COMP 540 HW 01

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0. Background Refresher

0.0 Samplers

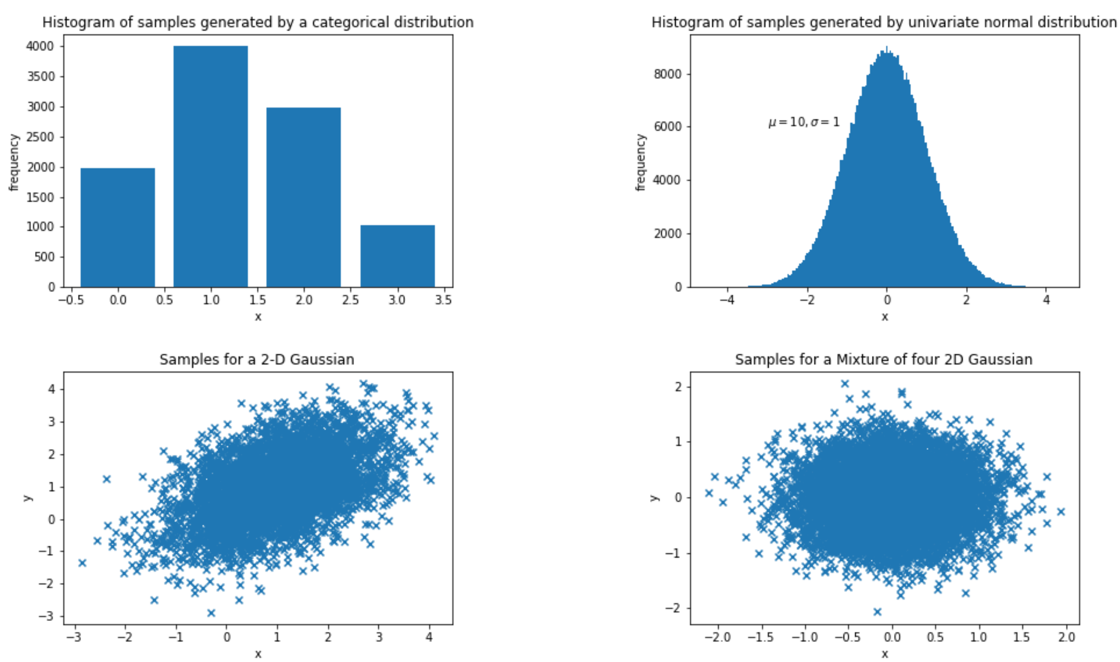

Figure 1: Visualization of four distributions

The probability that a sample from this distribution lies within the unit circle centered at $(0.1, 0.2)$ is $p =$.

0.1 Prove two independent Poisson random variables are also Poisson variable.

Proof:

Given two independent random variables $X_1 \sim P(\lambda_1)$ and $X \sim P(\lambda_2)$. i.e.,

$$P(X_1 = m) = e^{-\lambda_1} \frac{\lambda_1^m}{m!},\ m = 1, 2, ... \tag{1}$$

and

$$P(X_2 = n) = e^{-\lambda_2}\frac{\lambda_2^n}{n!},\ n = 0, 1, 2, ... \tag{2}$$

Denote sum of them are

$$X = X_1 + X_2 \tag{3}$$

then the probability distribution of X is:

$$\begin{aligned}
P(X = k) &= \sum_{m+n=k} P(X_1 = m, X_2 = n) \\
&\overset{X_1,X_2 indep.R.V.}{=} \sum_{m+n=k} P(X_1 = m)P(X_2 = n) \\
&= \sum_{m+n=k} e^{-\lambda_1}\frac{\lambda_1^m}{m!}e^{-\lambda_2}\frac{\lambda_2^n}{n!} \\
&= \frac{1}{k!}e^{-(\lambda_1+\lambda_2)} \sum_{m+n=k} \frac{k!}{m!n!}\lambda_1^m\lambda_2^n \\
&= \frac{1}{k!}e^{-(\lambda_1+\lambda_2)}(\lambda_1 + \lambda_2)^k \qquad (4)
\end{aligned}$$

that is, $X = X_1 + X_2 \sim P(\lambda_1 + \lambda_2)$. Q.E.D.

