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Stable Neo-Hookean Flesh Simulation

Bachelor Thesis

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Vorwort

Dies ist ein Vorwort

Abstract

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Contents

Chapter 1

Introduction

"Animation offers a medium of story telling and visual entertainment which can bring pleasure and information to people of all ages everywhere in the world."
- Walt Disney

1.1 Motivation

With steadily increasing computational power the demand of better results is constantly growing. Especially in the field of animation and simulation we are no longer happy with mediocre results. In the entertainment sector gaming industries and animation studios like Pixar© or Disney© brought us games and movies of highest quality. Both of them have made groundbreaking progress over the years. This is easily observed when we compare today's work with that from ten years ago.

As always we have different requirements for each use. In some cases we want to exaggerate a movement or a reaction in a certain way. We can for example create a massive explosion in a movie that would not be half as spectacular in the real world.

In other scenarios we want to come as close as possible to reality. For instance we want an animated character to move and physically interact with its environment as a real human being would. Otherwise our brain would immediately recognize that some things do not add up. The goal here is to bring characters quite literally to life. We can add small

details like visible breathing and small wrinkles to have an even more convincing effect. The goal is to create the illusion of a character with personality, thought and emotions. In order to achieve this effect we need the character to move and react nearly physically correct.

In the paper *Stable Neo-Hookean Flesh Simulation* [SGK18] the authors addressed exactly the problem of making an animated movement of a human-like character look as natural as possible. In order to animate a physical movement we first need to understand the physics behind it which lies in the field of continuum mechanics. Unfortunately for most of us it has yet to be learned. The goal of this thesis is to give the necessary physical and mathematical background for a regular computer science student to understand the field of animation.

1.2 Structure

Following up I will give a brief overview of the necessary mathematical background and deliver an introduction in continuum mechanics. Next I will go through the ideas made in the paper mentioned and include some calculations and visualisations that help for a better understanding.

TODO: Improve intorduction (goals etc.). Adjust according to additions in text. Improve quote at beginning. Maybe add other applications as well (physical simulation) and how to go over to it?

Chapter 2

Background

The goal we are striving for is to animate human-like characters. In order to narrow it down even further we concentrate on the behaviour of the flesh of the character. For understanding the thematics of simulating human-like flesh it is necessary to have a basic mathematical background and knowledge of continuum mechanics. The goal of this chapter is to deliver an understanding in the topics mentioned.

2.1 Notation and Convention

At first we will declare the notation used in this thesis to avoid misunderstandings. We will use the common notation used in continuum mechanics taken from the book *Continuum Mechanics* [Spe80]. Additionally we will include some more specific declarations formulated and used in the paper *Stable Neo-Hookean Flesh Simulation* [SGK18].

2.1.1 General Notation

Scalars are represented by regular, normal-weight variables such as a whereas tensors and matrices are represented by upper-case bold letters as for example $\mathbf{A}$. Vectors will be denoted by bold lower-case variables like $\mathbf{a}$.

2.1.2 Tensor Notation

Furthermore we will use the tensor notation used in the paper *Stable Neo-Hookean Flesh Simulation*. They decided to to define vectorization $\text{vec}(\cdot)$ as column-wise flattening of a matrix into a vector ([SGK18], 12:5) similar to Golub and Van Loan (2012) [GV12].

In order to indicate that we are dealing with a vectorized matrix we will use the symbol $\check{\cdot}$ as shown in the following equation:

$$\mathbf{A} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \qquad \text{vec}(\mathbf{A}) = \check{\mathbf{a}} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

Additionally we will have to deal with 4^{th} order tensors in a form of matrix-of-matrices. These matrices are denoted by using blackboard bold:

$$\mathbb{A} = \begin{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} & \begin{bmatrix} i & k \\ j & l \end{bmatrix} \\ \begin{bmatrix} e & g \\ f & h \end{bmatrix} & \begin{bmatrix} m & o \\ n & p \end{bmatrix} \end{bmatrix} = \begin{bmatrix} [\mathbf{A}_{00}] & [\mathbf{A}_{01}] \\ [\mathbf{A}_{10}] & [\mathbf{A}_{11}] \end{bmatrix}$$

If we now vectorize $\mathbb{A}$ we receive the form described in the following line:

$$\mathbb{A} = \text{vec}(\mathbb{A}) = \left[\text{vec}\left(\mathbf{A}_{00}\right) \,\middle|\, \text{vec}\left(\mathbf{A}_{10}\right) \,\middle|\, \text{vec}\left(\mathbf{A}_{01}\right) \,\middle|\, \text{vec}\left(\mathbf{A}_{11}\right) \right]$$

This term above is equivalent to the notation described here:

$$\mathbb{A} = \begin{bmatrix} a & e & i & m \\ b & f & j & n \\ c & g & k & o \\ d & h & l & p \end{bmatrix} = \check{\boldsymbol{A}}$$

The advantage of this form is that we can write several expressions as a cross product. We will need this property later to simplify complicated expressions and calculations.

2.1.3 Summary

A quick overview of the notation used so far:

a: Scalar
$\mathbf{A}$: Matrix or tensor
$\mathbf{a}$: Vector
$\text{vec}(\mathbf{A}) = \check{\mathbf{a}}$: Vectorized matrix
$\mathbb{A} = \check{\mathbf{A}}$: matrix-of-matrices

TODO: Check if all notations used later are declared. Make matrix of matrices better looking.

2.2 Mathematical Background

Since mathematics play an important role in our field of interest we need to build a solid background before we go further into more complex calculations. This chapter should cover all the important concepts used later in the calculations. A basic understanding of linear algebra is assumed.

2.2.1 Matrices

At first we will discuss the physical or geometrical meaning of some common matrix properties.

