Åbo Akademi University

Process Control Laboratory

5. Dynamics of Simple Systems

5. Dynamics of Simple Systems

In Chapter 3, we derived models for some simple technical systems. In all cases, the systems could be described by

- ordinary differential equations (ODEs) of relatively low order

In many cases, the ODEs were

- nonlinear, but they could be linearized around an operating point

In this chapter we shall study the properties of some

- simple, linear, dynamical systems

In particular, we shall derive the

- time response to well-defined inputs such as impulses and steps; analysis based on such responses is called transient analysis

We shall also study

- simple, graphical techniques for determining linear low-order models from step response data; these are simple examples of *system identification*

5.1 Integrating systems

An *integrating system* is the simplest dynamical system that can be described by a differential equation. The most typical process example of an integrating system is probably a liquid tank.

Example 5.1. A liquid Tank.

Consider the tank in figure 5.1.

- V = volume of liquid in the tank
- F_1 = volumetric flow rate of liquid into the tank
- F_2 = volumetric flow rate of liquid out of the tank

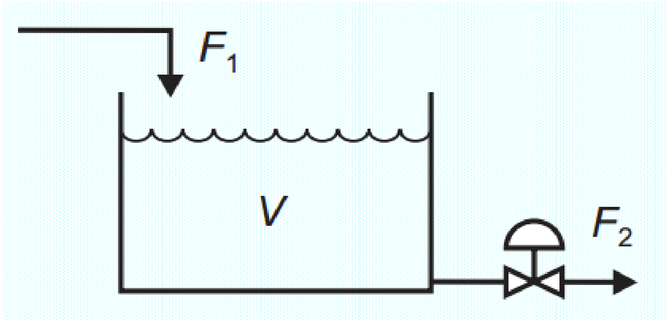

Figure 5.1. Liquid tank.

Notice that V is the output signal of the system (i.e., a dependent variable), whereas F_1 and F_2 are input signals (“independent” variables). See section 2.3.

5.1 Integrating systems

A mass balance around the tank gives under the assumption of constant density (which cancels out) the model

$$\frac{\mathrm{d}V}{\mathrm{d}t} = F_1 - F_2 \tag{1}$$

This equation is linear and we can directly replace the variables with Δ-variables to get

$$\frac{\mathrm{d}\Delta V}{\mathrm{d}t} = \Delta F_1 - \Delta F_2 \tag{2}$$

The Laplace transform considering that the initial state is zero gives

$$s\Delta V(s) = \Delta F_1(s) - \Delta F_2(s) \qquad \text{or} \qquad \Delta V(s) = \frac{1}{s}\Delta F_1(s) - \frac{1}{s}\Delta F_2(s) \tag{3}$$

The two transfer functions of the system are

$$\frac{\Delta V(s)}{\Delta F_1(s)} = \frac{1}{s} \qquad \text{and} \qquad \frac{\Delta V(s)}{\Delta F_2(s)} = -\frac{1}{s} \tag{4}$$

which, according to the Laplace transform , corresponds to integrals in the time domain.

5.1 Integrating systems

Generally, a linear integrating system with input signal u and output signal y can be described by the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = Ku \quad \text{or} \quad T\frac{\mathrm{d}y}{\mathrm{d}t} = u \tag{5.1}$$

The transfer function of the system is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{s} = \frac{1}{Ts} \tag{5.2}$$

Exercise 5.1.

Derive and sketch

(a) the impulse response

(b) the step response

(c) the ramp response

for $\Delta V(t)$ when a change in the input flow F_1 to the tank in Fig. 5.1 is done.

5.2 First-order systems

A linear first-order system can be described by differential equation

$$T\frac{\mathrm{d}y}{\mathrm{d}t}+y=Ku \tag{5.3}$$

where K is the system gain and T its time constant. The transfer function of the system is

$$G(s)=\frac{Y(s)}{U(s)}=\frac{1}{Ts+1} \tag{5.4}$$

5.2.1 Transient response

The system time response $y(t)$ of a given input signal $u(t)$ can be easily determined by finding the inverse Laplace transform by using the Table of Laplace transforms (see section 4.2). Two common inputs are the

- impulse function
- step function

5.2.1 Transient response

Impulse response

If the input signal of the system is an impulse with the time integral ("area") I, *i*.e.

$$u(t) = I\,\delta(t)$$

where $\delta(t)$ is the unit impulse (Dirac delta function), its Laplace transform (as listed in the table of Laplace transforms) is

$$U(s) = I$$

The inverse Laplace transform of

$$Y(s) = G(s)U(s) = \frac{KI}{Ts+1}$$

then gives the *impulse response*

$$y(t) = \frac{KI}{T}\mathrm{e}^{-t/T} \tag{5.5}$$

Step response

If the input signal is a step change of magnitude u_{step} , i.e.

$$u(t) = u_{\text{step}}\sigma(t)$$

where $\sigma(t)$ is the unit step, the Laplace transform is

$$U(s) = \frac{u_{\text{step}}}{s}$$

The inverse Laplace transform of

$$Y(s) = G(s)U(s) = \frac{Ku_{\text{step}}}{(Ts+1)s}$$

then gives the step response

$$y(t) = Ku_{\text{step}}(1 - \mathrm{e}^{-t/T}) \tag{5.6}$$

5.2.1 Transient response

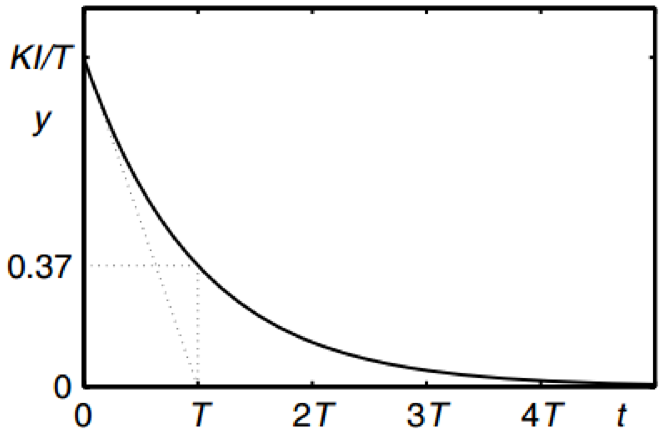

Figure 5.2. Impulse response of a first-order system.

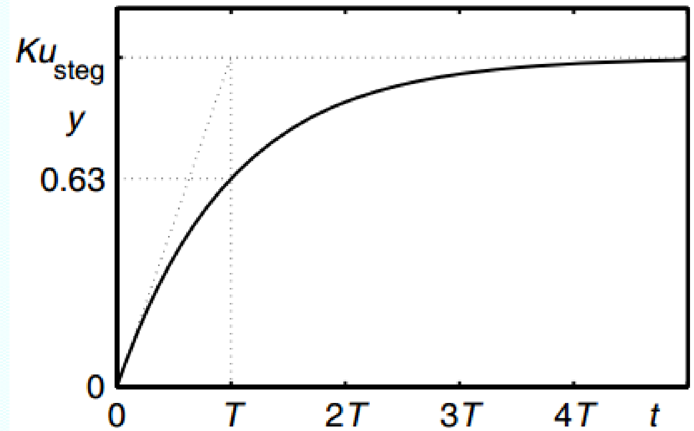

Figure 5.3. Step response of a first-order system.

– The initial slope of the curves (i.e. their derivative) is obtained by drawing a line from the start of the response so that it intersects the line representing the new steady-state value at $t = T$.
– The distance between the response and the point of intersection at $t = T$ is $1/\mathrm{e} = 0.368$ of the total change of the output signal.
– In practice, the new steady-state value (within 2%) is reached at $t \approx 4T$ (however, in theory it takes ∞ long time).

5.2.2 Identification from step response

From the information above it is evident that the system gain and time constant can be identified (i.e. determined) from the transient response, which can be generated experimentally by a suitable change in the input signal.

In this respect, the identification from step changes are normally used, particularly because a well defined impulse response is difficult to achieve.

The identification by a step input gives the system gain according

$$K = y_\infty / u_{\text{step}} \tag{5.7}$$

- u_{step} is the magnitude of the step change of the input signal
- y_∞ is the total change of the output signal when $t \to \infty$.

There are different methods for the determination of the system time constant and an eventual time-delay (see section 5.4). Some simple "graphical" methods will be studied in the following section.

