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Part I

The System Dimensioning Problem

Chapter 1

The System Dimensioning Problem

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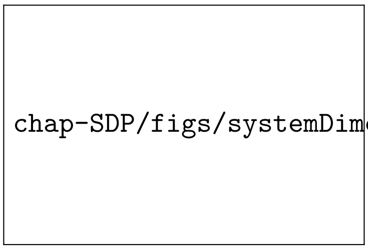

Abstract

This section aims at modelling our optimization problem using an integer linear program. The inputs and the outputs of the problem are first described. Second subsection is dedicated to the building of an oriented valued graph, namely the Timed Extended Graph. This graph, previously introduced by Ahuja *et al.* [?], allows to express all the constraints of the problem following the time and the space dimensions. The third subsection introduces the decision variables of our optimization problem. More precisely, it is shown that vehicles can be equivalently aggregated into flows to express all the constraints and the criteria of our optimization problem. Last subsection presents its formulation using integer linear programming.

1.1 Introduction : problem description

Considering potential one-way carsharing station locations and demands over time, what could be the optimal system configuration capturing the higher number of demands ?

1.2 Related work

1.3 Mathematical model

1.3.1 Inputs and outputs of the problem

In order to describe the mathematical problem, let us define first the required inputs. As discussed later, those data can be extracted from various sources such as simulation tools or real operating carsharing system data for example.

First of all, let $\mathcal{H} = \{1, \cdots, T\}$ be the set of time-steps considered in the study. For relevance issues, it has to cover a representative period of time, as an average weekday or an average week for instance.

The set of carsharing stations is defined as $\mathcal{N} = \{1, \cdots, N\}$ and comes with $Z(i)$ the maximum size (*i.e.* maximum capacity, in terms of number of vehicles) of station $i \in \mathcal{N}$. Then, the demand $D(i, j, t)$ contains the number of passengers wishing to borrow a car from station $i \in \mathcal{N}$ at time $t \in \mathcal{H}$ to station $j \in \mathcal{N}$. Note here that there is no condition on the station themselves, thus allowing the travel to be a "round trip" or a "one-way" travel. In this case, i is simply equal to j.

Finally, $\delta(i, j, t)$ represents the travel time it takes for a car-user from station $i \in \mathcal{N}$ to station $j \in \mathcal{N}$, when departure time from i is $t \in \mathcal{H}$. We suppose that for any triple $(i, j, t) \in \mathcal{N} \times \mathcal{N} \times \mathcal{H}$, $\delta(i, j, t) < T$. This assumption comes from the fact that the highest distance from different stations is quite low (usually less than 150 or 200 kilometres) and that the time-steps are covering at least a day.

A feasible solution of our problem consists on a set of vehicle tours, each of them modelling the situation of a car at each time step. The three criteria (the demand, the number of vehicles and the number of relocation operations) can be polynomially computed from any feasible solution.

1.3.2 Time Extended Graph

Including time in a network flow model can be done using Time Expanded Graphs (TEGs) as suggested by [**?**]. Carsharing stations are duplicated at every discrete time-step so that links (arcs) between stations could represent time-dependent vehicle operations (staying parked, satisfying a demand or being relocated).

To deal with discrete-time dynamic networks, Ahuja *et al.* [**?**] suggested the use of *time-space network*, also known as *Time Extended Graphs* (TEG). It is a static network constructed by expanding the original network in the time dimension, considering a separate copy of every node $i \in \mathcal{N}$ at every discrete time-step $t \in \mathcal{H}$. Thus, from the data listed above, let us consider $G = (\mathcal{X}, \mathcal{A}, u)$ as a TEG defined as follows:

Set of nodes (MOSIM)

Nodes are couples (i, t) with $i \in \mathcal{N}$ and $t \in \mathcal{H}$ associated with station i at time t. Formally, $\mathcal{X} = \mathcal{N} \times \mathcal{H}$.

Let η and θ the functions which reciprocally return the station and the time-step associated with every element of $\mathcal{X}$. More formally,

$$\eta : \mathcal{X} \to \mathcal{N} \text{ with } x = (i, t) \mapsto \eta(x) = i$$

$$\theta : \mathcal{X} \to \mathcal{H} \text{ with } x = (i, t) \mapsto \theta(x) = t$$

Set of arcs (MOSIM)

Edges are partitioned into 3 sets $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ defined as follows:

