

## Numerical Methods for Complex Systems

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## Table of Contents

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<b>1. Case 1: 1-D Non-Homogenous Linear Advection<sup>1</sup></b>	2
1.1. <u>Forward in Time Centered in Space (FTCS)</u>	2
1.2. <u>Upwind Method (<math>c &gt; 0</math>)</u>	2
1.3. <u>Downwind Method (<math>c &lt; 0</math>)</u>	4
1.4. <u>The Lax Method</u>	6
1.5. <u>The Lax-Wendroff Method<sup>2</sup></u>	9
1.6. Heat Exchanger as An Example	13
1.6.1. Two-Equation Model	13
1.6.2. Three-Equation Model	16
<b>2. Case 2: 1D-Rod<sup>3</sup></b>	17
2.1. <u>Explicit Method</u>	17
2.2. <u>Implicit Method</u>	18
<b>3. Case 3: Rectangular 2D Plane<sup>4</sup></b>	21
3.1. The Five-Point Difference Method	21
3.2. Energy Balance Model: Out-of-Plane Heat Transfer for a Thin Plate - Comparison with COMSOL	21

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## 1. Case 1: 1-D Non-Homogenous Linear Advection<sup>5</sup>

Given a hyperbolic PDE as follows.

$$u_t + cu_x = f(x, t) \quad (1)$$

with the following initial condition:

$$\begin{aligned} u(x, 0) &= u_0(x) \\ c &= \text{constant of real number} \end{aligned} \quad (2)$$

We want to find its solution numerically. When  $f(x, t) = 0$ , the analytical solution is given by:  $u(x, t) = u_0(x - ct)$ . This kind of PDE is widely used for real time modeling and control application. Therefore, it is important to investigate its non-homogenous version to further allow an implementation of a PDE system. A PDE system is composed of more than one interacting PDEs. For example, in a heat exchanger, there are at least two PDEs: one for the hot system and another one for the cold system. The two PDEs interact by exchanging heat one to another.

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### 1.1. Forward in Time Centered in Space (FTCS)

This method is a finite difference method with central difference for the distance to increase the approximation. To approximate  $u_t$ , we use the following approximation.

$$u_t \approx \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} \quad (3)$$

While to approach  $u_x$ , we use the following approximation.

$$u_x \approx \frac{u(x + \Delta x, t) - u(x - \Delta x, t)}{2\Delta x} \quad (4)$$

Substituting (3) and (4) to (1):

$$\begin{aligned} u_t &\approx -cu_x + f(x, t) \\ \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} &\approx -c \frac{u(x + \Delta x, t) - u(x - \Delta x, t)}{2\Delta x} + f(x, t) \\ u(x, t + \Delta t) - u(x, t) &\approx -\frac{c\Delta t}{2\Delta x} [u(x + \Delta x, t) - u(x - \Delta x, t)] + f(x, t)\Delta t \\ u(x, t + \Delta t) &\approx u(x, t) - \frac{c\Delta t}{2\Delta x} [u(x + \Delta x, t) - u(x - \Delta x, t)] + f(x, t)\Delta t \end{aligned} \quad (5)$$

FTCS method is unconditionally unstable. It should never be used. Nevertheless, FTCS is the fundamental building block for another methods, such as the Lax method.

---

### 1.2. Upwind Method ( $c > 0$ )

By using finite difference method, we can approximate both  $u_t$  and  $u_x$ . To approximate  $u_t$ , we use the following approximation.

$$u_t \approx \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} \quad (6)$$

While to approach  $u_x$ , we use the following approximation.

$$u_x \approx \frac{u(x, t) - u(x - \Delta x, t)}{\Delta x} \quad (7)$$

Substituting (6) and (7) to (1):

$$\begin{aligned}
u_t &\approx -cu_x + f(x, t) \\
\frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} &\approx -c \frac{u(x, t) - u(x - \Delta x, t)}{\Delta x} + f(x, t) \\
u(x, t + \Delta t) + u(x, t) &\approx -\frac{c\Delta t}{\Delta x} [u(x, t) - u(x - \Delta x, t)] + f(x, t) \\
u(x, t + \Delta t) &\approx u(x, t) - \frac{c\Delta t}{\Delta x} [u(x, t) - u(x - \Delta x, t)] + f(x, t)\Delta t
\end{aligned} \tag{8}$$

Consider  $\frac{c\Delta t}{\Delta x} = r$ ,  $U_{j,k} = u(x_j, t_k)$ ,  $F_{j,k} = f(x_j, t_k)$ ,  $x_j = j\Delta x$  and  $t_k = k\Delta t$ , we can have the following shorter terms:

$$U_{j,k+1} = U_{j,k} - r(U_{j,k} - U_{j-1,k}) + F_{j,k}\Delta t \tag{9}$$

MATLAB implementation of an upwind method (Eq. 9) is as follows.

```

1 function next_u_array = upwind(u_array, f_array, c, dt, dx)
2
3 N = length(u_array);
4 next_u_array = u_array;
5
6 for i = 2:N
7     next_u_array(i) = u_array(i) - c*dt/dx*(u_array(i) - u_array(i-1)) + f_array(i)*dt;
8 end
9
10 end

```

To increase its efficiency, the MATLAB code above can be vectorized as follows.

```

1 function next_u_array = upwind(u_array, f_array, c, dt, dx)
2
3 N = length(u_array);
4 next_u_array = u_array;
5
6 next_u_array(2:N) = u_array(2:N) - (c*dt/dx).* ...
7                 (u_array(2:N) - u_array(1:N-1)) + f_array(2:N).*dt;
8
9 end

```

The following lines of code is used to test the upwind function. Here we want to solve the following PDE.

$$\begin{aligned}
u_t + 0.5u_x &= 0 \\
u(x, 0) &= e^{-(x-2)^2} \\
0 \leq x \leq 10
\end{aligned} \tag{10}$$

The length is divided 100 segments ( $dx = 0.1$ ) and the time is divided into 400 segments ( $dt = 0.05$ ).

```

1 L = 10;
2 dx = 0.1;
3 x = 0:dx:L;
4
5 dt = 0.05;
6 T = 20;
7 t = 0:dt:T;

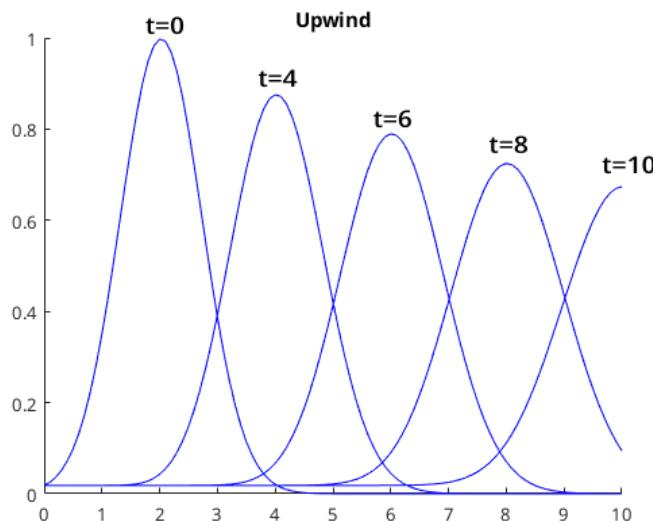
```

```

8
9 c = 0.5; % Upwind
10
11 u = zeros(length(x),1);
12 f = zeros(length(x),1);
13
14 for k = 1:length(x)
15     u(k) = exp(-(x(k)-2).^2);
16 end
17
18 figure;
19 h = plot(0,0);
20 title('Upwind')
21 ylim([0 1]);
22
23 for k = 1:length(t)
24     u = upwind(u,f, c, dt, dx);
25     set(h,'XData',x, 'Ydata',u);
26     drawnow;
27 end

