Project: Nonlinear Elliptic problem POD vs PINNs

Let us consider the two-dimensional spatial domain $\Omega = (0,1)^2$. We wan to solve the following parametrized problem: given $\boldsymbol{\mu} = (\mu_0, \mu_1) \in \mathcal{P} = [0.1, 1]^2$, find $u(\boldsymbol{\mu})$ such that

$$-\Delta u(\boldsymbol{\mu}) + \frac{\mu_0}{\mu_1}(e^{\mu_1 u(\boldsymbol{\mu})} - 1) = g(\boldsymbol{x}; \boldsymbol{\mu}),$$

with homogeneous Dirichlet condition on the boundary.

Part 1

The source term defined as

$$g(\boldsymbol{x}; \boldsymbol{\mu}) = g_1 = 100 \sin(2\pi x_0) cos(2\pi x_1) \quad \forall \boldsymbol{x} = (x_0, x_1) \in \Omega.$$

Tasks:

1. solve the problem by means of POD-Galerkin method over a Finite Element full order model
2. solve the problem with a parametric PINN
3. compare the two approaches in terms of computational costs and accuracy with respect to the full order model
4. **Optional**: solve the problem with the POD-NN approach and compare it to the other two strategies

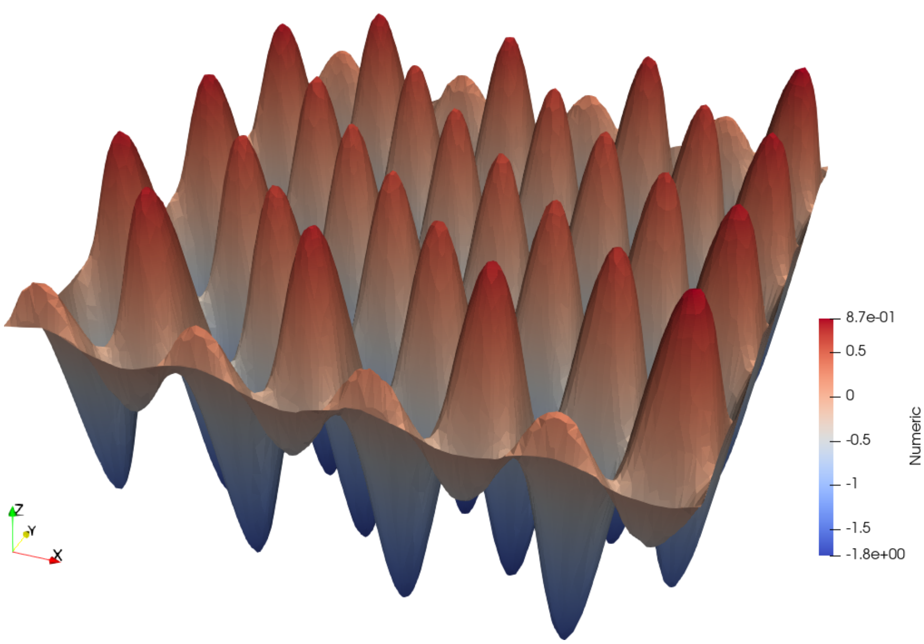

Figure 1 - *Solution on* $\tilde{\Omega} = [0,4]^2$ *for* $\mu = [3.34, 4.45]$ *with* g_1

Part 2

The source term defined as

$$g(\boldsymbol{x};\boldsymbol{\mu}) = g_2 = 100\sin(2\pi\mu_0 x_0)cos(2\pi\mu_0 x_1) \quad \forall \boldsymbol{x} = (x_0, x_1) \in \Omega.$$

Tasks:

1. solve the problem by means of POD-Galerkin method over a Finite Element full order model
2. solve the problem with a parametric PINN using the **same** network structure (number of layers and nodes per layer) of Part 1
3. compare the two approaches in terms of computational costs and accuracy with respect to the full order model
4. **Optional**: solve the problem with the POD-NN approach and compare it to the other two strategies

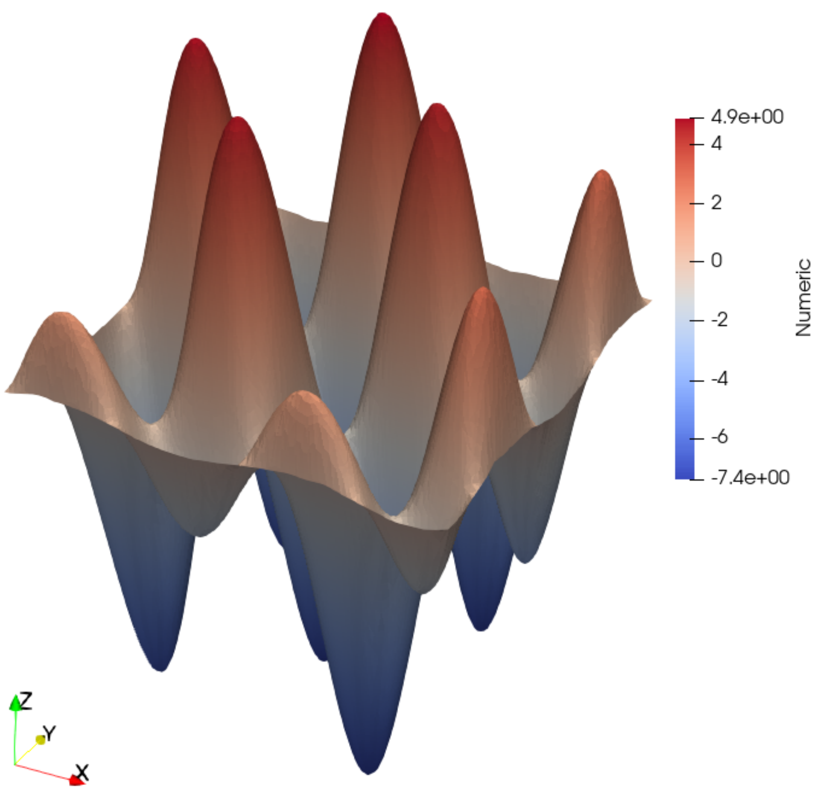

Figure 2 - *Solution on* $\tilde{\Omega} = [0,4]^2$ *for* $\mu = [0.455, 0.72]$ *with* g_2 *zoom* $0.5x$

Hints!

Given a neural network $\tilde{w}(\boldsymbol{x}, \boldsymbol{\mu})$, the loss will be of the form

$$MSE \doteq MSE_b^{\boldsymbol{\mu}} + \lambda MSE_p^{\boldsymbol{\mu}}.$$

where the *boundary* MSE is:

$$MSE_b^{\boldsymbol{\mu}} \doteq \frac{1}{N_b} \sum_{k=1}^{N_b} |\tilde{w}(\boldsymbol{x}_k^b, \boldsymbol{\mu}_k^b) - u(\boldsymbol{x}_k^b, \boldsymbol{\mu}_k^b)|^2,$$

for $(\boldsymbol{x}_k^b, \boldsymbol{\mu}_k^b) \in \partial\Omega \times \mathcal{P}$. While the *physical* MSE is

$$MSE_p^{\boldsymbol{\mu}} \doteq \frac{1}{N_p} \sum_{k=1}^{N_p} |\mathcal{R}(\tilde{w}(\boldsymbol{x}_k^p, \boldsymbol{\mu}_k^p))|^2,$$

where $(\boldsymbol{x}_k^p, \boldsymbol{\mu}_k^p) \in \Omega \times \mathcal{P}$. Finally, λ is the hyperparameter that can be used to balance the two different contributions.