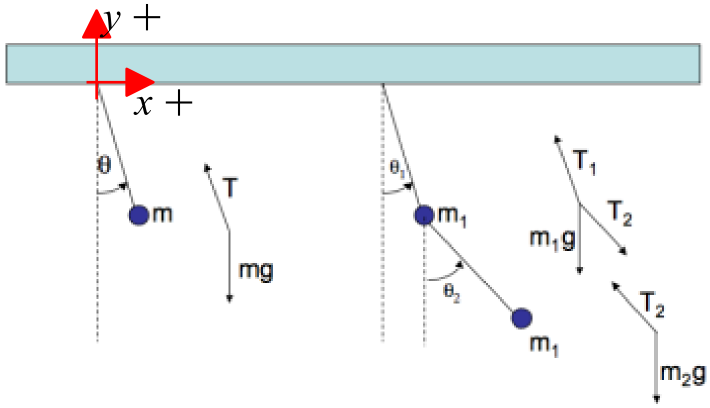
Pendulum's equation of motion

Auralius Manurung

from the note by of Gabriela González



Planar Single Pendulum

Newton's Equations

Constraints

These are the kinematics of a single pendulum:

$$r(t) = \begin{bmatrix} l\sin(\theta(t)) \\ -l\cos(\theta(t)) \end{bmatrix}$$

$$\frac{d}{dt}\, r(t) = \begin{bmatrix} l\left(\frac{d}{dt}\,\theta(t)\right)\cos(\theta(t)) \\ l\left(\frac{d}{dt}\,\theta(t)\right)\sin(\theta(t)) \end{bmatrix}$$

$$v = \frac{d}{dt}\, r(t)$$

$$a = \frac{d^2}{dt^2}\, r(t) \tag{1.1.1.1}$$

Forces

Forces acting on the pendulum F are contributed by the tension T and mg. This can be written as: $F = T\frac{-r}{|r|} - m\begin{pmatrix} 0 \\ g \end{pmatrix}$

$$F = \begin{bmatrix} -T\sin(\theta(t)) \\ T\cos(\theta(t)) - m\,g \end{bmatrix} \tag{1.1.2.1}$$

Equations of motion:

The Newton's law for motion: $F = ma = m\ddot{r}(t)$, we have:

$$\begin{bmatrix} -T\sin(\theta(t)) \\ T\cos(\theta(t)) - m\,g \end{bmatrix} \tag{1.1.3.1}$$

$$= \begin{bmatrix} m\left(l\left(\frac{d^2}{dt^2}\,\theta(t)\right)\cos(\theta(t)) - l\left(\frac{d}{dt}\,\theta(t)\right)^2\sin(\theta(t))\right) \\ m\left(l\left(\frac{d^2}{dt^2}\,\theta(t)\right)\sin(\theta(t)) + l\left(\frac{d}{dt}\,\theta(t)\right)^2\cos(\theta(t))\right) \end{bmatrix}$$

We now have two equations for the two unknowns, T and θ. However, we don't need T here. To remove T, we multiply the first equation with $\cos(\theta)$:

$$-T\sin(\theta(t))\cos(\theta(t)) = m\,l\left(-\left(\frac{d}{dt}\,\theta(t)\right)^2\sin(\theta(t)) + \left(\frac{d^2}{dt^2}\,\theta(t)\right)\cos(\theta(t))\right)\cos(\theta(t)) \tag{1.1.3.2}$$

and the second equation with $\sin(\theta)$:

$$(T\cos(\theta(t)) - m\,g)\sin(\theta(t)) = m\,l\left(\left(\frac{d}{dt}\,\theta(t)\right)^2\cos(\theta(t)) + \left(\frac{d^2}{dt^2}\,\theta(t)\right)\sin(\theta(t))\right)\sin(\theta(t)) \tag{1.1.3.3}$$

