COMP 540 HW 01

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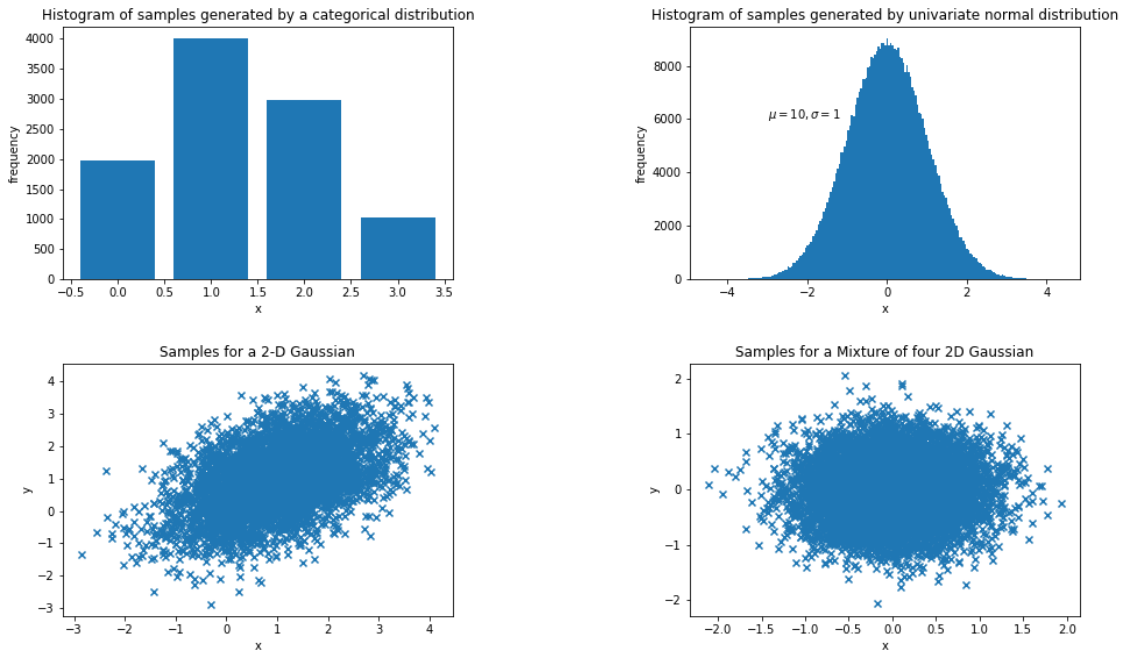January 16, 2018

0. Background Refresher

0.0 Samplers

Figure 1: Visualization of four distributions

The probability that a sample from this distribution lies within the unit circle centered at $(0.1, 0.2)$ is $p =$.

0.1   Prove two independent Poisson random variables are also Poisson variable.

Proof:

Given two independent random variables $X_1 \sim P(\lambda_1)$ and $X \sim P(\lambda_2)$. i.e.,

$$P(X_1 = m) = e^{-\lambda_1} \frac{\lambda_1^m}{m!}, \quad m = 1, 2, ... \tag{1}$$

and

$$P(X_2 = n) = e^{-\lambda_2} \frac{\lambda_2^n}{n!}, \ n = 0, 1, 2, ... \tag{2}$$

Denote sum of them are

$$X = X_1 + X_2 \tag{3}$$

then the probability distribution of X is:

$$
\begin{aligned}
P(X = k) \quad &= \quad \sum_{m+n=k} P(X_1 = m, X_2 = n) \\
&\overset{X_1, X_2 indep.R.V.}{=} \quad \sum_{m+n=k} P(X_1 = m) P(X_2 = n) \\
&= \quad \sum_{m+n=k} e^{-\lambda_1} \frac{\lambda_1^m}{m!} e^{-\lambda_2} \frac{\lambda_2^n}{n!} \\
&= \quad \frac{1}{k!} e^{-(\lambda_1+\lambda_2)} \sum_{m+n=k} \frac{k!}{m!n!} \lambda_1^m \lambda_2^n \\
&= \quad \frac{1}{k!} e^{-(\lambda_1+\lambda_2)} (\lambda_1 + \lambda_2)^k \tag{4}
\end{aligned}
$$

that is, $X = X_1 + X_2 \sim P(\lambda_1 + \lambda_2)$. Q.E.D.

