FOURTH WORKSHOP ON GEOMETRY AND GRAPHS: OPEN PROBLEMS

Bellairs 2016

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Abstract. A collection of open problems posed by participants of and studied at the Fourth Workshop on Geometry and Graphs.

Contents

1 A Grid Puzzle

Pat Morin

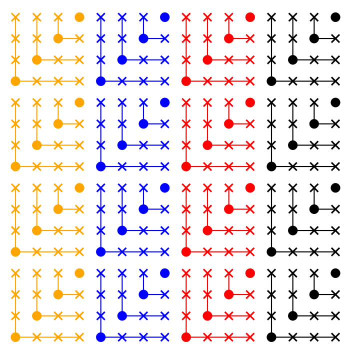
Consider the following game/puzzle played on the $n \times n$ integer grid $G = \{(i,j) : i,j \in \{1,\ldots,n\}\}$. Initially, all points in G are marked as *live*.

The game proceeds in n rounds. During the ith round, the player selects a monotonically increasing set, P_i, of live points in G.[1] Each point (x,y) selected in P_i *kills* the L-shape $\{(x,y') : y' \geq y\} \cup \{(x',y) : x' \geq x\}$ so that these points are no longer live in subsequent rounds.

The game ends after n rounds and the player's score is $\sum_{i=1}^{n} |P_i|$.

Open Problem 1. What is the maximum score a player can earn in this game?

By playing a cluster of $\sqrt{n}$ points in each round starting with a cluster in the top-right corner, a player can earn a score of $\Omega(n^{3/2})$:



This seems optimal, but I've been unable to prove it.

History: this game models a combinatorial geometry problem posed by Braß (2004) of determining the maximum number of triangles one can draw on a convex point set before creating a pair of triangles that (a) share an edge and cross () or (b) share a vertex but whose edges oppposite the shared vertex are "nested" ().

2 Facial Nonrepetitive Colouring of Trees

Vida Dujmović

A non-empty even-length sequence S is a *repetition* if it is the concatenation of two equal non-empty sequences; in other words, S is a repetition if $S = RR$ for some non-empty sequence R. A *block* in a sequence is a non-empty contiguous subsequence. A sequence S is *nonrepetitive* if none of its blocks is a repetition. (Nonrepetitive sequences are also sometimes called *square free* since they have no block of the form $RR = R^2$.)

[1] Here monotonically increasing means that each point in P_i has a unique x-coordinate and y-coordinate and, for every $(x_1,y_1),(x_2,y_2) \in P_i$, $x_1 < x_2 \Leftrightarrow y_1 < y_2$.

A classic result of Thue, from 1906, states that there are arbitrarily long nonrepetitive sequences on an alphabet of size 3. This notion has been generalized to graph colourings, where we say that a colouring of a graph G is *nonrepetitive* if, for every (simple) path, P, in G, the sequence of P's colours is not a repetition.

This question asks about nonrepetitive colourings of trees. It is known that every tree has a nonrepetitive 4-colouring and that this is optimal for some trees. Here is a warm-up question, that would only need a slight generalization of Thue's result:

Open Problem 2. Let T_n^3 be the tree having one vertex of degree 3 attached to three paths of length n. Does T_n^3 have a nonrepetitive 3-colouring?

Update: We managed to prove that T_n^3 does indeed have a nonrepetitive 3-colouring, though the proof does not look like it has much chance of generalizing. (If you're not worried about poisoning your own thoughts on this problem, see `http://cglab.ca/~morin/misc/bb2016/drafts/starcolour.pdf`.)

A *facial path* in an embedded graph, G, is a path that appears as a block in the facial walk of some face of G. A *facial nonrepetitive colouring* of G is a colouring where, for each facial path, P, of G, the sequence of colours encountered in P is not a repetition. It is known, for example, that every plane graph has a facial nonrepetitive 22-colouring.

Open Problem 3. Does every (ordered) tree have a facial nonrepetitive 3-colouring?

Note that the facial nonrepetitive 22-colouring of plane graphs uses nonrepetitive colouring of trees as a building block (twice), so solving Open Problem 3 would give a facial nonrepetitive 20-colouring of plane graphs.

