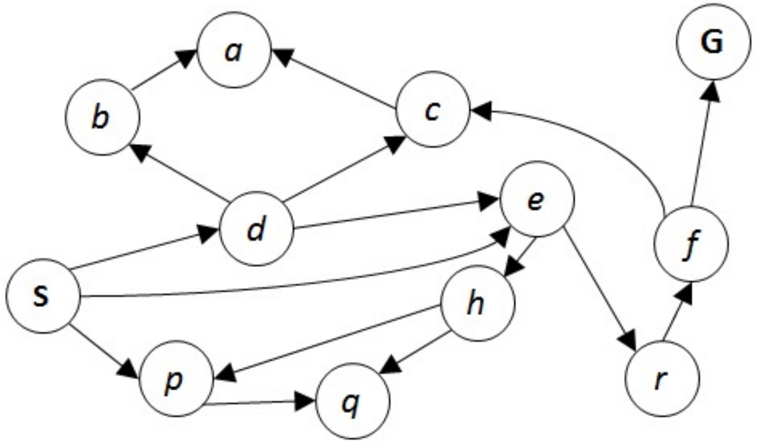
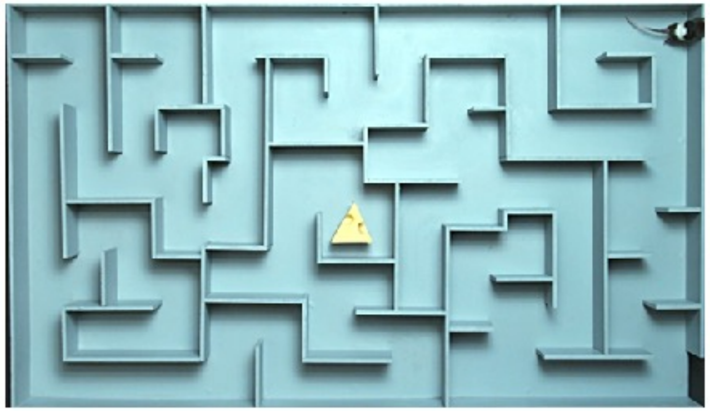
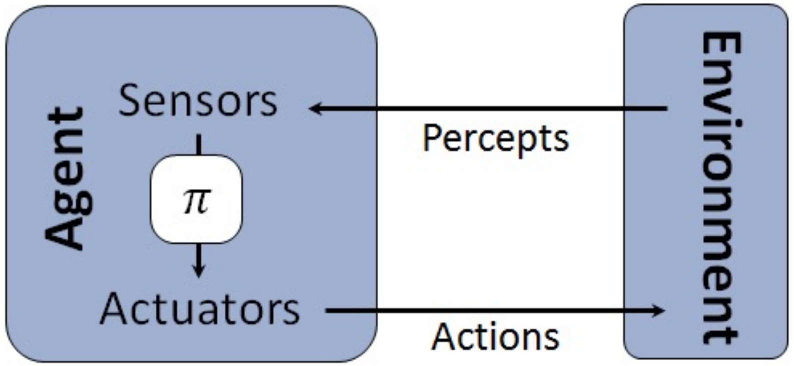
COMP 341 Intro to AI
Constraint Satisfaction Problems

Asst. Prof. Barış Akgün

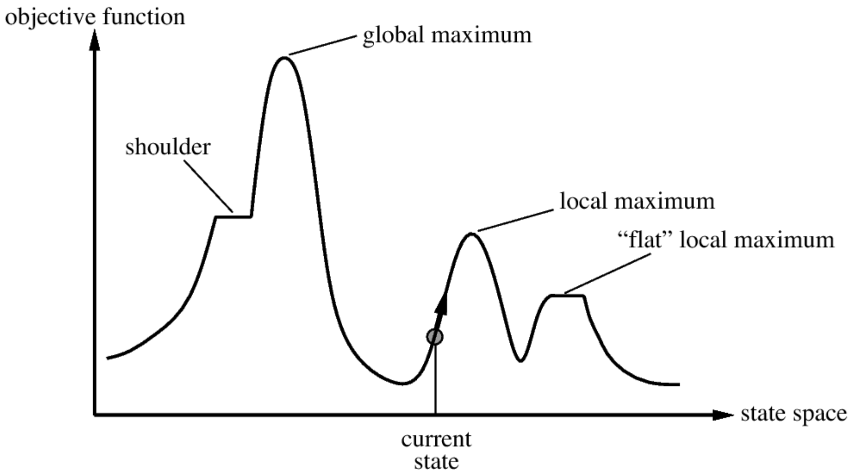
Koç University

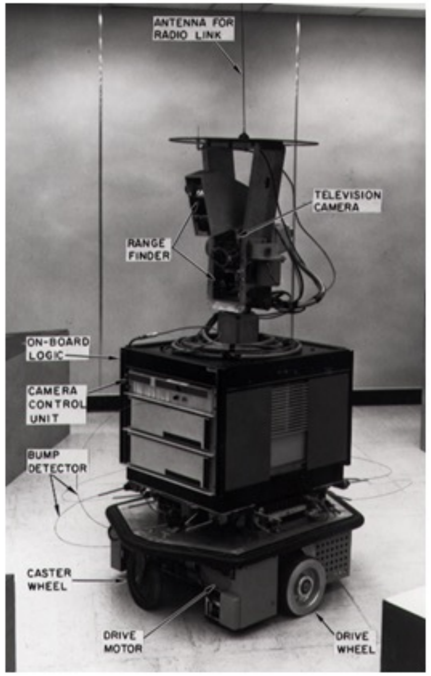






Previously on Intro to AI





Search

- Uninformed
  - DFS, BFS, UCS
  - No domain knowledge

- Informed
  - Greedy, A*
  - Heuristics based on domain/problem knowledge

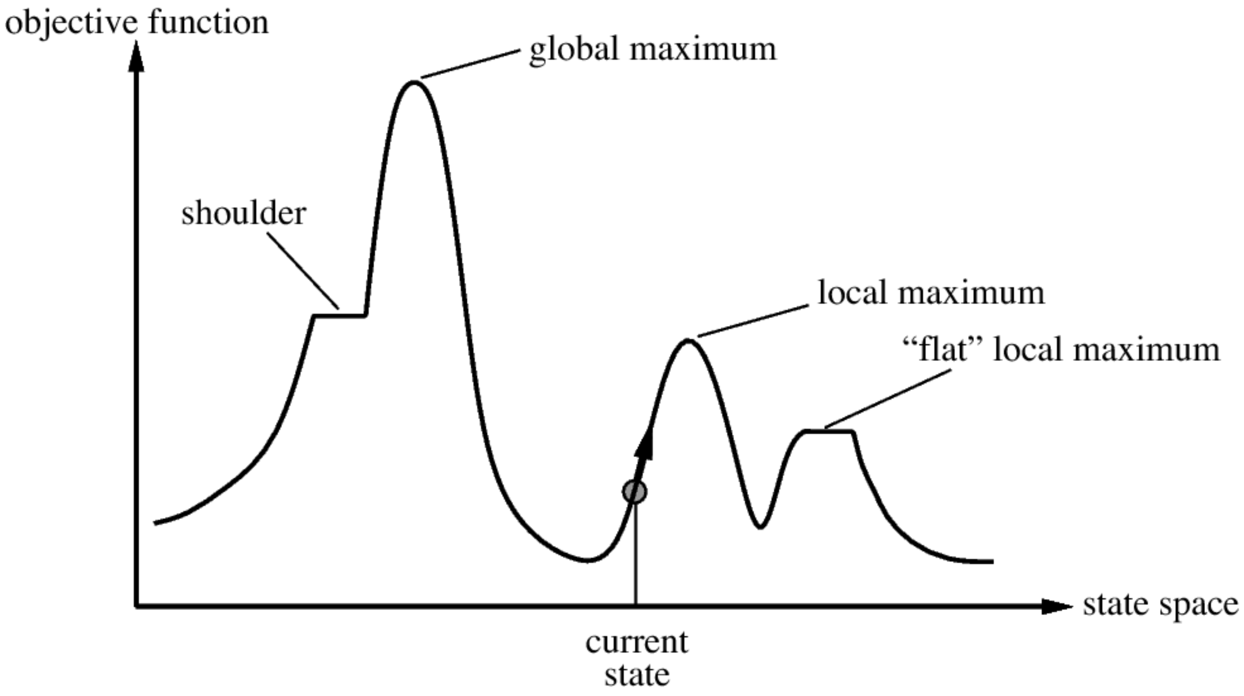
- Solution is a path to goal

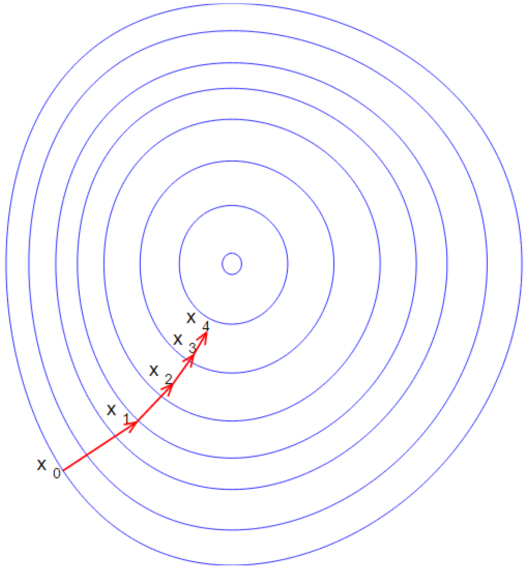
Local Search

- Solution is important, not the path!

- Hill Climbing
- Simulated Annealing
- Local Beam Search
- Genetic Algorithms

- Gradient Descent





Local Search

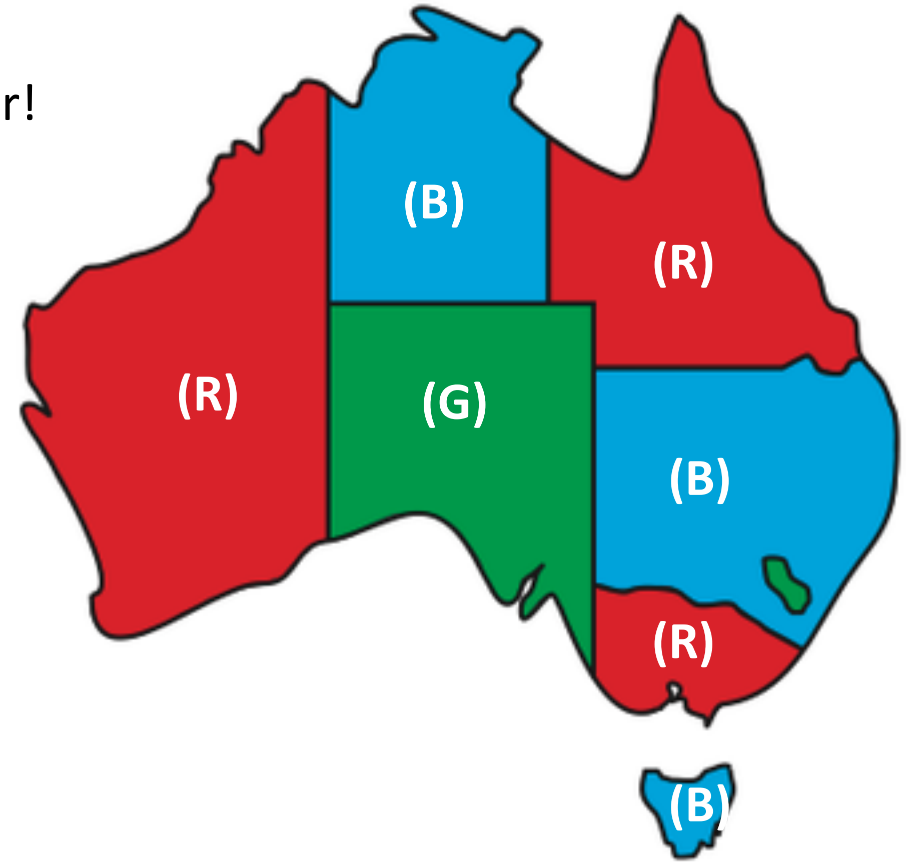
- Formulation:
  - Current State
  - Transition Function
  - Evaluation Function and State Space “Landscape”

- Algorithms: Move towards Better States (Where the “Local” comes from)
  - Complete: Find a solution if one exists
  - Optimal: Find the best state

- Usually easy to code!