```

The analytical solution of Eq. 10 is  $u(x,t) = e^{-((x-0.5t)-2)^2}$ . However, when executed, the code above will generate the following figure. As can be seen, the numerical solution is not exactly the same as its analytical solution. Overtime, its amplitude is actually decreasing.



### 1.3. Downwind Method ( $c < 0$ )

By using finite difference method, we can approximate both  $u_t$  and  $u_x$ . To approximate  $u_t$ , we use the following approximation.

$$u_t \approx \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} \quad (11)$$

While to approach  $u_x$ , we use the following approximation.

$$u_x \approx \frac{u(x + \Delta x, t) - u(x, t)}{\Delta x} \quad (12)$$

Substituting (11) and (12) to (1) yields the following.

$$\begin{aligned} u_t &\approx -cu_x + f(x, t) \\ \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} &\approx -c \frac{u(x + \Delta x, t) - u(x, t)}{\Delta x} + f(x, t) \\ u(x, t + \Delta t) + u(x, t) &\approx -\frac{c\Delta t}{\Delta x} [u(x + \Delta x, t) - u(x, t)] + f(x, t)\Delta t \\ u(x, t + \Delta t) &\approx u(x, t) - \frac{c\Delta t}{\Delta x} [u(x + \Delta x, t) - u(x, t)] + f(x, t)\Delta t \end{aligned}$$

Consider  $\frac{c\Delta t}{\Delta x} = r$ ,  $U_{j,k} = u(x_j, t_k)$ ,  $F_{j,k} = f(x_j, t_k)$ ,  $x_j = j\Delta x$  and  $t_k = k\Delta t$ , we can have the following shorter terms:

$$U_{j,k+1} = U_{j,k} - r(U_{j+1,k} - U_{j,k}) + F_{j,k}\Delta t \quad (13)$$

MATLAB implementation of an upwind method (Eq. 13) is as follows.

```

1 function next_u_array = downwind(u_array, f_array, c, dt, dx)
2
3 N = length(u_array);
4 next_u_array = u_array;
5
6 for i = 1:N-1
7     next_u_array(i) = u_array(i) - c*dt/dx*(u_array(i+1) - u_array(i)) + f_array(i)*dt;
8 end
9
10 end

```

To increase its efficiency, the MATLAB code above can be vectorized as follows.

```

1 function next_u_array = downwind(u_array, f_array, c, dt, dx)
2
3 N = length(u_array);
4 next_u_array = u_array;
5
6 next_u_array(1:N-1) = u_array(1:N-1) - (c*dt/dx).* ...
7                 (u_array(2:N) - u_array(1:N-1)) + f_array(1:N-1).*dt;
8
9 end

```

The following lines of code is used to test the upwind function. Here we want to solve the following PDE.

$$\begin{aligned} u_t - 0.5u_x &= 0 \\ u(x, 0) &= e^{-(x-8)^2} \\ 0 \leq x \leq 10 \end{aligned} \quad (14)$$

The length is divided 100 segments ( $dx = 0.1$ ) and the time is divided into 400 segments ( $dt = 0.05$ ).

```

1 L = 10;
2 dx = 0.1;
3 x = 0:dx:L;

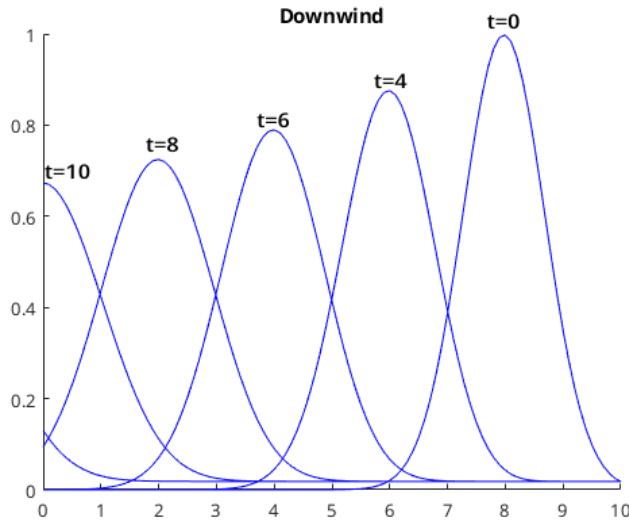
```

```

4
5 dt = 0.05;
6 T = 20;
7 t = 0:dt:T;
8
9 c = -0.5; % Downwind
10
11 u = zeros(length(x), 1);
12 f = zeros(length(x), 1);
13
14 for k = 1:length(x)
15     u(k) = exp(-(x(k)-8).^2);
16 end
17
18 figure;
19 h = plot(0,0);
20 title('Downwind')
21 ylim([0 1]);
22
23 for k = 1:length(t)
24     u = downwind(u,f, c, dt, dx);
25     set(h,'XData',x, 'Ydata',u);
26     drawnow;
27 end

```

The analytical solution of Eq. 14 is  $u(x,t) = e^{-(x+0.5t)-8)^2}$ . However, when executed, the code above will generate the following figure. As can be seen, the numerical solution is not exactly the same as its analytical solution. Overtime, its amplitude is actually decreasing.