- $\mathcal{A}_1$ is the set of arcs representing the possibility for the vehicles to stay at a station between two consecutive time-steps. Formally, $\mathcal{A}_1 = \{(x, y) \in \mathcal{X} \times \mathcal{X} \mid \eta(x) = \eta(y) \textit{ and } \theta(y) = \theta(x) + 1 \bmod T\}$.
- $\mathcal{A}_2$ are the arcs associated with a demand: each demand $D(i, j, t) > 0$ corresponds to an arc (x, y) with $x = (i, t)$ and $y = (j, t + \delta(i, j, t))$. Set $\mathcal{A}_2$ is then formally defined as $\mathcal{A}_2 = \{(x, y) \in \mathcal{X} \times \mathcal{X} \mid D(\eta(x), \eta(y), \theta(x)) \neq 0 \textit{ and } \theta(x) + \delta(\eta(x), \eta(y), \theta(x)) = \theta(y) \bmod T\}$.
- $\mathcal{A}_3$ represents all the possible relocation operations over time. Any triple $(i, j, t) \in \mathcal{N} \times \mathcal{N} \times \mathcal{H}$ with $i \neq j$ is associated to an arc from $x = (i, t)$ to $y = (j, t + \delta(i, j, t) \bmod T)$. Thus, $\mathcal{A}_3 = \{(x, y) \in \mathcal{X} \times \mathcal{X} \mid \eta(x) \neq \eta(y) \textit{ and } \theta(x) + \delta(\eta(x), \eta(y), \theta(x)) = \theta(y) \bmod T\}$.

The total number of arcs is given by:

$$|\mathcal{A}| = |\mathcal{A}_1| + |\mathcal{A}_2| + |\mathcal{A}_3| = N \times T + M + (N \times T) \cdot (N - 1)$$

where M is the number of requested demands. Since $M \ll N^2$, $|\mathcal{A}| = \Theta(N^2 \cdot T)$.

1.3.3 Basic definitions (EWGT)

The Time Expanded Graph is a valued directed graph $\mathcal{G} = (\mathcal{X}, \mathcal{A}, u)$ such that nodes represent stations states over the time period, *i.e.* $\mathcal{X} = \mathcal{N} \times \mathcal{H}$. Any arc $a = (x, y) \in \mathcal{A}$ is associated to a possible move of a vehicle from node x to y. The capacity $u(a)$ is the maximum number of vehicles allowed on a.

Let the function $\eta : \mathcal{X} \to \mathcal{N}$ with $x = (i, t) \mapsto \eta(x) = i$ referring to the station of a node $x \in \mathcal{X}$. Similarly, $\theta : \mathcal{X} \to \mathcal{H}$ with $x = (i, t) \mapsto \theta(x) = t$ is the step-time of x. Let $\Gamma^-(\mathcal{G}, x)$ and $\Gamma^+(\mathcal{G}, x)$ respectively denotes the set of immediate predecessors and successors of a node $x \in \mathcal{X}$ in $\mathcal{G}$, *i.e.* $\Gamma^-(\mathcal{G}, x) = \{y \in \mathcal{X} \mid (y, x) \in \mathcal{A}\}$ and $\Gamma^+(\mathcal{G}, x) = \{y \in \mathcal{X} \mid (x, y) \in \mathcal{A}\}$. We simply note $\Gamma^-(\mathcal{G}, x) = \Gamma^-(x)$ and $\Gamma^+(\mathcal{G}, x) = \Gamma^+(x)$ if no confusion is possible.

1.3.4 Set of arcs (EWGT)

Arcs set $\mathcal{A}$ is partitioned into three sets $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ defined as follows.

- $\mathcal{A}_1$ is a set of arcs associated to vehicles staying in a same station between two consecutive time steps. Formally,

  $$\mathcal{A}_1 = \{(x, y) \in \mathcal{X}^2 \mid \eta(x) = \eta(y) \textit{ and } \theta(y) = \theta(x) + 1 \bmod T\}. \tag{1.1}$$

  The capacity of any arc $a = (x, y) \in \mathcal{A}_1$ with $i = \eta(x) = \eta(y) \in \mathcal{N}$ is $u(a) = Z(i)$.

- Any arc $a = (x, y) \in \mathcal{A}_2$ corresponds to a positive demand from $\eta(x)$ to $\eta(y)$ at time $\theta(x)$. The arrival time is $\theta(x) + \delta(\eta(x), \eta(y), \theta(x)) \mod T$. Arcs set $\mathcal{A}_2$ is then formally defined as

  $$\mathcal{A}_2 = \{(x, y) \in \mathcal{X}^2 \mid D(\eta(x), \eta(y), \theta(x)) > 0 \textit{ and } \theta(x) + \delta(\eta(x), \eta(y), \theta(x)) = \theta(y) \bmod T\}. \tag{1.2}$$

  The capacity of any arc $a = (x, y) \in \mathcal{A}_2$ equals $u(a) = D(\eta(x), \eta(y), \theta(x))$.

