Data adaptive selection of the truncation level

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1 Problem setting

For a single time point intervention, assume W is univariate and continuous.

Denote $\Psi_0(\delta) = E_{P_0}\left[\frac{g_0(d(W)|W)}{g_{0,\delta}(d(W)|W)} Q_{Y,0}(d(W), W)\right]$.

Under the proper causal assumptions, $\Psi_0(0) = EY^d$.

Let $\delta_n = \text{argmin}_\delta MSE_n(\delta) \equiv \text{argmin}_\delta E_{P_0}\left(\Psi_{n,g_n}(\delta)(\delta) - EY^d\right)^2$.

We'd like to find a method that would data-adaptively select a truncation level $\hat{\delta}_n$ such that $MSE_n(\hat{\delta}_n) \sim MSE_n(\delta_n)$.

Minimizing $MSE_n(\delta)$ (w.r.t. δ) from the data is tough: we would have to estimate the $MSE_n'(\delta)$ which involves $(b_0^2)'(\delta) = 2b_0(\delta)b_0'(\delta)$. Since bias is necessarily as hard to estimate as EY^d itself, estimating $MSE_n'(\delta)$ should be hard too.

We have to find a surrogate risk that is easier to estimate from the data. Let $R_n(\delta) = b_0(\delta) + \frac{1}{\sqrt{n}}\sigma_0(\delta)$. This risk might be easier to estimate, since $b_0'(\delta)$ should be close to the finite difference $\frac{\Psi_0(\delta+\Delta)-\Psi_0(\delta)}{\Delta}$. Since for δ large enough $\Psi_0(\delta)$ and $\Psi_0(\delta+\Delta)$ are "easy" to estimate, there's hope we can estimate the finite difference.

We've convinced ourselves that $\delta_n^* \equiv$ argmin$_\delta \sim \delta_n$ (we sketched a proof of this over email).

2 A method to find $\hat{\delta}_n \sim \delta_n$

2.1 Estimating $b_0'(\delta)$

Define the "true finite difference" $\Delta b_0(\delta) = \frac{b_0(\delta+\Delta)-b_0(\delta)}{\Delta}$

Denote the estimated finite difference $\widehat{\Delta b}_n(\delta) = \frac{\hat{b}_n(\delta+\Delta)-\hat{b}_n(\delta)}{\Delta} = \frac{\hat{\Psi}_n(\delta+\Delta)-\hat{\Psi}_n(\delta)}{\Delta}$.

Assume there exist $1 > \beta \geq 0$ and $1/2 > \gamma \geq 0$ such that $b_0(\delta) \sim \delta^{1-\beta}$, $b_0'(\delta) \sim \delta^{-\beta}$, $\beta_0''(\delta) \sim \delta^{-\beta-1}$, $\sigma_0(\delta) \sim \delta^{-\gamma}$, and $\sigma_0'(\delta) \sim \delta^{-\gamma-1}$. Assume also that $\beta < \gamma + 1$.

Under these assumptions, $R'(\delta_n^*) = 0$ implies that $\delta^{-\beta} + \frac{1}{\sqrt{n}}\delta^{-\gamma} = 0$, i.e. $\delta_n^* \sim n^{-\frac{1}{2(\gamma+1-\beta)}}$.

The typical error we make in estimating $b_n'(\delta)$, which I'll denote $\sigma_{b_0'(\delta),n}$, is the statistical error plus the approximation error. The approximation error is $\Delta b''(\delta) + o(\Delta)$. The standard deviation of $\widehat{\Delta b_n}(\delta)$ can be upper bounded by $n^{-1/2}\Delta^{-1}\sigma_0(\delta)$. Therefore

$$\sigma_{b_0'(\delta),n} = \Delta b''(\delta) + n^{-1/2}\Delta^{-1}\sigma_0(\delta) \sim \Delta\delta^{-\beta-1} + \Delta^{-1}n^{-1/2}\delta^{-\gamma}.$$

For a given δ and a given n, let's optimize it wrt Δ. Setting $\frac{d}{d\Delta}\sigma_{b_0'(\delta),n,\Delta} = 0$ yields $\Delta(n,\delta) \sim n^{-1/4}\delta^{\frac{\beta+1-\gamma}{2}}$. For this choice of Δ, $\sigma_{b_0'(\delta_n^*),n,\Delta} \sim n^{-1/4}\delta^{-\frac{\gamma+1+\beta}{2}}$.