A matrix $\mathbf{A}$ is positive definite if $\mathbf{x}^T\mathbf{A}\mathbf{x}$ is positive for all of the non-zero values of the column matrix $\mathbf{x}$.

TODO: positive definite matrices definition and lemmas, orthogonal, etc.

2.2.2 Singular Value Decomposition

The singular value decomposition (SVD) will play an important role in formulating the deformation gradient. It is important for our application since it represents the best possible approximation of a given matrix by a matrix of low rank. This approximation can be looked at as a compression of the data given ([LM15], S. 295).

Definition 1 (**Singular Values**). *The singular values of a matrix* $\mathbf{A} \in \mathbb{R}^{mxn}$ *are the square roots of the eigenvalues of* $\mathbf{A}\mathbf{A}^\intercal$.

The theorem of the singular value decomposition tells us that we can factor every m-by-n matrix into one orthogonal m-by-m, one diagonal m-by-n and one orthogonal n-by-n matrix. More formally:

Theorem 1 (**The SVD Theorem**). *Let* $\mathbf{A} \in \mathbb{R}^{mxn}$ *be a matrix having r positive singular values,* $m \geq n$. *Then there exist orthogonal matrices* $\mathbf{U} \in \mathbb{R}^{mxm}$, $\mathbf{V} \in \mathbb{R}^{nxn}$ *and a diagonal matrix* $\tilde{\mathbf{\Sigma}} \in \mathbb{R}^{mxn}$ *such that*

$$\mathbf{A} = \mathbf{U}\tilde{\mathbf{\Sigma}}\mathbf{V}^\intercal$$

$$\tilde{\mathbf{\Sigma}} = \begin{bmatrix} \mathbf{\Sigma} & 0 \\ 0 & 0 \end{bmatrix}$$

where $\mathbf{\Sigma} = diag\ (\sigma 1, \sigma 2, \ldots, \sigma r)$, *and* $\sigma 1 \geq \sigma 2 \geq \cdots \geq \sigma r > 0$ *are the positive singular values of* $\mathbf{A}$.

The definition and theorem were taken from [For14] (def. 1 S.113, thm. 1 S.300).

TODO: Check reference. Make paragraphs work together. Add examples.

2.2.3 Polar Decomposition

Theorem 2 (**The Polar Decomposition Theorem**). *Let* $\mathbf{F}$ *be a non-singular square matrix. Then* $\mathbf{F}$ *can be decomposed uniquely into either of the following two products*

$$\mathbf{F} = \mathbf{RU}, \quad \mathbf{F} = \mathbf{VR}$$

where $\mathbf{R}$ *is orthogonal and* $\mathbf{U}$ *and* $\mathbf{V}$ *are positive definite symmetric matrices.*

The definition and theorem were taken from [Spe80] (thm. 2 S.12).

TODO: Complete this section. Add lemmas and theorems used afterwards in the calculations. Possible adjustments may come at the end. Add Frobenius norm?

2.3 Continuum Mechanics

In this section we will give a broad introduction into the field of Continuum Mechanics. In Continuum Mechanics we are less interested in small particles like atoms or molecules of an object but rather concentrate on pieces of matter which are in comparison very large. We are therefore concerned with the mechanical behavior of solids and fluids on the macroscopic scale ([Spe80], p. 1).

2.4 Deformation

When applying a force over an object naturally the object itself undergoes a deformation. In the following we will be consistent with most previous literature in continuum mechanics and use the term strain as a measure of deformation and stress as the force per unit area:

Strain = measure of deformation
Stress = force per unit area

TODO: Decide whether to include the above paragraph and how to connect it better to the paper.

Now let us have a look at a deformation in a rather mathematical sense. Graphically we can imagine a deformation with the help of a two dimensional deformation map as shown in Fig. 2.1. Here we have on the left side an ellipse that represents an object or material in its rest state. A function ϕ maps this rest state of the ellipse to a deformed state shown in Fig. 2.1 on the right side. Mathematically spoken this means that we can map each point of a chosen object from its rest state to a deformed one.

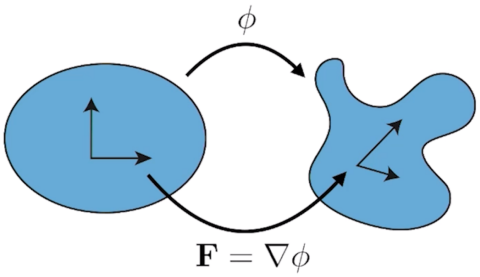


Figure 2.1: Deformation Map [Pix]

In conclusion the function ϕ describes our deformation. If we derive ϕ we can calculate the deformation gradient $\mathbf{F}$ which serves as a measure of the deformation in the following sense:

$$\begin{aligned} &\text{Deformation gradient:} && \mathbf{F} = \nabla\,\phi \\ &\text{Length changes:} && I_C = \operatorname{tr}\left(\mathbf{F}^T\mathbf{F}\right) \\ &\text{Volume changes:} && J = \det(\mathbf{F}) \end{aligned}$$

TODO: Adjust so the equal sign is at the same place in each column. Explain strain and stress and their meaning for later use in material constants (not explained in paper).