#### 1.4. The Lax Method

The Lax method is based on the FTCS method. However,  $u(x,t)$  is replaced by an average of its two neighbors. To

approximate  $u_t$ , we use the following approximation.

$$u_t \approx \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} \quad (15)$$

$$u_t \approx \frac{u(x, t + \Delta t) - \left[ \frac{u(x + \Delta x, t) + u(x - \Delta x, t)}{2} \right]}{\Delta t}$$

While to approach  $u_x$ , we use the following approximation.

$$u_x \approx \frac{u(x + \Delta x, t) - u(x - \Delta x, t)}{2\Delta x} \quad (16)$$

Substituting (3) and (4) to (1) yields the following.

$$\begin{aligned} u_t &\approx -cu_x + f(x, t) \\ \frac{u(x, t + \Delta t) - \left[ \frac{u(x + \Delta x, t) + u(x - \Delta x, t)}{2} \right]}{\Delta t} &\approx -c \frac{u(x + \Delta x, t) - u(x - \Delta x, t)}{2\Delta x} + f(x, t) \\ u(x, t + \Delta t) - \left[ \frac{u(x + \Delta x, t) + u(x - \Delta x, t)}{2} \right] &\approx -\frac{c\Delta t}{2\Delta x} [u(x + \Delta x, t) - u(x - \Delta x, t)] + f(x, t)\Delta t \quad (17) \\ u(x, t + \Delta t) &\approx \left[ \frac{u(x + \Delta x, t) + u(x - \Delta x, t)}{2} \right] - \\ &\quad \frac{c\Delta t}{2\Delta x} [u(x + \Delta x, t) - u(x - \Delta x, t)] + f(x, t)\Delta t \end{aligned}$$

Consider  $r = \frac{c\Delta t}{\Delta x}$ ,  $U_{j,k} = u(x_j, t_k)$ ,  $F_{j,k} = f(x_j, t_k)$ ,  $x_j = j\Delta x$  and  $t_k = k\Delta t$ , we can have the following shorter terms:

$$U_{j,k+1} = \frac{U_{j+1,k} + U_{j-1,k}}{2} - \frac{r}{2}(U_{j+1,k} - U_{j,k}) + F_{j,k}\Delta t \quad (18)$$

The Lax method can handle both upwind and downwind direction. There is no need to check the sign of  $c$  as in the previous methods. However, we still have the same issue since the amplitude is also decreasing overtime. MATLAB implementation of the Lax method is as follows.

```

1  function next_u_array = lax(u_array, f_array, c, dt, dx)
2
3  N = length(u_array);
4  next_u_array = u_array;
5
6  r = c*dt/dx;
7
8  for i = 2:N-1 % i = 1 and i = N are untouched !!!
9      next_u_array(i) = 0.5*(u_array(i+1)+u_array(i-1)) - ...
10                 0.5*r*(u_array(i+1)-u_array(i-1)) + ...
11                 f_array(i)*dt;
12 end
13
14 % fill up the two ends by using upwind or downwind method
15 if c > 0
16     next_u_array(N) = u_array(N) - r* ...
17                 (u_array(N)- u_array(N-1)) + f_array(N).*dt;
18
19 elseif c < 0
20     next_u_array(1) = u_array(1) - r* ...
21                 (u_array(2) - u_array(1)) + f_array(1).*dt;

```

```

22
23 end

```

There is one issue with the Lax method. The Lax method leaves the two end segments untouched. This can be seen from the for loop that starts from 2 to N-1 (line 8). To address this issue, we can apply the upwind and downwind method for the two ends only.

To increase its efficiency, the MATLAB code above can be vectorized as follows.

```

1 function next_u_array = lax(u_array, f_array, c, dt, dx)
2
3 N = length(u_array);
4 next_u_array = u_array;
5
6 r = c*dt/dx;
7
8 next_u_array(2:N-1) = 0.5.* (u_array(3:N)+(u_array(1:N-2))) - ...
9             (0.5*r).* (u_array(3:N)- u_array(1:N-2)) + ...
10            f_array(2:N-1).*dt;
11
12
13 % fill up the two ends by using upwind or downwind method
14 if c > 0
15     next_u_array(N) = u_array(N) - r* ...
16                     (u_array(N)- u_array(N-1)) + f_array(N).*dt;
17
18 elseif c < 0
19     next_u_array(1) = u_array(1) - r* ...
20                     (u_array(2) - u_array(1)) + f_array(1).*dt;
21
22 end

```

Next, we test the code above for both upwind and downwind direction. For the upwind direction, we will use the same system as in Eq. 10. As for the downwind direction, we will also use the same system as in Eq. 14.

```

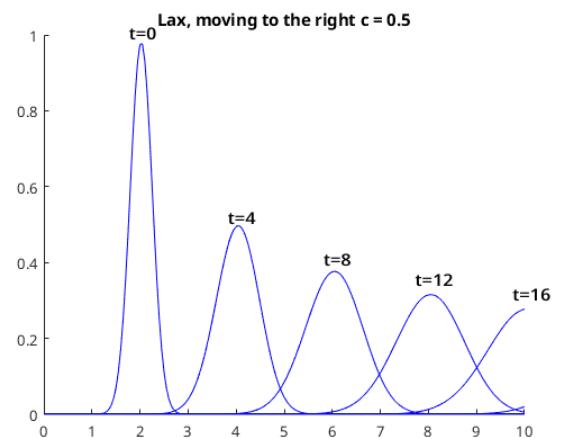
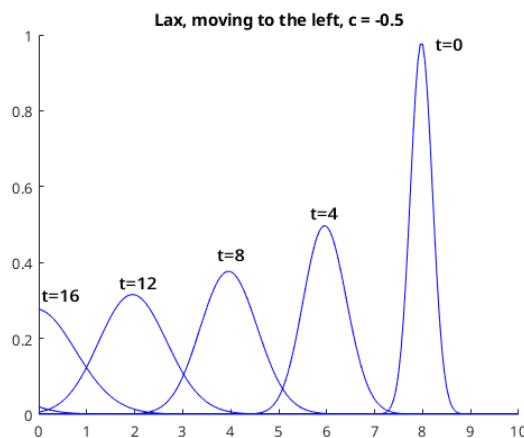
1 L = 10;
2 dx = 0.05;
3 x = 0:dx:L;
4
5 dt = 0.05;
6 T = 20;
7 t = 0:dt:T;
8
9 u = zeros(length(x), 1);
10 f = zeros(length(x), 1);
11
12 %% -----
13 c = 0.5; % moving to right
14
15 for k = 1:length(x)
16     u(k) = exp(-10*(x(k)-2).^2);
17 end
18

```

```

19 figure;
20 h1 = plot(0,0);
21 title('Lax, moving to the right c = 0.5')
22 ylim([0 1]);
23
24 for k = 1:length(t)
25     u = lax(u,f, c, dt, dx);
26     set(h1, 'XData',x, 'Ydata',u);
27     drawnow;
28 end
29
30 %% -----
31 c = -0.5; % moving to left
32
33 for k = 1:length(x)
34     u(k) = exp(-10*(x(k)-8).^2);
35 end
36
37 figure;
38 h2 = plot(0,0);
39 title('Lax, moving to the left, c = -0.5')
40 ylim([0 1]);
41
42 for k = 1:length(t)
43     u = lax(u,f, c, dt, dx);
44     set(h2, 'XData',x, 'Ydata',u);
45     drawnow;
46 end

```



## 1.5. The Lax-Wendroff Method<sup>6</sup>

We start from the Taylor series definition of  $f(x + \Delta x)$ :

$$f(x + \Delta x) = f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2!} f''(x) + \frac{\Delta x^3}{3!} f'''(x) + \dots \quad (19)$$

