Reinforcement Learning: Multi-Armed Bandits

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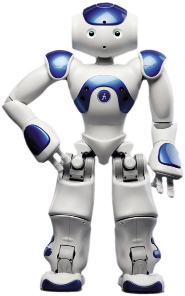
- Inspired by the behavior of animals (including humans!)
- The **exploration-exploitation** trade-off
- Many applications: robotics, games, advertising, content recommendation, medicine, etc.



Outline

1. Multi-armed bandits
2. Performance metrics
3. Main algorithms
4. Lower bound
5. Extensions

Multi-Armed Bandits

- A class of RL problems where the agent does **not** modify her environment
- At time $t = 1, 2, \ldots$, the agent selects an **action** a_t in some finite set A and receives some **reward** r_t

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- The objective is to find and to **exploit** the best action(s) on observing the rewards

Example: A/B testing

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Example: Obama 2008 campaign

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- The objective is to maximize the **cumulative reward** over some (possibly unknown) time horizon T:

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- The objective is to maximize the **cumulative reward** over some (possibly unknown) time horizon T:

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- If action a is always selected, the cumulative reward is approximately $Tq(a)$ for large T
- Maximum reward (per action):

$$q^\star = \max_a q(a)$$

- Best action(s)

$$a^\star = \arg\max_a q(a)$$

Performance metrics

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$$P = \frac{1}{T} \sum_{t=1}^{T} 1_{\{a_t = a^\star\}}$$

Performance metrics (in expectation)

Let $N_t(a)$ be # of times action a has been selected up to time t

1. **Cumulative regret** (expected gap to optimal reward)

$$
\begin{aligned}
\mathrm{E}(R) &= q^\star T - \sum_{t=1}^{T} \mathrm{E}(q(a_t)) \\
&= \sum_{a \in A} (q^\star - q(a)) \mathrm{E}(N_T(a))
\end{aligned}
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2. **Precision** (expected proportion of optimal actions)

$$\mathrm{E}(P) = \frac{1}{T} \sum_{t=1}^{T} P(a_t = a^\star)$$
$$= \frac{\mathrm{E}(N_T(a^\star))}{T}$$

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Action	A	B	C
Expected reward	1	9	10

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Table: Policy 1: Regret (per action) $= 1.3$, Precision $= 0.5$

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Table: Policy 1: Regret (per action) $= 1.3$, Precision $= 0.5$

Action	A	B	C
Distribution	20%	20%	60%

Table: Policy 2: Regret (per action) $= 2$, Precision $= 0.6$

Efficiency

- Efficient algorithm = **sublinear regret**

$$\frac{\mathrm{E}(R)}{T} \to 0 \quad \text{when} \quad T \to +\infty$$

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  this implies

$$\forall a \neq a^\star, \quad \frac{\mathrm{E}(N_T(a))}{T} \to 0$$

  and

$$P \to 1$$

A first algorithm

Greedy algorithm

Initialize, for all actions a:

- $N(a) \leftarrow 0$
- $Q(a) \leftarrow 0$

Repeat:

- $a \leftarrow \arg\max_a Q(a)$ (random tie breaking)
- $r \leftarrow \text{reward}(a)$
- $N(a) \leftarrow N(a) + 1$
- $Q(a) \leftarrow Q(a) + \frac{1}{N(a)}(r - Q(a))$

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Remark

- Importance of initial values!

A second algorithm

ε-greedy algorithm

Parameter: ε

Initialize, for all actions a:

- $N(a) \leftarrow 0$
- $Q(a) \leftarrow 0$

Repeat:

- $a \leftarrow \begin{cases} \text{random action} & \text{with probability } \varepsilon \\ \arg\max_a Q(a) & \text{with probability } 1-\varepsilon \end{cases}$
- $r \leftarrow \text{reward}(a)$
- $N(a) \leftarrow N(a) + 1$
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A third algorithm

Adaptive-greedy algorithm

Parameter: c

Initialize, for all actions a:

- $N(a) \leftarrow 0$
- $Q(a) \leftarrow 0$

Repeat for $t = 1, 2, \ldots$

- $\varepsilon \leftarrow \frac{c}{c+t}$
- $a \leftarrow \begin{cases} \text{random action} & \text{with probability } \varepsilon \\ \arg\max_a Q(a) & \text{with probability } 1 - \varepsilon \end{cases}$
- $r \leftarrow \text{reward}(a)$
- $N(a) \leftarrow N(a) + 1$
- $Q(a) \leftarrow Q(a) + \frac{1}{N(a)}(r - Q(a))$

Upper confident bound

Idea = **bonus** for uncertainty

UCB algorithm

Parameter: c

Initialize, for all actions a:

- $N(a) \leftarrow 0$
- $Q(a) \leftarrow 0$

Repeat for $t = 1, 2, \dots$

- $a \leftarrow \arg\max_a (Q(a) + c\sqrt{\frac{\log t}{N(a)}})$
- $r \leftarrow \text{reward}(a)$
- $N(a) \leftarrow N(a) + 1$
- $Q(a) \leftarrow Q(a) + \frac{1}{N(a)}(r - Q(a))$

Bayesian algorithm

Idea = replace uncertainty by... **randomness!**

Thompson sampling

Initialize, for all actions a:

- $P(a) \leftarrow$ prior

Repeat:

- for all actions a, $Q(a) \leftarrow \text{sample}(P(a))$
- $a \leftarrow \arg\max_a Q(a)$
- $r \leftarrow \text{reward}(a)$
- $P(a) \leftarrow \text{update}(r)$

Remark

- Proposed by Thompson in... 1933!
- Very efficient in practice, proved optimal only recently

Thompson sampling: binary rewards

- Prior = **uniform distribution**

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- Writing

$$p(r|q) = q^r(1-q)^{1-r}, \quad r = 0, 1,$$

the **posterior distribution** follows from Bayes' rule:

$$\begin{aligned} p(q|r_1, \ldots, r_N) &= \frac{p(r_1, \ldots, r_N|q)p(q)}{p(r_1, \ldots, r_N)} \\ &\propto q^{r_1+\ldots+r_N}(1-q)^{N-(r_1+\ldots+r_N)} \end{aligned}$$

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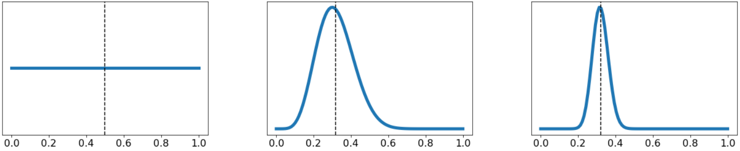
- This is a **Beta distribution**

Example

- True parameter $= 0.3$
- Beta distribution after $N = 0, 10, 100$ tries:



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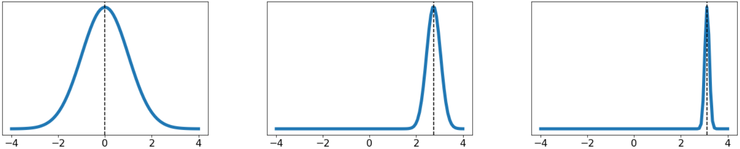
Example

- True parameter $= 3$
- Beta distribution after $N = 0, 10, 100$ tries:



Efficiency of UCB and TS (binary rewards)

- For UCB [Auer et al., 2002a]

$$\mathrm{E}(R) \leq 8 \sum_{a \neq a^\star} \frac{\log T}{q^\star - q(a)} + K \frac{\pi^2}{3}$$

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- For Thompson sampling [Kaufmann et al., 2012]

$\forall \varepsilon > 0,$

$$\mathrm{E}(R) \leq (1 + \varepsilon) \sum_{a \neq a^\star} \frac{q^\star - q(a)}{D(q(a) || q^\star)} (\log T + \log \log T) + C(\varepsilon)$$

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- In both cases, **quasi-optimal actions** seem to incur the highest cost!

