HOW TO THUE-COLOUR A 3-BRANCHED TREE

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1 Thue-Colouring T_n^3

Let T_n^3 be a tree having a root r of degree 3 and the three subtrees of r are each paths of length n. We will prove that T_n^3 has a nonrepetitivie 3-colouring. Our proof uses the Prouhet-Thue-Morse sequence and some of its properties, which we review here.

For each $i \in N$, define s_i as the parity of the binary representation of i. The sequence $s = s_0, s_1, \ldots$ is known as the Prouhet-Thue-Morse sequence. Alternatively, we can say that $s_0 = 0$ and, for every $i \geq 1$, $s_{2^i}, \ldots, s_{2^{i+1}-1} = \overline{s_0, \ldots, s_{2^i-1}}$. The first few bits of this sequence are:

$$01\ 10\ 1001\ 10010110\ 1001011001101001\ldots$$

The Prouhet-Thue-Morse string has the important property of being *overlap free*: It does not contain any block of the form $awawa$, where $a \in \{0,1\}$ and $w \in \{0,1\}^*$.

Using s, one can form an infinite sequence, r, over $\{0,1,2\}$ by taking the gaps between successive 0's:

$$s = 0\underbrace{\ 11\ }_{2}0\underbrace{\ 1\ }_{1}0\underbrace{\quad}_{0}0\underbrace{\ 11\ }_{2}0\underbrace{\quad}_{0}0\underbrace{\ 1\ }_{1}0\underbrace{\ 11\ }_{2}0\ldots$$

$$r = 2,1,0,2,0,1,2,1,0,1,2,0,2,1,0,2,0,1,2,0,2,1,0,1,\ldots$$

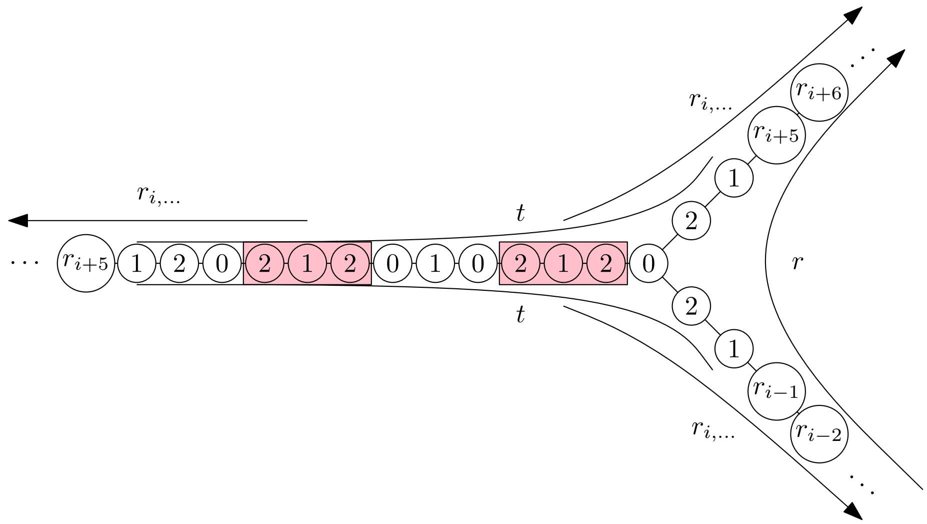
The overlap-freeness of s, in particular, the lack of any block of the form $0w0w0$ ensures that r is nonrepetitive. Another useful property of r is that, if it contains some block b then it contains infinitely many occurrences of b.

Note that the overlap-freeness of s also implies that r does not contain the block $2,1,2$ since this would mean that s contains the block

$$01\underbrace{10101}_{\text{overlap}}10\ .$$

Theorem 1. *T_n^3 has a nonrepetitive 3-colouring*

Proof. Let $t = 20\,212010212\,021$. Notice that t is nonrepetitive. Furthermore, if $r_{i,\ldots,i+4} = 12021$, then $t, r_{i+5}, r_{i+6}, \ldots$ is also nonrepetitive. Finally, if $r_{i,\ldots,i+3} = 1202$, then $\overleftarrow{t}, r_{i+4}, r_{i+5}, \ldots,$ is also nonrepetitive. Finally, we note that the block 12021 (and hence also 1202) appears in r so infinitely many such values of i exist. In particular, there is an $i > n + 20$ such that $r_{i,\ldots,i+4} = 12021$. Therefore, the following colouring of T_n^3 works:



Why does this work? Referring to the figure above, any path contained in the two right branches is coloured with a substring of r, so it is coloured nonrepetitively. Therefore, if there is a path that forms a repetition, it must use some of the left branch. The path must use some part of the left branch other than the trailing 120 since otherwise the colour sequence we obtain also appears in r. For the same reason, the path must use some part of the left branch to the right of 12021.

Therefore, any repetitively coloured path must include the colour sequence 212 that appears in t. Some straightforward verification (that we have to do anyway to ensure that t is nonrepetitive) then shows that the path must include both occurrence of 212 in t. Since the block 212 does not appear anywhere else in r, there must be one occurrence of 212 in each of the first and second halfs of the path, which leaves only these possibilities:

$$\begin{array}{cccccc|cccccc} 1 & 2 & 0 & 2 & 1 & 2 & 0 & 1 & 0 & 2 & 1 & 2 \\ 2 & 0 & 2 & 1 & 2 & 0 & 1 & 0 & 2 & 1 & 2 & 0 \\ 0 & 2 & 1 & 2 & 0 & 1 & 0 & 2 & 1 & 2 & 0 & 2 \\ 2 & 1 & 2 & 0 & 1 & 0 & 2 & 1 & 2 & 0 & 2 & 1 \end{array},$$

none of which is a repetition. □

2 Remarks

Unfortunately, because the sequence t is so long, this proof doesn't look like it will generalize to arbitray trees. If we continue with the current line of argument, we may be able to prove that any (binary) tree that has had all its edges subdivided k times can be (facially) nonrepetitively 3-coloured.