Thus, taking only up to the second order, we can write the following equations.

$$u(x + \Delta x, t) = u(x, t) + \Delta x u_x + \frac{\Delta x^2}{2} u_{xx} \quad (20)$$

The advection equation can then be approximated as follows

$$\begin{aligned}
 u_t &\approx -cu_x + f(x, t) \\
 \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} &\approx -c \frac{u(x + \Delta x, t) - u(x, t)}{\Delta x} + f(x, t) \\
 u(x, t + \Delta t) &\approx u(x, t) - c \frac{\Delta t}{\Delta x} [\textcolor{red}{u(x + \Delta x, t) - u(x, t)}] + f(x, t) \Delta t \\
 u(x, t + \Delta t) &\approx u(x, t) - c \frac{\Delta t}{\Delta x} \left[ \textcolor{red}{u(x, t) + \Delta x u_x + \frac{\Delta x^2}{2} u_{xx} - u(x, t)} \right] + f(x, t) \Delta t \\
 u(x, t + \Delta t) &\approx u(x, t) - c \Delta t u_x - c \frac{\Delta t \textcolor{green}{\Delta x}}{2} u_{xx} + f(x, t) \Delta t
 \end{aligned} \tag{21}$$

Since  $c$  is the propagation speed of  $u(x, t) = u_0(x - ct)$ , we can then take  $\Delta x = -c\Delta t$ . As the result, now we have the following equation.

$$u(x, t + \Delta t) \approx u(x, t) - c \Delta t u_x + c^2 \frac{\Delta t^2}{2} u_{xx} + f(x, t) \Delta t \tag{22}$$

Next, we need to approximate  $u_x$  and  $u_{xx}$ . For  $u_x$ , we use central difference method as follows.

$$u_x \approx \frac{u(x + \Delta x, t) - u(x - \Delta x, t)}{2\Delta x} \tag{23}$$

As for  $u_{xx}$ , we do not use the central difference method, instead, we take  $u_x \approx \frac{u(x + \Delta x, t) - u(x, t)}{\Delta x}$ . Therefore, we can express  $u_{xx}$  as follows.

$$\begin{aligned}
 u_{xx} &\approx \frac{u_x(x + \Delta x, t) - u_x(x, t)}{\Delta x} \\
 &\approx \frac{\frac{u(x + \Delta x, t) - u(x, t)}{\Delta x} - \frac{u(x, t) - u(x - \Delta x, t)}{\Delta x}}{\Delta x} \\
 &\approx \frac{u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t)}{(\Delta x)^2}
 \end{aligned} \tag{24}$$

Therefore, by substituting Eq. 23 and Eq. 24 to Eq. 22, we can get the following equations.

$$\begin{aligned}
 u(x, t + \Delta t) &\approx u(x, t) - c \Delta t \textcolor{red}{u_x} + c^2 \frac{\Delta t^2}{2} \textcolor{blue}{u_{xx}} + f(x, t) \Delta t \\
 &\approx u(x, t) - c \Delta t \left[ \frac{u(x + \Delta x, t) - u(x - \Delta x, t)}{2\Delta x} \right] + \\
 &\quad c^2 \frac{\Delta t^2}{2} \left[ \frac{u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t)}{\Delta x^2} \right] + f(x, t) \Delta t \\
 &\approx u(x, t) - \frac{c \Delta t}{2\Delta x} [u(x + \Delta x, t) - u(x - \Delta x, t)] + \\
 &\quad \frac{c^2 \Delta t^2}{2\Delta x^2} [u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t)] + f(x, t) \Delta t
 \end{aligned}$$

Consider  $r = \frac{c\Delta t}{\Delta x}$ ,  $U_{j,k} = u(x_j, t_k)$ ,  $F_{j,k} = f(x_j, t_k)$ ,  $x_j = j\Delta x$  and  $t_k = k\Delta t$ , we can have the following shorter terms:

$$U_{j,k+1} = U_{j,k} - \frac{r}{2}(U_{j+1,k} - U_{j-1,k}) + \frac{r^2}{2}(U_{j+1,k} - 2U_{j,k} + U_{j-1,k}) + F_{j,k} \Delta t \tag{25}$$

MATLAB implementation of Eq. 25 is as follows.

```

1 function next_u_array = lax_wendroff(u_array, f_array, c, dt, dx)
2
3 N = length(u_array);
4 next_u_array = u_array;
5
6 r = c*dt/dx;
7
8 for i = 2:N-1
9     next_u_array(i) = u_array(i)-0.5*r*(u_array(i+1)-u_array(i-1)) ...
10        + 0.5*r^2 * (u_array(i+1)-2*u_array(i)+u_array(i-1)) ...
11        + f_array(i)*dt;
12 end
13
14 % fill up the two ends by using upwind or downwind method
15 if c > 0
16     next_u_array(N) = u_array(N) - r* ...
17        (u_array(N)- u_array(N-1)) + f_array(N)*dt;
18
19 elseif c < 0
20     next_u_array(1) = u_array(1) - r* ...
21        (u_array(2) - u_array(1)) + f_array(1)*dt;
22
23 end

```

Here, we face the same issue as in the Lax method. The Lax-Wendroff method also leaves the two end segments untouched. This can be seen from the for loop that starts from 2 to N-1 (line 8). To address this issue, we can apply the upwind and downwind method for the two ends only.

To increase its efficiency, the MATLAB code above can be vectorized as follows.

```

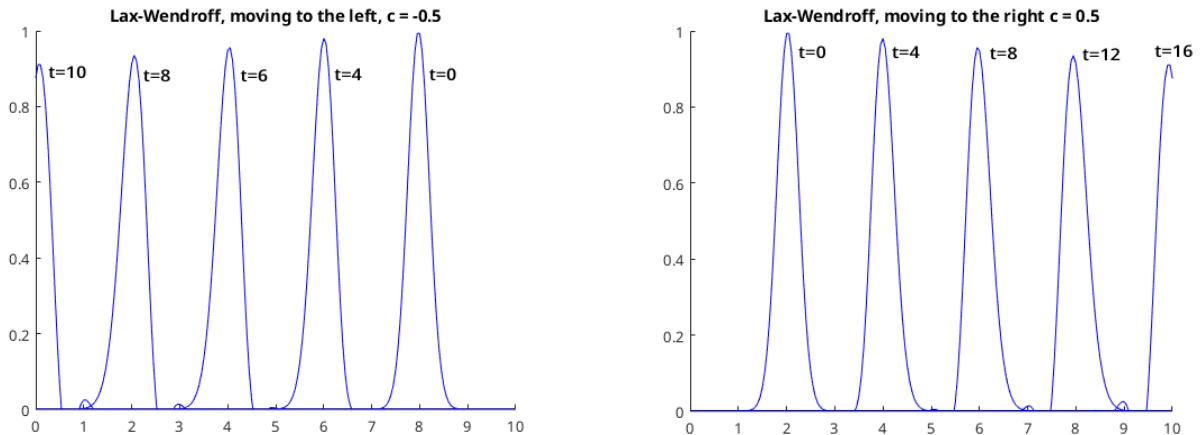
1 function next_u_array = lax_wendroff(u_array, f_array, c, dt, dx)
2
3 N = length(u_array);
4 next_u_array = u_array;
5
6 r = c*dt/dx;
7
8 next_u_array(2:N-1) = u_array(2:N-1)-0.5*r*(u_array(3:N)-u_array(1:N-2)) ...
9        + 0.5*r^2 * (u_array(3:N)-2*u_array(2:N-1)+u_array(1:N-2)) ...
10       + f_array(2:N-1)*dt;
11
12 % fill up the two ends by using upwind or downwind method
13 if c > 0
14     next_u_array(N) = u_array(N) - r* ...
15        (u_array(N)- u_array(N-1)) + f_array(N)*dt;
16
17 elseif c < 0
18     next_u_array(1) = u_array(1) - r* ...
19        (u_array(2) - u_array(1)) + f_array(1)*dt;
20
21 end