0.2 Proof question 2

$$\begin{aligned}
p(x_1,x_0) &= p(x_1|x_0)p(x_0) \\
&= \alpha e^{-\frac{(x_1-x_0)^2}{2\sigma^2}} \alpha_0 e^{-\frac{(x_0-\mu_0)^2}{2\sigma_0^2}} \qquad (5)
\end{aligned}$$

The probability distribution of X_1 is,

$$\begin{aligned}
p(x_1) &= \int_{-\infty}^{\infty} p(x_1, x_0)dx_0 \\
&= \int_{-\infty}^{\infty} \alpha e^{-\frac{(x_1-x_0)^2}{2\sigma^2}} \alpha_0 e^{-\frac{(x_0-\mu_0)^2}{2\sigma_0^2}} dx_0 \\
&= D\int_{-\infty}^{\infty} e^{-A[x_0-B]^2+C}dx_0 \qquad (6)
\end{aligned}$$

where A, B, C, D

$$D = \alpha\alpha_0 e^{-\frac{\mu_0^2}{2\sigma_0^2}} e^{-\frac{x_1^2}{2\sigma^2}} \tag{7}$$

$$A = \frac{\sigma_0^2 + \sigma^2}{2\sigma_0^2\sigma^2}, \tag{8}$$

$$B = \frac{x_1\sigma_0^2 + \mu_0\sigma^2}{\sigma_0^2 + \sigma^2}, \tag{9}$$

$$C = \frac{(x_1\sigma_0^2 + \mu_0\sigma^2)^2}{2\sigma_0^2\sigma^2(\sigma_0^2 + \sigma^2)}. \tag{10}$$

Making the substitution $t = \sqrt{2A}(x_0 - B)$ gives

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-A[x_0-B]^2+C} dx_0 &= \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}+C} d\frac{t}{\sqrt{2A}} \\
&= \frac{1}{\sqrt{2A}} e^{-C} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt \\
&= \sqrt{\frac{\pi}{A}} e^{-C} \qquad (11)
\end{aligned}$$

where in the second step we used $\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt = \sqrt{2\pi}$.

Substitute the above equation to the Eq. ():

$$\begin{aligned}
p(x_1) &= D\sqrt{\frac{\pi}{A}} e^{-C} \\
&= \alpha\alpha_0 e^{-\frac{\mu_0^2}{2\sigma_0^2}} e^{-\frac{x_1^2}{2\sigma^2}} \sqrt{\frac{\pi}{A}} e^{-C} \qquad (12)
\end{aligned}$$

0.3 question 4 eigenvalues

$$\boldsymbol{A} = \begin{bmatrix} 13 & 5 \\ 2 & 4 \end{bmatrix}$$

$$\begin{aligned}
\boldsymbol{AX} &= \lambda \boldsymbol{X} \\
(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{X} &= 0 \\
\begin{bmatrix} 13-\lambda & 5 \\ 2 & 4-\lambda \end{bmatrix} \boldsymbol{X} &= 0
\end{aligned}$$

$\lambda_1 = 3,\ \boldsymbol{X}_1 = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$

$\lambda_2 = 14,\ \boldsymbol{X}_2 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

0.4 question 5, matrix multiplication

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$, we have $(A+B)^2 \neq A^2 + 2AB + B^2$

$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $A, B \neq 0$ and we have $AB = 0$

0.5 question 6

$$\begin{aligned} A^TA &= (I-2uu^T)^T(I-2uu^T) \\ &= (I-2uu^T)(I-2uu^T) \\ &= I-2uu^T-2uu^T+4u(u^Tu)u^T \\ &= I \end{aligned} \tag{13}$$

0.6 convex function

0.7 entropy of categorical distribution

The entropy of a categorical distribution on K values is

$$H(p) \;=\; -\sum_{i=1}^{K} p_i \log(p_i), \tag{14}$$

with constraint that

$$\sum_{i=1}^{K} p_i = 1. \tag{15}$$

Using Lagrange Multiplier, one can combine the above two equations into:

$$L(p,\lambda) = -\sum_{i=1}^{K} p_i \log(p_i) + \lambda(\sum_{i=1}^{K} p_i - 1). \tag{16}$$

Taking derivative of all unknown variables gives:

$$\frac{\partial L}{\partial p_i} = -(\log p_i + 1) + \lambda = 0,\; i = 1, 2, ... \tag{17}$$

Substituting p_i with λ yields

$$p_i = \frac{1}{K},\; i = 1, 2, ..., K. \tag{18}$$

Q.E.D.

1. Locally weighted linear regression.

1.1 Expression of $J(\theta)$

Define $\boldsymbol{X}$, $\boldsymbol{W}$ in the following way:

$$\boldsymbol{X} = \begin{bmatrix} ----\left(x^{(1)}\right)^T---- \\ ----\left(x^{(2)}\right)^T---- \\ \vdots \\ ----\left(x^{(m)}\right)^T---- \end{bmatrix} \iff \boldsymbol{X}_{i,j} = (x^{(i)})_j \tag{19}$$

$$\boldsymbol{W} = \begin{bmatrix} w^{(1)} & & & \\ & w^{(2)} & & \\ & & \ddots & \\ & & & w^{(m)} \end{bmatrix} \Longleftrightarrow \boldsymbol{W}_{i,j} = \delta_{i,j} w^{(i)} \tag{20}$$

Substituting $\boldsymbol{X}$, $\boldsymbol{W}$ into $(\boldsymbol{X}\theta - \boldsymbol{y})^T \boldsymbol{W} (\boldsymbol{X}\theta - \boldsymbol{y})$ yields

$$\begin{aligned}
(\boldsymbol{X}\theta - \boldsymbol{y})^T \boldsymbol{W} (\boldsymbol{X}\theta - \boldsymbol{y}) &= \sum_{i,j} \left[(\boldsymbol{X}\theta - \boldsymbol{y})^T \right]_{1,i} \boldsymbol{W}_{i,j} (\boldsymbol{X}\theta - \boldsymbol{y})_{j,1} \\
&= \sum_{i,j} \delta_{i,j} w^{(i)} \left[(\boldsymbol{X}\theta - \boldsymbol{y})^T \right]_{1,i} (\boldsymbol{X}\theta - \boldsymbol{y})_{j,1} \\
&= \sum_{i} w^{(i)} \left[(\boldsymbol{X}\theta - \boldsymbol{y})_{i,1} \right]^2 \\
&= \sum_{i} w^{(i)} \left[\left(x^{(i)} \right)^T \theta - y^{(i)} \right]^2 \\
&= \sum_{i} w^{(i)} \left[\theta x^{(i)} - y^{(i)} \right]^2 .
\end{aligned}$$

So $J(\theta) = \frac{1}{2} (\boldsymbol{X}\theta - \boldsymbol{y})^T \boldsymbol{W} (\boldsymbol{X}\theta - \boldsymbol{y})$. Q.E.D.

1.2 Closed formed solution of θ

$$\begin{aligned}
\frac{\partial J(\theta)}{\partial \theta} &= \frac{1}{2} \frac{\partial}{\partial \theta} \left[(\boldsymbol{X}\theta - \boldsymbol{y})^T \boldsymbol{W} (\boldsymbol{X}\theta - \boldsymbol{y}) \right] \\
&= \frac{1}{2} \frac{\partial}{\partial \theta} tr \left[\boldsymbol{y}^T \boldsymbol{W} \boldsymbol{X} \boldsymbol{\theta} - \boldsymbol{y}^T \boldsymbol{W} \boldsymbol{y} - \boldsymbol{\theta}^T \boldsymbol{X}^T \boldsymbol{W} \boldsymbol{X} \boldsymbol{\theta} + \boldsymbol{\theta}^T \boldsymbol{X}^T \boldsymbol{W} \boldsymbol{y} \right] \\
&= \boldsymbol{X}^T \boldsymbol{W} \boldsymbol{y} - \boldsymbol{X}^T \boldsymbol{W} \boldsymbol{X} \boldsymbol{\theta}
\end{aligned}$$

where in the third line we used the following two equations[1]:

$$\frac{\partial tr(\boldsymbol{A}^T \boldsymbol{B})}{\partial \boldsymbol{A}} = \boldsymbol{B}^T \tag{21}$$

and

$$\frac{\partial tr(\boldsymbol{A}^T \boldsymbol{B} \boldsymbol{A})}{\partial \boldsymbol{A}} = \boldsymbol{B}\boldsymbol{A} + \boldsymbol{B}^T \boldsymbol{A} \tag{22}$$

Therefore the closed form solution for θ is

$$\theta = (\boldsymbol{X}^T \boldsymbol{W} \boldsymbol{X})^{-1} \boldsymbol{X}^T \boldsymbol{W} \boldsymbol{y}. \tag{23}$$

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[1] Take the second equation for example:

$$\begin{aligned}
\left[\frac{\partial tr(\boldsymbol{A}^T \boldsymbol{B} \boldsymbol{A})}{\partial \boldsymbol{A}} \right]_{i,j} &= \frac{\partial A^T_{l,m} B_{m,n} A_{n,l}}{\partial A_{i,j}} = \frac{\partial A_{m,l} B_{m,n} A_{n,l}}{\partial A_{i,j}} \\
&= \delta_{m,i} \delta_{l,j} B_{m,n} A_{n,l} + A_{m,l} B_{m,n} \delta_{n,i} \delta_{l,j} \\
&= B_{i,n} A_{n,j} + A_{m,j} B_{m,i} \\
&= (\boldsymbol{B}\boldsymbol{A})_{i,j} + (\boldsymbol{B}^T \boldsymbol{A})_{i,j},
\end{aligned}$$

hence leading to $\frac{\partial tr(\boldsymbol{A}^T \boldsymbol{B} \boldsymbol{A})}{\partial \boldsymbol{A}} = \boldsymbol{B}\boldsymbol{A} + \boldsymbol{B}^T \boldsymbol{A}$

1.3 Batch gradient descent for locally weighted linear regression

The derivative of J_θ is:

$$\frac{\partial}{\partial\theta}J(\theta) = \sum_{i=1}^{m} w^{(i)}(h_\theta(x^{(i)}) - y^{(i)})x_j^{(i)}. \tag{24}$$

Therefore we have the following algorithm:

Algorithm 1: Batch gradient descent algorithm for locally weighted linear regression

```
Result: The estimated parameters θ for locally weighted linear regression
while θ does not converge do
    for every j do
        θ_j = θ_j − α ∑_{i=1}^{m} w^{(i)}(h_θ(x^{(i)}) − y^{(i)})x_j^{(i)};
    end
end
```

2. Properties of linear regression estimator.

2.1 Prove $E[\theta] = \theta^*$

Proof:

The following facts:

1. $y^{(i)} = \theta^{*T}x^{(i)} + \epsilon^{(i)}$,
2. $\epsilon^{(i)}, i = 1, 2, ..., m$ are *i.i.d.* of $N(0, \sigma^2)$,

indicate that:

given fixed arbitrary $x^{(i)}$ and fixed unknown parameter θ^*, $y^{(i)}$, $1 \le i \le m$ are *i.i.d. of* $N(\theta^{*T}x^{(i)}, \sigma^2)$

$$p(y^{(i)}|x^{(i)}, \theta^*) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(y^{(i)}-\theta^{*T}x^{(i)})^2}{2\sigma^2}}. \tag{25}$$

The least-square estimate of θ^* is θ given by

$$\theta = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{y} \tag{26}$$

$$\overset{denote: (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T \equiv \boldsymbol{A}}{=} \boldsymbol{A}\boldsymbol{y} \tag{27}$$

The expectation of θ is

$$E[\theta] = E[\boldsymbol{A}\boldsymbol{y}] \tag{28}$$

$$= E\begin{bmatrix} \sum_j A_{1,j}y^{(j)} \\ \sum_j A_{2,j}y^{(j)} \\ \vdots \\ \sum_j A_{d+1,j}y^{(j)} \end{bmatrix}. \tag{29}$$