- Elements from $\mathcal{A}_3$ model relocations. Each arc $a = (x, y) \in \mathcal{A}_3$ is associated to a possible relocation from station $\eta(x)$ to $\eta(y)$ at time $\theta(x)$:

$$\mathcal{A}_3 = \{(x, y) \in \mathcal{X}^2 \mid \eta(x) \neq \eta(y) \textit{ and } \theta(x) + \delta(\eta(x), \eta(y), \theta(x)) = \theta(y) \bmod T\}. \quad (1.3)$$

  The capacity $u(a)$ of any arc $a = (x, y) \in \mathcal{A}_3$ is not bounded.

The total number of arcs is then $|\mathcal{A}| = \sum_{k=1}^{3} |\mathcal{A}_k| = N \cdot T + M + (N \cdot T) \cdot (N - 1) = N^2 \cdot T + M$. As $M \ll N^2$, $|\mathcal{A}| = \Theta(N^2 \cdot T)$. We observe that $|\mathcal{A}_3| \gg |\mathcal{A}_1 \cup \mathcal{A}_2|$ and that the number of arcs is proportional to $\mathcal{A}_3$.

Arcs Capacity

Associated with each arc $a = (x, y) \in \mathcal{A}$, is given a capacity function $u : \mathcal{A} \to \mathbb{N}$ corresponding to a maximum number of vehicles allowed on a. It is defined as follows for any arc $a \in \mathcal{A}$:

$$u(a = (x, y)) = \begin{cases} Z(\eta(x)) & \text{if } a \in \mathcal{A}_1 \\ D(\eta(x), \eta(y), \theta(x)) & \text{if } a \in \mathcal{A}_2 \\ +\infty & \text{if } a \in \mathcal{A}_3 \end{cases}$$

For any arc $a = (x, y) \in \mathcal{A}_1$, the maximum number of cars is the capacity of the station $\eta(x) = \eta(y)$. It corresponds to the demand for any arc $a \in \mathcal{A}_2$, and it is not bounded for relocation arcs $a \in \mathcal{A}_3$.

Additional notations

We denote by $\Gamma^-(x)$ and $\Gamma^+(x)$ respectively the set of immediate predecessors and successors of a node $x \in \mathcal{X}$, *i.e.*

$$\Gamma^-(x) = \{y \in \mathcal{X} \mid (y, x) \in \mathcal{A}\}$$
$$\Gamma^+(x) = \{y \in \mathcal{X} \mid (x, y) \in \mathcal{A}\}$$

For any couple of time instants $\forall (t, t') \in \mathcal{H}^2$, the number of time-steps between those two instants is defined through the following function

$$\begin{array}{rcl} \vartheta : \mathcal{H}^2 & \to & \mathbb{N} \\ (t, t') & \mapsto & \vartheta(t, t') \end{array}$$

with $\vartheta(t, t') = \begin{cases} t' - t & \text{if } t \leq t' \\ T + t' - t & \text{otherwise.} \end{cases}$

For each arc $a = (x, y) \in \mathcal{A}$, let us define the boolean value ϵ_a as:

$$\epsilon_a = \begin{cases} 0 & \text{if } \theta(x) \leq \theta(y) \\ 1 & \text{otherwise} \end{cases}$$

The time required for a movement from x to y is then given by the function

$$\begin{array}{rcl} \ell : \mathcal{A} & \to & \mathbb{N} \\ a = (x, y) & \mapsto & \ell(a) = \theta(y) - \theta(x) + \epsilon_a \cdot T \end{array}$$

By extension, if $\mu = (a_1, \cdots, a_p) \in \mathcal{A}^p$ is a path of the TEG from x to y, the value $\ell(\mu) = \sum_{i=1}^{p} \ell(a_i)$ is the total time required for a vehicle going from x to y following μ.

For any time value $t \in \mathcal{H}$, let us define the set $\mathcal{C}_t(\mu)$ as the arcs $a = (x, y)$ from μ starting at time t or earlier but ending after t. Formally, $\mathcal{C}_t(\mu) = \{a = (x, y) \in \mu \mid \vartheta(\theta(x), t) < \ell(a)\}$.

1.3.5 Decision variables (MOSIM)

The aim of our study is to compute the planning of each vehicles during the period. At any time, each vehicle is either parked in a station or in transit between two stations. Its position over the period can be modelled as a vehicle tour *i.e.* a circuit $c = (a_1, \cdots, a_p)$ in the TEG.

The size of any feasible solution may be highly reduced if we only consider the number of vehicles passing through each arc. For each arc $a = (x, y) \in \mathcal{A}$, we call $\varphi(a)$ the flow of vehicles transiting through the arc a. It can be interpreted as:

- the number of vehicle staying in station $\eta(x)$ between two consecutive time-steps $\theta(x)$ and $\theta(y)$, if $a \in \mathcal{A}_1$;
- the number of vehicle picked by users from station $\eta(x)$ at time $\theta(x)$ to station $\eta(y)$, if $a \in \mathcal{A}_2$;
- the number of vehicle relocated between stations $\eta(x)$ and $\eta(y)$ at time $\theta(x)$, if $a \in \mathcal{A}_3$.

