Pendulum's equation of motion

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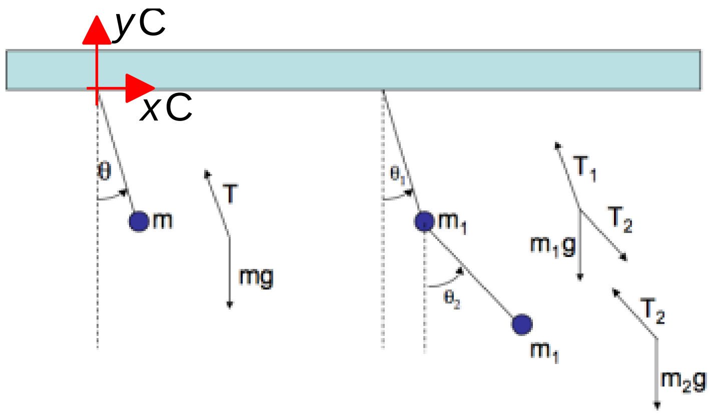


Planar Single Pendulum

Newton's Equations

Constraints

These are the kinematics of a single pendulum:

$$r(t) = \begin{bmatrix} l\sin(\theta(t)) \\ -l\cos(\theta(t)) \end{bmatrix}$$

$$\frac{d}{dt}\, r(t) = \begin{bmatrix} l\,\dot{\theta}(t)\,\cos(\theta(t)) \\ l\,\dot{\theta}(t)\,\sin(\theta(t)) \end{bmatrix}$$

$$v = \frac{d}{dt}\, r(t)$$

$$a = \frac{d^2}{dt^2}\, r(t)$$

Forces

Forces acting on the pendulum F are contributed by the tension T and mg. This can be written as: $F = T\frac{-r}{|r|} - m\begin{pmatrix} 0 \\ g \end{pmatrix}$

$$F = \begin{bmatrix} -T\sin(\theta(t)) \\ T\cos(\theta(t)) - m\,g \end{bmatrix}$$

Equations of motion:

The Newton's law for motion: $F = ma = m\ddot{r}(t)$, we have:

$$\begin{bmatrix} -T\sin(\theta(t)) \\ T\cos(\theta(t)) - m\,g \end{bmatrix} = \begin{bmatrix} m\left(l\,\ddot{\theta}(t)\cos(\theta(t)) - l\,\dot{\theta}(t)^2\sin(\theta(t))\right) \\ m\left(l\,\ddot{\theta}(t)\sin(\theta(t)) + l\,\dot{\theta}(t)^2\cos(\theta(t))\right) \end{bmatrix}$$

We now have two equations for the two unknowns, T and θ. However, we don't need T here. To remove T, we multiply the first equation with $\cos(\theta)$:

$$-T\sin(\theta(t))\cos(\theta(t)) = m\,l\left(-\dot{\theta}(t)^2\sin(\theta(t)) + \ddot{\theta}(t)\cos(\theta(t))\right)\cos(\theta(t))$$

and the second equation with $\sin(\theta)$:

$$\left(T\cos(\theta(t)) - m\,g\right)\sin(\theta(t)) = m\,l\left(\dot{\theta}(t)^2\cos(\theta(t)) + \ddot{\theta}(t)\sin(\theta(t))\right)\sin(\theta(t))$$

After that we add them both so that T is gone, and we write the final equation in $\ddot{\theta}(t)$:

$$-m\,g\sin(\theta(t)) = m\,l\,\ddot{\theta}(t)$$

$$\ddot{\theta}(t) = -\frac{g\sin(\theta(t))}{l}$$

Lagrange's Equations

Let us now try the Lagrange's method.
First, we compute the potential energy (V) and the kinetic energy (T). As for the kinetic energy, it can be expressed as: $T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{\theta}l)^2 = \frac{1}{2}ml^2\dot{\theta}^2$

Computing the Lagrangian:

$$T = \frac{m\,l^2\,\dot{\theta}(t)^2}{2}$$

$$V = -m\,g\,l\cos(\theta(t))$$

$$L = T - V$$

$$L = \frac{m\,l^2\,\dot{\theta}(t)^2}{2} + m\,g\,l\cos(\theta(t))$$

The Lagrange equation is:

$$\frac{\partial^2}{\partial \frac{\mathrm{d}}{\mathrm{d}t}\theta(t)\,\partial t} L - \frac{\partial}{\partial\theta(t)} L = 0$$

$$m\,l^2\,\ddot{\theta}(t) + m\,g\,l\sin(\theta(t)) = 0$$

Thus:

