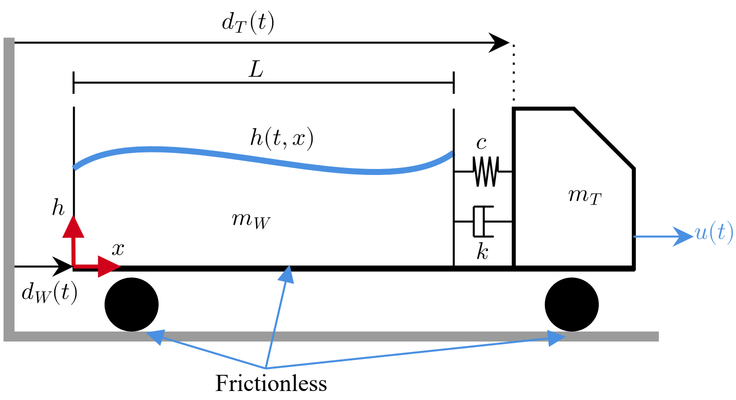
The Saint-Venant- Moving-Truck Toy Problem[1]

Auralius Manurung - ME Universitas Pertamina



Drawing 1: Truck with a fluid-filled basin

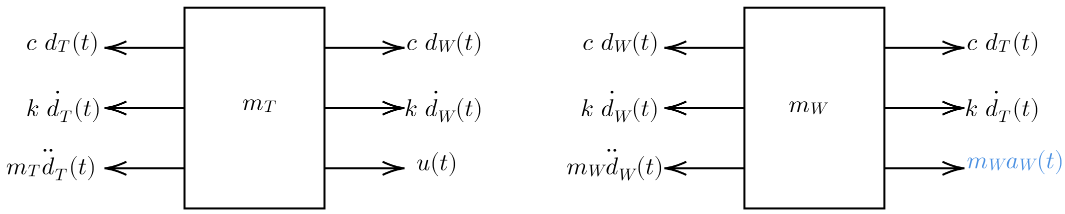
1. The Moving Truck Equations

The truck mass is m_T and the mass of the filled basin is m_W.

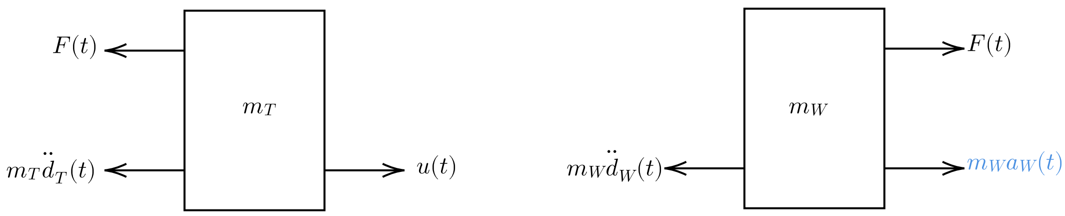
The force between the truck and the basin can be expressed as (see the free-body diagrams below):

$$F(t) = c(d_T(t) - d_W(t) - \bar{d}) + k(\dot{d}_T(t) - \dot{d}_W(t)) \tag{1}$$

where c is the spring force constant, k is the damper force constant, and $\bar{d}$ is an offset used in the spring force.



Drawing 2: Free body diagrams of the truck and the filled basin



$$F(t) = c(d_T(t) - d_W(t)) + k(\dot{d}_T(t) - \dot{d}_W(t))$$

Drawing 3: The simplified Free body diagrams by introducing $F(t)$

The equations of motion for the truck and the filled basin:

$$m_T\ddot{d}_T = u(t) - F(t) \tag{2}$$
$$m_W\ddot{d}_W = F(t) + m_W a_W(t) \tag{3}$$

[1] [Numerical optimal control of a coupled ODE-PDE model of a truck with a fluid basin]

where

$$a_W(t) = \frac{1}{L}\int_0^L \left[\frac{\partial v(t,x)}{\partial t}\right]\partial x \tag{4}$$

In (4), $\dfrac{\partial v(t,x)}{\partial t}$ is the horizontal acceleration of the fluid inside the basin at time t and at position x. The average of this horizontal acceleration acts as another external force applied to the basin.

