bostonexpt

January 16, 2018

0.1 Building regularized models for Boston data set

Perform a bias variance analysis of the Boston housing data set with the thirteen predictors, following the steps on the simple data set above. Use sklearn's built-in functions to split the data into training, validation and test sets. What is the lowest achievable error on the test set with $\lambda = 0$? Select the best value for λ and report the test set error with the best λ. Use the technique of adding features to extend each column of the Boston data set with powers of the values in the column. Repeat the bias-variance analysis with quadratic and cubic features. What is the test set error with quadratic features with the best λ chosen with the validation set? What is the test set error with cubic features with the best λ chosen with the validation set? Put your analysis code in a separate Python script or notebook called bostonexpt.py or bostonexpt.ipynb. Present your results analytically with plots to support your findings. Discuss the impact of regularization for building good models for the Boston housing data set.

0.2 Reading data

will start by loading and displaying some values from the full Boston housing dataset with thirteen features of census tracts that are believed to be predictive of the median home price in the tract (see **housing.names.txt** for a full description of these features). By looking at the values, you will note that the values of some of the features are about 1000 times the values of others.

```
In [8]: from sklearn.datasets import load_boston
        import numpy as np
        import pandas as pd
        import matplotlib.pyplot as plt
        import utils
        import plot_utils
        from reg_linear_regressor_multi import RegularizedLinearReg_SquaredLoss

        # This is a bit of magic to make matplotlib figures appear inline in the notebook
        # rather than in a new window.

        %matplotlib inline
        plt.rcParams['figure.figsize'] = (10.0, 8.0) # set default size of plots
        plt.rcParams['image.interpolation'] = 'nearest'
        plt.rcParams['image.cmap'] = 'gray'

        # Some more magic so that the notebook will reload external python modules;
```

```
        # see http://stackoverflow.com/questions/1907993/autoreload-of-modules-in-ipython

        %load_ext autoreload
        %autoreload 2

        print 'Reading data ...'
        bdata = load_boston()
        df = pd.DataFrame(data = bdata.data, columns = bdata.feature_names)
        df
```

```
The autoreload extension is already loaded. To reload it, use:
  %reload_ext autoreload
Reading data ...
```

Out[8]:

	CRIM	ZN	INDUS	CHAS	NOX	RM	AGE	DIS	RAD	TAX	\
0	0.00632	18.0	2.31	0.0	0.538	6.575	65.2	4.0900	1.0	296.0	
1	0.02731	0.0	7.07	0.0	0.469	6.421	78.9	4.9671	2.0	242.0	
2	0.02729	0.0	7.07	0.0	0.469	7.185	61.1	4.9671	2.0	242.0	
3	0.03237	0.0	2.18	0.0	0.458	6.998	45.8	6.0622	3.0	222.0	
4	0.06905	0.0	2.18	0.0	0.458	7.147	54.2	6.0622	3.0	222.0	
5	0.02985	0.0	2.18	0.0	0.458	6.430	58.7	6.0622	3.0	222.0	
6	0.08829	12.5	7.87	0.0	0.524	6.012	66.6	5.5605	5.0	311.0	
7	0.14455	12.5	7.87	0.0	0.524	6.172	96.1	5.9505	5.0	311.0	
8	0.21124	12.5	7.87	0.0	0.524	5.631	100.0	6.0821	5.0	311.0	
9	0.17004	12.5	7.87	0.0	0.524	6.004	85.9	6.5921	5.0	311.0	
10	0.22489	12.5	7.87	0.0	0.524	6.377	94.3	6.3467	5.0	311.0	
11	0.11747	12.5	7.87	0.0	0.524	6.009	82.9	6.2267	5.0	311.0	
12	0.09378	12.5	7.87	0.0	0.524	5.889	39.0	5.4509	5.0	311.0	
13	0.62976	0.0	8.14	0.0	0.538	5.949	61.8	4.7075	4.0	307.0	
14	0.63796	0.0	8.14	0.0	0.538	6.096	84.5	4.4619	4.0	307.0	
15	0.62739	0.0	8.14	0.0	0.538	5.834	56.5	4.4986	4.0	307.0	
16	1.05393	0.0	8.14	0.0	0.538	5.935	29.3	4.4986	4.0	307.0	
17	0.78420	0.0	8.14	0.0	0.538	5.990	81.7	4.2579	4.0	307.0	
18	0.80271	0.0	8.14	0.0	0.538	5.456	36.6	3.7965	4.0	307.0	
19	0.72580	0.0	8.14	0.0	0.538	5.727	69.5	3.7965	4.0	307.0	
20	1.25179	0.0	8.14	0.0	0.538	5.570	98.1	3.7979	4.0	307.0	
21	0.85204	0.0	8.14	0.0	0.538	5.965	89.2	4.0123	4.0	307.0	
22	1.23247	0.0	8.14	0.0	0.538	6.142	91.7	3.9769	4.0	307.0	
23	0.98843	0.0	8.14	0.0	0.538	5.813	100.0	4.0952	4.0	307.0	
24	0.75026	0.0	8.14	0.0	0.538	5.924	94.1	4.3996	4.0	307.0	
25	0.84054	0.0	8.14	0.0	0.538	5.599	85.7	4.4546	4.0	307.0	
26	0.67191	0.0	8.14	0.0	0.538	5.813	90.3	4.6820	4.0	307.0	
27	0.95577	0.0	8.14	0.0	0.538	6.047	88.8	4.4534	4.0	307.0	
28	0.77299	0.0	8.14	0.0	0.538	6.495	94.4	4.4547	4.0	307.0	
29	1.00245	0.0	8.14	0.0	0.538	6.674	87.3	4.2390	4.0	307.0	
..	...	...	...	...	...	...	...	...	...	...	
476	4.87141	0.0	18.10	0.0	0.614	6.484	93.6	2.3053	24.0	666.0	