63 % of the total change

– The time constant of a first order system can be determined starting from the point of intersection between the final asymptotic value ($y_\infty = Ku_{\text{step}}$) and the tangent (i.e. the derivative) to that point where the step response begins to change (see figure 5.3).

– However, it is difficult in practice to determine the tangent (i.e. the step response initial derivative) with good accuracy.

  It is better to use the point where the step response reaches 63,2 % of the total change.

– One can easily show that the step response of a first order system reaches this point when a time equal to the system time constant has elapsed since the beginning of the step response. In other words, the time constant is given by the time coordinate of that point where 63,2 % of the total change has been reached.

– Generally, the time constant that is obtained from the 63,2 % of the total change can be called *equivalent time constant* even if the system is not of first order.

5.2.2 Identification from step response

In practice, a system often contains a time delay, for example due to a transport delay. The step response delays then with the corresponding time, which should be considered when identifying the system .

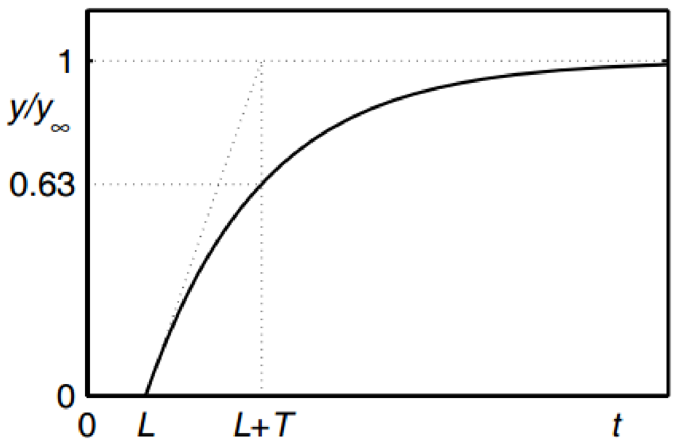

Figure 5.4. Identification of a 1st order system from the 63 % of the total change.

A first order system with a time delay L has the transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K\mathrm{e}^{-Ls}}{Ts+1} \qquad (5.8)$$

and the step response

$$y(t+L) = Ku_{\text{step}}(1-\mathrm{e}^{-t/T}) \qquad (5.9)$$

A step change at $t = 0$ gives a step response that starts at time L and reaches 63,2 % of the total change at time $L + T$. Both parameters are thus obtained from the same step response. In figure 5.4 the step response has been normalized by dividing the output signal by y_∞ .

The Tangent method

In practice, a perfect first-order system is hardly ever obtained (with or without time delay).

- Often the step response does not have its steepest slope immediately at the beginning, which a system of first order would have.
- It means that the system is of higher order than first order.

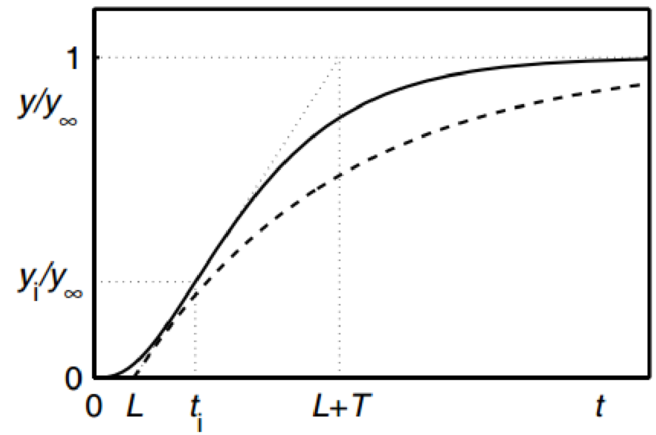

Figure 5.5. Identification of a 1st order system with the tangent method.

Sometimes we want anyways to approximate the system to a first-order system with time delay. Figure 5.5 illustrates this method.

- The system gain is calculated in the normal way according to equation (5.7).
- The time delay and time constant are obtained by drawing a tangent through the inflection point of the step response (t_i, y_i), i.e. the steepest part of a slope. The tangent point of intersection with the time-axis gives the time delay L , the intersection point with the final asymptotic value gives the time coordinate $L + T$.

5.2.2 Identification from step response

- Ziegler-Nichols' as well as some other step-response based recommendations for setting of PID controllers (see section 7.5) assume that the model parameters are determined according the tangent method.
- As shown in figure 5.5, the model step response (the dotted line) can, however, differ significantly from the actual step response. Because the step response of a first-order system has its steepest slope at the beginning, where it is the same as the slope of the drawn tangent, and then decreases, it is easy to understand that the model step response will always be below the line of the actual step response.
- In other words, the time constant determined by the tangent method is too large.
- However, this does not mean that the method is not ok for controller setting, but it is relatively poor for model identification.

5.2.2 Identification from step response

Modification of the tangent method

A considerable modification of the methods named above can be obtained if both of them are combined. We determine then

- the time delay according the tangent method
- the time constant = the equivalent time constant, i.e. the time it takes for the step response to reach 63,2 % of the total change.

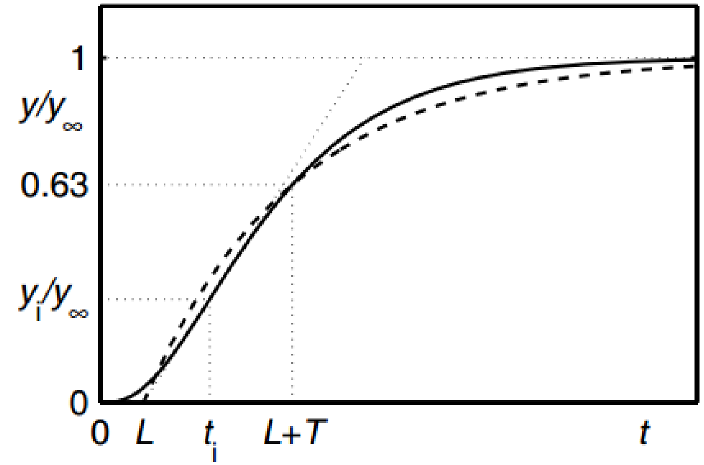

Figure 5.6. Identification of a 1st order system with the modified tangent method.

– This procedure gives a model whose step response (the dotted line in figure 5.6) matches much better the actual step response.

– This method is also less sensible to disturbances, because two points of the step response are used to determine the model parameters.

– The ordinary tangent method try to determine both time delay and time constant from the step response characteristics in only one point, the inflection point, which is also difficult to handle in practice.

5.2.2 Identification from step response

Sundaresan-Krishnaswamy method

- The inflection point and the point where the step response reaches 63,2 % of the total change are often close to each other.
- A better adaptation can be expected if we use two points that are slightly further apart.
- According Sundaresan and Krishnaswamy (1977) we will use the two points where the actual step response reaches 35% and 85% of the total change.

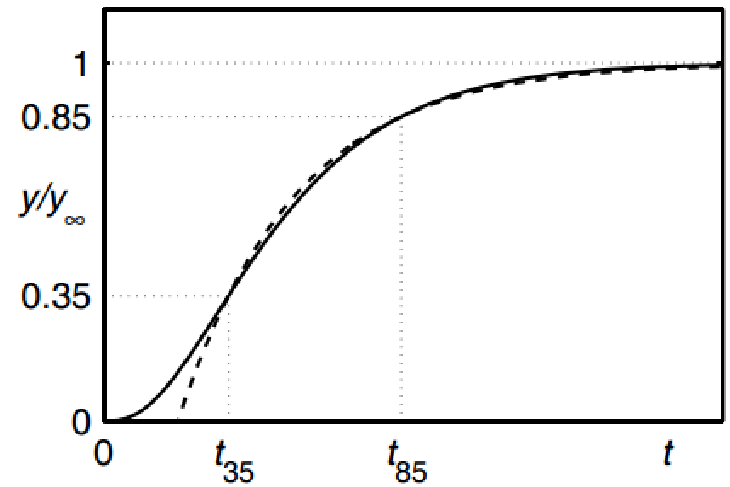

Figure 5.7. Identification of a 1st order system from 35% and 85% of the change.

If the time coordinate for each point is denoted t_{35} and t_{85}, we can then derive by using equation (5.9)

$$T = 0{,}682(t_{85} - t_{35}) \qquad (5.10)$$

$$L = t_{35} - 0{,}431T \qquad (5.11)$$

The gain K is calculated according equation (5.7)

5.2.2 Identification from step response

The Logarithmic method

There is a simple way to check how well an experimental step response is consistent with the step response of a first-order system (with or without time delay) without actually determining the model parameters.