Constraint Satisfaction Problems

No neighbors with the same color!



Search So Far

- Classical Search:
  - Solution is path to a goal state
- Local Search:
  - Solution is the goal state itself
- CSPs?
  - Goal matters
  - States and goal test have specific structure!
  - Allows for general heuristics

Constraint Satisfaction Problems

- Standard Search
  - State is a black box data structure
  - Goal test: Can be any Boolean function of states
  - Successor Function: Can be anything that returns valid states
  - Heuristic function: Can be anything that maps states to a non-negative scalar
- CSPs
  - State is defined by variables X_i with values from domain D_i.
    - Map Coloring Example: Variables are the color of each Australian state and the domain is the set of allowable colors
  - Goal test is a set of constraints specifying allowable combinations of values for subsets of variables
    - Map Coloring Example: All states are colored and neighboring states do not have the same color
- This structure allows useful general-purpose algorithms with more power than standard search algorithms

CSPs

- State is defined by variables X_i with values from domain D_i

X_1	X_2	…	X_n
$\{d_1, d_2, \ldots, d_m\}$	$\{d_1, d_2, \ldots, d_m\}$		$\{d_1, d_2, \ldots, d_m\}$

Domains of variables can be different!

- Goal test is a set of constraints specifying allowable combinations of values for subsets of variables. E.g.

$$\sum_{i \in A} X_i == k \qquad X_i \neq X_j \text{ for } i \neq j, i \in A, j \in A$$

Real Life Example (!) - Carpool

- Ahmet, Elif, Mehmet, Zeynep want to carpool to Bolu
  - Variables are A,E,M,Z
- There are only 2 cars
  - Domains = $\{C_1, C_2\}$
- The cars belong to Ahmet and Zeynep
  - Consraints: $A = C_1$ and $Z = C_2$
- Ahmet and Elif do not like each other
  - Constraint: $A \neq E$
- Mehmet has a crush on Zeynep
  - Constraint: $M = Z$
- A solution
  - $A=C_1$, $E=C_2$, $M= C_2$, $Z=C_2$

Side Note: They should just sit Elif in the front and Ahmet in the back and take 1 car

But the problem does not model that!

Solving CSPs

- Each state of the problem is a possible assignment to some or all the variables

- **Legal Assignment**: no violations
- **Complete Assignment**: every variable assigned

Map Coloring

- Color the map such that no two neighbors have the same color

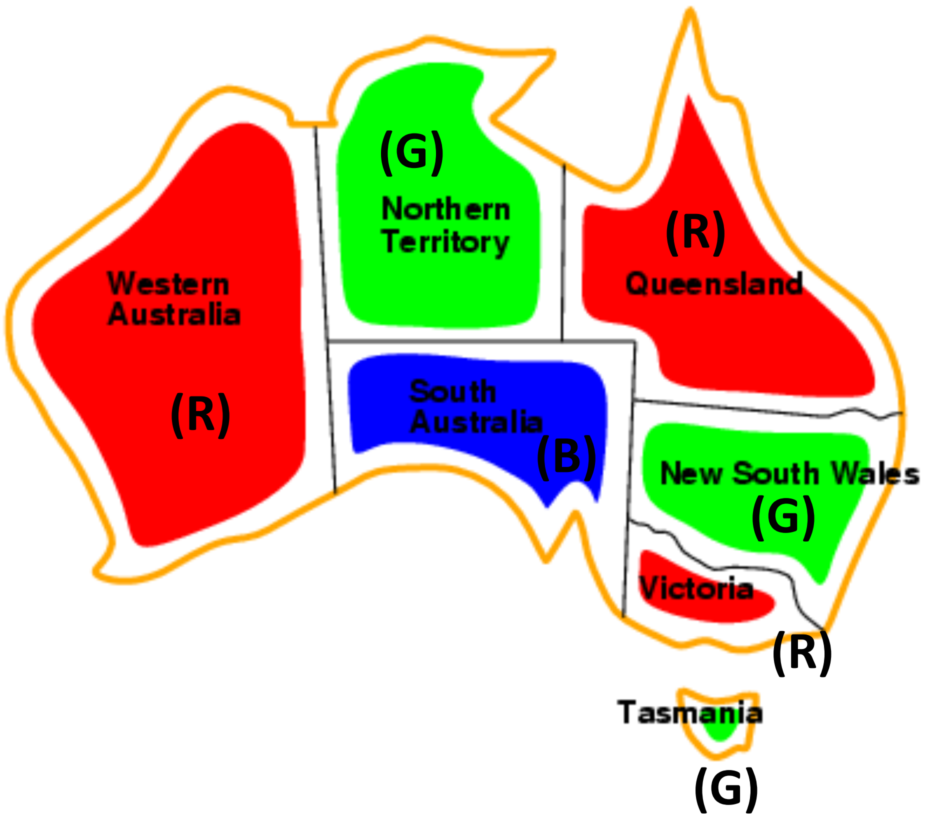
- Variables:
  - *WA, NT, Q, NSW, V, SA, T*
- Domains:
  - D_i = {red, green, blue}
- Constraints: adjacent regions must have different colors
  - Implicit: *WA* ≠ *NT*
  - Explicit: (*WA, NT*) ∈ {(red, green), (red, blue), (green, red), (green, blue), . . .}

Map Coloring

- Color the map such that no two neighbors have the same color

- Solutions are assignments satisfying all constraints, e.g.,

{WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green}

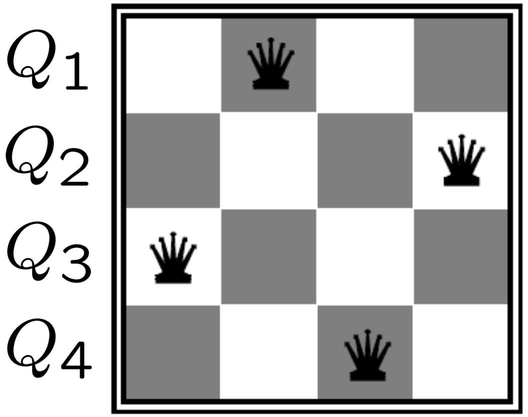


Example: N-Queens

- Variables: Q_k

- Domains: {1,2,…,N}

- Constraints:



Implicit: $\forall i, j \;\; \text{non-threatening}(Q_i, Q_j)$

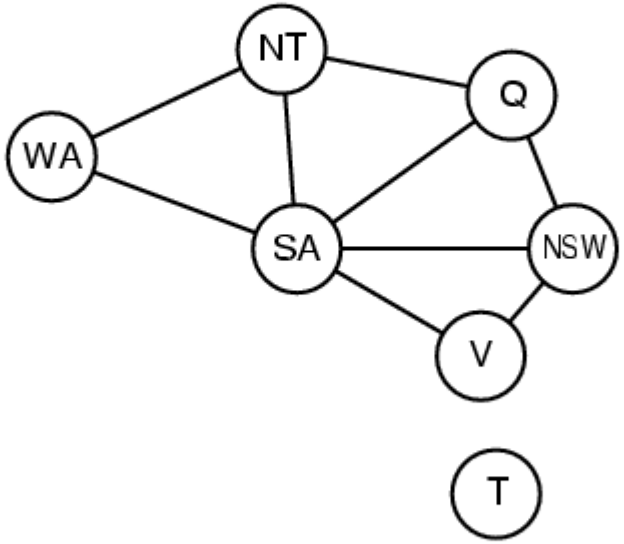
Explicit: $(Q_1, Q_2) \in \{(1,3), (1,4), \ldots\}$

$\cdots$

Constraint Graph

- Binary CSP: each constraint relates (at most) two variables

- **Binary Constraint Graph** is a data structure we use to represent the problem
  - Nodes are variables
  - Arcs show which variables are constrained



Example: Cryptarithmetic

- Variables:

  $$F \ T \ U \ W \ R \ O \ X_1 \ X_2 \ X_3$$

- Domains:

  $$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

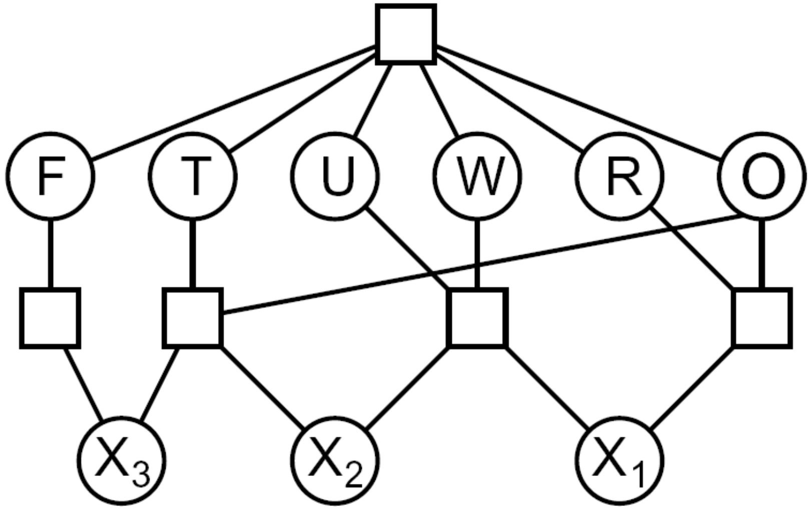
- Constraints:

  $$\text{alldiff}(F, T, U, W, R, O)$$

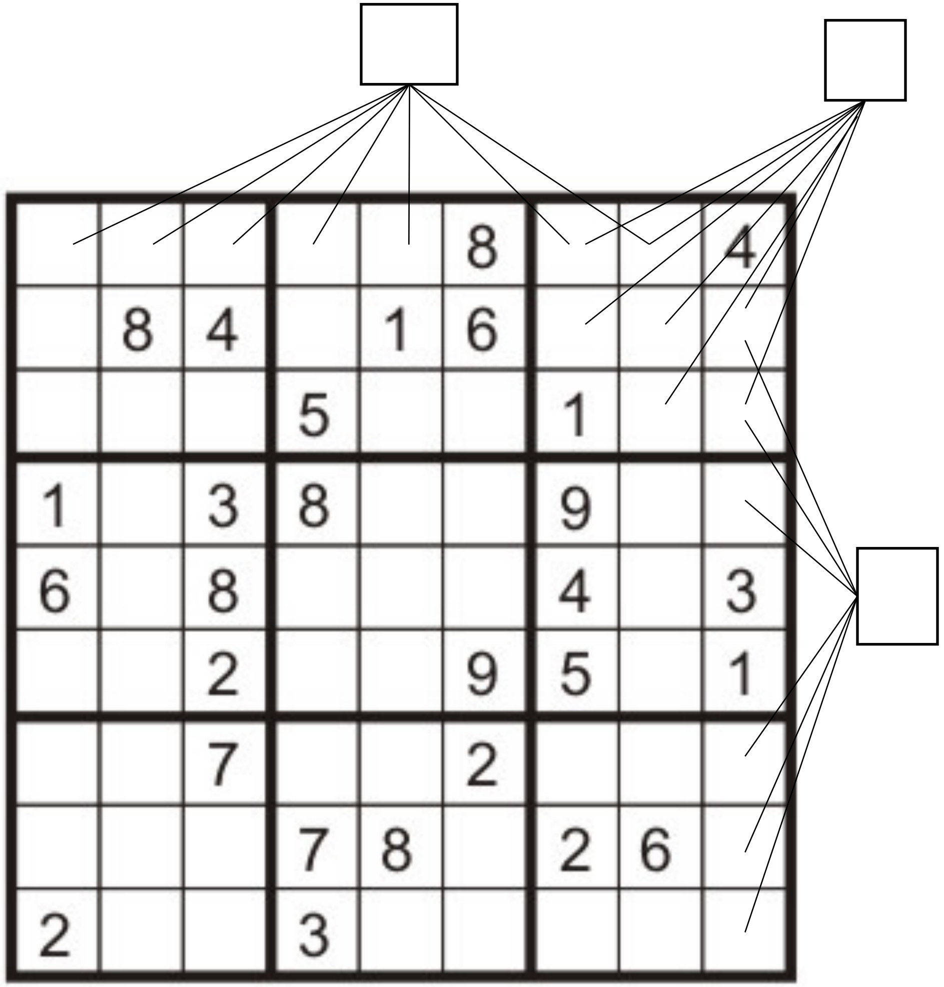
  $$O + O = R + 10 \cdot X_1$$

  $$\cdots$$

$$\begin{array}{cccc} & \mathrm{T} & \mathrm{W} & \mathrm{O} \\ + & \mathrm{T} & \mathrm{W} & \mathrm{O} \\ \hline \mathrm{F} & \mathrm{O} & \mathrm{U} & \mathrm{R} \end{array}$$



Most Famous CSP - Sudoku



- Variables:
  - Each (open) square
- Domains:
  - {1,2,…,9}
- Constraints:

9-way alldiff for each column

9-way alldiff for each row

9-way alldiff for each region

(or can have a bunch of pairwise inequality constraints)

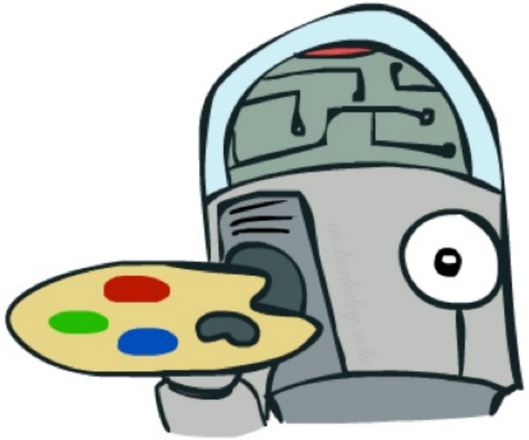
Other Real-World Examples

- Assignment problems: e.g., who teaches what class
- Timetabling problems: e.g., which class is offered when and where?
- Hardware configuration
- Transportation scheduling
- Factory scheduling
- Floor Planning
- …

- Many real-world problems involve real-valued variables…

Varieties of CSPs

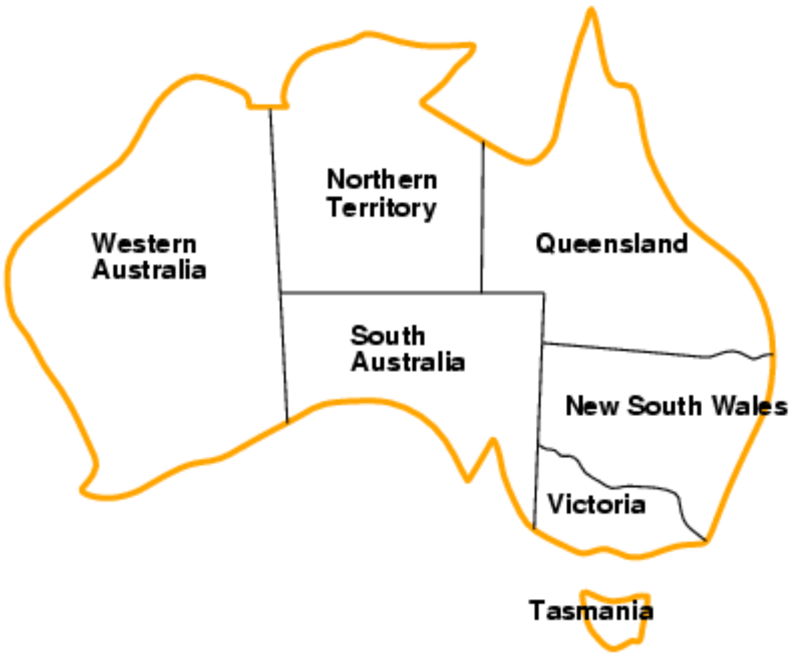
- Discrete Variables
  - Finite domains
    - Size d means $O(d^n)$ complete assignments
    - E.g., Boolean CSPs, including Boolean satisfiability (NP-complete)
  - Infinite domains (integers, strings, etc.)
    - E.g., job scheduling, variables are start/end times for each job
    - Need constraint language: *Job1 + 5 < Job2*
    - Linear constraints solvable, nonlinear undecidable



- Continuous variables
  - E.g., start/end times for Hubble Telescope observations
  - Linear constraints solvable in polynomial time by Linear Programming methods (ever heard of the Simplex Method?)

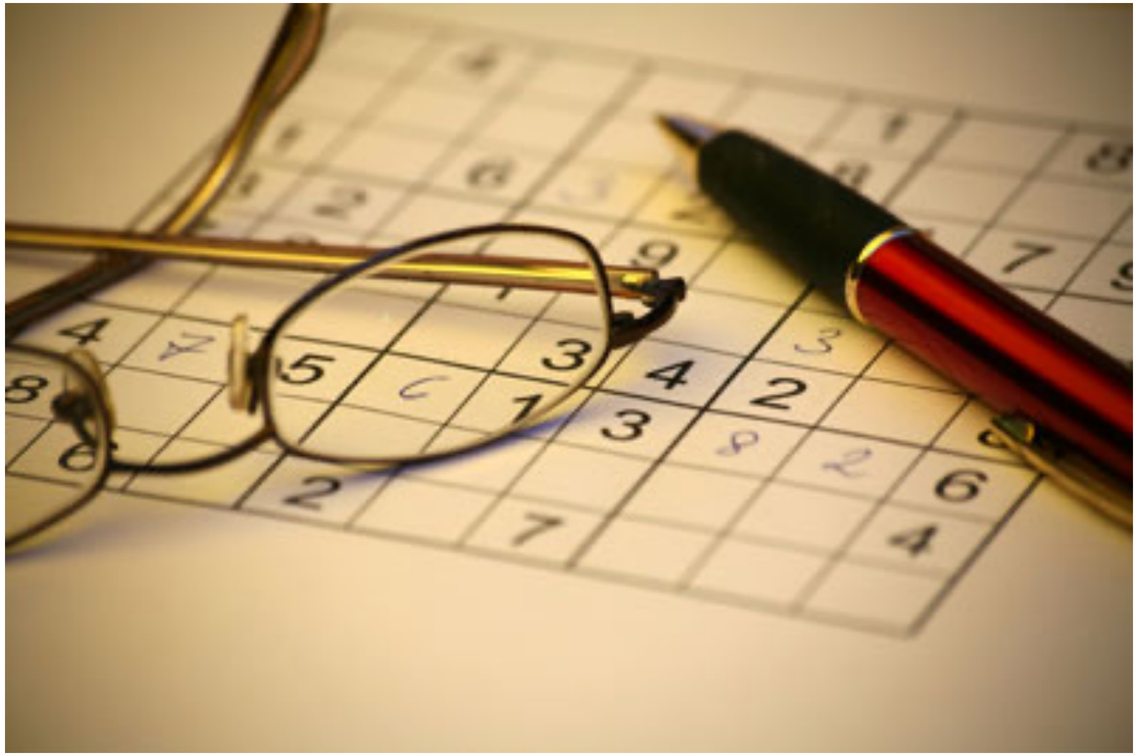
Varieties of Constraints

- Varieties of Constraints
  - Unary constraints involve a single variable (equivalent to reducing domains)
    e.g.: $SA \neq green$
  - Binary constraints involve pairs of variables,
    e.g.: $SA \neq WA$
  - Higher-order constraints involve 3 or more variables:
    e.g.: cryptarithmetic column constraints



- Preferences (soft constraints):
  - E.g., red is better than green
  - Often representable by a cost for each variable assignment
  - Gives rise to constrained optimization problems
  - We might come back to this later on in the course

Solving CSPs



Search Formulation

Local Search Formulation

Search Formulation for CSPs

- Initial State: {}
- Successor(): assign a value (consistent with constraints) to an unassigned variable
- Goal Test(): All variables are assigned and all constraints are satisfied
- Failure: No legal assignment to do

- This is the **same** for all CSPs!
- Path is irrelevant
- Every solution appears at depth ***n*** with ***n*** variables
  - DFS anyone
- Complexity (*n* vars, *d* values)
  - Branch factor: $(n-l)d$ at depth l
  - $n!d^n$ leaves!

Backtracking Search

- Backtracking search is the basic uninformed algorithm for solving CSPs
- Idea 1: One variable at a time
  - Variable assignments are **commutative**, so fix ordering
  - I.e., [WA = red then NT = green] same as [NT = green then WA = red]
  - Only need to consider assignments to a single variable at each step
- Idea 2: Check constraints as you go
  - I.e. consider only values which **do not conflict** previous assignments
  - Might have to do some computation to check the constraints
  - "Incremental goal test"
- Depth-first search with these two improvements is called *backtracking search* (not the best name)
- Can solve n-queens for $n \approx 25$

Detour: Recursive DFS

```
function RECURSIVE-DFS(problem) returns a solution, or failure
  return RECURSIVE-DFS_(MAKE-NODE(problem.INITIAL-STATE), problem)

function RECURSIVE-DFS_(node,problem) returns a solution, or failure
  if problem.GOAL-TEST(node.STATE) then return SOLUTION(node)
  for each action in problem.ACTIONS(node.STATE) do
    child ← CHILD-NODE(problem, node, action)
    result ← RECURSIVE-DFS_(child, problem)
    if result != failure then return result
  return failure
```

