

DIGITAL LOGIC

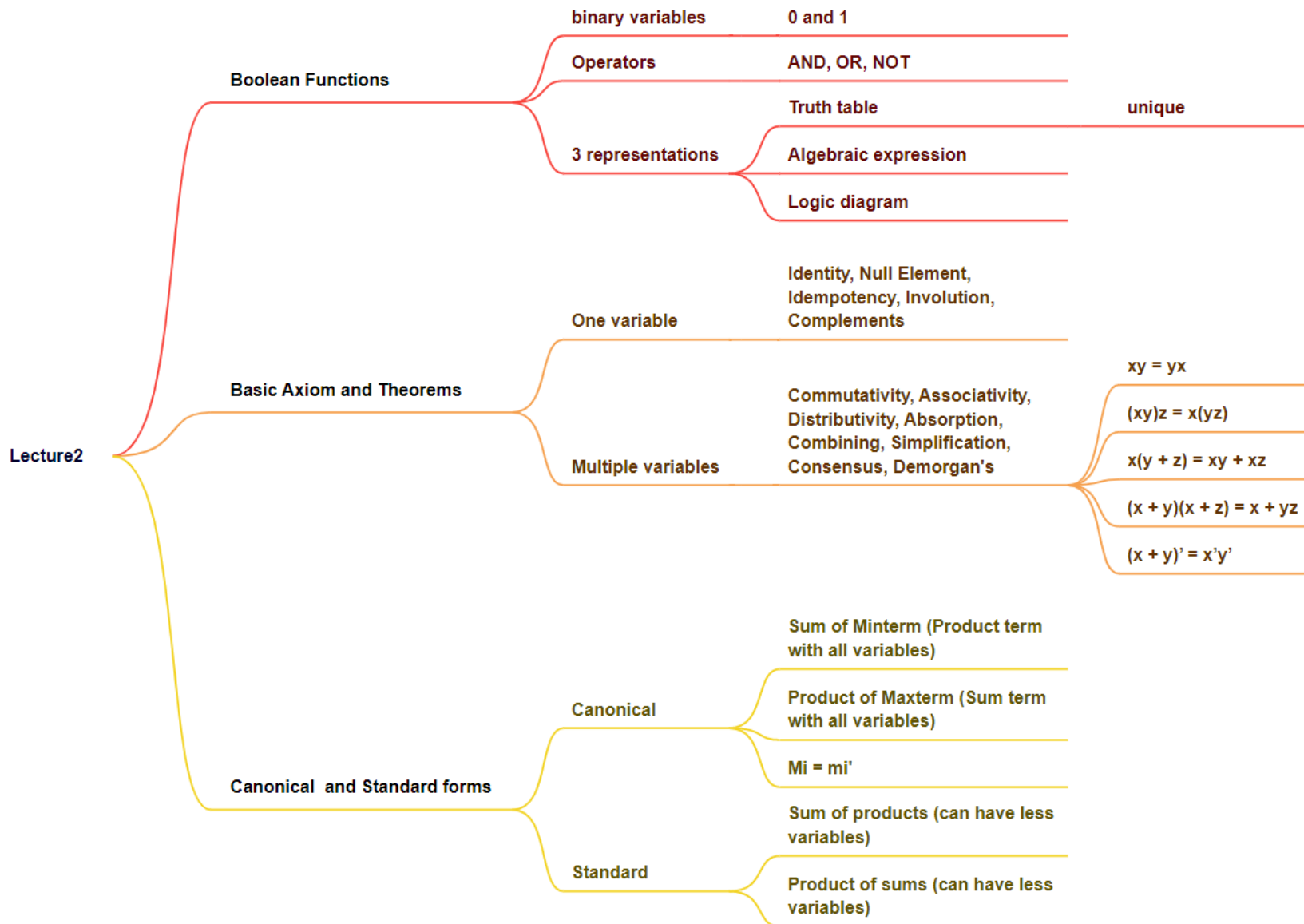
Chapter 3 part1: Gate-Level Minimization