0.2 Proof question 2

$$
\begin{aligned}
p(x_1, x_0) \quad &= \quad p(x_1|x_0) p(x_0) \\
&= \quad \alpha e^{-\frac{(x_1-x_0)^2}{2\sigma^2}} \alpha_0 e^{-\frac{(x_0-\mu_0)^2}{2\sigma_0^2}} \tag{5}
\end{aligned}
$$

The probability distribution of X_1 is,

$$
\begin{aligned}
p(x_1) \quad &= \quad \int_{-\infty}^{\infty} p(x_1, x_0) dx_0 \\
&= \quad \int_{-\infty}^{\infty} \alpha e^{-\frac{(x_1-x_0)^2}{2\sigma^2}} \alpha_0 e^{-\frac{(x_0-\mu_0)^2}{2\sigma_0^2}} dx_0 \\
&= \quad D \int_{-\infty}^{\infty} e^{-A[x_0-B]^2+C} dx_0 \tag{6}
\end{aligned}
$$

where A, B, C, D

$$D = \alpha\alpha_0 e^{-\frac{\mu_0^2}{2\sigma_0^2}} e^{-\frac{x_1^2}{2\sigma^2}} \tag{7}$$

$$A = \frac{\sigma_0^2 + \sigma^2}{2\sigma_0^2\sigma^2}, \tag{8}$$

$$B = \frac{x_1\sigma_0^2 + \mu_0\sigma^2}{\sigma_0^2 + \sigma^2}, \tag{9}$$

2

$$C = \frac{(x_1\sigma_0^2 + \mu_0\sigma^2)^2}{2\sigma_0^2\sigma^2(\sigma_0^2 + \sigma^2)}. \tag{10}$$

Making the substitution $t = \sqrt{2A}(x_0 - B)$ gives

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-A[x_0-B]^2+C} dx_0 &= \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}+C} d\frac{t}{\sqrt{2A}} \\
&= \frac{1}{\sqrt{2A}} e^{-C} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt \\
&= \sqrt{\frac{\pi}{A}} e^{-C} \tag{11}
\end{aligned}$$

where in the second step we used $\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt = \sqrt{2\pi}$.

Substitute the above equation to the Eq. ():

$$\begin{aligned}
p(x_1) &= D\sqrt{\frac{\pi}{A}} e^{-C} \\
&= \alpha\alpha_0 e^{-\frac{\mu_0^2}{2\sigma_0^2}} e^{-\frac{x_1^2}{2\sigma^2}} \sqrt{\frac{\pi}{A}} e^{-C} \tag{12}
\end{aligned}$$

0.3 question 4 eigenvalues

$$A = \begin{bmatrix} 13 & 5 \\ 2 & 4 \end{bmatrix}$$

$$\begin{aligned}
AX &= \lambda X \\
(A - \lambda I)X &= 0 \\
\begin{bmatrix} 13-\lambda & 5 \\ 2 & 4-\lambda \end{bmatrix} X &= 0
\end{aligned}$$

$\lambda_1 = 3,\ X_1 = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$

$\lambda_2 = 14,\ X_2 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

0.4 question 5, matrix multiplication

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$, we have $(A+B)^2 \neq A^2 + 2AB + B^2$

$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $A, B \neq 0$ and we have $AB = 0$

0.5   question 6

$$
\begin{aligned}
A^T A &= (I - 2uu^T)^T (I - 2uu^T) \\
&= (I - 2uu^T)(I - 2uu^T) \\
&= I - 2uu^T - 2uu^T + 4u(u^T u)u^T \\
&= I
\end{aligned} \tag{13}
$$

0.6   convex function

0.7   entropy of categorical distribution

The entropy of a categorical distribution on K values is

$$
H(p) = -\sum_{i=1}^{K} p_i \log(p_i), \tag{14}
$$

with constraint that

$$
\sum_{i=1}^{K} p_i = 1. \tag{15}
$$

Using Lagrange Multiplier, one can combine the above two equations into:

$$
L(p, \lambda) = -\sum_{i=1}^{K} p_i \log(p_i) + \lambda(\sum_{i=1}^{K} p_i - 1). \tag{16}
$$

Taking derivative of all unknown variables gives:

$$
\frac{\partial L}{\partial p_i} = -(\log p_i + 1) + \lambda = 0, \ \ i = 1, 2, ... \tag{17}
$$

Substituting p_i with λ yields

$$
p_i = \frac{1}{K}, \ \ i = 1, 2, ..., K. \tag{18}
$$

Q.E.D.