477	15.02340	0.0	18.10	0.0	0.614	5.304	97.3	2.1007	24.0	666.0
478	10.23300	0.0	18.10	0.0	0.614	6.185	96.7	2.1705	24.0	666.0
479	14.33370	0.0	18.10	0.0	0.614	6.229	88.0	1.9512	24.0	666.0
480	5.82401	0.0	18.10	0.0	0.532	6.242	64.7	3.4242	24.0	666.0
481	5.70818	0.0	18.10	0.0	0.532	6.750	74.9	3.3317	24.0	666.0
482	5.73116	0.0	18.10	0.0	0.532	7.061	77.0	3.4106	24.0	666.0
483	2.81838	0.0	18.10	0.0	0.532	5.762	40.3	4.0983	24.0	666.0
484	2.37857	0.0	18.10	0.0	0.583	5.871	41.9	3.7240	24.0	666.0
485	3.67367	0.0	18.10	0.0	0.583	6.312	51.9	3.9917	24.0	666.0
486	5.69175	0.0	18.10	0.0	0.583	6.114	79.8	3.5459	24.0	666.0
487	4.83567	0.0	18.10	0.0	0.583	5.905	53.2	3.1523	24.0	666.0
488	0.15086	0.0	27.74	0.0	0.609	5.454	92.7	1.8209	4.0	711.0
489	0.18337	0.0	27.74	0.0	0.609	5.414	98.3	1.7554	4.0	711.0
490	0.20746	0.0	27.74	0.0	0.609	5.093	98.0	1.8226	4.0	711.0
491	0.10574	0.0	27.74	0.0	0.609	5.983	98.8	1.8681	4.0	711.0
492	0.11132	0.0	27.74	0.0	0.609	5.983	83.5	2.1099	4.0	711.0
493	0.17331	0.0	9.69	0.0	0.585	5.707	54.0	2.3817	6.0	391.0
494	0.27957	0.0	9.69	0.0	0.585	5.926	42.6	2.3817	6.0	391.0
495	0.17899	0.0	9.69	0.0	0.585	5.670	28.8	2.7986	6.0	391.0
496	0.28960	0.0	9.69	0.0	0.585	5.390	72.9	2.7986	6.0	391.0
497	0.26838	0.0	9.69	0.0	0.585	5.794	70.6	2.8927	6.0	391.0
498	0.23912	0.0	9.69	0.0	0.585	6.019	65.3	2.4091	6.0	391.0
499	0.17783	0.0	9.69	0.0	0.585	5.569	73.5	2.3999	6.0	391.0
500	0.22438	0.0	9.69	0.0	0.585	6.027	79.7	2.4982	6.0	391.0
501	0.06263	0.0	11.93	0.0	0.573	6.593	69.1	2.4786	1.0	273.0
502	0.04527	0.0	11.93	0.0	0.573	6.120	76.7	2.2875	1.0	273.0
503	0.06076	0.0	11.93	0.0	0.573	6.976	91.0	2.1675	1.0	273.0
504	0.10959	0.0	11.93	0.0	0.573	6.794	89.3	2.3889	1.0	273.0
505	0.04741	0.0	11.93	0.0	0.573	6.030	80.8	2.5050	1.0	273.0

	PTRATIO	B	LSTAT
0	15.3	396.90	4.98
1	17.8	396.90	9.14
2	17.8	392.83	4.03
3	18.7	394.63	2.94
4	18.7	396.90	5.33
5	18.7	394.12	5.21
6	15.2	395.60	12.43
7	15.2	396.90	19.15
8	15.2	386.63	29.93
9	15.2	386.71	17.10
10	15.2	392.52	20.45
11	15.2	396.90	13.27
12	15.2	390.50	15.71
13	21.0	396.90	8.26
14	21.0	380.02	10.26
15	21.0	395.62	8.47
16	21.0	386.85	6.58

```
17      21.0  386.75  14.67
18      21.0  288.99  11.69
19      21.0  390.95  11.28
20      21.0  376.57  21.02
21      21.0  392.53  13.83
22      21.0  396.90  18.72
23      21.0  394.54  19.88
24      21.0  394.33  16.30
25      21.0  303.42  16.51
26      21.0  376.88  14.81
27      21.0  306.38  17.28
28      21.0  387.94  12.80
29      21.0  380.23  11.98
..       ...     ...    ...
476     20.2  396.21  18.68
477     20.2  349.48  24.91
478     20.2  379.70  18.03
479     20.2  383.32  13.11
480     20.2  396.90  10.74
481     20.2  393.07   7.74
482     20.2  395.28   7.01
483     20.2  392.92  10.42
484     20.2  370.73  13.34
485     20.2  388.62  10.58
486     20.2  392.68  14.98
487     20.2  388.22  11.45
488     20.1  395.09  18.06
489     20.1  344.05  23.97
490     20.1  318.43  29.68
491     20.1  390.11  18.07
492     20.1  396.90  13.35
493     19.2  396.90  12.01
494     19.2  396.90  13.59
495     19.2  393.29  17.60
496     19.2  396.90  21.14
497     19.2  396.90  14.10
498     19.2  396.90  12.92
499     19.2  395.77  15.10
500     19.2  396.90  14.33
501     21.0  391.99   9.67
502     21.0  396.90   9.08
503     21.0  396.90   5.64
504     21.0  393.45   6.48
505     21.0  396.90   7.88

[506 rows x 13 columns]
```

0.3 Divide data into training, validation and test sets

```
In [9]: from sklearn.model_selection import train_test_split

        X_train, X_test_val, y_train, y_test_val = train_test_split(df.values, bdata.target, te

        X_val, X_test, y_val, y_test = train_test_split(X_test_val, y_test_val, test_size=0.5)

        # X = df.values
        # y = bdata.target
        # X_train_val, X_test, y_train_val, y_test = train_test_split(X, y, test_size=0.25)
        # X_train, X_val, y_train, y_val = train_test_split(X_train_val, y_train_val, test_siz

        print 'Training set size ', X_train.shape
        print 'Validation set size ', X_val.shape
        print 'Test set size ', X_test.shape
```

```
Training set size  (303L, 13L)
Validation set size  (101L, 13L)
Test set size  (102L, 13L)
```

0.4 Regularized Linear Regression cost function and gradient (vectorized)

Regularized linear regression has the following cost function:

$$J(\theta) = \frac{1}{2m}\left(\sum_{i=1}^{m}(y^{(i)} - h_\theta(x^{(i)})^2\right) + \frac{\lambda}{2m}\left(\sum_{j=1}^{n}{\theta_j}^2\right)$$

where λ is a regularization parameter which controls the degree of regularization (thus, help preventing overfitting). The regularization term puts a penalty on the overall cost $J(\theta)$. As the magnitudes of the model parameters θ_j increase, the penalty increases as well. Note that you should not regularize the θ_0 term. You should now complete the code for the method loss in the class Reg_LinearRegression_SquaredLoss in the file reg_linear_regressor_multi.py to calculate $J(\theta)$. Vectorize your code and avoid writing for loops.

