y) = e^{-\left(\int \frac{2}{y}\, dy\right)}$$
$$\sigma(y) = \frac{1}{y^2}$$
$$\sigma M(x, y)\ dx + \sigma N(x, y)\ dy = 0$$
$$\frac{2x\,dx}{y} + \frac{(-x^2 + y^2)\ dy}{y^2} = 0$$

Now, we have a new ODE which is exact.
We then continue with the procedure for the exact equation.

$$\frac{2x\,dx}{y} + \frac{(-x^2 + y^2)\ dy}{y^2} = 0$$
$$M(x, y) = \frac{2x}{y}$$
$$N(x, y) = \frac{-x^2 + y^2}{y^2}$$

Test for exactness:

$$M_y = N_x$$
$$-\frac{2x}{y^2} = -\frac{2x}{y^2}$$

We need to find a function F(x,y) such that:

$$M(x, y) = \frac{\partial}{\partial x} F(x, y)$$
$$N(x, y) = \frac{\partial}{\partial y} F(x, y)$$

Therefore:

$$F(x,y) = \int M(x,y)\,dx$$

$$F(x,y) = \int \frac{2x}{y}\,dx + A(y)$$

$$F(x,y) = \frac{x^2}{y} + A(y)$$

Or:

$$F(x,y) = \int N(x,y)\,dy$$

$$F(x,y) = \int \frac{-x^2+y^2}{y^2}\,dy + B(x)$$

$$F(x,y) = \frac{x^2+y^2}{y} + B(x)$$

We have two F(x,y) functions.
Both should be the same
We show this by first calculating the unknown A(y) form the first F(x,y) and B(x) from the second F(x,y).

Calculating A(y):

$$\frac{\partial}{\partial y} F(x,y) = N(x,y)$$

$$-\frac{x^2}{y^2} + \frac{d}{dy} A(y) = \frac{-x^2+y^2}{y^2}$$

$$\frac{d}{dy} A(y) = 1$$

$$A(y) = \int \left(\frac{d}{dy} A(y)\right) dy$$

$$A(y) = \int 1\,dy$$

$$A(y) = y + C$$

Thus:

$$F(x,y) = \frac{x^2+y^2+C\,y}{y}$$

Calculating B(x):

$$\frac{\partial}{\partial x} F(x,y) = M(x,y)$$

$$\frac{2x}{y} + \frac{d}{dx} B(x) = \frac{2x}{y}$$

$$\frac{d}{dx} B(x) = 0$$

$$B(x) = \int \left(\frac{d}{dx} B(x) \right) dx$$

$$B(x) = \int 0 \, dx$$

$$B(x) = D$$

Thus:

$$F(x, y) = \frac{x^2 + y^2 + D y}{y}$$

Pay attention, the two F(x,y) functions MUST be the same.

The solution is the F(x,y) = constant and it is in an implicit form.
