Stochastic Control

Lecture 1: Optimal Control; deterministic case*

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1 Introduction

In this lecture we give an example of dynamic programming. We start by looking at a deterministic case and then, in other lectures, we extend our approach to more general setups under uncertainty. Most of the problems that we look at are seen from the point of view of an agent who needs to maximise his utility over a certain period (finite or infinite horizon) subject to budget constraints, or an investor who must decide how to diversify his investment. These examples help to develop an understanding of dynamic optimisation, which is widely used in financial economics. Particular applications of optimal control are seen in the field of Algorithmic and High-Frequency to design strategies such as: i) how does a market maker maximise expected profits by optimally choosing where to post (and when to cancel) orders in the limit order book, and ii) how to optimally liquidate/acquire a position which is too large to execute without having an adverse impact; that is, prices move down (up) when trying to sell (buy).

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2 Deterministic example

Let y denote the wealth process which satisfies the state equation

$$\frac{dy(t)}{dt} = r\,y(t) - c(t)\,, \qquad \text{with initial condition} \qquad y(t) = x\,, \tag{1}$$

where r is the riskfree rate and $c(t)$ is the amount the agent consumes at time t and x is his initial wealth – note that this equation is deterministic. Before proceeding, let us make sure that we understand the state equation. From the state equation we see that if the agent consumes $c(t)\,dt$ then his wealth declines by the the same amount. We also see that the agent earns the riskfree rate for the amount of wealth $y(t)$ he holds at time t. We can also see that this agent does not earn any other money in terms of wages.

Now we assume that the agent's objective is to

$$\max_{c(t)} \int_t^T h\left(y(s), c(s)\right)\, ds \, + B(y(T))\,, \tag{2}$$

where $h(y,c)$ is, for the time being, a 'nice' function, and $B(y(T))$ is a bequest. We could think of h as the agent's utility function. What we mean by 'nice' is that it satisfies any requirement we need, so that what follows is mathematically OK. Before proceeding, note that this is a finite horizon problem because everything ends at time $T < \infty$. Of course this is an oversimplification because most people would not know when 'it is over for them'.

When the agent solves problem (2) he should find a recipe or a formula that tells him how much to consume at every point in time, say $c^*(t)$, so that he is the happiest he can, given his initial wealth x.

From the state equation we also see that the agent is incentivised to postponing consumption because he earns the riskfree rate on the cash he holds, but on the other hand, for this problem to make sense, we expect the function $h(c,y)$ to be such that the agent is incentivised to consuming at least a little bit at every instant in time because the agent needs to survive. Therefore, we expect h to be increasing in consumption, $h'(c) > 0$.

The agent's **value function**

$$J(y,t) = \max_c \left[\int_t^T h(y(s), c(s))\, ds + B(y(T)) \right] \tag{3}$$

and by the optimality principle (we will be more precise about this later on)

$$J(y,t) = \max_c \left[\int_t^{t+\Delta t} h(y(s), c(s))\, ds + \max_c \left[\int_{t+\Delta t}^T h(y(s), c(s))\, ds + B(y(T)) \right] \right] . \tag{4}$$

The **control** in this equation is $c(t)$ because it is what we can change (optimally) to obtain maximal utility.

What the optimality principle allows us to do is to break up the problem into two parts. The idea is the following. If the agent wishes to maximise his utility over an interval $[t, T]$ this is the same as behaving optimally over the time step $t + \Delta t$ knowing that he has already done the right thing from $[t + \Delta t, T]$. ["Principle of Optimality: An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. (See Bellman, 1957, Chap. III.3.)"]

To find the **optimal policy** (by this we mean finding $c^*(t)$) we go through the motions in different steps by focusing on the two terms in (4).

We use the Intermediate Value Theorem (IVT) to write the first term inside the maximisation in (4):

$$\int_t^{t+\Delta t} h(y(s), c(s))\, ds = h\left(y + \Delta y, c(t + \Delta t')\right)\, \Delta t\,,$$

where $\Delta t' \in [0, \Delta t]$ and it is easy to see that the second term inside the maximisation in (4) can be written as

$$\max_c \left[\int_{t+\Delta t}^T h(y(s), c(s))\, ds + B(y(T)) \right] = J(y + \Delta y, t + \Delta t')\,.$$

Note that here we evaluate the value function at wealth $y + \Delta y$, which is the wealth of the agent at time $t + \Delta t$ and we use the state equation to calculate the change in wealth Δy. In other words, it is like starting the optimisation problem at time $t + \Delta t$ with initial wealth $y(t) + \Delta y = x + \Delta y$, because we know that the initial wealth $y(t) = x$.