```

Next, we test the code above for both upwind and downwind direction. For the upwind direction, we will use the same

system as in Eq. 10. As for the downwind direction, we will also use the same system as in Eq. 14.

```
1 L = 10;
2 dx = 0.05;
3 x = 0:dx:L;
4
5 dt = 0.05;
6 T = 20;
7 t = 0:dt:T;
8
9 u = zeros(length(x), 1);
10 f = zeros(length(x), 1);
11
12 %% -----
13 c = 0.5; % moving to right
14
15 for k = 1:length(x)
16     u(k) = exp(-10*(x(k)-2).^2);
17 end
18
19 h = figure;
20 h1 = plot(0,0);
21 title('Lax-Wendroff, moving to the right c = 0.5')
22 ylim([0 1]);
23
24 for k = 1:length(t)
25     u = lax_wendroff(u,f, c, dt, dx);
26     set(h1,'XData',x,'Ydata',u);
27     drawnow
28 end
29
30 %% -----
31 c = -0.5; % moving to left
32
33 for k = 1:length(x)
34     u(k) = exp(-10*(x(k)-8).^2);
35 end
36
37 h = figure;
38 h1 = plot(0,0);
39 title('Lax-Wendroff, moving to the left, c = -0.5')
40 ylim([0 1]);
41
42 for k = 1:length(t)
43     u = lax_wendroff(u,f, c, dt, dx);
44     set(h1,'XData',x,'Ydata',u);
45     drawnow
46 end
```



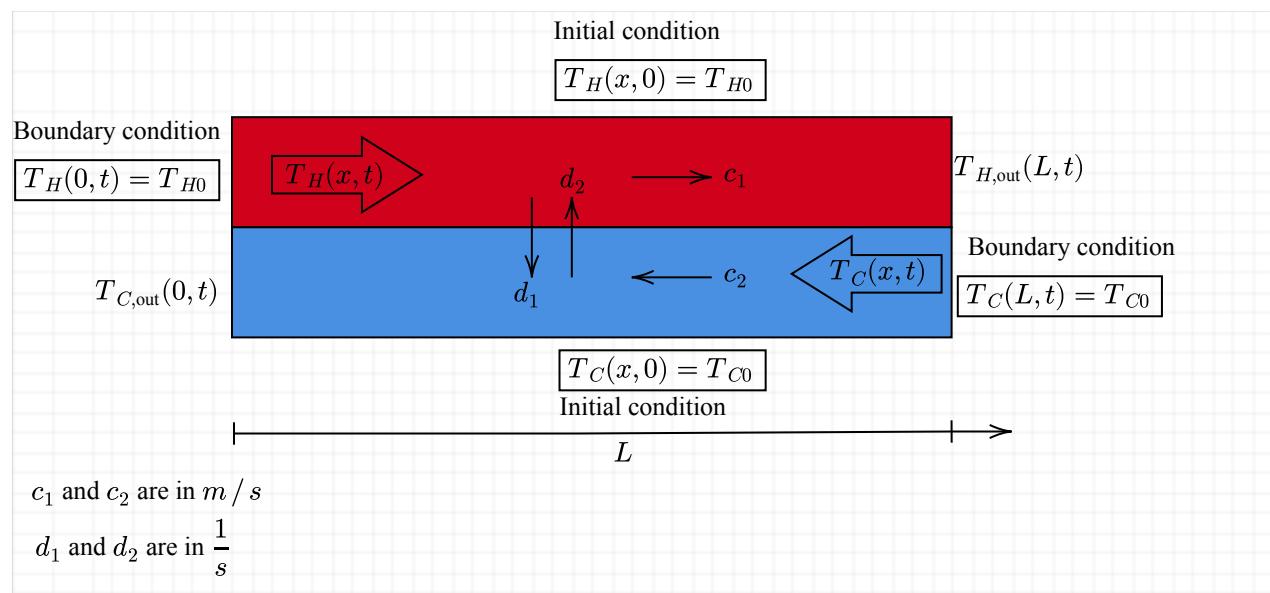
As can be seen from the two figures above, Lax-Wendroff method give a more stable solution. Overtime, the amplitude is still decreasing, however, the decrease in amplitude is much smaller than other methods that have been previously discussed.

## 1.6. Heat Exchanger as An Example

A heat exchanger is model with two or more interacting non-homogenous PDEs. The simplest model involves two PDEs only: the hot flow and the cold flow. Yet another simple method, three PDEs are involved: the hot flow, the cold flow, and the wall between them.

### 1.6.1. Two-Equation Model

Below is a diagram of a simple heat-exchanger. Hot fluid flows from left to right and cold fluid flows from left to right. Source temperature of the hot flow is locked at  $T_{H0}$ . While source temperature of the cold flow is locked at  $T_{C0}$ . Those two temperatures never change.