We need the $\sigma_{b_0'(\delta),n} << b_0'(\delta)$ to be able to estimate $b'0_\delta$ from the finite difference $\widehat{b_n}(\delta)$. For the optimal choice $\Delta(n,\delta)$, and **for δ small, $\sigma_{b_0'(\delta),n} << b_0'(\delta)$ is equivalent to $\delta >> \delta_n^*$.**

Thus, if we are to estimate anything related to $b_0'(\delta)$, we need to it at a $\tilde{\delta}_n$ such that $\tilde{\delta_n} = o(\delta_n^*)$

2.2 Estimating the rate in δ of $\sigma_0(\delta)$

Estimating the rate in δ of $\sigma_0(\delta)$ seems to be feasible. For various samples sizes and for a few target parameters, I plotted the $\log \sigma_n(\delta)$ against $log(\delta)$, where $\sigma_n(\delta)$ is the empirical variance of the influence curve.

In my simulations I worked with a family of data-generating distributions defined by:

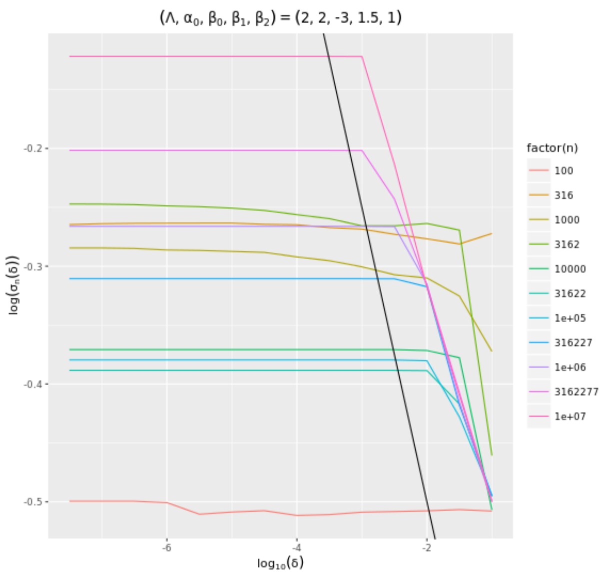
$$
\begin{aligned}
W &\sim Exp\left(\frac{1}{\lambda}\right) \\
A &\sim Bernouilli(expit(\alpha_0 W)) \\
Y &\sim Bernouilli(expit(\beta_0 + \beta_1 A + \beta_2 W))
\end{aligned}
$$

True rates for these data generating distributions:

If $\beta_2 > 0$, $b_0(\delta) \sim \delta^{-1+\frac{\lambda^{-1}+\alpha_0+\beta_2}{\alpha_0}}$ and thus $\beta = 2 - \frac{\lambda^{-1}+\alpha_0+\beta_2}{\alpha_0}$. Also, $\sigma_0^2(\delta) = \delta^{-2+\frac{\lambda^{-1}+\alpha_0+\beta_2}{\alpha_0}}$ and thus $\gamma = 1 - \frac{\lambda^{-1}+\alpha_0+\beta_2}{\alpha_0}$. The optimal rate is then $\frac{\alpha_0}{\alpha_0+\lambda^{-1}+\beta_2}$.

If $\beta_2 < 0$, $b_0(\delta) \sim \delta^{-1+\frac{\lambda^{-1}+\alpha_0}{\alpha_0}}$ and thus $\beta = 2 - \frac{\lambda^{-1}+\alpha_0}{\alpha_0}$. Also, $\sigma_0^2(\delta) = \delta^{-2+\frac{\lambda^{-1}+\alpha_0+|\beta_2|}{\alpha_0}}$ and thus $\gamma = 1 - \frac{\lambda^{-1}+\alpha_0+|\beta_2|}{\alpha_0}$. The optimal rate is then $\frac{\alpha_0}{\alpha_0+\lambda^{-1}-|\beta_2|}$.

Here is what I got for $(\lambda, \alpha_0, \beta_0, \beta_1, \beta_2) = (2, 2, -3, 1.5, 1)$ and $(\lambda, \alpha_0, \beta_0, \beta_1, \beta_2) = (2, 4, -3, 1.5, 0.5)$. These specifications of the data-generating distribution make EY_1 weakly identifiable.



The black line has the true rate γ as slope. Seems like for $n > 1e3$ we should be able to estimate γ decently well from our data, just by fitting a line to $\sigma_n(\delta)$ in a region where δ is neither too big (so that the behavior of $\sigma_0(\delta)$ is asymptotic) and neither too small (so that we have asymptotic linearity of our TMLE).