The total number of vehicles transiting to any node $x \in \mathcal{X}$ is clearly constant, thus

$$\sum_{y \in \Gamma^-(x)} \varphi((y, x)) = \sum_{y \in \Gamma^+(x)} \varphi((x, y)).$$

It is immediate that a feasible flow may be obtained from any feasible set of vehicle tours. We prove in the following that vehicle tours may be easily computed from a feasible flow.

1.3.6 Decision variables (EWGT)

The aim of our study is to compute the planning of each vehicles during the period. At any time, each of them is either parked in a station or in transit between two stations. Its position over the period can be modelled as a vehicle tour *i.e.* a circuit $c = (a_1, \cdots, a_p)$ in the TEG.

We show in the following that a feasible solution can be described by only considering the number of vehicles passing through each arc. For each arc $a = (x, y) \in \mathcal{A}$, we call $\varphi(a)$ the flow of vehicles transiting through the arc a. It can be interpreted as the number of vehicle staying in station $\eta(x)$ between two consecutive time-steps $\theta(x)$ and $\theta(y)$ if $a \in \mathcal{A}_1$, or the number of vehicle moving from station $\eta(x)$ at time $\theta(x)$ to station $\eta(y)$ otherwise.

Since the total number of vehicles transiting to any node $x \in \mathcal{X}$ is constant,

$$\sum_{y \in \Gamma^-(x)} \varphi((y, x)) = \sum_{y \in \Gamma^+(x)} \varphi((x, y)). \tag{1.4}$$

A flow $\varphi : \mathcal{A} \mapsto \mathbb{N}$ is said to be feasible if $\forall a \in \mathcal{A}$, $\varphi(a) \leq u(a)$ and $\forall x \in \mathcal{X}$, the flow conservation equation (1.4) is true. A feasible flow may be easily obtained from any feasible set of vehicle tours.

We prove in the following that the reverse is also true, with the consequence that any feasible solution of our problem can be described using a flow. Next lemma computes the exact number of vehicles associated to a constant unitary flow over a circuit c.

Lemma 1 *Let c be a circuit and φ_c a feasible flow such that:*

$$\varphi_c(a) = \begin{cases} 1 & \textit{if } a \textit{ belongs to } c \\ 0 & \textit{otherwise.} \end{cases}$$

The minimum number of vehicles to insure φ_c is $\frac{\ell(c)}{T}$.

Proof. For any time value $t \in \mathcal{H}$, let us define the set $\mathcal{C}_t(c)$ as the arcs $a = (x, y)$ from c starting at time t or earlier but ending after t. Since $\vartheta(\theta(x), t)$ equals the number of time steps from $\theta(x)$ to t, we get $\mathcal{C}_t(c) = \{a = (x, y) \in c \mid \vartheta(\theta(x), t) < \ell(a)\}$.

Now, since c is a circuit, the value $|\mathcal{C}_t(c)|$ is a constant $\forall t \in \mathcal{H}$ and corresponds to the total number of vehicles needed to insure a unitary flow over c. Let us prove that $|\mathcal{C}_T(c)| = \sum\limits_{a\in c} \epsilon_a$. For that purpose, setting $B(c) = \{a = (x, y) \in c \mid \epsilon_a = 1\}$, we show that $B(c) = \mathcal{C}_T(c)$.

- $B(c) \subseteq \mathcal{C}_T(c)$: if $a = (x, y) \in B(c)$, then as $\theta(x) \leq T$, $\vartheta(\theta(x), T) = T - \theta(x)$. Now, since $\epsilon_a = 1$ and $\theta(y) \geq 1$, $\ell(a) = \theta(y) - \theta(x) + T \geq 1 - \theta(x) + T > \vartheta(\theta(x), T)$ and $a \in \mathcal{C}_T(c)$.

- $\mathcal{C}_T(c) \subseteq B(c)$: let consider now an arc $a = (x, y) \in \mathcal{C}_T(c)$. Since $\vartheta(\theta(x), T) = T - \theta(x) < \ell(a)$ we get that $\theta(y) - \theta(x) + \epsilon_a \cdot T > T - \theta(x)$ and thus $\theta(y) + \epsilon_a \cdot T > T$. As $\theta(y) \leq T$, we necessarily have $\epsilon_a = 1$ and thus $a \in B(c)$.

Now, by Lemma 3, $|\mathcal{C}_T(c)| = \sum\limits_{a\in c} \epsilon_a = \frac{\ell(c)}{T}$, the lemma. ∎

Theorem 1 *Any feasible solution φ can be decomposed into a set of circuits $\mathcal{S}$ such that, for any arc $a \in \mathcal{A}$, $\varphi(a) = \sum\limits_{c\in\mathcal{S}} \varphi_c(a)$.*

Proof. The proof is by recurrence on $n(\varphi) = \sum\limits_{a\in\mathcal{A}} \varphi(a)$. The theorem is trivially true if $n(\varphi) = 0$.