The system model of a heat exchanger based on the figure above can be described as follows.

$$\frac{\partial}{\partial t} T_H(x, t) = -c_1 \left( \frac{\partial}{\partial x} T_H(x, t) \right) - d_1(T_H(x, t) - T_C(x, t))$$

$$\frac{\partial}{\partial t} T_C(x, t) = c_2 \left( \frac{\partial}{\partial x} T_C(x, t) \right) + d_2(T_H(x, t) - T_C(x, t))$$

For the simulation, we will set the parameters according to<sup>7</sup>. First, we will solve the system above by using the upwind and downwind method. Next, we will use the Lax-Wendroff method and compare the results.

```

1  L = 1;
2  dx = 0.2;
3  x = 0:dx:L;
4
5  dt = 0.01;
6  T = 20;
7  t = 0:dt:T;
8
9  % Define the parameters
10 c1 = 0.001711593407;
11 c2 = -0.01785371429;
12 d1 = -0.03802197802;
13 d2 = 0.3954285714;
14
15 %% =====
16 % Upwind-downwind method
17 % =====
18
19 TH = zeros(length(x), 1);
20 TC = zeros(length(x), 1);
21 f = zeros(length(x), 1);
22
23 % Intial condition
24 for k = 1 : length(x)
25     TH(k) = 273+30;
26     TC(k) = 273+10;
27 end
28
29 figure;
30 hold on
31 h1 = plot(0,0,'r');
32 h2 = plot(0,0,'b');
33 title('Cross-current heat exchanger')
34 ylim([270 320])
35 xlabel('x')
36 ylabel('Temperature')
37 legend('Hot flow','Cold flow')
38
39 for k = 1 : length(t)
40     set(h1,'XData',x,'Ydata',TH);
41     set(h2,'XData',x,'Ydata',TC);
42     drawnow;
43
44     f = d1.* (TH-TC);
45     TH = upwind(TH, f, c1, dt, dx);

```

<sup>7</sup> PDE Observer Design for Counter-Current Heat Flows in a Heat-Exchange by F. Zobiri, et. al., published at IFAC-PapersOnLine

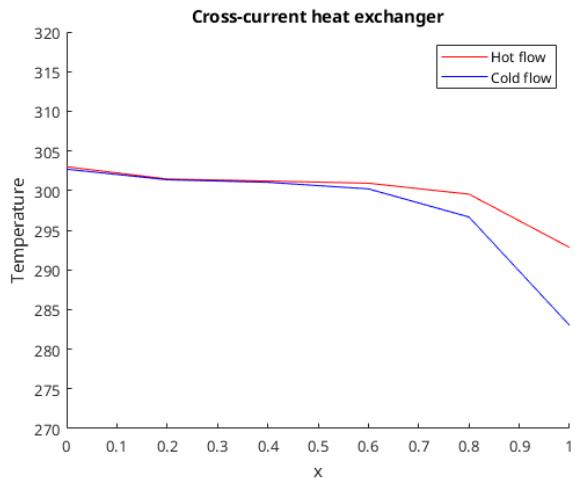
```

46
47     f = d2.* (TH-TC);
48     TC = downwind(TC, f, c2, dt, dx);
49 end
50
51 %% =====
52 % Lax-Wnderoff
53 % =====
54 TH = zeros(length(x), 1);
55 TC = zeros(length(x), 1);
56 f = zeros(length(x), 1);
57
58 % Intial conditoin
59 for k = 1 : length(x)
60     TH(k) = 273+30;
61     TC(k) = 273+10;
62 end
63
64 figure;
65 hold on
66 h3 = plot(0,0,'r');
67 h4 = plot(0,0,'b');
68 title('Cross-current heat exchanger')
69 ylim([270 320])
70 xlabel('x')
71 ylabel('Temperature')
72 legend('Hot flow','Cold flow')
73
74 for k = 1:length(t)
75
76     set(h3, 'XData',x, 'Ydata',TH);
77     set(h4, 'XData',x, 'Ydata',TC);
78     drawnow;
79
80     f = d1.* (TH-TC);
81     TH = lax_wendroff(TH, f, c1, dt, dx);
82     f = d2.* (TH-TC);
83     TC = lax_wendroff(TC, f, c2, dt, dx);
84 end

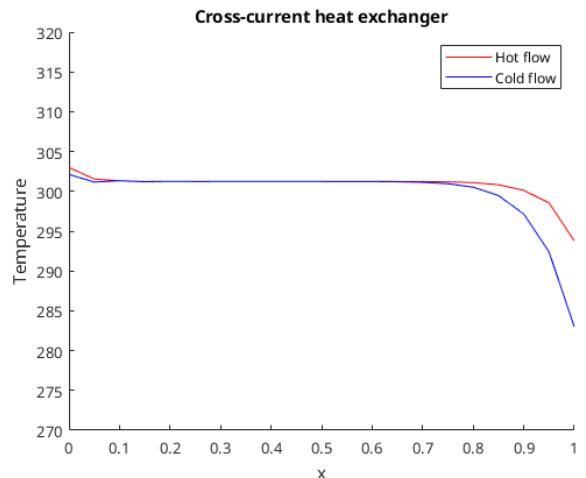
```

As can be seen in line 44 to 48, the hot flow uses the upwind method since the flow moves to the left to right. As for the cold flow, we use the downwind method since the flow moves from right to left. As for the Lax-Wendroff method, there is only one function that we need to call (line 83 to 86). However, we need to call that function twice, one for the cold flow, another one is for the hot flow. When executed, the code above will generate animations of how the temperatures evolve over time.

The two figures below show the temperature distribution of the heat exchanger after 20 seconds for the two different methods. As can be seen, Lax-Wendroff method generates flat curve in the mid-area of the heat exchanger. Theoretically, this makes more sense when compared with the result from the upwind-downwind method. However, if we think practically, upwind-and downwind method might be more desirable since a theoretically perfect advection might be impossible in real world.