We can derive the following relationship from equation (5.9)

$$\ln\left(\frac{y_\infty - y(t)}{y_\infty}\right) = -\frac{t-L}{T} \quad , \quad y_\infty = Ku_{\text{step}} \tag{5.12}$$

– If the expression to the left of the equation plots as function of t , we get then for a first-order system a straight line that has slope coefficient $-1/T$ and intersects the time axis (i.e. the value is 0) in the point $t = L$.

Same expression can be calculated and plotted for an arbitrary experimental step response.

– If the obtained relationship is sufficiently linear the system is then of first order.
  - At the same time, the system time constant is obtained based on the slope coefficient of the straight line, and its possible time delay from the intersection point of the line with the time axis.

5.2.2 Identification from step response

– If the relationship is moderately nonlinear we might consider to determine an approximate model of first order by fitting the relationship with a straight line.
  - One is tempted to draw a line so that the final asymptotic value coincides with the plotted experimental relationship when t tends to infinity.
  - However, this gives a too large time delay and too small time constant. A smaller slope must be chosen.

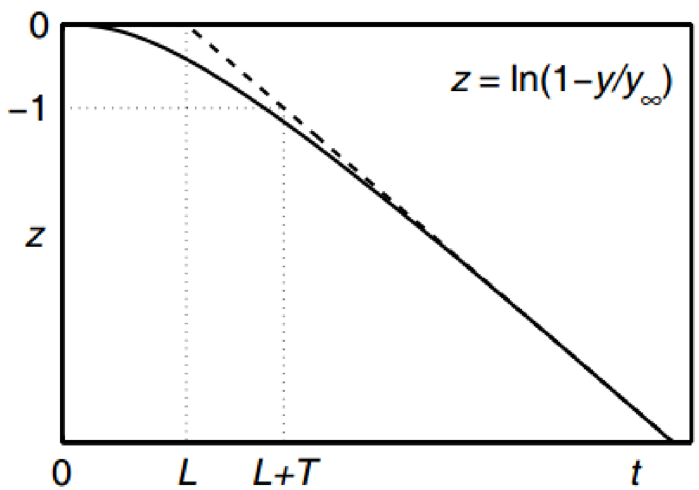

Figure 5.8. Identification of a 1st order system with the logarithmic method.

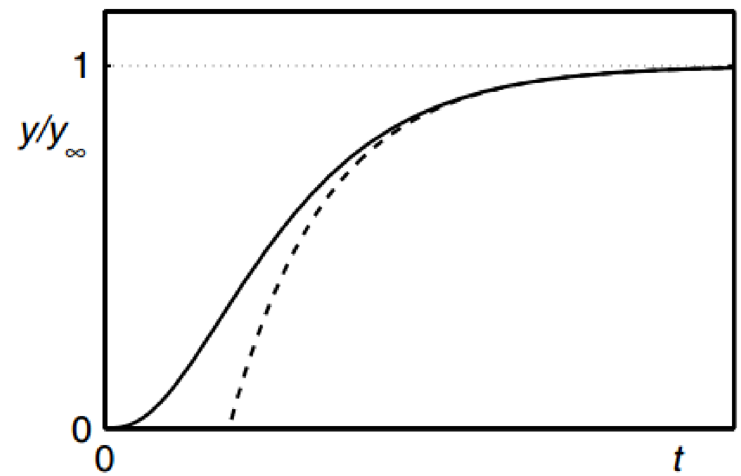

Figure 5.9. Step response of a 1st order system identified with the logarithmic method.

5.2.2 Identification from step response

In brief, we can say that

- the modified tangent method
- the method proposed by Sundaresan and Krishnaswamy (the points 35% and 85%) are clearly the bests of the simple graphics methods here presented for identification of a first-order system with time delay.

5.3 Second-order systems

A strictly linear system of second order can be described with the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + a_1 \frac{\mathrm{d}y}{\mathrm{d}t} + a_2 y = b_1 \frac{\mathrm{d}u}{\mathrm{d}t} + b_2 u \tag{5.13}$$

and the transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_1 s + b_2}{s^2 + a_1 s + a_2} \tag{5.14}$$

We will only consider systems with $b_2 \neq 0$ and in this section only the case where $b_1 = 0$, i.e. systems with transfer functions *without zeros*.

In section 5.5 we will consider systems with $b_1 \neq 0$.

5.3 Second-order systems

To emphasize the overall properties of the system the transfer function is often written in the form

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5.15}$$

where K is the *system gain*, ζ is called *relative damping* and ω_n is a constant called *undamped natural frequency*. Sometimes, the following form is also used

$$G(s) = \frac{K}{T_n^2 s^2 + 2\zeta T_n s + 1} \tag{5.14}$$

where $T_n = 1/\omega_n$. There is no universally accepted term for T_n, both *natural period* and *second-order time constant* are used.

The system is said to be *underdamped* if $0 \leq \zeta < 1$, *critically damped* if $\zeta = 1$ and *overdamped* if $\zeta > 1$. If $\zeta < 0$ the system is *unstable*.

5.3.1 Transient response

The transient response to a given input signal change can be determined in a normal way by using the Laplace transform inversion.

Here should be considered that the form of the solution is different depending on if the system is

- underdamped
- overdamped
- critically damped

The reason is that

- an underdamped system $(0 \le \zeta < 1)$ has complex poles (i.e. roots of the characteristic equation are complex)
- an overdamped system $(\zeta > 1)$ has real poles
- a critically damped system $(\zeta = 1)$ has a real double pole

Critically damped systems

The transfer function of a critically damped system is often written in the form

$$G(s) = \frac{K}{(T_\mathrm{n} s + 1)^2} \tag{5.17}$$

where $T_\mathrm{n} = 1/\omega_\mathrm{n}$. The impulse and step response are obtained by using the Laplace transform inversion of the expression $Y(s) = G(s)U(s)$.

For an impulse of magnitude I , is then $U(s) = I$, which gives the *impulse response*

$$y(t) = \frac{K\,I\,t}{T_\mathrm{n}^2} e^{-t/T_\mathrm{n}} \tag{5.18}$$

For a step change of magnitude u_step corresponds $U(s) = u_\mathrm{step} / s$, which gives the *step response*

$$y(t) = K u_\mathrm{step}\,(1 - (1 + t / T_\mathrm{n})\,\mathrm{e}^{-t/T_\mathrm{n}}) \tag{5.19}$$

The responses are represented in figure 5.10 and 5.11 ($\zeta = 1$) .

Overdamped systems

The transfer function of a overdamped system is written often in the form

$$G(s) = \frac{K}{(T_1 s + 1)(T_2 s + 1)} \tag{5.20}$$

where T_1 and T_2 are related to ω_n and ζ according (assuming $T_1 > T_2$)

$$T_1 = \frac{\zeta + \sqrt{\zeta^2 - 1}}{\omega_\mathrm{n}} \;,\; T_2 = \frac{\zeta - \sqrt{\zeta^2 - 1}}{\omega_\mathrm{n}} \tag{5.21}$$

$$\omega_n = \frac{1}{\sqrt{T_1 T_2}} \;,\; \zeta = \frac{T_1 + T_2}{2\sqrt{T_1 T_2}} \tag{5.22}$$

The system has the *impulse response*

$$y(t) = \frac{K\,I}{T_1 - T_2}\left(\mathrm{e}^{-t/T_1} - \mathrm{e}^{-t/T_2}\right) \tag{5.23}$$

and the *step response*

$$y(t) = K u_\mathrm{step}\left(1 - \frac{1}{T_1 - T_2}\left(T_1 \mathrm{e}^{-t/T_1} - T_2 \mathrm{e}^{-t/T_2}\right)\right) \tag{5.24}$$

The responses are represented in figure 5.10 and 5.11 ($\zeta > 1$) .

Underdamped systems

The fact that the characteristic equation of an underdamped system has complex roots makes the analytical expression of the system transient response to contain trigonometric functions.

The *impulse response* is given by

$$y(t) = K I \omega_n \beta^{-1} e^{-\zeta \omega_n t} \sin(\beta \omega_n t) \tag{5.25}$$

where

$$\beta = \sqrt{1-\zeta^2} \quad , \quad 0 \le \zeta < 1 \tag{5.26}$$

The *step response* is given by

$$y(t) = K u_{step} (1 - \beta^{-1} e^{-\zeta \omega_n t} \sin(\beta \omega_n t + \phi)) \tag{5.25}$$

Alternatively, the step response can be expressed by using trigonometric relationship as $\sin(\beta\omega_n t)$ and $\cos(\beta\omega_n t)$ (or another form of the Laplace transform) . The responses are represented in figure 5.10 and 5.11 ($\zeta < 1$) .