Kullback-Leibler divergence (Bernoulli distribution)

- **Divergence** of $\mathcal{B}(q)$ with respect to $\mathcal{B}(p)$

$$D(p||q) = p \log \frac{p}{q} + (1-p) \log \frac{1-p}{1-q}$$

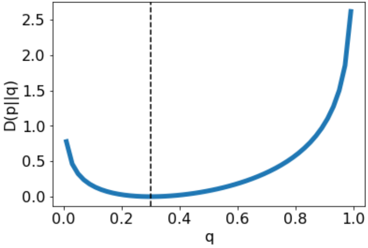

Figure: Example with $p = 0.3$

- **Cost** of coding a sequence of i.i.d. random variables assuming $\mathcal{B}(q)$ instead of $\mathcal{B}(p)$

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- In particular,

$$\liminf_{T \to +\infty} \frac{\mathrm{E}(R)}{\log T} \geq \sum_{a \neq a^\star} \frac{q^\star - q(a)}{D(q(a)||q^\star)}$$

Combinatorial bandits

- Selection of k items among n
- A large number $\binom{n}{k}$ of correlated actions
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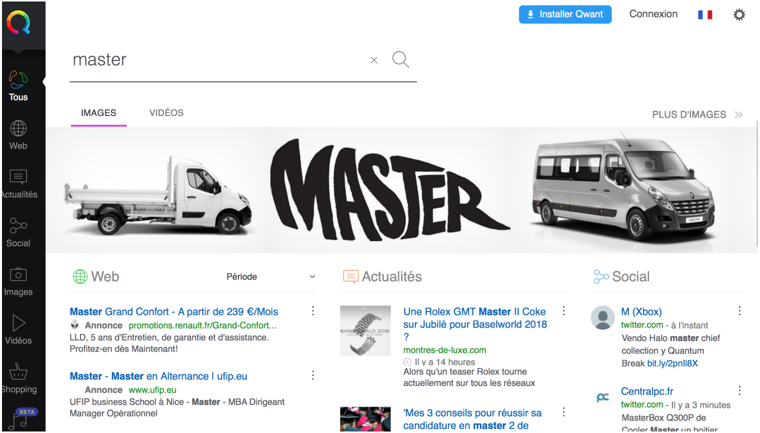

Contextual bandits

- Some **context** associated with each action
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- At time $t = 1, 2, \ldots$, choose action a_t based on state s_t (context) and receive reward r_t that depends on both a_t and s_t
- Case of **linear bandits**: $r = s^T a + \text{noise}$, with $a, s \in \mathbb{R}^d$
  LinUCB algorithm combines linear regression and UCB [Li et al., 2010]

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Adversarial bandits

- Assume unknown or time-varying reward statistics
- Now the rewards are **arbitrary** sequences, and the regret is for the **worst** scenario
- Learning with **sublinear regret** is still possible! (now in $O(\sqrt{T})$)
- For instance, Exp3 (Exponential-weight algorithm for Exploration and Exploitation) **randomly** selects an action in proportion to some weights, which are adapted to the received rewards [Auer et al., 2002b]

References

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From Hoeffding to UCB

- Hoeffding's **concentration inequality**
  (for Bernoulli distribution with parameter p)

$$P\left(\frac{1}{N}(X_1 + \ldots + X_N) < p - \delta\right) \leq e^{-2\delta^2 N}$$

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$$P\left(\frac{1}{N}(X_1 + \ldots + X_N) < p - \delta\right) \leq e^{-2\delta^2 N}$$

- Thus for a target error rate of $1/t$, we get

$$\delta = \sqrt{\frac{\log t}{2N}}$$

From Tsybakov to the lower bound

- Tsybakov's **estimation bound**
  (for Bernoulli distributions with parameters p, q with $|p - q| \ = \varepsilon$)

$$P_p\left(\left|\frac{1}{N}(X_1 + \ldots + X_N) - p\right| > \frac{\varepsilon}{2}\right) \geq \frac{1}{4}e^{-ND(p||q)}$$

$$P_q\left(\left|\frac{1}{N}(X_1 + \ldots + X_N) - q\right| > \frac{\varepsilon}{2}\right) \geq \frac{1}{4}e^{-ND(p||q)}$$

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- Any sub-optimal action a needs to be tested at least

$$N = \frac{\log t}{D(q(a)||q^\star)}$$