2.4.1 Deformation Gradient

The deformation gradient $\mathbf{F}$ offers us as said previously a measurement of the deformation. With its help we can amongst other things calculate

the volume and length change an object undergoes during a deformation. For our needs we define the deformation gradient exactely as the authors of the paper [SGK18] did:

$$\mathbf{F} = \left[f_0 \middle| f_1 \middle| f_2 \right] = \begin{bmatrix} f_0 & f_3 & f_6 \\ f_1 & f_4 & f_7 \\ f_2 & f_5 & f_8 \end{bmatrix}. \tag{2.1}$$

Using the SVD theorem mentioned in theorem 1 $\mathbf{F}$ can be written in the form of

$$\mathbf{F} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}_\intercal \tag{2.2}$$

in which $\mathbf{\Sigma}$ stands for

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_0 & 0 & 0 \\ 0 & \sigma_1 & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix}. \tag{2.3}$$

The σ_i denote the singular values of $\mathbf{\Sigma}$. Unlike the normal convention the authors of the paper *Stable Neo-Hookean Flesh Simulation* the rotation variant moves reflections to $\mathbf{\Sigma}$. This results into $\det(\mathbf{U}) = \det(\mathbf{V}) = 1$ and the property that $\mathbf{\Sigma}$ is allowed to have negative entries.

TODO: Add some examples. Already include definitions of the paper?
Some more sources:
http://www.continuummechanics.org/deformationgradient.html

2.4.2 Polar Decomposition of the Deformation Gradient

With the help of mathematics we can decompose the deformation gradient in the following form:

$$\mathbf{F} = \mathbf{RS}$$

2.4.3 Material Constants

When we look at a deformation of an object we need to consider the material the object consists of. A material can be very stiff like steel or easily deformable like rubber. In order to measure the deformation in a specific material we need the *Poisson's Ratio* of said material. The poisson's ratio is a material constant that is defined as

$$\sigma = -\frac{\epsilon_{11}}{\epsilon_{22}} \in [-1, 0.5] \tag{2.4}$$

where ϵ_{11} is the lateral and ϵ_{22} the axial strain. The range in which σ lies in starts at -1 and goes up to 0.5 [MR09]. Usually the poisson's ratio of a material is positive. A negative value would mean that the material becomes wider in the cross section when it is stretched. This behaviour is very uncommon in nature. Examples of materials with a negative poisson's ratio are for instance discussed in *Foam structures with a negative Poisson's ratio* [Lak87] or *Advances in negative Poisson's ratio materials* [Lak93].

Here are some examples of the poisson's ratio of various materials:

Material	**Poisson's ratio**
C (graphite)	0.31
Sn (metal)	0.357
Cu	0.355
Zn	0.25
Ag	0.36
Au	0.45
Concrete	0.20–0.37
Titanium (dental alloy)	0.30–0.31
Bronze	0.34
18–8 Stainless steel	0.305
Natural rubber	0.4999
B_2O_3 glass	0.30
GeO_2 glass	0.20

Table 2.1: Different materials with their poisson's ratio taken from [MR09]

In the context of flesh simulations we use the poisson's ratio as a characterization for the resistance to volume change of flesh. Biological tissues

such as flesh, fat and muscles have one important property: Volume preservation. As a result the poisson's ratio is a higher value, meaning it is found in the range between 0.45 and 0.5 [SGK18].

The calculation of the poisson's ratio defined as in equation 2.4 is a challenge. Fortunately we can make use of the *Lamé Parameters*. These are the two constants μ and λ that are specific for a material. With the help of these two constants we can transform equation 2.4 into the following form:

$$\sigma = \frac{\lambda}{2(\lambda + \mu)} \tag{2.5}$$

This equation allows us to calculate the poisson's ratio much easier.

TODO: Topic generally covered, maybe adjust the following: Explain why volume preservation => high poisson's ratio. Explain how to get formula with lamé parameters, explain lamé parameters. Explain more on negative poisson's value?
Further reading: http://silver.neep.wisc.edu/ lakes/PoissonIntro.html

2.4.4 Deformation Energy

In order to get a convincing simulation of high quality we must choose an appropriate energy. In the case of modelling deformations on human-like characters we have to choose an elastic energy. The key property that makes an energy elastic is that if all the forces that are applied over an object add up to zero the object must come back into its rest shape.
The energy then has to be minimized in order to get the results we want.

Definition 2. *This is a definition.*

TODO: Add some exaples and visualisations. What exactly is it and what role does it play in a deformation process? Hyperelastic energy, connect with paper.
To include: Piola-Kirchhoff Stress, Cauchy Green invariant, polar decomposition, cauchy green tensor

Chapter 3

Paper

In this chapter we will examine the topic of the paper *Stable Neo-Hookean Flesh Simulation* [SGK18]. In the interest of understanding the thought process of the authors I will include some of their calculations a bit more detailed. In addition examples and visualisations should help for a better perception.

3.1 Deformation Gradient

In the following calculations some properties of the deformation gradient $\mathbf{F}$ covered in the background chapter will be used. These properties are summarized in table 3.1 for a better overview:

Symbol	**Definition**
$\mathbf{F} = \mathbf{RS}$	Polar decomposition
$J = \det(\mathbf{F})$	Relative volume change
$\mathbf{C} = \mathbf{F}^\intercal\mathbf{F}$	Right Cauchy-Green
$I_C = \mathrm{tr}(\mathbf{C})$	First right Cauchy-Green invariant

Table 3.1: Quantities Derived from the Deformation Gradient $\mathbf{F}$ taken from [SGK18]

TODO: Check if everything is explained enough in Background.

