Statistical Method for Finance Coursework

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The Fractal Market Hypothesis

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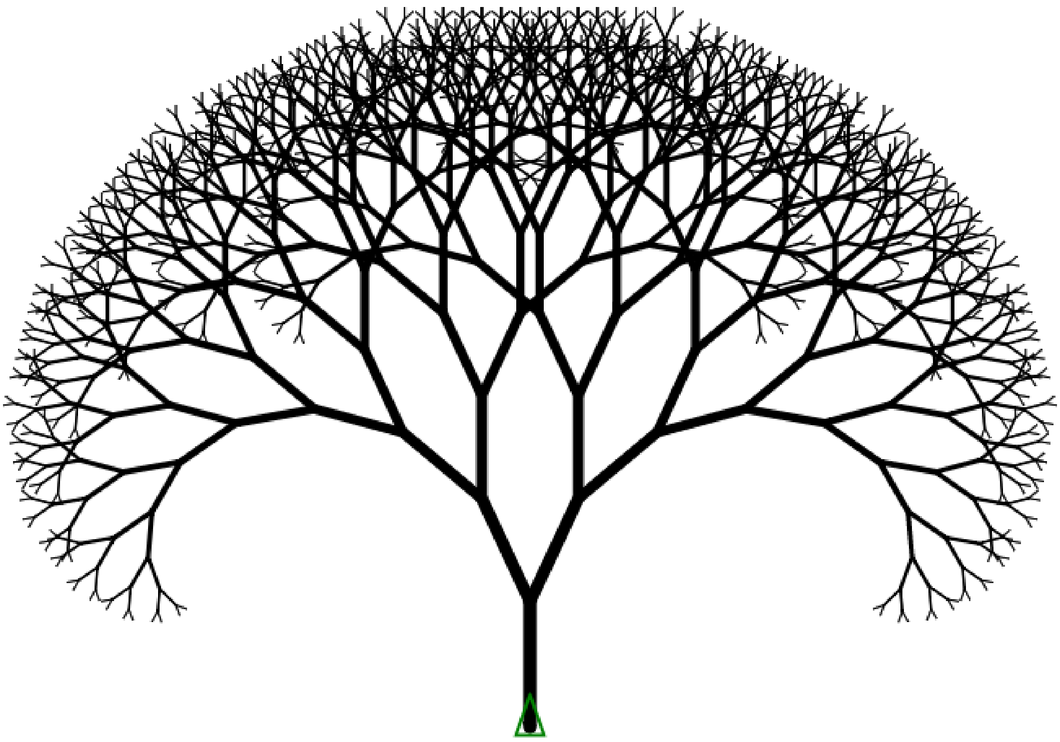

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1 Introduction

The following pictures show the famous example of self-similarity in statistics. Consider that you have a task to measure the coastline of Great Britain. If you use the unit of 100KM to measure it, it will look like the first picture, and the coastline is 2800km. Now, if more accuracy is required, one can measure it with a smaller unit, say, 50km. It will now look like the second picture and the result is 3400km. This practice can go on and on with the infinitely smaller unit. In another word, there is no smallest measure for this. One can view this as ”magnification” of a structure. Under different degrees of magnification, a similar structure can be observed. This kind of structure exhibits ”self-similarity”.

(a) 100km scale **(b)** 50km scale

Figure 1: Britain Fractal Coastline

Self-similarity is a structure found in the natural world. You may find this by observing the edges of a leaf. Under magnification, the same pattern repeats and increases its complexity. Surprisingly, one can observe this property in time series of the financial market price.

Inspired by this observation, Nicola Anderson and Joseph Noss from Bank of England (BoE) published a paper, The Fractal Market Hypothesis and its Implications for the stability of financial markets, in 2013.

The purpose of this report is to verify the findings from the BoE paper by replicating the model is used. In particular, the errors, limitations and bias of the models and assumptions will be discussed. Revision of models will be given and assessed.

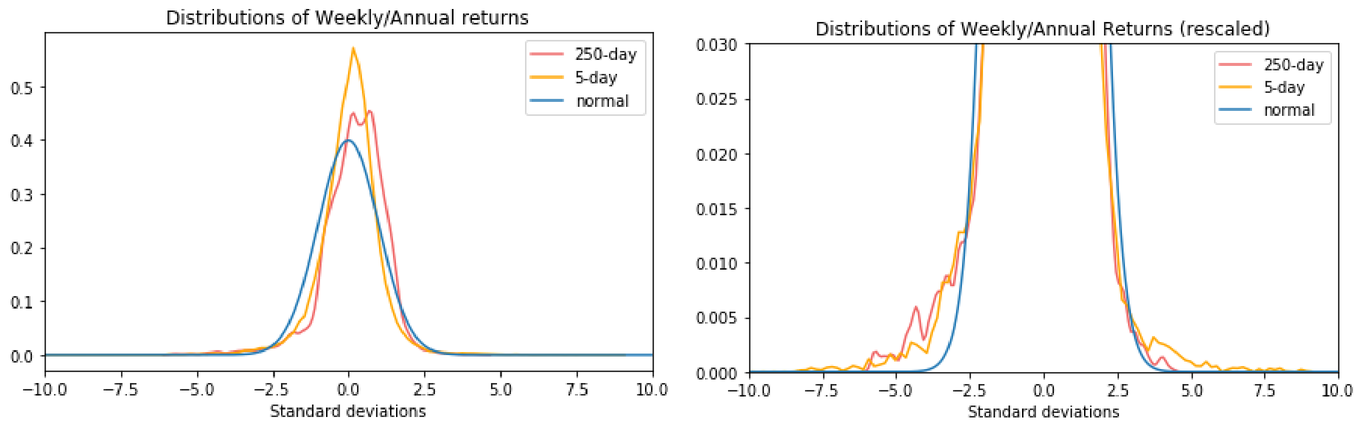
The report will start with discussing the limitations of the widely used Efficient Market Hypothesis (EMH) by examining its underlying assumptions, and further explain why it is not applicable in real-world financial markets. Then Fractal Markets Hypothesis (FMH), whose properties address some limitations of EMH, will be introduced. It will be assessed first from a qualitative perspective, and then from a quantitative perspective by replicating the model introduced by BoE. Revision of their models will also be included and elaborated. The measurement of fractals using the Hurst exponent estimated from rescaled range analysis will be examined and expanded with more insights.

2 Efficient Market Hypothesis and its limitation

The Efficient Market Hypothesis(EMH) is widely accepted even in modern financial markets. The market is said to be "efficient" because it integrates all available information in price. To realise this, first, it is assumed that all investors are rational, and they react to new information rationally as soon as it goes public, given that they all have the same level of access to information. For example, if there is negative news about company A, investors will expect a decrease in price, and therefore sell the stocks they hold. Supply will increase while demand decreases, resulting in a decrease in price, up to a point where the price reflects all past information including the latest negative news. Now, if investors analyse the past information of company A, they will not find any arbitrage opportunity. This implies that there is no way to predict the price since all information is fully reflected in price already, and future events are unknown. No one can "beat" the market and generate returns consistently.

Future events are unknown and can be seen as independent overtime under EMH. Inspired by this assumption, one may argue that the future price distribution can be modelled based on Central Limit Theorem (CLT) (Rice(1995)). Rice suggested that stock prices are log-normal distributed, which is still one of the most popular assumptions in the industry. By CLT, given random variables (not necessarily to be normally distributed) in small sub-intervals are independent and identically distributed (iid), with finite variance.

This proposal is subject to dispute. We examined this assumption using the Dow Jones Industrial Index.



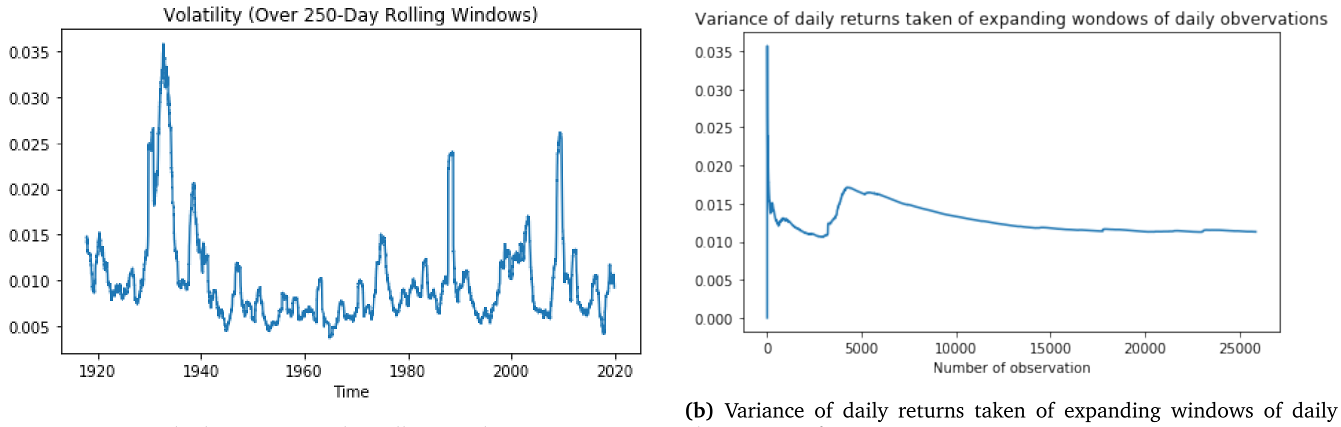
(a) Distributions of weekly/annual returns

(b) Distributions of weekly/annual returns (rescaled)

Figure 2: Distribution of DJ return

From **Figure 2a**, we compared the distribution of weekly and annual returns with a normal distribution. It can be seen that the actual returns are not normally distributed, with higher peaks. If we have a closer look to the tails from **Figure 2b**, fatter tails are observed. Further, **Figure 3a** demonstrates volatility clustering, where volatility is positively correlated on consecutive days. Besides, **Figure 3b** indicates that variance is not necessarily finite, given the plot does not converge to a single number, rather, it fluctuates to some degree over time. All the instances above are against the assumption that the returns follow a log- normal distribution.