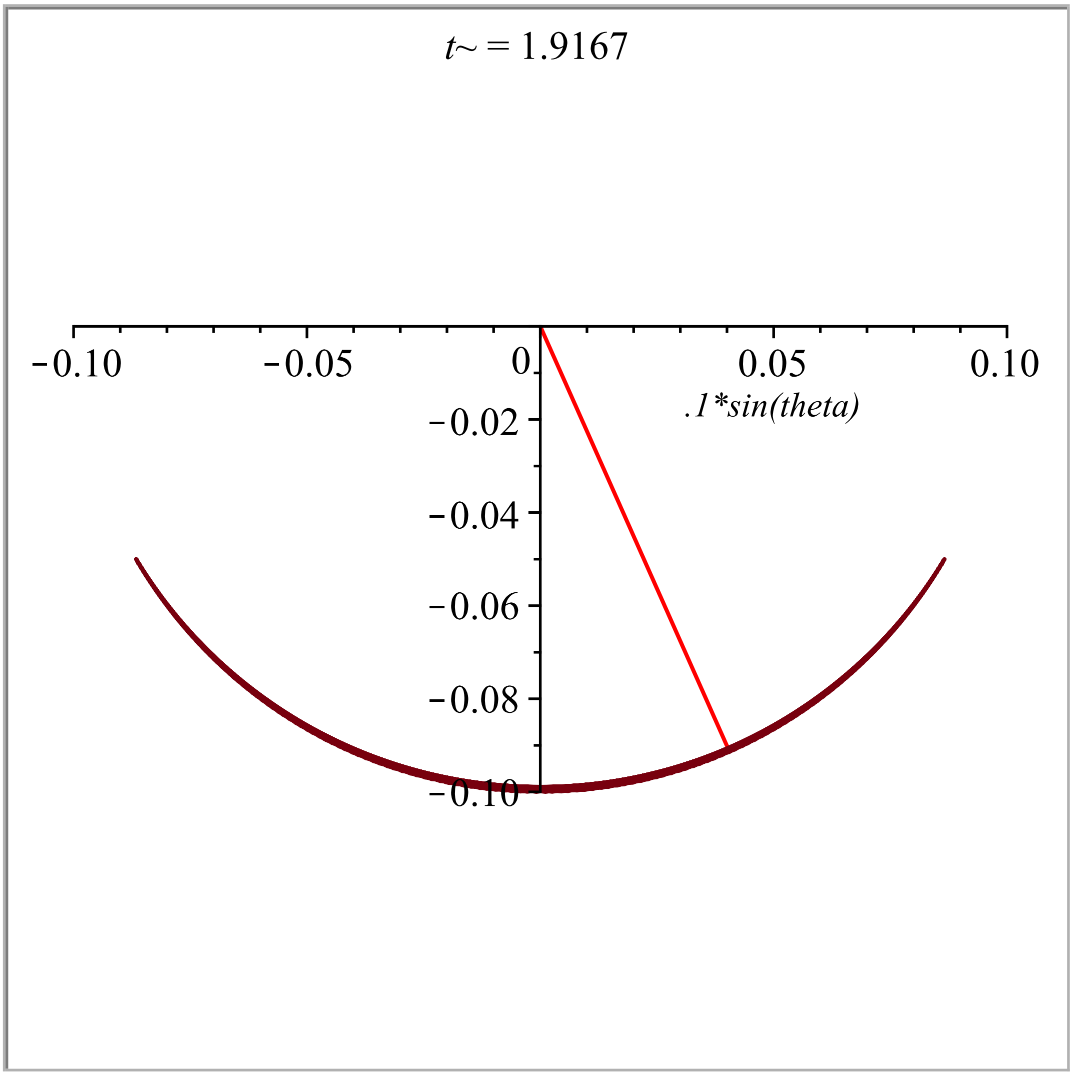
$$\ddot{\theta}(t) = -\frac{g\sin(\theta(t))}{l}$$

The result is exactly the same as with the Newton's equation.