After that we add them both so that T is gone, and we write the final equation in $\ddot{\theta}(t)$:

$$-m\,g\sin(\theta(t)) = m\,l\left(\frac{d^2}{dt^2}\,\theta(t)\right)$$

$$\frac{d^2}{dt^2}\,\theta(t) = -\frac{g\sin(\theta(t))}{l} \tag{1.1.3.4}$$

Lagrange's Equations

Let us now try the Lagrange's method.
First, we compute the potential energy (V) and the kinetic energy (T). As for the kinetic energy, it can be expressed as: $T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{\theta}l)^2 = \frac{1}{2}ml^2\dot{\theta}^2$

Computing the Lagrangian:

$$T = \frac{m\,l^2\left(\frac{d}{dt}\,\theta(t)\right)^2}{2}$$

$$V = -m\,g\,l\cos(\theta(t))$$

$$L = T - V$$

$$L = \frac{m\, l^2 \left(\frac{\mathrm{d}}{\mathrm{d}t}\, \theta(t)\right)^2}{2} + m\, g\, l \cos(\theta(t))$$

The Lagrange equation is:

$$\frac{\partial}{\partial t} \frac{\partial}{\partial \frac{\mathrm{d}}{\mathrm{d}t}\, \theta(t)} L - \frac{\partial}{\partial \theta(t)} L = 0$$

$$m\, l^2 \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\, \theta(t)\right) + m\, g\, l \sin(\theta(t)) = 0$$

Thus:

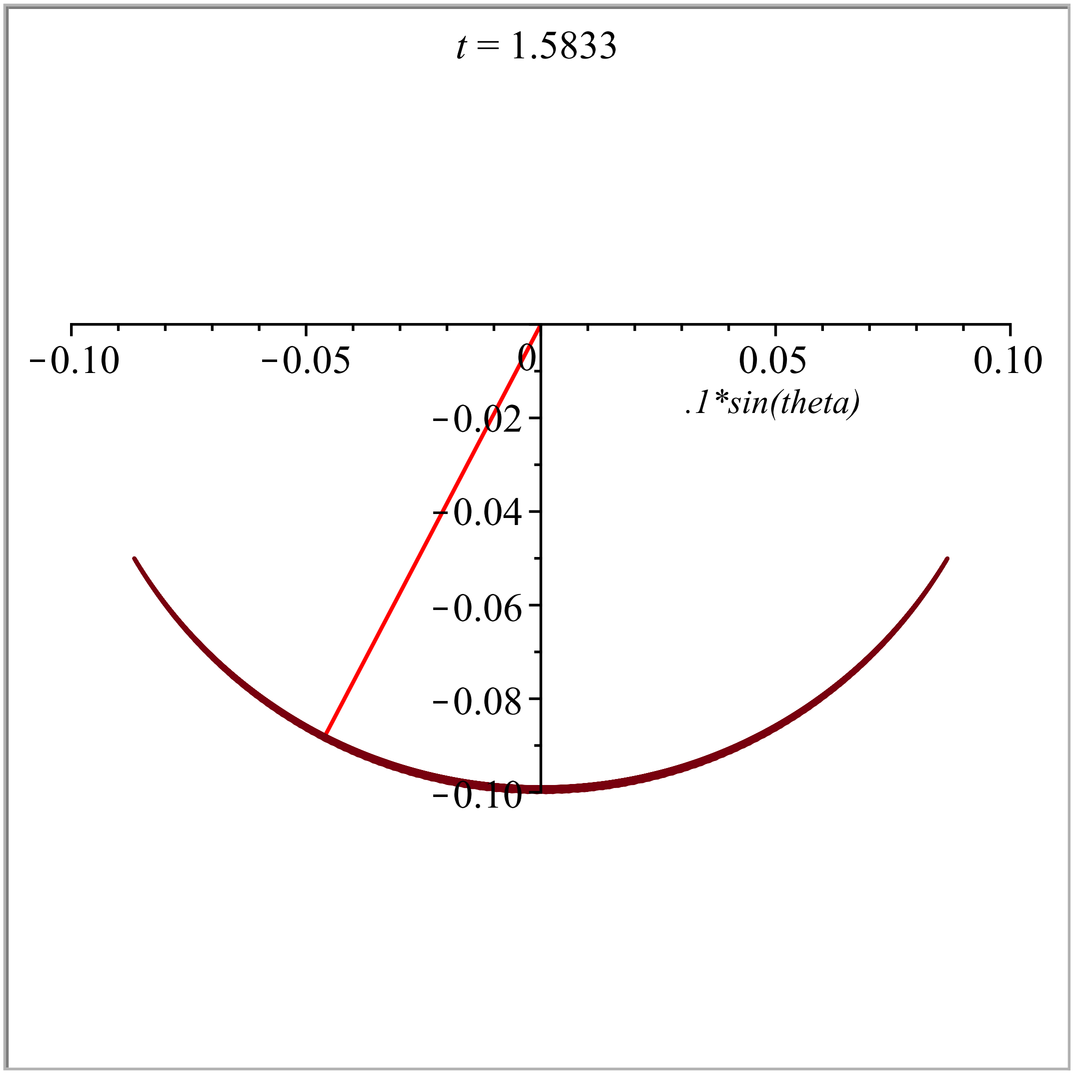
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\, \theta(t) = -\frac{g \sin(\theta(t))}{l} \tag{1.2.1}$$

