An Efficient Algorithm for Partial Order Production


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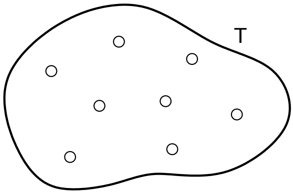
Sorting by Comparisons

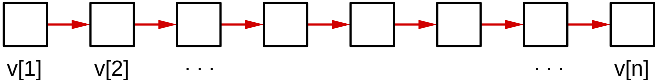
Input: a set T of size n, totally ordered by $\leqslant$

Goal: place the elements of T in a vector v in such a way that

$$v[1] \leqslant_T v[2] \leqslant_T \cdots \leqslant_T v[n]$$

after asking a min number of questions of the form "is $t \leqslant_T t'$?"





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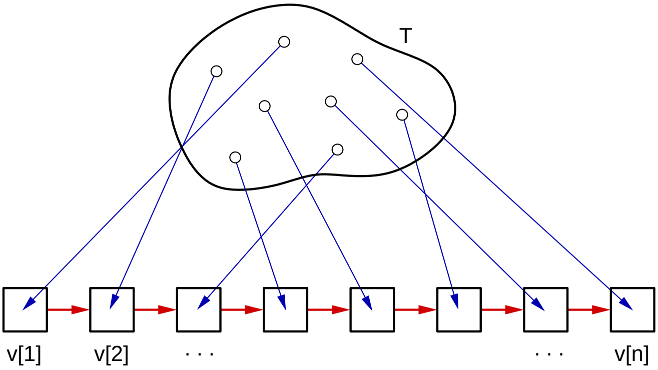


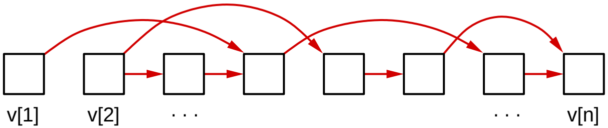
Partial Order Production ("Partial Sorting")

Input: a set T of size n, totally ordered by $\leqslant_T$
a partial order $\leqslant_P$ on the set of positions $[n]$

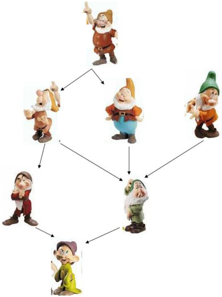
Goal: place the elements of T in a vector v in such a way that

$$v[i] \leqslant_T v[j] \quad \text{whenever } i \leqslant_P j$$

after asking a min number of questions of the form "is $t \leqslant_T t'$?"




HARDWOOD
LUMP
CHARCOAL

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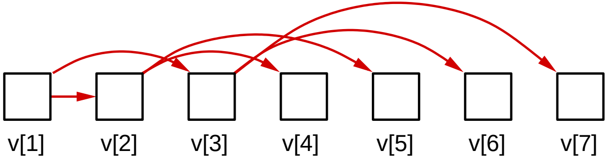
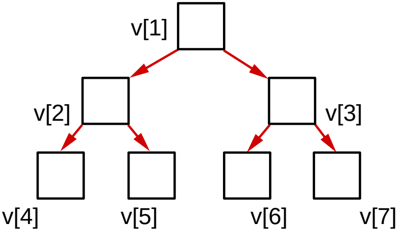
Particular Cases (1/2)

Heap Construction

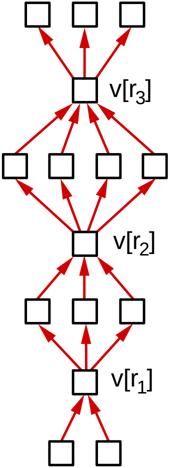


or



Particular Cases (2/2)

Multiple Selection



Find the elements of rank $r_1, r_2, \ldots, r_k$

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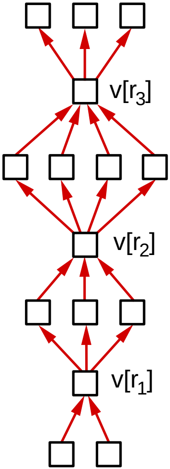

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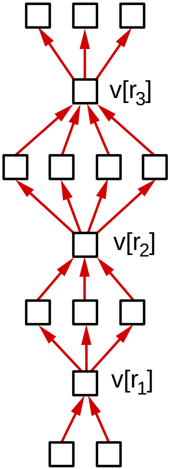