Numerical Simulation

We are going to simulate a single pendulum with the following initial condition:

$\theta(0) = \frac{\pi}{3}, \dot{\theta}(0) = 0$



Planar Double Pendulum

Newton's Equations

Constraints

These are the kinematic equations of a double pendulum:

The first link:

$$r_1(t) = \begin{bmatrix} l_1 \sin(\theta_1(t)) \\ -l_1 \cos(\theta_1(t)) \end{bmatrix}$$

$$\frac{d}{dt} r_1(t) = \begin{bmatrix} l_1 \dot{\theta}_1(t) \cos(\theta_1(t)) \\ l_1 \dot{\theta}_1(t) \sin(\theta_1(t)) \end{bmatrix}$$

$$\frac{d^2}{dt^2} r_1(t) = \begin{bmatrix} l_1 \ddot{\theta}_1(t) \cos(\theta_1(t)) - l_1 \dot{\theta}_1(t)^2 \sin(\theta_1(t)) \\ l_1 \ddot{\theta}_1(t) \sin(\theta_1(t)) + l_1 \dot{\theta}_1(t)^2 \cos(\theta_1(t)) \end{bmatrix}$$

$$v_1 = \frac{d}{dt} r_1(t)$$

$$a_1 = \frac{d^2}{dt^2} r_1(t)$$

The second link:

$$r_2(t) = \begin{bmatrix} l_1 \sin(\theta_1(t)) + l_2 \sin(\theta_2(t)) \\ -l_1 \cos(\theta_1(t)) - l_2 \cos(\theta_2(t)) \end{bmatrix}$$

$$\frac{d}{dt} r2(t) = \begin{bmatrix} l_1 \dot{\theta}_1(t) \cos(\theta_1(t)) + l_2 \dot{\theta}_2(t) \cos(\theta_2(t)) \\ l_1 \dot{\theta}_1(t) \sin(\theta_1(t)) + l_2 \dot{\theta}_2(t) \sin(\theta_2(t)) \end{bmatrix}$$

$$\frac{d^2}{dt^2} r2(t)$$

$$= \Big[\Big[l_1 \ddot{\theta}_1(t) \cos(\theta_1(t)) - l_1 \dot{\theta}_1(t)^2 \sin(\theta_1(t)) + l_2 \ddot{\theta}_2(t) \cos(\theta_2(t)) - l_2 \dot{\theta}_2(t)^2 \sin(\theta_2(t)) \Big], \Big[l_1 \ddot{\theta}_1(t) \sin(\theta_1(t)) + l_1 \dot{\theta}_1(t)^2 \cos(\theta_1(t)) + l_2 \ddot{\theta}_2(t) \sin(\theta_2(t)) + l_2 \dot{\theta}_2(t)^2 \cos(\theta_2(t)) \Big] \Big]$$

$$v_2 = \frac{d}{dt} r_2(t)$$

$$a_2 = \frac{d^2}{dt^2} r_2(t)$$

Forces

Forces acting on the pendulum F are contributed by the tension T_1, T_2 and mg. This can be written as: $F_1 = T_1 \frac{-r_1}{|r_1|} + T_2 \frac{r_2 - r_1}{|r_2 - r_1|} - m_1 \begin{pmatrix} 0 \\ g \end{pmatrix}$ and $F_2 = -T_2 \frac{r_2 - r_1}{|r_2 - r_1|} - m_2 \begin{pmatrix} 0 \\ g \end{pmatrix}$.

$$F_1 = \begin{bmatrix} -T_1 \sin(\theta_1(t)) + T_2 \sin(\theta_2(t)) \\ T_1 \cos(\theta_1(t)) - T_2 \cos(\theta_2(t)) - m_1 g \end{bmatrix}$$

$$F_2 = \begin{bmatrix} -T_2 \sin(\theta_2(t)) \\ T_2 \cos(\theta_2(t)) - m_2 g \end{bmatrix}$$

$$Since\,, F = m \left(\frac{\partial^2}{\partial t^2} r \right)$$

$$\begin{bmatrix} -T_1 \sin(\theta_1(t)) + T_2 \sin(\theta_2(t)) \\ T_1 \cos(\theta_1(t)) - T_2 \cos(\theta_2(t)) - m_1 g \end{bmatrix}$$