Correspondingly, the partial derivative of the regularized linear regression cost function with respect to θ_j is defined as:

$$\begin{aligned}
\frac{\partial J(\theta)}{\partial \theta_0} &= \frac{1}{m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})x_j{}^{(i)} \\
\frac{\partial J(\theta)}{\partial \theta_j} &= \left(\frac{1}{m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})x_j{}^{(i)}\right) + \frac{\lambda}{m}\theta_j \quad \text{for } j \geq 1
\end{aligned}$$

This training function uses scipy's fmin_bfgs to optimize the cost function.
Here we have set the regularization parameter λ to zero.

```
In [10]: reg = 0.0
```

```
        # Scale training features
        XX_train, mu, sigma = utils.feature_normalize(X_train)
        # Add a column to training set
        XX = np.c_[np.ones(XX_train.shape[0]), XX_train]

        XX_val = (X_val - mu)/sigma
        XX_val = np.c_[np.ones(XX_val.shape[0]), XX_val]
        yy = y_train

        #  Train linear regression with lambda = 0
        reglinear_reg1 = RegularizedLinearReg_SquaredLoss()
        theta_opt0 = reglinear_reg1.train(XX,yy,reg,num_iters=1000)
        print 'Theta at lambda = 0 is ', theta_opt0

        theta_opt_normal = reglinear_reg1.normal_equation(X_train, y_train, reg)
        print 'Theta solved using normal equation is ', theta_opt_normal
        print 'Error on the test set with lambda = 0 is', reglinear_reg1.loss(theta_opt_normal

        # Compute error on the test set
        XX_test = (X_test - mu)/sigma
        XX_test = np.c_[np.ones(XX_test.shape[0]), XX_test]
        print 'Error on the test set with lambda = 0 is', reglinear_reg1.loss(theta_opt0, XX_t

        # Plot learning curve
        error_train, error_val = utils.learning_curve(XX,yy,XX_val,y_val,reg)
        plot_utils.plot_learning_curve(error_train[50:], error_val[50:],reg)
        plt.show()

Optimization terminated successfully.
        Current function value: 11.937549
        Iterations: 25
        Function evaluations: 26
        Gradient evaluations: 26
Theta at lambda = 0 is  [ 22.31056014  -0.67875414   1.75668422  -0.43426183   1.0605493
  -2.47713647   1.52681658   0.2777398   -4.11093516   3.3143865
  -2.30087759  -1.90126959   0.88046535  -4.42039277]
Theta solved using normal equation is  [-0.02251358  0.08188825 -0.1034787   3.99231191 -1.8868
  0.00940891 -1.28609075  0.15469184 -0.00865322 -0.07358001  0.01726806
 -0.47256703]
Error on the test set with lambda = 0 is 11.2459245333
Error on the test set with lambda = 0 is 12.2291502445
Optimization terminated successfully.
        Current function value: 0.000000
        Iterations: 2
        Function evaluations: 4
        Gradient evaluations: 4
Optimization terminated successfully.
        Current function value: 0.000000
```

```
         Iterations: 6
         Function evaluations: 7
         Gradient evaluations: 7
Optimization terminated successfully.
         Current function value: 0.000000
         Iterations: 9
         Function evaluations: 12
         Gradient evaluations: 12
Optimization terminated successfully.
         Current function value: 0.000000
         Iterations: 12
         Function evaluations: 13
         Gradient evaluations: 13
Optimization terminated successfully.
         Current function value: 0.000000
         Iterations: 12
         Function evaluations: 16
         Gradient evaluations: 16
Optimization terminated successfully.
         Current function value: 0.000000
         Iterations: 15
         Function evaluations: 16
         Gradient evaluations: 16
Optimization terminated successfully.
         Current function value: 0.000000
         Iterations: 18
         Function evaluations: 22
         Gradient evaluations: 22
Optimization terminated successfully.
         Current function value: 0.000000
         Iterations: 20
         Function evaluations: 24
         Gradient evaluations: 24
Optimization terminated successfully.
         Current function value: 0.000000
         Iterations: 24
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 0.000000
         Iterations: 27
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 0.000000
         Iterations: 34
         Function evaluations: 36
         Gradient evaluations: 36
```

```
Optimization terminated successfully.
         Current function value: 0.000000
         Iterations: 35
         Function evaluations: 37
         Gradient evaluations: 37
Optimization terminated successfully.
         Current function value: 0.000000
         Iterations: 45
         Function evaluations: 48
         Gradient evaluations: 48
Optimization terminated successfully.
         Current function value: 0.000000
         Iterations: 53
         Function evaluations: 55
         Gradient evaluations: 55
Optimization terminated successfully.
         Current function value: 1.276784
         Iterations: 48
         Function evaluations: 50
         Gradient evaluations: 50
Optimization terminated successfully.
         Current function value: 1.200136
         Iterations: 43
         Function evaluations: 45
         Gradient evaluations: 45
Optimization terminated successfully.
         Current function value: 2.133199
         Iterations: 39
         Function evaluations: 42
         Gradient evaluations: 42
Optimization terminated successfully.
         Current function value: 2.291844
         Iterations: 37
         Function evaluations: 39
         Gradient evaluations: 39
Optimization terminated successfully.
         Current function value: 3.641263
         Iterations: 35
         Function evaluations: 38
         Gradient evaluations: 38
Optimization terminated successfully.
         Current function value: 3.469783
         Iterations: 35
         Function evaluations: 38
         Gradient evaluations: 38
Optimization terminated successfully.
         Current function value: 4.533796
         Iterations: 35
```

```
         Function evaluations: 37
         Gradient evaluations: 37
Optimization terminated successfully.
         Current function value: 4.375438
         Iterations: 35
         Function evaluations: 37
         Gradient evaluations: 37
Optimization terminated successfully.
         Current function value: 4.357115
         Iterations: 34
         Function evaluations: 36
         Gradient evaluations: 36
Optimization terminated successfully.
         Current function value: 4.354328
         Iterations: 32
         Function evaluations: 34
         Gradient evaluations: 34
Optimization terminated successfully.
         Current function value: 4.223515
         Iterations: 25
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 4.177341
         Iterations: 30
         Function evaluations: 32
         Gradient evaluations: 32
Optimization terminated successfully.
         Current function value: 4.054893
         Iterations: 31
         Function evaluations: 33
         Gradient evaluations: 33
Optimization terminated successfully.
         Current function value: 3.958688
         Iterations: 30
         Function evaluations: 32
         Gradient evaluations: 32
Optimization terminated successfully.
         Current function value: 3.870704
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 4.147715
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
```

```
         Current function value: 4.267108
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 4.969571
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 4.832999
         Iterations: 28
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 4.841958
         Iterations: 29
         Function evaluations: 32
         Gradient evaluations: 32
Optimization terminated successfully.
         Current function value: 4.839875
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 4.708806
         Iterations: 29
         Function evaluations: 32
         Gradient evaluations: 32
Optimization terminated successfully.
         Current function value: 6.031931
         Iterations: 29
         Function evaluations: 32
         Gradient evaluations: 32
Optimization terminated successfully.
         Current function value: 6.031782
         Iterations: 30
         Function evaluations: 32
         Gradient evaluations: 32
Optimization terminated successfully.
         Current function value: 6.159925
         Iterations: 30
         Function evaluations: 32
         Gradient evaluations: 32
Optimization terminated successfully.
         Current function value: 6.484172
         Iterations: 28
         Function evaluations: 31
```

```
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 6.440118
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 6.323095
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 6.185245
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 6.338102
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 6.253426
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 6.328402
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 7.636368
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 7.509535
         Iterations: 26
         Function evaluations: 28
         Gradient evaluations: 28
Optimization terminated successfully.
         Current function value: 7.561752
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 7.513317
```

```
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 7.381989
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 7.693661
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 7.761719
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 7.644524
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 7.507279
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 7.854079
         Iterations: 27
         Function evaluations: 29
         Gradient evaluations: 29
Optimization terminated successfully.
         Current function value: 7.721997
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 7.750332
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 7.623571
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
```

```
Optimization terminated successfully.
         Current function value: 7.577260
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 7.459687
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 7.354471
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 7.243039
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 7.130193
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 7.023525
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 7.602935
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 7.489463
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 7.580119
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 8.368023
         Iterations: 27
```

```
         Function evaluations: 29
         Gradient evaluations: 29
Optimization terminated successfully.
         Current function value: 8.251813
         Iterations: 27
         Function evaluations: 29
         Gradient evaluations: 29
Optimization terminated successfully.
         Current function value: 8.416581
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 8.387721
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 8.286621
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 8.215667
         Iterations: 27
         Function evaluations: 29
         Gradient evaluations: 29
Optimization terminated successfully.
         Current function value: 9.039910
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 8.947907
         Iterations: 27
         Function evaluations: 29
         Gradient evaluations: 29
Optimization terminated successfully.
         Current function value: 8.989603
         Iterations: 27
         Function evaluations: 29
         Gradient evaluations: 29
Optimization terminated successfully.
         Current function value: 8.977303
         Iterations: 27
         Function evaluations: 29
         Gradient evaluations: 29
Optimization terminated successfully.
```

```
         Current function value: 8.885489
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 8.785031
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 9.218177
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 9.573417
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 9.465485
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 9.525525
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 9.532262
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 9.790792
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 9.678535
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 10.002150
         Iterations: 29
         Function evaluations: 31
```

```
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 9.964458
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 9.869021
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 9.772880
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 9.688261
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 10.202817
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 10.142811
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 10.113517
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 10.018688
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 9.921880
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 9.984664
```

```
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 9.887634
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 9.792407
         Iterations: 27
         Function evaluations: 29
         Gradient evaluations: 29
Optimization terminated successfully.
         Current function value: 9.753133
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 10.332459
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 10.251066
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 10.170103
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 10.089869
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 9.994945
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 10.219684
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
```

```
Optimization terminated successfully.
         Current function value: 10.802002
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 10.750686
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 10.931284
         Iterations: 27
         Function evaluations: 29
         Gradient evaluations: 29
