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Convex Optimization

Homework 1

Convex Sets

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Problem 1: Convexity of the Set S

Is the set

$$S = \left\{ a \in \mathbb{R}^{k+1} \mid p(0) = 1, |p(t)| \leq 1 \text{ for } \alpha \leq t \leq \beta \right\} \tag{1}$$

where

$$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_k t^k \tag{2}$$

convex? Using MATLAB or Python, visualize the set for $k = 2$ *as an intersection of slabs.*
Note that this problem is similar to the one in Lecture 02 Convex Sets*, slide 7, where we visualized the set of Fourier coefficients for* $m = 2$*.*

To begin with, let us separate the convexity analysis into the two different conditions the subset S presents: $p(0) = 1$ and $p(t)| \leq 1$ for $\alpha \leq t \leq \beta$. For both, convexity is proven when

$$x, y \in S, \ \lambda, \mu \geq 0, \ \lambda + \mu = 1 \implies \lambda x + \mu y \in S \tag{3}$$

is satisfied. Starting from the first one, let us define $a, b \in S$. It is implied that $p_a(0) = 1$ from the definition of S. Then, $a_0 = 1$. Similarly, for $b \in S_b \in S$, $p_b(0) = 1$, then $b_0 = 1$.

From the convexity definition (equation 3), and using a and b, c needs to be part of S in:

$$c = a\lambda + b\mu \tag{4}$$

Taking the polynomial at $t = 0$:

$$p_c(0) = \lambda p_a(0) + \mu p_b(0) = \lambda \cdot 1 + \mu \cdot 1 = \lambda + \mu = 1 \tag{5}$$

thus $c \in S$, proving this condition.

For the second one, and again, for $a, b \in S$, it is known that $|p_a(t)| \leq 1$ and $|p_b(t)| \leq 1$ for the interval $\alpha \leq t \leq \beta$. Taking again c and trying to prove it is in S as per equation 4,

$$p_c(t) = \lambda p_a(t) + \mu p_b(t) \tag{6}$$

applying the inequality:

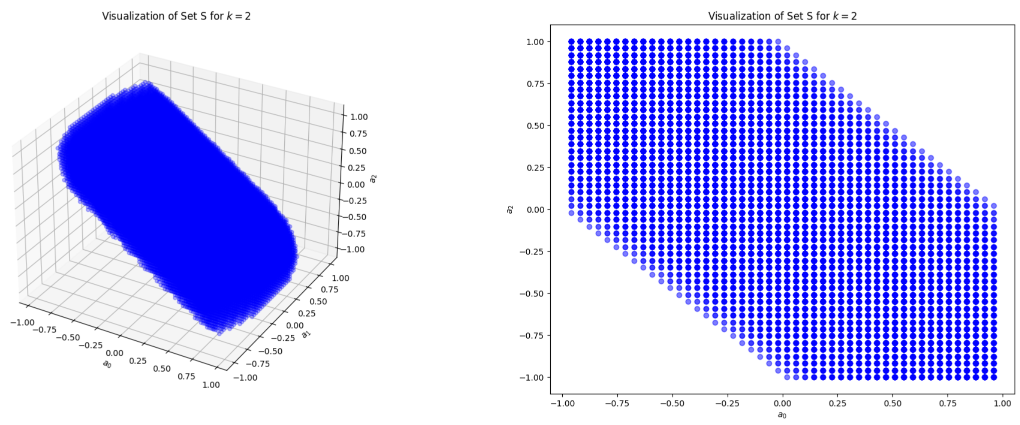
$$|p_c(t)| = |\lambda p_a(t) + \mu p_b(t)| \leq \lambda |p_a(t)| + \mu |p_b(t)| \tag{7}$$

on the right-hand side, it is possible to use the fact that $|p_a(t)|, |p_b(t)| \leq 1$, thus:

$$|p_c(t)| \leq \lambda \cdot 1 + \mu \cdot 1 = \lambda + \mu = 1 \tag{8}$$

Therefore, $|p_c(t)| \leq 1$ for $\alpha \leq t \leq \beta$, fulfilling the second and last condition. Necessarily, then, S is convex.