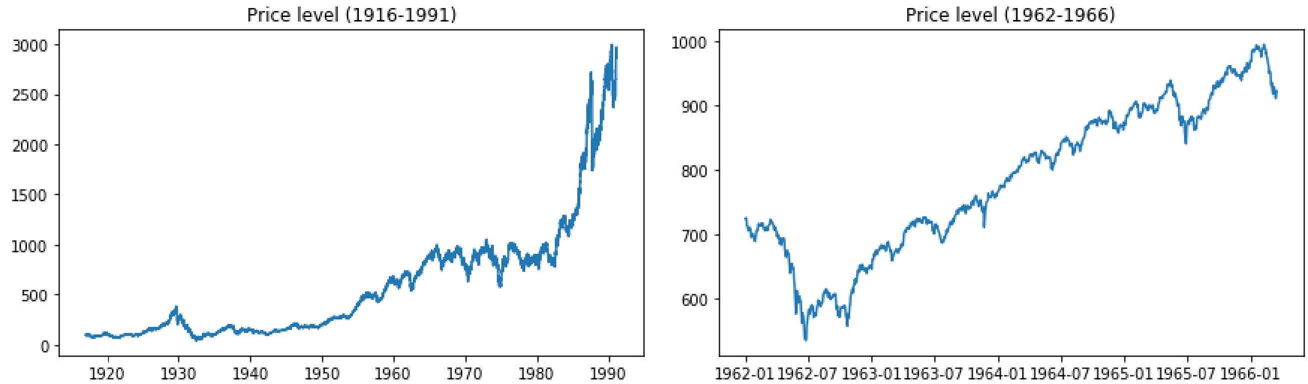
Are the returns iid? **Figure 4a** and **Figure 4b** may provide some clues. **Figure 4a** is the price level of Dows from 1916 to 2019, and **Figure 4b** is the price level of Dows from 1962 to 1966. We can view **Figure 4b** as a "magnification" from **Figure 4a** under the same time period. From **Figure 4a**, the graph is not as "rough" as **Figure 4b**. The claim is that the markets price may have self-similarity if we view them under different magnitudes.



(a) Volatility (over 250-day rolling windows)

(b) Variance of daily returns taken of expanding windows of daily observations (from 1 1916)

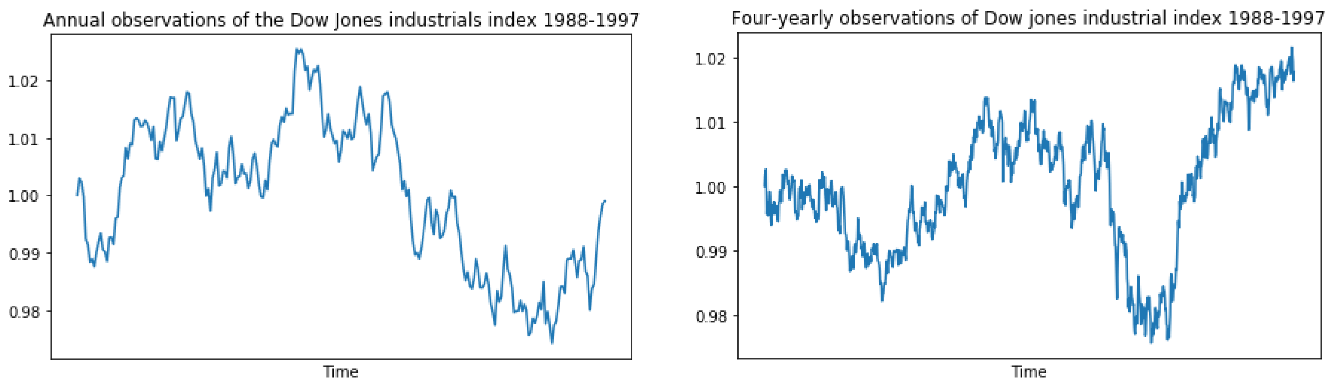
Figure 3: Volatility and variance



(a) Prices 1916-2019

(b) Prices 1962-1966

Figure 4: Dow Jones Index Movement

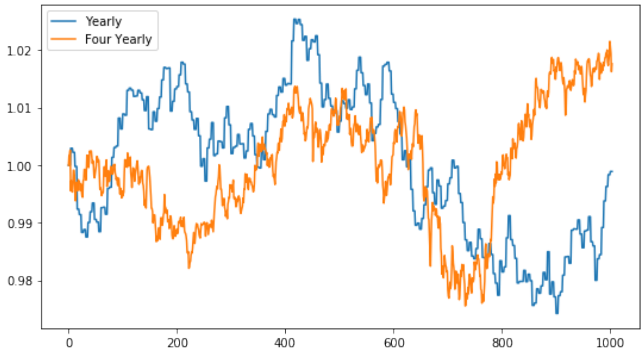


(a) Annual observations (1988–97)

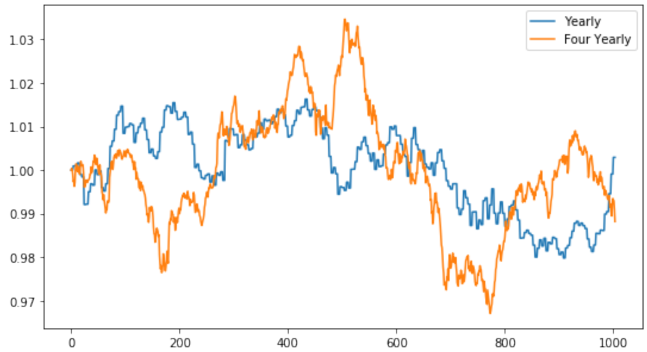
(b) Four-yearly observations (1988–97)

Figure 5: Observation over different horizon

We can continue to examine this by looking into the price from different frequency. **Figure 5a** is the annual observations of the Dows from 1988-1997, while **Figure 5b** is the 4-yearly observations under the same period. For comparison purpose, the data was rescaled to have the same mean and start at 100. **Figure 6a** is a consolidated graph of same data from **Figure 5a** and **Figure 5b**. It is easy to observe that the 2 curves demonstrate some qualitative similarities, although they are 2 different time series. For example, for the first half of the graph, both time series have a generic increasing trend, while they both experienced dramatic drop and then reversal in the second half of the 10-year period. This behaviour exhibits self-similarity, which will have more complexity if we "zoom-in" any particular sub period. Self-similarity can also be found in other stocks and indexes. Take an example of "FTSE all share index", in the same period of 1988-1997, we can observe similar property as Dow Jones as shown in **Figure 6b**.



(a) Annual/four-yearly observations of the Dow Jones industrials index (1988–97)



(b) Annual/four-yearly observations of the FTSE All-Share index (1988–97)

Figure 6: Self-similarity in both Dow Jones and FTSE

With the observation above, the analysis casts some doubts on the iid assumption of the market price. Hence, EMH is challenged and there may be another hypothesis that will better explain the behaviours. Inspired by the self-similarity which is a property of fractals. Peters(1991) proposed the Fractal Markets Hypothesis (FMH), and explored what caused this particular structure in the markets.

3 Fractal Market Hypothesis

In this chapter, we will explain Fractal Market hypothesis both qualitatively and quantitatively. We will also point out several critical errors in the model and its calibration that makes the result of this paper absolutely not replicable. We aim to analyse Fractal Market Hypothesis in three aspects:

- Qualitative analysis on "Role of liquidity and information"
- Quantitative model on Fractal Market Hypothesis (FMH) and current critical flaws
- Improvement on quantitative model to better replicate market volatility

3.1 The Role of Liquidity and Information

FMH was proposed to explain the self-similarity of the market price, which EMH fails to address. In EMH, all information (or at least, public information and price history under its semi-strong form) is reflected in the price, and no one can take advantage of information to arbitrage. Also, it is assumed that there is no constraint on liquidity; however, both information and liquidity are crucial in the markets and should be taken into account. It is generally accepted that the markets follow the semi-strong form of EMH, where non-public information will have an impact on price. In reality, it is inevitable that some investors have access to insider information and create an advantage for themselves. Regarding public information which is accessible to all investors, does it have the same impact to all investors?