Backtracking Search

select a variable

Find a value consistent with constraints

Recurse to assign another

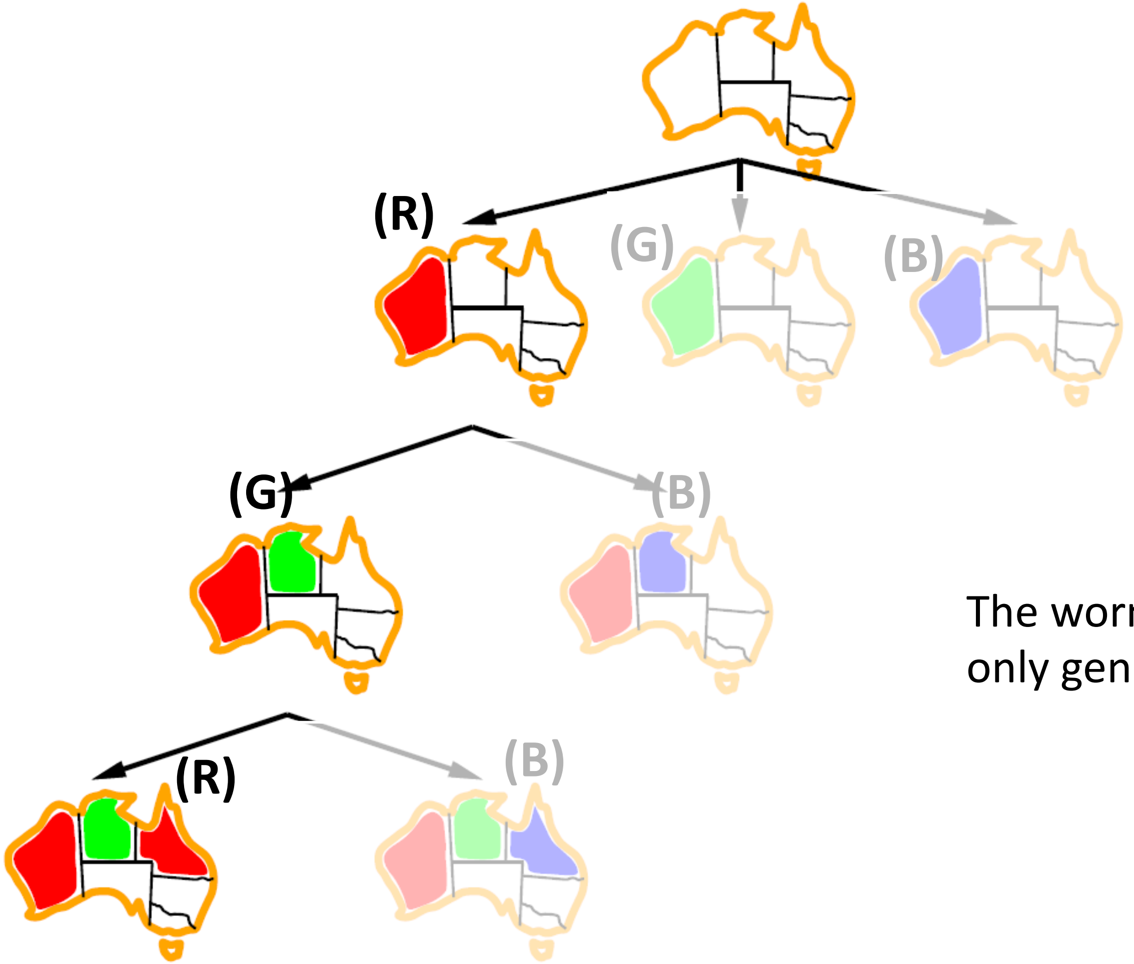
only keeps a single representation of the assigned state!

```
function BACKTRACKING-SEARCH(csp) returns a solution, or failure
    return BACKTRACK({ }, csp)

function BACKTRACK(assignment , csp) returns a solution, or failure
    if assignment is complete then return assignment
    var ←SELECT-UNASSIGNED-VARIABLE(csp)
    for each value in ORDER-DOMAIN-VALUES(var , assignment , csp) do
        if value is consistent with assignment then
            add {var = value} to assignment
            inferences ←INFERENCE(csp, var , value)
            if inferences ≠ failure then
                add inferences to assignment
                result ←BACKTRACK(assignment , csp)
                if result ≠ failure then
                    return result
        remove {var = value} and inferences from assignment
    return failure
```

If no consistent assignment exists, return failure, which causes another value to be tried

Backtracking Example



The worn-out states are only generated if needed

CSPs Recap

- Variables (e.g. Australian States)
- Domains of Variables (e.g. map colors)
- Assign values to variables from the corresponding domains (WA = red)
- Constraints (WA $\neq$ SA)

- Solution Method: Backtracking Search
  - DFS
  - One variable at a time
  - Check constraints along the way

Improvements

- Backtracking: DFS + variable ordering + constraint checking
- Uninformed: Add heuristics to improve

- **General Purpose Heuristics** thanks to the structure of CSPs
- Ordering:
  - Which variable should be assigned next?
  - In what order should its values be tried?
- Filtering: What inference can be made to detect failures early ?
- Structure: Can we exploit the problem structure?

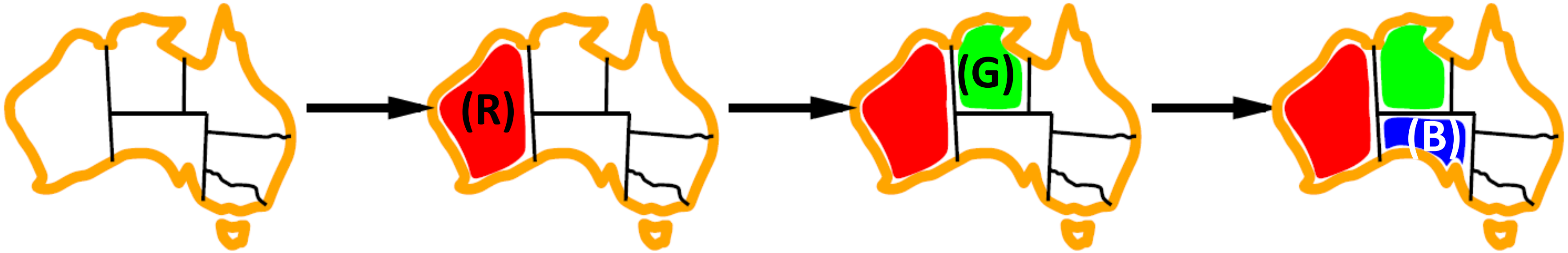
Ordering: What variable to assign next?

- Fixed order
- Random

- Other ideas?
  - Let's look at the number of constraints per variable!

Minimum Remaining Values (MRV)

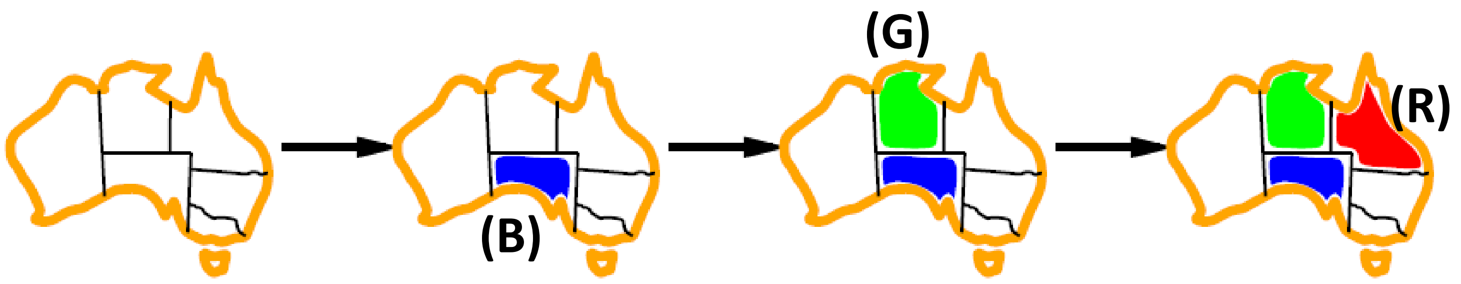
- Variable Ordering: Minimum remaining values (MRV):
  - Choose the variable with the fewest legal left values in its domain

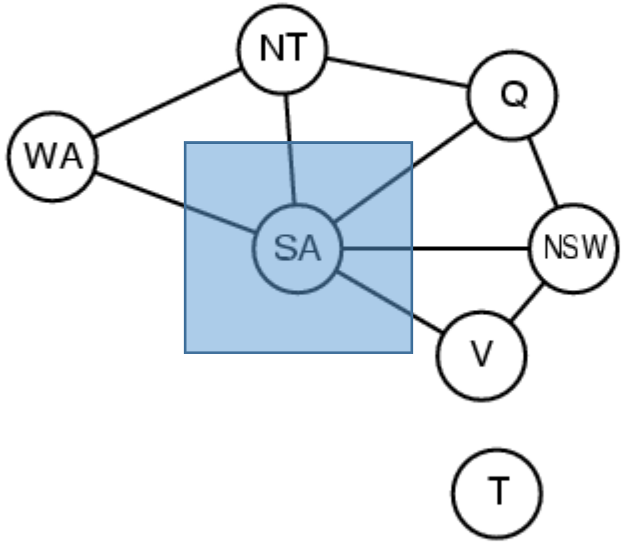


- Why min rather than max?
- Also called “most constrained variable”
- “Fail-fast” ordering + not running out of options

Degree Heuristic

- Tie breaker among MRV variables
- Choose the variable with most constraints on remaining variables

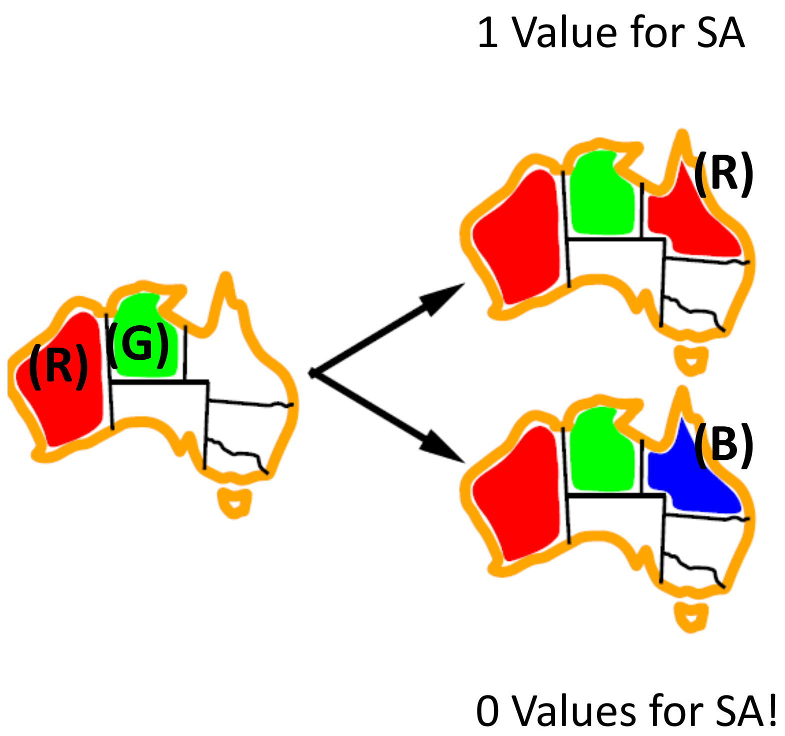




Least Constraining Value

- Which value to assign next?
- Value Ordering: Least Constraining Value
  - Given a choice of variable, choose the **least constraining value**
  - I.e., the one that **rules out the fewest values** in the remaining variables
  - Note that it may take some computation
- Why least rather than most?

- Combining these ordering ideas makes 1000 queens feasible



Summary of Ordering

- Which variable to select next and which value to assign to it?
- Detect failures early (MRV + DH)
- Enter the most promising branch (LCV)
- Note that the heuristics do not change the theoretical bounds!