3 Optimal Isotopic Embedding

Günter Rote

We are given a (small) undirected *guest graph* $G = (V_0, E_0)$ and a larger undirected *host graph* $H = (V, E)$. Both graphs are embedded in the plane. We look for an isotopic embedding from G to H with a given vertex assignment $V_0 \mapsto V$. This means that the edges $e \in E_0$ have to be mapped to edge-disjoint paths $p(e)$ in H such that the resulting image of $p(G)$ is obtained from G by a deformation of the plane. The face structure, orientation, etc. of G must be preserved, except that different edges $p(e)$ and $p(e')$ are allowed to "touch" (but not cross) at vertices. (As an alternative, we might also ask for vertex-disjoint embeddings of edges.)

In addition, we are given a nonnegative cost function $c_e \colon E \to R$, separately for each edge $e \in E_0$ that is to be embedded. The objective is to minimize the total cost $\sum_{e \in E_0} \sum_{f \in p(e)} c_e(f)$.

What is the complexity of this problem for small fixed graphs G? For example, when G is a triangle?

When $|V_0| = 2$ and G consists just of parallel edges,

I can model the problem as a dual network flow problem and solve it in polynomial time.

4 Straightening Monotone Drawings

Günter Rote

We are given a plane drawing of a graph where every edge is both x-monotone and y-monotone.

Open Problem 4. Can such a drawing always be straightened in such a way that the edges are x-monotone and y-monotone in the same sense as before?

It is know than it can be done with at most one bend per edge.

5 Free Edge in Planar UDGs

Ahmad Biniaz

A *free edge* in a unit disk graph is an edge which is not intersected by other edges. There are some UDGs that have no free edge. The question is:

Open Problem 5. Does any planar UDG have a free edge?

(Note the qualifier *planar* (as opposed to *plane*) in the preceding problem.)

6 Plane Red Blue Matchings

Ahmad Biniaz

We have a set R of red points and a set B of blue points in the plane such that $(R \cup B)$ is in general position and $|R|+|B|$ is a big number. We want to match the red points together and the blue points together so that the union of these two matchings is non-crossing. It may not be possible to compute a perfect non-crossing matching even if both $|R|$ and $|B|$ are even numbers. In 2001, Dumitrescu and Kaye [1] showed that it is possible to match 85.71% of the total number of points. They also showed there are large point sets which do not admit a plane matching which matches more than 98.95% of the points.

Open Problem 6. Improve these lower and upper bounds.

Adrian Dumitrescu, Rick Kaye: Matching colored points in the plane: Some new results. Comput. Geom. 19(1): 69-85 (2001)

7 Grid Point Sets With No Convex n-gons

Ruy Fabila

Open Problem 7. Does there exist, for every $\gamma \geq 1$ and every $n > 0$, a suitable constant $\varepsilon(\gamma) > 0$ and a set S of n points in general position in the plane with the following property: S has positive integer coordinates not exceeding n^γ, and S does not contain a convex $\varepsilon(\gamma)\log_2(n)$-gon?

Open Problem 8. Does the following holds for every constant $\gamma \geq 0$? Every sufficiently large set of n points in the general position in the plane with positive integer coordinates that do not exceed n^γ contains an empty convex 7-gon.

8 Packing and Covering K_4's in a Graph

Abdul Basit

Given a graph G, we would like to make G K_4-free by deleting a small number of edges. An obvious approach to do this is the following: Let F be a maximal family of pairwise edge disjoint K_4's in G. Clearly we need to delete at least one edge from every element of F, and it suffices to delete all edges from every element of F. Let $\nu(G)$ be the maximum size of an edge-disjoint family of K_4's in G, and $\tau(G)$ be the minimum number of edges required to delete every K_4 in G. The above argument shows that $\nu(G) \le \tau(G) \le \binom{4}{2}\nu(G)$. The lower bound is sharp, as seen by taking G to be the disjoint union of K_4's. Taking G to be K_8 shows that $\tau(G)$ can be as large as $3.5\nu(G)$.

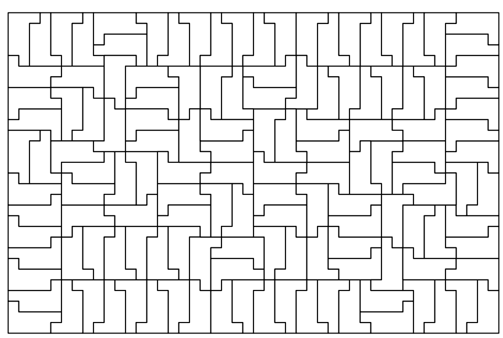
Open Problem 9. Does there exist a constant c such that $\tau(G) \le (\binom{4}{2} - c)\nu(G)$?