Optimization terminated successfully.
         Current function value: 11.348481
         Iterations: 27
         Function evaluations: 29
         Gradient evaluations: 29
Optimization terminated successfully.
         Current function value: 11.869518
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 11.904834
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 11.801889
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 11.714702
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 11.617159
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 11.522043
         Iterations: 29
```

```
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 11.442505
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 11.378897
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 11.286276
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 11.227193
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 11.241632
         Iterations: 27
         Function evaluations: 29
         Gradient evaluations: 29
Optimization terminated successfully.
         Current function value: 11.183326
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 11.093651
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 11.006225
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 10.919319
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
```

```
         Current function value: 10.840289
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 10.872422
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 10.878950
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 10.812399
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 10.731451
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 10.654012
         Iterations: 28
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 10.579887
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 10.542436
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 10.464381
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 10.434459
         Iterations: 29
         Function evaluations: 31
```

```
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 10.438281
         Iterations: 29
         Function evaluations: 31
         Gradient evaluations: 31
Optimization terminated successfully.
         Current function value: 10.535589
         Iterations: 29
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 10.538362
         Iterations: 29
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 10.489486
         Iterations: 29
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 10.472369
         Iterations: 28
         Function evaluations: 29
         Gradient evaluations: 29
Optimization terminated successfully.
         Current function value: 10.400934
         Iterations: 27
         Function evaluations: 28
         Gradient evaluations: 28
Optimization terminated successfully.
         Current function value: 10.354241
         Iterations: 27
         Function evaluations: 28
         Gradient evaluations: 28
Optimization terminated successfully.
         Current function value: 10.318963
         Iterations: 27
         Function evaluations: 28
         Gradient evaluations: 28
Optimization terminated successfully.
         Current function value: 10.310437
         Iterations: 27
         Function evaluations: 28
         Gradient evaluations: 28
Optimization terminated successfully.
         Current function value: 10.364041
```

```
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 10.404223
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 10.489432
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 10.421209
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 10.376993
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 10.382256
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 10.369042
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 10.349780
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 10.352411
         Iterations: 27
         Function evaluations: 28
         Gradient evaluations: 28
Optimization terminated successfully.
         Current function value: 10.401307
         Iterations: 27
         Function evaluations: 28
         Gradient evaluations: 28
```

```
Optimization terminated successfully.
         Current function value: 10.359311
         Iterations: 27
         Function evaluations: 28
         Gradient evaluations: 28
Optimization terminated successfully.
         Current function value: 10.403161
         Iterations: 27
         Function evaluations: 28
         Gradient evaluations: 28
Optimization terminated successfully.
         Current function value: 10.342130
         Iterations: 27
         Function evaluations: 28
         Gradient evaluations: 28
Optimization terminated successfully.
         Current function value: 10.282740
         Iterations: 27
         Function evaluations: 28
         Gradient evaluations: 28
Optimization terminated successfully.
         Current function value: 10.916380
         Iterations: 27
         Function evaluations: 28
         Gradient evaluations: 28
Optimization terminated successfully.
         Current function value: 10.910667
         Iterations: 27
         Function evaluations: 28
         Gradient evaluations: 28
Optimization terminated successfully.
         Current function value: 11.131543
         Iterations: 28
         Function evaluations: 29
         Gradient evaluations: 29
Optimization terminated successfully.
         Current function value: 11.121391
         Iterations: 28
         Function evaluations: 29
         Gradient evaluations: 29
Optimization terminated successfully.
         Current function value: 11.170851
         Iterations: 28
         Function evaluations: 29
         Gradient evaluations: 29
Optimization terminated successfully.
         Current function value: 11.124668
         Iterations: 28
```

```
         Function evaluations: 29
         Gradient evaluations: 29
Optimization terminated successfully.
         Current function value: 11.058638
         Iterations: 28
         Function evaluations: 29
         Gradient evaluations: 29
Optimization terminated successfully.
         Current function value: 11.142470
         Iterations: 28
         Function evaluations: 29
         Gradient evaluations: 29
Optimization terminated successfully.
         Current function value: 11.077375
         Iterations: 28
         Function evaluations: 29
         Gradient evaluations: 29
Optimization terminated successfully.
         Current function value: 11.014255
         Iterations: 29
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 10.950010
         Iterations: 29
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 11.157530
         Iterations: 29
         Function evaluations: 30
         Gradient evaluations: 30
Optimization terminated successfully.
         Current function value: 11.093167
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 11.355026
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 11.290458
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
```

```
         Current function value: 11.247680
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.710197
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.724834
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 12.655908
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.590816
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.529467
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.553418
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.487028
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 12.418793
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 12.555970
         Iterations: 26
         Function evaluations: 27
```

```
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 12.490931
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 12.495988
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 12.515999
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 12.473451
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 13.540914
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 13.532247
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 13.464277
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 13.397168
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 13.344636
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 13.281948
```

```
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 13.215573
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 13.149940
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 13.166352
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 13.123399
         Iterations: 24
         Function evaluations: 25
         Gradient evaluations: 25
Optimization terminated successfully.
         Current function value: 13.066082
         Iterations: 24
         Function evaluations: 25
         Gradient evaluations: 25
Optimization terminated successfully.
         Current function value: 13.001411
         Iterations: 24
         Function evaluations: 25
         Gradient evaluations: 25
Optimization terminated successfully.
         Current function value: 12.971528
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.973890
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.957293
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
```

```
Optimization terminated successfully.
         Current function value: 12.893995
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.866302
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 12.906580
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 12.888557
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 12.840838
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 12.782063
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 12.721408
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 12.665036
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.676294
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.843274
         Iterations: 24
```

```
         Function evaluations: 25
         Gradient evaluations: 25
Optimization terminated successfully.
         Current function value: 12.888692
         Iterations: 24
         Function evaluations: 25
         Gradient evaluations: 25
Optimization terminated successfully.
         Current function value: 12.912779
         Iterations: 24
         Function evaluations: 25
         Gradient evaluations: 25
Optimization terminated successfully.
         Current function value: 12.855553
         Iterations: 24
         Function evaluations: 25
         Gradient evaluations: 25
Optimization terminated successfully.
         Current function value: 12.812290
         Iterations: 24
         Function evaluations: 25
         Gradient evaluations: 25
Optimization terminated successfully.
         Current function value: 12.861046
         Iterations: 24
         Function evaluations: 25
         Gradient evaluations: 25
Optimization terminated successfully.
         Current function value: 12.820793
         Iterations: 24
         Function evaluations: 25
         Gradient evaluations: 25
Optimization terminated successfully.
         Current function value: 12.811636
         Iterations: 24
         Function evaluations: 25
         Gradient evaluations: 25
Optimization terminated successfully.
         Current function value: 13.053960
         Iterations: 24
         Function evaluations: 25
         Gradient evaluations: 25
Optimization terminated successfully.
         Current function value: 13.054916
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
```

```
         Current function value: 12.996376
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.954043
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.913539
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.860071
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.808118
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.753548
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.724441
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.669119
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.641439
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.711572
         Iterations: 25
         Function evaluations: 26
```

```
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.657154
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.723629
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.670527
         Iterations: 24
         Function evaluations: 25
         Gradient evaluations: 25
Optimization terminated successfully.
         Current function value: 12.621287
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.568885
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.601493
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.561523
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.510456
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.485495
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.483347
```

```
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.441038
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.450808
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.422204
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.374196
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.341317
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.291723
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.242393
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.195353
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.231207
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
```

```
Optimization terminated successfully.
         Current function value: 12.259937
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.211495
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.313397
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.269218
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.221404
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.218527
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.173332
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.240038
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.614652
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.566386
         Iterations: 25
```