Normalized transient response

Figure 5.10 shows the impulse response and figure 5.11 the step response of different systems of second order without zeros. When the response and the time are normalized, as shown in the figures, the responses are clearly determined by the damping factor ζ.

The transient responses of an underdamped system oscillate, while the responses of a critically damped system and an overdamped system are monotone.

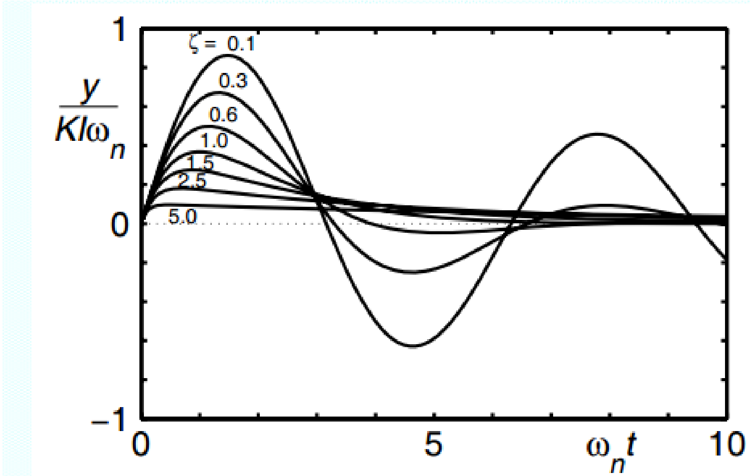

Figure 5.10. Impulse response of a system of second order without zeros.

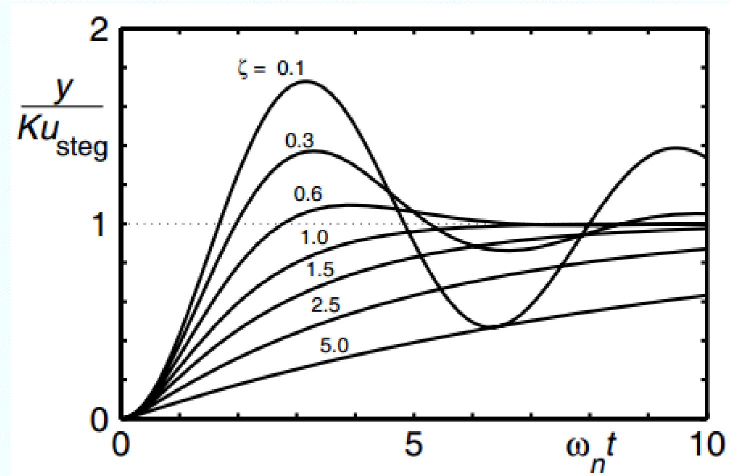

Figure 5.11. Step response of a system of second order without zeros.

5.3.2 Identification of overdamped systems

A simple method is described here for identification of a overdamped second order system without zeros by using its step response.

The system transfer function is given by equation (5.20), or if a time delay is include

$$G(s) = \frac{K\,\mathrm{e}^{-Ls}}{(T_1 s + 1)(T_2 s + 1)} \tag{5.29}$$

- The system gain K is determined as previously according equation (5.7).
- Any time delay is given by the time that the initial value of the step response is delayed in relation to the step change.
- The main problem is thus to determine the system time constants T_1 and T_2.

We will consider the case $T_1 \geq T_2$. As a limiting case, the system can be

- critically damped ($T_1 = T_2$),
- of first order ($T_2 = 0$).

5.3.2 Identification of overdamped systems

Modified Harriott's method

Harriott (1964) has developed a relatively simple graphic method for determining the transfer functions of the type (5.29) by using the step response.

Since numerical calculations of the type that form the basis of this method is not a problem nowadays, we will introduce here a slightly improved version of the Harriott's method.

General description

– All systems of the type (5.29) have a step response that reaches 72 % of the final total change at a time $t = L + 1{,}25(T_1 + T_2)$. If we first estimate the time delay L we can easily get the sum of the time constants from the step response at this point.

– The step responses of systems with different values of the parameter $z = T_1/(T_1 + T_2)$ are well separated at the time $t = L + 0{,}5(T_1 + T_2)$. The parameter z provides a good characteristic of the system properties because
  - A first order system has $z = 1$,
  - A critically damped second order system has $z = 0{,}5$,
  - for an overdamped second order system applies $0{,}5 < z < 1$.

5.3.2 Identification of overdamped systems

Figure 5.12 shows the step response of a first order system, a critically damped second order system and a overdamped second order system with $z = 0{,}8$.

– The step responses are normalized so that the output signal y is divided by the final change y_∞ and the time is indicated by the variable $\tau = (t - L)/(T_1 + T_2)$.

– The step responses reach 72% of the total change at $\tau = \tau_{72} \approx 1{,}25$ and they are well separated at $\tau = \tau_z \approx 0{,}5$.

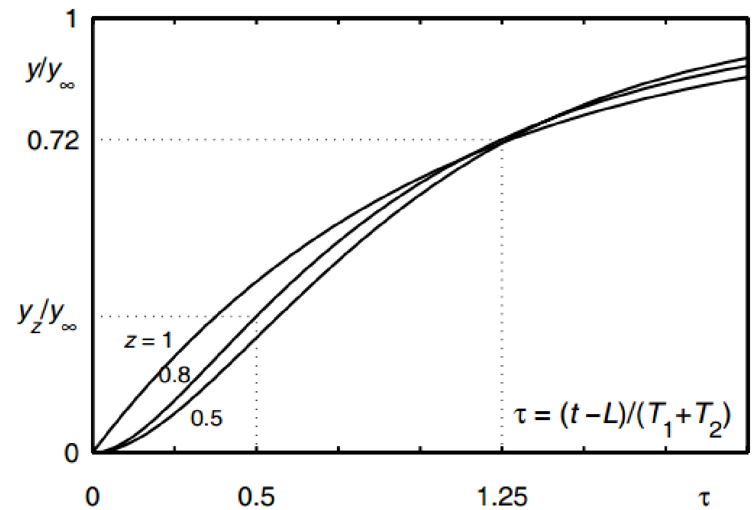

Figure 5.12. Step response of a overdamped second order system with different values of z.

– The sum of the time constants $\sum T_\mathrm{i}$ can be estimated by using the time τ_{72} , and the step response value at τ_z can be used for the estimation of the parameter z according to the graph in figure 5.13.

– When $\sum T_\mathrm{i} = T_1 + T_2$ and $z = T_\mathrm{i} / \sum T_\mathrm{i}$ are known the time constants T_1 and T_2 can then be calculated.

5.3.2 Identification of overdamped systems

– It is thanks to the time axis normalization in figure 5.12 that the step responses reach 72 % at the same normalized time τ_{72} and they have a good separation at another normalized time τ_z .

– This normalization requires that we know the sum of the time constants and the possible time delay, which we do not know, however, when identifying the system.

Procedure

Fortunately, the procedure is transformed so that it can be used with the actual time variable t .

– Indicate the time when the step response reaches 72 % of the total change according to the actual time scale with t_{72} .

– Estimate the sum of the time constants $\sum T_i$ according

$$\sum T_i = 0{,}8(t_{72} - L) \tag{5.30}$$

where L is the time delay, which is estimated separately. One can often (to start with) assume that $L = 0$, unless, it is obvious from the step response that some other value would be better.

5.3.2 Identification of overdamped systems

– Let t_z denote the time in the actual time scale which is $\tau_z = 0{,}4\tau_{72}$, where the step responses from different types of systems deviate most from each other. This time is given by

$$t_z = 0{,}4t_{72} + 0{,}6L \qquad (5.31)$$

– Denote the value of the output signal at t_z with y_z ; find it from the step response.

– Calculate the ratio $y_z \,/\, y_\infty$, where the final change y_∞ , as y_z , can be read from the step response.

– Read z from the diagram in figure 5.13 which corresponds to $y_z \,/\, y_\infty$
  – If $y_z \,/\, y_\infty < 0{,}27$, the time delay L has to be increased
  – If $y_z \,/\, y_\infty > 0{,}4$, the time delay L has to be decreased
  – If a new value for L was selected, calculate a new t_z etc.