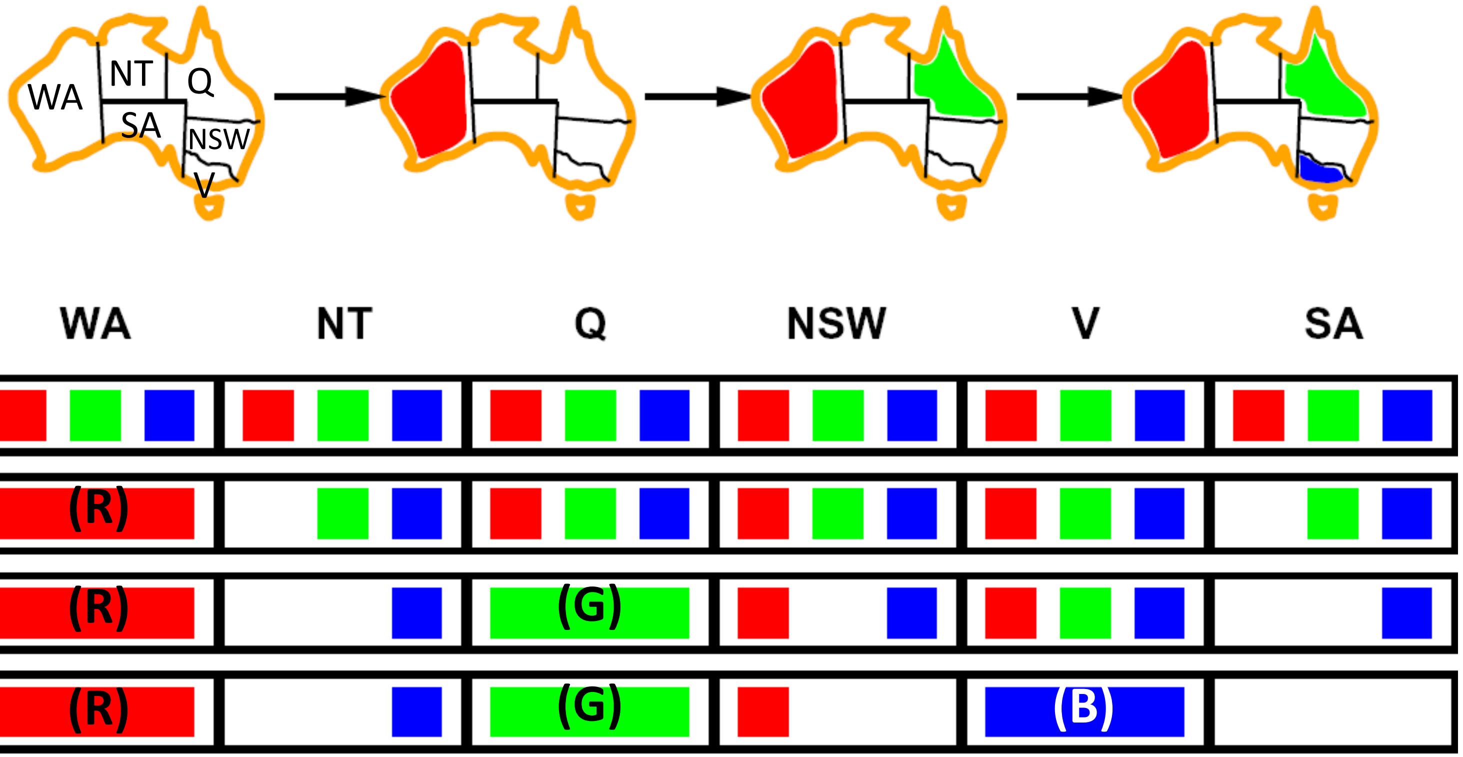
Filtering

- How to detect failures early?
- Filtering: Keep track of domains for unassigned variables and cross off bad options

- Forward Checking
- Constraint Propagation - Arc consistency

Forward Checking

- Forward checking: Cross off values that violate a constraint when added to the existing assignment
- Backtrack when no assignments left

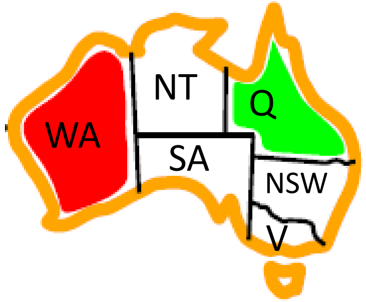


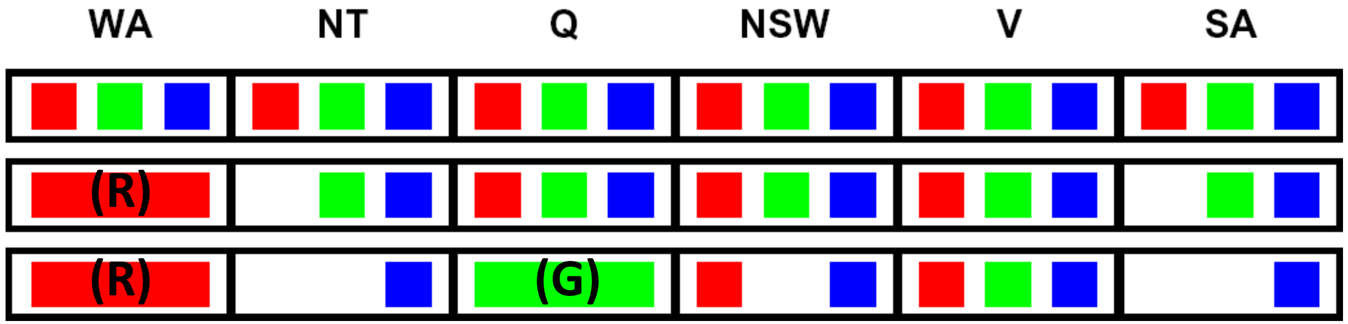
Legend:
Left: Red
Middle: Green
Right: Blue

MRV + Forward Checking: FC can be used to compute what MRV needs!

Constraint Propagation

- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures
- After deleting neighbors, check constraints for all other variables





- NT and SA cannot both be blue!
- Why didn't we detect this yet?
- *Constraint propagation:* reason from constraint to constraint

Legend:
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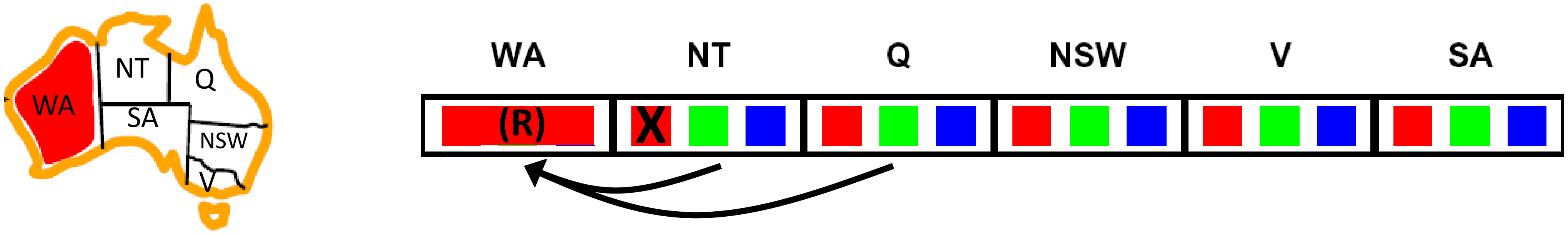
Consistency of A Single Arc

- An arc X $\rightarrow$ Y is consistent iff for *every* x in the tail there is *some* y in the head which could be assigned without violating a constraint



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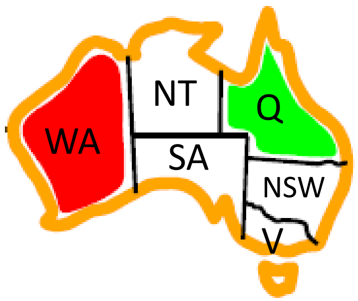
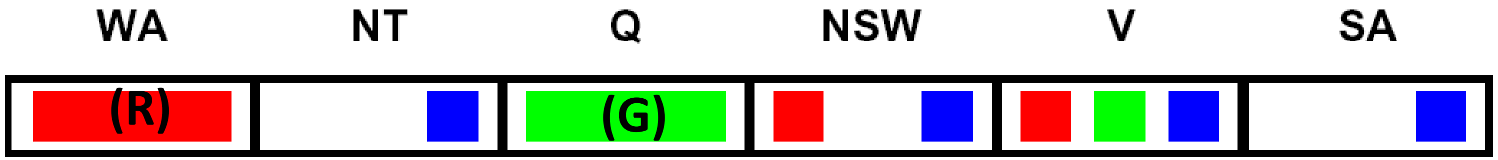
- Delete from tail!
- Forward checking: Enforcing consistency of arcs pointing to each new assignment

Arc Consistency of an Entire CSP

- A simple form of propagation makes sure all arcs are consistent:





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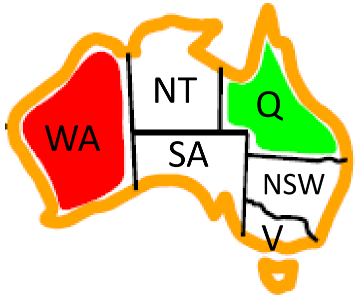

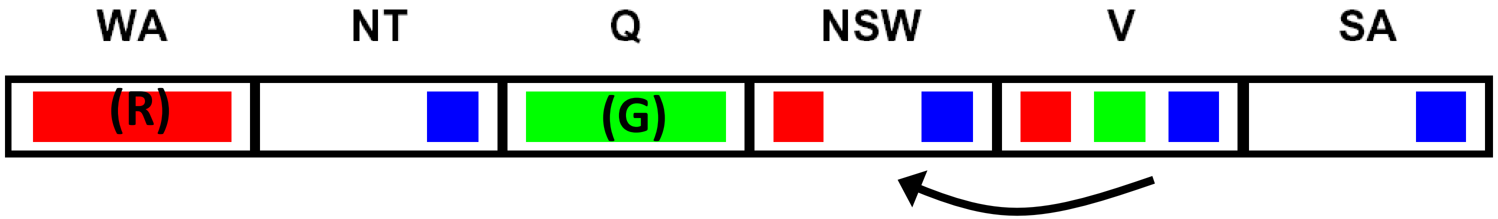

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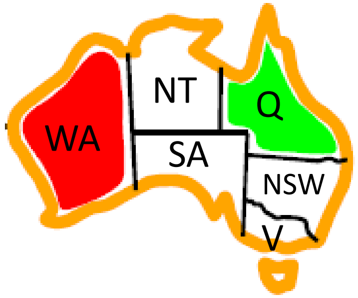

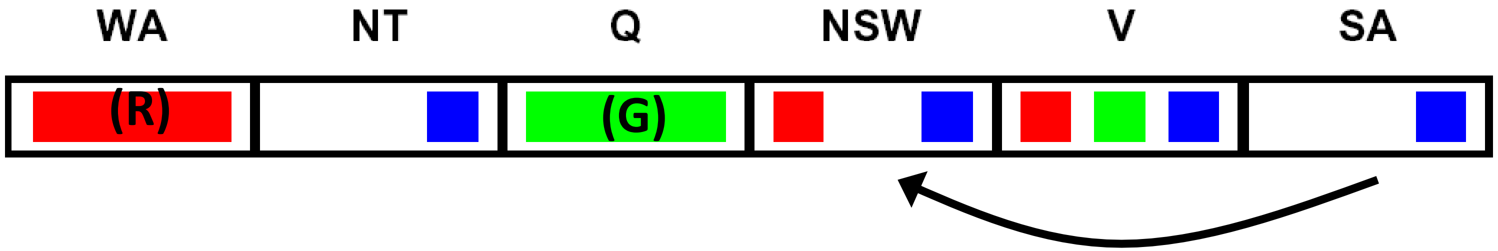

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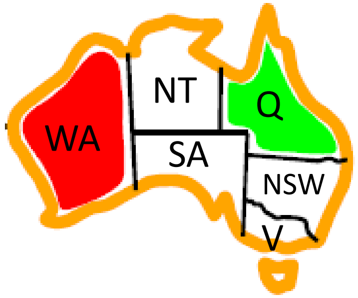