Taylor expanding $J(y+\Delta y, t+\Delta t')$ about $(y+\Delta y, t+\Delta t') = (y,t)$:

$$J(y+\Delta y, t+\Delta t') = J(y,t) + J_t(y,t)\,\Delta t + J_y(y,t)\,\Delta y + \text{ error terms} \tag{5}$$

we write

$$J(y,t) = \max_c \Big[h(y+\Delta y, c(t+\Delta t'))\,\Delta t + J(y,t) + J_t(y,t)\,\Delta t + J_y(y,t)\,(r\,y - c)\,\Delta t + \cdots \Big] \tag{6}$$

dividing through by Δt and letting $\Delta t \to 0$

$$0 = \max_c \Big[h(y,c) + J_t(y,t) + J_y(y,t)(r\,y - c) \Big]\,. \tag{7}$$

Here J_t , J_y denote partial derivatives with respect to t and y respectively.

Equation (7) is known as Bellman equation or Hamilton-Jacobi-Equation (HJB). In this course we will refer to it either way. In general, economists prefer the term Bellman equation and mathematicians prefer HJB.

We have come a long way because we already know that solving (7) should give us the optimal consumption policy the agent is looking for. However, we need to specify the function $h(c,y)$ and see how far we get.

Specifying $h(y,c)$. Let us assume that

$$h(y, c(t)) = e^{-\rho t} U(c(t))\,, \tag{8}$$

where $\rho \geq 0$ is the discount rate and the utility function U is given by

$$U(c(t)) = c(t)^\gamma \tag{9}$$

with $0 < \gamma < 1$ to ensure that $U' > 0$ and $U'' < 0$, i.e. the utility function is increasing and concave in consumption, and for simplicity we assume that there is no bequest, i.e. $B(y(T)) = 0$. (The concavity assumption is quite important because it determines how risk averse the agent is.)

Then the Bellman equation becomes

$$0 = \max_c \Big[e^{-\rho t} c^\gamma + J_t(y,t) + J_y(y,t)\,(r\,y - c) \Big]\,. \tag{10}$$

It is easy to calculate the optimal consumption

$$c^* = \left(\frac{J_y(y,t)\, e^{\rho t}}{\gamma} \right)^{\frac{1}{\gamma-1}}, \tag{11}$$

and substituting back into (10) we obtain

$$0 = e^{-\rho t} \left(\frac{J_y(y,t)\, e^{\rho t}}{\gamma} \right)^{\frac{\gamma}{\gamma-1}} + J_t(y,t) + J_y(y,t) \left(r\, y(t) - \left(\frac{J_y(y,t)\, e^{\rho t}}{\gamma} \right)^{\frac{1}{\gamma-1}} \right). \tag{12}$$

So far so good, but we are not there yet because we need to solve this PDE to obtain (if possible) the value function $J(y,t)$ in closed-form, so that we can have an analytical expression for the optimal consumption path $c^*(t)$. Note that (11) is a control in feedback form because the control is a function of the value function.

One can always resort to numerical techniques but in this case we can obtain an analytical solution to the HJB equation. Moreover, note that the optimal consumption c^* depends on wealth y at time t. So at every point in time t (assuming you know the function J) the agent uses (11), plugs in his wealth $y(t)$, and then decides how much to consume. This particular example is rather straightforward because the state equation is deterministic (of course we are assuming that we know J which is not that straightforward to find!).