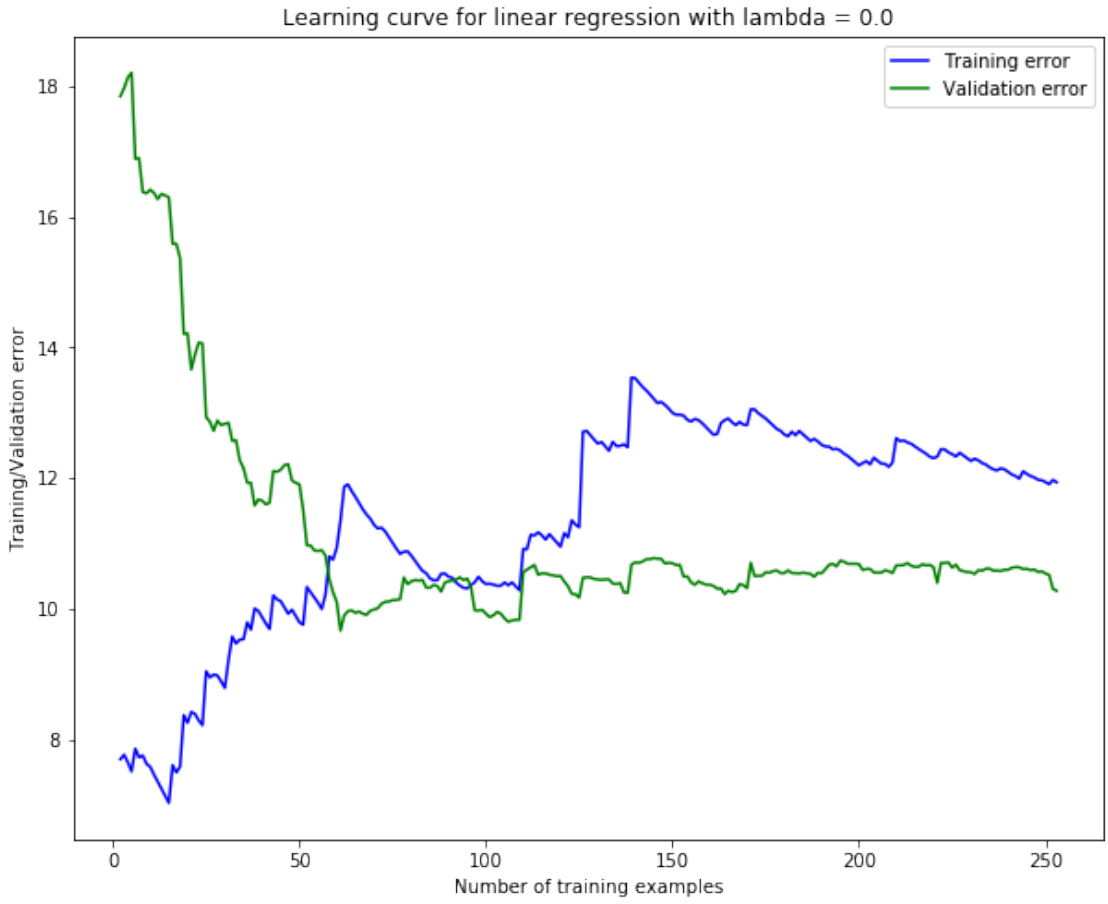
```
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.577748
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 12.547694
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 12.526701
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 12.485515
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 12.448641
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 12.413197
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.373066
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.328393
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 12.305969
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
```

```
         Current function value: 12.330145
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.443563
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.439469
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 12.394860
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.366765
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.331271
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.388911
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.346708
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 12.308226
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 12.264769
         Iterations: 26
         Function evaluations: 27
```

```
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 12.297606
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 12.273902
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.231921
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.208777
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.166319
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.135535
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.117999
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.149603
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.135425
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.093823
```

```
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.055420
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.031523
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 11.992716
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.105610
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.065081
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.032253
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.010544
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 11.975404
         Iterations: 26
         Function evaluations: 27
         Gradient evaluations: 27
Optimization terminated successfully.
         Current function value: 11.966776
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
```

```
Optimization terminated successfully.
         Current function value: 11.941057
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 11.906390
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 11.972491
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 11.937549
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
```