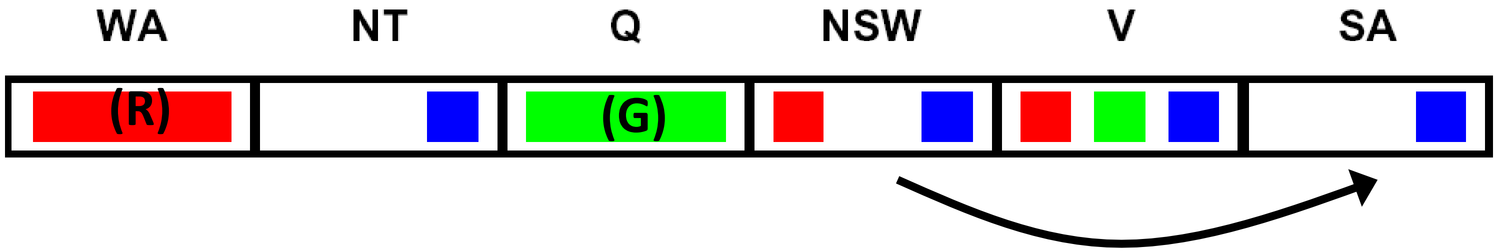

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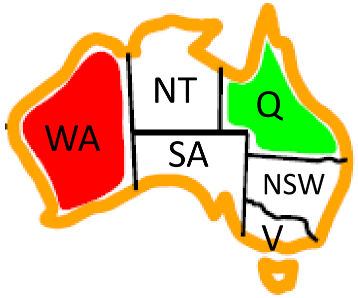

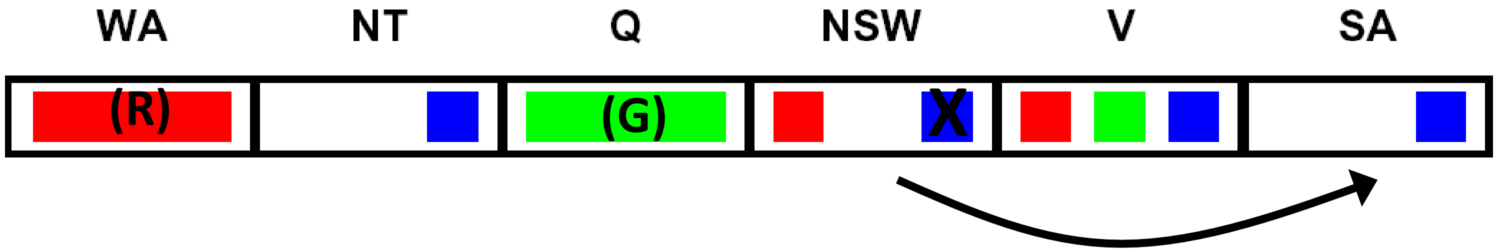

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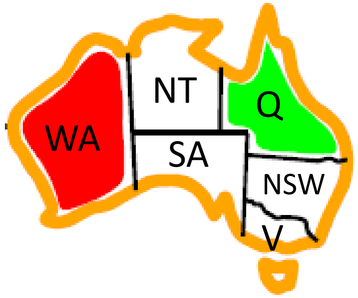

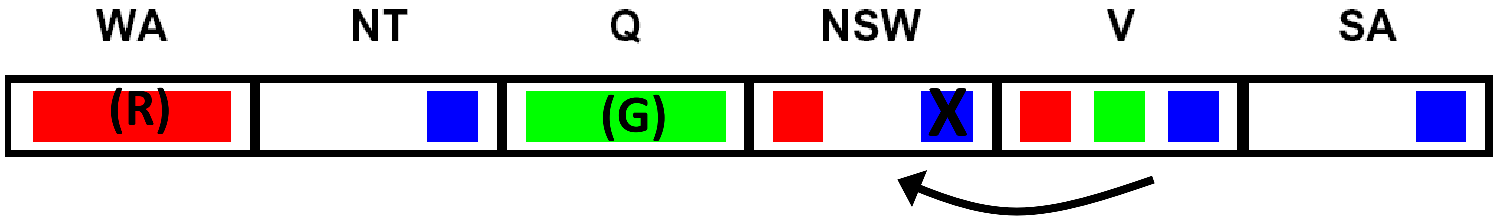

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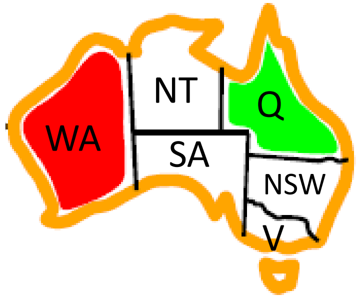

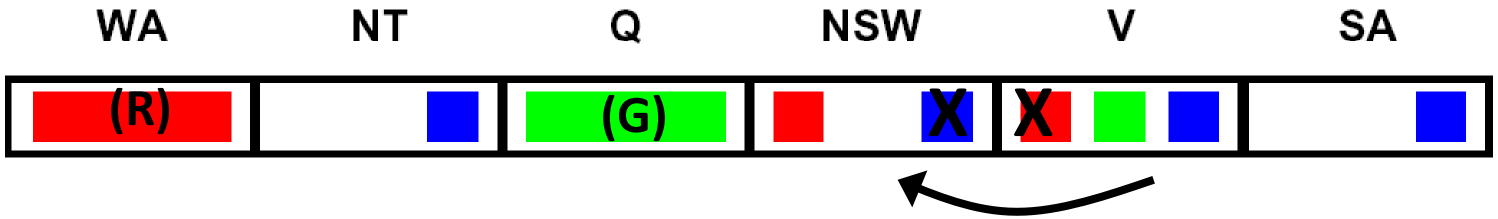

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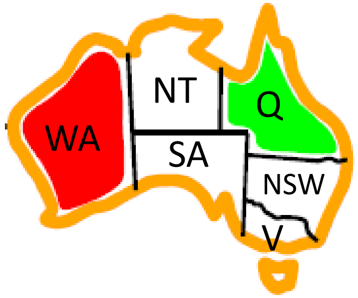

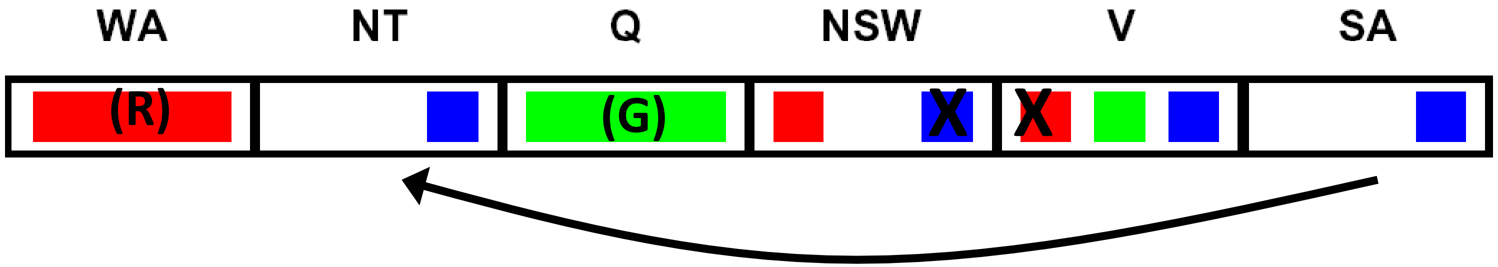

Legend:
Left: Red
Middle: Green
Right: Blue

Remember: Delete from the tail!

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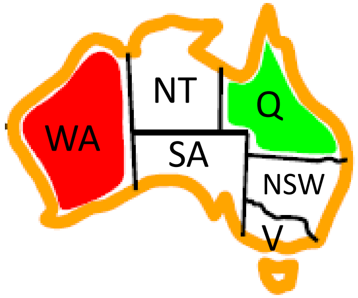

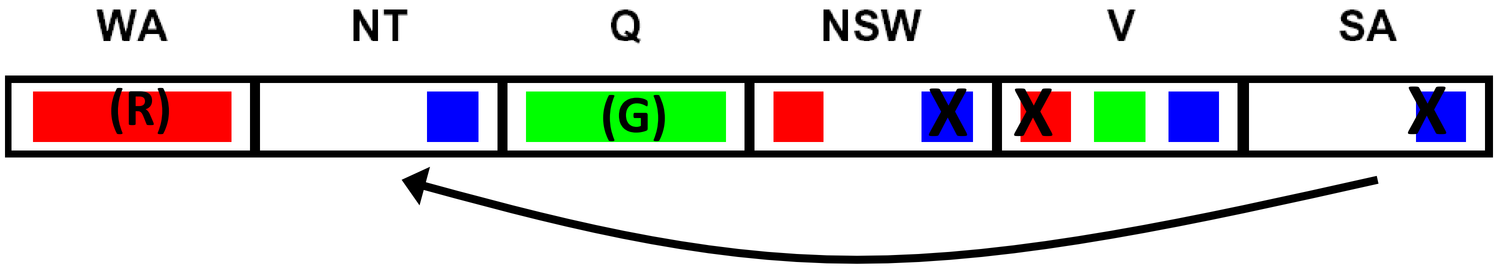

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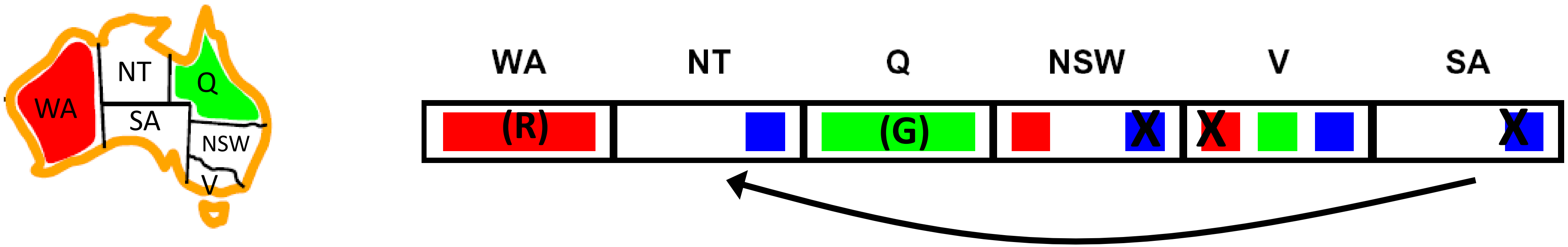

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- Important: If X loses a value, neighbors of X need to be rechecked!
- Arc consistency detects failure earlier than forward checking
- Can be run as before or after each assignment
- What's the downside of enforcing arc consistency?

Remember: Delete from the tail!