Solving for J looks like 'hocus pocus', but the usual way to proceed is to 'guess' the functional form of the value function $J(y,t)$. This is not an easy task. In our case we observe that if $y \to \lambda\, y$, where λ is a constant, and we also scale consumption, i.e. $c \to \lambda\, c$, we can show that the value function is homogeneous of degree γ

$$J(\lambda y, t) = \lambda^\gamma\, J(y,t)\,. \tag{13}$$

But how does this scaling property of the value function help us? Note that if we choose $\lambda = 1/y$, then

$$J(1,t) = y^{-\gamma} J(y,t)\,, \tag{14}$$

which, after a simple rearrangement, leads us to propose the **trial solution** (also known as **Ansatz**)

$$J(y,t) = g(t) y^\gamma\,. \tag{15}$$

Later we also need to check that it is true that if we scale the initial condition of our wealth y then the optimal trajectory is the same as before but also scaled, i.e. that the optimal policy is to have $\lambda\, c^*$.

So let us recall that we need to solve (if possible) HJB (12). Our first step is to substitute (15) in (12) and calculate the derivatives

$$J_y(y,t) = \gamma g(t) y^{\gamma-1}\,, \tag{16}$$

and

$$J_t(y,t) = g_t(t) y^{\gamma}\,. \tag{17}$$

And after substituting the trial solution $J(y,t)$ and its derivatives in (12) we obtain

$$g_t(t) + r\,\gamma\, g(t) + e^{-\rho t}\left(g(t)\, e^{\rho t}\right)^{\frac{\gamma}{\gamma-1}} - \gamma\, g(t)\left(g(t)\, e^{\rho t}\right)^{\frac{1}{\gamma-1}} = 0\,, \tag{18}$$

with boundary condition $B(T) = 0$ because there is no bequest.

To solve (18) we first multiply it through by $e^{\rho t}$, and noting that

$$\frac{d}{dt} e^{\rho t} g(t) = \rho e^{\rho t} g(t) + e^{\rho t} g_t(t)\,, \tag{19}$$

we proceed and write

$$\begin{aligned}
e^{\rho t} g_t(t) + e^{\rho t}\, r\,\gamma\, g(t) + \left(g(t)\, e^{\rho t}\right)^{\frac{\gamma}{\gamma-1}} - \gamma\, e^{\rho t}\, g(t)\left(g(t)\, e^{\rho t}\right)^{\frac{1}{\gamma-1}} &= 0 \\
\frac{d}{dt} e^{\rho t}\, g(t) + (r\,\gamma - \rho)\, e^{\rho t}\, g(t) + \left(g(t)\, e^{\rho t}\right)^{\frac{\gamma}{\gamma-1}} - \gamma\, e^{\rho t}\, g(t)\left(g(t)\, e^{\rho t}\right)^{\frac{1}{\gamma-1}} &= 0 \\
\frac{d}{dt} e^{\rho t}\, g(t) + (r\,\gamma - \rho) e^{\rho t}\, g(t) + \left(g(t)\, e^{\rho t}\right)^{\frac{\gamma}{\gamma-1}} - \gamma\, \left(g(t)\, e^{\rho t}\right)^{\frac{\gamma}{\gamma-1}} &= 0\,.
\end{aligned}$$

Now let $G(t) = e^{\rho t}\, g(t)$

$$G_t(t) + (r\,\gamma - \rho)\, G(t) + (1-\gamma)\, G(t)^{\frac{\gamma}{\gamma-1}} \quad = \quad 0\,,$$

and dividing through by $(1-\gamma)\, G(t)^{\frac{\gamma}{\gamma-1}}$

$$\frac{G_t(t)}{(1-\gamma)\, G(t)^{\frac{\gamma}{\gamma-1}}} + \frac{r\,\gamma - \rho}{1-\gamma}\, G(t)^{1-\frac{\gamma}{\gamma-1}} + 1 \quad = \quad 0\,.$$

