1 Bayes Risk with Absolute Loss

1.1 Question 0

1. Let $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = x^3$$

$$f'(x) = 3x^2, \quad f'(0) = 0, \quad f''(x) = 6x, \quad f''(0) = 0$$

$$For x < 0 : f(x) = x < 0, For x > 0 : f(x) = x > 0$$

Therefore, f is strictly increasing around x = 0, making it an inflection point rather than a local extremum

1.2 Question 1

1. Let Y be a random variable with

$$P(Y = 0) = 0.1, \quad P(Y = 1) = 0.9$$

$$f^*_{\text{squared}} = \mathbb{E}[Y] = 0.1 \times 0 + 0.9 \times 1 = 0.9$$

2. Set

$$h_1 = 0.9 \quad \text{(conditional mean)}$$

$$h_2 = 1 \quad \text{(empirical median)}$$

3. We want to calculate the absolute risk $R_{\text{absolute}}(h) = \mathbb{E}[|Y - h|]$

$$R_{\text{absolute}}(h_1) = 0.1 \times |0 - 0.9| + 0.9 \times |1 - 0.9| = 0.09 + 0.09 = 0.18 = R_{\text{absolute}}(f^*_{\text{squared}})$$

$$R_{\text{absolute}}(h_2) = 0.1 \times |0 - 1| + 0.9 \times |1 - 1| = 0.1$$

4. Conclusion

$$R_{\text{absolute}}(h_2) < R_{\text{absolute}}(f^*_{\text{squared}}) = 0.18$$

5. The estimator h is 1

6.

$$f^*_{\text{absolute}}(x) = \arg\min_{z \in \mathbb{R}} g(z)$$

1.3 Question 2

We want to show that the minimizer of $g(z)$ is the median of the distribution $Y|X = x$.

$g(z)$ is a convex function: it is a weighted average of convex functions $|y - z|$ (see part 2.3.1)

$\Rightarrow$ every local minimum is a global minimum

So we can find the minimum of $g(z)$

$$g(z) = \int_{-\infty}^{+\infty} |y - z| \, p_{Y|X=x}(y) \, dy$$

$$\frac{d}{dz}|y - z| = \begin{cases} -1 & y > z \\ 1 & y < z \\ 0 & y = z \end{cases} \quad \text{with} \quad |y - z| = \begin{cases} y - z & y > z, \quad (y - z)' = -1 \\ z - y & z > y, \quad (z - y)' = 1 \end{cases}$$

$$\Rightarrow \delta'(z) = \int_{y<z} p(y) \, dy - \int_{y>z} p(y) \, dy = P(Y < z|X = x) - P(Y > z|X = x)$$

$$= P(Y < z|X = x) - 1 + P(Y < z|X = x) = 2P(Y < z|X = x) - 1$$

$$= 2F(z) - 1$$

$$\delta'(z^*) = 0$$

$$\Rightarrow 2F(z^*) - 1 = 0$$

$$\Rightarrow F(z^*) = \frac{1}{2} = 0.5$$

$\Rightarrow z^*$ is the median of $Y|X = x$

$\delta''(z) = 2p(Y = z|X = x) > 0 \Rightarrow z^*$ is indeed a local minimum

$$\Rightarrow f^*_{\text{absolute}}(x) = \arg\min_{z \in \mathbb{R}} g(z) = z = \text{median}(Y|X = x)$$