Let suppose now that $n(\varphi) > 0$, thus there exists at least one arc $a = (x, y) \in \mathcal{A}$ with $\varphi(a) > 0$. Set $\mu_0 = (x, y)$ and let consider the sequence of paths μ_i built as follows:

1. Stop the sequence as soon as μ_i contains a circuit c;

2. Otherwise, let $\tilde{a} = (\tilde{x}, \tilde{y})$ the last arc of μ_i. Since $\varphi(\tilde{a}) > 0$, the flow conservation equation (1.4) insures that there exists an arc a starting at $\tilde{y}$ with $\varphi(a) > 0$. We then set $\mu_{i+1} = \mu_i{\cdot}a$.

As $\mathcal{G}$ has a finite number of nodes, the algorithm stops and a non empty circuit c is returned. The flow $\hat{\varphi}$ defined as

$$\hat{\varphi}(a) = \begin{cases} \varphi(a) - 1 & \text{if } a \in c \\ \varphi(a) & \text{otherwise.} \end{cases}$$

is feasible with $n(\hat{\varphi}) < n(\varphi)$, thus the theorem. ∎

Note that the number of flow variables is a polynomial function on the size of the problem. This is not true anymore for vehicle tours, which number can be exponential. The consequence is that the determination of a flow is in $\mathcal{NP}$, which is not the case for the determination of vehicle tours.

1.3.7 Solution and Objectives

A feasible solution of our optimization problem is given by a set of vehicles, each of them associated with its position in the system at any-time during the period studied. The first objective is to maximize the total number of satisfied demands, *i.e.* for which a vehicle is allocated. However, two other objectives must be taken into account: each vehicle in the system is associated to a fixed cost, so that the total number of vehicles must be minimized. In the same way, vehicle relocations are fundamental for increasing the number of satisfied demands with a fixed number of vehicles. However, they cost an extra charge for the operator, and thus their number should also be limited. In the following, the total number of vehicles and relocations are referred respectively by C and R.

1.3.8 Effective duration of a path or a circuit

The duration of any path of a TEG may be easily evaluated. Indeed, for any couple of time instants $(t, t') \in \mathcal{H}^2$, let the function $\vartheta : \mathcal{H}^2 \mapsto \mathbb{N}^\star$ that computes the number of time-steps between those two instants. It is defined formally as

$$\vartheta(t, t') = \begin{cases} t' - t & \text{if } t \leq t' \\ T + t' - t & \text{otherwise.} \end{cases}$$

For each arc $a = (x, y) \in \mathcal{A}$, let us define and set the boolean value ϵ_a to true if $\theta(x) > \theta(y)$. For any arc $a = (x, y) \in \mathcal{A}$, the effective time required for a move from x to y is then equal to $\ell(a) = \theta(y) - \theta(x) + \epsilon_a \cdot T$. By extension, if $\mu = (a_1, \cdots, a_p) \in \mathcal{A}^p$ is a path of the TEG from x to y, the value $\ell(\mu) = \sum_{i=1}^{p} \ell(a_i)$ is the total time required for a vehicle going from x to y following μ. Next lemma evaluates the total time of any circuit c.

Lemma 2 *The total time of any circuit $c = (a_1, \cdots, a_p)$ is $\ell(c) = T \times \sum\limits_{i=1}^{p} \epsilon_{a_i}$.*

Proof. Let x_i, $i \in \{1, \cdots, p+1\}$ be the sequence of elements from $\mathcal{X}$ such that, $x_{p+1} = x_1$ and $\forall i \in \{1, \cdots, p\}$, $a_i = (x_i, x_{i+1})$. The total time of c is then

$$\ell(c) = \sum_{i=1}^{p} \ell(a_i) = \sum_{i=1}^{p} (\theta(x_{i+1}) - \theta(x_i) + \epsilon_{a_i} \cdot T) = T \times \sum_{i=1}^{p} \epsilon_{a_i},$$

the result. ■

1.3.9 Formal problem statements

A formal definition of our main optimization problem, referred as the basic carsharing problem with relocations [BCPR] can be stated as follows:

Basic carsharing problem with relocations [bcpr]:

Inputs: A set of stations $\mathcal{N}$ with their capacity $Z(i)$, $i \in \mathcal{N}$, time periods set $\mathcal{H} = \{1, \cdots, T\}$, travel times $\delta(i, j, t)$ for each triplet $(i, j, t) \in \mathcal{N}^2 \times \mathcal{H}$, a set of M demands, fixed number of vehicles C and relocation operations R.

Question: What is the maximum number of demands $m \leq M$ that can be captured by a vehicle routing of at most C vehicles and R vehicle relocation operations during the considered period $\mathcal{H}$?