Now we let $H(t) = G(t)^{\frac{1}{1-\gamma}}$ so that $G_t(t) = (1-\gamma)\, H(t)^{-\gamma}\, H_t(t)$ and write

$$\begin{aligned} H_t(t)\,(1-\gamma)\,H(t)^{-\gamma} + (r\,\gamma-\rho)\,H(t)^{1-\gamma} + (1-\gamma)\,H(t)^{-\gamma} &= 0 \\ H_t(t) + \frac{r\,\gamma-\rho}{(1-\gamma)}\,H(t) + 1 &= 0\,, \end{aligned}$$

with boundary condition $H(T) = 0$. This is an easy linear equation to solve. Let $v(t) = -\left(\mu\, H(t) + 1\right)$, with $v(T) = 1$ and $\mu = \frac{r\,\gamma-\rho}{1-\gamma}$, and write

$$\begin{aligned} v_t(t) &= -\mu\, v(t) \\ \int_t^T \frac{dv(s)}{v(s)} &= -\mu\,(T-t) \\ -1 &= v(t)\, e^{-\mu\,(T-t)}\,, \end{aligned}$$

hence

$$H(t) = -\frac{1}{\mu}\left(1 - e^{\mu(T-t)}\right)\,.$$

Now we can write (note that instead of having $r\,\gamma-\rho$ we have written $-(\rho - r\gamma)$ since it is more natural for the discount rare ρ to appear in this way)

$$g(t) = e^{-\rho t}\left[\frac{1-\gamma}{\rho - r\,\gamma}\left(1 - e^{-\frac{(\rho-r\,\gamma)(T-t)}{1-\gamma}}\right)\right]^{1-\gamma}\,,$$

and the value function

$$J(y,t) = e^{-\rho t}\left[\frac{1-\gamma}{\rho - r\,\gamma}\left(1 - e^{-\frac{(\rho-r\,\gamma)(T-t)}{1-\gamma}}\right)\right]^{1-\gamma} y^{\gamma}\,.$$

This allows us to calculate the optimal consumption

$$\begin{aligned} c^* &= \left(\frac{J_y(y,t)\,e^{\rho t}}{\gamma}\right)^{\frac{1}{\gamma-1}} \\ &= \left(\frac{\gamma\, e^{-\rho t}\left[\frac{1-\gamma}{\rho-r\,\gamma}\left(1 - e^{-\frac{(\rho-r\gamma)(T-t)}{1-\gamma}}\right)\right]^{1-\gamma} y^{\gamma-1} e^{\rho t}}{\gamma}\right)^{\frac{1}{\gamma-1}} \\ &= \left(\left[\frac{1-\gamma}{\rho - r\,\gamma}\left(1 - e^{-\frac{(\rho)-r\,\gamma(T-t)}{1-\gamma}}\right)\right]^{1-\gamma} y^{\gamma-1}\right)^{\frac{1}{\gamma-1}} \\ &= \left[\frac{1-\gamma}{\rho - r\,\gamma}\left(1 - e^{-\frac{(\rho-r\,\gamma)(T-t)}{1-\gamma}}\right)\right]^{-1} y \\ &= \left(1 - e^{-\frac{(\rho-r\,\gamma)(T-t)}{1-\gamma}}\right)^{-1} \frac{\rho - r\,\gamma}{1-\gamma}\, y\,. \end{aligned} \tag{20}$$

It is interesting (and useful) to stare at the optimal consumption to see if it makes sense. It is always good to check that the maths is OK but sometimes financial intuition is also a good common sense check... For example, we know that at expiry T it is optimal to have consumed everything because the agent derives no utility from leaving any bequests. Thus, one can check that $c^*(t) \to y(t)$ as $t \to T$. This makes sense because whatever wealth there is at time T it must be consumed.

3 Deterministic Example II

Let the state equation be

$$\frac{dy(t)}{dt} = r\,y(t) + w - c(t) \tag{21}$$

where $y(t)$ is wealth at time t, r is the risk-free rate, w denotes wages, and $c(t)$ consumption. The agents's problem is

$$\max_c \int_t^T e^{-\rho s}\, h(c(s))\, ds\,, \tag{22}$$

and he faces the constraint $y(T) \geq 0$, i.e. the agent is restricted to end with non-negative cash at time T.

Before trying to solve the optimisation problem we ask ourselves:

1. What amount of initial cash does the restriction $y(T) \geq 0$ implies?
2. What is the maximum amount of cash that the agent can end up with at time T?

The answers to these two questions will help solve the optimisation problem.

Before getting down to the maths we can think about the second question in simple financial terms. First, if the agent does not consume any of the cash between t and T then he maximises the amount of cash that is accumulated up until and including time T. Thus, if we set $c(s) = 0$ for $s \in [t, T]$ then the state equation (21) becomes

$$\frac{dy(s)}{ds} = ry(s) + w\,. \tag{23}$$

To solve (23) we integrate it between t and T

$$\begin{aligned}
\int_t^T \frac{dy}{ry+w} &= \int_t^T ds \\
\int_t^T \frac{dv}{v} &= \int_t^T ds \\
J(T) &= J(t)e^{r(T-t)} \\
ry(T) + w &= (ry(t) + w)e^{r(T-t)} \\
y(T) &= y(t)e^{r(T-t)} + \frac{w}{r}\left(e^{r(T-t)} - 1\right)\,. \qquad (24)
\end{aligned}$$

The last expression, which tells us how much cash we end up with at time T if we start with $y(t)$ and do not consume anything, is very simple to understand. Note that the first term on the right-hand side of (24) is the cash that we started with, which is invested for a period of time $T - t$ in a bank account at the risk-free rate r. The second term represents the wages w that we earn at every instant in time, which have been invested in a bank account and left there earning interest – see that in this case the amount of cash we end up with, which comes from wages and interest, is given by $\int_t^T w\, e^{r(T-s)}\, ds$.

Below we will see that knowing the maximum amount of cash the agent can end up with comes in handy when looking for a trial solution to the HJB we derive. So let us first turn the crank as before and show that the value function $J(y, t)$ satisfies:

$$0 = \max_c \left[e^{-\rho t}c^\gamma - \rho J(y,t) + J_t(y,t) + J_y(y,t)(ry(t) + w - c(t))\right]\,. \tag{25}$$

To show this, we proceed as usual:

$$
\begin{aligned}
J(y,t) &= \max_c \int_t^T e^{-\rho s} c^\gamma ds \\
&= \max_c \left[e^{-\rho(t+\Delta t')} c^\gamma \Delta t + \int_{t+\Delta t}^T e^{-\rho s} c^\gamma ds \right] \\
&= \max_c \left[e^{-\rho(t+\Delta t')} c^\gamma \Delta t + J(y+\Delta y, t+\Delta t) \right] \\
&= \max_c \left[e^{-\rho(t+\Delta t')} c^\gamma \Delta t + J(y,t) + J_t(y,t)\Delta t + J_y(y,t)\Delta y + \text{error terms} \right] \\
0 &= \max_c \left[e^{-\rho(t+\Delta t')} c^\gamma + J_t(y,t) + J_y(y,t)(ry(t) + w - c(t)) + \text{error terms} \right] \\
0 &= \max_c \left[e^{-\rho t} c^\gamma + J_t(y,t) + J_y(y,t)(ry(t) + w - c(t)) \right],
\end{aligned}
$$

where it is simple to calculate the optimal consumption

$$
c^*(t) = \left(\frac{e^{\rho t} J_y(y,t)}{\gamma} \right)^{\frac{1}{\gamma-1}} .
$$

As before, we are not there yet because we still do not know the value function $J(y,t)$. We need to guess the form of the value function. Above we solved a similar problem where the agent maximised utility but did not earn any wages. In that case the trial solution was of the form $g(t)y^\gamma$ which, unfortunately, does not work when the agent earns wages. The problem is that earning wages, which affects the state equation, does make a difference to the value function. Somehow, we need to see how wages appear in the trial solution.

To this end, we go back to our discussion of what was the maximum amount of cash we could have after $T-t$ which is given by (24). Now we can ask ourselves, what about if the agent's initial endowment $y(t)$ of cash is such that

$$
-y(t)e^{r(T-t)} + \frac{w}{r}\left(e^{r(T-t)} - 1\right) = 0\,, \tag{26}
$$

in other words the agent has

$$
y(t) = \frac{w}{r}\left(1 - e^{-r(T-t)}\right) = 0\,. \tag{27}
$$

Thus, the agent starts owing money and should repay it. Also, recall that one of the conditions of the problem is that the agent cannot end up owing any money, i.e.

cannot end up with $y(T) < 0$. This means that the agent must use all his wages to pay off the debt, consume nothing, and end up with $y(T) = 0$. Not consuming means that the agent will, at all times, derive zero utility, $J(y,t) = 0$ for all $t \leq T$.

So why do we want to know all of this? Well, this is important when writing down the trial solution. Note that the trial solution $J(y,t) = g(t)y^\gamma$ will not satisfy

$$g(t)y^\gamma(t) = 0\,, \qquad \text{for all } t \leq T\,,$$

if the initial endowment $y(t)$ is such that equation (26) holds. Thus all we need is to tweak the trial solution:

$$J(y,t) = g(t)\left(y + \frac{w}{r}\left(1 - e^{-r(T-t)}\right)\right)^\gamma\,, \tag{28}$$

where it is obvious that if $y(t)$ is such that equation (26) is satisfied, the value function is zero over the period $[t,T]$ as required.

Now we proceed to calculate the derivatives of the value function:

$$J_t(y,t) = g_t(t)\left(y(t) + \frac{w}{r}\left(1 - e^{-r(T-t)}\right)\right)^\gamma - \gamma w g(t) e^{-r(T-t)}\left(y(t) + \frac{w}{r}\left(1 - e^{-r(T-t)}\right)\right)^{\gamma-1}\,, \tag{29}$$

and

$$J_y(y,t) = \gamma g(t)\left(y(t) + \frac{w}{r}\left(1 - e^{-r(T-t)}\right)\right)^{\gamma-1}\,. \tag{30}$$

Thus, the optimal consumption is given by

$$c^*(t) = \left(e^{\rho t}g(t)\right)^{\frac{1}{\gamma-1}}\left(y(t) + \frac{w}{r}\left(1 - e^{-r(T-t)}\right)\right)\,. \tag{31}$$

Putting all of these into the HJB:

$$
\begin{aligned}
0 &= e^{-\rho t}\left(e^{\rho t}g(t)\right)^{\frac{\gamma}{\gamma-1}}\left(y(t)+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)^{\gamma} \\
&\quad +g_t(t)\left(y(t)+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)^{\gamma}-\gamma w g(t)e^{-r(T-t)}\left(y+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)^{\gamma-1} \\
&\quad +\gamma g(t)\left(y(t)+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)^{\gamma-1} \\
&\quad \times\left(ry(t)+w-\left(e^{\rho t}g(t)\right)^{\frac{1}{\gamma-1}}\left(y(t)+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)\right) \\
0 &= \left[g_t(t)+e^{-\rho t}\left(e^{\rho t}g(t)\right)^{\frac{\gamma}{\gamma-1}}\right]\left(y(t)+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)^{\gamma} \\
&\quad +\gamma g(t)\left[\left(ry(t)+w-\left(e^{\rho t}g(t)\right)^{\frac{1}{\gamma-1}}\left(y(t)+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)\right)-we^{-r(T-t)}\right] \\
&\quad \times\left(y(t)+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)^{\gamma-1} \\
0 &= \left[g_t(t)+e^{-\rho t}\left(e^{\rho t}g(t)\right)^{\frac{\gamma}{\gamma-1}}\right]\left(y(t)+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)^{\gamma} \\
&\quad +\gamma g(t)\left[ry(t)+w\left(1-e^{-r(T-t)}\right)-\left(e^{\rho t}g(t)\right)^{\frac{1}{\gamma-1}}\left(y(t)+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)\right] \\
&\quad \times\left(y(t)+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)^{\gamma-1} \\
0 &= \left[g_t(t)+e^{-\rho t}\left(e^{\rho t}g(t)\right)^{\frac{\gamma}{\gamma-1}})\right]\left(y+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)^{\gamma} \\
&\quad +\gamma r g(t)\left[y+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right]\left(y+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)^{\gamma-1} \\
&\quad -\gamma g(t)\left(e^{\rho t}g(t)\right)^{\frac{1}{\gamma-1}}\left(y+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)^{\gamma} \\
0 &= g_t(t)+e^{-\rho t}\left(e^{\rho t}g(t)\right)^{\frac{\gamma}{\gamma-1}}+\gamma r g(t)-\gamma g(t)\left(e^{\rho t}g(t)\right)^{\frac{1}{\gamma-1}} \\
0 &= e^{\rho t}g_t(t)+\left(e^{\rho t}g(t)\right)^{\frac{\gamma}{\gamma-1}}-\gamma\left(e^{\rho t}g(t)\right)^{\frac{\gamma}{\gamma-1}}+\gamma r e^{\rho t}g(t)\,.
\end{aligned}
$$

