Edge detection

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Blabla

We use multi-scale wavelets at scales $s < 12$. Each wavelet high-frequency HF is computed from the subtraction between the previous scale (or the original image if $s = 0$) and $\bar{u}$, the output of a low-pass filter, assumed to be some kind of local moving average, applied on the previous image scale (or the original image if $s = 0$). Most applications simply assume $HF = (\bar{u}_s - \bar{u}_{s-1}) = \Delta u = div(\nabla^2 u)$, but this is actually inaccurate. Let us define :

$$\Delta u_s \approx K(\bar{u}_s - \bar{u}_{s-1}) \tag{1}$$

with :

$$\begin{aligned} \sigma_s &= 2\sigma_{s-1} \\ \bar{u}_s(i,j) &= \frac{1}{2\pi\sigma^2} \sum_{x=-\lfloor 4\sigma_s \rfloor}^{+\lfloor 4\sigma_s \rfloor} \sum_{y=-\lfloor 4\sigma_s \rfloor}^{+\lfloor 4\sigma_s \rfloor} \bar{u}_{s-1}(i+x, j+y) \cdot e^{-\frac{x^2}{2\sigma_s^2}} \cdot e^{-\frac{y^2}{2\sigma_s^2}} \end{aligned} \tag{2}$$

and, for $s = 0$, we initialize :

$$\bar{u}_{s-1}(i,j) = u_0(i,j) \tag{3}$$

If we expand u in Taylor's series, we get :

$$u(i,j) = u_0 + \left(\frac{\partial u}{\partial i}\right)_0 i + \left(\frac{\partial u}{\partial j}\right)_0 j + \frac{1}{2}\left[\left(\frac{\partial^2 u}{\partial i^2}\right)_0 i^2 + \left(\frac{\partial^2 u}{\partial j^2}\right)_0 j^2\right] + \left(\frac{\partial^2 u}{\partial i \partial j}\right)_0 ij + \dots \tag{4}$$

The continuous gaussian filter over a window $a \times a$ gives us :

$$\bar{u}_0 = \frac{1}{2\pi\sigma^2} \iint_{-a/2}^{a/2} u(i,j) \cdot e^{-\frac{i^2+j^2}{2\sigma^2}} \,\mathrm{d}i\,\mathrm{d}j \tag{5}$$

To find K, we simply need to replace (4) in (5). For odd functions like $\iint_{-a/2}^{a/2} i \,\mathrm{d}i\,\mathrm{d}j = 0$, so the contribution of the odd terms is zero. For even functions, the computation is not funny, and assuming $a \to \infty$:

$$\begin{aligned} \iint_{-a/2}^{a/2} i^2 \cdot e^{-\frac{i^2}{2\sigma^2}} \cdot e^{-\frac{j^2}{2\sigma^2}} \,\mathrm{d}i\,\mathrm{d}j &= \int_{-a/2}^{a/2} e^{-\frac{j^2}{2\sigma^2}} \,\mathrm{d}j \cdot \int_{-a/2}^{a/2} i^2 \cdot e^{-\frac{i^2}{2\sigma^2}} \,\mathrm{d}i && (6) \\ &= \sqrt{2\pi\sigma^2} \cdot \frac{1}{2}\sqrt{8\pi\sigma^6} && (7) \\ &= \frac{\sqrt{16\pi}}{2}\sigma^4 && (8) \end{aligned}$$

So, the filter (5) applied on $u(i,j)$ as defined in (4) yields :

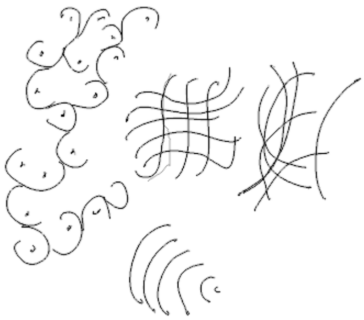
$$\bar{u}_0 = u_0 + \frac{\sqrt{16\pi}\sigma^4}{2} \times \frac{1}{2\pi\sigma^2} \times \frac{1}{2}\left[\left(\frac{\partial^2 u}{\partial i^2}\right)_0 + \left(\frac{\partial^2 u}{\partial j^2}\right)_0\right] = u_0 + \frac{\sqrt{\pi}\sigma^2}{2\pi}\Delta u_0 \Rightarrow \Delta u_0 = \frac{2\pi}{\sqrt{\pi}\sigma^2}(\bar{u}_0 - u_0) \tag{9}$$