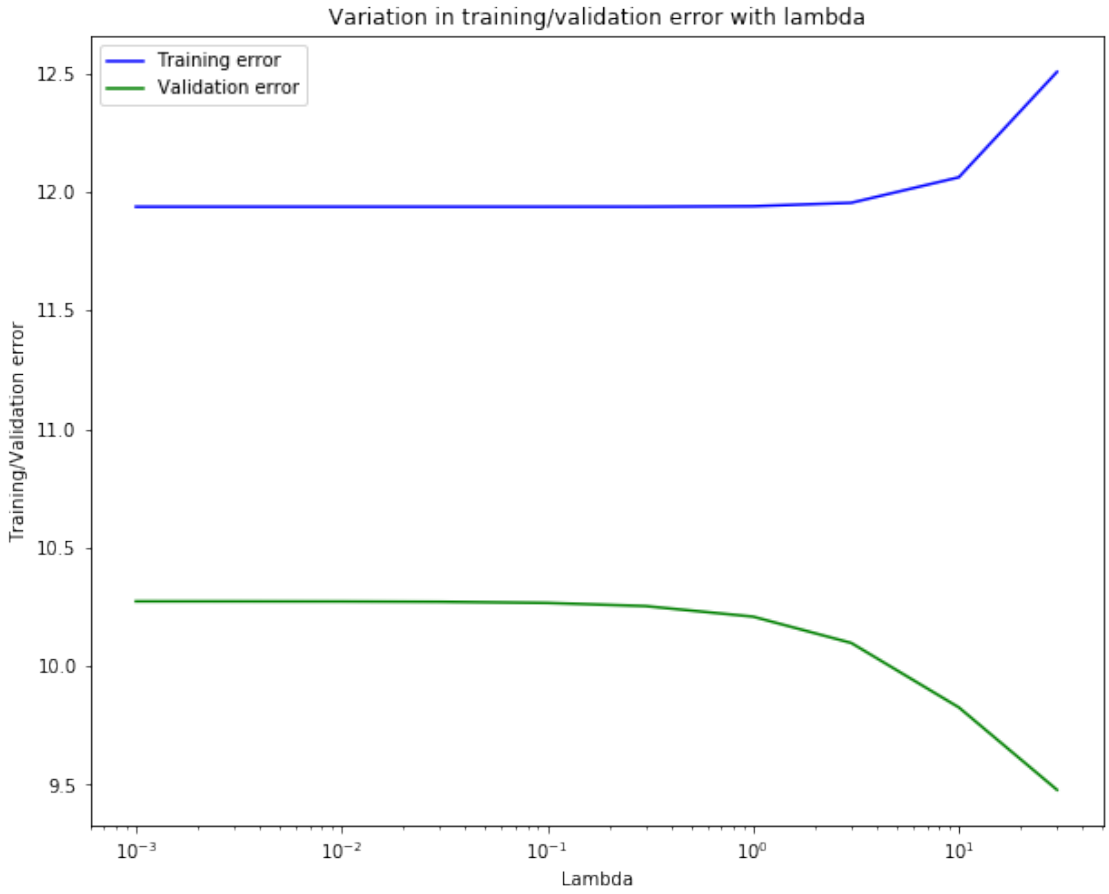
0.5 Selecting λ using a validation set

Train the model using different values of λ and to compute the training error and validation error. Try λ in the following range: {0, 0.001, 0.003, 0.01, 0.03, 0.1, 0.3, 1, 3, 10}.