9 Minimal Rectangle Tilings

Andrew Winslow

Klarner [1969] defined the *order* of a polyomino to be the smallest number of copies that tile a rectangle. Polyominoes with order $4s$ are known for all s [Golomb 1989], as well as a small number of other even orders [Marshall 1997]. It was proved that no polyomino has order 3 [Stewart, Wormstein 1992] using a long case analysis, but results for larger odd n remain open:

Open Problem 10. Does there exist a polyomino with odd order $n > 3$?



The smallest rectangle tiled by this polyomino (of order 138).

A similar problem has been considered for general polygons. For every prime number n, a square can be tiled using n congruent copies of a polygon using rectangles arranged in a single column. In the 1980s, Danzer conjectured that no other tilings exist; this conjecture was confirmed for $n = 3$ [Maltby 1994] and $n = 5$ for the special case of convex polygons [Yuan, Zamfirescu, Zamfirescu 2016].

Open Problem 11. Does there exist a tiling of a square using a prime number of congruent copies of a non-rectangular polygon?

10 Approximation Algorithms for Slope Number

David Eppstein

The slope number of a graph is the minimum number of slopes of edges needed in a drawing of the graph, with the edges drawn as straight line segments, but allowing edges to cross. It is known that the slope number is at least $\Delta/2$, where Δ is the maximum degree, but that there exist bounded-degree graphs with unbounded slope number. It is also known that planar graphs of bounded degree have bounded slope number (exponential in the degree) and that testing whether the slope number is two is NP-complete. It remains NP-complete for testing whether a planar graph has a planar drawing with slope number two.

Open Problem 12. What can we say about approximating the slope number, either of all graphs, of all planar drawings of planar graphs, or of some interesting subclasses of graphs. Are there graph classes with unbounded slope number for which we can approximate this number accurately? Alternatively, can we prove any inapproximability results for slope number?

11 Plane spanners of points in convex position

Michiel Smid

Let S be a finite and non-empty set of points in the plane that is in convex position, and let $CH(S)$ denote the convex hull of S. For two points p and q of S,

- let $|pq|$ denote their Euclidean distance and
- let $|pq|_{\partial CH(S)}$ denote their shortest-path distance along the boundary of $CH(S)$.

We say that a point p of S is *t-good in S*, if for each point q of S,

$$|pq|_{\partial CH(S)} \leq t|pq|.$$

Define

$$t^* = \inf\{t : \begin{array}{l}\text{each finite and non-empty set of points in convex position} \\ \text{has at least one } t\text{-good point}\}.\end{array}$$

For any $t > t^*$, the following algorithm constructs a plane t-spanner for any finite and non-empty set S of points in convex position:

- Initialize $E = \emptyset$.
- While $|S| \geq 4$:
  - Let p be any t-good point in S.
  - Let q and r be p's neighbors along $CH(S)$.
  - Add the edges pq, pr, and qr to E.

- Let $S = S \setminus \{p\}$.

- Add the edges of $CH(S)$ to E.
- Return the graph with edge set E.

Open Problem 13. What is the value of t^*?

Let pq be a diametral pair in the point set S. We have recently shown that both p and q are 1.88-good. In fact, this remains true if we take for pq a sufficiently good approximation to the diameter of S.

Consider the set S consisting of the three vertices of an equilateral triangle, together with the three points at the midpoints of the edges. It is easy to verify that none of the points in S is $(\sqrt{3} - \epsilon)$-good for any $\epsilon > 0$.

Thus, the currently best known bounds for t^* are

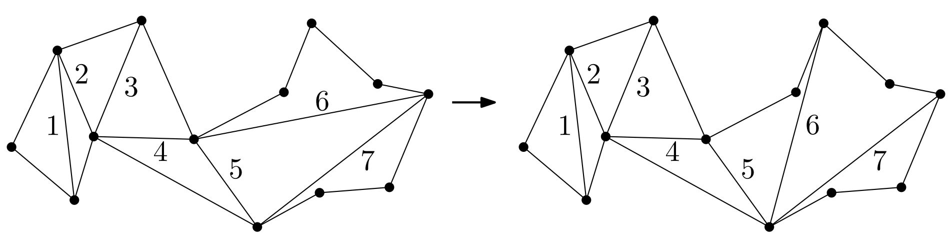
$$\sqrt{3} \leq t^* \leq 1.88.$$

As a result, we obtain, for any set of points in convex position, a plane spanner with stretch factor 1.88. The previously best known stretch factor for plane spanners was 1.998, which is attained by the Delaunay triangulation and, in fact, also holds for point sets that are not in convex position.