- First, even information is public, investors may not receive it simultaneously, or they may not act on it at the same time. Even all investors are rational and make the right decision, the time lag will differentiate their returns. This is also applicable to non-public information because it will eventually become public, although the value of this information may be fully utilised already.
- Second, consider the case where two investors (short-term horizon vs long-term horizon) receive the same fundamental information without any time difference. Depending on the type of investors, they may have different actions. For a short-term investor, they may buy/sell based on the information and realize gains for a better daily PnL. But for a long-investor like a pension fund, they care about steady long-term growth more, i.e., fundamental values of their investments. Information that may fluctuate market price under a short period is out of concern to them. Thus, the long-term investor may not take action on this information as a short-term investor does. Different investors interpret information differently.

In the context, the same information will have different results from different investors because of their time lag on information arrivals, or their interpretation on it. Price is then not entirely random and the distribution of return is then not independent, which violates the assumption of EMH. Instead, price is driven by the behaviours among different type of investors. The interactions of short-term investors and long-term investors mix local randomness within a short period and the global determinism over a long period. Thus, price then demonstrates self-similarity when viewed at a different frequency, a fractal structure.

The findings above explain how the financial markets remain stable under normal market conditions and drive liquidity. When a short-term investor experiences a price fall that is greater than expected, he may sell the stock to stop further loss; but a long-term investor may see this as a great chance to buy, given a good outlook on its fundamentals. Under this interaction, the trade provides liquidity and stabilise the market. The fractal structure ensures stability. Consider a market with numerous interactions from short/long-term investors. If one of the trades malfunctions, say, the execution price is not reasonable, there would be other trades to compensate. **Hence, the local randomness and global determinism of the structure ensure the overall stability of the market with a certain level of error tolerance.**

But what if the fractal structure breaks, for example, the long-term investor does not buy when a short-term investor sells? Consider the following scenarios:

1. The information causes short-term investors to sell, and it also affects the fundamental views of long-term investors. Now the long-term investors may revise their views and expect no growth in the long term, hence, they sell as well, or they become short-term investors themselves to stop loss by trading overwhelmingly. This phenomenon stresses the market further and leads to a crash or crisis.

2. Long-term investors view price less frequently than short-term investors, they may doubt the information derived from price and hold their trading until they have a higher level of confidence. This will also break the fractal structure since now the long-term investor is out of the market.

In either scenario, without the role of long-term investors, there is no longer heterogeneity that facilities trading in the market and liquidity evaporates. Next section will provide a quantitative model to fractal structure in the market and visualise the effects without long-term investors, to further examine FMH.

3.2 Quantitative model on FMH

As explained in the previous section, the fractal property of financial market arises from two sectors:

1. Different understanding and accessibility of information between long/short term investors
2. Different reactions to same price movement between long/short term investors

Now we can develop a model that seeks to capture how different interpretations of information by investors with different investment horizons affect the stability of the resulting price series. In doing so, and through the subsequent effect that investors' buying/selling behaviour have on prices, it is able to replicate many of the non-Gaussian properties of markets (including fat tails, stochastic volatility and self-similarity)

3.2.1 Derive the quantitative model

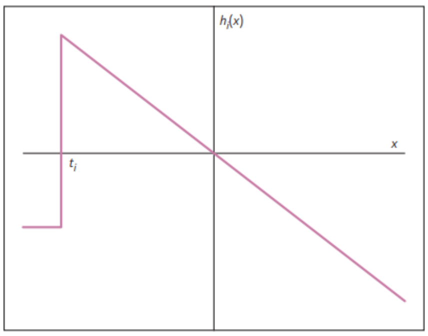


Figure 7: An illustration of the function governing the demand of the mean-reversionist investors

Both investors invest based on the information gleaned from the change in the price they witness in the previous period, but the distribution on which they condition their behaviour varies depending on their horizon. Normally both long and short term investors are **'mean reversionists'**: buying if this relative price movement is negative (exhibiting positive demand), or selling if it is positive (exhibiting negative demand). But if the decline in price in the previous period that is particularly extreme, their demand becomes negative and they sell in some fixed (large) quantity. For both long and short term investors, their response to prices of the demand can be represented by functions h_S and h_L respectively, where:

$$h_i(x) = \begin{cases} -a_j X & \text{if } x > t_i \\ -d_i & \text{otherwise} \end{cases}$$

As shown in **Figure 10c** t_i is the 'threshold amount' which, if the return is above it, causes the investor to buy (sell) if the price change is negative (positive); d_i is the amount by which they 'force sell' otherwise; and a_i is the 'aggression' of their mean reversion . It also requires a larger upward (downward) price deviation from fundamentals, in absolute terms, to provoke as strong a selling (buying) reaction from the long term investor compared to the short-term investor. This is because the variance of the long-term investors' price distribution, to which they compare any change, is far larger. Hence $t_L < t_S$.

There is also a 'momentum trader', who follows bull runs. They exert positive buying pressure when the price change in the previous period is positive, but do not trade when it is negative.

$$M(x) = \begin{cases} a_M X & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

In summary, if we combine the two types of investors and momentum trader together, we could have the following function of price change rate (return rate) R_t:

$$R_t = r + \epsilon_t + h_s(x) + h_t(x) + M(x) \quad with \quad x = R_{t-1} - r$$

3.2.2 The critical errors in current parameters setting based on Bank of England paper:

Error in r:

Currently, the r and σ are both set to 0.05 as they accord roughly with the long-term 'risk-free' rate of interest rates on one-year UK government bonds, and the volatility of the Dow Jones series used in Section 1. However, notice that all the description in the paper indicates that t is the in the unit of one day. R_t represent the price change between day t and $t-1$. Hence, the r and σ should be both derived from daily return of Dow Jones Industrial Index. Working with $r = 0.05$ creates huge error.

The reason we claim the increment in R_t being daily is that if we refer to **Figure 8**, by the end of the period, price R_t only moves from 100 to 115. And there are more than 100 price movement in the graph. With so many data points, it must represent daily price change.

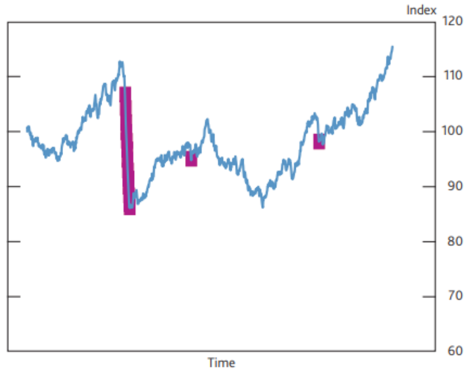


Figure 8: Simulated price movement given in paper

Error in t_i:

Firstly, the formula of $t_i = E(R_t) - 3 \times \sqrt{varR_t}$ should be $t_i = E(R_{t-1}) - 3 \times \sqrt{varR_{t-1}}$. It is because at time= t, we have no idea of what R_t is. The investor can only make decision based on information until day $t-1$

Secondly, the thresholds at which the short-horizon investors force sell t_s are set to a fixed point on the distribution of price returns as observed at their time horizons, which is 1 day for short-horizon investor. Both investors force sell when they witness a price move three standard deviations below the mean of the distribution of returns at their horizon. Mathematically, we are comparing $x = R_{t-1} - r$ and $t_i = E(R_{t-1}) - 3 \times \sqrt{varR_{t-1}}$. Here comes the contradiction in understanding in formula:

- For horizon being one, if follow the description in the paper — **The return rate observed at their horizon**, then R_t only consist of one number, there is no way to calculate variance of R_t $E(R_{t-1})$.
- However if we assume $var(R_t)$ being a constant —- the variance of 1-day historical return, then as $E(R_t) = R_t$, we have $x - t_i = 3 \times \sqrt{varR_{t-1}} - r$ which is constant, the condition will not change at all.

Hence, model parameter calibration is fundamentally wrong. Next section is our approach for calibration.

3.2.3 Our approach in parameter calibration:

- **r:** We used historical daily returns of Dow Jones Industrial Index to represent the fundamentals. (r=0.000218)
- σ: We used historical standard deviation of Dow Jones daily returns. ($\sigma = 0.01129$)
- a_s a_L a: We used the given parameter of $a_s = 0.4$ $a_L = 0.4$ $a_M = 1$
- **d:** For now, we follow the paper to set d same for both long and short horizon investor with the objective function below. In the next part, we will try to improve the current model to better represent the fractal property of the market. In our case, with our choice of r and simulation of 1500 days, we aim to achieve the final index going from 1 to $(1+r)^{1500} = 1.3870$

$$d = \arg\min \left| E\left(\frac{P_{t+1500} - P_t}{P_t}\right) - 0.3870 \right|$$