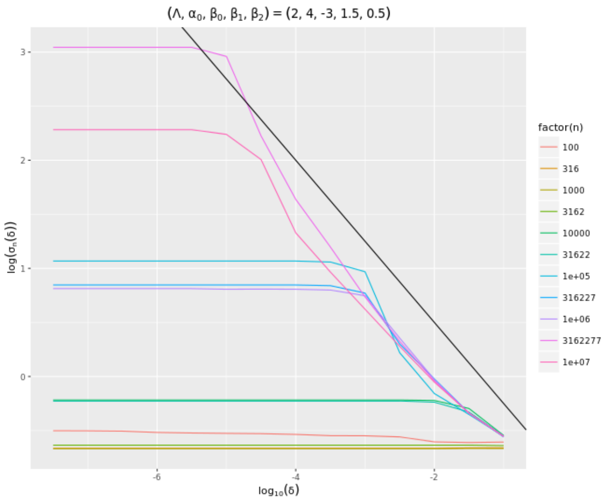
2.3 Estimating the rate of δ_n

Assume that we can estimate the rate $-\gamma$ of $\sigma_0(\delta)$ (see above section regarding feasibility of this).

For $\eta > 1$, let $\delta_{n,\eta} = n^{-\frac{1}{2\eta(\gamma+1-\beta)}}$.

Assume that $\Psi_n(\delta_{\eta,n}) \sim \Psi_0(\delta_n \eta, n) + P_n D_\delta(P_0)(O)$. This should be fine for $\delta_n \eta, n$ slow enough (i.e. for η large enough).

Let $g_{\eta,n}(\delta) = \sqrt{n}\left(\widehat{\Delta b_n}(\delta)\delta^{\gamma+1}\right)^{\eta}$.



We have that $g_{\eta,n}(\delta_{n,\eta-\epsilon}) \sim C^{\eta} n^{-\frac{\epsilon}{2(\eta-\epsilon)}} \to 0$, $g_{\eta,n}(\delta_{n,\eta+\epsilon}) \sim C^{\eta} n^{\frac{\epsilon}{2(\eta+\epsilon)}} \to \infty$, and $g_{\eta,n}(\delta_{n,\eta}) \to C^{\eta}$, for a certain constant C.

This suggests two different methods to estimate the optimal rate $\frac{-1}{2(\gamma+1-\beta)}$.

First method. Let $\eta > 1$. Solve $g_{\eta,n}(\delta) = 1$. The solution is $\sim n^{-\frac{1}{2(\gamma+1-\beta)}}$. Simulations show that the existence of a solution can require an extremely high n, since the constant $\eta|\log C|$ can be large.

Second method. Since the constant C is problematic, another option is to check for different rates $r_1, ..., r_q$ whether $g_{\eta,n}(n^{-r_i})$ is increasing, decreasing or stationary as n increases.

To be able to do this, we need to compare $g_{n,n^{-r_i}}$ for different values of n. This suggest the following method. Let $\mathcal{O}_n$ our sample. Let $\mathcal{O}_{\tilde{n}}^1, ..., \mathcal{O}_{\tilde{n}}^m$ m subsamples of $\mathcal{O}$ of size $\tilde{n}$ (in my simulations I worked with $\tilde{n} = n/3$ and $\tilde{n} = n/10$). Compute the median of $\{g_{\eta,n,\mathcal{O}_n}(n^{-r_i}) - g_{\eta,\tilde{n},\mathcal{O}_{\tilde{n}}^k}(\tilde{n}^{-r_i}) : k \in \{1, ..., m\}\}$. If for the rates r_i and r_{i+1} this median is respectively positive and negative, then we estimate the optimal rate by a value between r_i and r_{i+1}.

We can refine the interval $[r_i, r_{i+1}]$ such that we find a rate r for which we have stationarity of $g_{\eta,n}(n^{-r}$ from our subsamples to our full sample.

I did a bunch of simulations for the family of data-generating distribution specified above.

Here are some encouraging plots (figures 1 and 2).

2.4 Remark on selection of the step of the finite difference

We need to know β and γ to know the optimal $\Delta(n, \delta)$. We can probably access γ (the rate of $\sigma_0(\delta)$) directly from the data, but it seems much harder to do so for β.

We probably don't need Δ to be optimal to have $\sigma_{b_0'(\delta_n^*),n} = o(b_0'(\delta_n^*))$.

A suggestion is to act as if β was $\gamma + 1 - \tilde{\epsilon}$ for a small $\epsilon \in (0, \gamma + 1 - \beta)$ (but closer to zero than to $\gamma + 1 - \beta$). We assume that γ is known. Let's see where this leads:

The "optimal" Δ we would pick is then given by

$$\Delta^2 = \frac{n^{-1/2}\delta^{-\gamma}}{\delta^{-(\gamma+1-\tilde{\epsilon})-1}} = n^{-1/2}\delta^{2-\tilde{\epsilon}}$$

i.e. $\Delta = n^{-1/4}\delta^{1-\tilde{\epsilon}/2}$.