We will show that [BCPR] belongs to $\mathcal{NP}$ by modelling feasible solutions as a non classical flow problem.

1.3.10 How to recover the number of vehicles?

Next lemmas characterize the total time of any circuit c and the exact number of vehicles required for a unitary flow on c.

Lemma 3 *The total time of any circuit $c = (a_1, \cdots, a_p)$ is $\ell(c) = T \times \sum_{i=1}^{p} \epsilon_{a_i}$.*

Proof. Let x_i, $i \in \{1, \cdots, p+1\}$ be the sequence of elements from $\mathcal{X}$ such that, $x_{p+1} = x_1$ and $\forall i \in \{1, \cdots, p\}$, $a_i = (x_i, x_{i+1})$. The total time of c is then

$$\begin{aligned} \ell(c) &= \textstyle\sum_{i=1}^{p} \ell(a_i) \\ &= \textstyle\sum_{i=1}^{p} (\theta(x_{i+1}) - \theta(x_i) + \epsilon_{a_i} \cdot T) \\ &= T \times \textstyle\sum_{i=1}^{p} \epsilon_{a_i}. \end{aligned}$$

the result. ■

Lemma 4 *Let c be a circuit and φ_c a feasible flow such that*

$$\varphi_c(a) = \begin{cases} 1 & \textit{if a belongs to c} \\ 0 & \textit{otherwise} \end{cases}$$

The minimum number of vehicles to insure φ_c is $\frac{\ell(c)}{T}$.

Proof. The total number of vehicles needed at time $t \in \mathcal{H}$ for c is clearly $|\mathcal{C}_t(c)|$.

We first show that $|\mathcal{C}_T(c)| = \sum_{a \in c} \epsilon_a$. For that purpose, setting $B(c) = \{a = (x, y) \in c \mid \epsilon_a = 1\}$, we prove that $B(c) = \mathcal{C}_T(c)$.

- $B(c) \subseteq \mathcal{C}_T(c)$: If $a = (x, y) \in B(c)$, then as $\theta(x) \leq T$, $\vartheta(\theta(x), T) = T - \theta(x)$. Now, since $\epsilon_a = 1$, $\ell(a) = \theta(y) - \theta(x) + T \geq \vartheta(\theta(x), T)$ and $a \in \mathcal{C}_T(c)$.
- $\mathcal{C}_T(c) \subseteq B(c)$: Let consider now an arc $a = (x, y) \in \mathcal{C}_T(c)$. Since $\vartheta(\theta(x), T) = T - \theta(x) < \ell(a)$, we get $\theta(y) + \epsilon_a \cdot T > T$. As $\theta(y) \leq T$, we necessarily have $\epsilon_a = 1$ and thus $a \in B(c)$.

Now, by Lemma 3, $|\mathcal{C}_T(c)| = \sum_{a \in c} \epsilon_a = \frac{\ell(c)}{T}$. Lastly, the minimum number of vehicles to insure φ_c is constant over $\mathcal{H}$ and thus $\forall t \in \mathcal{H}$, the lemma. ■

Theorem 2 *Any feasible solution φ can be decomposed into a set of circuits $\mathcal{S}$ such that, for any arc $a \in G$, $\varphi(a) = \sum_{c \in \mathcal{S}} \varphi_c(a)$.*

Proof. The proof is by recurrence on $n(\varphi) = \sum_{a \in \mathcal{A}} \varphi(a)$. The theorem is trivially true if $n(\varphi) = 0$.

Let suppose now that $n(\varphi) > 0$, thus there exists at least one arc $a = (x, y)$ with $\varphi(a) > 0$. Set $\mu_0 = (x, y)$ and let consider the sequence of paths μ_i built as follows:

1. Stop the sequence as soon as μ_i contains a circuit c;
2. Otherwise, let $\tilde{a} = (\tilde{x}, \tilde{y})$ the last arc of μ_i. Since $\varphi(\tilde{a}) > 0$, following the conservation low of φ over nodes of G, there exists an arc a starting at $\tilde{y}$ with $\varphi(a) > 0$. We then set $\mu_{i+1} = \mu_i \cdot a$.

As G has a finite number of nodes, the algorithm stops and a non empty circuit is then returned. Let now define the flow $\hat{\varphi}$ as follows

$$\hat{\varphi}(a) = \begin{cases} \varphi(a) - 1 & \text{if } a \in c \\ \varphi(a) & \text{otherwise} \end{cases}$$

$\hat{\varphi}$ is feasible with $n(\hat{\varphi}) < n(\varphi)$, thus the theorem. ■

Note that the number of flow variables is a polynomial function on the size of the problem. It is not the case for the vehicle tours, which number can be of exponential size. The consequence is that the determination of a flow is in $\mathcal{NP}$, which is not the case for the determination of vehicle tours.