- t_i: For long horizon investor, we calculate V_l being the standard deviation of historical ten-day return, and V_s being the standard deviation of historical one-day return. For $E(R_t)$, we define short-horizon investor's observation period: $R_s = E(R_{t-1}) = R_{t-1}$ and $R_l = E(R_{t-1}, ..., R_{t-10})$. Hence, their decision threshold is:

$$V_l = std(log(R_{t+10}) - log(R_t)) \text{ for all t} \qquad V_s = std(log(R_{t+1}) - log(R_t)) \text{ for all t}$$

$$t_l = R_l - 1.12 \times V_l \qquad t_s = R_s - 1.12 \times V_s$$

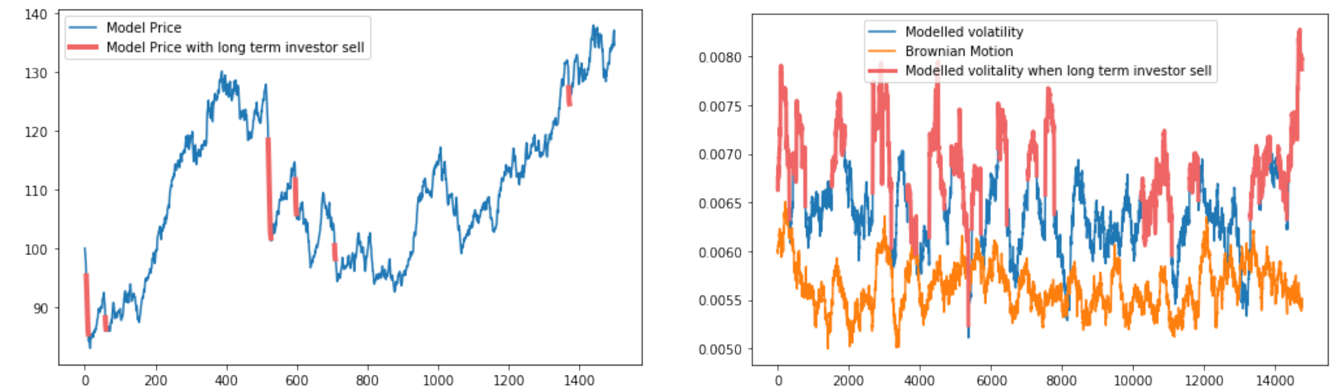
During the current calibration, we are changing d to calibrate only mean of the observed return. In the improved version of the model, we are also going to replicate the shape of the rolling volatility.

3.2.4 Empirical result in full and restricted model

Full model with both long and short term investor:

The ability of this model and its interacting investors to match the market dynamic postulated by the FMH is clear from the resulting price time series. Consider **Figure 9a** of a modelled time series with random innovations. The blue illustrates normal market movements and the red highlighted the times when the long-term investor's force sell. As suggested by the FMH, under normal market conditions, the interactions between short-term investors buying and long-term investors selling stabilised the markets and price does not change abruptly. But in some periods, when long term investors force sell, the price drops dramatically, resulting in further price pressure. This makes the markets unstable. Followed by the "red" periods, the price recovers slowly, which means new information may reverse long-term investors' views and they stop selling further.

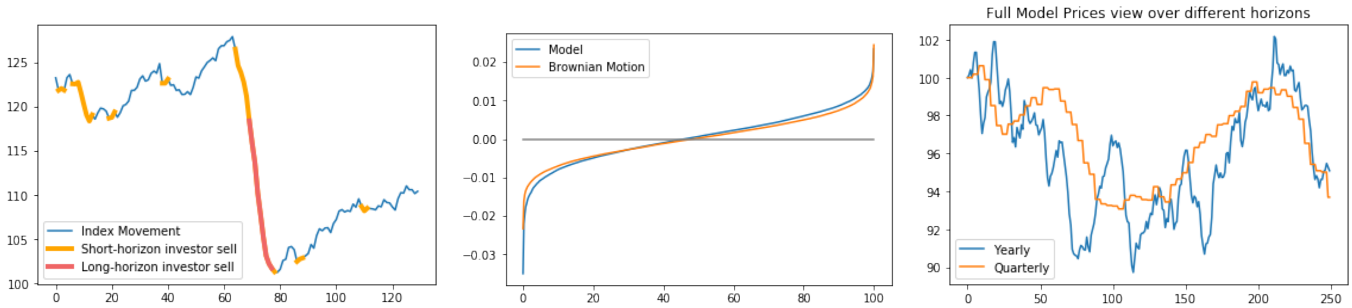
Figure 9b shows the volatility compared to a Brownian Motion. Under normal condition, the volatility (blue line) does not deviate much from the Brownian Motion; however, when the long-term investor's force sell (red line), the price becomes highly volatile. This shows our model successfully replicates the shape of the high spike of the Volatility graph (over 250-day rolling windows) in Dow Jones index **Figure 3a**, proving the interaction between investors leads to the limitation of Efficient Market Hypothesis.



(a) Model price movements with long term investor behaviour

(b) Volatility from modelled price

Figure 9: Details of full quantitative model



(a) Modelled prices over shorter time scale

(b) Percentile plot on distribution of daily return

(c) Modelled price over different horizons

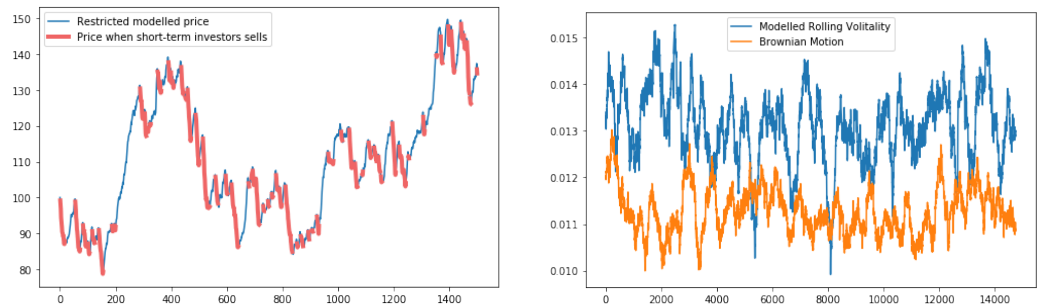
Figure 10: Details of full quantitative model

Figure 10a shows a closer look to the markets. The orange parts are those force-sell from short-term investors. By FMH, the long-term investors step in since the downward movement caused by short-term investors force-sell is seen as buying opportunities for long-term investors. Thus, the price then recovers. The red part is where both short-term and long-term investors force sell and causes instability of the market. Without interactions of opposite investment decisions, it takes longer for the price to recover than usual.

Figure 10b shows a quartile-quartile plot of daily modelled returns, compared to a Brownian Motion generated from normal distribution. It can be observed that the modelled price has a fatter tail, but the 20 percentile to 80 percentile do not deviate much from a Brownian Motion. Under normal market condition, the market is stable, but when event occur and cause long-term investor to force sell, it creates instability of the market and hence, fatter tail. This is in line with FMH. The model hence successfully replicates the fat tail distribution of the log returns, with more than expected number of extreme events.

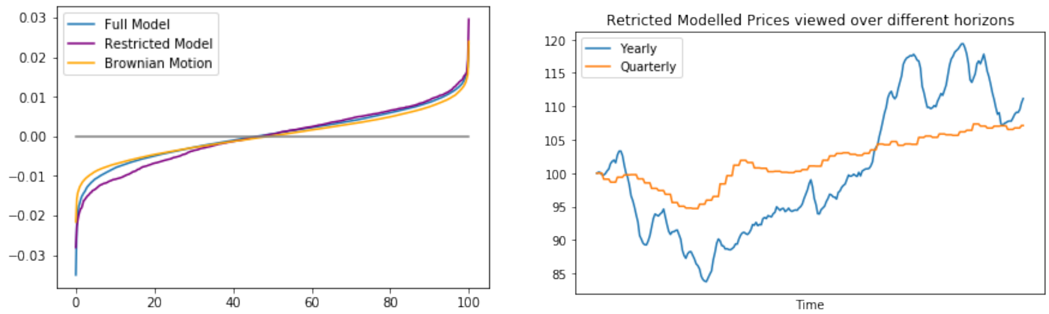
Finally, by comparing the modelled price viewed over different time horizons (annual vs four-yearly) in **Figure 10c**, we find that the time series are qualitatively similar. They all decrease at the first half of the period, increase at the third quarter, and decrease again at fourth quarter. Note that both series were rescaled to have the same mean and volatility, starting at the same index level.

Restricted model with only short term investor and momentum trader:
Since we claim that long-term investor help stabilise the market, it is important to investigate the market without long-term investor to prove our claim. We calibrated the model, the short-term investors will force sell when they observe a price drops 3 standard deviations below the mean of returns.



(a) Model price movements (restricted model) **(b)** Volatility from modelled price (restricted model)

Figure 11: Details of restricted quantitative model



(a) Percentile plot on distribution of daily return **(b)** Modelled price viewed over different horizons

Figure 12: Details of restricted quantitative model

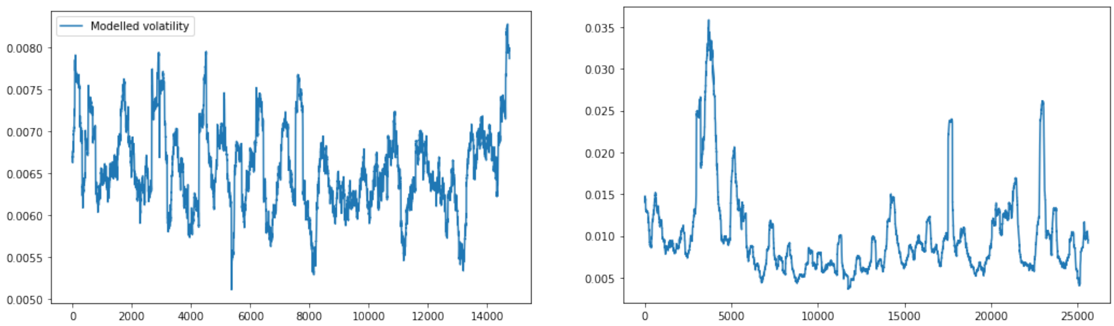
- The resulting time series is shown in **Figure 11a**. With the absence of long-term investors, now the price is significantly unstable, indicated by the red parts of the time series.
- Next, we look into the quartile-quartile plot with restricted model price in **Figure 13a**. It can be seen that the restricted model has a fatter tail than both Brownian Motion and full model price, implying more frequent extreme events. This coincide with FHM, since there is no long-term investor to ensure the stability of the market.
- **Figure 11b** shows the volatility of restricted model price (blue) compared to Brownian Motion (orange). The volatility is greater than Brownian Motion in magnitude, and more volatile. This follows FMH.
- Lastly, we view the restricted modelled price with different time horizons in **Figure 12b**. Again, when we view the annual returns in comparison to four-yearly returns, they look similar overall. It preserves self-similarity.