$$= \begin{bmatrix} m_1 \left(l_1 \ddot{\theta}_1(t) \cos(\theta_1(t)) - l_1 \dot{\theta}_1(t)^2 \sin(\theta_1(t)) \right) \\ m_1 \left(l_1 \ddot{\theta}_1(t) \sin(\theta_1(t)) + l_1 \dot{\theta}_1(t)^2 \cos(\theta_1(t)) \right) \end{bmatrix}$$

$$\begin{bmatrix} -T_2 \sin(\theta_2(t)) \\ T_2 \cos(\theta_2(t)) - m_2 g \end{bmatrix} = \Big[\Big[m_2 \left(l_1 \ddot{\theta}_1(t) \cos(\theta_1(t)) - l_1 \dot{\theta}_1(t)^2 \sin(\theta_1(t)) + l_2 \ddot{\theta}_2(t) \cos(\theta_2(t)) - l_2 \dot{\theta}_2(t)^2 \sin(\theta_2(t)) \right)\Big],$$
$$\Big[m_2 \left(l_1 \ddot{\theta}_1(t) \sin(\theta_1(t)) + l_1 \dot{\theta}_1(t)^2 \cos(\theta_1(t)) + l_2 \ddot{\theta}_2(t) \sin(\theta_2(t)) + l_2 \dot{\theta}_2(t)^2 \cos(\theta_2(t)) \right)\Big]\Big]$$

We now have 4 sets of equations **Function Call**:

-- Multiply the first equation with $\cos(\theta_1)$ and the second equation with $\sin(\theta_1)$. Add both results.

-- Multiply the first equation with $\sin(\theta_1)$ and the second equation with $\cos(\theta_1)$. Substract both results.

-- Multiply the first equation with $\sin(\theta_2)$ and the second equation with $\cos(\theta_2)$. Substract both results.

-- Multiply the first equation with $\sin(\theta_2)$ and the second equation with $\cos(\theta_2)$. Substract both results.

$$\Big[\Big[-T_2 \sin(-\theta_2(t) + \theta_1(t)) - \sin(\theta_1(t))\, g\, m_1 = m_1\, \ddot{\theta}_1(t)\, l_1\Big],$$
$$\Big[T_2 \cos(-\theta_2(t) + \theta_1(t)) - T_1 + \cos(\theta_1(t))\, g\, m_1 = -m_1\, \dot{\theta}_1(t)^2\, l_1\Big],$$
$$\Big[-\sin(\theta_2(t))\, m_2\, g = -\dot{\theta}_1(t)^2 \sin(-\theta_2(t) + \theta_1(t))\, l_1\, m_2 + \cos(-\theta_2(t) + \theta_1(t))\, \ddot{\theta}_1(t)\, l_1\, m_2 + \ddot{\theta}_2(t)\, l_2\, m_2\Big],$$
$$\Big[-T_2 + \cos(\theta_2(t))\, g\, m_2 = -\cos(-\theta_2(t) + \theta_1(t))\, \dot{\theta}_1(t)^2\, l_1\, m_2 - \ddot{\theta}_1(t) \sin(-\theta_2(t) + \theta_1(t))\, l_1\, m_2 - \dot{\theta}_2(t)^2\, l_2\, m_2\Big]\Big]$$

The first and the second equation above **Function Call** can be rewritten in $\frac{d^2}{dt^2}\, \theta_1(t)$ and in $\frac{d^2}{dt^2}\, \theta_1(t)$. The results are then substituted to the other two equations:

$$\left[\begin{array}{c}
\ddot{\theta}_1(t) = -\frac{T_2 \sin\left(-\theta_2(t)+\theta_1(t)\right) + \sin\left(\theta_1(t)\right) g\, m_1}{m_1\, l_1} \\
\dot{\theta}_1(t)^2 = \frac{-T_2 \cos\left(-\theta_2(t)+\theta_1(t)\right) + T_1 - \cos\left(\theta_1(t)\right) g\, m_1}{m_1\, l_1} \\
\ddot{\theta}_2(t) = \frac{\sin\left(-\theta_2(t)+\theta_1(t)\right) T_1}{l_2\, m_1} \\
\dot{\theta}_2(t)^2 = \frac{-\cos\left(-\theta_2(t)+\theta_1(t)\right) T_1\, m_2 + T_2\, m_1 + T_2\, m_2}{m_2\, l_2\, m_1}
\end{array}\right]$$

From

$$\left[\begin{array}{c}
\ddot{\theta}_1(t) = -\frac{T_2 \sin\left(-\theta_2(t)+\theta_1(t)\right) + \sin\left(\theta_1(t)\right) g\, m_1}{m_1\, l_1} \\
\dot{\theta}_1(t)^2 = \frac{-T_2 \cos\left(-\theta_2(t)+\theta_1(t)\right) + T_1 - \cos\left(\theta_1(t)\right) g\, m_1}{m_1\, l_1} \\
\ddot{\theta}_2(t) = \frac{\sin\left(-\theta_2(t)+\theta_1(t)\right) T_1}{l_2\, m_1} \\
\dot{\theta}_2(t)^2 = \frac{-\cos\left(-\theta_2(t)+\theta_1(t)\right) T_1\, m_2 + T_2\, m_1 + T_2\, m_2}{m_2\, l_2\, m_1}
\end{array}\right]$$

, the first and the third equations can be rewritten as follows to get T_1 and T_2:

$$T_2 = \frac{-m_1\, \ddot{\theta}_1(t)\, l_1 - \sin\left(\theta_1(t)\right) g\, m_1}{\sin\left(-\theta_2(t)+\theta_1(t)\right)}$$

$$T_1 = \frac{\ddot{\theta}_2(t)\, l_2\, m_1}{\sin\left(-\theta_2(t)+\theta_1(t)\right)}$$

Equations of Motion

T_1 and T_2 are then sustituted to the second and fourth equations of

$$\left[\begin{array}{c}
\ddot{\theta}_1(t) = -\frac{T_2 \sin\left(-\theta_2(t)+\theta_1(t)\right) + \sin\left(\theta_1(t)\right) g\, m_1}{m_1\, l_1} \\
\dot{\theta}_1(t)^2 = \frac{-T_2 \cos\left(-\theta_2(t)+\theta_1(t)\right) + T_1 - \cos\left(\theta_1(t)\right) g\, m_1}{m_1\, l_1} \\
\ddot{\theta}_2(t) = \frac{\sin\left(-\theta_2(t)+\theta_1(t)\right) T_1}{l_2\, m_1} \\
\dot{\theta}_2(t)^2 = \frac{-\cos\left(-\theta_2(t)+\theta_1(t)\right) T_1\, m_2 + T_2\, m_1 + T_2\, m_2}{m_2\, l_2\, m_1}
\end{array}\right]$$

. These two equations are the final equations of motion of a planar double pendulum:

$$\dot{\theta}_1(t)^2 = \frac{\sin(\theta_2(t))\, g + \cos(-\theta_2(t) + \theta_1(t))\, \ddot{\theta}_1(t)\, l_1 + \ddot{\theta}_2(t)\, l_2}{l_1 \sin(-\theta_2(t) + \theta_1(t))}$$

$$\dot{\theta}_2(t)^2 = \frac{-\cos(-\theta_2(t) + \theta_1(t))\, \ddot{\theta}_2(t)\, l_2\, m_2 - (m_1 + m_2)\, (g \sin(\theta_1(t)) + \ddot{\theta}_1(t)\, l_1)}{m_2\, l_2 \sin(-\theta_2(t) + \theta_1(t))}$$

Lagrange's equation

We now relize how complicated it is to use the Newton's method to derive the equations of motion of a double pendulum. Let us now try the Lagrange's method.
First, we compute the potential energy (V) and the kinetic energy (K).

$$K = \frac{1}{2}\, m_1\, v_1^2 + \frac{1}{2}\, m_2\, v_2^2$$

Where:

$$v_1 = \left\| \left(\frac{d}{dt}\, r_1(t) \right) \right\|_2$$

$$v_1 = \sqrt{l_1^2\, \dot{\theta}_1(t)^2 \cos(\theta_1(t))^2 + l_1^2\, \dot{\theta}_1(t)^2 \sin(\theta_1(t))^2}$$

and

$$v_2 = \left\| \left(\frac{d}{dt}\, r_2(t) \right) \right\|_2$$

$$v_2 = \left(\left(l_1\, \dot{\theta}_1(t) \cos(\theta_1(t)) + l_2\, \dot{\theta}_2(t) \cos(\theta_2(t)) \right)^2 + \left(l_1\, \dot{\theta}_1(t) \sin(\theta_1(t)) + l_2\, \dot{\theta}_2(t) \sin(\theta_2(t)) \right)^2 \right)^{1/2}$$