The result is exactly the same as with the Newton's equation.

Numerical Simulation

We are going to simulate a single pendulum with the following initial condition:

$\theta(0) = \frac{\pi}{3}, \dot{\theta}(0) = 0$



t = 1.5833
−0.10
−0.05
0
0.05
0.10
.1*sin(theta)
−0.02
−0.04
−0.06
−0.08
−0.10

Planar Double Pendulum

Newton's Equations

Constraints

These are the kinematic equations of a double pendulum:

The first link:

$$r_1(t) = \begin{bmatrix} l_1 \sin(\theta_1(t)) \\ -l_1 \cos(\theta_1(t)) \end{bmatrix}$$

$$\frac{d}{dt}\, r1(t) = \begin{bmatrix} l_1 \left(\frac{d}{dt}\,\theta_1(t)\right) \cos(\theta_1(t)) \\ l_1 \left(\frac{d}{dt}\,\theta_1(t)\right) \sin(\theta_1(t)) \end{bmatrix}$$

$$\frac{d^2}{dt^2}\, r1(t) = \begin{bmatrix} l_1 \left(\frac{d^2}{dt^2}\,\theta_1(t)\right) \cos(\theta_1(t)) - l_1 \left(\frac{d}{dt}\,\theta_1(t)\right)^2 \sin(\theta_1(t)) \\ l_1 \left(\frac{d^2}{dt^2}\,\theta_1(t)\right) \sin(\theta_1(t)) + l_1 \left(\frac{d}{dt}\,\theta_1(t)\right)^2 \cos(\theta_1(t)) \end{bmatrix}$$

$$v_1 = \frac{d}{dt}\, r_1(t)$$

$$a_1 = \frac{d^2}{dt^2}\, r_1(t)$$

The second link:

$$r_2(t) = \begin{bmatrix} l_1 \sin(\theta_1(t)) + l_2 \sin(\theta_2(t)) \\ -l_1 \cos(\theta_1(t)) - l_2 \cos(\theta_2(t)) \end{bmatrix}$$

$$\frac{d}{dt}\, r2(t) = \begin{bmatrix} l_1 \left(\frac{d}{dt}\,\theta_1(t)\right) \cos(\theta_1(t)) + l_2 \left(\frac{d}{dt}\,\theta_2(t)\right) \cos(\theta_2(t)) \\ l_1 \left(\frac{d}{dt}\,\theta_1(t)\right) \sin(\theta_1(t)) + l_2 \left(\frac{d}{dt}\,\theta_2(t)\right) \sin(\theta_2(t)) \end{bmatrix}$$