2023 Fall

Today's Agenda

- Recap
- Context
 - Gate level minimization using the Map Method
 - Product of sums simplification
 - Don't Care Conditions
- Reading: Textbook, Chapter 3.1-3.5

Recap



Outline

- **Map Method Simplification**
- Product of sums simplification
- Don't Care Conditions

Boolean function simplification

- A function's truth-table representation is unique, while its algebraic expression is not unique.
- Complexity of digital circuit (gate count) \propto complexity of algebraic expression (literal count)
 - $F = x'y'z + x'yz + xy'$ (3 AND, 1 OR term, 8 literals)
 - $F = x'z + xy'$ (2 AND terms, 1 OR terms, 4 literals)
- The simplest algebraic expression is one that has minimum number of terms with the smallest possible number of literals in each term
- Methods for gate-level minimization:
 - **Algebraic method**(逻辑代数): Boolean algebra (Last lecture)
 - **Karnaugh map**(卡诺图): the map method (This lecture)

Karnaugh Map (K-map)

- An array of squares each representing one minterm to be minimized
- Each K-map defines a unique Boolean function
 - A Boolean function can be represented by a truth table, a Boolean expression, or a map
- K-map is a visual diagram of all possible ways a function may be expressed
- Used for manual minimization of Boolean functions

Merging Minterms

- In function F, m1 and m3 in the truth table differ only in one position

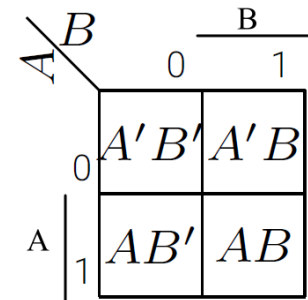
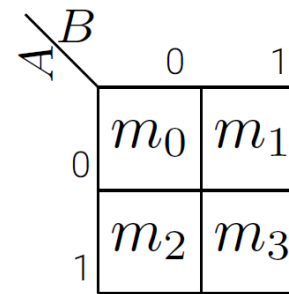
$\begin{matrix} 001 \\ 011 \end{matrix} \rightarrow 0?1$

- ?: matches either 0 or 1
- The minterms in a function can be merged to form a simpler product term
- $F_{0X1} = x'y'z + x'yz = x'z(y' + y) = x'z$

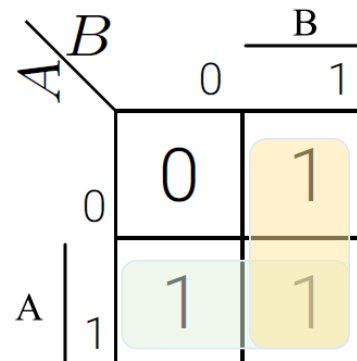
x	y	z	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	0

Two-Variable Map

- A two-variable map
 - Four minterms
 - A' = row 0; A = row 1
 - B' = column 0; B = column 1
 - A truth table in square diagram