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$\exists$ near-optimal algorithm (Kaligosi, Mehlhorn, Munro and Sanders, 05)

Worst Case Lower Bounds

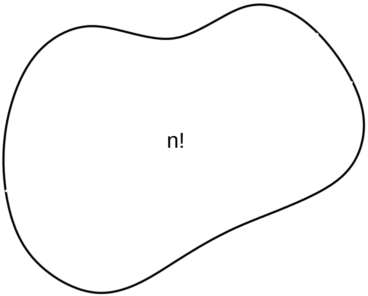
Well known fact. *For Sorting by Comparisons:*

$$\text{worst case } \#\text{comparisons} \geq \lg n!$$

Fact. (Schönage 76, Aigner 81) *For Partial Order Production:*

$$\text{worst case \#comparisons} \geq \underbrace{\lg n! - \lg e(P)}_{=:\, LB}$$

where $e(P) := \#$ linear extensions of P



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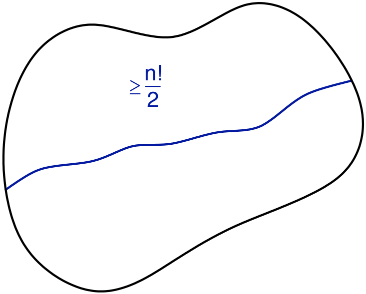

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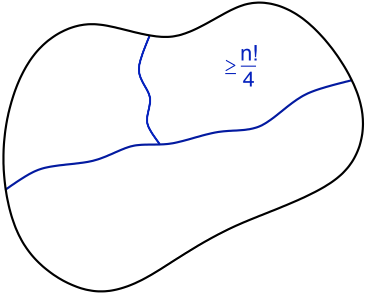

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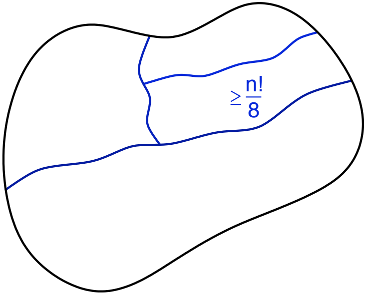

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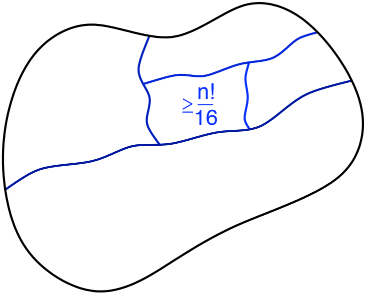

$$|\text{leaf set}| \leq e(P) \Longrightarrow \#\text{comparisons} \geq \lg \frac{n!}{e(P)} = LB$$

Problem History

1976 Schönage defined POP problem

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1989 Yao solved Saks' conjecture, stated open problems

Our Result

There exists an algorithm for the POP problem with

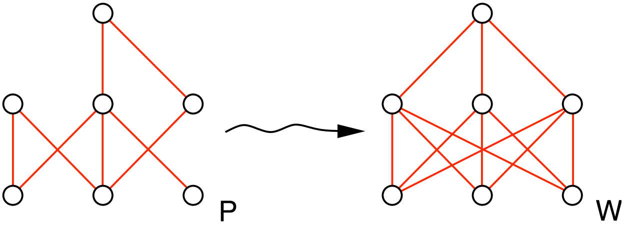
- preprocessing time $O(n^3)$
- worst case $\#$comparisons $= LB + o(LB) + O(n)$

Improvements over Yao's algorithm:

- overall complexity is polynomial
- smaller number of comparisons

A Simple Plan

1. Extend the target poset P to a weak order W
2. Solve the problem for W using Multiple Selection algorithm



Key Tool: the Entropy of a Graph

The entropy of $G = (V, E)$ equals:

$$H(G) := \min_{x \in STAB(G)} -\frac{1}{n} \sum_{v \in V} \lg x_v$$

where $STAB(G) :=$ stable set polytope of G

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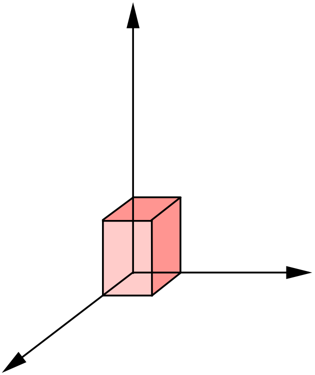
- Introduced in information theory by J. Körner (73)
- Graph invariant with lots of applications (mostly in TCS)
  - lower bounds for perfect hashing
  - lower bounds for monotone Boolean functions
  - sorting under partial information (Kahn and Kim 95)
  - ...