$$\frac{d^2}{dt^2}\, r2(t) = \Bigg[\Bigg[l_1 \left(\frac{d^2}{dt^2}\,\theta_1(t)\right) \cos(\theta_1(t)) - l_1 \left(\frac{d}{dt}\,\theta_1(t)\right)^2 \sin(\theta_1(t)) + l_2 \left(\frac{d^2}{dt^2}\right.$$
$$\left.\theta_2(t)\right) \cos(\theta_2(t)) - l_2 \left(\frac{d}{dt}\,\theta_2(t)\right)^2 \sin(\theta_2(t)) \Bigg],$$
$$\Bigg[l_1 \left(\frac{d^2}{dt^2}\,\theta_1(t)\right) \sin(\theta_1(t)) + l_1 \left(\frac{d}{dt}\,\theta_1(t)\right)^2 \cos(\theta_1(t)) + l_2 \left(\frac{d^2}{dt^2}\right.$$

$$\theta_2(t)\Big) \sin(\theta_2(t)) + l_2 \left(\frac{d}{dt}\theta_2(t)\right)^2 \cos(\theta_2(t))\Big]\Big]$$

$$v_2 = \frac{d}{dt} r_2(t)$$

$$a_2 = \frac{d^2}{dt^2} r_2(t) \qquad \textbf{(2.1.1.1)}$$

Forces

Forces acting on the pendulum F are contributed by the tension T_1, T_2 and mg. This can be written as: $F_1 = T_1 \frac{-r_1}{|r_1|} + T_2 \frac{r_2 - r_1}{|r_2 - r_1|} - m_1 \begin{pmatrix} 0 \\ g \end{pmatrix}$ and $F_2 = -T_2 \frac{r_2 - r_1}{|r_2 - r_1|} - m_2 \begin{pmatrix} 0 \\ g \end{pmatrix}$.

$$F_1 = \begin{bmatrix} -T_1 \sin(\theta_1(t)) + T_2 \sin(\theta_2(t)) \\ T_1 \cos(\theta_1(t)) - T_2 \cos(\theta_2(t)) - m_1 g \end{bmatrix}$$

$$F_2 = \begin{bmatrix} -T_2 \sin(\theta_2(t)) \\ T_2 \cos(\theta_2(t)) - m_2 g \end{bmatrix}$$

$$Since\,,\, F = m \left(\frac{\partial^2}{\partial t^2} r\right)$$

$$\begin{bmatrix} -T_1 \sin(\theta_1(t)) + T_2 \sin(\theta_2(t)) \\ T_1 \cos(\theta_1(t)) - T_2 \cos(\theta_2(t)) - m_1 g \end{bmatrix}$$

$$= \begin{bmatrix} m_1 \left(l_1 \left(\frac{d^2}{dt^2}\theta_1(t)\right) \cos(\theta_1(t)) - l_1 \left(\frac{d}{dt}\theta_1(t)\right)^2 \sin(\theta_1(t))\right) \\ m_1 \left(l_1 \left(\frac{d^2}{dt^2}\theta_1(t)\right) \sin(\theta_1(t)) + l_1 \left(\frac{d}{dt}\theta_1(t)\right)^2 \cos(\theta_1(t))\right) \end{bmatrix}$$

$$\begin{bmatrix} -T_2 \sin(\theta_2(t)) \\ T_2 \cos(\theta_2(t)) - m_2 g \end{bmatrix} = \Big[\Big[m_2 \Big(l_1 \left(\frac{d^2}{dt^2}\theta_1(t)\right) \cos(\theta_1(t)) - l_1 \Big(\frac{d}{dt} \qquad \textbf{(2.1.2.1)}$$

$$\theta_1(t)\Big)^2 \sin(\theta_1(t)) + l_2 \left(\frac{d^2}{dt^2}\theta_2(t)\right) \cos(\theta_2(t)) - l_2 \Big(\frac{d}{dt}$$

$$\theta_2(t)\Big)^2 \sin(\theta_2(t))\Big)\Big],$$

$$\Big[m_2 \Big(l_1 \left(\frac{d^2}{dt^2}\theta_1(t)\right) \sin(\theta_1(t)) + l_1 \left(\frac{d}{dt}\theta_1(t)\right)^2 \cos(\theta_1(t)) + l_2 \Big(\frac{d^2}{dt^2}$$

$$\left.\left.\left.\theta_2(t)\right)\sin\left(\theta_2(t)\right)+l_2\left(\frac{d}{dt}\theta_2(t)\right)^2\cos\left(\theta_2(t)\right)\right)\right]\right]$$