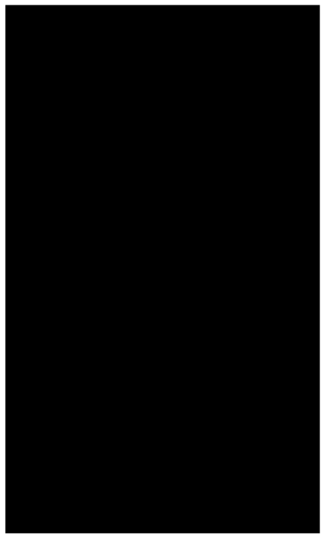
By identification, $K = \frac{2\pi}{\sqrt{\pi}\sigma^2}$. Then, we collapse the contribution of each scale into an energy term that will represent the detail density :

$$E_c(i,j) = \sum_s |\Delta u_s(i,j)|^2 \tag{10}$$

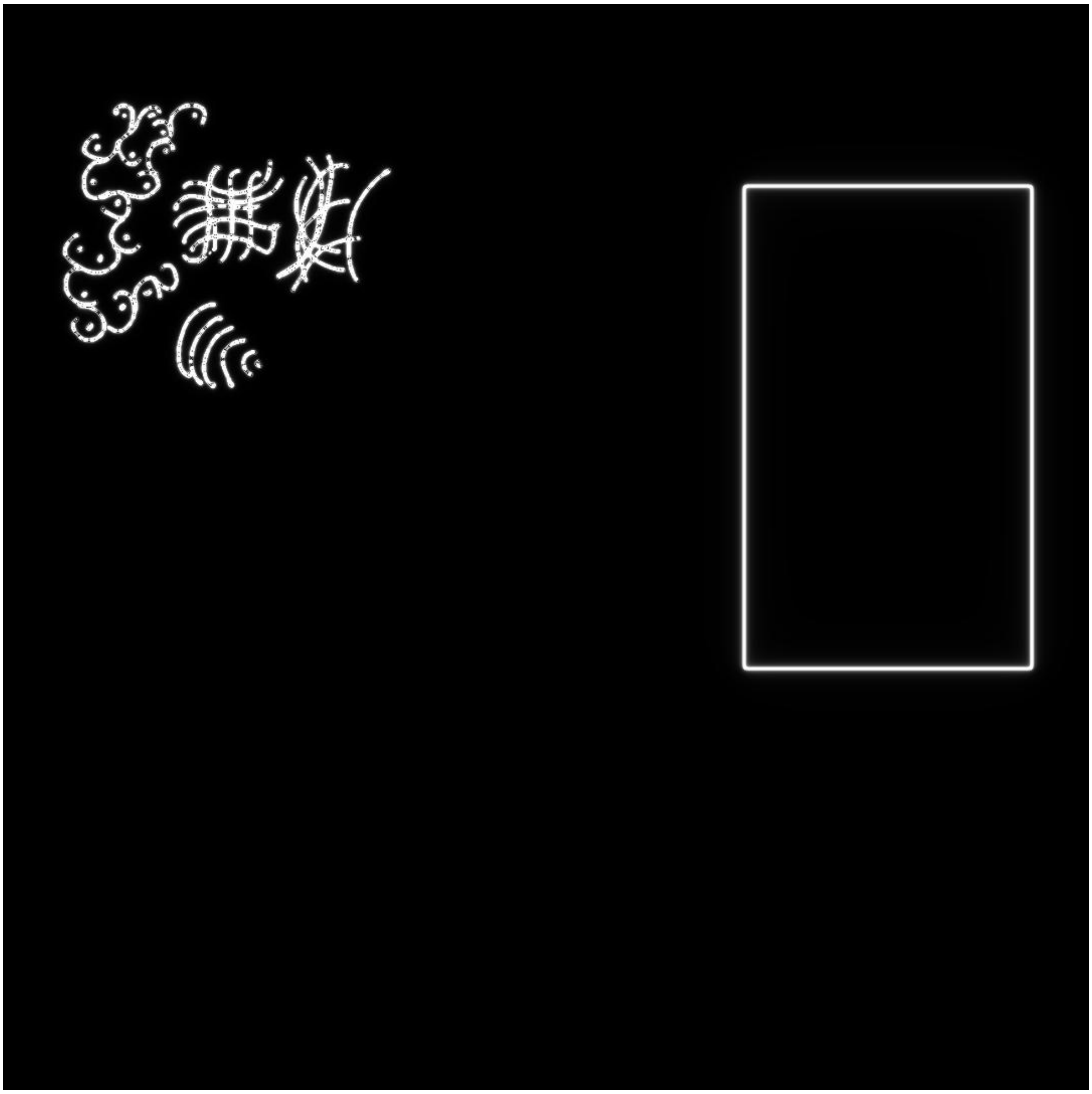
Pretty pictures

Test pic :