Upwind-downwind method

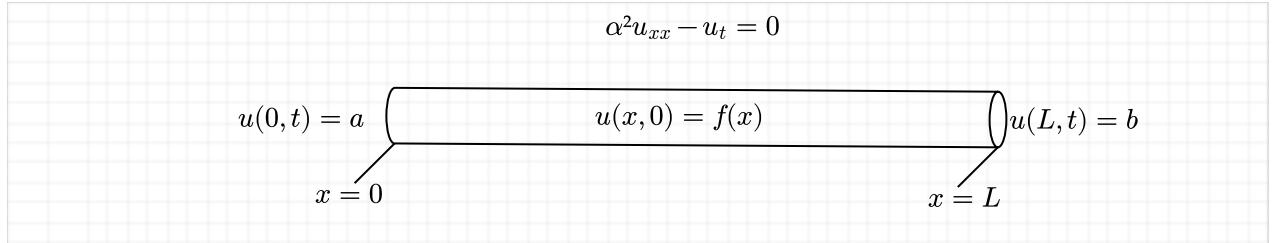


Lax-Wendroff method

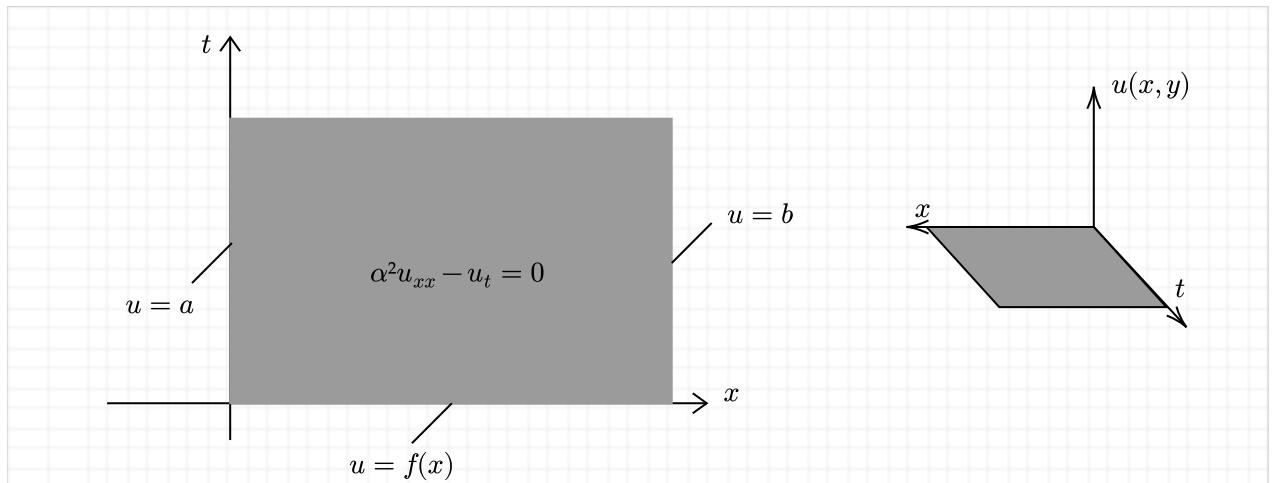
### 1.6.2. Three-Equation Model

## 2. Case 2: 1D-Rod<sup>8</sup>

Temperature distribution ( $u$ ) in a one-dimensional rod over a certain distance ( $x$ ) and time period ( $t$ ). Certain conditions ( $a, b$ ) are applied at both ends. These conditions remain true forever.



Top view of the temperature distribution along the rod over a certain period of time can be presented in a 2D-Cartesian plane as follows. This kind of plot typically done with color plot where every different color represents different temperature.



Given a temperature function of a 1-dimensional rod  $u(x, t)$  where  $u$  is the temperature at location  $x$  and at time  $t$ .  $\alpha$  is the diffusivity coefficient.

(26)

### 2.1. Explicit Method

With finite difference method, we can approach  $u_t$  and  $u_{xx}$ . To approximate  $u_t$ , we use the following approximation.

$$u_t \approx \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} \quad (27)$$

$$u_x \approx \frac{u(x + \Delta x, t) - u(x, t)}{\Delta x} \quad (28)$$

While to approach  $u_{xx}$ , we use the following approximation.

$$\begin{aligned} u_{xx} &\approx \frac{u_x(x + \Delta x, t) - u_x(x, t)}{\Delta x} \\ &\approx \frac{\frac{u(x + \Delta x, t) - u(x, t)}{\Delta x} - \frac{u(x, t) - u(x - \Delta x, t)}{\Delta x}}{\Delta x} \end{aligned}$$

$$\approx \frac{u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t)}{(\Delta x)^2} \quad (29)$$

Substituting (4) to (1), we can get the following equations.

$$\begin{aligned} \frac{u_t}{\Delta t} &\approx \alpha^2 u_{xx} \\ \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} &\approx \alpha^2 \frac{u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t)}{(\Delta x)^2} \\ u(x, t + \Delta t) - u(x, t) &\approx \frac{\alpha^2 \Delta t}{(\Delta x)^2} (u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t)) \\ u(x, t + \Delta t) &\approx \frac{\alpha^2 \Delta t}{(\Delta x)^2} u(x + \Delta x, t) + \left(1 - 2 \frac{\alpha^2 \Delta t}{(\Delta x)^2}\right) u(x, t) + \frac{\alpha^2 \Delta t}{(\Delta x)^2} u(x - \Delta x, t) \\ u(x, t + \Delta t) &\approx r u(x + \Delta x, t) + (1 - 2r) u(x, t) + r u(x - \Delta x, t), \quad r = \frac{\alpha^2 \Delta t}{(\Delta x)^2} \end{aligned} \quad (30)$$

Consider  $U_{jk} = u(x_j, t_k)$ ,  $x_j = j\Delta x$ , and  $t_k = k\Delta t$ , we can have the following shorter terms.

$$U_{j,k+1} = u(x, t + \Delta t) \quad (31)$$

$$U_{j-1,k} = u(x - \Delta x, t) \quad (32)$$

$$U_{j,k} = u(x, t) \quad (33)$$

Therefore, additionally taking  $r = \alpha^2 \frac{\Delta t}{(\Delta x)^2}$ , we can shorten the equation (21) into as follows.

$$\boxed{U_{j,k+1} = r U_{j-1,k} + (1 - 2r) U_{j,k} + r U_{j+1,k}} \quad (34)$$

## 2.2. Implicit Method

The principal idea of an implicit method is to calculate  $u_{xx}$  not only at the current time, but also at the time one step ahead.

$$u_{xx}(x, t) = (1 - \theta) u_{xx}(x, t) + \theta u_{xx}(x, t + \Delta t), \quad 0 \leq \theta \leq 1 \quad (35)$$

where  $\theta$  acts as a weighting factor. When  $\theta = 0$ , (10) becomes similar to (4) and what we have here is an explicit method that is similar to Forward Euler Method. On the other hand, when  $\theta = 1$ , what we have here is a very common explicit method that is similar to Backward Euler method. When  $\theta = 0.5$ , the method is called Crank-Nicolson method. As in the Greenberg's book, we will focus the derivation of the Crank-Nicolson method.

$$\begin{aligned} u_{xx} &= \frac{1}{2} \frac{u_x(x + \Delta x, t) - u_x(x, t)}{\Delta x} + \frac{1}{2} \frac{u_x(x + \Delta x, t + \Delta t) - u_x(x, t + \Delta t)}{\Delta x} \\ &= \frac{\frac{u(x + \Delta x, t) - u(x, t)}{\Delta x} - \frac{u(x, t) - u(x - \Delta x, t)}{\Delta x}}{2\Delta x} + \frac{\frac{u(x + \Delta x, t + \Delta t) - u(x, t + \Delta t)}{\Delta x} - \frac{u(x, t + \Delta t) - u(x - \Delta x, t + \Delta t)}{\Delta x}}{2\Delta x} \\ &= \frac{u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t)}{2(\Delta x)^2} \\ &\quad + \frac{u(x + \Delta x, t + \Delta t) - 2u(x, t + \Delta t) + u(x - \Delta x, t + \Delta t)}{2(\Delta x)^2} \\ &= \frac{U_{j+1,k} - 2U_{j,k} + U_{j-1,k} + U_{j+1,k+1} - 2U_{j,k+1} + U_{j-1,k+1}}{2(\Delta x)^2} \end{aligned} \quad (36)$$

Substituting (29) back to  $\alpha^2 u_{xx} - u_t = 0$  gives us following equation.

$$\frac{u_t = \alpha^2 u_{xx}}{\frac{U_{j,k+1} - U_{j,k}}{\Delta t} = \alpha^2 \frac{U_{j+1,k} - 2U_{j,k} + U_{j-1,k} + U_{j+1,k+1} - 2U_{j,k+1} + U_{j-1,k+1}}{2(\Delta x)^2}}$$
(37)

