Lower Bounds and Algorithms in the Linear Decision Tree Model

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MÉMOIRE PRÉSENTÉ EN VUE DE L'OBTENTION
DU DIPLÔME DU MASTER EN SCIENCES INFORMATIQUES
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1. Introduction

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- Linial's algorithm (Linial [4] + efficient implementation)

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- Sorting $X + Y$ (naive approach + Fredman [2])

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2. Decision Tree Model and ITLB

Definition (Decision Tree Model)

We are allowed to ask questions to an oracle "$\leq$" that are answered "yes" or "no". Each question asked to the oracle costs us a single unit. Every other operation can be carried out for free.

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input $\mathcal{S} = \{x_1, \ldots, x_n\}$

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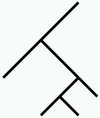

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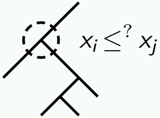

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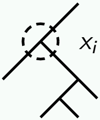

$x_i - x_j \leq^? 0$

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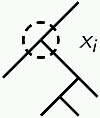

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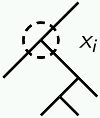

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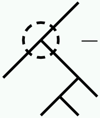

$-4 + x_i - x_j - 3x_k \leq^? 0$

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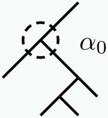

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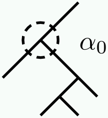

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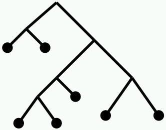

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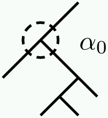

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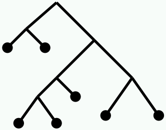

• = feasible solutions

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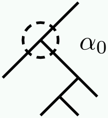

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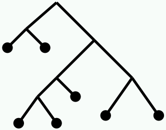

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$\mathrm{ITLB} = \log(\# \bullet)$

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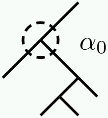

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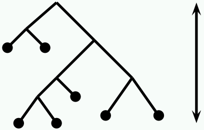

$h \geq \mathrm{ITLB}$

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Merging under Partial Information

input $\mathcal{P} = (\mathcal{S} = \{x_1, \ldots, x_N\}, <_{\mathcal{P}})$

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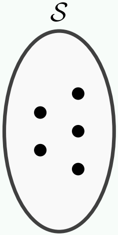

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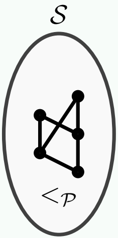

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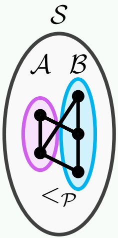

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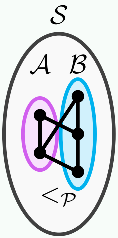
input $\mathcal{P} = (\mathcal{S} = \{a_1 <_\mathcal{P} \cdots <_\mathcal{P} a_m\} \cup \{b_1 <_\mathcal{P} \cdots <_\mathcal{P} b_n\}, <_\mathcal{P})$



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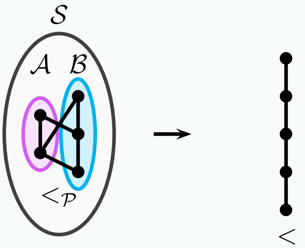

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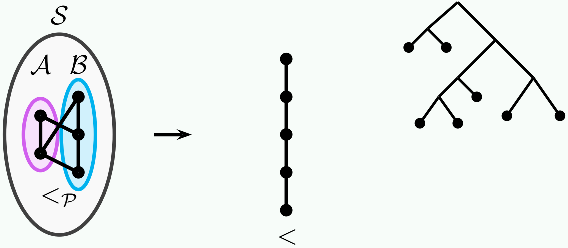

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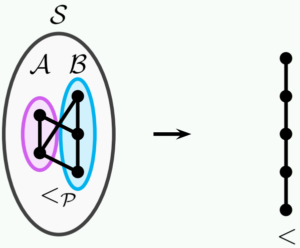

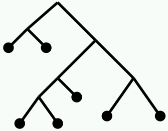

$\# \bullet = e(\mathcal{P}) \leq n!$

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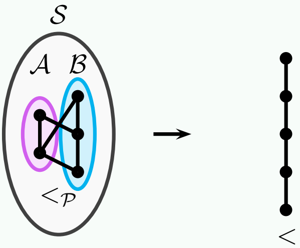

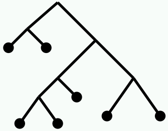

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$\mathrm{ITLB} = \log e(\mathcal{P})$

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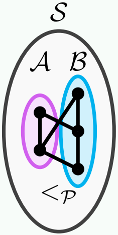

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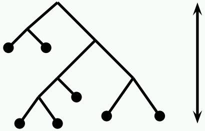

$h \geq \log e(\mathcal{P})$

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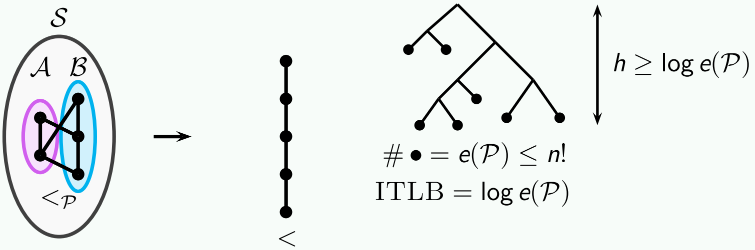

Theorem (Linial [4])

Given a poset $\mathcal{P} = (\{x_1, \ldots, x_N\}, <_{\mathcal{P}})$ covered by two chains $\mathcal{A}$ and $\mathcal{B}$, we can always find a query $x_i \overset{?}{<} x_j$ with $x_i \in \mathcal{A}, x_j \in \mathcal{B}$ such that the probability that $x_i < x_j$ lies in the interval $[1/3, 2/3]$.

3. Efficient Implementation of Linial's Algorithm

Using dynamic programming

$$\mathcal{A} = \{a_1 < \cdots < a_m\}, \mathcal{B} = \{b_1 < \cdots < b_n\}, \mathcal{P} = (\mathcal{A} \cup \mathcal{B}, <_{\mathcal{P}})$$

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$$e(\mathcal{P}) = \begin{cases} 1 & \text{if } |\mathcal{A}| = 0 \text{ or } |\mathcal{B}| = 0 \\ e(\mathcal{P} \setminus \{a_1\}) & \text{if } a_1 <_{\mathcal{P}} b_1 \\ e(\mathcal{P} \setminus \{b_1\}) & \text{if } b_1 <_{\mathcal{P}} a_1 \\ e(\mathcal{P} \setminus \{a_1\}) + e(\mathcal{P} \setminus \{b_1\}) & \text{if } a_1 \text{ and } b_1 \text{ are incomparable in } \mathcal{P} \end{cases}$$

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$e(\mathcal{P})$	$e(\mathcal{P} \setminus \{b_1\})$	$e(\mathcal{P} \setminus \{b_1, b_2\})$	$\cdots$	1
$e(\mathcal{P} \setminus \{a_1\})$	$\ddots$			$\vdots$
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$e(\mathcal{P})$	$e(\mathcal{P}(b_1 < a_1))$	$e(\mathcal{P}(b_2 < a_1))$	$\cdots$	1
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$e(\mathcal{P}(a_2 < b_1))$		$\ddots$		$\vdots$
$\vdots$			$\ddots$	$\vdots$
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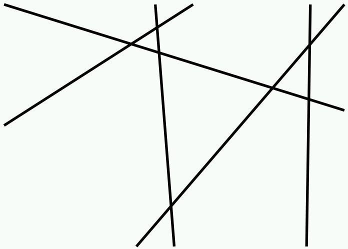
4. Application of Meiser's Algorithm

Point Location in an Arrangement of Hyperplanes



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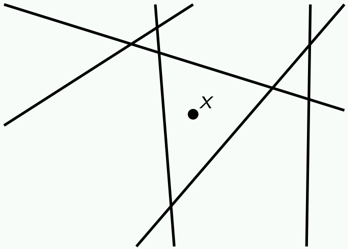

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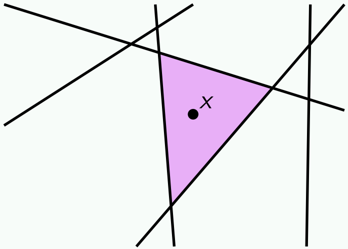

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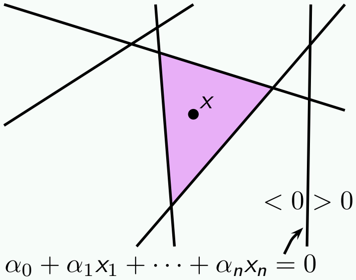

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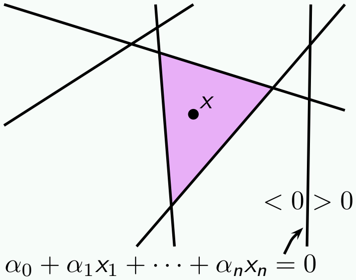
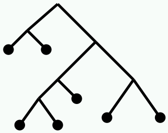
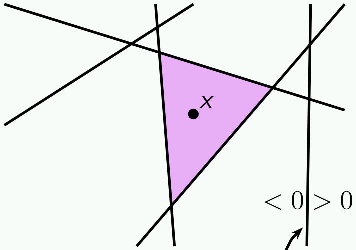


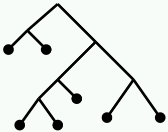


4. Application of Meiser's Algorithm

Point Location in an Arrangement of Hyperplanes



$\alpha_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n = 0$



$\# \bullet = \Phi_n(m) = O(m^n)$

4. Application of Meiser's Algorithm

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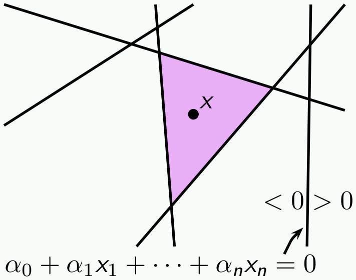

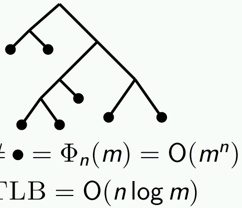

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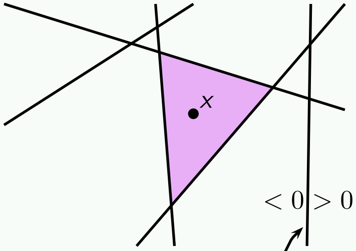

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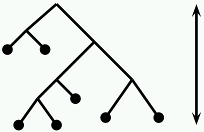

$\# \bullet = \Phi_n(m) = O(m^n)$

$\mathrm{ITLB} = O(n \log m)$

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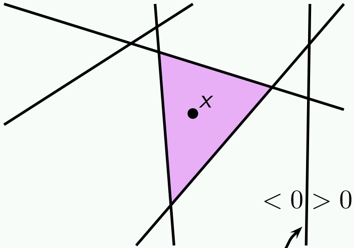

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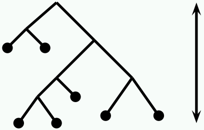

$h = \Omega(n \log m)$

$\# \bullet = \Phi_n(m) = O(m^n)$

$\mathrm{ITLB} = O(n \log m)$

k-SUM

Given a n-tuple $\mathcal{S} = (x_1, \ldots, x_n)$, $x_i \in \mathbb{R}$, decide whether there exists a k-tuple $(x_{i_1}, \ldots, x_{i_k})$ such that $\sum_{j=1}^{k} x_{i_j} = 0$.

4. Application of Meiser's Algorithm

Algorithm (Idea of the algorithm)

input $x \in \mathbb{R}^n$, the point to be located.

1. Compute the position of x relative to a subset $\mathcal{H}^*$ of $\mathcal{H}$, finding the cell C of $\mathcal{H}^*$ containing x.
2. For any hyperplane H_i not meeting C, deduce $pv_i(x)$ then discard the hyperplane.
3. Recurse on hyperplanes that are left.

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Theorem (Bürgisser et al. [1])

If we choose $\mathrm{O}(\frac{n^2}{\epsilon} \log^2 \frac{n}{\epsilon})$ hyperplanes uniformly at random from $\mathcal{H}$ and denote this selection $\mathcal{H}^*$ and if there is no hyperplane in $\mathcal{H}^*$ intersecting a given simplex, then, with high probability, the number of hyperplanes of $\mathcal{H}$ intersecting the simplex is less or equal to $\epsilon|\mathcal{H}|$.

4. Application of Meiser's Algorithm

Algorithm (Meiser's algorithm)

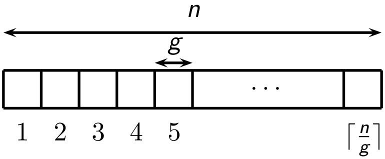
input $x \in \mathbb{R}^n$, the point to be located.

1. Compute the position of x relative to a subset $\mathcal{H}^*$, $|\mathcal{H}^*| = O(n^2 \log^2 n)$, of $\mathcal{H}$, finding the cell C of $\mathcal{H}^*$ containing x.
2. Build simplex S containing x and inscribed in C.
3. For any hyperplane H_i not meeting S, deduce $pv_i(x)$ then discard the hyperplane.
4. Recurse on the hyperplanes that are left.

Theorem (Bürgisser et al. [1])

If we choose $O(\frac{n^2}{\epsilon} \log^2 \frac{n}{\epsilon})$ hyperplanes uniformly at random from $\mathcal{H}$ and denote this selection $\mathcal{H}^*$ and if there is no hyperplane in $\mathcal{H}^*$ intersecting a given simplex, then, with high probability, the number of hyperplanes of $\mathcal{H}$ intersecting the simplex is less or equal to $\epsilon|\mathcal{H}|$.

5. Solve k-SUM using only $o(n)$-linear queries



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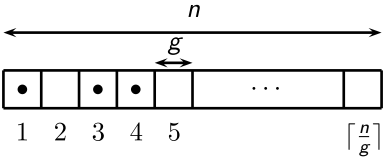

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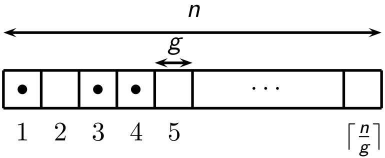

For any g

Complexity

$$O\left(\left(\frac{n}{g}\right)^k (kg)^3 \log^3 (kg)\right)$$

Query size

$$kg$$

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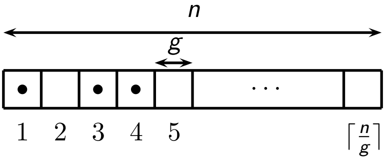

For $g = n^{\frac{k-1}{k}} = n^{1-\frac{1}{k}}$

Complexity

$$O\left(n(kn^{1-\frac{1}{k}})^3 \log^3(kn^{1-\frac{1}{k}})\right)$$

Query size

$$kn^{1-\frac{1}{k}}$$

6. Conclusion

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- Lots of related problems

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An Open Question

- Is there an $O(n^{2-\epsilon})$ algorithm for 3SUM?

7. References

[1] Bürgisser, P., Clausen, M., and Shokrollahi, M. A. (1997). *Algebraic complexity theory*, volume 315 of *Grundlehren der mathematischen Wissenschaften*. Springer.

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[5] Meiser, S. (1993). Point location in arrangements of hyperplanes. *Information and Computation*, 106(2):286–303.