We now have 4 sets of equations **(2.1.2.1)**:

-- Multiply the first equation with $\cos(\theta_1)$ and the second equation with $\sin(\theta_1)$. Add both results.

-- Multiply the first equation with $\sin(\theta_1)$ and the second equation with $\cos(\theta_1)$. Substract both results.

-- Multiply the first equation with $\sin(\theta_2)$ and the second equation with $\cos(\theta_2)$. Substract both results.

-- Multiply the first equation with $\sin(\theta_2)$ and the second equation with $\cos(\theta_2)$. Substract both results.

$$\left[\left[-T_2\sin\left(-\theta_2(t)+\theta_1(t)\right)-\sin\left(\theta_1(t)\right)g\,m_1=m_1\left(\frac{d^2}{dt^2}\theta_1(t)\right)l_1\right],\right. \tag{2.1.2.2}$$

$$\left[T_2\cos\left(-\theta_2(t)+\theta_1(t)\right)-T_1+\cos\left(\theta_1(t)\right)g\,m_1=-m_1\left(\frac{d}{dt}\theta_1(t)\right)^2 l_1\right],$$

$$\left[-\sin\left(\theta_2(t)\right)m_2\,g=-\left(\frac{d}{dt}\theta_1(t)\right)^2\sin\left(-\theta_2(t)+\theta_1(t)\right)l_1\,m_2+\cos\left(-\theta_2(t)\right.\right.$$

$$\left.\left.+\theta_1(t)\right)\left(\frac{d^2}{dt^2}\theta_1(t)\right)l_1\,m_2+\left(\frac{d^2}{dt^2}\theta_2(t)\right)l_2\,m_2\right],$$

$$\left[-T_2+\cos\left(\theta_2(t)\right)g\,m_2=-\cos\left(-\theta_2(t)+\theta_1(t)\right)\left(\frac{d}{dt}\theta_1(t)\right)^2 l_1\,m_2-\left(\frac{d^2}{dt^2}\right.\right.$$

$$\left.\left.\theta_1(t)\right)\sin\left(-\theta_2(t)+\theta_1(t)\right)l_1\,m_2-\left(\frac{d}{dt}\theta_2(t)\right)^2 l_2\,m_2\right]$$

The first and the second equation above **(2.1.2.2)** can be rewritten in $\frac{d^2}{dt^2}\theta_1(t)$ and in $\frac{d^2}{dt^2}\theta_1(t)$. The results are then substituted to the other two equations:

$$\left[\begin{array}{c}
\frac{d^2}{dt^2}\theta_1(t)=-\frac{T_2\sin\left(-\theta_2(t)+\theta_1(t)\right)+\sin\left(\theta_1(t)\right)g\,m_1}{m_1\,l_1}\\
\left(\frac{d}{dt}\theta_1(t)\right)^2=\frac{-T_2\cos\left(-\theta_2(t)+\theta_1(t)\right)+T_1-\cos\left(\theta_1(t)\right)g\,m_1}{m_1\,l_1}\\
\frac{d^2}{dt^2}\theta_2(t)=\frac{\sin\left(-\theta_2(t)+\theta_1(t)\right)T_1}{m_1\,l_2}\\
\left(\frac{d}{dt}\theta_2(t)\right)^2=\frac{-\cos\left(-\theta_2(t)+\theta_1(t)\right)T_1\,m_2+T_2\,m_1+T_2\,m_2}{m_2\,m_1\,l_2}
\end{array}\right] \tag{2.1.2.3}$$