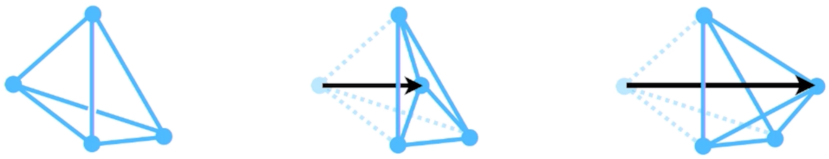
3.2 Energy Formulation

3.2.1 Stability

The core goal of the paper was to model deformations for virtual characters that have human-like features. They concentrated on the deformation energy. In order to achieve a convincing result as a first step we need to specify some requirements. For our needs in this case the stability of the energy is important. More precisely we need a hyperelastic energy that is stable in the following four important ways:

1. Inversion Stability: Given some arbitrary object it is possible that while deforming the object we can arrive at a zero volume state or even an entire inversion. Take for example the tetrahedron shown in figure 3.1a. In figure 3.1b we see a deformed state of this tetrahedron where the volume is scaled down to zero and we are left with a simple triangle. In figure 3.1c image we have an inversion of the object. The needed deformation energy has to be able to deal with both cases. That means that the energy has to be singularity-free and does not need any filters or threshold ([SGK18], 12:3).

TODO: Explain what last sentence means. Reference correct like this?



(a) Rest state (b) Zero-volume state (c) Inversed state

Figure 3.1: Inversion of a tetrahedron [Pix]

2. Reflection stability: A reflection is rotation around the coordinate origin. For the deformation energy we need it to be well behaved regardless of the reflection convention used in the singular value decomposition.

3. Rest stability: When deforming an object in a certain way we apply one or multiple forces over that object. With rest stability we want that

if the sum of forces is equal to zero the object must be back in its rest state.

4. Meta-stability under degeneracy: We can crush an object into an arbitrary shape like a plane, line or point. That process is illustrated for a cube in figure 3.2. The cube should now be able to recover to its actual shape after the deformation.

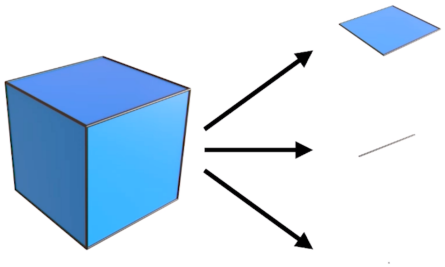


Figure 3.2: Illustration of meta stability [Pix]

Based on these four requirements we will in the following determine if a deformation energy is suited for our needs.

TODO: Finish reflection stability. Maybe add own images? Some more sources?

3.2.2 Existing Neo-Hookean Energies

In previous literature a few energies were proposed that should be analysed properly.

Energy	**Author(s)**
$\Psi_{Neo} = \frac{\mu}{2}(I_C - 3) - \mu \log J + \frac{\lambda}{2}(\log J)^2$	e.g. Bonet and Wood 2008
$\Psi_{\mathrm{A}} = \frac{\mu}{2}(I_C - 3) - \mu \log J + \frac{\lambda}{2}(J-1)^2$	Odgen 1997
$\Psi_{\mathrm{B}} = \frac{\mu}{2}\left(J^{-2/3} I_C - 3\right) + \frac{\lambda}{2}(J-1)^2$	Bower 2009
$\Psi_{\mathrm{C}} = \frac{\mu}{2}\left(J^{-2/3} I_C - 3\right) + \frac{\lambda}{2}(J-1)^2$	Wang and Yang 2016

Table 3.2: Summary of proposed energies taken from [SGK18]

Each energy formulation can be split up into a 1D length term and a 3D volume term. The 1D length term penalizes length changes the object undergoes whereas the 3D volume term is a volume-preserving penalty term.

Energy	**1D length term**	**3D volume term**
Ψ_{Neo}	$\frac{\mu}{2}(I_C - 3)$	$-\mu \log J + \frac{\lambda}{2}(\log J)^2$
Ψ_{A}	$\frac{\mu}{2}(I_C - 3)$	$-\mu \log J + \frac{\lambda}{2}(J-1)^2$
Ψ_{B}	$\frac{\mu}{2}\left(J^{-2/3} I_C - 3\right)$	$\frac{\lambda}{2}(J-1)^2$
Ψ_{C}	$\frac{\mu}{2}\left(J^{-2/3} I_C - 3\right)$	$\frac{\lambda}{2}(J-1)^2$

Table 3.3: Energies split up into its 1D length and 3D volume term

1D Length Term

Mooney ([Moo40]) originally proposed the 1D length term

$$\Psi_M = \frac{\mu}{2}(I_C - 3)$$

that is used in Ψ_{Neo} and Ψ_A. If we expand the energy with the singular values of the deformation gradient $\mathbf{F}$ we get the following term:

$$\Psi_M = \frac{\mu}{2}\left(\sigma_0^2 + \sigma_1^2 + \sigma_2^2 - 3\right)$$

The energy as it is formulated reaches its minimum at a zero volume state, meaning $I_C = 0$ which results in $\Psi_M = -3$. Mooney added the hard constraint that $J = 1$ so that the energy is minimized at the volume preserving configuration that is closest to the stretch space origin. Note that the energy is singularity free and well defined under inversion.

The second term

$$\Psi_R = \frac{\mu}{2}\left(J^{-2/3} I_C - 3\right)$$

is used in Ψ_B and Ψ_C and was introduced by Rivlin in 1948. Using the singular values of $\mathbf{F}$ we get the following term:

$$\Psi_R = \frac{\mu}{2}\left(\frac{\sigma_0^2 + \sigma_1^2 + \sigma_2^2}{(\sigma_0\sigma_1\sigma_2)^{\frac{2}{3}}} - 3\right)$$

Unfortunately this term is not singularity free. If either σ_0, σ_1 or σ_2 is equal to zero the result is not defined anymore.