2. The Saint-Venant Equations

The equations of motion for the water basin:

$$\frac{\partial h(t,x)}{\partial t} + \frac{\partial (h(t,x)v(t,x))}{\partial x} = 0 \tag{5}$$

$$\frac{\partial v(t,x)}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}(v(t,x))^2 + gh(t,x)\right) = -\frac{1}{m_W}F(t), \quad (t,x) \in \Omega \tag{6}$$

with $\Omega := (0,T)\times(0,L)$ and initial and boundary conditions are:

$$v(t,0) = v(t,L) = 0, \qquad t \in [0,T] \tag{7}$$

since $v(t,0) = v(t,L) = 0$, it follows $\left.\dfrac{\partial v}{\partial t}\right|_{x=0} = 0$ and $\left.\dfrac{\partial v}{\partial t}\right|_{x=L} = 0$.

Hence:

$$\begin{aligned} \cancel{\frac{\partial v(t,x)}{\partial t}} + \frac{\partial}{\partial x}\left(\cancel{\frac{1}{2}(v(t,x))^2} + gh(t,x)\right) &= -\frac{1}{m_W}F(t),\ x = \{0,L\} \\ \frac{\partial}{\partial x}h(t,x) &= -\frac{1}{g\ m_W}F(t),\ x = \{0,L\} \end{aligned} \tag{8}$$

Discretization rules:

- Forward in time:

$$\frac{\partial h}{\partial t} = \frac{1}{\Delta t}\left[h(t+\Delta t,x) - \frac{1}{2}(h(t,x+\Delta x) + h(t,x-\Delta))\right] \tag{9}$$

- Centered in space:

$$\frac{\partial (hv)}{\partial x} = \frac{h(t,x+\Delta x)v(t,x+\Delta x) - h(t,x-\Delta x)v(t,x-\Delta x)}{2\Delta x} \tag{10}$$

- Forward in time:

$$\frac{\partial v}{\partial t} = \frac{1}{\Delta t}\left[v(t+\Delta t,x) - \frac{1}{2}(v(t,x+\Delta x) + v(t,x-\Delta x))\right] \tag{11}$$

- Centered in space:

$$\frac{\partial}{\partial x}\left(\frac{1}{2}v^2 + gh\right) = \frac{1}{2\Delta x}\left(\left[\frac{1}{2}(v(t,x+\Delta x))^2 + gh(t,x+\Delta x)\right] - \left[\frac{1}{2}v(t,x-\Delta x)^2 + gh(t,x-\Delta x)\right]\right) \tag{12}$$

Hence, we can rewrite and rearrange (5) as:

$$\boxed{h(t+\Delta t,x) = -\frac{\Delta t}{2\Delta x}[h(t,x+\Delta x)v(t,x+\Delta x) - h(t,x-\Delta x)v(t,x-\Delta x)] + \frac{1}{2}[h(t,x+\Delta x) + h(t,x-\Delta x)]} \tag{13}$$

We can also rewrite and rearrange and (6) as:

$$\boxed{\begin{aligned}v(t+\Delta t,x) = &-\frac{\Delta t}{m_W}F(t)-\frac{\Delta t}{2\Delta x}\left(\left[\frac{1}{2}(v(t,x+\Delta x))^2+gh(t,x+\Delta x)\right]-\left[\frac{1}{2}(v(t,x-\Delta x))^2+gh(t,x-\Delta x)\right]\right)\\&+\frac{1}{2}[v(t,x+\Delta v)+v(t,x-\Delta x)]\end{aligned}} \tag{14}$$

We also need to discretize the boundaries ((7) and (8)). Discretizing (7) gives us:

$$\boxed{\frac{h(t,\Delta x)-h(t,0)}{\Delta x} = -\frac{1}{gm_W}F(t), \quad \frac{h(t,L)-h(t,L-\Delta x)}{\Delta x} = -\frac{1}{gm_W}F(t)} \tag{15}$$

Rewriting (15) gives us:

$$\boxed{h(t,0) = h(t,\Delta x)+\frac{\Delta x}{gm_W}F(t), \quad h(t,L) = h(t,L-\Delta x)-\frac{\Delta x}{gm_W}F(t)} \tag{16}$$