Therefore:

$$K = \frac{\dot{\theta}_1(t)^2\, l_1^2\, m_1}{2} + \frac{\dot{\theta}_1(t)^2\, l_1^2\, m_2}{2} + \dot{\theta}_1(t)\, \dot{\theta}_2(t)\, l_1\, l_2\, m_2 \cos(-\theta_2(t) + \theta_1(t)) + \frac{\dot{\theta}_2(t)^2\, l_2^2\, m_2}{2}$$

and

$$V = -m_1\, g\, l_1 \cos(\theta_1(t)) - m_2\, g\, (l_1 \cos(\theta_1(t)) + l_2 \cos(\theta_2(t)))$$

And then we compute the Lagrangian (L):

$$L = K - V$$

$$L = \frac{\dot{\theta}_1(t)^2\, l_1^2\, m_1}{2} + \frac{\dot{\theta}_1(t)^2\, l_1^2\, m_2}{2} + \dot{\theta}_1(t)\, \dot{\theta}_2(t)\, l_1\, l_2\, m_2 \cos(-\theta_2(t) + \theta_1(t)) + \frac{\dot{\theta}_2(t)^2\, l_2^2\, m_2}{2} + m_1\, g\, l_1 \cos(\theta_1(t)) + m_2\, g\, (l_1 \cos(\theta_1(t)) + l_2 \cos(\theta_2(t)))$$

Finally, we can calculate the Lagrange's equations:

The Lagrange equations are:

$$\frac{\partial^2}{\partial \frac{d}{dt}\theta(t)\partial t} L - \frac{\partial}{\partial\theta(t)} L = 0$$

$$\Big[\Big[\Big(\cos\big(-\theta_2(t) + \theta_1(t)\big)\ \ddot{\theta}_2(t)\ l_2\, m_2 + \dot{\theta}_2(t)^2\ l_2\, m_2 \sin\big(-\theta_2(t) + \theta_1(t)\big) + (m_1 + m_2)\ \big(g \sin\big(\theta_1(t)\big) + \ddot{\theta}_1(t)\ l_1\big)\Big)\ l_1 = 0\Big],$$
$$\Big[m_2\, l_2\ \Big(-\dot{\theta}_1(t)^2\ l_1 \sin\big(-\theta_2(t) + \theta_1(t)\big) + \cos\big(-\theta_2(t) + \theta_1(t)\big)\ \ddot{\theta}_1(t)\ l_1 + \sin\big(\theta_2(t)\big)\ g + \ddot{\theta}_2(t)\ l_2\Big) = 0\Big]\Big]$$

Let's rewrite the equations in form of: $\left(\frac{d}{dt}\theta_2(t)\right)^2$ and $\left(\frac{d}{dt}\theta_1(t)\right)^2$:

$$\dot{\theta}_2(t)^2 = \frac{-\cos\big(-\theta_2(t) + \theta_1(t)\big)\ \ddot{\theta}_2(t)\ l_2\, m_2 - (m_1 + m_2)\ \big(g \sin\big(\theta_1(t)\big) + \ddot{\theta}_1(t)\ l_1\big)}{m_2\, l_2 \sin\big(-\theta_2(t) + \theta_1(t)\big)}$$

$$\dot{\theta}_1(t)^2 = -\frac{-\cos\big(-\theta_2(t) + \theta_1(t)\big)\ \ddot{\theta}_1(t)\ l_1 - \sin\big(\theta_2(t)\big)\ g - \ddot{\theta}_2(t)\ l_2}{l_1 \sin\big(-\theta_2(t) + \theta_1(t)\big)}$$