Let $(\delta_n^+)_{n\geq 1}$ be such that $\delta_n^+ = o(\delta_n)$. Now let's check if we have $\sigma_{b_0'(\delta_n^+),n} = o(b_0'(\delta_n^+))$.

We have

Figure 1: $\log g_{n,\eta}(n^{-r/\eta})$ (y axis) for $r \in \{0.95r^{optimal}, r^{optimal}, 1.05r^{optimal}\}$, for various values of η. We observe that the stationary curves are the ones that correspond to $r^{optimal}$. x axis is $log(\delta)$

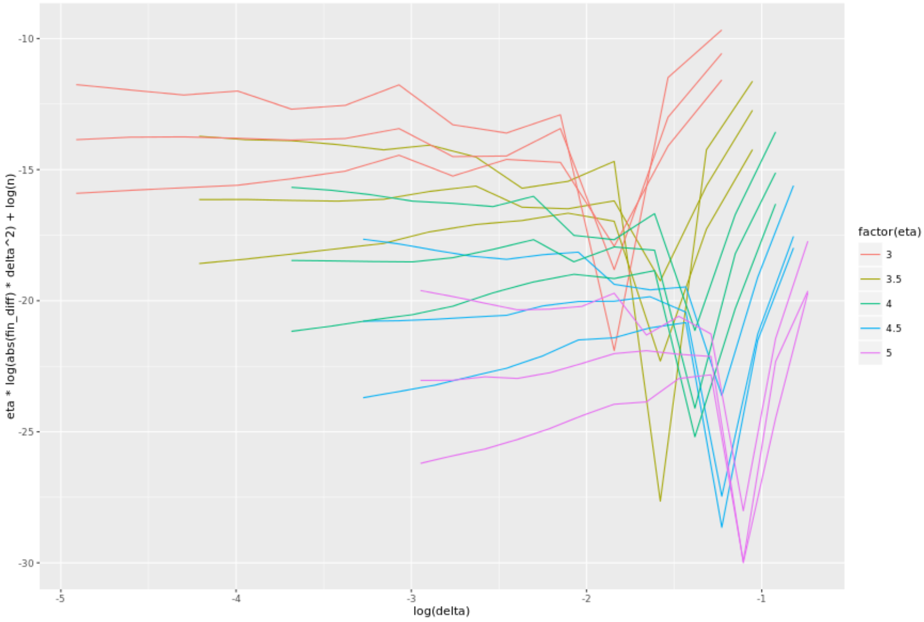
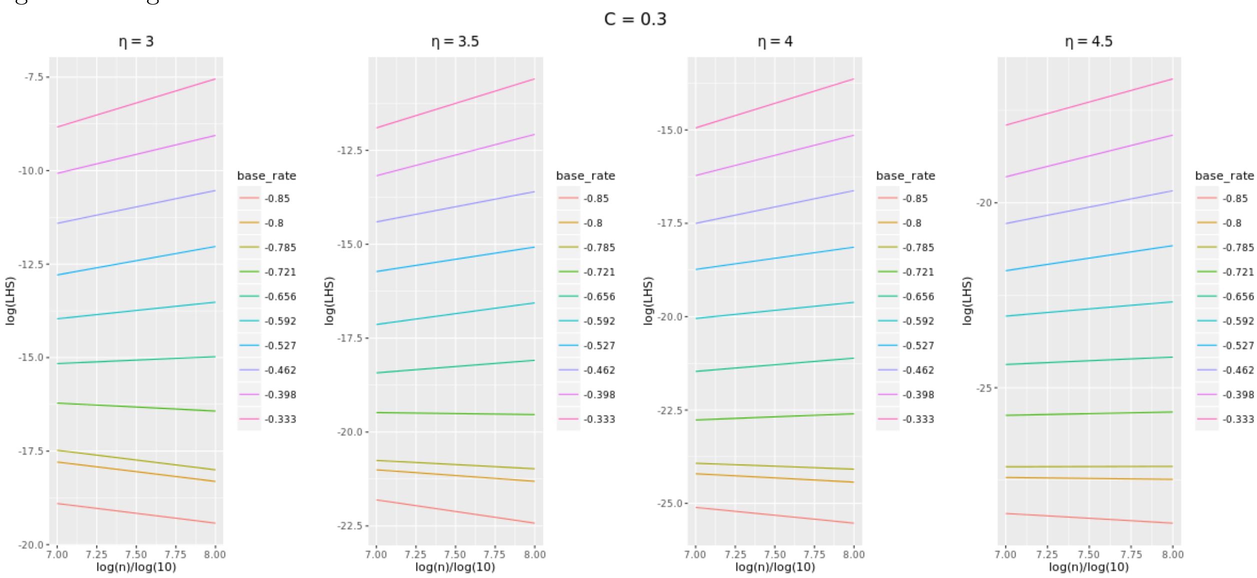