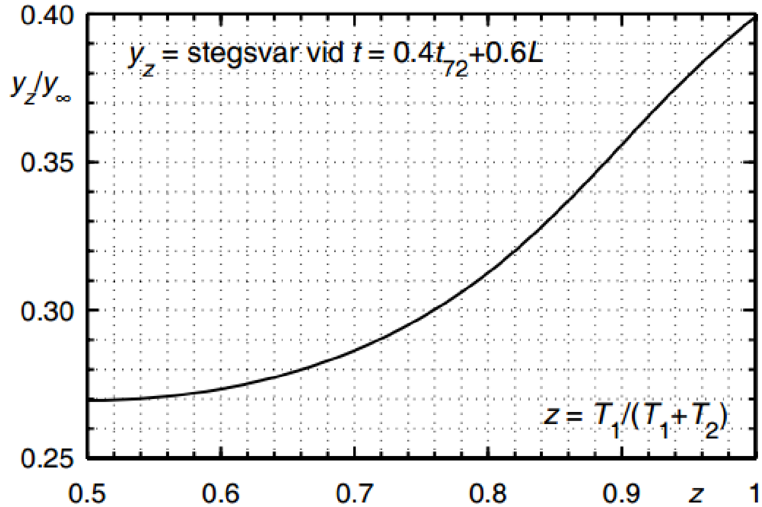

Figure 5.13. y_z for different values of z .

– Calculate the time constants of the system

$$T_1 = z\sum T_i \;\; , \;\; T_2 = \sum T_i - T_1 \qquad (5.32)$$

Iterative improvement

The procedure described above generally gives sufficient accurate estimate of T_1 and T_2. The method is based on the assumption that $t_{72} = 1{,}25$, but t_{72} varies slightly with z as shown in figure 5.14.

If t_{72}, according to figure 5.14, deviates from the value 1,25, the estimate of the time constants can be improved as follows:

– Read τ_{72} corresponding to z from figure 5.14.

– Calculate the new estimate of $\sum T_i$ according to

$$\sum T_i = (t_{72} - L) / \tau_{72} \qquad (5.33)$$

– Calculate new estimates of the time constants according (5.32).

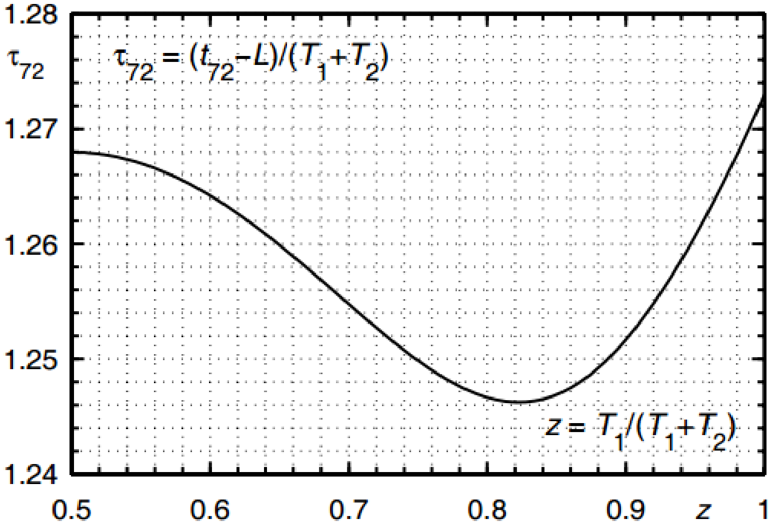

Figure 5.14. τ_{72} for different values of z .

Notice that the estimate of z , according to figure 5.13, is not affected because y_z does not change.

Example 5.2. Approximate identification of 1st and 2nd order systems.

By using the unit step response of a system described by the transfer function

$$G(s) = \frac{1}{(6s+1)(4s+1)(2s+1)} \tag{1}$$

we will determine

a) an approximate model of first order with time delay according to the modified tangent method;

b) an approximate model of second order with a possible time delay according to the modified Harriott's method.

For comparison's sake, we will also determine models optimally adapted to first and second order and compare the different step responses of the models with the exact step response.

For simplicity sake, we will calculate dimensionless times (i.e. we will not use any unit)

5.3.2 Identification of overdamped systems

The input signal u is an unit step, i.e. $U(s) = 1 / s$. We have, then, the unit step response

$$Y(s) = G(s)U(s) = \frac{1}{(6s+1)(4s+1)(2s+1)s} \tag{2}$$

This expression is not in our table of Laplace transforms, which means that we need to do partial fractions. We omit details and we find out that the inverse transform of the generic expression

$$F(s) = \frac{1}{(T_1 s+1)(T_2 s+1)(T_3 s+1)s} \tag{3}$$

gives the time function

$$f(t) = 1 - \frac{T_1^2}{(T_1 - T_2)(T_1 - T_3)} e^{-t/T_1} - \frac{T_2^2}{(T_2 - T_1)(T_2 - T_3)} e^{-t/T_2} - \frac{T_3^2}{(T_3 - T_1)(T_3 - T_2)} e^{-t/T_3} \tag{4}$$

In our case, with $T_1 = 6$, $T_2 = 4$ and $T_3 = 2$, we get the step response

$$y(t) = 1 - \frac{9}{2} e^{-t/6} + 4e^{-t/4} - \frac{1}{2} e^{-t/2} \tag{5}$$

as it is plotted in figure 5.15.

5.3.2 Identification of overdamped systems

For both, a) and b) case, the gain K is needed.

For an unit step we have $u_{\text{step}} = 1$, and according to figure 5.15 we have $y_\infty = 1$. Equation (5.7) gives then the gain $K = 1$.

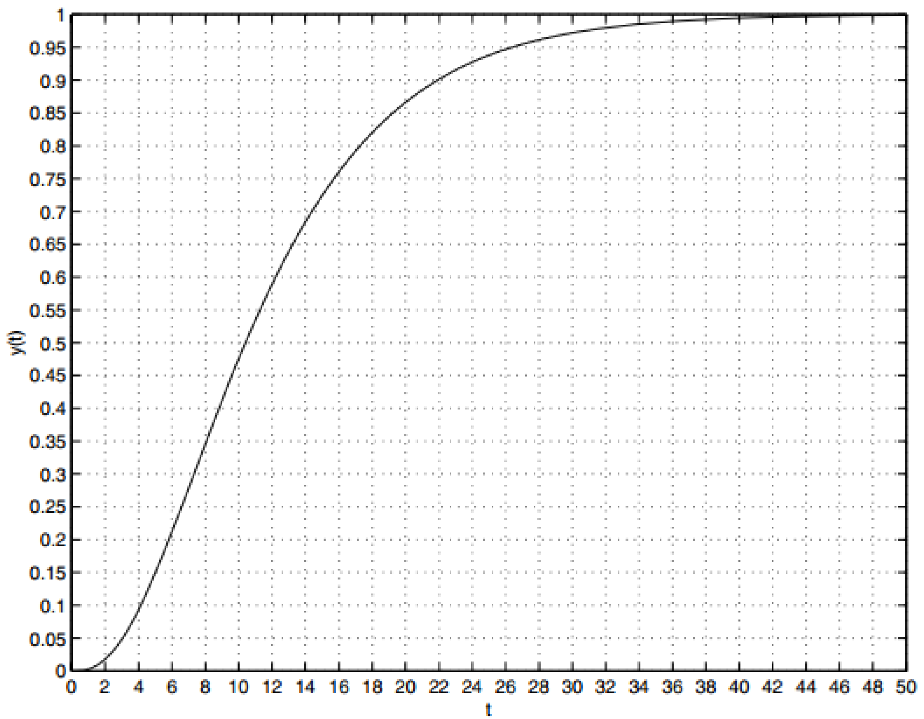

Figure 5.15. Unit step response of the system $G(s)$.

5.3.2 Identification of overdamped systems

a) We will determine a model of first order with time delay according to the modified tangent method.

- We begin by drawing a tangent through the point where the step response has its steepest slope, and by reading the point where the tangent intersects the time axis. The intersection point has the time coordinate $t \approx 2{,}5$, which gives the time delay $L \approx 2{,}5$.
- At 63 % of the total change y_∞ is $y \;= y_{63} = 0{,}63\, y_\infty = 0{,}63$. This value is obtained at $t \approx 12{,}5$ (too approximately), which means that $L + T \approx 12{,}5$.