1.   Locally weighted linear regression.

1.1 Expression of $J(\theta)$

Define $\boldsymbol{X}$, $\boldsymbol{W}$ in the following way:

$$
\boldsymbol{X} = \begin{bmatrix} ---- \left(x^{(1)}\right)^T ---- \\ ---- \left(x^{(2)}\right)^T ---- \\ \vdots \\ ---- \left(x^{(m)}\right)^T ---- \end{bmatrix} \iff \boldsymbol{X}_{i,j} = \left(x^{(i)}\right)_j \tag{19}
$$

$$\boldsymbol{W} = \begin{bmatrix} w^{(1)} & & & \\ & w^{(2)} & & \\ & & \ddots & \\ & & & w^{(m)} \end{bmatrix} \iff \boldsymbol{W}_{i,j} = \delta_{i,j} w^{(i)} \tag{20}$$

Substituting $\boldsymbol{X}$, $\boldsymbol{W}$ into $(\boldsymbol{X}\theta - \boldsymbol{y})^T \boldsymbol{W}(\boldsymbol{X}\theta - \boldsymbol{y})$ yields

$$
\begin{aligned}
(\boldsymbol{X}\theta - \boldsymbol{y})^T \boldsymbol{W}(\boldsymbol{X}\theta - \boldsymbol{y}) &= \sum_{i,j} \left[(\boldsymbol{X}\theta - \boldsymbol{y})^T\right]_{1,i} \boldsymbol{W}_{i,j}(\boldsymbol{X}\theta - \boldsymbol{y})_{j,1} \\
&= \sum_{i,j} \delta_{i,j} w^{(i)} \left[(\boldsymbol{X}\theta - \boldsymbol{y})^T\right]_{1,i} (\boldsymbol{X}\theta - \boldsymbol{y})_{j,1} \\
&= \sum_i w^{(i)} \left[(\boldsymbol{X}\theta - \boldsymbol{y})_{i,1}\right]^2 \\
&= \sum_i w^{(i)} \left[\left(x^{(i)}\right)^T \theta - y^{(i)}\right]^2 \\
&= \sum_i w^{(i)} \left[\theta x^{(i)} - y^{(i)}\right]^2.
\end{aligned}
$$

So $J(\theta) = \frac{1}{2}(\boldsymbol{X}\theta - \boldsymbol{y})^T \boldsymbol{W}(\boldsymbol{X}\theta - \boldsymbol{y})$. Q.E.D.

1.2 Closed formed solution of θ

$$
\begin{aligned}
\frac{\partial J(\theta)}{\partial \theta} &= \frac{1}{2} \frac{\partial}{\partial \theta} \left[(\boldsymbol{X}\theta - \boldsymbol{y})^T \boldsymbol{W}(\boldsymbol{X}\theta - \boldsymbol{y})\right] \\
&= \frac{1}{2} \frac{\partial}{\partial \theta} tr \left[\boldsymbol{y}^T \boldsymbol{W} \boldsymbol{X}\theta - \boldsymbol{y}^T \boldsymbol{W} \boldsymbol{y} - \theta^T \boldsymbol{X}^T \boldsymbol{W} \boldsymbol{X}\theta + \theta^T \boldsymbol{X}^T \boldsymbol{W} \boldsymbol{y}\right] \\
&= \boldsymbol{X}^T \boldsymbol{W} \boldsymbol{y} - \boldsymbol{X}^T \boldsymbol{W} \boldsymbol{X}\theta
\end{aligned}
$$

where in the third line we used the following two equations[1]:

$$\frac{\partial tr(\boldsymbol{A}^T \boldsymbol{B})}{\partial \boldsymbol{A}} = \boldsymbol{B}^T \tag{21}$$

and

$$\frac{\partial tr(\boldsymbol{A}^T \boldsymbol{B} \boldsymbol{A})}{\partial \boldsymbol{A}} = \boldsymbol{B}\boldsymbol{A} + \boldsymbol{B}^T \boldsymbol{A} \tag{22}$$