Figure 2: $\log g_{n,\eta}(n^{-r/\eta})$ (y-axis) from $n = 10^7$ to $n = 10^8$, for various candidate rates. x-axis: $\log \delta$. The optimal rates lies between two consecutive candidates rates for which we observe the slopes to change sign. That's good.



$$\begin{aligned}\sigma_{b_0'(\delta_n^+),n} &= \Delta \delta_n^{+-\beta-1} + n^{-1/2}\delta_n^{+-\gamma}\Delta^{-1} \\ &= n^{-1/4}\delta_n^{+-\beta-\tilde{\epsilon}/2} + n^{-1/4}\delta_n^{+-\gamma-\tilde{\epsilon}/2-1} \\ &= n^{-1/4}\left(\delta_n^{+-(\beta+\tilde{\epsilon}/2)} + \delta_n^{+-(\gamma+1-\tilde{\epsilon}/2)}\right)\end{aligned}$$

Since we pick $\tilde{\epsilon}$ relatively small we have $\gamma + 1 - \tilde{\epsilon}/2 > \beta + \tilde{\epsilon}/2$.
Thus $\sigma_{b_0'(\delta_n^+),n} \sim n^{-1/4}\delta_n^{+-(\beta+\tilde{\epsilon}/2)} \sim \left(\frac{\delta_n}{\delta_n^+}\right)^{\beta+\tilde{\epsilon}/2} n^{\frac{2\beta-((\gamma+1-\beta)-\tilde{\epsilon})}{4(\gamma+1-\beta)}}$.
We have $b_0'(\delta_n^+) \sim \left(\frac{\delta_n}{\delta_n^+}\right)^{\beta} n^{\frac{\beta}{2(\gamma+1-\beta)}}$.
Since $\tilde{\epsilon} < \gamma + 1 - \beta$ we have $\sigma_{b_0'(\delta_n^+)} = o(b_0'(\delta_n^+))$.

That's hopeful since picking a small $\tilde{\epsilon}$ is probably practically feasible.

3 Theoeretical result

Given the above, I guess we can prove the following theorem:

Theorem 1. *Assume there exist $1 > \beta \geq 0$ and $1/2 > \gamma \geq 0$ such that $b_0(\delta) \sim \delta^{1-\beta}$, $b_0'(\delta) \sim \delta^{-\beta}$, $\beta_0''(\delta) \sim \delta^{-\beta-1}$, $\sigma_0(\delta) \sim \delta^{-\gamma}$, and $\sigma_0'(\delta) \sim \delta^{-\gamma-1}$.*
Assume also that $\beta < \gamma + 1$.
Denote δ_n the solution to $MSE_n(\delta) = 0$.
Assume that there exist $\delta_n^+ \to 0$, such that for any $\tilde{\delta}_n$ that goes to zero slower than δ_n^+,

$$\hat{\Psi}_n(\tilde{\delta}_n) \sim \Psi_0(\tilde{\delta}_n) + P_n D^*_{\Psi_0(\tilde{\delta}_n)}.$$

Let $\tilde{\epsilon} \in (0, \gamma + 1 - \beta)$, and $\Delta(\delta, n) = n^{-1/4}\delta^{1-\tilde{\epsilon}/2}$. Let $\eta > 1$ large enough so that $n^{-\frac{1}{2\eta(\gamma+1-\beta)}}$ goes to zero slower than δ_n^+.
Denote r_n the rate we find using the "second method" above. Then $n^{-r_n} \sim C\delta_n^{1/\eta}$ for a certain constant C, and $MSE_n(n^{-r_n}) \sim C' MSE_n(\delta_n)$ for a certain constant C'.

4 Discussion

I need to make this method work well for small sample sizes. Checking asymptotic linearity is key to this.
Shapiro-Wilks test for normality of $Psi_n^(\delta)$ for which I bootstrap the targeting step does not seem to be stringent enough.
For low sample sizes, it's kind of possible to check that things aren't behaving as expected from the asymptotics: the plots of the type of figure 1 and 2 then look messy.