Using $\frac{d}{dt}e^{\rho t}g(t)=\rho e^{\rho t}g(t)+e^{\rho t}g_t(t)$

$$0=\frac{d}{dt}e^{\rho t}g(t)+(\gamma r-\rho)e^{\rho t}g(t)+(1-\gamma)\left(e^{\rho t}g(t)\right)^{\frac{\gamma}{\gamma-1}}\ ,$$

letting $G(t)=e^{\rho t}g(t)$

$$G_t(t)+(r\gamma-\rho)G(t)+(1-\gamma)G(t)^{\frac{\gamma}{\gamma-1}}=0\,,$$

and dividing through by $(1-\gamma)G(t)^{\frac{\gamma}{\gamma-1}}$

$$\frac{G_t(t)}{(1-\gamma)G(t)^{\frac{\gamma}{\gamma-1}}}+\frac{r\gamma-\rho}{(1-\gamma)}G(t)^{1-\frac{\gamma}{\gamma-1}}+1=0\,.$$

Now we let $H(t) = G(t)^{\frac{1}{1-\gamma}}$ so that $G_t(t) = (1-\gamma)H(t)^{-\gamma}H_t(t)$ and write

$$\begin{aligned} H_t(t)(1-\gamma)H(t)^{-\gamma} + (r\gamma-\rho)H(t)^{1-\gamma} + (1-\gamma)H(t)^{-\gamma} &= 0 \\ H_t(t) + \frac{r\gamma-\rho}{(1-\gamma)}H(t) + 1 &= 0\,, \end{aligned}$$

with boundary condition $H(T) = 0$. This is an easy linear equation to solve. Let $v(t) = -\left(\mu H(t) + 1\right)$, with $v(T) = 1$ and $\mu = \frac{r\gamma-\rho}{1-\gamma}$, and write

$$\begin{aligned} v_t(t) &= -\mu v(t) \\ \int_t^T \frac{dv(s)}{v(s)} &= -\mu(T-t) \\ -1 &= v(t)e^{-\mu(T-t)}\,. \end{aligned}$$

Hence

$$H(t) = -\frac{1}{\mu}\left(1 - e^{\mu(T-t)}\right)\,.$$

Now we can write (note that instead of having $r\gamma-\rho$ we have written $-(\rho-r\gamma)$ since it is more natural for the discount rare ρ to appear in this way)

$$g(t) = e^{-\rho t}\left[\frac{1-\gamma}{\rho-r\gamma}\left(1-e^{-\frac{(\rho-r\gamma)(T-t)}{1-\gamma}}\right)\right]^{1-\gamma}\,,$$

and the value function

$$J(y,t) = e^{-\rho t}\left[\frac{1-\gamma}{\rho-r\gamma}\left(1-e^{-\frac{(\rho-r\gamma)(T-t)}{1-\gamma}}\right)\right]^{1-\gamma}\left(y(t)+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)^{\gamma}\,,$$

and the optimal consumption

$$c^*(t) = \frac{1-\gamma}{\rho-r\gamma}\left(1-e^{-\frac{(\rho-r\gamma-)(T-t)}{1-\gamma}}\right)^{-1}\left(y(t)+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)\,.$$

As a sanity check you can verify that if we let $w = 0$ in the equation above we obtain the same optimal consumption derived above.

Exercise 1

Let $y(t)$ denote the wealth process which satisfies the state equation

$$\frac{dy(t)}{dt} = -c(t), \qquad \text{with initial condition} \qquad y(t) = x\,, \tag{32}$$

where $c(t)$ is the amount that the agent consumes at time t and x is his initial wealth. (Note that we are assuming that the riskfree rate is zero).

Assume that the agent's objective is

$$\max_{c(t)} \int_t^T c(s)^\gamma \, ds \,, \tag{33}$$

where $0 < \gamma < 1$

i) What is the optimal consumption path?

ii) Show that

$$\int_t^T c(s) \, ds = y(t) \,.$$