The comparison between the full model and restricted model highlights the importance of heterogeneity in investors. With the full model, the interactions between long-term investors and short-term investors help stabilise the market under normal market condition, but it also amplify severity of adverse market event, thus, that left tail is fatter than normal distribution; whereas the restricted model, with the absence of long-term investors, creates a market with instability. Also the volatility seems less volatile (fewer spikes) than the full model, but it generally has higher volatility (number-wise). This is because the restricted price contains more force-selling by short-term investors.

3.3 A more sophisticated quantitative model

3.3.1 Current limitation of the model

Though the current quantitative model successfully capture the observed returns, the model is unsuccessful in term of replicating the spikes in volatility of the Dow Jones index. It also took us to try 200 different random seeds in order to replicate similar sharp decrease in modelled price. Shown as figure below, We could see the spike shape in the volatility is not well represented.



(a) Full model Volatility (250-day rolling windows) **(b)** Dow Jones Volatility (250-day rolling windows)

Figure 13: Comparing Rolling window volatility

3.3.2 The area of improvement

We improve the model by diversifying the parameter space. In the previous model, we assumed the volume of force-selling is same for both long and short-horizon investors. But realistically, long term investors usually have a higher holdings. When they force sell, the volume should be typically higher than the short-horizon investors. Hence, we assume d_l to be the force sell volume for long-horizon investors and d_s to be the force sell volume for short-term investors. In addition, we quantify the similarity in spike shape into measuring the difference in percentile plot. (Calculating the mean square error between two graphs in **Figure 15b**). Example will be shown below.

Now our parameter space are:

- **r:** We used historical daily returns of Dow Jones Industrial Index to represent the fundamentals. (r=0.000218)
- σ: We used historical standard deviation of Dow Jones daily return. ($\sigma = 0.01129$)
- a_s a_L a: We used the given parameter of $a_s = 0.4$ $a_L = 0.4$ $a_M = 1$
- **d:** As described earlier, we set d_l larger than d_s. Hence, We take the parameter k to represent the relation: $d_l = k \times d_s \quad k > 1$. Finally, with our choice of r and simulation of 2500 days, we aim to achieve the final index going from 1 to $(1+r)^{2500} = 1.725$

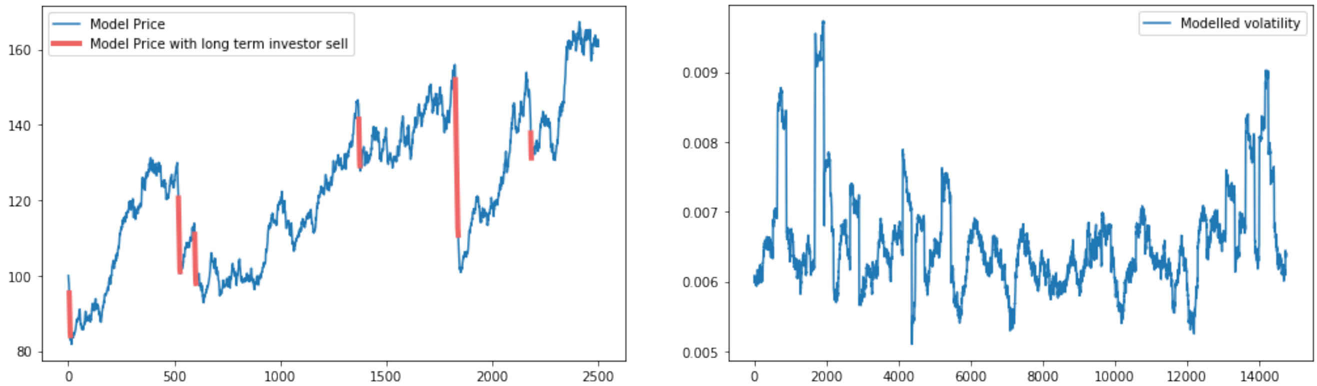
$$d_s, (d_l = k * d_s) = \arg\min \left| E(\frac{P_{t+2500} - P_t}{P_t}) - 0.7250 \right| \quad \text{with } k \text{ minimise MSE in percentile plot of volatility}$$

- t_i: For long horizon investor, we calculate V_l as the standard deviation of historical ten days return, and V_s as the standard deviation of historical one days return. For $E(R_t)$, we define short-horizon investor's observation period: $R_s = E(R_{t-1}) = R_{t-1}$ and $R_l = E(R_{t-1}, ..., R_{t-10})$. Hence, their decision threshold is:

$$t_l = R_l - 1.12 \times V_l \qquad t_s = R_s - 1.12 \times V_s$$

3.3.3 Example of new model

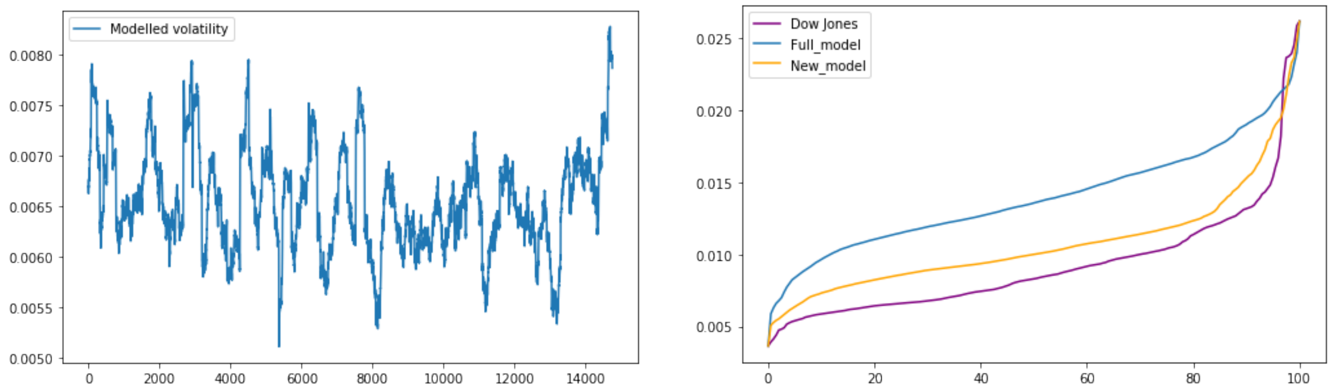
To better illustrate how the revised model helps, here we give an example. We run two simulation, one runs over 2500 days to illustrate price movement, the other runs over 15000 days to illustrate the shape of rolling volatility. If we take $d_s = 0.0095$ and $k = 1.5$ for both simulation, we will have the following simulated price movement.



(a) Modified model price movement over 2500 days simulation

(b) Modified model rolling volatility over 15000 days simulation

Figure 14: Illustration of new model



(a) Full model Volatility (over 250-day rolling windows)

(b) Comparing percentile of volatility

Figure 15: Details of full quantitative model

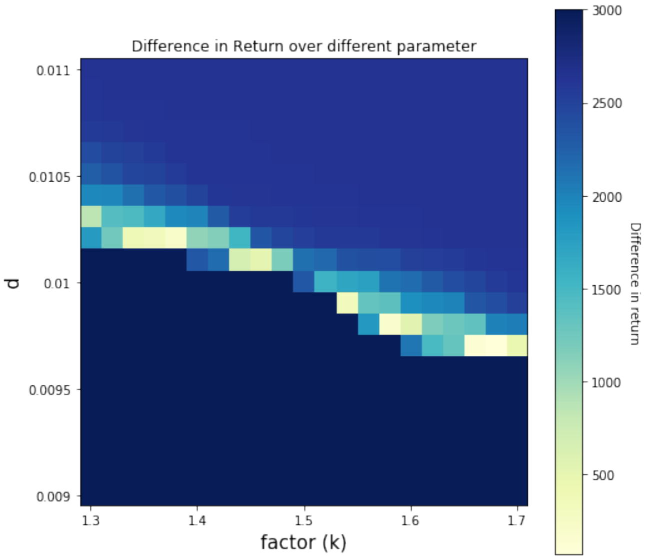
Looking at **Figure 14a**, we may feel the shape of modelled price is still similar. But if we look at **Figure 14b**, we can see we have a surprisingly better representation of volatility as we see significant spikes in the volatility distribution similar to Dow Jones volatility spikes illustrated in **Figure 13b**. It is definitely better than the original model shown as in **Figure 15a**. Finally, we can compare the percentile distribution **Figure 15b** of the two models to the actual volatility of Dow Jones. We could see the revised model is much better than the original model. But there is still significant disparity in the distribution. It is because here relatively random parameters were chosen for illustration purposes. It leads to the next step: optimisation of parameters.