From **(2.1.2.3)**, the first and the third equations can be rewritten as follows to get

T_1 and T_2:

$$T_2 = \frac{-m_1 \left(\frac{d^2}{dt^2}\, \theta_1(t)\right) l_1 - \sin\left(\theta_1(t)\right) g\, m_1}{\sin\left(-\theta_2(t) + \theta_1(t)\right)}$$

$$T_1 = \frac{\left(\frac{d^2}{dt^2}\, \theta_2(t)\right) m_1\, l_2}{\sin\left(-\theta_2(t) + \theta_1(t)\right)} \tag{2.1.2.4}$$

Equations of Motion

T_1 and T_2 are then sustituted to the second and fourth equations of **(2.1.2.3)**. These two equations are the final equations of motion of a planar double pendulum:

$$\left(\frac{d}{dt}\, \theta_1(t)\right)^2 = \frac{1}{l_1 \sin\left(-\theta_2(t) + \theta_1(t)\right)} \left(\left(\frac{d^2}{dt^2}\, \theta_1(t)\right) l_1 \cos\left(-\theta_2(t) + \theta_1(t)\right) + g \sin\left(\theta_1(t)\right) \cos\left(-\theta_2(t) + \theta_1(t)\right) - \sin\left(-\theta_2(t) + \theta_1(t)\right) \cos\left(\theta_1(t)\right) g + \left(\frac{d^2}{dt^2}\, \theta_2(t)\right) l_2\right)$$

$$\left(\frac{d}{dt}\, \theta_2(t)\right)^2 = \frac{1}{m_2\, l_2 \sin\left(-\theta_2(t) + \theta_1(t)\right)} \left(-\cos\left(-\theta_2(t) + \theta_1(t)\right) \left(\frac{d^2}{dt^2}\, \theta_2(t)\right) l_2\, m_2 - \left(m_1 + m_2\right) \left(g \sin\left(\theta_1(t)\right) + l_1 \left(\frac{d^2}{dt^2}\, \theta_1(t)\right)\right)\right) \tag{2.1.3.1}$$

Lagrange's equation

We now relize how complicated it is to use the Newton's method to derive the equations of motion of a double pendulum. Let us now try the Lagrange's method.
First, we compute the potential energy (V) and the kinetic energy (K).

$$K = \frac{1}{2}\, m_1\, v_1^2 + \frac{1}{2}\, m_2\, v_2^2$$

Where:

$$v_1 = \left\| \left(\frac{d}{dt}\, r_1(t)\right) \right\|_2$$

$$v_1 = \sqrt{l_1^2 \left(\frac{d}{dt}\, \theta_1(t)\right)^2 \cos\left(\theta_1(t)\right)^2 + l_1^2 \left(\frac{d}{dt}\, \theta_1(t)\right)^2 \sin\left(\theta_1(t)\right)^2}$$

and

$$v_2 = \left\| \left(\frac{d}{dt}\, r_2(t)\right) \right\|_2$$

v_2

$$= \left(\left(l_1 \left(\frac{d}{dt} \theta_1(t) \right) \cos(\theta_1(t)) + l_2 \left(\frac{d}{dt} \theta_2(t) \right) \cos(\theta_2(t)) \right)^2 + \left(l_1 \left(\frac{d}{dt} \theta_1(t) \right) \sin(\theta_1(t)) + l_2 \left(\frac{d}{dt} \theta_2(t) \right) \sin(\theta_2(t)) \right)^2 \right)^{1/2}$$

Therefore:

$$K = \frac{\left(\frac{d}{dt} \theta_1(t) \right)^2 l_1^2 m_1}{2} + \frac{\left(\frac{d}{dt} \theta_1(t) \right)^2 l_1^2 m_2}{2} + \left(\frac{d}{dt} \theta_1(t) \right) \left(\frac{d}{dt} \theta_2(t) \right) l_1 l_2 m_2 \cos(-\theta_2(t) + \theta_1(t)) + \frac{\left(\frac{d}{dt} \theta_2(t) \right)^2 l_2^2 m_2}{2}$$

and

$$V = -m_1 g l_1 \cos(\theta_1(t)) - m_2 g \left(l_1 \cos(\theta_1(t)) + l_2 \cos(\theta_2(t)) \right) \quad \textbf{(2.1.4.1)}$$

