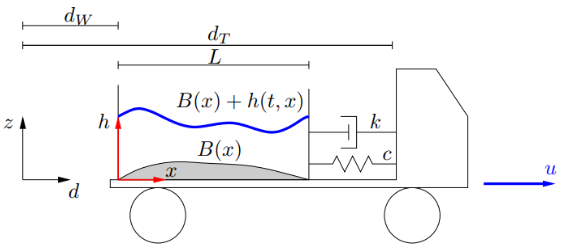
The Saint-Venant- Moving-Truck Toy Problem[1]



1. The Moving Truck Equations

The truck mass is m_T and the mass of the filled basin is m_W.

The force between the truck and the basin:

$$F(t) = c(d_T - d_W + \overline{d}) + k(\dot{d}_T - \dot{d}_W)$$

where:
where c is the spring force constant, k is the damper force constant, and $\overline{d}$ is an offset used in the spring force.

The equation of motion for the truck:

2. The Saint-Venant Equations

The equation of motion for the water basin:

$$\frac{\partial h(t,x)}{\partial t} + \frac{\partial (h(t,x)v(t,x))}{\partial x} = 0 \tag{1}$$

$$\frac{\partial v(t,x)}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}(v(t,x))^2 + gh(t,x)\right) = -\frac{1}{m_W}F(t), \quad (t,x) \in \Omega \tag{2}$$

with $\Omega := (0,T) \times (0,L)$ and initial and boundary conditions are:

$$v(t,0) = v(t,L) = 0, \qquad t \in [0,T] \tag{3}$$

since $v(t,0) = v(t,L) = 0$, it follows $\left.\frac{\partial v}{\partial t}\right|_{x=0} = 0$ and $\left.\frac{\partial v}{\partial t}\right|_{x=L} = 0$.

Hence:

$$\begin{aligned} \cancel{\frac{\partial v(t,x)}{\partial t}} + \frac{\partial}{\partial x}\left(\cancel{\frac{1}{2}(v(t,x))^2} + gh(t,x)\right) &= -\frac{1}{m_W}F(t), \;\; x = \{0, L\} \\ \frac{\partial}{\partial x}h(t,x) &= -\frac{1}{g\; m_W}F(t), \;\; x = \{0, L\} \end{aligned} \tag{4}$$

Discretization rules:

- Forward in time:

$$\frac{\partial h}{\partial t} = \frac{1}{\Delta t}\left[h(t+\Delta t, x) - \frac{1}{2}(h(t, x+\Delta x) + h(t, x-\Delta))\right] \tag{5}$$

- Centered in space:

$$\frac{\partial (hv)}{\partial x} = \frac{h(t,x+\Delta x)v(t,x+\Delta x) - h(t,x-\Delta x)v(t,x-\Delta x)}{2\Delta x} \tag{6}$$

- Forward in time:

[1] http://www.aimsciences.org/journals/displayPaperPro.jsp?paperID=11915

$$\frac{\partial v}{\partial t} = \frac{1}{\Delta t}\left[v(t+\Delta t, x) - \frac{1}{2}(v(t, x+\Delta x) + v(t, x-\Delta x))\right] \tag{7}$$

- Centered in space:

$$\frac{\partial}{\partial x}\left(\frac{1}{2}v^2 + gh\right) = \frac{1}{2\Delta x}\left(\left[\frac{1}{2}(v(t, x+\Delta x))^2 + gh(t, x+\Delta x)\right] - \left[\frac{1}{2}v(t, x-\Delta x)^2 + gh(t, x-\Delta x)\right]\right) \tag{8}$$

Hence, we can rewrite and rearrange (1) as:

$$\boxed{h(t+\Delta t, x) = -\frac{\Delta t}{2\Delta x}[h(t, x+\Delta x)v(t, x+\Delta x) - h(t, x-\Delta x)v(t, x-\Delta x)] + \frac{1}{2}[h(t, x+\Delta x) + h(t, x-\Delta x)]} \tag{9}$$

We can also rewrite and rearrange and (2) as:

$$\boxed{\begin{aligned} v(t+\Delta t, x) = &-\frac{\Delta t}{m_W}F(t) - \frac{\Delta t}{2\Delta x}\left(\left[\frac{1}{2}(v(t, x+\Delta x))^2 + gh(t, x+\Delta x)\right] - \left[\frac{1}{2}(v(t, x-\Delta x))^2 + gh(t, x-\Delta x)\right]\right) \\ &+\frac{1}{2}[v(t, x+\Delta v) + v(t, x-\Delta x)] \end{aligned}} \tag{10}$$

We also need to discretize the boundaries ((3) and (4)). Discretizing (3) gives us:

$$\boxed{\frac{h(t, \Delta x) - h(t, 0)}{\Delta x} = -\frac{1}{gm_W}F(t), \quad \frac{h(t, L) - h(t, L-\Delta x)}{\Delta x} = -\frac{1}{gm_W}F(t)} \tag{11}$$

Rewriting (11) gives us:

$$\boxed{h(t, 0) = h(t, \Delta x) + \frac{\Delta x}{gm_W}F(t), \quad h(t, L) = h(t, L-\Delta x) - \frac{\Delta x}{gm_W}F(t)} \tag{12}$$