Since  $r = \alpha^2 \frac{\Delta t}{(\Delta x)^2}$ , (12) can then be simplified as:

$$\begin{aligned} u_t &= \alpha^2 u_{xx} \\ U_{j,k+1} - U_{j,k} &= \frac{r}{2}(U_{j+1,k} - 2U_{j,k} + U_{j-1,k} + U_{j+1,k+1} - 2U_{j,k+1} + U_{j-1,k+1}) \\ U_{j,k+1} + rU_{j,k+1} &= \frac{r}{2}U_{j+1,k} + (1-r)U_{j,k} + \frac{r}{2}U_{j-1,k} + \frac{r}{2}U_{j+1,k+1} + \frac{r}{2}U_{j-1,k+1} \\ 2(1+r)U_{j,k+1} &= rU_{j+1,k} + 2(1-r)U_{j,k} + rU_{j-1,k} + rU_{j+1,k+1} + rU_{j-1,k+1} \\ -rU_{j-1,k+1} + 2(1+r)U_{j,k+1} - rU_{j+1,k+1} &= rU_{j-1,k} + 2(1-r)U_{j,k} + rU_{j+1,k} \end{aligned}$$
(38)

Thus, the well-known Crank-Nicolson scheme can be expressed as follows.

$$[-rU_{j-1,k+1} + 2(1+r)U_{j,k+1} - rU_{j+1,k+1} = rU_{j-1,k} + 2(1-r)U_{j,k} + rU_{j+1,k}]$$
(39)

In matrix form:

$$\underbrace{\begin{bmatrix} -r & 2(1+r) & -r & & \cdots & 0 \\ 0 & -r & 2(1+r) & -r & & \vdots \\ & & -r & 2(1+r) & -r & \\ & & & \ddots & & \\ \vdots & & & & -r & 2(1+r) \\ 0 & \cdots & & & 0 & -r \end{bmatrix}}_A \underbrace{\begin{bmatrix} p((k+1)\Delta t) \\ U_{1,k+1} \\ U_{2,k+1} \\ \vdots \\ U_{N-2,k+1} \\ q((k+1)\Delta t) \end{bmatrix}}_U =$$
  

$$\underbrace{\begin{bmatrix} r & 2(1-r) & r & & \cdots & 0 \\ 0 & r & 2(1-r) & r & & \vdots \\ & & r & 2(1-r) & r & \\ & & & \ddots & & \\ \vdots & & & & r & 2(1-r) \\ 0 & \cdots & & & 0 & r \end{bmatrix}}_B \underbrace{\begin{bmatrix} p(k\Delta t) \\ U_{1,k} \\ U_{2,k} \\ \vdots \\ U_{N-2,k} \\ q(k\Delta t) \end{bmatrix}}_U$$
(40)

Red means the values are known. In (15), we can notice that some known values reside in  $U$ . We must remove these known values from  $U$  and put them to the right side. The goal is to have a perfect  $AU = B$  system where all the unknowns are all in  $U$ .

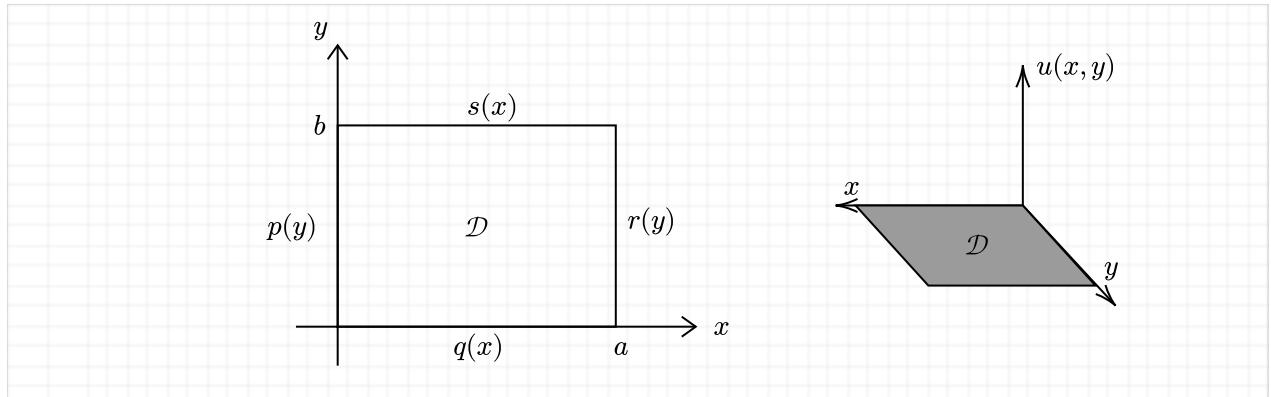
$$\underbrace{\begin{bmatrix} 2(1+r) & -r & & \cdots & 0 \\ -r & 2(1+r) & -r & & \vdots \\ & & \ddots & & \\ & & -r & 2(1+r) & -r \\ & & & -r & 2(1+r) \end{bmatrix}}_A \underbrace{\begin{bmatrix} U_{1,k+1} \\ U_{2,k+1} \\ \vdots \\ U_{N-2,k+1} \\ U_{N-1,k+1} \end{bmatrix}}_{U_{new}} =$$

$$\underbrace{\begin{bmatrix} 2(1-r) & r & & \cdots & & \cdots \\ r & 2(1-r) & r & & & \\ & \ddots & & & & \\ & & r & 2(1-r) & r & \\ & & & r & 2(1-r) & \\ \cdots & & & & & 2(1-r) \end{bmatrix}}_b \underbrace{\begin{bmatrix} U_{1,k} \\ U_{2,k} \\ \vdots \\ U_{N-2,k} \\ U_{N-1,k} \end{bmatrix}}_{U_{now}} + \underbrace{\begin{bmatrix} rp((k+1)\Delta t) + rp(k\Delta t) \\ 0 \\ \vdots \\ 0 \\ rq((k+1)\Delta t) + rq(k\Delta t) \end{bmatrix}}_c$$
(41)

Now, we can find  $U_{new} = A^{-1}B$ . In real implementation, this might not be plausible and we have to perform an iteration-based algorithm to find the optimal  $U_{new}$ . In Maple, we can use the  $U_{new} = \text{LinearSolve}(A,B)$  command. In MATLAB, we can use the popular  $U_{new} = A \setminus B$  command. Other alternatives are using the Jacobi method and the Gauss-Siedel method as explained the Greenberg's book.

### 3. Case 3: Rectangular 2D Plane<sup>9</sup>

#### 3.1. The Five-Point Difference Method



Finite difference method for the system above is as follows.

#### 3.2. Energy Balance Model: Out-of-Plane Heat Transfer for a Thin Plate - Comparison with COMSOL