And then we compute the Lagrangian (L):

$$L = K - V$$

$$L = \frac{\left(\frac{d}{dt} \theta_1(t) \right)^2 l_1^2 m_1}{2} + \frac{\left(\frac{d}{dt} \theta_1(t) \right)^2 l_1^2 m_2}{2} + \left(\frac{d}{dt} \theta_1(t) \right) \left(\frac{d}{dt} \theta_2(t) \right) l_1 l_2 m_2 \cos(-\theta_2(t) + \theta_1(t)) + \frac{\left(\frac{d}{dt} \theta_2(t) \right)^2 l_2^2 m_2}{2} + m_1 g l_1 \cos(\theta_1(t)) + m_2 g \left(l_1 \cos(\theta_1(t)) + l_2 \cos(\theta_2(t)) \right) \quad \textbf{(2.1.4.2)}$$

Finally, we can calculate the Lagrange's equations:

The Lagrange equations are:

$$\frac{\partial}{\partial t} \frac{\partial}{\partial \frac{d}{dt} \theta(t)} L - \frac{\partial}{\partial \theta(t)} L = 0$$

$$\left[\left[\left(\cos(-\theta_2(t) + \theta_1(t)) \left(\frac{d^2}{dt^2} \theta_2(t) \right) l_2 m_2 + \left(\frac{d}{dt} \theta_2(t) \right)^2 l_2 m_2 \sin(-\theta_2(t) + \theta_1(t)) + (m_1 + m_2) \left(g \sin(\theta_1(t)) + l_1 \left(\frac{d^2}{dt^2} \theta_1(t) \right) \right) \right) l_1 = 0 \right], \right.$$
$$\left. \left[l_2 m_2 \left(-\left(\frac{d}{dt} \theta_1(t) \right)^2 l_1 \sin(-\theta_2(t) + \theta_1(t)) + \left(\frac{d^2}{dt^2} \theta_1(t) \right) l_1 \cos(-\theta_2(t) + \theta_1(t)) + g \sin(\theta_2(t)) + \left(\frac{d^2}{dt^2} \theta_2(t) \right) l_2 \right) = 0 \right] \right] \quad \textbf{(2.1.4.3)}$$

Let's rewrite the equations in form of: $\left(\frac{d}{dt} \theta_2(t) \right)^2$ and $\left(\frac{d}{dt} \theta_1(t) \right)^2$:

$$\left(\frac{d}{dt}\theta_2(t)\right)^2 = \frac{1}{m_2 l_2 \sin(-\theta_2(t)+\theta_1(t))}\left(-\cos(-\theta_2(t)+\theta_1(t))\left(\frac{d^2}{dt^2}\theta_2(t)\right) l_2 m_2 - (m_1+m_2)\left(g\sin(\theta_1(t)) + l_1\left(\frac{d^2}{dt^2}\theta_1(t)\right)\right)\right)$$

$$\left(\frac{d}{dt}\theta_1(t)\right)^2 = -\frac{-\left(\frac{d^2}{dt^2}\theta_1(t)\right) l_1 \cos(-\theta_2(t)+\theta_1(t)) - g\sin(\theta_2(t)) - \left(\frac{d^2}{dt^2}\theta_2(t)\right) l_2}{l_1 \sin(-\theta_2(t)+\theta_1(t))} \tag{2.1.4.4}$$

These are the same equations as in **(2.1.3.1)**.

Numerical Simulation

We are going to simulate a double pendulum with the following initial condition:

$\theta_1(0) = \frac{\pi}{3}$, $\dot{\theta}_1(0) = 0$ and $\theta_2(0) = 0$, $\dot{\theta}_2(0) = 0$