3D Volume Term

The volume term of Ψ_{Neo}, meaning

$$\Psi_{Neo,volume} = -\mu \log J + \frac{\lambda}{2}(\log J)^2$$

results in some numerical problems since the logarithmic function is not defined for $J < 0$ and grows unbounded for $J \to 0$. In conclusion $\Psi_{Neo,volume}$ is not singularity free. The same applies for the 3D volume term of Ψ_A, namely

$$\Psi_{A,volume} = -\mu \log J + \frac{\lambda}{2}(J-1)^2.$$

The term of Ψ_A and Ψ_B which is of the form

$$\Psi_M = \frac{\lambda}{2}(J-1)^2$$

does not have these noted difficulties. It is bounded, well defined and invertible. After these observations we combine the robust length with the robust volume term and receive Ψ_D that is defined as

$$\Psi_D = \frac{\mu}{2}(I_C - 3) + \frac{\lambda}{2}(J-1)^2$$

and is singularity free and well defined under inversion. Unfortunately Ψ_D does not satisfy the requirement of being rest stable which will be discussed in the next section.

TODO: Bonet and Wood 2008 or 1997? Table looks ugly, improve and add resources. Overall okay?

3.2.3 Rest Stabilization

Although Ψ_D meets almost all stated requirements it is not rest stable. This can be shown with the Piola-Kirchhoff (PK1) stress tensor for Ψ_D

depending on the identity matrix $\mathbf{I}$:

$$\begin{aligned} P_D(\mathbf{I}) = \frac{\partial \Psi_D}{\partial \mathbf{F}}(\mathbf{I}) &= \frac{\partial \Psi_D}{\partial \mathbf{F}} \left[\frac{\mu}{2} (I_C - 3) + \frac{\lambda}{2}(J-1)^2 \right] \\ &= \frac{\partial \Psi_D}{\partial \mathbf{F}} \frac{\mu}{2} (\mathrm{tr}(\mathbf{I}^\intercal \mathbf{I}) - 3) + \frac{\partial \Psi_D}{\partial \mathbf{F}} \frac{\lambda}{2} (\mathrm{tr}(\mathbf{I}) - 1)^2 \\ &= \mu I + \lambda(\det(\mathbf{I}) - 1) \frac{\partial}{\partial \mathbf{F}} \det(\mathbf{I}) = \mu \mathbf{I} \neq 0 \end{aligned}$$

If the energy had rest stability $P_D(\mathbf{I})$ would resolve to zero. Unfortunately this is not the case here. In order to solve that problem the authors modified $(J-1)^2$ to $(J-\alpha)^2$. Using this modification the energy shifts to

$$\Psi_E = \frac{\mu}{2} (I_C - 3) + \frac{\lambda}{2}(J-\alpha)^2.$$

Taking Ψ_E in the PK1 equation from before we get

$$\begin{aligned} P_E(\mathbf{F}) &= \frac{\partial \Psi_E}{\partial \mathbf{F}} \left[\frac{\mu}{2} (I_C - 3) + \frac{\lambda}{2}(J-\alpha)^2 \right] \\ &= \mu \mathbf{F} + \lambda(\det(\mathbf{F}) - \alpha) \frac{\partial}{\partial \mathbf{F}} \det(\mathbf{F}). \end{aligned}$$

Solving for an alpha that satisfies $P_E(\mathbf{I}) = 0$ gives us $\alpha = 1 + \frac{\mu}{\lambda}$. Now Ψ_E has to be changed accordingly:

$$\begin{aligned} \Psi_E &= \frac{\mu}{2} (I_C - 3) + \frac{\lambda}{2}(J - 1 - \frac{\mu}{\lambda})^2 \\ &= \frac{\mu}{2} (I_C - 3) - \mu (J-1) + \frac{\lambda}{2}(J-1)^2 + \left(\frac{\mu}{\lambda}\right)^2. \end{aligned}$$

Since constant disappear under differentiation this expression is functionally equivalent to

$$\Psi_E = \frac{\mu}{2} (I_C - 3) - \mu (J-1) + \frac{\lambda}{2}(J-1)^2.$$

Now one can notice that Ψ_E looks very similar to Ψ_{Neo}. The difference is that $log(J)$ is replaced with $(J-1)$ in Ψ_E. Remember that $(J-1)$ is

the first term in the taylor approximation of $\log(J)$ at $J = 1$:

$$\begin{aligned}\sum_{n=0}^{\infty} &= \frac{f^{(n)}(1)}{n!}(J-1)^n \\ &= \boldsymbol{(J-1)} - \frac{1}{2}(J-1)^2 + \frac{1}{3}(J-1)^3 - \frac{1}{4}(J-1)^4 + ...\end{aligned}$$

TODO: Add "Zwischenschritte"? Include all calculations? Do we need a bit of a better conclusion of what we did (taylor, neo, sing. free, rest stable)? Check consistency of calculations (when trace when I_C etc.).

3.2.4 Meta-Stability under Degeneracy

The final energy is

$$\Psi_{new} = \frac{\mu}{2}(I_C - 3) + \frac{\lambda}{2}(J-\alpha)^2 - \frac{\mu}{2}\log(I_C + 1). \tag{3.1}$$

With that adjustment the rest stability term shifts to $\alpha = 1 + \frac{\mu}{\lambda} - \left(\frac{\mu}{4}\right)\lambda$.

TODO: How much needs to be explained here?

3.2.5 Lamé Reparametrization

TODO: How much needs to be explained here?

3.3 Energy Analysis

The goal of this chapter is to show that a complete eigenanalysis can be performed on equation 3.1.