Enforcing Arc Consistency in a CSP

Check consistency and remove value if necessary

Add all the neighbors to the queue if something is removed

function AC-3(*csp*) **returns** false if an inconsistency is found and true otherwise
    inputs: *csp*, a binary CSP with components *(X, D, C)*
    local variables: *queue*, a queue of arcs, initially all the arcs in *csp*
    while *queue* is not empty **do**
        $(X_i, X_j) \leftarrow$ REMOVE-FIRST(*queue*)
        if REVISE(*csp*, X_i, X_j) **then**
            if size of $D_i = 0$ **then return** *false*
            for each X_k **in** X_i.NEIGHBORS - $\{X_j\}$ **do**
                add (X_k, X_i) to queue
    return *true*

Delete from tail to enforce consistency

function REVISE(*csp*, X_i, X_j) **returns** true iff we revise the domain of X_i
    revised $\leftarrow$ *false*
    for each *x* **in** D_i **do**
        if no value *y* in D_j allows (*x*,*y*) to satisfy the constraint between X_i and X_j **then**
            delete *x* from D_i
            revised $\leftarrow$ *true*
return *revised*

- Runtime: $O(n^2d^3)$, can be reduced to $O(n^2d^2)$
- … but detecting all possible future problems is NP-hard

Arc Consistency and Forward Checking

- FC is essentially an arc consistency check for a single node!
  - Head is the assigned node, tails are its neighbors
- If you ran AC, no need to run FC

- FC is faster per value-assignment
- AC can catch failures earlier

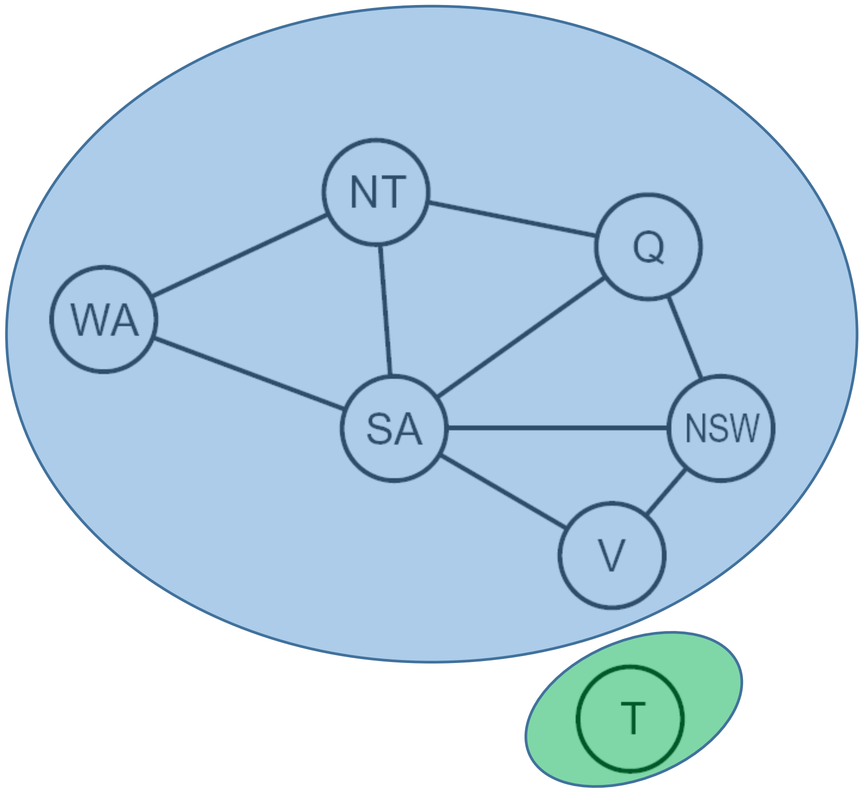
Summary of Filtering

- FC: Remove values from the domains neighboring nodes
  - Fast to compute
  - Plays well with MRV
  - Does not catch some failures early

- Arc Consistency: Make all arcs consistent after an assignment
  - Keep checking arcs until there is no change
  - Entails FC
  - Earlier failure detection
  - Costly to Compute (especially if there are a lot of constraints)

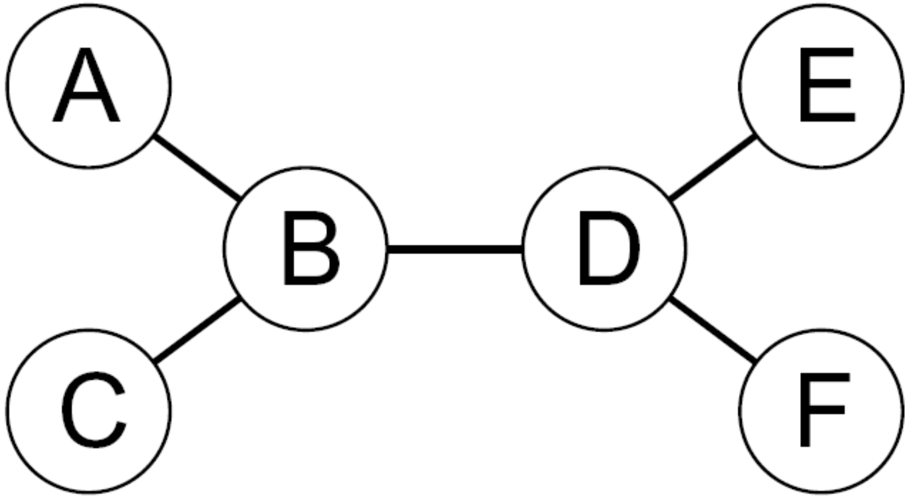
Problem Structure

- Extreme case: independent subproblems
  - Example: Tasmania and mainland do not interact
- Independent subproblems are identifiable as connected components of constraint graph
- Suppose a graph of n variables can be broken into subproblems of only c variables:
  - Worst-case solution cost is $O((n/c)(d^c))$
  - E.g., n = 80, d = 2, c =20
  - 2^{80} = 4 billion years at 10 million nodes/sec
  - $(4)(2^{20})$ = 0.4 seconds at 10 million nodes/sec



It's rare to find unconnected parts of the graph

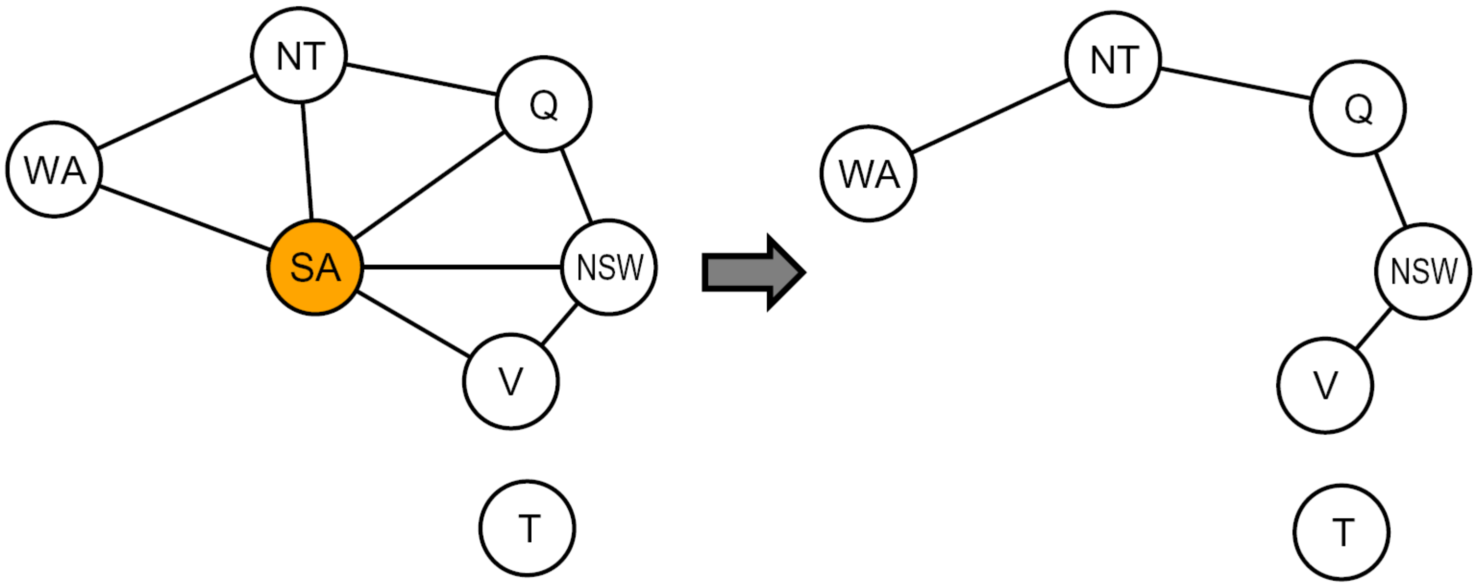
Tree-Structured CSPs



- Theorem: if the constraint graph has no loops, the CSP can be solved in $O(nd^2)$ time
- Compare to general CSPs, where worst-case time is $O(d^n)$

(Skipping the algorithm this semester)

Nearly Tree-Structured CSPs



- **Conditioning**: instantiate a variable, prune its neighbors' domains
- **Cutset conditioning**: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree
- <u>Cutset size *c*</u> gives runtime $O((d^c)\ (n\text{-}c)\ d^2)$, very fast for small *c*

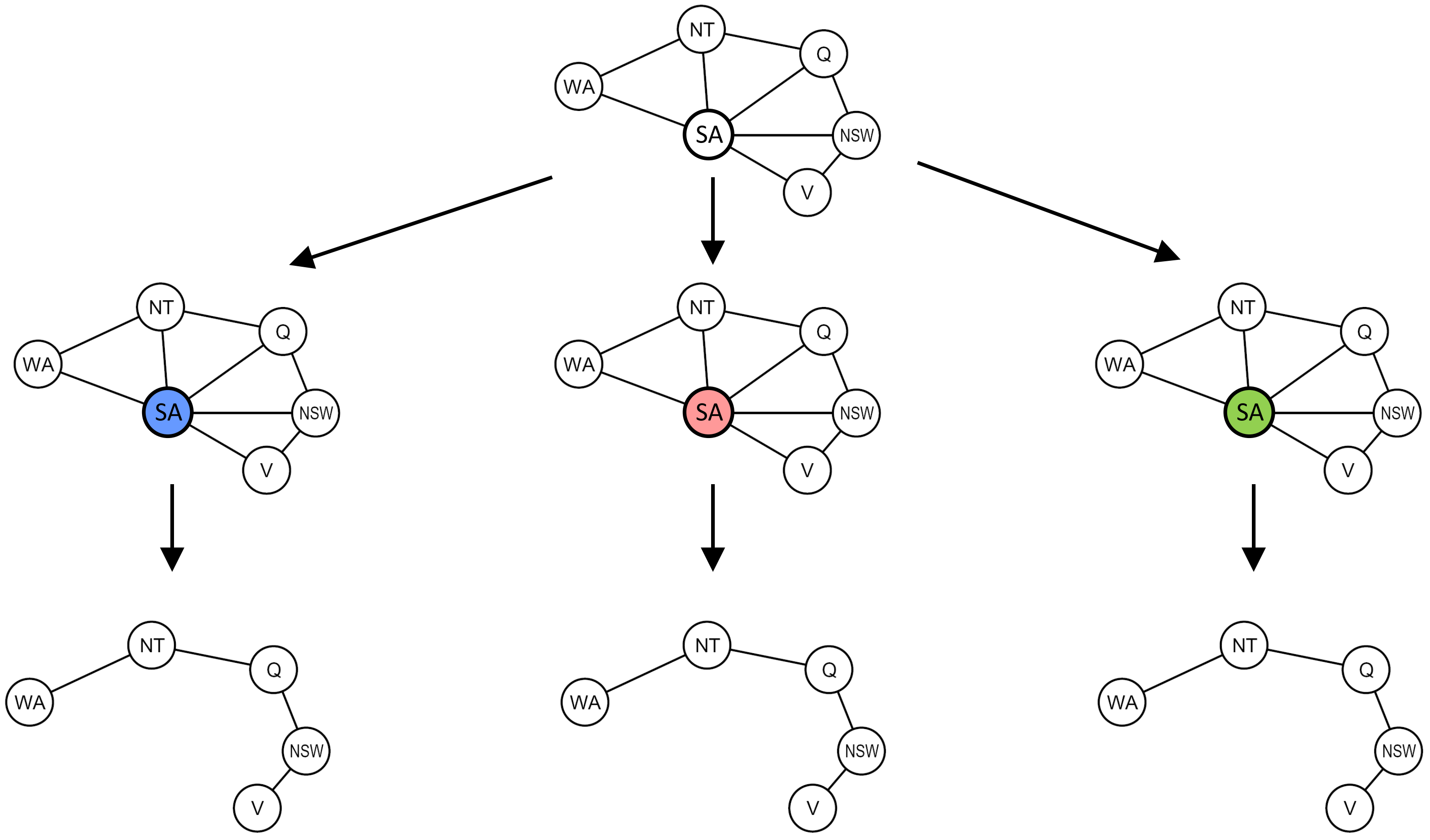
Cutset Conditioning

Choose a cutset

Instantiate the cutset (all possible ways)