Since expectations has the following property (regardless of the independence of Z_k):

$$E[Z_1 + Z_2 + ... + Z_l] = \sum_k Z_k, \tag{30}$$

we have

$$\begin{aligned}
E[\theta] &= E\begin{bmatrix} \sum_j A_{1,j} y^{(j)} \\ \sum_j A_{2,j} y^{(j)} \\ \vdots \\ \sum_j A_{d+1,j} y^{(j)} \end{bmatrix} = \begin{bmatrix} \sum_j A_{1,j} E[y^{(j)}] \\ \sum_j A_{2,j} E[y^{(j)}] \\ \vdots \\ \sum_j A_{d+1,j} E[y^{(j)}] \end{bmatrix} &(31)\\
&= \boldsymbol{A}\begin{bmatrix} E[y^{(1)}] \\ E[y^{(2)}] \\ \vdots \\ E[y^{(m)}] \end{bmatrix} = \boldsymbol{A}\begin{bmatrix} \theta^{*T} x^{(1)} \\ \theta^{*T} x^{(2)} \\ \vdots \\ \theta^{*T} x^{(m)} \end{bmatrix} &(32)\\
&= \boldsymbol{A}\boldsymbol{X}\theta^* = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{X}\theta^* &(33)\\
&= \theta^* &(34)
\end{aligned}$$

where in the second step the following equation is used: $E[y^{(i)}] = \theta^{*T} x^{(i)}$ (trivial to obtain as $y^{(i)}$ observe normal distribution). Therefore $E[\theta] = \theta^*$, implying that the estimation θ is unbiased. Q.E.D

2.2 Prove $Var(\theta) = \left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T\sigma^2$

Denote $\boldsymbol{A} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T$, therefore $\theta = \boldsymbol{A}\boldsymbol{y}$.

$$\begin{aligned}
Var(\theta) &= Var(\boldsymbol{A}\boldsymbol{y}) &(35)\\
&= Var(\begin{bmatrix} \sum_j A_{1,j} y^{(j)} \\ \sum_j A_{2,j} y^{(j)} \\ \vdots \\ \sum_j A_{m,j} y^{(j)} \end{bmatrix}) = \begin{bmatrix} \sum_j A_{1,j} Var(y^{(j)}) \\ \sum_j A_{2,j} Var(y^{(j)}) \\ \vdots \\ \sum_j A_{m,j} Var(y^{(j)}) \end{bmatrix} = \boldsymbol{A}\begin{bmatrix} Var(y^{(1)}) \\ Var(y^{(2)}) \\ \vdots \\ Var(y^{(m)}) \end{bmatrix}, &(36)
\end{aligned}$$

where in the second line we made used of the property of variance:

$$Var(\sum_k Z_k) = \sum Var(Z_k),\ Z_k\ is\ independent\ from\ each\ other. \tag{37}$$

Substituting $Var(y^{(i)}) = \sigma^2\ i = 1, 2, ..., m$, we have $Var(\theta) = \boldsymbol{A}\sigma^2 = \left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T\sigma^2$. Q.E.D.

3. Implementing linear regression and regularized linear regression

3.1.A1

No plot for this question.

3.1.A2

Figure 3 plots the linear fit using parameter obtained through training.

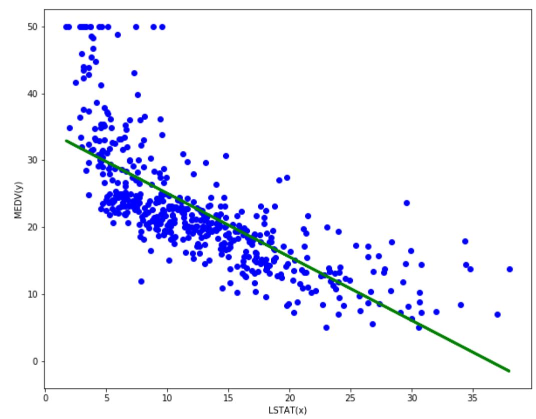


Figure 2: Fitting a linear model to the data

Problem 3.1.A3: Predicting on unseen data

For average home in Boston suburbs, we predict a median home value of 225328.063241 (USD)

3.1.B1: Feature Normalization

No plot for this question.

3.1.B2: Loss function and gradient descent

There should be a figure for this question.