Using python, this set can be visualized for $k = 2$, as shown in Figure 1.



(a) Set S, k = 2, in 3D.

(b) Set S, k = 2, in 2D.

Figure 1: Visualizations for the Set S.

Problem 2: Representations of Ellipsoid

Show analytically (by derivations or formal arguments) the equivalence between the following three representations of an ellipsoid (See also Lecture 02 Convex Sets, *slide 9):*

- $E = \{x \mid (x - x_c)^T A^{-1} (x - x_c) \leq 1\}$, *where* A *is a symmetric positive definite matrix and* $x_c \in \mathbb{R}^n$ *is the center of the ellipsoid.*
- $E = \{Bu + x_c \mid \|u\|_2 \leq 1\}$, *where* $\|u\|_2$ *is the Euclidean norm of* u.
- $E = \{x \mid f(x) \leq 0\}$, *where* $f(x) = x^T C x + 2d^T x + e$, C *is a symmetric positive definite matrix, and* $e - d^T C^{-1} d < 0$.

Beginning with the first two, one can start by substituting A^{-1} by $B^T B$, where B is an invertible matrix, like:

$$(x - x_c)^T B^T B (x - x_c) = ||B(x - x_c)||_2^2 \tag{9}$$

which in turn means

$$||B(x - x_c)||_2 \leq 1 \tag{10}$$

Now, let $u = B(x - x_c)$, which allows an approach to the second representation, as $||u||_2 \leq 1$. In fact, this is just a step away, for now one only needs to rearrange terms and set $B^{-1} = B$, to get:

$$E = \{Bu + x_c \mid ||u||_2 \leq 1\} \tag{11}$$

Now, to find the third representation, it is useful to execute the substitution $x = Bu + x_c$, so:

$$(Bu + x_c)^T C (Bu + x_c) + 2d^T (Bu + x_c) + e \leq 0 \tag{12}$$

and, by aplying the different deffinitions for C, d, and e:

$$u^T B^T B u - 2u^T B^T x_c + x_c^T B^T B x_c - 1 \leq 0 \tag{13}$$

which can be simplified to get $u^T u - 1 \leq 0$, bringing it back to definition 2.

Problem 3: Conic Hull of Outer Products

Consider the set of rank-k outer products, defined as $\{XX^T \mid X \in \mathbb{R}^{n \times k}, \text{rank}(X) = k\}$. Describe its conic hull in simple terms.

The defined conic hull of a set S describes the set corpsing all **non-negative** linear combinations of elements in S; which is to mean it is the set of points that can be reached by scaling elements in S by non-negative numbers and adding them.

This is further limited by two constraints. First, the outer product XX^T is always positive semidefinite; which means that for any vector v, $v^T(XX^T)v \leq 0$. Secondly, its rank cannot be more than k. Thus, from these restrictions, it is possible to understand that this conic hull is the combination of positive semidefinite matrices in $\mathbb{R}^{n \times k}$ whose rank never exceeds k. This necessarily includes every element of S, and no larger cone exists inside these restrictions, for any positive semidefinite matrix with rank greater than k cannot be expressed as a non-negative linear combination of rank-k outer products.

Problem 4: Generalized Inequalities

To better understand how the generalized inequalities work, go through the proofs of the following properties (nonstrict and strict generalized inequalities for cones as they are defined in Lecture 02 *Convex Sets*, slide 18: closed, non-empty interior, pointed):