3.3.4 Optimisation of parameters

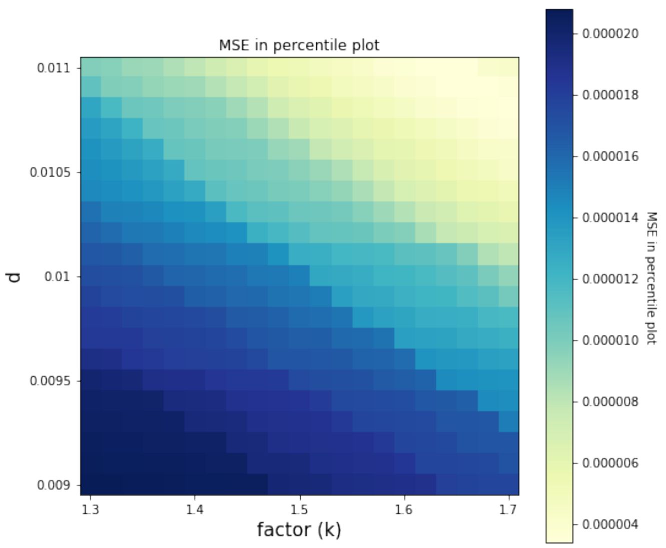
This optimisation problem over d_s and k consists of two objectives (we can represent $d_l = d_s k$):

- Optimise d,k to ensure modelled price has similar return ratio of Dow Johns
- Optimise K to ensure modelled volatility shared similar distribution as that of Dow Johns

As a result, we tried a range of k (1.3 - 1.7) and d (0.009 - 0.011), calculating the difference in return and MSE in volatility percentile plot. Here is the calculation result.



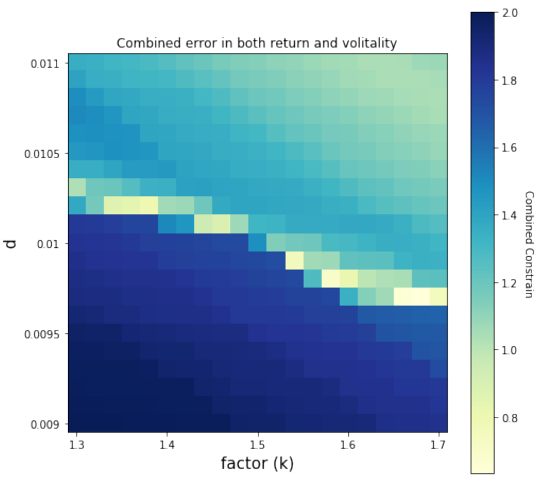
(a) Difference in expected return



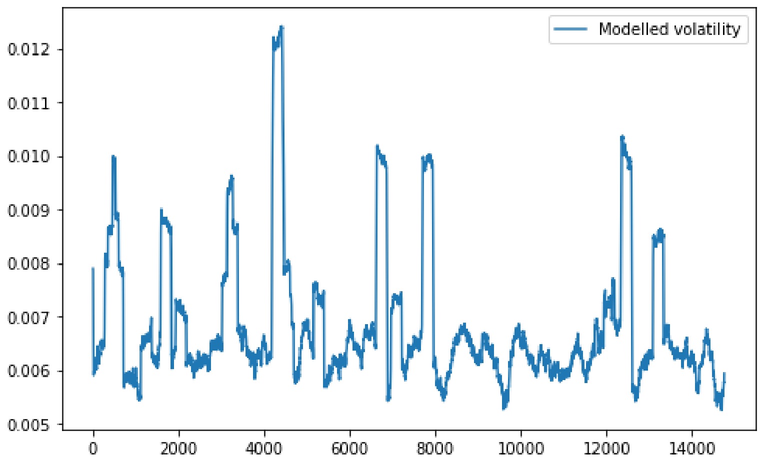
(b) MSE between two volatility percentile plot

Figure 16: Optimise parameters by looking at difference in mean and volatility between model and Dow Jones

By **Figure 16a**, we could see that the difference in expected return is only small for a small range of parameters for d between 0.0097 to 0.0102. By **Figure 17b**, we could see the difference in percentile plot between model and Dow Jones is smaller when k and d increase. Combining the both heatmap together, we get the following conclusion:



(a) Combined difference measure for parameters d and k

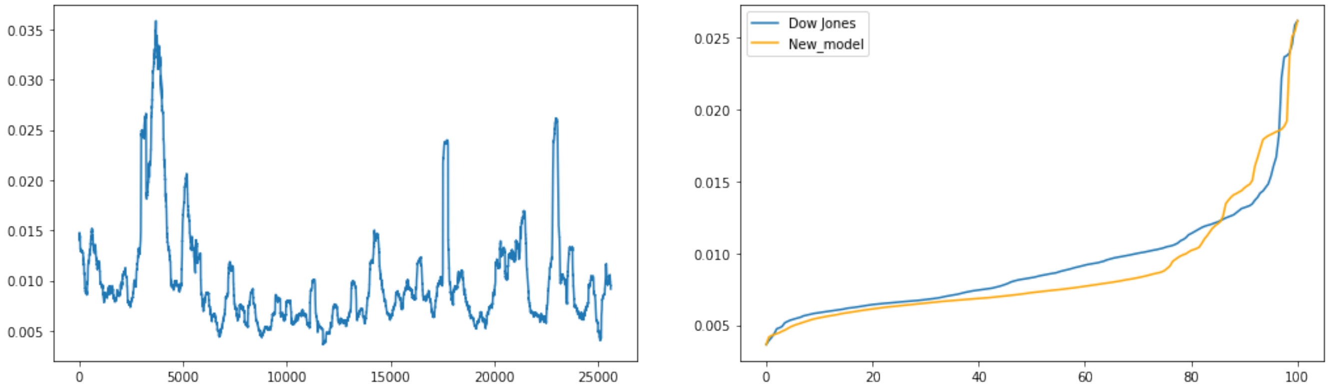


(b) Volatility of optimised modelled price (250 rolling window)

Figure 17: Optimise parameters by looking at difference in mean and volatility between model and Dow Jones

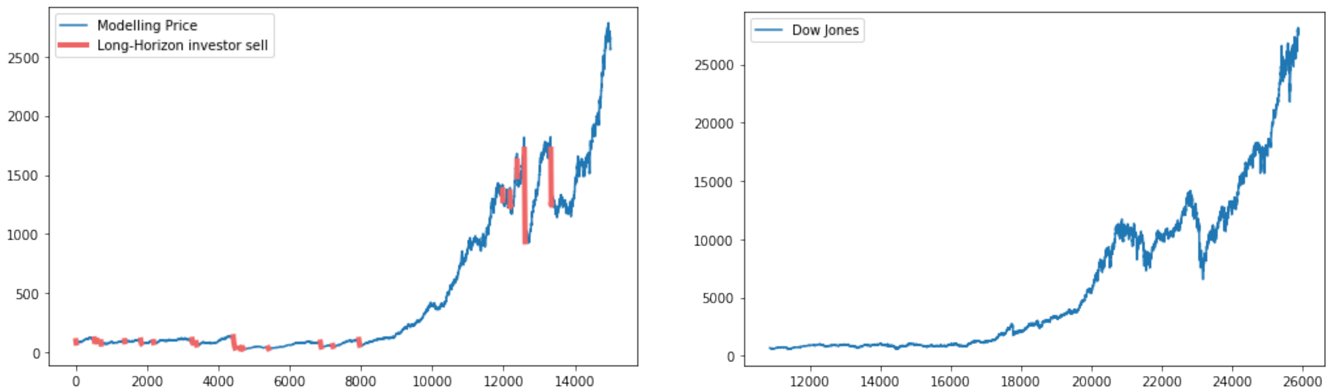
By looking the **Figure 17a**, we can easily conclude that the best set of parameters are; $d = 0.0097$ and $k = 1.68$. Then we can see some amazing price behavior replication of the Dow Jones Industrial Index.