We have determined a system of first order with $K = 1$, $T = 10$ and $L = 2{,}5$, i.e. a system with the transfer function

$$G_1(s) = \frac{1}{10s+1} \mathrm{e}^{-2{,}5s} \tag{6}$$

The unit step $U(s) = 1/s$ and the inverse transform of $Y_1(s) = G_1(s)U(s)$ gives the unit step response

$$y_1(t+2{,}5) = 1 - \mathrm{e}^{-t/10} \tag{7}$$

Figure 5.16 shows this step response together with the step response of the actual system.

5.3.2 Identification of overdamped systems

We can even determine the model parameters numerically by minimizing the square sum of the difference between the model step response and the step response of the actual system. Such optimization gives $K = 1{,}02$, $T = 9{,}05$ and $L = 3{,}83$.

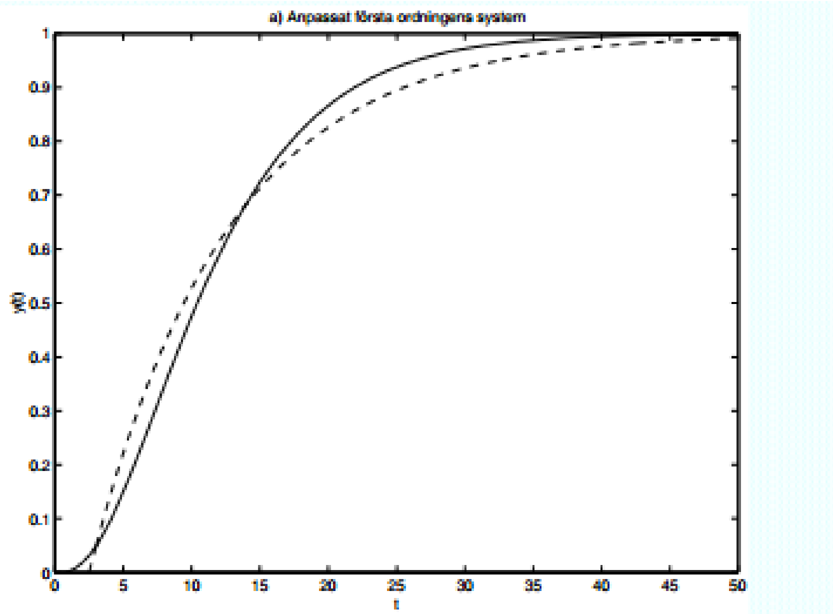

Figure 5.16. Unit step response of $G(s)$ (solid line) and $G_1(s)$ (dotted line).

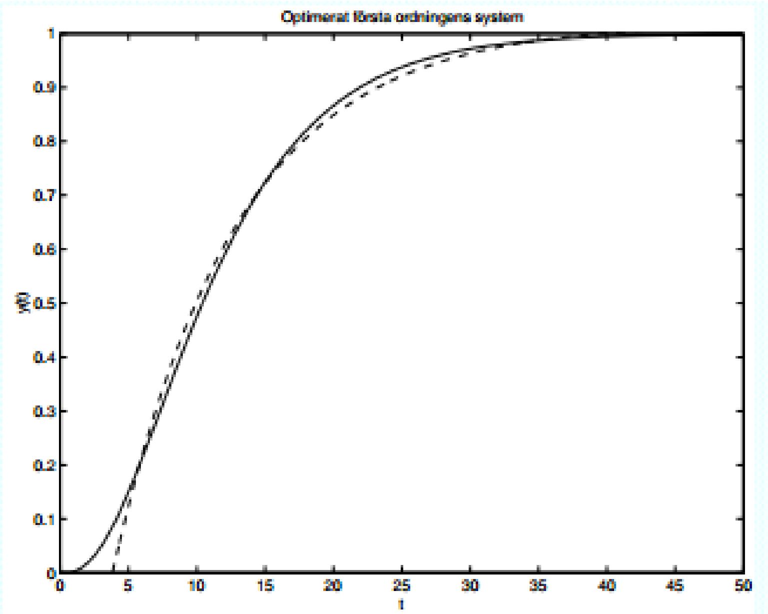

Figure 5.17. Unit step response of $G(s)$ (solid line) and the optimized adjustment (dotted line).

5.3.2 Identification of overdamped systems

b) We will determine a model of second order with the modified Harriott's method.

- We begin by determining the time when the system reaches 72 % of the total change. According to 5.15, we get $t_{72} \approx 15$.
- Based on the step response, it looks like if a time delay would be needed $L \approx 1$. However, we get a better adjustment by choosing a time delay that is slightly larger than the "actual", as is clear from the a)-case. Let us choose $L = 1{,}5$. According to equation (5.30), we get then $\sum T_i \approx 10{,}8$.
- The next step is to determine t_z according to equation (5.31), which gives here $t_z \approx 6{,}9$. At this time, according to the step response in figure 5.15, is $y = y_z \approx 0{,}275$, and $z \approx 0{,}6$ (too approximately) is given by $y_\infty = 1$.
- From figure 5.14, we can read then a revised $\tau_{72} \approx 1{,}264$, which gives, according to equation (5.33), a sum of the time constants $\sum T_i \approx 10{,}68$.
- Equation (5.32) gives $T_1 = z \sum T_i \approx 6{,}41$ and $T_2 = \sum T_i - T_1 \approx 4{,}27$.

5.3.2 Identification of overdamped systems

We have determined a system of second order with the transfer function

$$G_2(s) = \frac{1}{(6{,}41s+1)(4{,}27s+1)}\mathrm{e}^{-1{,}5s} \qquad (8)$$

that has the unit step response

$$y_2(t+1{,}5) = 1 - \frac{1}{2{,}14}\left(6{,}41e^{-t/6{,}41} - 4{,}27e^{-t/4{,}27}\right) \qquad (9)$$

This step response is represented in figure 5.18 together with the actual system step response.

– According to the figure the adjustment appear to be very god.

The adjustment to the real step response, for optimizing the parameters of a 2nd order system with time delay, gives $K = 1{,}002$, $T_1 = T_2 = 5{,}35$ and $L = 1{,}38$.

Figure 5.19 shows the step response of this system and the step response of the actual system.

– The adjustment is only marginally better than the one that was obtained with the modified Harriott's method.

5.3.2 Identification of overdamped systems

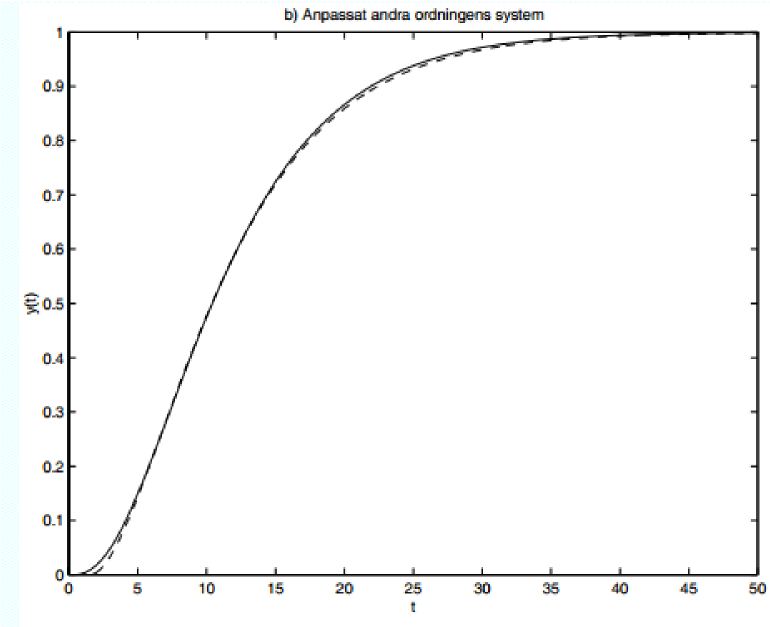

Figure 5.18. Unit step response of $G(s)$ (solid line) and $G_2(s)$ (dotted line).