Then plot a validation curve of λ versus the error, which allows us to select which best λ value to use.

```
In [11]: import myutils
         reg_vec = [0, 0.001, 0.003, 0.01, 0.03, 0.1, 0.3, 1, 3, 10, 30]
         reg_vec, error_train, error_val = myutils.validation_curve(XX,yy,XX_val,y_val, reg_ve
         plot_utils.plot_lambda_selection(reg_vec,error_train,error_val)
         plt.show()


         best_reg = 1.0
         #  Train linear regression with lambda = 50
         reglinear_reg2 = RegularizedLinearReg_SquaredLoss()
         theta_opt2 = reglinear_reg1.train(XX,yy,best_reg,num_iters=1000)
         # print 'Theta at lambda = ' + str(best_reg) +' is ', theta_opt2

         # Compute error on the test set
         print 'Error on the test set with lambda = ' + str(best_reg)+ ' is', reglinear_reg2.l

Optimization terminated successfully.
         Current function value: 11.937549
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 11.937665
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 11.937898
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 11.938712
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 11.941038
```

```
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 11.949162
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 11.972250
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.051681
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Optimization terminated successfully.
         Current function value: 12.268034
         Iterations: 24
         Function evaluations: 25
         Gradient evaluations: 25
Optimization terminated successfully.
         Current function value: 12.933812
         Iterations: 22
         Function evaluations: 23
         Gradient evaluations: 23
Optimization terminated successfully.
         Current function value: 14.403153
         Iterations: 18
         Function evaluations: 19
         Gradient evaluations: 19
```




```
Optimization terminated successfully.
         Current function value: 12.051681
         Iterations: 25
         Function evaluations: 26
         Gradient evaluations: 26
Error on the test set with lambda = 1.0 is 12.1573639285
```

0.6 Selecting λ with square features

Train the model using different values of λ and to compute the training error and validation error. Try λ in the following range: {0, 0.001, 0.003, 0.01, 0.03, 0.1, 0.3, 1, 3, 10}.