3.3.1 First Piola-Kirchhoff Stress (PK1)

In order to analyse the energy PK1 can be used for equation 3.1:

$$\begin{aligned} P(\mathbf{F}) &= \frac{\partial \Psi_D}{\partial \mathbf{F}} \left[\frac{\mu}{2} (I_C - 3) + \frac{\lambda}{2} (J - \alpha)^2 - \frac{\mu}{2} (I_C + 1) \right] \\ &= \mu \mathbf{F} + \lambda (\det(\mathbf{F}) - \alpha) \frac{\partial}{\partial \mathbf{F}} \det(\mathbf{F}) - \frac{\partial}{\partial \mathbf{F}} \left[\log \left(\operatorname{tr}(\mathbf{F}^\intercal \mathbf{F}) + 1 \right) \right] \\ &= \mu \mathbf{F} + \lambda (\det(\mathbf{F}) - \alpha) \frac{\partial}{\partial \mathbf{F}} \det(\mathbf{F}) - \mu \mathbf{F} \frac{1}{\operatorname{tr}(\mathbf{F}^\intercal \mathbf{F}) + 1} \\ &= \mu \left(1 - \frac{1}{\operatorname{tr}(\mathbf{F}^\intercal \mathbf{F}) + 1} \right) \mathbf{F} + \lambda (J - \alpha) \frac{\partial J}{\partial \mathbf{F}} \end{aligned}$$

with $\alpha = 1 + \frac{\mu}{\lambda} - \left(\frac{\mu}{4} \right) \lambda$. For future use it can be more practical to write $\frac{\partial J}{\partial \mathbf{F}}$ as a result of cross products:

$$\frac{\partial J}{\partial \mathbf{F}} = \left[f_1 \times f_2 \,\middle|\, f_2 \times f_0 \,\middle|\, f_0 \times f_1 \right]. \tag{3.2}$$

TODO: Is it easy to follow? Maybe add some explanations of some steps? Maybe move cross product into background instead of here.

3.3.2 The Energy Hessian Terms

Using the scalar notation for $\mathbf{F}$ the hessian of the energy can be written in a fourth-order matrix-of-matrices:

$$\frac{\partial^2 \psi}{\partial \mathbf{F}^2} = \frac{\partial P(\mathbf{F})}{\partial \mathbf{F}} = \begin{bmatrix} \left[\frac{\partial P(\mathbf{F})}{\partial f_0} \right] & \left[\frac{\partial P(\mathbf{F})}{\partial f_3} \right] & \left[\frac{\partial P(\mathbf{F})}{\partial f_6} \right] \\ \left[\frac{\partial P(\mathbf{F})}{\partial f_1} \right] & \left[\frac{\partial P(\mathbf{F})}{\partial f_4} \right] & \left[\frac{\partial P(\mathbf{F})}{\partial f_7} \right] \\ \left[\frac{\partial P(\mathbf{F})}{\partial f_2} \right] & \left[\frac{\partial P(\mathbf{F})}{\partial f_5} \right] & \left[\frac{\partial P(\mathbf{F})}{\partial f_8} \right] \end{bmatrix}$$

in which each entry is defined as

$$\frac{\partial P(\mathbf{F})}{\partial f_i} = \frac{\partial}{\partial f_i} \left[\mu \left(1 - \frac{1}{\operatorname{tr}(\mathbf{F}^\intercal \mathbf{F}) + 1} \right) \mathbf{F} + \lambda (J - \alpha) \frac{\partial J}{\partial \mathbf{F}} \right]$$

$$\overset{\text{prod.rule}}{=} \underbrace{\frac{\partial \mathbf{F}}{\partial f_i}\mu\left(1-\frac{1}{\text{tr}(\mathbf{F}^\intercal\mathbf{F})+1}\right)}_{\text{T}_i} + \underbrace{\mathbf{F}\mu\frac{2}{(\text{tr}(\mathbf{F}^\intercal\mathbf{F})+1)^2}f_i}_{\text{M}_i}$$

$$r + \underbrace{\lambda\frac{\partial J}{\partial f_i}\frac{\partial J}{\partial \mathbf{F}}}_{\text{G}_i} + \underbrace{\lambda(J-\alpha)\frac{\partial^2 J}{\partial \mathbf{F}\partial f_i}}_{\text{H}_i} \in \mathbb{R}^3.$$

Each term in the final equation T_i, M_i, G_i, H_i respectively the Tikohonov, Mu, volume Gradient and volume Hessian term will be examined separately in the following sections.

3.3.3 The Tikhonov, Mu, and Gradient Terms

Tikhonov

The Tikhonov term can be viewed as a fourth-order matrix-of-matrices

$$\mathbb{T} = \frac{\partial \mathbf{F}}{\partial f_i} = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{bmatrix}.$$

If we vectorize $\mathbb{T}$ we get the identity matrix $\mathbf{I} \in \mathbb{R}^{9x9}$ which is full rank, positive definite and independent of the values in F. It serves as a regularizer for the rest of the energy

$$\text{vec}(\mathbb{T}) = \check{\mathbf{T}} = \mathbf{I} =\in \mathbb{R}^{9x9}.$$

Mu

The Mu term has the same structure with different entries

$$\mathbb{M} = \mathbf{F} f_i = \begin{bmatrix} \begin{bmatrix} f_0^2 & f_0 f_3 & f_0 f_6 \\ f_0 f_1 & f_0 f_4 & f_0 f_7 \\ f_0 f_2 & f_0 f_5 & f_0 f_8 \end{bmatrix} & \begin{bmatrix} f_3 f_0 & f_3^2 & f_3 f_6 \\ f_3 f_1 & f_3 f_4 & f_3 f_7 \\ f_3 f_2 & f_3 f_5 & f_3 f_8 \end{bmatrix} & \begin{bmatrix} f_6 f_0 & f_6 f_3 & f_6 f_6 \\ f_6 f_1 & f_6 f_4 & f_6 f_7 \\ f_6 f_2 & f_6 f_5 & f_6 f_8 \end{bmatrix} \\ \begin{bmatrix} f_1 f_0 & f_1 f_3 & f_1 f_6 \\ f_1^2 & f_1 f_4 & f_1 f_7 \\ f_1 f_2 & f_1 f_5 & f_1 f_8 \end{bmatrix} & \begin{bmatrix} f_4 f_0 & f_4 f_3 & f_4 f_6 \\ f_4 f_1 & f_4^2 & f_4 f_7 \\ f_4 f_2 & f_4 f_5 & f_4 f_8 \end{bmatrix} & \begin{bmatrix} f_7 f_0 & f_7 f_3 & f_7 f_6 \\ f_7 f_1 & f_7 f_4 & f_7^2 \\ f_7 f_2 & f_7 f_5 & f_7 f_8 \end{bmatrix} \\ \begin{bmatrix} f_2 f_0 & f_2 f_3 & f_2 f_6 \\ f_2 f_1 & f_2 f_4 & f_2 f_7 \\ f_2^2 & f_2 f_5 & f_2 f_8 \end{bmatrix} & \begin{bmatrix} f_5 f_0 & f_5 f_3 & f_5 f_6 \\ f_5 f_1 & f_5 f_4 & f_5 f_7 \\ f_5 f_2 & f_5^2 & f_5 f_8 \end{bmatrix} & \begin{bmatrix} f_8 f_0 & f_8 f_3 & f_8 f_6 \\ f_8 f_1 & f_8 f_4 & f_8 f_7 \\ f_8 f_2 & f_8 f_5 & f_8^2 \end{bmatrix} \end{bmatrix}.$$

When vectorizing $\mathbb{M}$ the resulting matrix has the squared values of f_i placed on the diagonal

$$\mathrm{vec}(\mathbb{M}) = \check{\mathbf{M}} = \begin{bmatrix} f_0^2 & f_1 f_0 & f_2 f_0 & f_3 f_0 & f_4 f_0 & f_5 f_0 & f_6 f_0 & f_7 f_0 & f_8 f_0 \\ f_0 f_1 & f_1^2 & f_2 f_1 & f_3 f_1 & f_4 f_1 & f_5 f_1 & f_6 f_1 & f_7 f_1 & f_8 f_1 \\ f_0 f_2 & f_1 f_2 & f_2^2 & f_3 f_2 & f_4 f_2 & f_5 f_2 & f_6 f_2 & f_7 f_2 & f_8 f_2 \\ f_0 f_3 & f_1 f_3 & f_2 f_3 & f_3^2 & f_4 f_3 & f_5 f_3 & f_6 f_3 & f_7 f_3 & f_8 f_3 \\ f_0 f_4 & f_1 f_4 & f_2 f_4 & f_3 f_4 & f_4^2 & f_5 f_4 & f_6 f_4 & f_7 f_4 & f_8 f_4 \\ f_0 f_5 & f_1 f_5 & f_2 f_5 & f_3 f_5 & f_4 f_5 & f_5^2 & f_6 f_5 & f_7 f_5 & f_8 f_5 \\ f_0 f_6 & f_1 f_6 & f_2 f_6 & f_3 f_6 & f_4 f_6 & f_5 f_6 & f_6^2 & f_7 f_6 & f_8 f_6 \\ f_0 f_7 & f_1 f_7 & f_2 f_7 & f_3 f_7 & f_4 f_7 & f_5 f_7 & f_6 f_7 & f_7^2 & f_8 f_7 \\ f_0 f_8 & f_1 f_8 & f_2 f_8 & f_3 f_8 & f_4 f_8 & f_5 f_8 & f_6 f_8 & f_7 f_8 & f_8^2 \end{bmatrix}.$$

This structure makes it possible to write $\check{\mathbf{M}}$ as an outer product of $\mathrm{vec}(\mathbf{F})$

$$\check{\mathbf{M}} = \mathrm{vec}(\mathbf{F})\,\mathrm{vec}(\mathbf{F})^\intercal = \check{\mathbf{f}}\check{\mathbf{f}}^\intercal.$$

This matrix has rank one and has a single non-zero eigenvalue. In order to examine the eigenvalues one can calculate

$$\|\check{\mathbf{f}}\|_2^2 = \sum_{n=0}^{8} |f_n|^2 = \|\mathbf{F}\|_F^2 = \sum_{n=0}^{3} \sigma_i^2 = \left(\sigma_0^2 + \sigma_1^2 + \sigma_2^2\right)$$

in which $\|\cdot\|_F$ stands for the Frobenius norm and σ_i for the singular values from $\mathbf{\Sigma}$ in the SVD of $\mathbf{F}$ stated in eq. 2.3. The eigenvector is

$\check{\mathbf{f}}/\|\check{\mathbf{f}}\|$. The eigenvalue is always non-negative and large if $\mathbf{F}$ contains a large strech.

Gradient

The gradient term also has the same structure as the two terms before with different entries defined by

$$\mathbb{G}(\mathbf{F}) = \frac{\partial J}{\partial \mathbf{F}} \frac{\partial J}{\partial fi}.$$

The vectorized matrix $\check{\mathbf{G}}$ can be written in a similar form as for the Mu term. With the help of the cross product of $\partial J/\partial \mathbf{F}$ stated in eq. 3.2 $\check{\mathbf{G}}$ can be written as an outer product in the following way:

$$\text{vec}(\mathbb{G}(\mathbf{F})) = \check{\mathbf{G}} = \text{vec}\left(\frac{\partial J}{\partial \mathbf{F}}\right) \text{vec}\left(\frac{\partial J}{\partial \mathbf{F}}\right)^{\intercal} = \check{\mathbf{g}}\check{\mathbf{g}}^{\intercal}.$$

As in the Mu term one can notice a single non-zero, non-negative:

$$\|\check{\mathbf{g}}\|_2^2 = \left\|\frac{\partial J}{\partial \mathbf{F}}\right\|_F^2 = \left[(\sigma_0\sigma_1)^2 + (\sigma_0\sigma_2)^2 + (\sigma_1\sigma_2)^2\right].$$

The eigenvector is $\check{\mathbf{g}}/\|\check{\mathbf{g}}\|$.