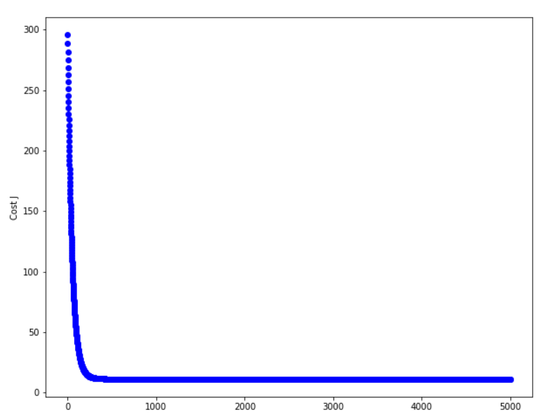


Figure 3: Fitting a linear model to the data

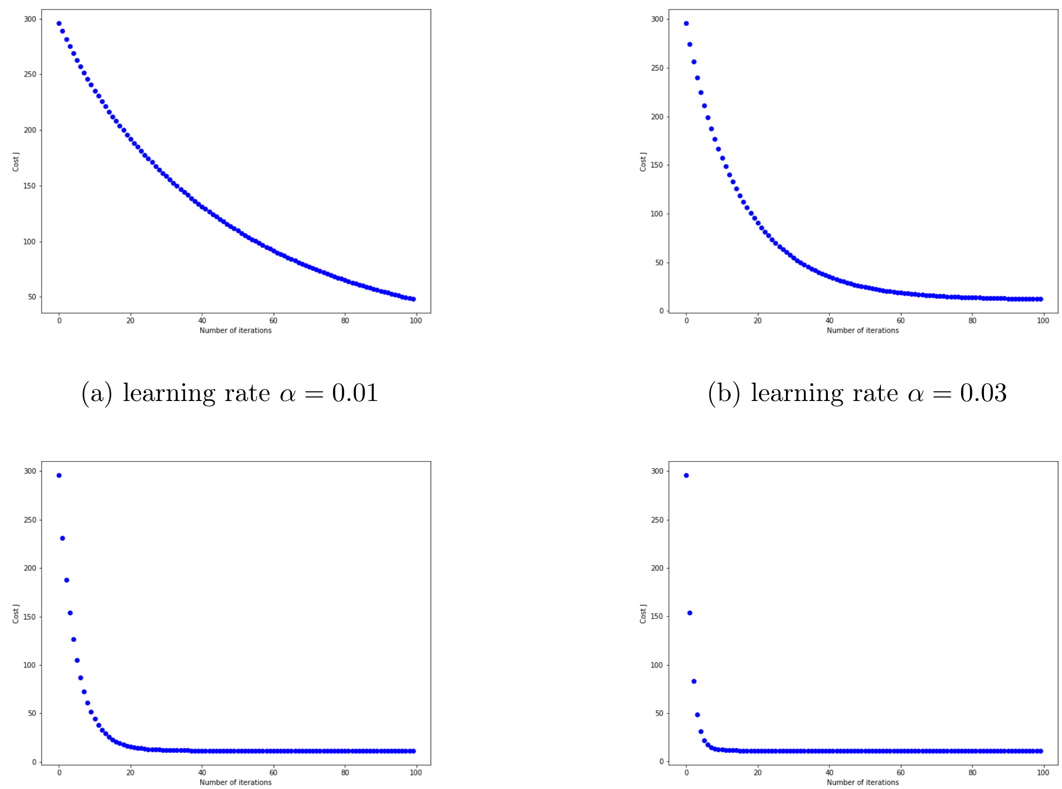
3.1.B3 Making predictions on unseen data

For average home in Boston suburbs, we predict a median home value of 225328.063241

3.1.B4: Normal equations

For average home in Boston suburbs, we predict a median home value of 225328.063241, which is the same as we obtained in subsec. 3.1.B3.

Problem 3.1.B5: Exploring convergence of gradient descent



(a) learning rate $\alpha = 0.01$

(b) learning rate $\alpha = 0.03$

(c) learning rate $\alpha = 0.1$

(d) learning rate $\alpha = 0.3$

Figure 4: Convergence of gradient descent for linear regression with multiple variables using different learning rate.

$\alpha = 0.1, 0.3$ and $N_{iteration} = 80$ are good trade off between accuracy and efficiency. By observing Fig. 4d , one can easily find that small α (i.e. $\alpha = 0.01, 0.03$) leads to very slow convergence rate. $\alpha = 0.1, 0.3$ on the other hands, converges swiftly.

3.2.A1: Regularized linear regression cost function

No figures for this question.

3.2.A2: Gradient of the regularized linear regression cost function

There should be a fitting here.

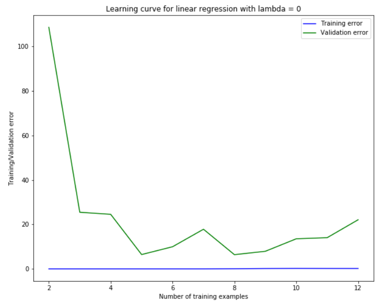
3.2.A3: Learning curves

there should be a learning curve here.

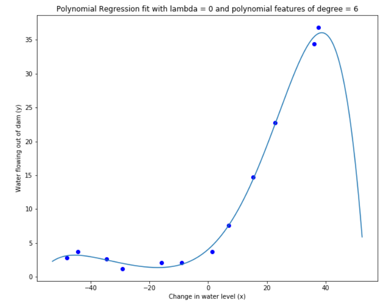
3.2.A4: Adjusting the regularization parameter

Figure 5 plots the polynomial fit and learning curves for each value of λ, from which we draw the following conclusions:

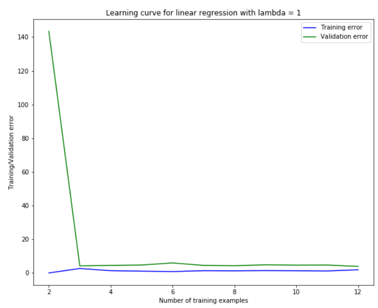
If λ is too small, the effect of penalty term can be neglected so that the regulation will not be implemented effectively – the training model is still troubled by high-variation/over-fitting issue (as can be seen in Fig....). When λ is too large, the loss function is in fact dominated by the penalty term, which is a slightly similar to the biased issue that we have too strong assumptions on the model. Therefore, the training model turns to be under-fit. Only when λ takes appropriate value that the regulation can works effectively.



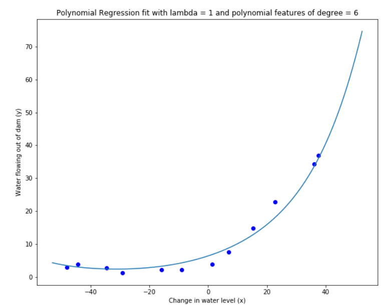
(a) learning rate $\lambda = 0$



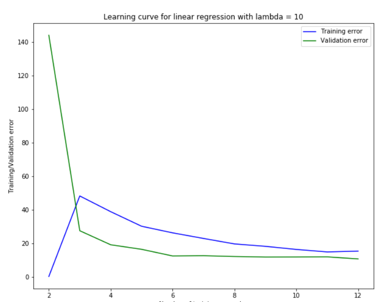
(b) Polynomial fit $\lambda = 0$



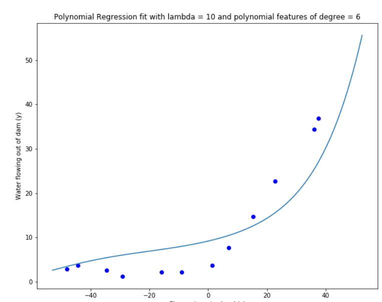
(c) learning rate $\lambda = 1$



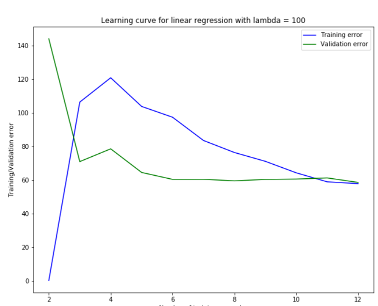
(d) Polynomial fit $\lambda = 1$



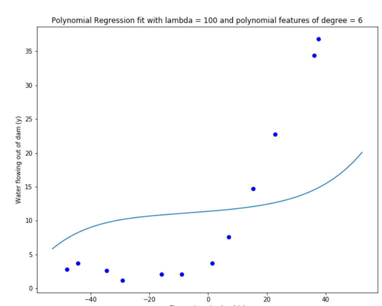
(e) learning rate $\lambda = 10$



(f) Polynomial fit $\lambda = 10$



(g) learning rate $\lambda = 100$



(h) Polynomial fit $\lambda = 100$

Figure 5: Convergence of gradient descent for linear regression with multiple variables using different learning rate.

3.2.A5 Selecting λ using a validation set

Figure 6 shows the variation in trainng/validation error with respect of regulation parameter λ. For the current model, $\lambda = 1$ is approximately a good choice for the training. We list several reasons to justify our choice:

- The difference between training/validation error is too large when λ is small, implying an over-fitting problem – this is true because the penalty term characterized by λ is too small to regulate the effect of excessive features.
- The difference between training/validation error becomes relatively small, but the absolute training/validation error becomes too large when λ gets larger than 1. This indicates that the penalty term dominates the loss function and we actually are making a strong assumption of the target function (of similar form as that of penalty term), i.e., too much bias / not enough variation.
- $\lambda = 1$ makes a good balance to avoid either overfitting or underfitting, i.e. the validation/training error is small (prediction performance); the difference between validation/training error is small (implying the prediction model is reliable).

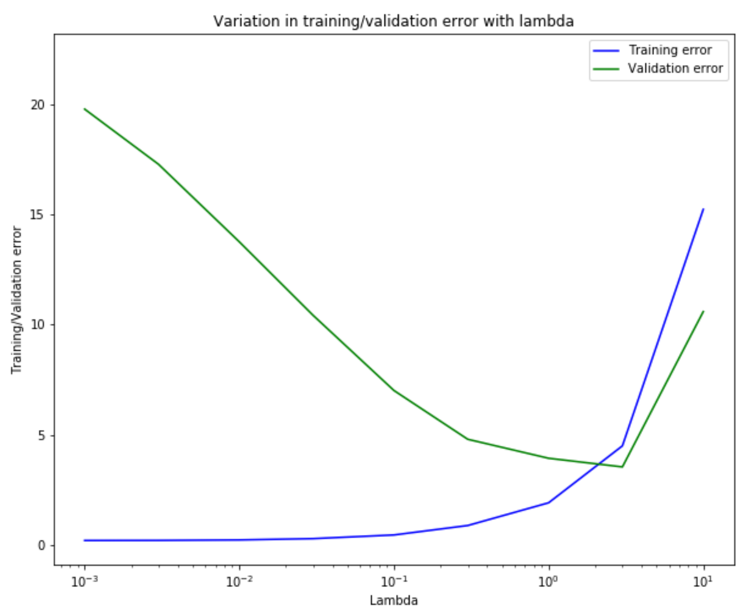


Figure 6: Variation in training/validation error with λ

3.2.A6 Computing test set error

Should make prediction on test set and calculate the test-error / loss.

3.2.A7 Plotting learning curves with randomly selected examples

Figure 7 plot the averaged learning curve for $\lambda = 1$.

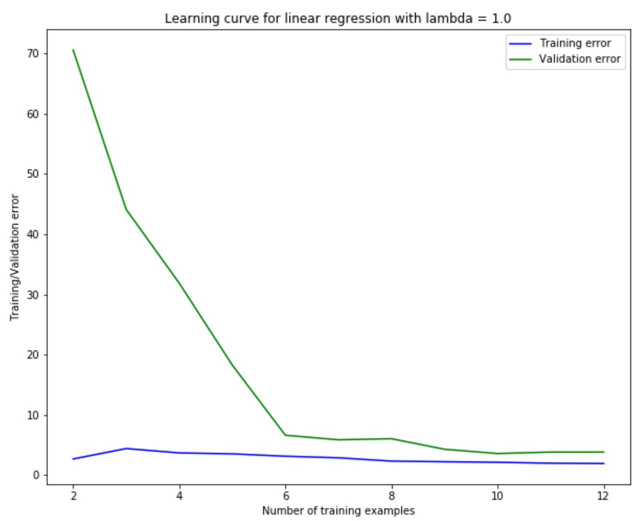


Figure 7: Averaged learning curve for $\lambda = 1$