12 Flipping Edge-Labelled Pseudotriangulations

Sander Verdonschot

A pseudo-triangle is a simple polygon with three convex interior angles, called *corners*, connected by reflex chains. Given a simple polygon P, a *pseudo-triangulation* of P is a subdivision of its interior into pseudo-triangles, using only the vertices of P. Pseudo-triangulations support a flip operation analogous to the one for triangulations. It is known that any pseudo-triangulation of P can be transformed into any other by flips. An *edge-labeled* pseudo-triangulation of P is one in which every interior edge has a unique label. Flipping an edge transfers the label from the removed edge to the new edge.



Open Problem 14. Can every edge-labeled pseudo-triangulation be transformed into every other via flips? If not, can we decide which pairs can be transformed into each other and which cannot?

13 Constant-Degree Spanners

Mirela Damian

A graph is a spanner if a shortest path between any pair of points in the graph is no longer than a constant times the Euclidean distance between the two points. The Yao graph Y_k and the Theta graph Θ_k are defined for a given point set and a fixed integer $k > 0$. The space around each point is divided into k cones of equal angle, and each point is connected to a nearest neighbor in each cone. The difference between Yao and Theta graphs is in the way the nearest neighbor is defined: Yao graphs minimize the Euclidean distance, whereas Theta graphs minimize the *projective* distance, measured from the cone origin to the orthogonal projection of the neighbor on the cone bisector.

The graph *half*-Θ_6 is composed of the edges of Θ_6 that lie in non-consecutive cones. Thus Θ_6 is the union of two edge-disjoint half-Θ_6 graphs. Each half-Θ_6 graph is a triangular distance Delaunay graph and is a 2-spanner of the complete Euclidean graph (Bonichon et al 2010).

Each of Θ_k and Y_k has out-degree k, but in-degree $n-1$. To reduce the in-degrees, the Yao-Yao graph $YY_k \subseteq Y_k$ is obtained from Y_k by discarding from each cone all incoming edges, with the exception of a shortest one; ties are broken arbitrarily. The graphs $\Theta\Theta_k \subseteq \Theta_k$ and half-$\Theta\Theta_6 \subseteq$ half-Θ_6 are similarly defined.

Open Problem 15. At last year's workshop we showed that $\Theta\Theta_6$ is not a spanner (writeup available online at `http://csc.villanova.edu/~mdamian/tt6.pdf`). Is $\Theta\Theta_6$ a spanner for points in convex position? What about half-$\Theta\Theta_6$ (this would have the advantage of being planar)? These problems might not be too difficult to tackle.

Open Problem 16. The smallest known k for which $\Theta\Theta_k$ and YY_k are spanners is $k = 30$. Is there a value $k < 30$ for which $\Theta\Theta_k$ and/or YY_k is a spanner? The case $k = 12$ appears to be the simplest (just split the Θ_6 cones in half), because we might be able to use the planarity (and low spanning ratio) of half-Θ_6.

14 Universal Drawings of K_m

Fabrizio Frati

Open Problem 17. For every n we ask: What is the smallest $m = f(n)$ such that there exists a drawing Γ of the complete graph K_m such that, for every n-vertex planar graph G, drawing Γ contains a planar drawing of G?

This problem is a purely topological version of the well-known "universal point set" problem for planar graphs. Indeed, if the drawing of K_m were required to be straight-line, then the problem would be exactly the same as finding the cardinality of a smallest set U of m points such that every n-vertex planar graph admits a straight-line drawing on an n-vertex subset of U – the "universal point set" problem for planar graphs. In Problem 17, the drawing Γ needs not to be straight-line.

The problem clearly derives known upper bounds from the universal point set problem, so:

- The answer is $m = n$ if we are only interested in finding a drawing Γ that contains a planar drawing of every n-vertex outer-planar graph or every n-vertex tree.
- The answer is $m = O(n^{1.5} \log n)$ if we are only interested in finding a drawing Γ that contains a planar drawing of every n-vertex series-parallel graph or every n-vertex planar 3-tree.
- The answer is $m = O(n^2)$ in general.