3.3.5 Optimised model and its properties



(a) Dow Jones Volatility (250-day rolling windows) **(b)** Percentile plot of rolling volatility

Figure 18: Comparing Rolling window volatility between optimised model and Dow Jones Index

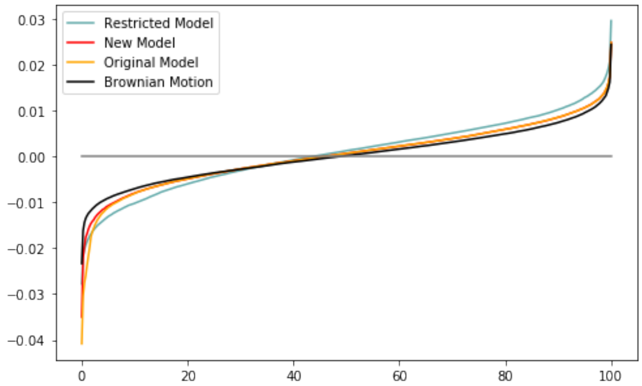


(a) Modelling price over 15000 days **(b)** Last 15000 observations of Dow Johns Index

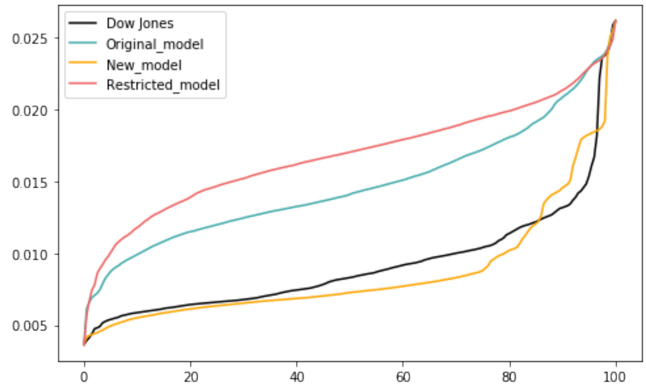
Figure 19: Comparing 15000 data points of modelled price movement and Dow Jones Index

By comparing **Figure 17b**, **Figure 18a** and **Figure 18b**, we can see the new optimised model has much better representation of volatility than the original model as it almost perfectly fits to the distribution of Dow Jones volatility. If we look at the price movement in **Figure 19a** and **Figure 19b**, we can see the trend in the original Dow Jones data is well preserved. It successfully replicate the price oscillation during the middle periods. This new model introduces more stability into the market. If we look at **Figure 18b**, we can see the volatility is low at 80 percent of the time and the rest 20 percents are the market fluctuations when long-horizon investors force sell.

3.3.6 Summary



(a) Percentile plot on distribution of return in different model



(b) Percentile plot on distribution of rolling volatility in different model

Figure 20: Comparing Rolling window volatility between optimised model and Dow Jones Index

The 3 different models (restricted model, full model, and new optimised model) all show that the interacting agents of multiple horizons provide consistent liquidity. But it also has occasional 'spikes' of high volatility, illiquidity and discontinuous price movements when prices crash. On the other hand, the restricted model has a price series that is fundamentally less stable in that it contains frequent periods of illiquid forced selling by the short-term investor. In addition, with the model optimisation, we can see that usually long-horizon investors force-sell more than short term investors. And it even makes the market more stable with occasional spikes. Seen from **Figure 20b**, the new model has lower volatility for most of the time (80 percent), similar to the behavior of Dow Jones, as well as more stable than any other models. The finding are summarised below:

Model	Distribution of returns	Volatility of returns
Modified model with forced-selling volume different for long and short-horizon investors.	Fat tailed, similar to original model	Stochastic; lower, with more spikes, similar to actual situation
Full model of the FMH with long and short-horizon investors.	Fat tailed.	Stochastic; low, with a spike
Restricted investor interaction: only a short-horizon investor (and a momentum trader)	Fat tailed, and more so than full model.	Stochastic; higher without a spike

Figure 21: Summary of 3 different models

4 Explained Fractals with Range Analysis

In **section 2**, we examined the price series with different horizons (e.g. annual vs four-yearly) and observed similar patterns. This signals that the market price may follow a fractal structure. In this section, we will look into the fractal property of the markets and quantify it.

4.1 H statistics and its implication

Self-similarity is the core of fractal structure. As examined earlier, market price series does not simply follow normal distribution. Here Hurst exponent estimated from rescaled range analysis was purposed. By Mandelbrot(1968), the Hurst exponent, H, can be used to measure the roughness of data series in fractal dimension.

$$H \begin{cases} < 0.5 & \text{Anti-persistence (mean-reverting): a high value in one period is likely to be followed by a lower value} \\ = 0.5 & \text{Brownian Motion} \\ > 0.5 & \text{Persistence (trending): a high value in one period is likely to be followed by a higher value in a later period} \end{cases}$$

To visualise this, the following plots with H = 0.3, 0.5 and 0.7 respectively:

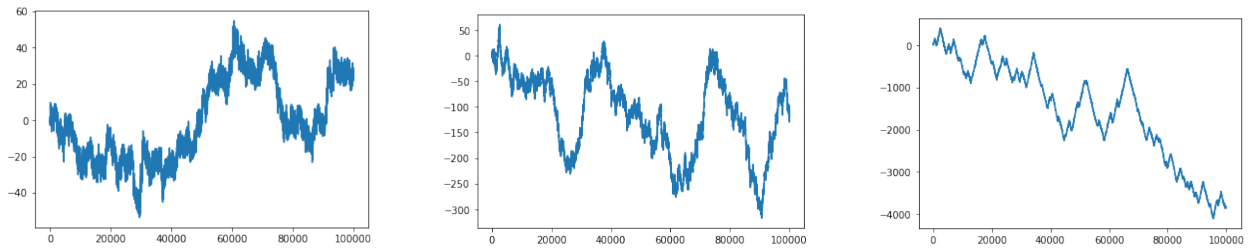


Figure 22: Illustration of different H statistics

As shown above, when $H = 0.7$, the plots is the smoothest, i.e most persistent.

4.2 Rescaled Range Analysis

To estimate H, we applied Rescaled Range Analysis: $(R/S)_n = cn^H$

H can then be estimated by regressing log of R/S on log n (c is a constant). Note that this method does not require any assumption of the distribution of the time series.

Based on this estimation method, BoE calculated the H exponent with 20-day returns between window size of 10-50. Actual time series, full modelled price and restricted model price were used as input, and respective H exponent of 0.72,0.66,0.73 were yielded. This demonstrated that the real time series and modelled price are persistent respectively. BoE then claimed that the persistence is irrespective of the scale, that is, daily returns and monthly returns are correlative to it corresponding future daily/monthly returns. But this scale can not expand infinitely, BoE claimed that the effect only lasts when the frequency is not longer than 4 years. We use this result as a benchmark to investigate the solidness of this claim.

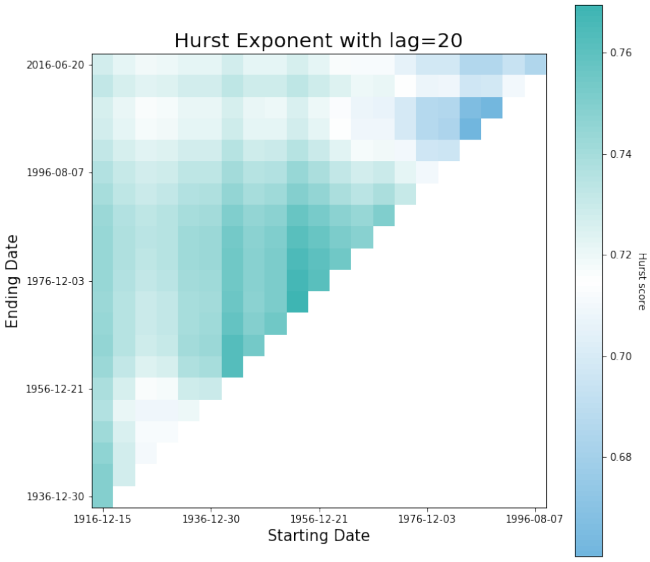
Table B Estimated Hurst coefficients

Series	Window size	Estimated value	Expected value under null hypothesis that *H*=0.5	Standard deviation of estimated value
Dow Jones (20-day returns)	10–50	0.72	0.62	0.025
Full model	10–50	0.66	0.53	0.032
Restricted model	10–50	0.73	0.53	0.032

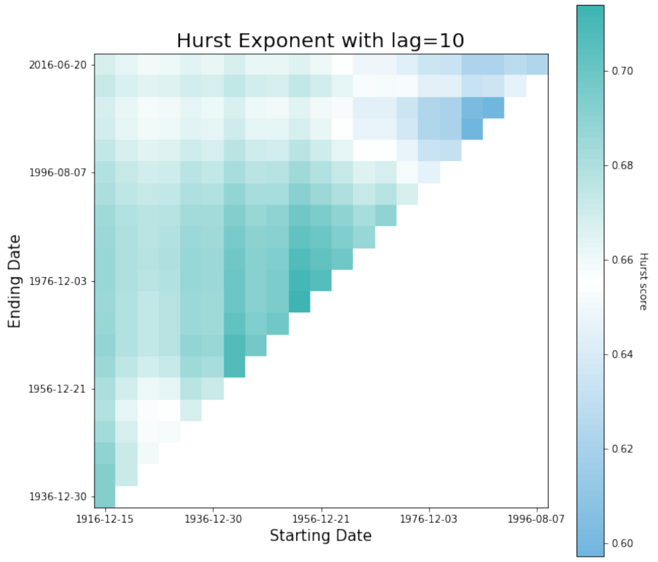
4.3 Change in H statistics over different periods and lags

Consistency of Hurst exponent over time

The first graph **Figure 23a** shows the path of change of H exponent when different periods of the actual Dows time series are used, with lag = 20 (i.e. monthly returns). We can see that H exponent fluctuates based on which period of data was used. The BoE paper used the time series from 1896 to present day, and yielded H = 0.72. Given our data, we compare this value (0.72) to the value (around 0.73) from our earliest period (1916) to present day. The values are close. However, if we look into other periods, say measuring H exponent for Dow Jones starting from 1980s and at any ending dates (the top right portion of the heatmap), the values fall below 0.7 in majority; and from 1950s to 1980s, the H exponent is generally higher than 0.74. It seems the selection of period of input data matters to the value of H exponent. And Dow Johns is getting less and less persistent over time.