	x	y	F
m_0	0	0	0
m_1	0	1	1
m_2	1	0	1
m_3	1	1	1



It's ok for groups to overlap, if that makes them larger

$$m_1 + m_2 + m_3 = A'B + AB' + AB = A + B$$

Three-variable Map

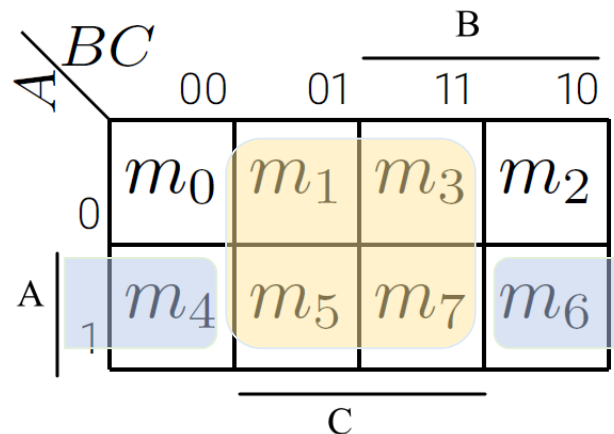
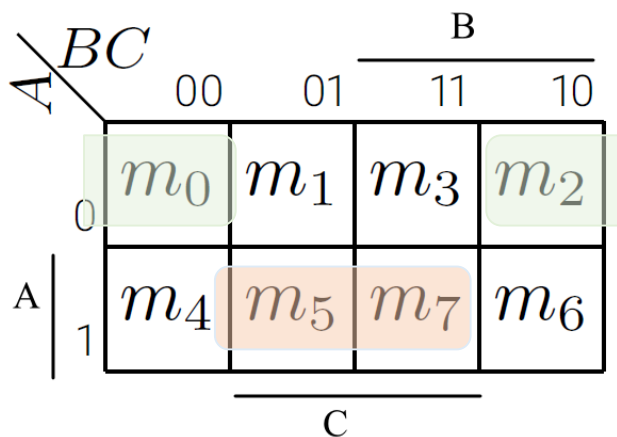
- Minterms are arranged in the Gray-code sequence
- Any 2 (horizontally or vertically) adjacent squares differ by exactly 1 variable, which is complemented in one square and uncomplemented in the other.
- Any 2 minterms in adjacent squares that are ORed together will cause a removal of the different variable

		BC			
		00	01	11	10
A	0	m_0	m_1	m_3	m_2
	1	m_4	m_5	m_7	m_6

C

Three-variable Map

- Example (adjacent squares)
- $m_5 + m_7 = AB'C + ABC = AC(B+B') = AC$
- $m_0 + m_2 = A'B'C' + A'BC' = A'C'(B+B') = A'C'$
- $m_4 + m_6 = AB'C' + ABC' = AC'(B'+B) = AC'$
- $m_1 + m_3 + m_5 + m_7$
 $= A'B'C + A'BC + AB'C + ABC = A'C(B+B') + AC(B+B')$
 $= A'C + AC = C$



Example

- Simplify the following Boolean functions.

$$F = A'BC + A'BC' + AB'C' + AB'C$$

$$= A'B + AB'$$

		BC			
		00	01	11	10
A	0	0	0	1	1
	1	1	1	0	0

Green circle highlights the 1s in the top row (A=0).
Red circle highlights the 1s in the bottom row (A=1).

Green circle: $A'BC + A'BC' = A'B$
Red circle: $AB'C' + AB'C = AB'$

$$F = A'BC + AB'C' + ABC' + ABC$$

$$= BC + AC'$$

		BC			
		00	01	11	10
A	0	0	0	1	0
	1	1	0	1	1

Green circle highlights the 1s in the column where B=1 and C=1.
Red circle highlights the 1s in the column where B=1 and C=0.

Green circle: $A'BC + ABC = BC$
Red circle: $AB'C' + ABC' = AC'$

Example

- Simplify the following Boolean functions.

$$F = \sum(1, 2, 3, 5, 7) = C + A'B$$

		B			
		BC			
A		00	01	11	10
0		0	1	1	1
1		0	1	1	0

Groups: C (columns 01, 11), $A'B$ (row 0, columns 01, 11, 10)

$$F = \sum(0, 2, 4, 5, 6) = C' + AB'$$

		B			
		BC			
A		00	01	11	10
0		1	0	0	1
1		1	1	0	1

Groups: C' (columns 00, 01), AB' (row 0, columns 00, 10)

It's ok for groups to overlap,
if that makes them larger

Exercise

- Simplify the following Boolean function.
 - $F = A'C + A'B + AB'C + BC$
 $= ?$

Exercise

- Simplify the following Boolean function.

- $$F = A'C + A'B + AB'C + BC = ?$$

- Solution:

- Express it in sum of minterms.

- Find the minimal sum of products expression.

- $$F = A'C + A'B + AB'C + BC$$

$$= A'C(B+B') + A'B(C+C') + AB'C + (A+A')BC$$

$$= A'BC + A'B'C + A'BC + A'BC' + AB'C + ABC + A'BC$$

$$= \sum(1, 2, 3, 5, 7) = C + A'B$$

		B			
		00	01	11	10
A	0	0	1	1	1
	1	0	1	1	0

Diagram illustrating the Karnaugh map for the Boolean function $F = A'C + A'B + AB'C + BC$. The map shows the function values for variables A, B, and C. The map is a 2x4 grid with rows labeled A (0, 1) and columns labeled BC (00, 01, 11, 10). The function values are 0, 1, 1, 1 for A=0 and 0, 1, 1, 0 for A=1. The map is simplified by grouping the 1s into two groups: a group of four 1s (A=0, B=01, 11, 10) and a group of four 1s (A=1, B=01, 11, 10). The simplified expression is $F = C + A'B$.

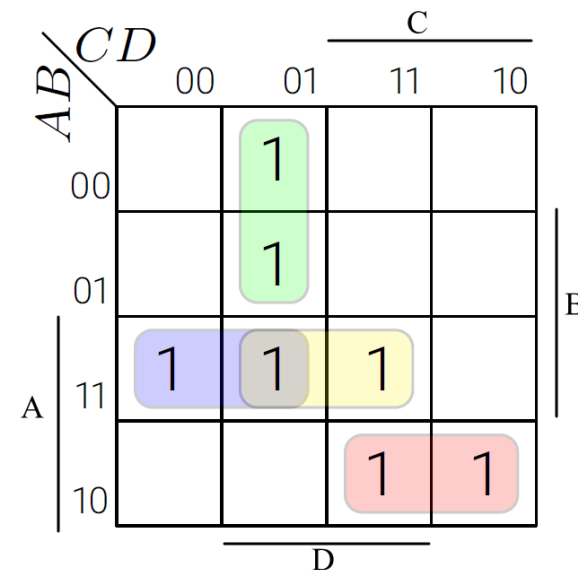
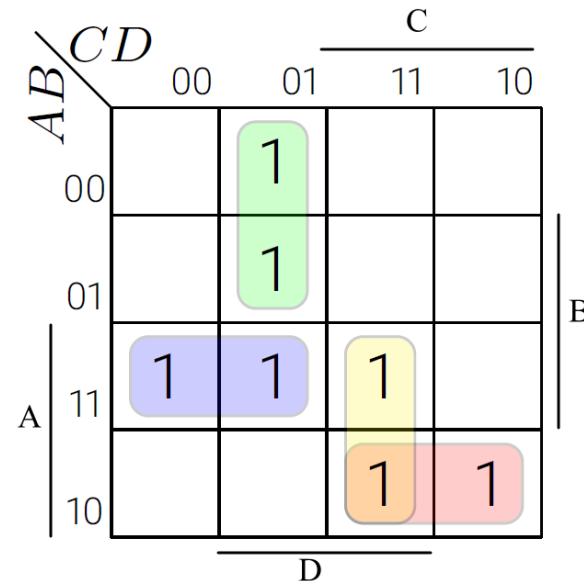
Four-Variable Map

- The map
 - 16 minterms
 - Combinations of 2, 4, 8, and 16 adjacent squares

		C			
		00	01	11	10
A	00	m_0	m_1	m_3	m_2
	01	m_4	m_5	m_7	m_6
	11	m_{12}	m_{13}	m_{15}	m_{14}
	10	m_8	m_9	m_{11}	m_{10}
		D			

Example

- Simplify the following Boolean functions
- $F = m_1 + m_5 + m_{10} + m_{11} + m_{12} + m_{13} + m_{15}$
 - Green circle: $A'B'C'D + A'BC'D = A'C'D$
 - Purple circle: $ABC'D' + ABC'D = ABC'$
 - ...
- $F = A'C'D + ABC' + \textcolor{red}{ACD} + AB'C$
- This reduced expression is not a unique one
 - If pairs are formed in different ways, the simplified expression will be different.
- $F = A'C'D + ABC' + \textcolor{red}{ABD} + AB'C$

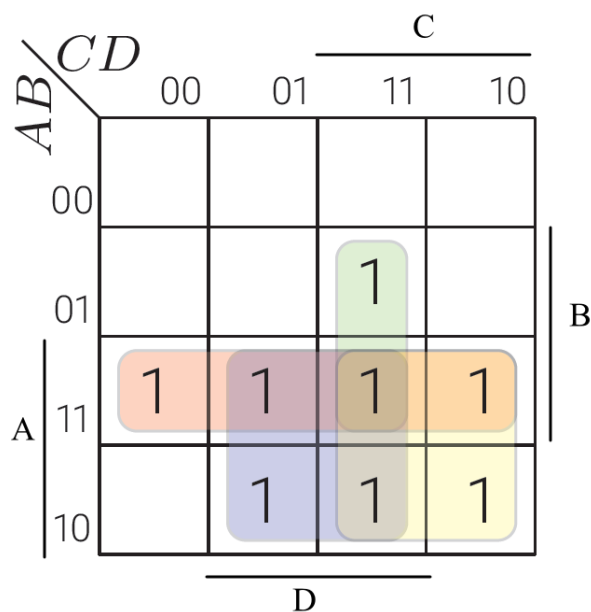


Example

- Simplify the following Boolean functions.

$$F = \sum(7, 9, 10, 11, 12, 13, 14, 15)$$

$$= AB + AC + AD + BCD$$



$$F(A,B,C,D) = ABCD + AB'C'D' + AB'C + AB$$

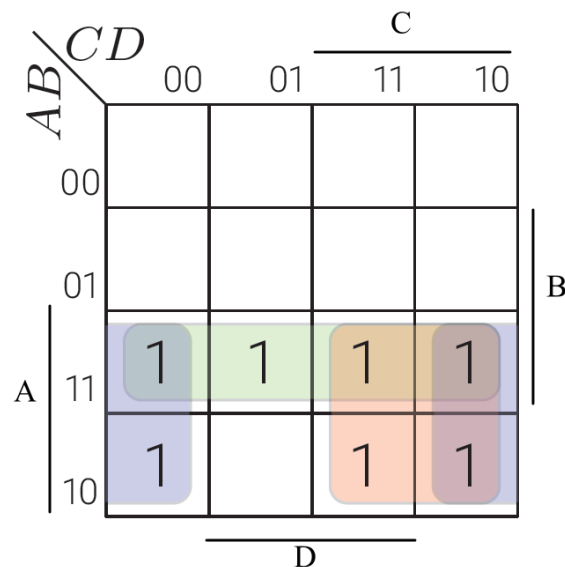
$$= ABCD + AB'C'D' + AB'C(D + D')$$

$$+ AB(C + C')(D + D')$$

$$= \dots$$

$$= \sum(8, 10, 11, 12, 13, 14, 15)$$

$$= AB + AC + AD'$$



Exercise

- Simplify the following Boolean functions.

$$F(A,B,C,D)$$

$$= \sum(0, 1, 2, 4, 5, 6, 8, 9, 12, 13, 14)$$

$$= ?$$

$$F(A,B,C,D)$$

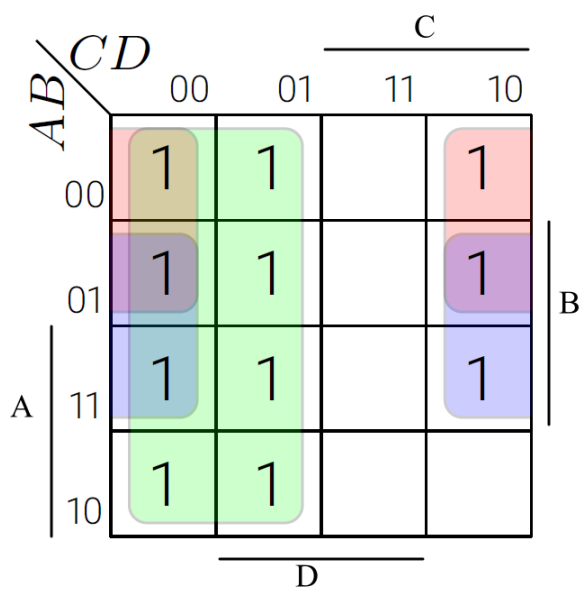
$$= A'B'C' + B'CD' + A'BCD' + AB'C'$$

$$= ?$$

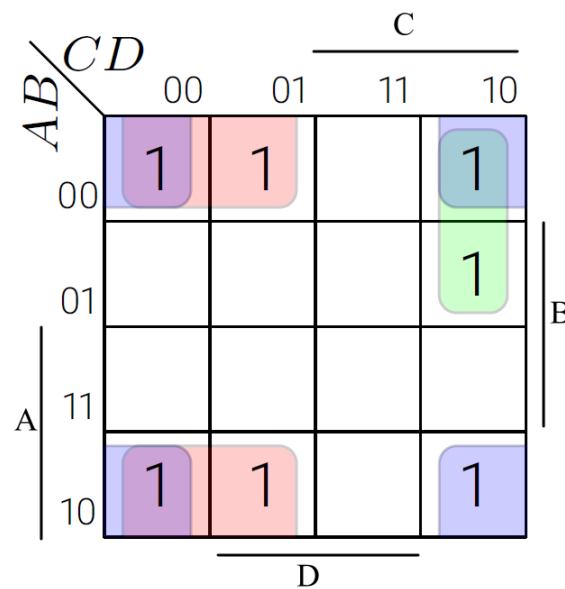
Exercise

- Simplify the following Boolean functions.

$$\begin{aligned}
 &F(A,B,C,D) \\
 &= \sum(0, 1, 2, 4, 5, 6, 8, 9, 12, 13, 14) \\
 &= C' + A'D' + BD'
 \end{aligned}$$



$$\begin{aligned}
 &F = A'B'C' + B'CD' + A'BCD' + AB'C' \\
 &= A'B'C'(D + D') + B'CD'(A + A') + \\
 &A'BCD' + AB'C'(D + D') \\
 &= A'B'C'D + A'B'C'D' + AB'CD' + A'B'CD' + \\
 &A'BCD' + AB'C'D + AB'C'D' \\
 &= \sum(0, 1, 2, 6, 8, 9, 10) \\
 &= B'C' + B'D' + A'CD'
 \end{aligned}$$



K-map Summary

- Any 2^k adjacent squares, $k=0,1,\dots,n$, in an n -variable map represent an area that gives a product term of $n-k$ literals

K	# of adjacent squares	# of literals left in a term in an n-variable map		
		n=2	n=3	n=4
0	1	2	3	4
1	2	1	2	3
2	4	0	1	2
3	8		0	1
4	16			0

- Five-Variable Map
 - Map for more than four variables becomes complicated
 - Five-variable map: two four-variable map (one on the top of the other), contains 2^5 or 32 cells.

Outline

- Map Method Simplification
- **Product of sums simplification**
- Don't Care Conditions

Product of Sums Simplification

- Previous Examples are Sum of Product Simplification
 - E.g. $F = AB + A'D + AB'C$ (Product of sum form)
- How to find Product of Sum simplification
 - E.g. $F = (A+B)(B+C')$ (Sum of Product form)
- POS simplification Steps
 - Simplified F' in the form of sum of products
 - Group adjacent 0-minterms squares together
 - Apply DeMorgan's theorem $F = (F')'$
 - F' : sum of products \rightarrow F : product of sums

Example

- Simplify the Boolean function:
 - $F(A,B,C,D) = \sum(2, 3, 7, 10, 11, 15)$
- Solution
 - Step1: group the 0-minterms to find **F complement**

$$F' = \sum(0, 1, 4, 5, 6, 8, 9, 12, 13, 14)$$

$$= C' + BD' \quad (\text{Group 0 minterms})$$
 - Step2: find the complement

$$F = (F')' = (C' + BD')'$$

$$= C(B' + D) \quad (\text{DeMorgan's})$$

		C			
		00	01	11	10
A	00	0	0	1	1
	01	0	0	1	0
	11	0	0	1	0
	10	0	0	1	1
		D			
		B			

Exercise

- simplify $F(A, B, C, D) = \sum(0, 1, 2, 5, 8, 9, 10)$ into
 - sum-of-products form
 - $F = ?$
 - product-of-sums form
 - $F = ?$

Exercise

- simplify $F(A, B, C, D) = \sum(0, 1, 2, 5, 8, 9, 10)$ into
 - sum-of-products form
 - $F = B'D' + B'C' + A'C'D$ (Group 1-minterms)
 - product-of-sums form
 - $F' = AB + CD + BD'$ (Group 0-minterms)
 - $F = (A'+B')(C'+D')(B'+D)$ (DeMorgan's)

		CD		C	
		00	01	11	10
A	AB	00	01	11	10
	00	1	1	0	1
	01	0	1	0	0
	11	0	0	0	0
	10	1	1	0	1
	D				

B

		CD		C	
		00	01	11	10
A	AB	00	01	11	10
	00	1	1	0	1
	01	0	1	0	0
	11	0	0	0	0
	10	1	1	0	1
	D				

B

Outline

- Map Method Simplification
- Product of sums simplification
- **Don't Care Conditions**

Don't care conditions

- Incompletely specified functions
 - Functions that have unspecified outputs for some input combinations
 - E.g. output are unspecified for 1010 to 1111 in 4-bit BCD code
- Don't-care conditions
 - Unspecified minterms of a function, don't-cares, Xs
 - Can be used on a map to provide further simplifications of the Boolean expression
 - Each X can be assigned an arbitrary value, 0 or 1, to help simplification procedure

Example

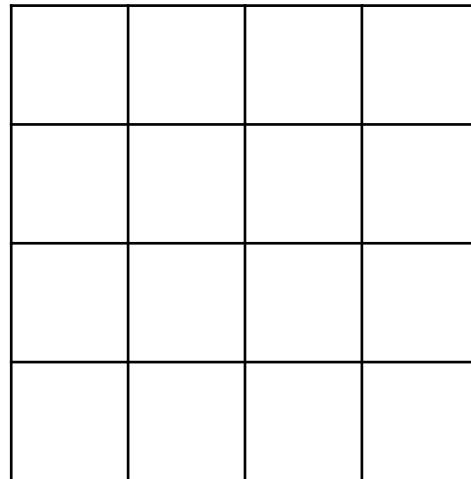
- Simplify $F(A, B, C, D) = \sum(1, 3, 7, 11, 15)$ with don't-care conditions $d(A, B, C, D) = \sum(0, 2, 5)$.
 - $F = A'B' + CD$
 - or $F = A'D + CD$
 - Either expression is acceptable

		C			
		00	01	11	10
A	00	X	1	1	X
	01		X	1	
	11			1	
	10			1	
		D			
		B			

		C			
		00	01	11	10
A	00	X	1	1	X
	01		X	1	
	11			1	
	10			1	
		D			
		B			

Exercise

- Using the Karnaugh map method obtain the minimal sum of the products expression for the function $F(A,B,C,D) = \Sigma(0, 2, 3, 6, 7) + d(8, 10, 11, 15)$



Exercise

- Using the Karnaugh map method obtain the minimal sum of the products expression for the function $F(A,B,C,D) = \Sigma(0, 2, 3, 6, 7) + d(8, 10, 11, 15)$
- $F = A'C + B'D'$

		CD					
		00	01	11	10	C	
A	AB	00	01	11	10	B	
	00	1	0	1	1		
	01	0	0	1	1		
	11	0	0	X	0		
A	10	X	0	X	X	D	

Implicants

- Implicant of a function: any product term that implies the function
 - A product term that is only true when a function is true
- Example: in F function

	minterm	implicant
m_1	✓	✓
m_2	✓	X
$0?1$	X	✓

1-minterm

0-minterm

x	y	z	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

Prime and Essential Prime Implicants

- Prime implicant (PI) (质蕴含)
 - A 1-product term obtained by combining the maximum possible number of adjacent squares in the map.
- Essential prime implicant (EPI) (基本质蕴含)
 - If a minterm in a square is covered by only one prime implicant'
- Simplification Steps:
 - Determine all essential prime implicants.
 - Find other prime implicants that cover remaining minterms.
 - Logical sum all prime implicants.
- Tips:
 - Minimize the number of groups
 - Maximize the group size
 - It's ok for groups to overlap, if that makes them larger