1.3.11 Modelling of the optimization problem

Three objectives are to be considered for our optimization problem: the main objective is to maximize the number of demands. The two other ones are to minimize both the number of relocations and the total number of vehicles.

As we shall see later, the number of demands and relocations are easily linearly expressed using the flows. Next theorem shows that it is also the case for the total number of vehicles:

Theorem 3 *The minimum total number of cars required for a feasible flow φ equals $\sum_{a\in\mathcal{A}} \varphi(a)\cdot \epsilon_a$.*

Proof. Let $\mathcal{S}$ be a set of circuits obtained from the decomposition of φ following Theorem 2 and let V be the minimum number of cars associated with φ. By Lemmas 3 and 4, the total number of car of any circuit $c \in \mathcal{S}$ is

$$\sum_{a\in c} \epsilon_a = \sum_{a\in\mathcal{A}} \epsilon_a \cdot \varphi_c(a)$$

and thus

$$V \leq \sum_{c\in\mathcal{S}} \sum_{a\in\mathcal{A}} \epsilon_a \cdot \varphi_c(a).$$

Now, from Theorem 2, $\varphi(a) = \sum_{c\in\mathcal{S}} \varphi_c(a)$. Thus,

$$\sum_{c\in\mathcal{S}} \sum_{a\in\mathcal{A}} \epsilon_a \cdot \varphi_c(a) = \sum_{a\in\mathcal{A}} \epsilon_a \cdot \sum_{c\in\mathcal{S}} \varphi_c(a) = \sum_{a\in\mathcal{A}} \varphi(a) \cdot \epsilon_a.$$

Lastly, the total number of vehicles required at time T to reach φ is exactly $\sum_{a\in\mathcal{A}} \varphi(a)\cdot\epsilon_a$, the theorem. ■

The modelling of our optimization problem follows. R and C are fixed bounds for respectively the total number of relocation operations and vehicles. Equation (1) is the maximization of the total demand. Equation (2) expresses the bound on the total number of relocation. Equation (3) expresses these on the total number of vehicles. Equations (4), (5) and (6) are lastly flow constraints.

$$\max \sum_{a\in\mathcal{A}_2} \varphi(a) \tag{1.5}$$

$$s.t. \begin{cases} \sum_{a\in\mathcal{A}_3} \varphi(a) \leq R & & (1.6)\\ \sum_{a\in\mathcal{A}} \varphi(a)\cdot\epsilon_a \leq C & & (1.7)\\ \varphi(a) \leq u(a) & \forall a \in \mathcal{A} & (1.8)\\ \sum_{y\in\Gamma^-(x)} \varphi((y,x)) = \sum_{y\in\Gamma^+(x)} \varphi((x,y)) & \forall x \in \mathcal{X} & (1.9)\\ \varphi(a) \in \mathbb{N} & \forall a \in \mathcal{A} & (1.10) \end{cases}$$

The total number of equations is around $2|\mathcal{A}| + N \times T = \Theta(N^2 \cdot T)$.

1.4 Theoretical result

1.4.1 Problem complexity

intro.

1.4.2 Polynomial sub-case

This section aims to prove that the determination of a flow that satisfies all the demands without a constraint on the total number of relocations or vehicles is a polynomial problem. The formal definition of this problem, designated by ALL-DEMANDS, follows.

[**all-demands**]:

Inputs : A Time Expanded Graph $\mathcal{G} = (\mathcal{X}, \mathcal{A}, u)$.

Question : Is there a feasible flow φ such that all the demands are fufilled, *i.e* $\forall a \in \mathcal{A}_2, \varphi(a) = u(a)$?

Let I be an instance of ALL-DEMANDS. We associate an instance of a max-flow problem $f(I)$ which network $\widehat{\mathcal{G}} = (\widehat{\mathcal{X}}, \widehat{\mathcal{A}}, \widehat{w})$ is defined as follows:

1. Vertices are $\widehat{\mathcal{X}} = \mathcal{X} \cup \{s^\star, t^\star\} \cup \{s_a, t_a, a \in \mathcal{A}_2\}$. $s^\star$ and $t^\star$ are respectively the source and the sink of $\widehat{\mathcal{G}}$, while s_a and t_a are two additional vertices associated to any demand arc $a \in \mathcal{A}_2$.

2. Arcs set is $\widehat{\mathcal{A}} = \mathcal{A}_1 \cup \mathcal{A}_3 \cup \{(x, t_a), (t_a, t^\star), (s^\star, s_a), (s_a, y), \forall a = (x, y) \in \mathcal{A}_2\}$.

3. Maximum capacity of arcs are $\widehat{w}(a) = u(a)$ for $a \in \mathcal{A}_1 \cup \mathcal{A}_3$. Otherwise, for any arc $a = (x, y) \in \mathcal{A}_2$, $\widehat{w}((x, t_a)) = \widehat{w}((t_a, t^\star)) = \widehat{w}((s^\star, s_a)) = \widehat{w}((s_a, y)) = u(a)$.

