Convex Optimization: HW 1 (Convex Sets)

Due to (23:59) Oct. 1, 2024

You may also need to check Chapter 2 in

Textbook: https://web.stanford.edu/~boyd/cvxbook/bv_cvxbook.pdf

Problem 1: Is the set

$$\mathcal{S} = \{\mathbf{a} \in R^{k+1} \mid p(0) = 1,\ |p(t)| \le 1 \text{ for } \alpha \le t \le \beta\},$$

where

$$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_k t^k,$$

convex?

Using MATLAB or Python, visualize the set for $k = 2$ as an intersection of slabs.

Note that this problem is similar to the one in Lec02 'Convex Sets', slide 7, where we visualized the set of Fourier coefficients for $m = 2$.

Problem 2: *Representations of ellipsoid.* Show analytically (by derivations or formal arguments) the equivalence between the following three representations of ellipsoid (See also Lec02 'Convex Sets', slide 9.):

- $\mathcal{E} = \left\{x \mid (x - x_c)^T A^{-1}(x - x_c) \le 1\right\}$, where A is a symmetric positive definite matrix and $x_c \in R^n$ is the center of ellipsoid.
- $\mathcal{E} = \{Bu + x_c \mid \|u\|_2 \le 1\}$, where $\|u\|_2$ is the Euclidean norm of u.
- $\mathcal{E} = \{x \mid f(x) \le 0\}$, where $f(x) = x^T Cx + 2d^T x + e$, C is a symmetric positive definite matrix, and $e - d^T C^{-1} d < 0$.

Problem 3: *Conic hull of outer products.* Consider the set of rank-k outer products, defined as $\{\mathbf{X}\mathbf{X}^T \mid \mathbf{X} \in R^{n\times k},\ \text{rank}(\mathbf{X}) = k\}$. Describe its conic hull in simple terms.

Problem 4: *Generalized inequalities.* To better understand how the generalized inequities work go through the proves of the following the properties (nonstrict and strict generalized inequalities for cones as they are defined in Lec02 'Convex Sets', slide 18: closed, non-empty interior, pointed):

- $\preceq_K$ is preserved under addition: if $x \preceq_K y$ and $u \preceq_K v$, then $x + u \preceq_K y + v$.
- $\preceq_K$ is transitive: if $x \preceq_K y$ and $y \preceq_K z$, then $x \preceq_K z$.
- $\preceq_K$ is preserved under nonnegative scaling: if $x \preceq_K y$ and $\alpha \geq 0$, then $\alpha x \preceq_K \alpha y$.
- $\preceq_K$ is reflexive: $x \preceq_K x$.
- $\preceq_K$ is antisymmetric: if $x \preceq_K y$ and $y \preceq_K x$, then $x = y$.
- $\preceq_K$ is preserved under limits: if $x_i \preceq_K y_i$ for $i = 1, 2, \cdots$, $x_i \to x$ and $y_i \to y$ as $i \to \infty$, then $x \preceq_K y$.
- if $x \prec_K y$ then $x \preceq_K y$.
- if $x \prec_K y$ and $u \preceq_K v$, then $x + u \prec_K y + v$.
- if $x \prec_K y$ and $\alpha > 0$, then $\alpha x \prec_K \alpha y$.
- $x \prec_K x$ is not true.
- if $x \prec_K y$, then for u and v small enough, $x + u \prec_K y + v$.

Problem 5: *Dual cones.*

- Prove that Second-Order Cone (SOC) (see Lec02 'Convex Sets', slide 12)
$$\mathcal{K} = \{(x, t) \mid \|x\|_2 \leq t\}.$$
is self-dual.

- Find a dual cone for First-Order Cone (the cone where l1-norm is used instead of Euclidian norm as in SOC), that is,

$$\mathcal{K} = \{(x, t) \mid \|x\|_1 \leq t\}.$$

Problem 6: Suppose $K \subseteq R^2$ is a closed convex cone.

Give a simple description of K in terms of the polar coordinates of its elements ($x = r(\cos\phi, \sin\phi)$ with $r \geq 0$).

Give a simple description of the dual cone K^*, and draw a plot illustrating the relation between K and K^* using Matlab or Python, or it is easy to draw it by hands as well.

Problem 7: Express the convex set (we will revisit this set often in the future)

$$\mathcal{S} = \left\{ [x_1\, x_2]^T \in R_+^2 \;\middle|\; x_1 x_2 \geq 1 \right\}$$

as an intersection of halfspaces.

Possible Hint. There are several different ways to express a line that cuts a halfspace. One way is to use the following inequality (the meaning of it will become more clear later when we talk about Jensen's inequality and log-convex functions): if a and b are non-negative and $0 \leq \lambda \leq 1$, then $a^\lambda b^{1-\lambda} \leq \lambda a + (1 - \lambda) b$.

Using MATLAB or Python, visualize this set as an intersection of halfspaces.