Asymptotic normality of our data-adaptive estimator

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Let $\hat{P}_0 = \hat{Q}_0 \hat{g}_0$ be the limit of our initial estimator P_n^0. Let $\delta_n(\hat{P}_0)$ the optimal truncation level under $\hat{P}_0$.

Assume that we have an estimate $\hat{\delta}_n(\hat{P}_0)$ of $\delta_n(\hat{P}_0)$ such that $\hat{\delta}_n(\hat{P}_0) \sim C\delta_n(\hat{P}_0)$.

With the usual notations,

$$\begin{aligned}\Psi_{\delta_n}(P_n^*) - \Psi_{\delta_n}(P_0) &= -P_0 D^*_{\delta_n}(P_n^*) + R_{\delta_n}(P_n^*, P_0)\\ &= (P_n - P_0) D^*_{\delta_n}(P_n^*) - P_n D^*_{\delta_n}(P_n^*) + R_{\delta_n}(P_n^*, P_0).\end{aligned}$$

The targeting step ensures that that $P_n D^*_{\delta_n}(P_n^*) = 0$.

Let $\sigma^2(P)(\delta) = PD^*_\delta(P)^2$ and $b(P)(\delta) = \Psi_\delta(P) - \Psi_0(P)$. Assume that $\sigma(P)^2(\delta) \sim C_0 \delta^{-2\gamma(P)}$, $\gamma(P) \geq 0$, and $b(P)(\delta) = C_1 \delta^{1-\beta(P)}$, $\beta < 1$. Under these assumptions, the optimal truncation level under P is $\delta_n(P) = n^{-\frac{1}{2(\gamma(P)+1-\beta(P))}}$.

We have that

$$\begin{aligned}\Psi_{\delta_n}(P_n^*) - \Psi_{\delta_n}(P_0) &= \delta_n(\hat{P}_0)^{-\gamma(\hat{P}_0)} (P_n - P_0)(\delta_n(\hat{P}_0)^{\gamma(\hat{P}_0)} D^*_{\delta_n}(P_n^*)) + R_{\delta_n(\hat{P}_0)}(P_n^*, P_0)\\ &= \delta_n(\hat{P}_0)^{-\gamma(\hat{P}_0)} (P_n - P_0) Z_n(P_n^*) + R_{\delta_n(\hat{P}_0)}(P_n^*, P_0),\end{aligned}$$

where I define $Z_n(P) = \delta_n(\hat{P}_0)^{\gamma(\hat{P}_0)} D^*_{\delta_n(\hat{P}_0)}(P)$.

Assume that

1. $P_0 \left(Z_n(P_n^*) - Z_\infty(\hat{P}_0) \right)^2 \to 0$ for a $Z_\infty(\hat{P}_0)$ such that $\hat{P}_0 Z_\infty(\hat{P}_0)^2 < \infty$,
2. $Z_n(P_n^*)$ falls in a P_0-Donsker class $\mathcal{F}$ with probability tending to one,
3. $R_{\delta_n}(P_n^*, P_0) = o_P(\delta_n(\hat{P}_0)^{-\gamma(\hat{P}_0)}/\sqrt{n}) = o_P\left(n^{-\frac{1-\beta(\hat{P}_0)}{2(\gamma+1-\beta(\hat{P}_0))}} \right)$.

From (1) and (2) above, by equicontinuity of the empirical process on $\mathcal{F}$, $(P_n - P_0)(Z_n(P_n^*) - Z_\infty(\hat{P}_0)) = o_P(1/\sqrt{n})$.

Therefore

$$\Psi_{\delta_n}(P_n^*) - \Psi_{\delta_n}(P_0) = n^{\frac{\gamma(\hat{P}_0)}{2(\gamma+1-\beta(\hat{P}_0))}} P_n \left(Z_\infty(\hat{P}_0) - P_0 Z_\infty(\hat{P}_0) \right) + o_P\left(n^{-\frac{1-\beta(\hat{P}_0)}{2(\gamma(\hat{P}_0)+1-\beta(\hat{P}_0))}} \right),$$

and thus

$$n^{\frac{1-\beta(\hat{P}_0)}{2(\gamma(\hat{P}_0)+1-\beta(\hat{P}_0))}} \left(\Psi_{\delta_n}(P_n^*) - \Psi_{\delta_n}(P_0) \right) \xrightarrow{d} \mathcal{N}\left(0, P_0 \left(Z_\infty(\hat{P}_0) - P_0 Z_\infty(\hat{P}_0) \right)^2 \right).$$