Therefore the closed form solution for θ is

$$\theta = (\boldsymbol{X}^T \boldsymbol{W} \boldsymbol{X})^{-1} \boldsymbol{X}^T \boldsymbol{W} \boldsymbol{y}. \tag{23}$$

---

[1]Take the second equation for example:

$$
\begin{aligned}
\left[\frac{\partial tr(\boldsymbol{A}^T \boldsymbol{B} \boldsymbol{A})}{\partial \boldsymbol{A}}\right]_{i,j} &= \frac{\partial A^T_{l,m} B_{m,n} A_{n,l}}{\partial A_{i,j}} = \frac{\partial A_{m,l} B_{m,n} A_{n,l}}{\partial A_{i,j}} \\
&= \delta_{m,i} \delta_{l,j} B_{m,n} A_{n,l} + A_{m,l} B_{m,n} \delta_{n,i} \delta_{l,j} \\
&= B_{i,n} A_{n,j} + A_{m,j} B_{m,i} \\
&= (\boldsymbol{B}\boldsymbol{A})_{i,j} + (\boldsymbol{B}^T \boldsymbol{A})_{i,j},
\end{aligned}
$$

hence leading to $\frac{\partial tr(\boldsymbol{A}^T \boldsymbol{B} \boldsymbol{A})}{\partial \boldsymbol{A}} = \boldsymbol{B}\boldsymbol{A} + \boldsymbol{B}^T \boldsymbol{A}$

1.3 Batch gradient descent for locally weighted linear regression

The derivative of J_θ is:

$$\frac{\partial}{\partial \theta} J(\theta) = \sum_{i=1}^{m} w^{(i)}(h_\theta(x^{(i)}) - y^{(i)})x_j^{(i)}. \tag{24}$$

Therefore we have the following algorithm:

Algorithm 1: Batch gradient descent algorithm for locally weighted linear regression
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Result: The estimated parameters θ for locally weighted linear regression

1 **while** θ *does not converge* **do**

2   **for** *every j* **do**

3          $\theta_j = \theta_j - \alpha \sum_{i=1}^{m} w^{(i)}(h_\theta(x^{(i)}) - y^{(i)})x_j^{(i)}$;

4   **end**

5 **end**

---

2. Properties of linear regression estimator.

2.1 Prove $E[\theta] = \theta^*$

Proof:

The following facts:

1. $y^{(i)} = \theta^{*T}x^{(i)} + \epsilon^{(i)}$,

2. $\epsilon^{(i)}, i = 1, 2, ..., m$ are *i.i.d.* of $N(0, \sigma^2)$,

indicate that:

given fixed arbitrary $x^{(i)}$ and fixed unknown parameter θ^*, $y^{(i)}$, $1 \le i \le m$ are *i.i.d.* of $N(\theta^{*T}x^{(i)}, \sigma^2)$

$$p(y^{(i)}|x^{(i)}, \theta^*) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(y^{(i)} - \theta^{*T}x^{(i)})^2}{2\sigma^2}}. \tag{25}$$

The least-square estimate of θ^* is θ given by

$$\theta \qquad = \qquad (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{y} \tag{26}$$

$$\underset{denote:(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T \equiv \boldsymbol{A}}{=} \qquad \boldsymbol{Ay} \tag{27}$$

The expectation of θ is

$$E[\theta] \quad = \quad E[\boldsymbol{Ay}] \tag{28}$$

$$= \quad E\begin{bmatrix} \sum_j A_{1,j}y^{(j)} \\ \sum_j A_{2,j}y^{(j)} \\ \vdots \\ \sum_j A_{d+1,j}y^{(j)} \end{bmatrix}. \tag{29}$$