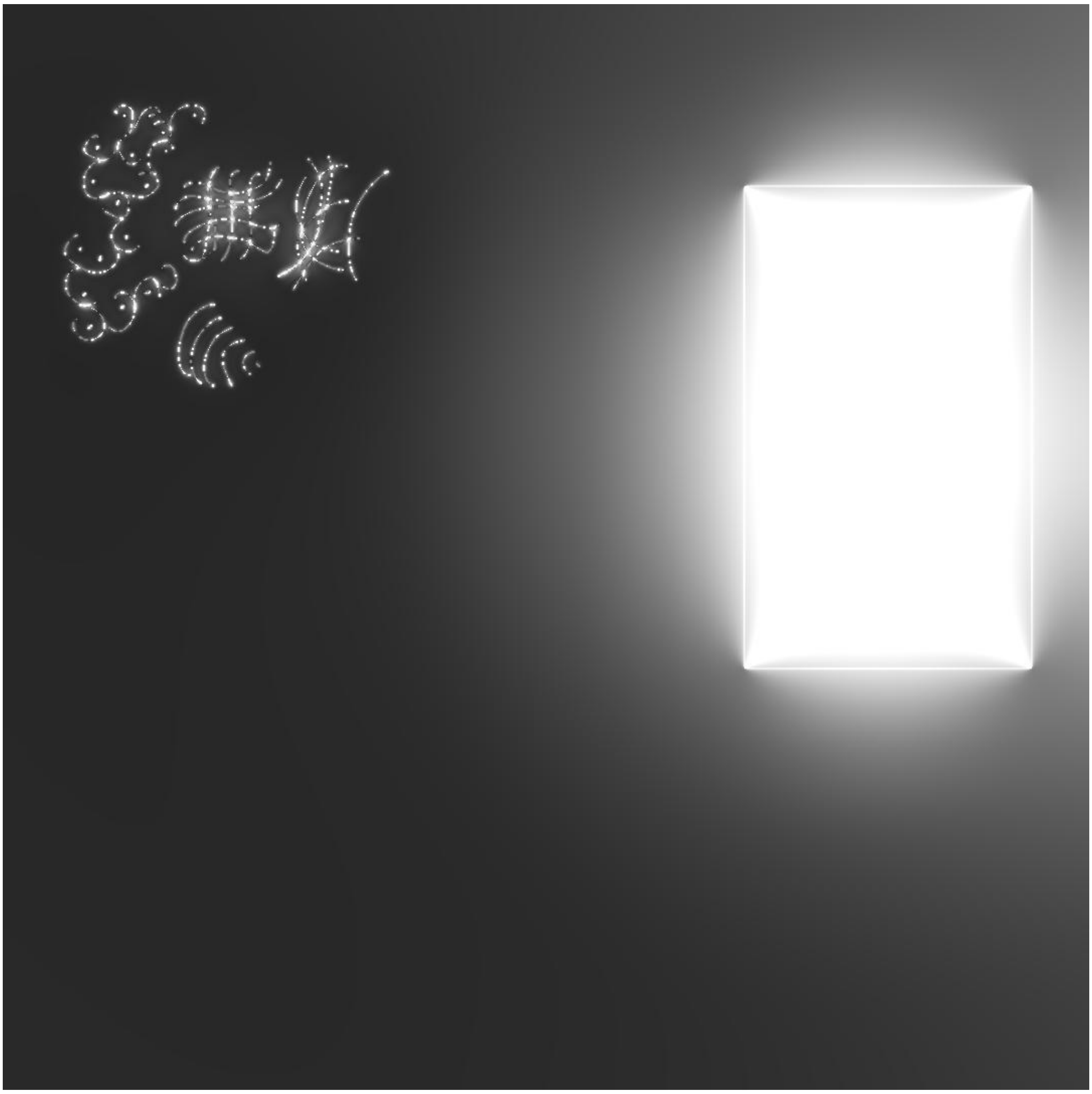




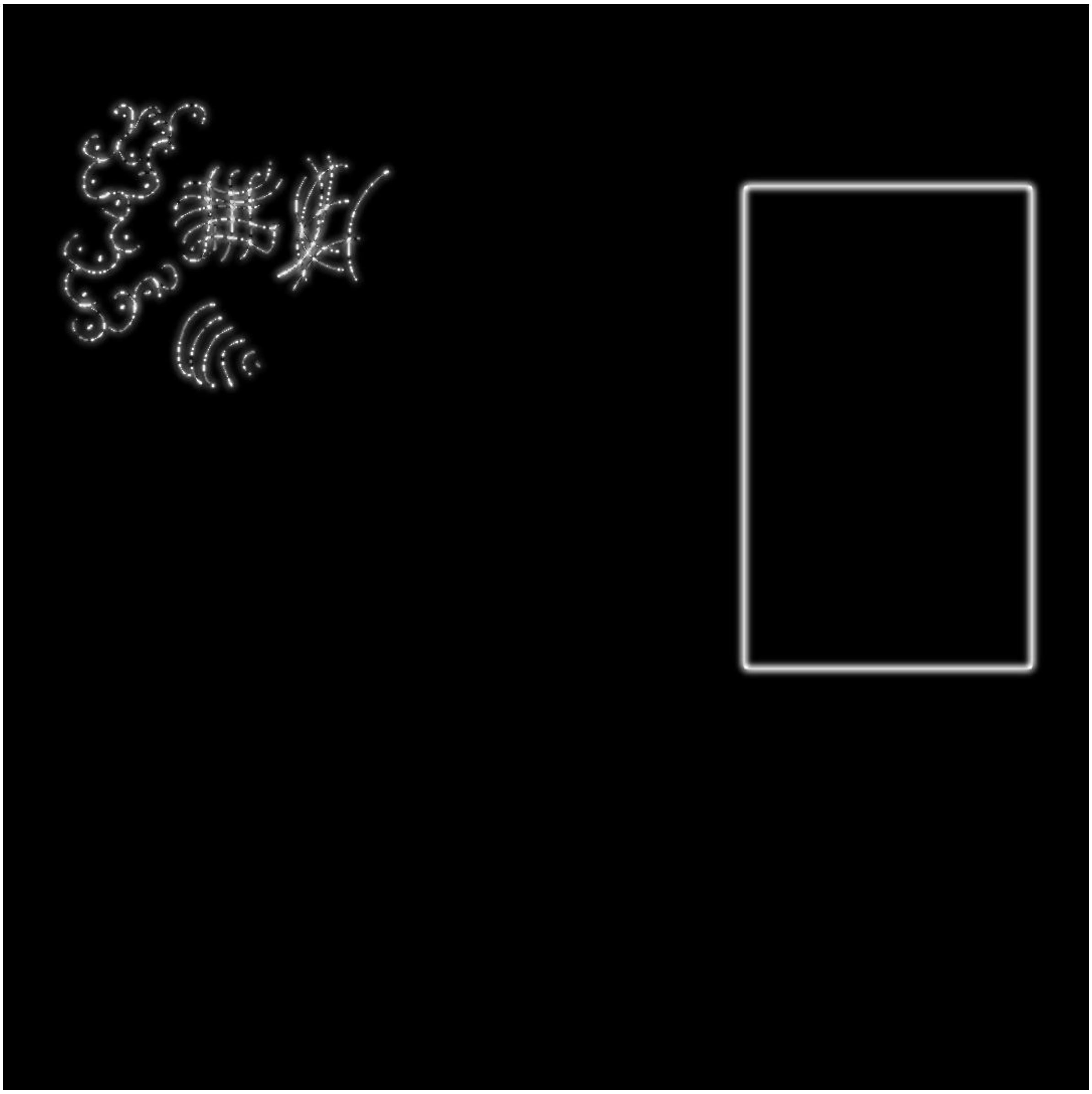
Find edges with $s = 12$ and $K = \frac{2\pi}{\sqrt{\pi}\sigma^2}$:



Find edges with $s = 12$ and $K = 1$:



Find edges with $K = 1$ and $s = 3$:



Find edges with discrete laplacian by orthogonal 2nd order finite differences of the gaussian scales with $s = 12$: