DIGITAL LOGIC

Lecture 2
Boolean Algebra

2023 Fall

Today's Agenda

- Recap
- Context
  - Boolean Algebra (布尔代数)
  - Axioms (公理) and Theorems(定理)
  - Boolean Functions (布尔方程)
  - Canonical (范式) and Standard form(标准式)
- Reading: Textbook, Chapter 2

- Chapter1
  - Number Systems
    - Decimal
    - Binary
    - Octal
    - Hexadecimal
  - Number based Conversion
    - Base-r to Decimal: $$D = \sum_{i=-m}^{n-1} d_i r^i$$
    - Decimal to Base-r: division for integer part, multiplication for fraction
    - Base-r to Base-r: group bits from radix point
  - Complements
    - r's complement
    - r−1's complement
    - Binary
      - 2's complement
      - 1's complement
    - subtraction
    - signed binary numbers
  - Representation of data: code
    - BCD
    - Gray code
    - ASCII
    - Parity

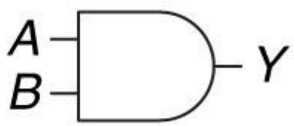
Outline

- **Axioms and Theorems of Boolean Algebra**
- Simplify Boolean Functions
- Canonical and Standard form
- Other Logic Operations

Binary Logic

- Deal with Variables like A, B… taking two values:
  - ‘0’, ‘1’; ‘L’, ‘H’; ‘T’, ‘F’

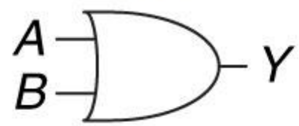
AND



$$Y = AB$$

A	B	Y
0	0	0
0	1	0
1	0	0
1	1	1

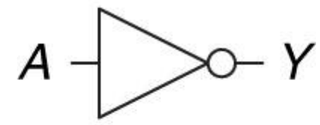
OR



$$Y = A + B$$

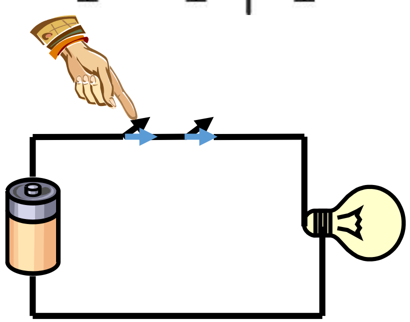
A	B	Y
0	0	0
0	1	1
1	0	1
1	1	1

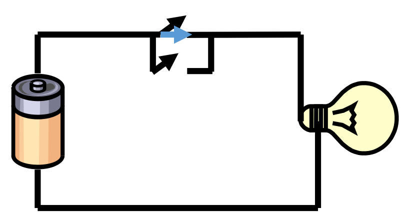
NOT



$$Y = \overline{A}$$

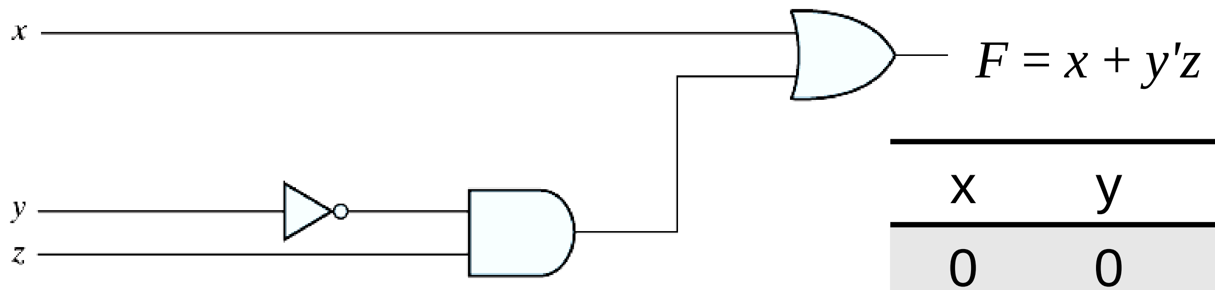
A	Y
0	1
1	0





Boolean Equation and Truth Table

- Boolean Equation: F = x + y'z
- Logic diagram:



- if x = y = 0, z = 1
  - F = 0 + 1•1 = 1
- Truth table (真值表)
  - The truth table of F has 2^n entries (n = num of inputs)

x	y	z	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Boolean Algebra

- Boolean algebra(逻辑代数), a deductive mathematical system developed by George Boole in 1854, deals with the rules by which logical operations are carried out.
- Boolean algebra is an algebraic structure defined by
  - a set of elements S: binary variables;
  - a set of binary operators: AND(•), OR(+) and NOT(’);
  - and a number of Axioms/theorems.

Boolean Axioms and Theorems of One Variable

- Axioms and theorems to simplify Boolean equations
- Duality (对偶性) in Axioms and theorems:
  - Replace • with +, 0 with 1

	Theorem	Dual	Name
1	$x + 0 = x$	$x \cdot 1 = x$	Identity
2	$x + 1 = 1$	$x \cdot 0 = 0$	Null Element
3	$x + x = x$	$x \cdot x = x$	Idempotency
4	$(x')' = x$		Involution
5	$x + x' = 1$	$x \cdot x' = 0$	Complements

- Operator precedence
  - Parentheses > NOT > AND > OR

Boolean Axioms and Theorems of Several Variables

- Dual: Replace • with +, 0 with 1

	Theorem	Dual	Name
6	xy = yx	x + y = y + x	Commutativity
7	(xy)z = x(yz)	(x + y) + z = x + (y + z)	Associativity
8	x(y + z) = xy + xz	x + yz = (x + y)(x + z)	Distributivity
9	x + xy = x	x(x + y) = x	Absorption
10	xy + xy' = x	(x + y)(x + y') = x	Combining
11	(x+y')y = xy	xy' + y = x + y	Simplification
12	xy + x'z + yz = xy + x'z	(x + y)(x' + z)(y + z) = (x + y)(x' + z)	Consensus
13	(x + y)' = x'y'	(xy)' = x' + y'	DeMorgan's law

Note: 8's Dual differs from traditional algebra: OR (+) distributes over AND (•)

Proofs (1)

Algebraic method

- **Absorption**
- $x + xy = x$
- *pf:* $x + xy = x(1+y) = x$

- **Combining**
- $(x + y)(x + y') = x$
- *pf:* $(x + y)(x + y') = x + yy' = x + 0 = x$

- **Simplification**
- $xy' + y = x + y$
- pf: $xy' + y = xy' + (x+x')y = xy' + xy + x'y$
  $= xy' + xy + xy + x'y = x(y'+y) + y(x+x') = x+ y$

- **Consensu**
- $xy + x'z + yz = xy + x'z$
- *pf:* $xy + x'z + yz = xy + x'z + (x+x')yz$
  $= xy + x'z + xyz + x'yz$
  $= (xy + xyz) + (x'z + x'zy) = xy + x'z$

Proofs (2)

Truth table method

- **DeMorgan's Law**
  - $(x + y)' = x'y'$ $(xy)' = x' + y'$

pf:

x	y	x'	y'	(x+y)'	x'y'	x'+y'	(xy)'
0	0	1	1	1	1	1	1
0	1	1	0	0	0	1	1
1	0	0	1	0	0	1	1
1	1	0	0	0	0	0	0

- **Associativity**
  - $(xy)z = x(yz)$
  - $(x + y) + z = x + (y + z)$

x	y	z	(xy)z	x(yz)	(x+y)+z	x+(y+z)
0	0	0	0	0	0	0
0	0	1	0	0	1	1
0	1	0	0	0	1	1
0	1	1	0	0	1	1
1	0	0	0	0	1	1
1	0	1	0	0	1	1
1	1	0	0	0	1	1
1	1	1	1	1	1	1

Outline

- Axioms and Theorems of Boolean Algebra
- **Simplify Boolean Functions**
- Canonical and Standard form
- Other Logic Operations

Boolean Functions

- A Boolean function from an algebraic expression can be realized to a logic diagram composed of logic gates.
  - Binary variables
  - operators OR, AND, NOT
  - Parentheses
- Terminology:
  - Literal: A variable or its complement
  - Product term: literals connected by •
  - Sum term: literals connected by +
- Example:
  - $A'B'C + A'BC + AB'$
    - 8 literals
    - 3 product terms
    - 1 sum term

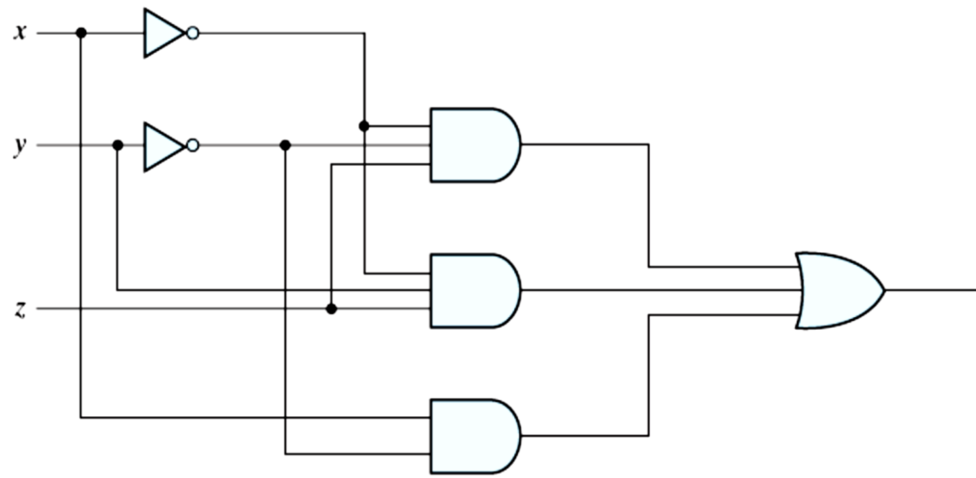
Boolean Functions

- Each Boolean function has
  - only one representation in truth table
  - but a variety of ways in algebraic form/gate implementation.
- Examples
  - $F_1 = x' y' z + x' y z + x y'$
  - $F_2 = x y' + x' z$
  - $F_1 = F_2$
    - Same truth table
    - Different algebraic expression

x	y	z	F_1	F_2
0	0	0	0	0
0	0	1	1	1
0	1	0	0	0
0	1	1	1	1
1	0	0	1	1
1	0	1	1	1
1	1	0	0	0
1	1	1	0	0

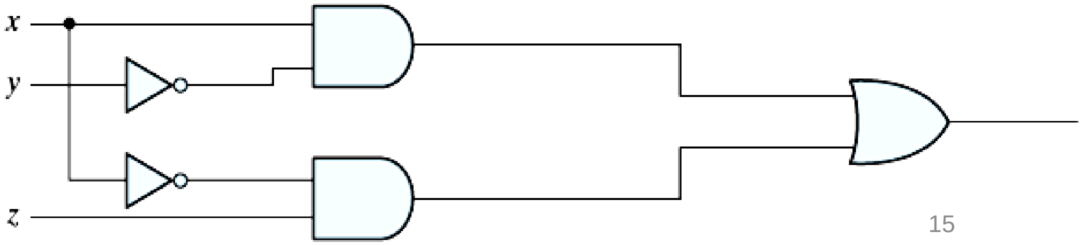
Gate Implementation

- $F_1 = x'y'z + x'yz + xy'$
  - 8 literals
  - 1 OR term (sum term) and 3 AND terms (product terms)
  - **literal**: a variable or its complement in a Boolean expression (a input to a gate)
  - **term**: implementation with a gate



F_1	$= x'y'z + x'yz + xy'$	
	$= x'z(y' + y) + xy'$	Distributivity
	$= x'z + xy' = F_2$	Complements

- $F_2 = x'z + xy'$
  - 4 literals
  - 1 OR term and 2 AND terms
  - Simpler circuit, more economical



Algebraic Simplification

- Minimize the number of literals and terms for a simpler circuits (less expensive)
- Algebraic simplification can minimize literals and terms. However, no specific rules to guarantee the optimal results
- Usually not possible by hand for complex functions, use computer minimization program
- More advanced techniques in the next lectures (K-Map)
- Useful rules
    - Distributivity
    - Idempotency
    - Complements
    - DeMorgan’s
    - etc

Example

- Examples:

$$
\begin{aligned}
F &= A'BC + A' \\
&= A'(BC + 1) && \text{Distributivity} \\
&= A' && \text{Null Element}
\end{aligned}
$$

Exercise:

$$
\begin{aligned}
F &= A'B'C + A'BC + AB' \\
&= ? \\
&= ?
\end{aligned}
$$

$$
\begin{aligned}
F &= XYZ + XY'Z + XYZ' \\
&= XYZ + XY'Z + XYZ + XYZ' && \text{Idempotency} \\
&= XZ(Y + Y') + XY(Z + Z') && \text{Distributivity} \\
&= XZ + XY && \text{Complements} \\
&= X(Y + Z) && \text{Distributivity}
\end{aligned}
$$

Boolean Function complement

- The complement of any function F is F', which can be obtained by DeMorgan's Theorem
  - Take the dual of expression, and then complement each literal in F
- Example: $F_3 = x'y'z+x'yz+xy'$
  - Step1, Dual: Replace • with +, 0 with 1

$$x'y'z + x'yz + xy' \xrightarrow{\text{Dual}} (x'+y'+z)(x'+y+z)(x+y')$$

  - Step2, complement each literal in F

$$F_3' = (x'y'z + x'yz + xy')'$$

$$= (x+y+z')(x+y'+z')(x'+y) \qquad \text{DeMorgan}$$

Outline

- Axioms and Theorems of Boolean Algebra
- Simplify Boolean Functions
- **Canonical and Standard form**
- Other Logic Operations

Minterms and Maxterms

- Minterms and Maxterms
- A **minterm**(最小项): an AND term consists of all literals in their normal form or in their complement form.
  - For example, two binary variables x and y,
    - xy, xy', x'y, x'y'
  - n variables can be combined to form 2^n minterms
- A **maxterm**(最大项): an OR term
  - For example, two binary variables x and y,
    - x+y, x+y', x'+y, x'+y'
  - 2^n maxterms
- Each maxterm is the complement of its corresponding minterm and vice versa. ($M_i = m_i'$)

Minterms and Maxterms

- Canonical forms
  - sum-of-minterms (som)
  - product-of-maxterms (pom)
  - Minterms and maxterms for three binary variables

			Minterms		Maxterms	
x	**y**	**z**	**Term**	**Designation**	**Term**	**Designation**
0	0	0	$x'y'z'$	m_0	$x + y + z$	M_0
0	0	1	$x'y'z$	m_1	$x + y + z'$	M_1
0	1	0	$x'yz'$	m_2	$x + y' + z$	M_2
0	1	1	$x'yz$	m_3	$x + y' + z'$	M_3
1	0	0	$xy'z'$	m_4	$x' + y + z$	M_4
1	0	1	$xy'z$	m_5	$x' + y + z'$	M_5
1	1	0	xyz'	m_6	$x' + y' + z$	M_6
1	1	1	xyz	m_7	$x' + y' + z'$	M_7

Canonical forms

x	y	z	f_1	f_2	f_1'	f_2'
0	0	0	0	0	1	1
0	0	1	1	0	0	1
0	1	0	0	0	1	1
0	1	1	0	1	1	0
1	0	0	1	0	0	1
1	0	1	0	1	1	0
1	1	0	0	1	1	0
1	1	1	1	1	0	0

- A Boolean function can be expressed using canonical forms:
  - **sum-of-minterms**
    - $f_1 = x'y'z + xy'z' + xyz$
      $= m_1 + m_4 + m_7 = \sum(1,4,7)$
    - $f_2 = x'yz + xy'z + xyz' + xyz$
      $= m_3 + m_5 + m_6 + m_7 = \sum(3,5,6,7)$
  - **product-of-maxterms**
    - $f_1 = (x + y + z)(x + y' + z)(x + y' + z')(x' + y + z')(x' + y' + z)$
      $= M_0 \cdot M_2 \cdot M_3 \cdot M_5 \cdot M_6 = \prod(0,2,3,5,6)$
    - $f_2 = (x + y + z)(x + y + z')(x + y' + z)(x' + y + z)$
      $= M_0 \cdot M_1 \cdot M_2 \cdot M_4 = \prod(0,1,2,4)$
  - $F_1 = \sum(1,4,7) = \prod(0,2,3,5,6)$, $F_2 = \sum(3,5,6,7) = \prod(0,1,2,4)$

Conversion between Canonical Forms

- To convert from one canonical from to another, interchange ∑ and ∏, and list the numbers that were excluded from the original form
- For example: F = xy+x'z
  - Sum of minterms:
  - F = ∑(1, 3, 6, 7) = $m_1 + m_3 + m_6 + m_7$ = x'y'z + x'yz + xyz' + xyz
- $M_i = m_i'$:

  F' = ∑(0, 2, 4, 5) = $m_0 + m_2 + m_4 + m_5$

  F = (F')' = $(m_0 + m_2 + m_4 + m_5)'$

  = $m'_0m'_2m'_4m'_5 = M_0M_2M_4M_5$

  Product of Maxterms:

  F = ∏(0, 2, 4, 5)

  = (x+y+z)(x+y'+z)(x'+y+z)(x'+y+z')

Truth Table for $F = xy + x'z$

x	*y*	*z*	*F*
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Example

- Example: Express $F = A+B'C$ as a sum of minterms.
  - Hint: by expanding the missing variables in each term, using 1=x+x', 0=xx'
- To expand to Sum of minterms: using complements and Distributivity to expand.
  - xy = xy(z+z') = xyz + xyz'

$$
\begin{aligned}
F &= A+B'C \\
&= A\,(B+B') + B'C \\
&= AB + AB' + B'C \\
&= \boxed{AB(C+C')} + AB'(C+C') + (A+A')B'C \\
&= ABC+ABC'+AB'C+AB'C'+A'B'C \\
&= m_1 + m_4 + m_5 + m_6 + m_7 \\
&= \sum(1, 4, 5, 6, 7)
\end{aligned}
$$

Truth Table for $F = A + B'C$

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Example

- Example: Express F = xy + x'z as a product of maxterms.
  - Hints: First convert to product of sum form, then expand
- To expand to Product of maxterms: using Complements and distributivity to expand.
  - x + y= (x + y + zz') = (x+y+z)(x+y+z')

$$
\begin{aligned}
F &= xy + x'z \\
&= (xy + x')(xy + z) \\
&= (x+x')(y+x')(x+z)(y+z) \\
&= (x'+y)(x+z)(y+z) \\
&= \boxed{(x'+y+zz')}(x+z+yy')(y+z+xx') \\
&= (x'+y+z)(x'+y+z')(x+z+y)(x+z+y')(\cancel{y+z+x})(\cancel{y+z+x'}) \\
&= (x+y+z)(x+y'+z)(x'+y+z)(x'+y+z') \\
&= M_0M_2M_4M_5 \\
&= \prod(0, 2, 4, 5)
\end{aligned}
$$

x + yz = (x + y)(x + z)	Distributivity

Tips: You can also use DeMorgan's Law (Involution first)

Canonical Forms

- Any function can be represented by either of the 2 canonical forms
- How to convert $f=x+y'z$ into canonical form?
  - by truth table
  - or by expanding the missing variables in each term, using $1=x+x'$, $0=xx'$

$$
\begin{aligned}
f &= x+y'z \\
&= ?
\end{aligned}
$$

Example

- Any function can be represented by either of the 2 canonical forms (Sum of Minterms/Product of Maxterms)
- How to convert f=x+y'z into canonical form?
  - by truth table
  - or by expanding the missing variables in each term, using 1=x+x', 0=xx'

$$
\begin{aligned}
f &= x+y'z \\
&= x(y+y') + y'z \\
&= xy + xy' + y'z \\
&= xy(z+z') + xy'(z+z') + (x+x')y'z \\
&= xyz + xyz' + xy'z + xy'z' + xy'z + x'y'z \\
&= xyz + xyz' + xy'z + xy'z' + x'y'z \\
&= m_7 + m_6 + m_5 + m_4 + m_1 = \sum(1,4,5,6,7) \\
&= M_0 \cdot M_2 \cdot M_3 = \prod(0,2,3)
\end{aligned}
$$

We can first find Sum of Minterms form, then easily convert into Product of Maxterm form

Standard Forms

- Canonical forms are very seldom the ones with the least number of literals.
- Standard forms: the terms that form the function may have fewer literals than the minterms.
  - Sum of products(sop): $F_1 = y' + xy + x'yz'$
  - Product of sums(pos): $F_2 = x(y'+z)(x'+y+z')$
  - $F_3 = A'B'CD+ABC'D'$
- Standard forms are not unique!

Outline

- Axioms and Theorems of Boolean Algebra
- Simplify Boolean Functions
- Canonical and Standard form
- **Other Logic Operations**

Other Logic Operations

- 2^n rows in the truth table of n binary variables.
- 2^{2n} functions for n binary variables.
- 16 functions of two binary variables.

Truth Tables for the 16 Functions of Two Binary Variables

x	y	F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

- All the new symbols except for the exclusive-OR symbol are not in common use by digital designers.

Boolean Expressions

- When the three operators AND, OR, and NOT are applied on two variables A and B, they form 16 Boolean functions:

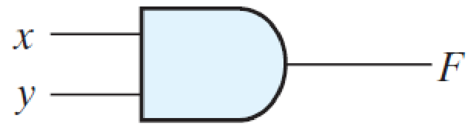
Boolean Functions	**Operator Symbol**	**Name**	**Comments**
$F_0 = 0$		Null	Binary constant 0
$F_1 = xy$	$x \cdot y$	AND	x and y
$F_2 = xy'$	x/y	Inhibition	x, but not y
$F_3 = x$		Transfer	x
$F_4 = x'y$	y/x	Inhibition	y, but not x
$F_5 = y$		Transfer	y
$F_6 = xy' + x'y$	$x \oplus y$	Exclusive-OR	x or y, but not both
$F_7 = x + y$	$x + y$	OR	x or y
$F_8 = (x + y)'$	$x \downarrow y$	NOR	Not-OR
$F_9 = xy + x'y'$	$(x \oplus y)'$	Equivalence	x equals y
$F_{10} = y'$	y'	Complement	Not y
$F_{11} = x + y'$	$x \subset y$	Implication	If y, then x
$F_{12} = x'$	x'	Complement	Not x
$F_{13} = x' + y$	$x \supset y$	Implication	If x, then y
$F_{14} = (xy)'$	$x \uparrow y$	NAND	Not-AND
$F_{15} = 1$		Identity	Binary constant 1

Digital Logic Gates

- Consider the 16 functions in previous Table
  - Two are equal to a constant (F_0 and F_{15}).
  - Four are repeated twice (F_4, F_5, F_{10} and F_{11}).
  - Inhibition (F_2) and implication (F_{13}) are not commutative or associative.
  - The other eight are used as standard gates:
    - complement (F_{12})
    - transfer (F_3)
    - AND (F_1)
    - OR (F_7)
    - NAND (F_{14})
    - NOR (F_8)
    - XOR (F_6)
    - equivalence (XNOR) (F_9)
  - Complement: inverter.
  - Transfer: buffer (increasing drive strength).
  - Equivalence: XNOR.

Summary of Logic Gates

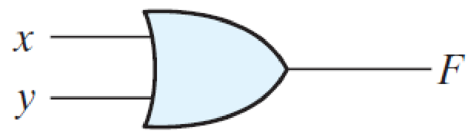
AND



$F = x \cdot y$

x	y	F
0	0	0
0	1	0
1	0	0
1	1	1

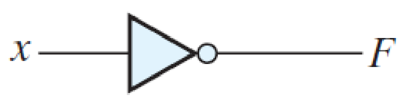
OR



$F = x + y$

x	y	F
0	0	0
0	1	1
1	0	1
1	1	1

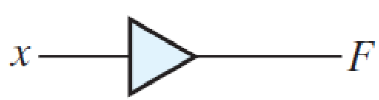
Inverter



$F = x'$

x	F
0	1
1	0

Buffer

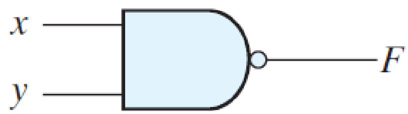


$F = x$

x	F
0	0
1	1

Summary of Logic Gates

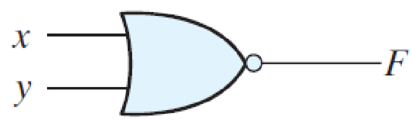
NAND



$F = (xy)'$

x	y	F
0	0	1
0	1	1
1	0	1
1	1	0

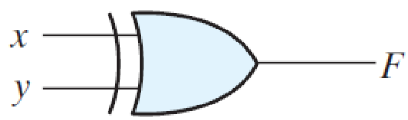
NOR



$F = (x + y)'$

x	y	F
0	0	1
0	1	0
1	0	0
1	1	0

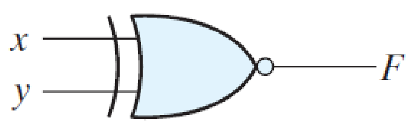
Exclusive-OR (XOR)



$$F = xy' + x'y = x \oplus y$$

x	y	F
0	0	0
0	1	1
1	0	1
1	1	0

Exclusive-NOR or equivalence

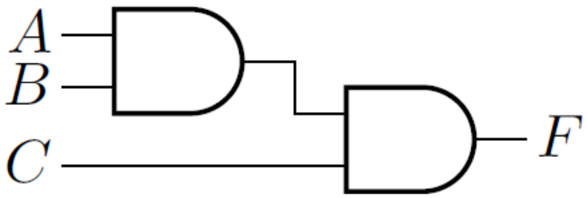


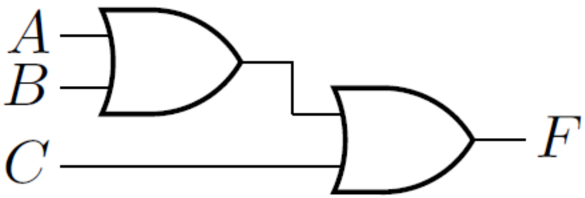
$$F = xy + x'y' = (x \oplus y)'$$

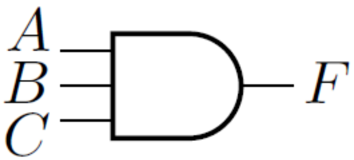
x	y	F
0	0	1
0	1	0
1	0	0
1	1	1

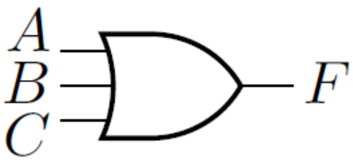
Multiple Inputs

- Extension to multiple inputs
  - A gate can be extended to multiple inputs.
  - AND and OR are commutative and associative.
    - F = ABC = (AB)C
    - F = A + B + C = (A + B) + C



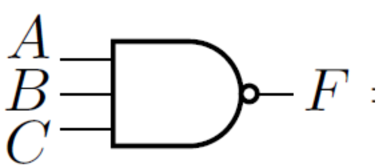


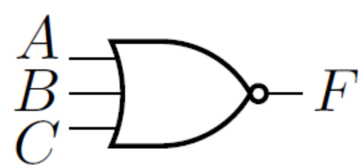




Multiple Inputs

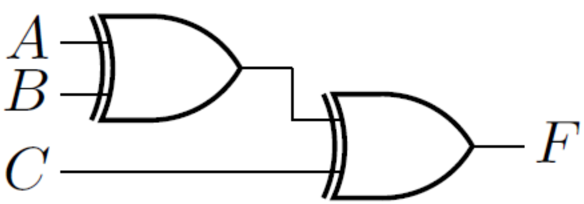
- NAND and NOR are commutative but not associative
  - $((AB)'C)' \neq (A(BC)')'$: does not follow associativity.
  - $((A + B)' + C)' \neq (A + (B + C)')'$: does not follow associativity.

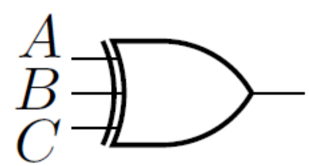




Multiple Inputs

- The XOR gates and equivalence gates both possess commutative and associative properties.
  - Gate output is low when even numbers of 1’s are applied to the inputs, and when the number of 1’s is odd the output is logic 0.
  - Multiple-input exclusive-OR and equivalence gates are uncommon in practice.





$F = A \oplus B \oplus C$