Since expectations has the following property (regardless of the independence of Z_k):

$$E[Z_1 + Z_2 + ... + Z_l] = \sum_k Z_k, \tag{30}$$

we have

$$
\begin{aligned}
E[\theta] &= E\begin{bmatrix} \sum_j A_{1,j} y^{(j)} \\ \sum_j A_{2,j} y^{(j)} \\ \vdots \\ \sum_j A_{d+1,j} y^{(j)} \end{bmatrix} = \begin{bmatrix} \sum_j A_{1,j} E[y^{(j)}] \\ \sum_j A_{2,j} E[y^{(j)}] \\ \vdots \\ \sum_j A_{d+1,j} E[y^{(j)}] \end{bmatrix} \tag{31} \\
&= \boldsymbol{A}\begin{bmatrix} E[y^{(1)}] \\ E[y^{(2)}] \\ \vdots \\ E[y^{(m)}] \end{bmatrix} = \boldsymbol{A}\begin{bmatrix} \theta^{*T} x^{(1)} \\ \theta^{*T} x^{(2)} \\ \vdots \\ \theta^{*T} x^{(m)} \end{bmatrix} \tag{32} \\
&= \boldsymbol{A}\boldsymbol{X}\theta^* = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{X}\theta^* \tag{33} \\
&= \theta^* \tag{34}
\end{aligned}
$$

where in the second step the following equation is used: $E[y^{(i)}] = \theta^{*T} x^{(i)}$ (trivial to obtain as $y^{(i)}$ observe normal distribution). Therefore $E[\theta] = \theta^*$, implying that the estimation θ is unbiased. Q.E.D

2.2 Prove $Var(\theta) = \left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T\sigma^2$

Denote $\boldsymbol{A} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T$, therefore $\theta = \boldsymbol{A}\boldsymbol{y}$.

$$
\begin{aligned}
Var(\boldsymbol{\theta}) &= Var(\boldsymbol{A}\boldsymbol{y}) \tag{35} \\
&= Var(\begin{bmatrix} \sum_j A_{1,j} y^{(j)} \\ \sum_j A_{2,j} y^{(j)} \\ \vdots \\ \sum_j A_{m,j} y^{(j)} \end{bmatrix}) = \begin{bmatrix} \sum_j A_{1,j} Var(y^{(j)}) \\ \sum_j A_{2,j} Var(y^{(j)}) \\ \vdots \\ \sum_j A_{m,j} Var(y^{(j)}) \end{bmatrix} = \boldsymbol{A}\begin{bmatrix} Var(y^{(1)}) \\ Var(y^{(2)}) \\ \vdots \\ Var(y^{(m)}) \end{bmatrix}, \tag{36}
\end{aligned}
$$

where in the second line we made used of the property of variance:

$$Var(\sum_k Z_k) = \sum Var(Z_k), \ Z_k \ is \ independent \ from \ each \ other. \tag{37}$$

Substituting $Var(y^{(i)}) = \sigma^2 \ i = 1, 2, ..., m$, we have $Var(\theta) = \boldsymbol{A}\sigma^2 = \left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T\sigma^2$. Q.E.D.

3. Implementing linear regression and regularized linear regression

Problem 3.1.A3: Predicting on unseen data

For average home in Boston suburbs, we predict a median home value of 225328.063241

Problem 3.1.B5: Exploring convergence of gradient descent

Plots 4 figures here.

$\alpha = 0.1, 0.3$ and $N_{iteration} = 80$ are good trade off between accuracy and efficiency. One can easily find that small α (i.e. $\alpha = 0.01, 0.03$) leads to very slow convergence rate. $\alpha = 0.1, 0.3$ on the other hands, converges swiftly.

Problem 3.2.A4: Adjusting the regularization parameter

draw two figures of polynomial fit and learning curves.

If λ is too small, the effect of penalty term can be neglected so that the regulation will not be implemented effectively – the training model is still troubled by high-variation/over-fitting issue (as can be seen in Fig....). When λ is too large, the loss function is in fact dominated by the penalty term, which is a slightly similar to the biased issue that we have too strong assumptions on the model. Therefore, the training model turns to be under-fit.