Exercise 1. Prove the following:

Lemma. (Kahn and Kim 95)

$$\underbrace{-n\,H(G)}_{=\lg Vol(Box)} \leq \lg \mathrm{Vol}(STAB(G)) \leq \underbrace{n \lg n - \lg n! - n\,H(G)}_{=\lg Vol(Simplex)}$$



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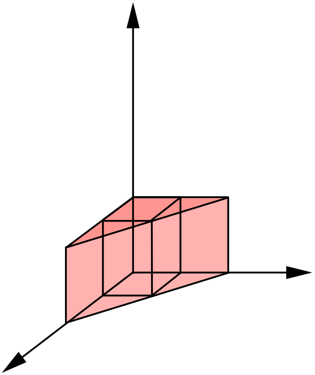

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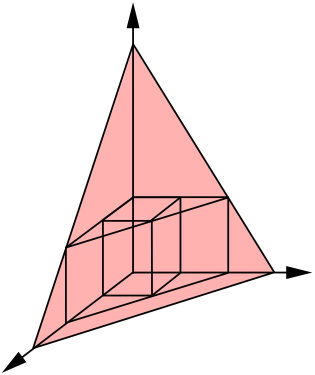


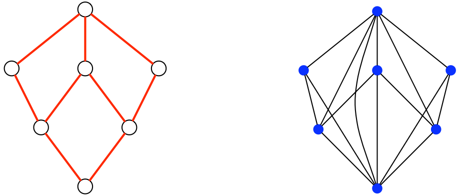
Exercise 2. Prove the min-max relation, for G perfect:

$$H(G) + H(\overline{G}) = \lg n$$

Comparability Graphs and Entropy

$G(P) :=$ comparability graph of target poset P

$H(P) := H(G(P))$



P $\qquad$ $G(P)$

Lemma. (Stanley 86) $\quad \mathrm{Vol}\Big(STAB\big(G(P)\big)\Big) = \dfrac{e(P)}{n!}$

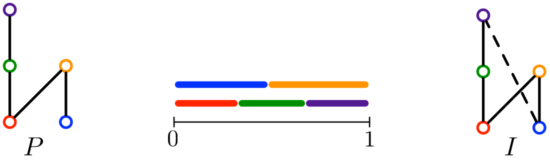
Corollary. $\quad n\,H(P) - n \lg e \le LB \le n\,H(P)$

Entropy of a Poset

A more intuitive definition!? (The magic of extended formulations.)

$\{(y_{v^-}, y_{v^+})\}_{v \in V}$ is *consistent* with P if

- $\forall v \in V$: (y_{v^-}, y_{v^+}) open interval $\subseteq (0, 1)$
- $v \leqslant_P w \implies y_{v^+} \leq y_{w^-}$



$$\boxed{H(P) := \min \left\{ -\frac{1}{n} \sum_{v \in V} \lg x_v \;\middle|\; \exists \{(y_{v^-}, y_{v^+})\}_{v \in V} \text{ consistent with } P \text{ s.t. } x_v = y_{v^+} - y_{v^-} \quad \forall v \in V \right\}}$$

$$H(P) = \tfrac{1}{n} \times \text{“information” in } P$$

Working out the Extended Formulation

Exercise 3. Check that

$$\begin{cases} x_v = y_{v^+} - y_{v^-} & \forall v \in V \\ y_{v^+} \leq y_{w^-} & \forall v \leqslant_P w \\ y_{v^\pm} \geq 0 & \forall v \in V \\ y_{v^\pm} \leq 1 & \forall v \in V \end{cases}$$

is an extended formulation of $STAB(G(P))$, that is, we get $STAB(G(P))$ when we eliminate the $y_{v^\pm}$ variables.

Weak Order Extensions $\to$ Colorings

Observation.

Every weak order extension W of P gives a coloring of $G(P)$

$$\Downarrow$$

Want: "good" coloring of $G(P)$

$$\boxed{W \text{ extends } P \Longrightarrow H(W) \geq H(P)}$$

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Entropy of distribution of class sizes should be as small as possible

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Entropy of distribution of class sizes should be as small as possible

Exercise 4. For W a weak order with classes of sizes $s_1, \ldots, s_k$:

$$H(W) = H\left(\frac{s_1}{n}, \ldots, \frac{s_k}{n}\right) = -\sum_{i=1}^{k} \frac{s_i}{n} \lg \frac{s_i}{n}$$

Greedy Colorings and Greedy Points

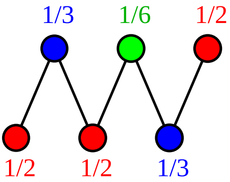
For G = perfect graph

Iterativelу remove a maximum stable set from G

$\rightsquigarrow$ sequence $S_1, S_2, \ldots, S_k$ of stable sets

- Gives greedy coloring (k colors, ith color class $= S_i$)
- Also gives greedy point:

$$\tilde{x} := \sum_{i=1}^{k} \frac{|S_i|}{n} \cdot \chi^{S_i} \quad \in \quad STAB(G)$$



Theorem. *Let G be a perfect graph on n vertices and denote by $\tilde{g}$ the entropy of an arbitrary greedy point $\tilde{x} \in STAB(G)$. Then*

$$\tilde{g} \leq \frac{1}{1-\delta}\left(H(G) + \lg \frac{1}{\delta}\right)$$

for all $\delta > 0$, and in particular (optimizing on δ)

$$\tilde{g} \leq H(G) + \lg H(G) + O(1)$$

Proof idea. Define

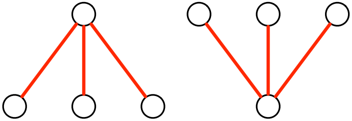
$$z_v := \frac{\delta}{n^{\delta}} \left(\frac{1}{\tilde{x}_v}\right)^{1-\delta}$$

check $z \in STAB(\overline{G})$. Now

$$H(G) = \lg n - H(\overline{G}) \geq \lg n - \left(-\frac{1}{n}\sum_{v \in V} \lg z_v\right)$$

$\square$

Colorings $\not\to$ Weak Order Extensions



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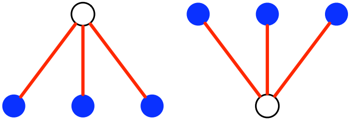

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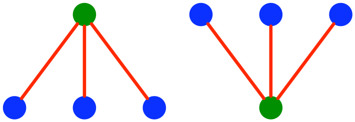

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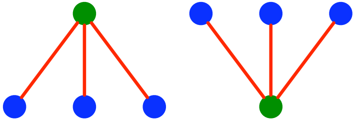
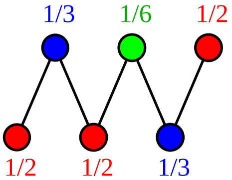


Weak order extensions of $P \to$ colorings of $G(P)$

$\not\leftarrow$

$\Longrightarrow$ need to "uncross" our greedy colorings

Uncrossing a Greedy Coloring



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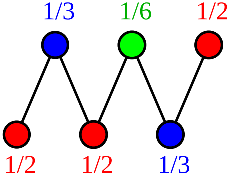

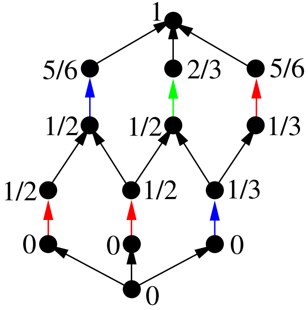

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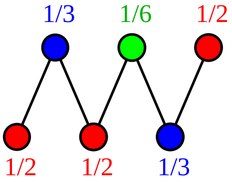

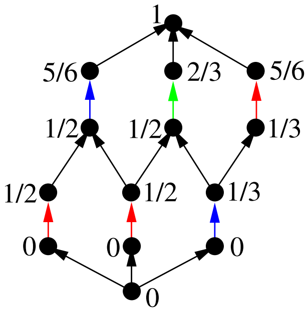

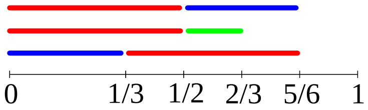

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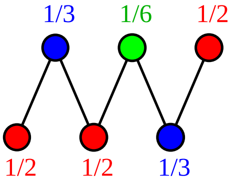

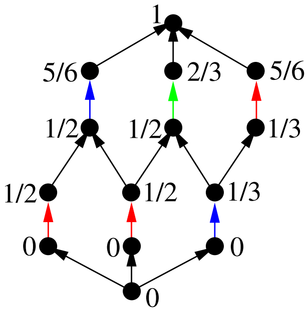

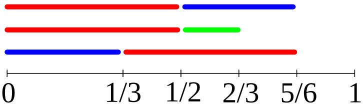

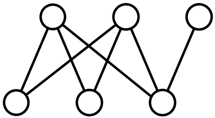

Main Steps of our Algorithm

1. $P \overset{greedy+DP}{\hookrightarrow} I$

2. $I \overset{greedy}{\hookrightarrow} W$

3. Use Multiple Selection algorithm of Kaligosi *et al.* on W

Theorem. *The algorithm above solves the POP problem, in $O(n^3)$ time, after performing at most*

$$LB + o(LB) + O(n)$$

comparisons

Sorting under partial information (without the ellipsoid algorithm)

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Sorting by comparisons under partial information

Input:

- a set $V = \{v_1, \ldots, v_n\}$, totally ordered by an unknown linear order $\leqslant$
- a poset $P = (V, \leqslant_P)$ compatible with $(V, \leqslant)$

Goal: Discover $\leqslant$ by making queries "is $v_i \leqslant v_j$?"

Objective function: #queries

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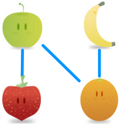

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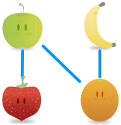

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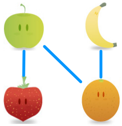

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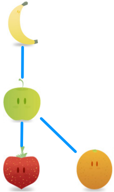

Sorting by comparisons under partial information

Input:

- a set $V = \{v_1, \ldots, v_n\}$, totally ordered by an unknown linear order $\leqslant$
- a poset $P = (V, \leqslant_P)$ compatible with $(V, \leqslant)$

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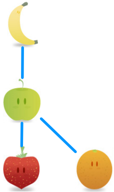

 OR  ?

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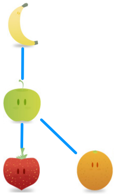

OR

?

Sorting by comparisons under partial information

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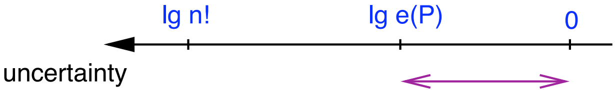
Objective function: #queries

Lower bound on #queries

$e(P) :=$ #linear extensions of P

Every algorithm can be forced to a #queries that is *at least*

$$\lg e(P)$$



$\Rightarrow$ Interested in algorithms that perform close to $\lg e(P)$

Results 

- **Known results**

	#queries	complexity
Fredman 1976	$\lg e(P) + 2n$	super-polynomial
Kahn & Saks 1984	$O(\lg e(P))$	super-polynomial
Kahn & Kim 1992	$O(\lg e(P))$	polynomial (ellipsoid alg.)

Results 

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- **Our contribution**: two ellipsoid-free algorithms

	#queries	complexity
Algorithm 1	$(1+\varepsilon)\lg e(P) + O_\varepsilon(n) \quad \forall \varepsilon > 0$	$O(n^{2.5})$
Algorithm 2	$O(\lg e(P))$	$O(n^{2.5})$

Bad news and a cure

Computing $e(P)$ is #P-complete — Brightwell & Winkler 1991

Bad news and a cure

Computing $e(P)$ is #P-complete — Brightwell & Winkler 1991

As Kahn & Kim 1992, use the *entropy* $H(\bar{P})$

Why?

- $\boxed{\lg e(P) = \Theta(nH(\bar{P}))}$ — Kahn & Kim 1992
- $H(\bar{P})$ can be "computed" in poly-time using the ellipsoid algorithm

Bounds

"additive" and "multiplicative"

$$nH(\bar{P}) \leq \lg e(P) + n \lg e$$

K&K 1992

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"additive" and "multiplicative"

$$nH(\bar{P}) \leq \lg e(P) + n \lg e$$ K&K 1992

$$\lg e(P) \leq nH(\bar{P}) \leq c \cdot \lg e(P)$$ K&K 1992

where $c = 1 + 7 \lg e \simeq 11.1$

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Rmk: Lower bound tight, but not upper bound

- K&K conjectured

$$nH(\bar{P}) \leq (1 + \lg e) \cdot \lg e(P)$$ $(1 + \lg e \simeq 2.44)$

Bounds

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Rmk: Lower bound tight, but not upper bound

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$$nH(\bar{P}) \leq (1 + \lg e) \cdot \lg e(P)$$ $(1 + \lg e \simeq 2.44)$

- We prove

$$nH(\bar{P}) \leq 2 \cdot \lg e(P)$$

(tight)

Exercise 1. Prove that

$$nH(\bar{P}) \leq 4 \cdot \lg e(P)$$

Hint: induct on n, then on $e(P)$. For $n = 2$, the result is trivial. Assume $n \geq 3$. For $e(P) = 1$, the result is trivial (P is a chain). Now consider an optimal set of intervals for P, and let $a \in V$ be such that the length of the interval for a (i.e., x_a) is maximum. If a is comparable to every $b \in V - a$ then $\lg e(P) = \lg e(P - a)$ and $nH(\bar{P}) = (n-1)H(\overline{P - a})$. Otherwise let $b \in V - a$ be such that the interval for b contains the midpoint of the interval for a. Then find a new poset P' on V with $\lg e(P) - \lg e(P') \geq 1$ and $nH(\bar{P}) - nH(\overline{P'}) \leq 4$, by carefully modifying the intervals for a and b.

K&K's algorithm

Key lemma:

$\exists$ incomparable pair a, b s.t.

$$\max\left\{nH(\bar{P}(a<b)), nH(\bar{P}(a>b))\right\} \leq nH(\bar{P}) - c,$$

where $c \simeq 0.2$

Algorithm:

1. Repeat:
   1.1 Compute $H(P)$ and optimal solution x^*
   1.2 Find good incomparable pair a, b using x^*
   1.3 Compare a and b
   1.4 Update P

$\#\text{steps} = O(nH(\bar{P})) = O(\lg e(P))$

Algorithms

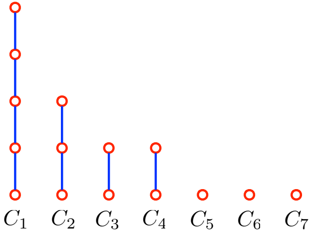
	#queries		complexity
Algorithm 1	$(1+\varepsilon)\lg e(P) + O_\varepsilon(n)$	$\forall \varepsilon > 0$	$O(n^{2.5})$
Algorithm 2	$O(\lg e(P))$		$O(n^{2.5})$

Algorithm 1: greedy + merge sort

Algorithm 2: greedy + "cautious" merge sort

Greedy

Greedy chain decomposition of $P \quad \rightarrow \quad U := C_1 \cup \cdots \cup C_k$



$$H(\bar{U}) = \lg n - H(U) = \sum_{i=1}^{k} -\frac{|C_i|}{n} \lg \frac{|C_i|}{n}$$

Greedy

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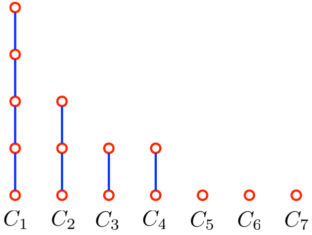


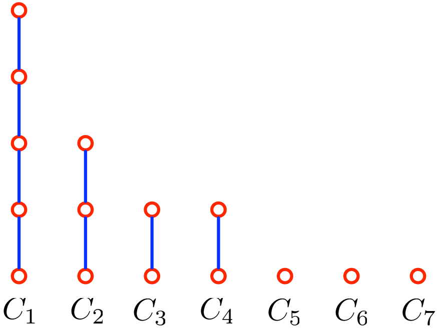
$$H(\bar{U}) = \lg n - H(U) = \sum_{i=1}^{k} -\frac{|C_i|}{n} \lg \frac{|C_i|}{n}$$

From perfectness of incomparability graph of P:

$$\boxed{H(\bar{U}) \leq (1+\varepsilon)H(\bar{P}) + O_\varepsilon(1) \qquad \forall \varepsilon > 0}$$

Algorithm 1

1. Compute greedy chain decomposition of P
2. Iteratively merge two smallest chains



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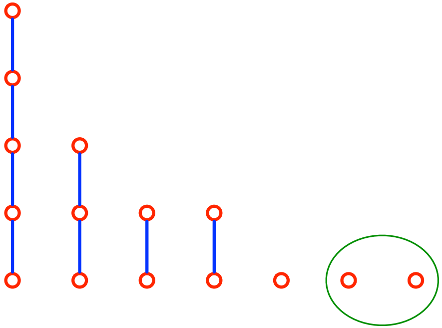

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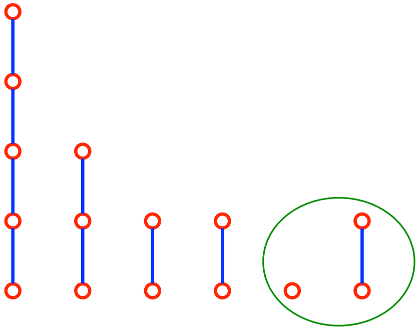

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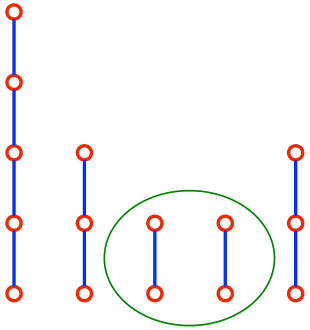

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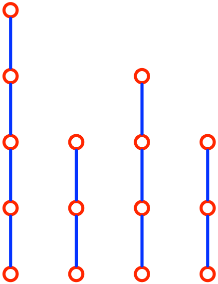
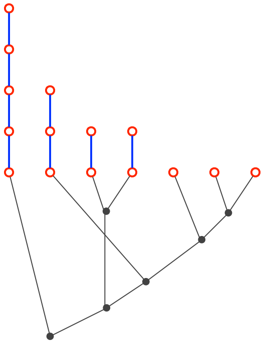


ETC.

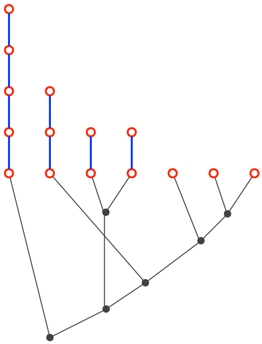
Algorithm 1



Huffman trees: average root-to-leaf distance in tree at most

$$\left(\sum_{i=1}^{k} -\frac{|C_i|}{n} \lg \frac{|C_i|}{n} \right) + 1 = H(\bar{U}) + 1$$

Algorithm 1



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$\Rightarrow \ldots \Rightarrow$ at most $(H(\bar{U}) + 1)n$ comparisons

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Huffman trees: average root-to-leaf distance in tree at most

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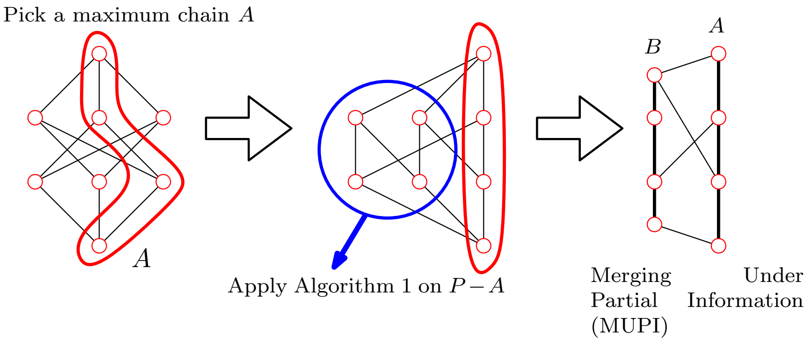
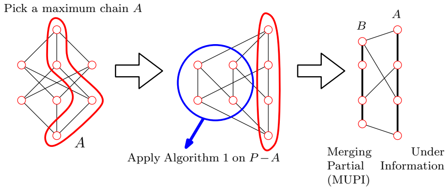
$\Rightarrow \ldots \Rightarrow$ at most $(H(\bar{U}) + 1)n$ comparisons

$$\begin{aligned} (H(\bar{U}) + 1)n &\leq (1 + \varepsilon)nH(\bar{P}) + O_\varepsilon(n) && \text{greedy} \\ &\leq (1 + \varepsilon)\big(\lg e(P) + n \lg e\big) + O_\varepsilon(n) && \text{K\&K's additive bd} \\ &= (1 + \varepsilon)\lg e(P) + O_\varepsilon(n) \end{aligned}$$

Algorithm 2

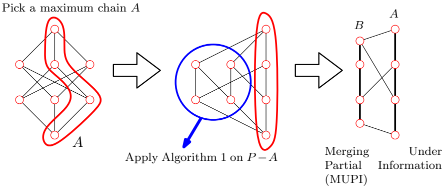

Pick a maximum chain A
A
Apply Algorithm 1 on $P - A$
B
A
Merging Under Partial Information (MUPI)

Algorithm 2



#comparisons in step 2 at most

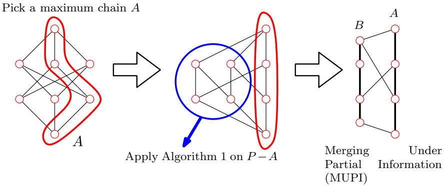
Algorithm 2



#comparisons in step 2 at most

$$(1+\varepsilon)\lg e(P-A) + O_\varepsilon(|P-A|)$$

Algorithm 2



#comparisons in step 2 at most

$$(1+\varepsilon)\lg e(P-A) + O_\varepsilon(|P-A|)$$

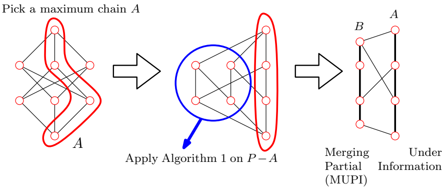
[Interlude] An easy lemma (take all intervals of length $x_v = \frac{1}{|A|}$):

$$H(\bar{P}) \geq -\lg \frac{|A|}{n}$$

$\Rightarrow |A| \geq 2^{-H(\bar{P})} n$

$\Rightarrow |P-A| \leq n\left(1 - 2^{-H(\bar{P})}\right) \leq \ln 2 \cdot nH(\bar{P})$ (using $1-2^{-x} \leq \ln 2 \cdot x$)

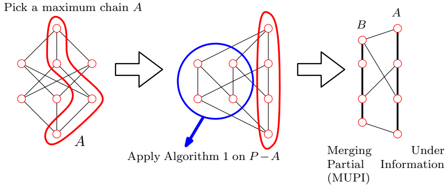
Algorithm 2



#comparisons in step 2 at most

$$
\begin{aligned}
& (1+\varepsilon)\lg e(P-A) + O_\varepsilon(|P-A|) \\
\leq\; & (1+\varepsilon)\lg e(P-A) + O_\varepsilon\left(\ln 2 \cdot nH(\bar{P})\right) \\
\leq\; & (1+\varepsilon)\lg e(P) + O_\varepsilon\left(\lg e(P)\right) \qquad \text{K\&K's multiplicative bd} \\
=\; & O_\varepsilon\left(\lg e(P)\right)
\end{aligned}
$$

Algorithm 2



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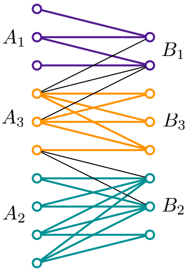
\le\ &(1+\varepsilon)\lg e(P) + O_\varepsilon\left(\lg e(P)\right) \qquad \text{K\&K's multiplicative bd} \\
=\ &O_\varepsilon\left(\lg e(P)\right)
\end{aligned}
$$

$\Rightarrow$ enough to solve MUPI = Merging under Partial Information!


MUPI

Merging under Partial Information

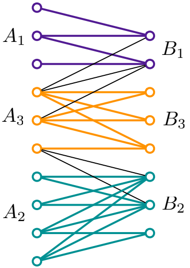
In that special case, the incomparability graph of P is bipartite



Körner and Marton 1988: optimal solution for entropy has "block structure"

Merging under Partial Information

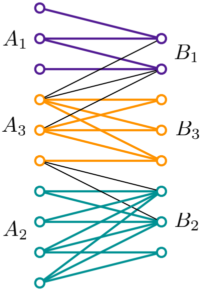
bipartite incomparability graphs $\Longrightarrow$ x^* defining $H(\bar{P})$ has an even nicer structure



- A_i interval of A, B_i interval of B, same ordering
- $x^*_v = (|A_i| + |B_i|)/n|A_i|$ whenever $v \in A_i$

Can compute $H(\bar{P})$ and x^* in time $O(n^2 \log^2 n)$

Solving MUPI - general ideas



Compute entropy and x^*

Apply Hwang-Ling merging algorithm on each component $A_i \cup B_i$ with $|A_i| \geq |B_i|$, in a certain order

Update x^* locally after each merging (details omitted)

Overall #comparisons is $\leq 3nH(\bar{P})$

Thank You!