Note that this transformation is a polynomial function and does not depend on the structure of $\mathcal{G}$.

Theorem 4 *Let an instance of* ALL-DEMANDS *expressed by a TEG $\mathcal{G}$. There exists a feasible flow fulfilling all the demands of $\mathcal{G}$ if and only if there exists a maximum flow in $\widehat{\mathcal{G}}$ of value $\sum_{a \in \mathcal{A}_2} u(a)$.*

Proof. Let suppose that φ is a feasible flow of $\mathcal{G}$ that fulfils all the demands, *i.e.* for any arc $a \in \mathcal{A}_2$, $\varphi(a) = u(a)$. A flow $\widehat{\varphi}$ of $\widehat{\mathcal{G}}$ may be built as follows:

1. $\forall a \in \mathcal{A}_1 \cup \mathcal{A}_3$, $\widehat{\varphi}(a) = \varphi(a)$;

2. For any arc $a = (x, y) \in \mathcal{A}_2$, $\widehat{\varphi}((x, t_a)) = \widehat{\varphi}((t_a, t^\star)) = \widehat{\varphi}((s^\star, s_a)) = \widehat{\varphi}((s_a, y)) = u(a)$.

We prove that $\widehat{\varphi}$ is a feasible flow of $\widehat{\mathcal{G}}$ of value $\sum_{a \in \mathcal{A}_2} u(a)$. Indeed, let consider a node $x \in \widehat{\mathcal{X}}$.

1. Let suppose first that $x \in \mathcal{X}$. Then any demand arc $a = (y, x) \in \mathcal{A}_2$ (*resp.* $a = (x, y) \in \mathcal{A}_2$) of flow $\varphi(a)$ is associated in $\widehat{A}$ to an arc $e = (s_a, x)$ (*resp.* $e = (x, t_a)$) with $\widehat{\varphi}(e) = \varphi(a)$. Thus,
$$\sum_{a \in \Gamma^-(\widehat{\mathcal{G}}, x)} \widehat{\varphi}(a) = \sum_{a \in \Gamma^-(\mathcal{G}, x)} \varphi(a) = \sum_{a \in \Gamma^+(\mathcal{G}, x)} \varphi(a) = \sum_{a \in \Gamma^+(\widehat{\mathcal{G}}, x)} \widehat{\varphi}(a).$$

2. For any arc $a = (z, y) \in \mathcal{A}_2$, the two vertices t_a and s_a are such that

$$\sum_{e \in \Gamma^-(\widehat{\mathcal{G}}, t_a)} \widehat{\varphi}(e) = \widehat{\varphi}((x, t_a)) = \widehat{\varphi}((t_a, t^\star)) = \sum_{e \in \Gamma^+(\widehat{\mathcal{G}}, t_a)} \widehat{\varphi}(e) \text{ and } \sum_{e \in \Gamma^-(\widehat{\mathcal{G}}, s_a)} \widehat{\varphi}(e) = \widehat{\varphi}((s^\star, s_a)) = \widehat{\varphi}((s_a, y)) =$$

3. Lastly,

$$\sum_{e \in \Gamma^-(\widehat{\mathcal{G}}, t^\star)} \widehat{\varphi}(e) = \sum_{a \in \mathcal{A}_2} u(a) \text{ and } \sum_{e \in \Gamma^+(\widehat{\mathcal{G}}, s^\star)} \widehat{\varphi}(e) = \sum_{a \in \mathcal{A}_2} u(a).$$

The consequence is that $\widehat{\varphi}$ is a feasible flow of $\widehat{\mathcal{G}}$ of value $\sum_{a \in \mathcal{A}_2} u(a)$.

Conversely, any feasible flow of $\widehat{\mathcal{G}}$ of value $\sum_{a \in \mathcal{A}_2} u(a)$ verifies that, for any arc $a = (x, y) \in \mathcal{A}_2$, $\widehat{\varphi}((x, t_a)) = \widehat{\varphi}((t_a, t^\star)) = \widehat{\varphi}((s^\star, s_a)) = \widehat{\varphi}((s_a, y)) = u(a)$. A feasible flow for $\mathcal{G}$ can be easily obtained by setting:

1. $\forall a \in \mathcal{A}_1 \cup \mathcal{A}_3$, $\varphi(a) = \widehat{\varphi}(a)$;

2. For any arc $a = (x, y) \in \mathcal{A}_2$, $\varphi(a) = u(a)$,

the theorem. ■

According to [**?**], the existence of a maximum-flow of a fixed value is a polynomial problem. The following corollary is thus a consequence of Theorem 4:

Corollary 1 ALL-DEMANDS *is polynomial.*