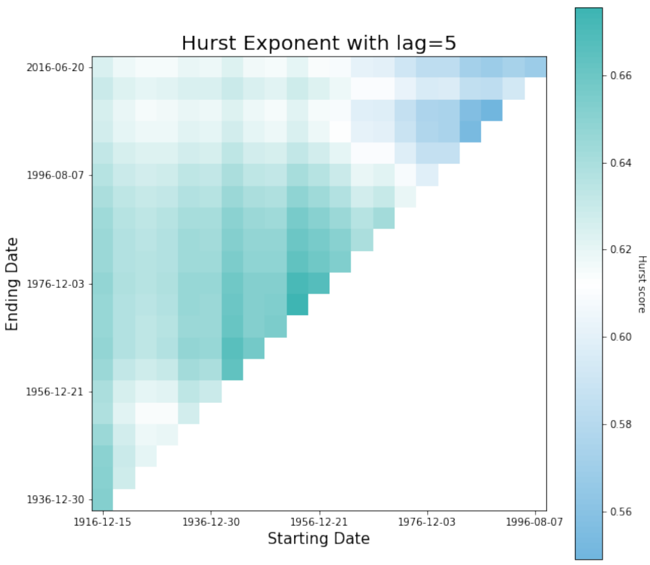


(a) H exponent at different starting and ending date

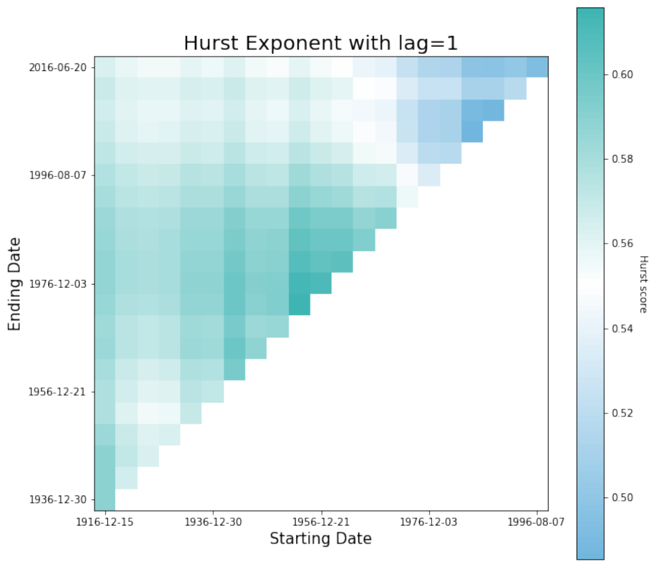


(b) H exponent with lag 10

Figure 23: H exponent calculated with Dow Jones index at different range (varying staring and ending date)



(a) H exponent with lag 5



(b) H exponent with lag 1

Figure 24: H exponent calculated with Dow Jones index at different range (varying staring and ending date)

We further examine this claim by plotting the H exponent with lag = 1,5,10 respectively, using data from different time period.

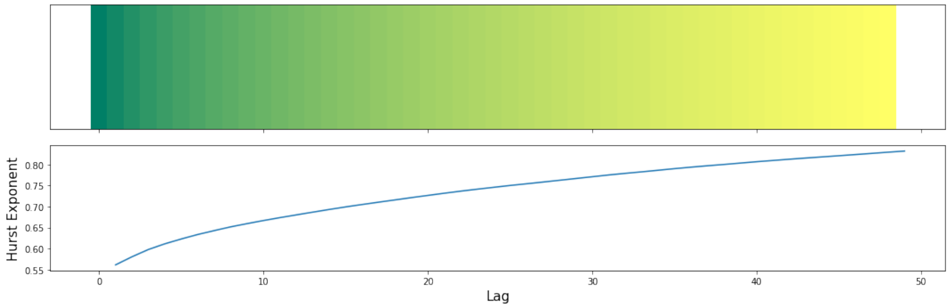


Figure 25: Change of H exponent of Dow Jones over different Lag

If we look into the graphs with different lags, which represent bi-weekly, weekly, and daily returns, we can observe the the H exponent will be smaller, positively related to the lag (frequency). For example, compare lag 20 to lag 1, we can see that the range of H exponent decreases from (0.68,0.76) to (0.5,0.6). This trend is obvious when we compare the graph with different lags. Also, it is worth to point out that the more recent time period we used, the smaller H exponent will be produced. If we look at the top right corners of the 4 graphs (the blue and white areas), which represents data from 1980s to present days, we will find that the H exponent is generally at the bottom of the range, smaller than other time period. Similarly, data from 1950s to 1980s generally produces higher H exponent.

Looking back to the BoE paper, their claim was concluded from the data set of the whole time series with lag 20. This selection returned a ”nice” value of 0.72. Without trying other period of the time series and other lags, this result shows some degree of bias. If they use more recent data (1980 to present days), their claim may be questionable. For comparison purpose, we create a graph to exhibit the trend of H exponent vs lag using all Dows data from 1916 **Figure 25**. A positively correlated curve indicates that the time series are more persistent if viewed at longer frequency.

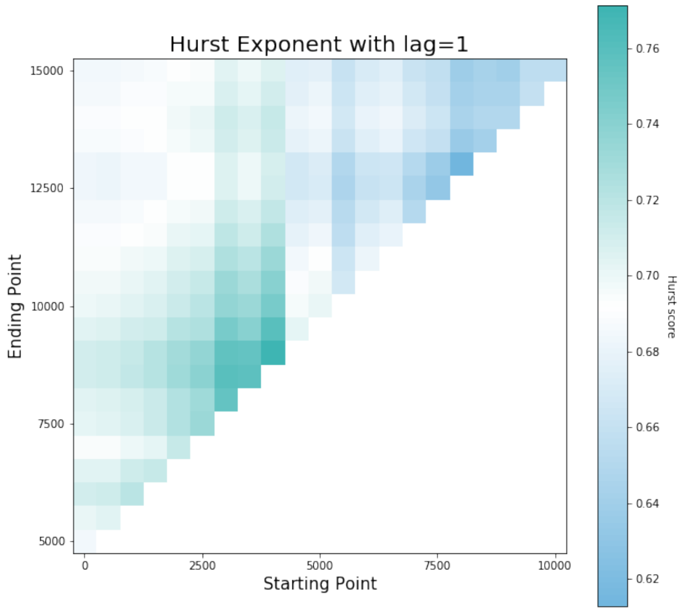


Figure 26: Change of H exponent of modelled price over different Lag

In addition, I apply the same algorithm on the modified modelled price I generated in the last section. I found both the Dow Jones industrials index — and the price series produced by the models in the previous section — seem to be characterised by the sort of persistence present in Hurst processes. They even share the similar pattern with the different starting and ending date. Both of the figures have lower H exponent when starting at the second half section of the price series. And both of them have high H exponent throughout the process. Put another way, financial markets show a systemic deviation from the dynamic predicted by the Gaussian distribution. In particular, they display persistence irrespective of the scale over which they are viewed: daily returns are correlated with future daily returns, monthly with monthly, and so on.

5 Conclusion

This report reproduces the BoE paper written by Nicola and Joseph (2013) and also points out the errors in it. Next, we improve the original model with adjusted results. Particularly, we modify the quantitative model that captures the interpretation of information by investors with different time horizons under FHM. By doing this, we can formalised predictions of prices which can be matched with some observed properties in the financial market. This report also offered a more sophisticated model to replicate the spikes in volatility, which the first model failed to cover. We summaries our finding of these three models in a table **Figure 21**. After that, it explains ”measuring” fractals with H exponent estimated from Range analysis to interpret the properties of ”self-similarity”.

As discussed in the paper, EMH fails to address the property of self-similarity, therefore FMH was proposed. FMH focuses on the role of market liquidity and heterogeneous interpretation by different investors with the same information. It helps us to explore the dynamics of financial markets and the factors that cause instability. Under normal market conditions, the interactions between long-term investors and short-term investors stabilised the markets, but reinforced the severity of adverse market event. However the fractal structure contains fragility to some extent. That means with only short term investor and momentum trader, or a change in long-term behavior, the market becomes unstable. Sometimes it can cause liquidity to evaporate, and in some cases cause panic selling and related markets crashes.

Since market price usually does not follow the normal distribution, Hurst exponent experiment with rescaled range analysis was used to measure the fractal structure. The interactions between investors increase the persistence in the fractal market for different time scales. Hence this nature of self-similarity depend on the the behaviours of different investors and the degree they interact.

In the end, FMH is crucial to evaluate the properties of self-similarity in financial market. To a practical point of view, it has important implications on securities regulation in maintaining stability in the market. To further discuss the application in effective securities regulation and a well-functioning financial systems will be the future work.