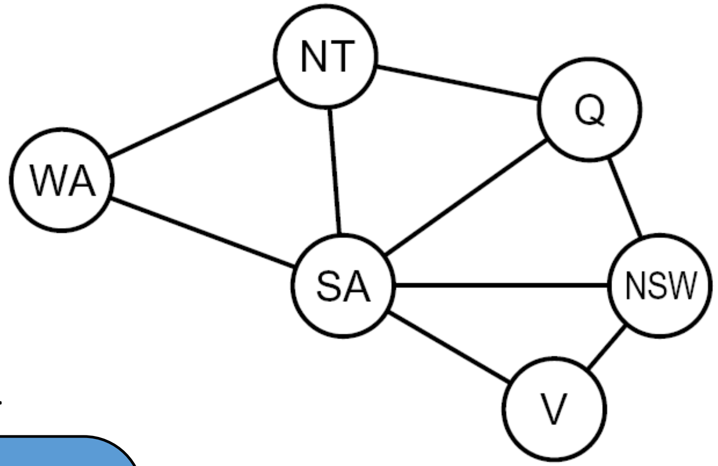
Compute residual CSP for each assignment

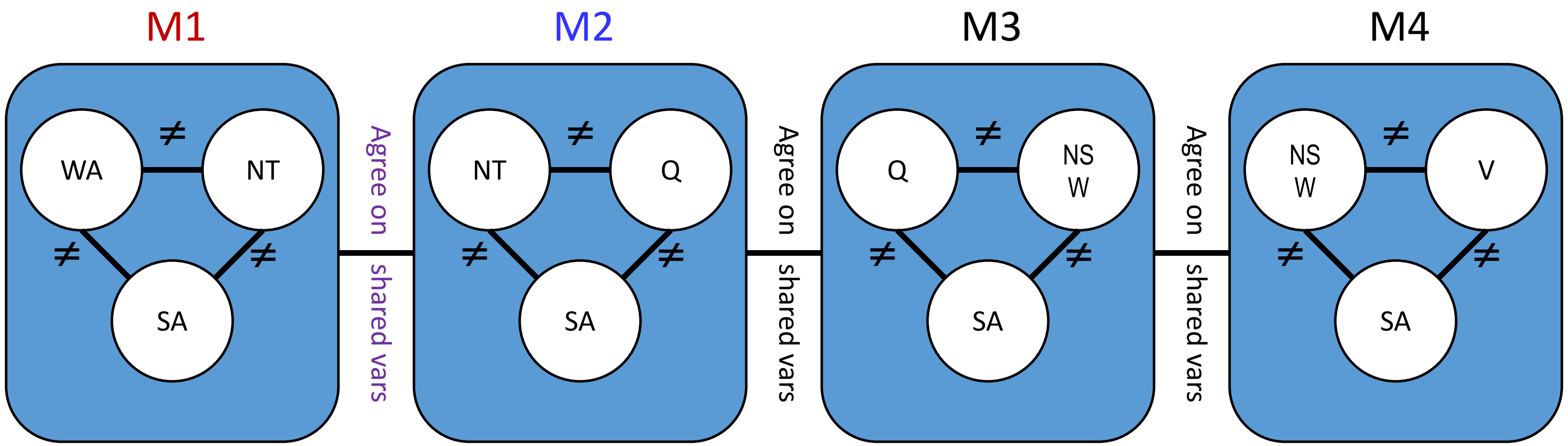
Solve the residual CSPs (tree structured)



Tree Decomposition*

- Idea: create a tree-structured graph of mega-variables
- Each mega-variable encodes part of the original CSP
- Subproblems overlap to ensure consistent solutions





{(WA=r,SA=g,NT=b),
(WA=b,SA=r,NT=g),
…}

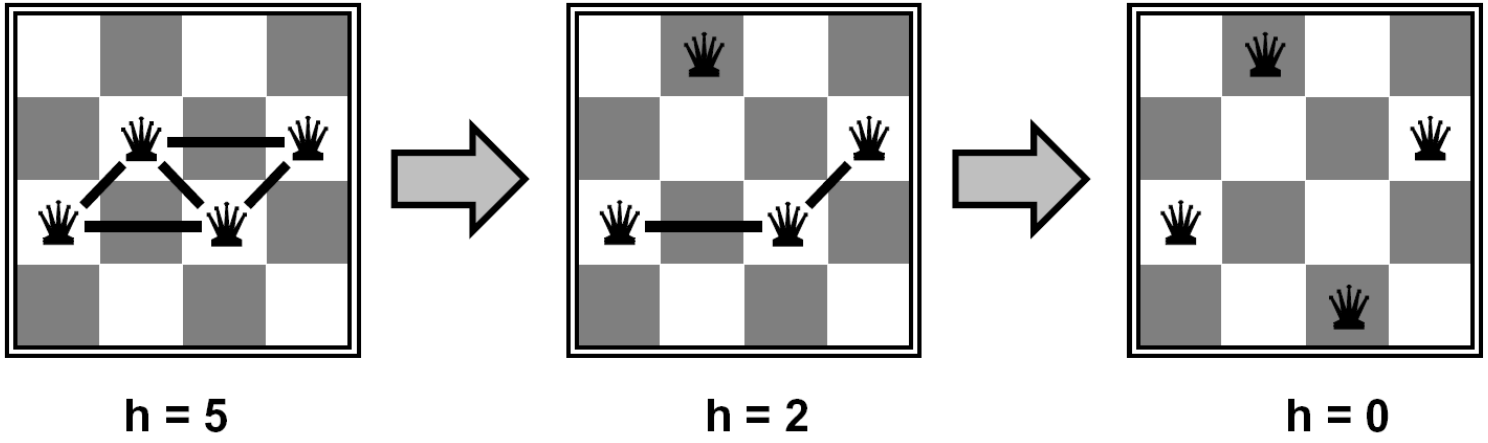
{(NT=r,SA=g,Q=b),
(NT=b,SA=g,Q=r),
…}

Agree: (M1,M2) $\in$
{((WA=g,SA=g,NT=g), (NT=g,SA=g,Q=g)), …}

Local Search For CSPs – MIN-CONFLICTS

function MIN-CONFLICTS(*csp*, *max_steps*) **returns** a solution or failure
  inputs: *csp*, a constraint satisfaction problem
  max_steps, the number of steps allowed before giving up
  current ←an initial complete assignment for *csp*
  for *i = 1* to *max_steps* **do**
    if *current* is a solution for *csp* **then return** *current*
    var ←a randomly chosen conflicted variable from *csp*.VARIABLES
    value←the value v for var that minimizes CONFLICTS(*var* , *v*, *current* , *csp*)
    set *var* =*value* in *current*
  return failure

Example: 4-Queens

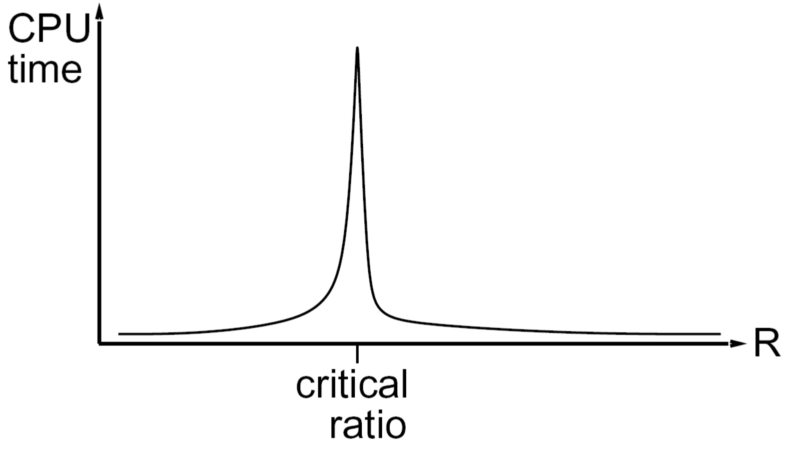


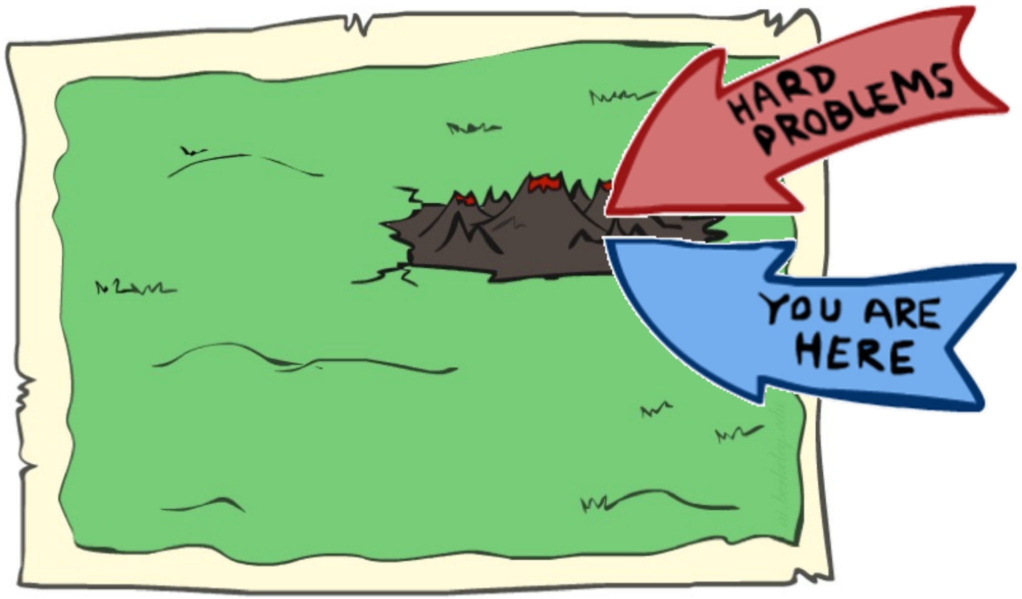
- States: 4 queens in 4 columns (4^4 = 256 states)
- Operators: move queen in column
- Goal test: no attacks
- Evaluation: c(n) = number of attacks

Performance of Min-Conflicts

- Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)!
- The same appears to be true for any randomly-generated CSP *except* in a narrow range of the ratio

$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$





Related to the "Phase Transition" phenomenon

Summary of CSPs

- CSPs are a special kind of search problem:
  - States are partial assignments
  - Goal test defined by constraints
- Basic solution: backtracking search
- Speed-ups:
  - Ordering
  - Filtering
  - Structure
- Iterative min-conflicts is often effective in practice