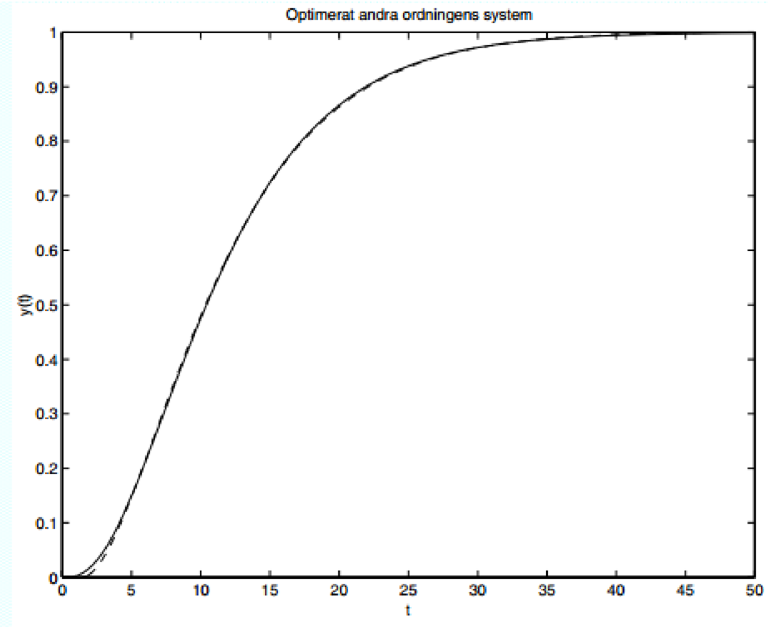

Figure 5.19. Unit step response of $G(s)$ (solid line) and the optimized adjustment (dotted line).

5.3.3 Identification of underdamped systems

As shown in figure 5.11, a step response of an underdamped system $(0 \leq \zeta < 1)$ is characterized by oscillations.

Obviously, the amplitude and frequency of the oscillations are used for identification of a second order underdamped system.

Systems with oscillatory step response can be characterized by different parameters that can be deduced from the step response. A number of such parameters are indicated in figure 5.20.

To facilitate the definition of the parameters, we assume that

- the initial value of the output signal is zero (i.e. we use deviation variable)
- the final value of the step response is positive (i.e. an input signal change so that the output signal will increase).

5.3.3 Identification of underdamped systems

y_∞ Output signal final value (>0)

$y_{\max}$ Output signal largest value, i.e. first peak of the overshoot.

M Maximum overshoot, $M = (y_{\max} - y_\infty) / y_\infty$.

P Time period of the oscillations (specially the first period).

t_r Rise time = time required for the output signal to pass y_∞ . Typically, is defined as the time that takes to change from 10 % to 90 % of y_∞ .

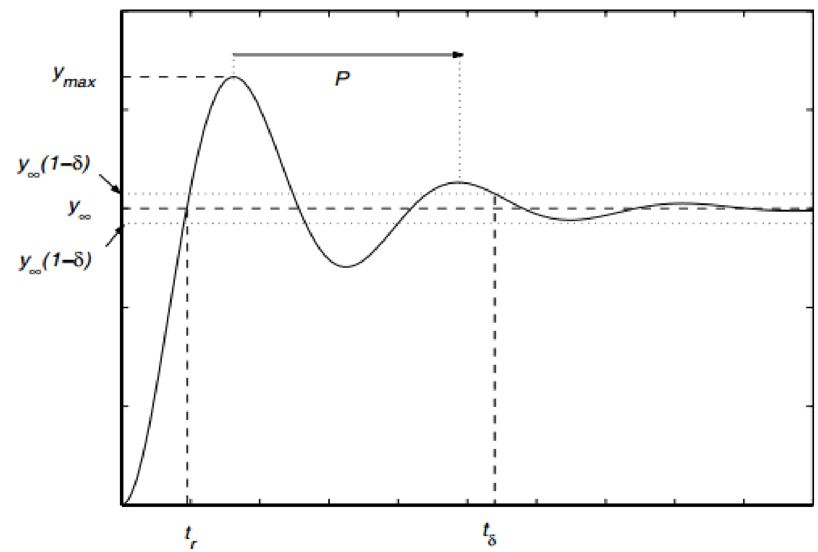

Figure 5.19. Step response of a underdamped system.

t_δ Settling time, is the time it takes until the output signal has entered and remained within $(1-\delta)y_\infty$ and $(1+\delta)y_\infty$, i.e. $(1-\delta)y_\infty \leq y(t) \leq (1+\delta)y_\infty$.

Normally, $\delta = 0{,}05 = 5\%$ or $\delta = 0{,}02 = 2\%$ are used.

5.3.3 Identification of underdamped systems

Based on the analytical solution of the system step response, we can derive an expression that relates those parameters to the parameters in the system transfer function

$$G_2(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad , \; 0 \le \zeta < 1 \tag{5.34}$$

By using the notation:

$$\beta = \sqrt{1-\zeta^2} \tag{5.35}$$

for the *maximum overshoot* we get:

$$M = \frac{y_{\max} - y_\infty}{y_\infty} = e^{-\pi\zeta/\beta} \tag{5.36}$$

for *the time period*:

$$P = \frac{2\pi}{\beta\omega_n} \tag{5.37}$$

for *the rise time*:

$$t_r = \frac{\pi - \arctan(\beta/\zeta)}{\beta\omega_n} \tag{5.38}$$

These expressions are exactly derived. For *the settling time* applies approximately to

$$t_\delta \approx -\frac{\ln(\delta)}{\zeta\omega_n} \quad , \qquad M > \delta \tag{5.39}$$

5.3.3 Identification of underdamped systems

Identification

It is easiest, and sufficient, to measure M and P .

– The system relative damping ζ can be determined by the equations (5.35) and (5.36).
– The undamped natural frequency ω_n is obtained from equation (5.37).
– The rise time and eq. (5.38) can even be used instead of (5.36) and (5.37).

The system gain K is determined according to equation (5.7).

Generally, the step response of a substantially underdamped system is sensitive to disturbances, variations in the parameters and deviations from assumptions of an ideal system.

- mostly affects the initial response of the system and thus the first overshoot
- better results if the identification is based on several oscillations

We denote the n^{th} overshoot's maximum value with $y_{\max,n}$ and the n^{th} undershoot's minimum value with $y_{\min,n}$. Using equation (5.27), we can derive

$$\frac{y_{\max,n+k} - y_\infty}{y_{\max,n} - y_\infty} = \frac{y_\infty - y_{\min,n+k}}{y_\infty - y_{\min,n}} = M_\mathrm{R}^k = \mathrm{e}^{-2\pi k\zeta/\beta} \qquad (5.40)$$

where M_R^k represents the ratio between the $(n+k)^{\mathrm{th}}$ and the n^{th} relative overshoot (or undershoot).

Example 5.3. Identification of an underdamped second order system.

We will identify an underdamped second order system based on the step response in figure 5.20. In the figure The time axis in the figure goes from 0 to 20 seconds and the output signal axis from 0 to 2,5.

From the figure, we obtain

$$M = \frac{y_{\max} - y_\infty}{y_\infty} \approx \frac{2{,}17 - 1{,}5}{1{,}5} = 0{,}447 \quad \text{and} \quad P \approx 9{,}75 - 3{,}25 = 6{,}5$$

Equation (5.35) and (5.36) can be solved with respect to ζ, which gives

$$\zeta = \frac{-\ln(M)}{\sqrt{\pi^2 + \ln^2(M)}} \tag{1}$$

Numerically, we get ζ = 0,2485. The undamped natural frequency is given equation (5.37) $\omega_{\mathrm{n}} = 0{,}998$.

The gain K can not be determined because the magnitude of the input signal step is not given.

The correct values are $\zeta = 0{,}25$ and $\omega_{\mathrm{n}} = 0{,}1$.

5.4 Time-delay systems

The output signal from a system consisting only of a time delay L looks exactly like the input signal, but delayed with L time units.

If the output signal is denoted $y(t)$ and the input signal $u(t)$, the response for a pure time delay is thus

$$y(t + L) = u(t) \tag{5.41}$$

In practice, a time delay depends often on a transport delay.

- A typical example is a conveyor belt.
- Even with liquid flows and gas flows in a pipeline occurs time delays on the properties of the flowing medium such as temperature and concentration.
- Measuring instruments can sometimes lead to a time delay, for example when analysing measuring samples.

The transfer function of a time delay of magnitude L is

$$G(s) = e^{-Ls} \tag{5.42}$$

This function is basically simple, but as we know, time delays cause problems in control engineering.

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5.4 Time-delay systems

- Systems with time delay belong to *non-minimum phase* systems (see chapter 8).
- That is why time delays give analytical problems and computational problems, especially in combination with other system elements.
- The reason is that the transfers functions for other types of system elements are rational functions, while the transfer function of a time delay is an irrational function. Therefore, we have a reason to use rational approximations of (5.42).