3.3.4 The Volume Hessian

The volume Hessian term is of the form

$$\mathbb{H} = \frac{\partial^2 J}{\partial \mathbf{F} \partial f_i} \in \mathbb{R}^{9\times 9}.$$

Vectorizing $\mathbb{H}$ reveals the structure

$$\text{vec}(\mathbb{H}) = \check{\mathbf{H}} = \begin{bmatrix} 0 & -\widehat{\mathbf{f_2}} & \widehat{\mathbf{f_1}} \\ \widehat{\mathbf{f_2}} & 0 & -\widehat{\mathbf{f_0}} \\ -\widehat{\mathbf{f_1}} & \widehat{\mathbf{f_0}} & 0 \end{bmatrix}$$

where $\widehat{\mathbf{f_1}}$ stands for a cross-product matrix:

$$\widehat{\mathbf{x}} = \begin{bmatrix} 0 & -x_2 & x_1 \\ x_2 & 0 & -x_0 \\ -x_1 & x_0 & 0 \end{bmatrix}.$$

One can easily observe that $\check{\mathbf{H}}$ is a *self-similar* cross-product matrix. Meaning that the macro structure of the matrix is the same as the micro structure.

Volume Hessian Eigenvalues

For determining the eigenvalues of $\check{\mathbf{H}}$ one needs to examine the two characteristic polynomials:

$$\epsilon^3 - \mathrm{tr}(\mathbf{C})\epsilon - 2J = 0 \tag{1}$$

$$\epsilon^3 - \mathrm{tr}(\mathbf{C})\epsilon^2 + \frac{1}{2}\left(\mathrm{tr}^2(\mathbf{C}) - \mathrm{tr}(\mathbf{C}^2)\right)\epsilon - \det(\mathbf{C}) = 0 \tag{2}$$

where ϵ denote the eigenvalues of $\check{\mathbf{H}}$ and $\mathbf{C}$ is taken from table 3.1. Equation (1) is easier to solve and

3.3.5 The Complete Eigensystem

Calculations

TODO: Make notation consistent (bold, cursive etc.) Explain better each step. Include all calculations? Make matrix of matrices more beautiful (more space). Eigenvalues of Mu and Gradient term better explain? Check each equation if correctly written.

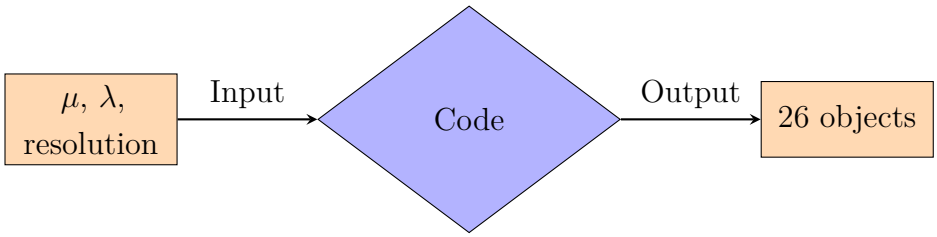
3.4 Experiments with the Code

The authors of the paper *Stable Neo-Hookean Flesh Simulation* [SGK18] provided the implementation for an application of their formulated energy. In said code they implemented the stretch test on a cube. Their

implementation demands a directory into which the output files should be saved, the two lamé parameters μ and λ and a value for defining the desired resolution as input data. Here is an example for a command with resolution = 10, $\mu = 1.0$, $\lambda = 10.0$ and asks the output files to be written into the directory *output*:

```
1 $ ./tetcli 10 stable_neo_hookean 1.0 10.0 output
```

The algorithm then calculates the deformation in 25 steps and the deformation increases in each step. The outputs are 26 static objects that show the object in its rest state and the 25 steps of deformation.



TODO: Explain how the code is implemented in simple words and how the energy is taken in account with the poisson's ratio. Do I have to reference code? Explain tetcli and Hexcli.

For starters let us take common values for μ and λ. We first start with $\mu = 1.0$, $\lambda = 10.0$ and a resolution of 10.0. For the poisson's ratio we get the value 0.4545:

$$\sigma = \frac{10.0}{2(10.0 + 1.0)} = 0.4545 \in [-1, 0.5]$$

The following images in figure 3.3 show the stretch test with $\mu = 1.0$, $\lambda = 10.0$ and a resolution of 10.0 on a tetrahedral and a hexahedral mesh.

TODO: Load into OpenFlipper and screenshot results. Include more examples and what went right and what went wrong.

(a) Stretch test on a hexahedral mesh

(b) Stretch test on a tetrahedral mesh

Figure 3.3: Stretch test performed on a cube with (a) a hexahedral mesh and (b) a tetrahedral mesh

3.5 Discussion

Stuff, Taylor approx.

List of Figures

List of Tables

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Online Sources

[Pix] Pixar. *Deformation Map.* URL: https://dl.acm.org/ft_gateway.cfm?id=3180491&ftid=2009597.

Figure Sources

[Pix] Pixar. *Deformation Map.* URL: https://dl.acm.org/ft_gateway.cfm?id=3180491&ftid=2009597.

Erklärung

gemäss Art. 28 Abs. 2 RSL 05

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