No lower bound greater than n seems to emerge from the literature. A strong relationship is also evident with other problems such as the universal point set problem for polyline drawings of planar graphs.

15 Finding the Least Area Rectangle Containing k Points

Luis Barba

For this problem we consider only parallel axis rectangles.

Open Problem 18. Given a set P of n points in the plane and a parameter $0 \le k \le n$, how fast can we find the smallest area rectangle that contains exactly k points of P?

It is known that we can solve the problem in $O(n \log n + (n-k)^3)$ time (Ahn et al 2011). This algorithm achieves optimal performance when k is really large. However, for values like $k = \alpha n$, the running time of the algorithm is cubic.

In another approach, we can think of this problem as an LP-type problem of dimension 4. In this case, a technique by Chan (Chan 2005) for linear programming with outliers allows us to solve the problem in $O(n \log n + (n-k)^{11/4} n^{1/4} \log^{O(1)} n)$ expected time, which is also cubic for $k = n/2$, but provides a different way of looking to the problem. Is there any sub-cubic algorithm to solve Problem 18?

The following problem is related to the above question and could be useful as a subroutine if the preprocessing time is near linear.

Open Problem 19. Given an integer $k > 0$, preprocess a set P of n points in the plane such that given a point $x \in \mathbb{R}^2$, we can find the smallest area rectangle having x as a corner and containing exactly k points of P.

16 Orthogeodesic Point Set Embeddings of Trees

Michael Hoffmann

Given a tree T on n vertices, we want to embed T on an $N \times N$ grid, for some $N \ge n$. In fact, we consider a more restricted setting where possible locations for vertices are specified in form of a set $P \subset \mathbb{R}^2$ of N points. Denote by Γ_P the arrangement induced by all horizontal and vertical lines that pass through a point from P. To embed a graph on the *grid defined by P*, vertices are mapped to points from P and edges are mapped to arcs that are polylines that bend at vertices of Γ_P only. A point set P is in *general position* if no two points have the same x- or y-coordinate.

A common theme in the study of metric graph embeddings is the desire to control the length of edges. For instance, can every edge be realized as a shortest path? In the Euclidean plane, we arrive at straight line embeddings. A natural counterpart of these embeddings on the grid is called an *orthogeodesic embedding*. In an orthogeodesic embedding, every edge is realized as an *orthogeodesic arc*, that is, a polyline that consists of axis-parallel line segments and forms a shortest path in the ℓ_1 metric. An *L-shaped* arc is an orthogeodesic arc with exactly one bend.

Clearly an orthogeodesic plane embedding can exist only for trees of degree at most four. As it is straightforward to find orthogeodesic embeddings for paths, the only interesting cases are maximum degree three and maximum degree four. For integers n and Δ, denote by $t_\Delta(n)$ the minimum number such that for every set P of $t_\Delta(n)$ points in general position, every tree on n vertices with degree at most Δ admits an orthogeodesic plane embedding on the grid defined by P. We know that $t_4(n) < 11n/8$ and $t_3(n) < 9n/8$ (to appear at EuroCG).

Open Problem 20. Prove or disprove $t_3(n) = n$.

Open Problem 21. Give a non-trivial lower bound for t_3 or t_4. Nothing seems to be known, even if the embedding is restricted to use L-shaped arcs only.

17 Embedding Planar Graphs into $\mathbb{R}^2$

Tasos Sidiropoulos

For an embedding $f : V(G) \to \mathbb{R}^2$ of a graph G onto $\mathbb{R}^2$, the *distortion* of f is defined as:

$$\mathrm{distortion}(f) = \left(\max_{x \neq y \in V(G)} \frac{\|f(x) - f(y)\|_2}{d_G(x,y)} \right) \cdot \left(\max_{x \neq y \in V(G)} \frac{d_G(x,y)}{\|f(x) - f(y)\|_2} \right) .$$

It is known that every n-vertex planar graph G has an embedding with distortion $O(n)$ (even in $\mathbb{R}^1$) and that some planar graphs require distortion $\Omega(n^{2/3})$ ($\Omega(n^{3/4})$ for weighted planar graphs).

Open Problem 22. Does every planar graph have an embedding with distortion $o(n)$?