Simple rational approximations can be derived from the Taylor series of e^{-Ls}, i.e.

$$\mathrm{e}^{-Ls} = 1 - Ls + \frac{(Ls)^2}{2!} - \frac{(Ls)^3}{3!} + \ldots \tag{5.43}$$

The first two terms give the simple, but relatively imprecise, approximation

$$\mathrm{e}^{-Ls} \approx 1 - Ls \tag{5.44}$$

The more terms there are, the better the approximation. But, in practice, the handling of the expression becomes more difficult when the degree of the polynomial increases.

5.4 Time-delay systems

Another possibility is to rewrite the series expansion

$$e^{-Ls} = \frac{1}{e^{Ls}} = \frac{1}{1 + Ls + \frac{(Ls)^2}{2!} + \frac{(Ls)^3}{3!} + \ldots} \tag{5.45}$$

If only the first two terms in the denominator are considered, we get then the approximation

$$e^{-Ls} \approx \frac{1}{1 + Ls} \tag{5.46}$$

which means that the time delay L is approximated to a first order system with time constant L . If more terms are included, we get then approximations with systems of higher order.

5.4 Time-delay systems

We can combine the methods in different ways. One way is to rewrite

$$e^{-Ls} = \frac{e^{-\frac{1}{2}Ls}}{e^{\frac{1}{2}Ls}} \tag{5.47}$$

and the Taylor series expansions of the nominator and the denominator. If only the first two terms of the Taylor series expansions are considered

$$e^{-Ls} \approx \frac{1-\dfrac{1}{2}Ls}{1+\dfrac{1}{2}Ls} = -1 + \frac{2}{1+\dfrac{1}{2}Ls} \tag{5.48}$$

i.e. a proper, but not strictly proper, first order system that has quite special properties, as shown by the right side of the expression above

Padé approximations are another type of approximations that are derived under some optimized conditions.

The first-order Padé approximation is identical to (5.48) while the second-order Padé approximation is

$$\mathrm{e}^{-Ls} \approx \frac{1 - \frac{1}{2}Ls + \frac{1}{12}(Ls)^2}{1 + \frac{1}{2}Ls + \frac{1}{12}(Ls)^2} \tag{5.49}$$

Note that (5.49) is *not* obtained starting from the cut-off Taylor series.

Padé approximations –there are also approximations of higher order – are derived so that their frequency responses (see chapter 8) are similar to the frequency response of the time delay (both have the gain $K = 1$ at all frequencies), while the time response deviates more.

5.4 Time-delay systems

The exponential function can be defined by using the limit

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \tag{5.50}$$

If we rewrite $e^{-Ls} = 1/e^{Ls}$, we then obtain the following approximation

$$e^{-Ls} \approx \frac{1}{\left(1+\frac{1}{n}Ls\right)^n} \tag{5.51}$$

i.e. a n^{th} order system where we can choose the order.

The order $n = 1$ gives the same approximation as (5.46). Higher order gives of course a better approximation.

5.5 Inverse-response systems

– Systems with inverse response show step responses whose direction changes one or several times at the beginning of the step response.

– This shall not be confused with the oscillations of an underdamped system, whose step response oscillates around the desired value where the output signal reaches a steady state condition.

– Systems with inverse response belong to the group of systems that are known as *non-minimum phase systems* (see chapter 8).

Systems with inverse response are not uncommon.

– A simple example is the quicksilver thermometer. The glass tube expands first when the ambient temperature increases, causing the column of quicksilver to fall. The quicksilver shortly begins to expand (density decreases) and the level change starts going to right direction.

– Another example with the same type of behavior is the liquid level in a boiler when the feed water supply increases.

5.5 Inverse-response systems

Systems with inverse response are difficult to control because sometimes they give out misleading information.

Such systems are characterized by a transfer function with (one or more) positive zeros, which are equivalent to negative time constants in the numerator.

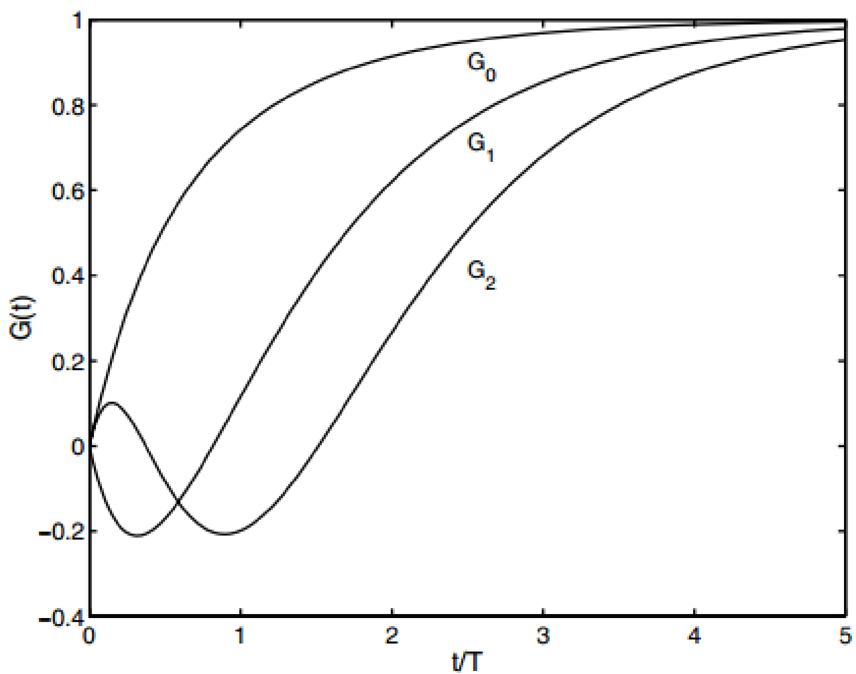

Figure 5.21. Step responses with different number of negative time constants in the numerator.

$$G_0 = \frac{\left(\frac{T}{1,5}s+1\right)\left(\frac{T}{2,5}s+1\right)}{\left(\frac{T}{1}s+1\right)\left(\frac{T}{2}s+1\right)\left(\frac{T}{3}s+1\right)}$$

$$G_1 = \frac{\left(-\frac{T}{1,5}s+1\right)\left(\frac{T}{2,5}s+1\right)}{\left(\frac{T}{1}s+1\right)\left(\frac{T}{2}s+1\right)\left(\frac{T}{3}s+1\right)}$$

$$G_2 = \frac{\left(-\frac{T}{1,5}s+1\right)\left(-\frac{T}{2,5}s+1\right)}{\left(\frac{T}{1}s+1\right)\left(\frac{T}{2}s+1\right)\left(\frac{T}{3}s+1\right)}$$

5.5 Inverse-response systems

Such examples suggest that inverse-response systems occur when two subsystems are connected in parallel whose gains have different signs.

Exercise 5.2.

Two subsystems with transfer functions

$$G_1 = \frac{K_1}{(T_1 s + 1)} \quad \text{and} \quad G_2 = \frac{K_2}{(T_2 s + 1)}$$

are connected in parallel so that a system with transfer function $G = G_1 + G_2$ is obtained.

Assume that $T_1 > T_2 > 0$ and show that G is a non-minimum phase system if

$$\frac{T_1}{T_2} > -\frac{K_1}{K_2} > 1$$

5.6 Systems in series

In the analysis of series-connected systems it is important to know if the systems are *interfering* or *non-interfering*.

- in an interfering system one subsystem is affected by the subsequent subsystem in the series
- in an non-interfering system each subsystem is only affected by previous subsystems in the series.

For instance, if we connect in series two copies of the lowpass filter in example 3.1, they will then

– interfere because the subsequent circuit charges the previous one.

– On the other hand, if we provide the first filter with a gain on the output side, then they will not interfere with each other.

5.6 Systems in series

A similar situation can be obtained if we connect in series two container for liquids, where the outflow occurs due to gravity (comp. with example 3.5).

- If the outflow from the first container flows freely into the second one, then no interference occurs.
- If the containers are connected so that the outflow from the first container flows through a tube to the bottom of the second container, then interference occurs due to the counter-pressure of the liquid level exerted on the inflow in the second container.

To summarize, we can conclude the following about systems in series:

- Non-interfering subsystems in series are easy to handle. Their transfer functions can be derived separately and then combined by multiplication as shown in section 4.3.3.
- Interfering subsystems are difficult to handle, since the individual characteristics of each subsystem are modified by the interference. In such cases we must model and consider the subsystems as a whole.