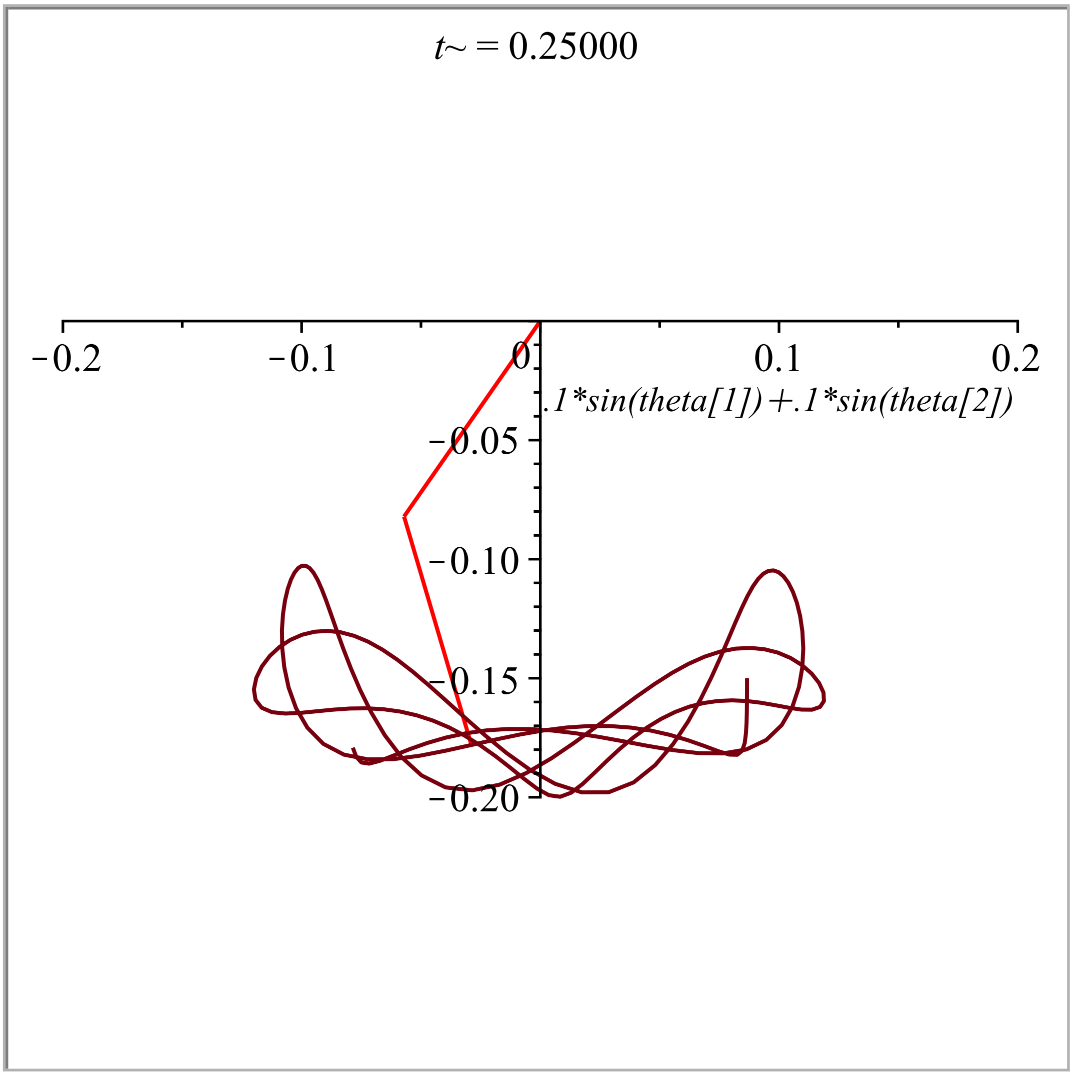
These are the same equations as in

$$\dot{\theta}_2(t)^2 = \frac{-\cos\big(-\theta_2(t) + \theta_1(t)\big)\ \ddot{\theta}_2(t)\ l_2\, m_2 - (m_1 + m_2)\ \big(g \sin\big(\theta_1(t)\big) + \ddot{\theta}_1(t)\ l_1\big)}{m_2\, l_2 \sin\big(-\theta_2(t) + \theta_1(t)\big)}.$$

Numerical Simulation

We are going to simulate a double pendulum with the following initial condition:

$\theta_1(0) = \frac{\pi}{3}$, $\dot{\theta}_1(0) = 0$ and $\theta_2(0) = 0$, $\dot{\theta}_2(0) = 0$



t~ = 0.25000
-0.2
-0.1
0
0.1
0.2
.1*sin(theta[1])+.1*sin(theta[2])
-0.05
-0.10
-0.15
-0.20