- $\preceq_K$ is preserved under addition: if $x \preceq_K y$ and $u \preceq_K v$, then $x + u \preceq_K y + v$.
- $\preceq_K$ is transitive: if $x \preceq_K y$ and $y \preceq_K z$, then $x \preceq_K z$.
- $\preceq_K$ is preserved under nonnegative scaling: if $x \preceq_K y$ and $\alpha \geq 0$, then $\alpha x \preceq_K \alpha y$.
- $\preceq_K$ is reflexive: $x \preceq_K x$.
- $\preceq_K$ is antisymmetric: if $x \preceq_K y$ and $y \preceq_K x$, then $x = y$.
- $\preceq_K$ is preserved under limits: if $x_i \preceq_K y_i$ for $i = 1, 2, \dots$, $x_i \to x$ and $y_i \to y$ as $i \to \infty$, then $x \preceq_K y$.
- If $x \prec_K y$, then $x \preceq_K y$.
- If $x \prec_K y$ and $u \preceq_K v$, then $x + u \prec_K y + v$.
- If $x \prec_K y$ and $\alpha > 0$, then $\alpha x \prec_K \alpha y$.
- $x \prec_K x$ is not true.
- If $x \prec_K y$, then for u and v small enough, $x + u \prec_K y + v$.

Problem 5: Dual Cones

Prove that the Second-Order Cone (SOC) (see Lecture 02 Convex Sets*, slide 12)*

$$K = \{(x, t) \mid \|x\|_2 \leq t\}$$

is self-dual.

The main goal here is to see whether the dual cone K^* is the same as the first one, K. K^* can be understood as:

$$K^* = \{y| < x, y > \geq 0 \ \forall x \in K\} \tag{14}$$

To show that K is a subset of K^*, it is necessary to show that for any $(x, t) \in K$, $< (x, t), (y, s) > \geq 0 \ \forall (y, s) \in K^*$. For that, let $(x, t) \in K$, and $(y, s) \in K^*$. Then,

$$< (x, t), (y, s) > = < x, y > + ts \geq ||x||_2 \cdot ||y||_2 + ts \geq 0 \tag{15}$$

Inversely, to show that K^* is a subset of K (and thus, that both are the same), we need to show that for any $(y, s) \in K^*$, $||y||_2 \leq s$. Let $(y, s) \in K^*$. Then, for any $(x, t) \in K$, necessarily $< x, y > + ts \geq 0$.
If $x = y/||y||_2$ and $t = ||y||_2$. Then,

$$< x, y > + ts = ||y||_2 + ||y||_2^2 \geq 0 \tag{16}$$

which implies that $||y||_2 \leq s$.

Appendix A: Python Code - Visualizations for the Set S

```python
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

alpha = -1
beta = 1
t_values = np.linspace(alpha, beta, 100)

# Grid
a0_values = np.linspace(-1, 1, 50)
a1_values = np.linspace(-1, 1, 50)
a2_values = np.linspace(-1, 1, 50)
a0, a1, a2 = np.meshgrid(a0_values, a1_values, a2_values)

# Polynomial p(t) = a0 + a1 * t + a2 * t^2
def p(t, a0, a1, a2):
    return a0 + a1 * t + a2 * t**2

# |p(t)| <= 1 for all t in [alpha, beta]
feasible = np.ones(a0.shape, dtype=bool)

for t in t_values:
    feasible &= (np.abs(p(t, a0, a1, a2)) <= 1)

# Plot the feasible region
fig = plt.figure(figsize=(10, 8))
ax = fig.add_subplot(111, projection='3d')

# Plot settings
ax.scatter(a0[feasible], a1[feasible], a2[feasible], c='blue', alpha=0.3)

ax.set_xlabel('$a_0$')
ax.set_ylabel('$a_1$')
ax.set_zlabel('$a_2$')
ax.set_title('Visualization of Set S for $k=2$')

plt.show()

# Now only visualize a_0 and a_2 in 2D
fig = plt.figure(figsize=(10, 8))
ax = fig.add_subplot(111)

# Plot settings
ax.scatter(a0[feasible], a2[feasible], c='blue', alpha=0.3)

ax.set_xlabel('$a_0$')
ax.set_ylabel('$a_2$')
ax.set_title('Visualization of Set S for $k=2$')

plt.show()
```