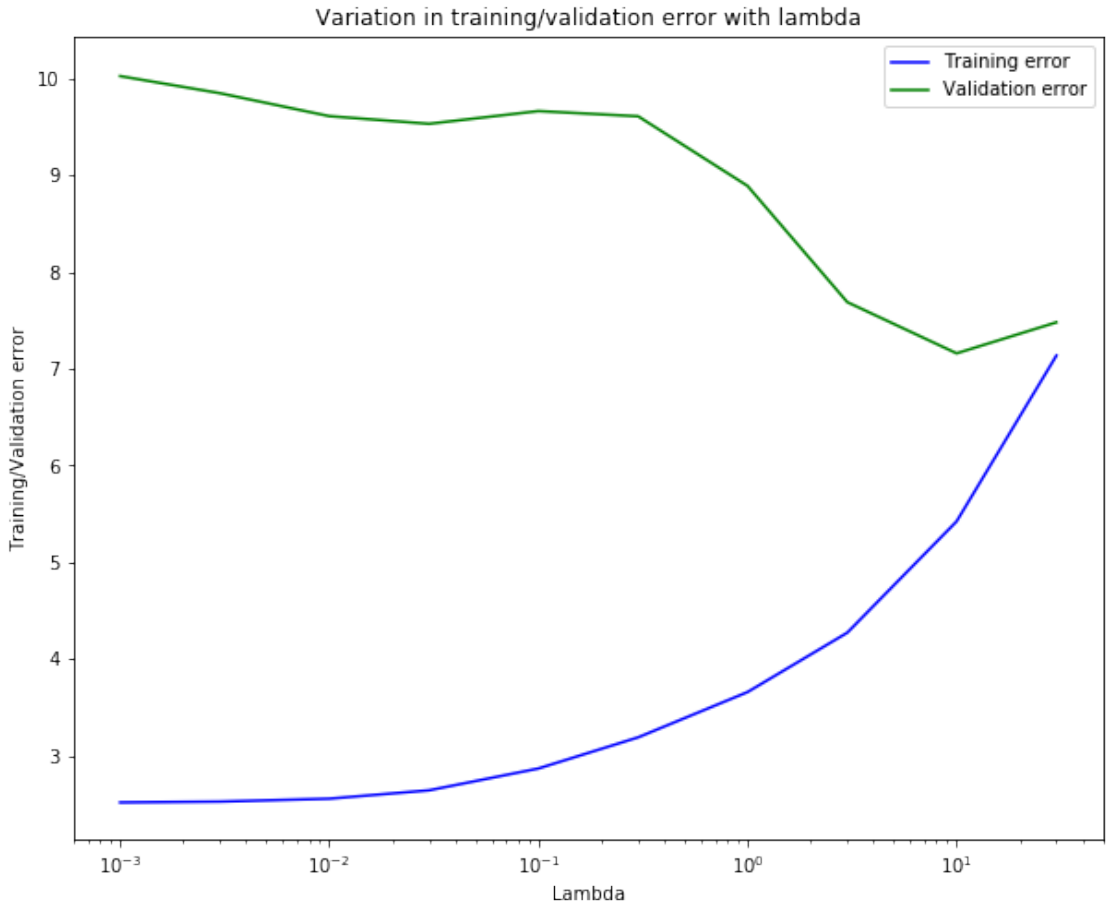
Then plot a validation curve of λ versus the error, which allows us to select which best λ value to use.

```
In [ ]: from sklearn.preprocessing import PolynomialFeatures

        # Map X onto polynomial features and normalize
        # We will consider a 6th order polynomial fit for the data
```

```
        p = 2
        poly = PolynomialFeatures(degree=p,include_bias=False)
        X_poly_train = poly.fit_transform(X_train)
        X_poly_train, mu, sigma = utils.feature_normalize(X_poly_train)

        # add a column of ones to X_poly
        XX_poly_train = np.c_[np.ones(X_poly_train.shape[0]), X_poly_train]

        # map Xtest and Xval into the same polynomial features

        X_poly_test = poly.fit_transform(X_test)
        X_poly_val = poly.fit_transform(X_val)

        # normalize these two sets with the same mu and sigma

        X_poly_test = (X_poly_test - mu) / sigma
        X_poly_val = (X_poly_val - mu) / sigma

        # add a column of ones to both X_poly_test and X_poly_val
        XX_poly_test = np.c_[np.ones(X_poly_test.shape[0]),X_poly_test]
        XX_poly_val = np.c_[np.ones(X_poly_val.shape[0]),X_poly_val]

        #
        reg_vec = [0, 0.001, 0.003, 0.01, 0.03, 0.1, 0.3, 1, 3, 10, 30]
        reg_vec, error_train, error_val = myutils.validation_curve(XX_poly_train,yy,XX_poly_val
        plot_utils.plot_lambda_selection(reg_vec,error_train,error_val)
        plt.show()

        best_reg = 1.0

        # Plot learning curve
        # error_train, error_val = utils.learning_curve(X_poly_train,yy,X_poly_val,y_val,best_
        # plot_utils.plot_learning_curve(error_train, error_val,best_reg)
        # plt.show()

        #  Train linear regression with lambda = 50
        reglinear_reg2 = RegularizedLinearReg_SquaredLoss()
        theta_opt2 = reglinear_reg1.train(XX_poly_train,yy,best_reg,num_iters=1000)
        # print 'Theta at lambda = ' + str(best_reg) +' is ', theta_opt2

        # Compute error on the test set
        print 'Error on the test set with lambda = ' + str(best_reg)+ ' is', reglinear_reg2.los

Optimization terminated successfully.
         Current function value: 2.513795
         Iterations: 943
         Function evaluations: 952
```

```
         Gradient evaluations: 952
Optimization terminated successfully.
         Current function value: 2.541516
         Iterations: 678
         Function evaluations: 686
         Gradient evaluations: 686
Optimization terminated successfully.
         Current function value: 2.580495
         Iterations: 618
         Function evaluations: 626
         Gradient evaluations: 626
Optimization terminated successfully.
         Current function value: 2.681674
         Iterations: 499
         Function evaluations: 507
         Gradient evaluations: 507
Optimization terminated successfully.
         Current function value: 2.872348
         Iterations: 397
         Function evaluations: 405
         Gradient evaluations: 405
Optimization terminated successfully.
         Current function value: 3.221602
         Iterations: 277
         Function evaluations: 285
         Gradient evaluations: 285
Optimization terminated successfully.
         Current function value: 3.675619
         Iterations: 190
         Function evaluations: 198
         Gradient evaluations: 198
Optimization terminated successfully.
         Current function value: 4.392680
         Iterations: 124
         Function evaluations: 132
         Gradient evaluations: 132
Optimization terminated successfully.
         Current function value: 5.396193
         Iterations: 82
         Function evaluations: 90
         Gradient evaluations: 90
Optimization terminated successfully.
         Current function value: 7.070152
         Iterations: 49
         Function evaluations: 57
         Gradient evaluations: 57
Optimization terminated successfully.
         Current function value: 9.047373
```

```
         Iterations: 33
         Function evaluations: 41
         Gradient evaluations: 41
```




```
Optimization terminated successfully.
         Current function value: 4.392680
         Iterations: 124
         Function evaluations: 132
         Gradient evaluations: 132
Error on the test set with lambda = 1.0 is 5.53596313915
```

0.7 Selecting λ with cubic features

```
In [ ]: from sklearn.preprocessing import PolynomialFeatures

        # Map X onto polynomial features and normalize
        # We will consider a 6th order polynomial fit for the data
```

```
        p = 3
        poly = PolynomialFeatures(degree=p,include_bias=False)
        X_poly_train = poly.fit_transform(X_train)
        X_poly_train, mu, sigma = utils.feature_normalize(X_poly_train)

        # add a column of ones to X_poly
        XX_poly_train = np.c_[np.ones(X_poly_train.shape[0]), X_poly_train]

        # map Xtest and Xval into the same polynomial features

        X_poly_test = poly.fit_transform(X_test)
        X_poly_val = poly.fit_transform(X_val)

        # normalize these two sets with the same mu and sigma

        X_poly_test = (X_poly_test - mu) / sigma
        X_poly_val = (X_poly_val - mu) / sigma

        # add a column of ones to both X_poly_test and X_poly_val
        XX_poly_test = np.c_[np.ones(X_poly_test.shape[0]),X_poly_test]
        XX_poly_val = np.c_[np.ones(X_poly_val.shape[0]),X_poly_val]

        #
        reg_vec = [0.01, 0.03, 0.1, 0.3, 1, 3, 10, 30]
        reg_vec, error_train, error_val = myutils.validation_curve(XX_poly_train,yy,XX_poly_val
        plot_utils.plot_lambda_selection(reg_vec,error_train,error_val)
        plt.show()

        best_reg = 1.0

        # Plot learning curve
        # error_train, error_val = utils.learning_curve(X_poly_train,yy,X_poly_val,y_val,best_
        # plot_utils.plot_learning_curve(error_train, error_val,best_reg)
        # plt.show()

        #  Train linear regression with lambda = 50
        reglinear_reg2 = RegularizedLinearReg_SquaredLoss()
        theta_opt2 = reglinear_reg1.train(XX_poly_train,yy,best_reg,num_iters=1000)

        # Compute error on the test set
        print 'Error on the test set with lambda = ' + str(best_reg) +' is', reglinear_reg1.los

Optimization terminated successfully.
         Current function value: 0.812402
         Iterations: 661
         Function evaluations: 672
         Gradient evaluations: 672
```

```
Optimization terminated successfully.
         Current function value: 1.074318
         Iterations: 491
         Function evaluations: 502
         Gradient evaluations: 502
```