Aalto-yliopisto
Aalto-universitetet
Aalto University

CS-E4850
Computer Vision

Homework 6

Affinity and Similarity

Aitor Urruticoechea Puig
aitor.urruticoecheapuig@aalto.fi
Student N°101444219
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Contents

Exercise 1: Least squares fitting for affine transformation

A brief overview of affine transformation estimation is presented on slides 17-19 of Lecture 5. Present a derivation and compute an example by performing the following stages:

1.1 Compute the gradient of the least squares error $E = \sum_{i=1}^{n} ||x'_i - Mx_i - t||^2$ with respect to the parameters of the transformation

(i.e. elements of matrix M and vector t).

Let us start by defining the residual $r_i = x'_i - Mx_i - t$. This will allow for the derivation with respect to M, as now E can be rewritten as:

$$E = \sum_{i=1}^{n} r_i^T r_i \tag{1}$$

which allows for:

$$\frac{\delta}{\delta M} r_i^T r_i = -2 r_i x_i^T \tag{2}$$

$$\frac{\delta}{\delta M} E = -2 \sum_{i=1}^{n} (x'_i - Mx_i - t) x_i^T \tag{3}$$

Similarly, for the derivation with respect to the vector t,

$$\frac{\delta}{\delta t} r_i^T r_i = -2r_i \tag{4}$$

$$\frac{\delta}{\delta t} E = -2\sum_{i=1}^{n}(x_i' - Mx_i - t) \tag{5}$$

1.2 Show that by setting the aforementioned gradient to zero you will get an equation of the form $Sh = u$, where vector h contains the unknown parameters of the transformation, and 6×6 matrix S and 6×1 vector u depend on the coordinates of the point correspondences $\{x_i', x_i\}$, $i = 1, ..., n$.

Let us start by setting the gradient to zero:

$$\begin{cases} \frac{\delta}{\delta M} E = -2\sum_{i=1}^{n}(x_i' - Mx_i - t)x_i^T = 0 \\ \frac{\delta}{\delta t} E = -2\sum_{i=1}^{n}(x_i' - Mx_i - t) = 0 \end{cases} \tag{6}$$

Which allows for a re-arrangement and simplification like:

$$\begin{cases} \sum_{i=1}^{n} x_i' x_i^T = \sum_{i=1}^{n} Mx_i x_i^T + t\sum_{i=1}^{n} x_i^T \\ \sum_{i=1}^{n} x_i' = -2\sum_{i=1}^{n} Mx_i + nt = 0 \end{cases} \tag{7}$$

This results in the expected four equations for the variables contained in M and two for those in t. This can then be used to construct the required system with the form $Sh = u$, as h can be defined by joining all the unknowns in a single vector, and S and u serve to complete the equation in matrix form:

$$h = \begin{bmatrix} m_{11} \\ m_{12} \\ m_{21} \\ m_{22} \\ t_1 \\ t_2 \end{bmatrix}, \quad S = \begin{bmatrix} \sum_{i=1}^{n} x_{i1}^2 & \sum_{i=1}^{n} x_{i1}x_{i2} & 0 & 0 & \sum_{i=1}^{n} x_{i1} & 0 \\ \sum_{i=1}^{n} x_{i1}x_{i2} & \sum_{i=1}^{n} x_{i2}^2 & 0 & 0 & \sum_{i=1}^{n} x_{i2} & 0 \\ 0 & 0 & \sum_{i=1}^{n} x_{i1}^2 & \sum_{i=1}^{n} x_{i1}x_{i2} & 0 & \sum_{i=1}^{n} x_{i1} \\ 0 & 0 & \sum_{i=1}^{n} x_{i1}x_{i2} & \sum_{i=1}^{n} x_{i2}^2 & 0 & \sum_{i=1}^{n} x_{i2} \\ \sum_{i=1}^{n} x_{i1} & \sum_{i=1}^{n} x_{i2} & 0 & 0 & n & 0 \\ 0 & 0 & \sum_{i=1}^{n} x_{i1} & \sum_{i=1}^{n} x_{i2} & 0 & n \end{bmatrix}, \quad U = \begin{bmatrix} \sum_{i=1}^{n} x_{i1}' x_{i1} \\ \sum_{i=1}^{n} x_{i1}' x_{i2} \\ \sum_{i=1}^{n} x_{i2}' x_{i1} \\ \sum_{i=1}^{n} x_{i2}' x_{i2} \\ \sum_{i=1}^{n} x_{i1}' \\ \sum_{i=1}^{n} x_{i2}' \end{bmatrix} \tag{8}$$

1.3 Thus, one may solve the transformation by computing $h = S^{-1}u$. Compute the affine transformation from the following point correspondences $\{(0,0) \implies (1,2)\}$, $\{(1,0) \implies (3,2)\}$, and $\{(0,1) \implies (1,4)\}$.

Hint: This calculation can be done with pen and paper but you may check the correct answer by running the function affinefit *given for the third exercise round in Exercise03.zip. Another way to check the answer is to draw the point correspondences on paper and visually determine the correct solution.*

Let us start by filling up the S and U matrices with the given values:

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \quad U = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \\ 5 \\ 8 \end{bmatrix} \tag{9}$$

which results in:

$$h = S^{-1}u = \begin{bmatrix} 1.33 \\ -0.67 \\ -0.67 \\ 1.33 \\ 1.67 \\ 2.67 \end{bmatrix} \tag{10}$$

Exercise 2: Similarity transformation from two point correspondences

A similarity transformation consists of rotation, scaling and translation and is defined in two dimensions as follows:

$$x' = sRx + t \leftrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = s \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix} \tag{11}$$

Describe a method for solving the parameters s, θ, t_x, t_y of a similarity transformation from two point correspondences $\{x_1 \rightarrow x_1'\}$, $\{x_2 \rightarrow x_2'\}$ using the following stages:

2.1 Compute the vectors $v' = x'_2 - x'_1$ and $v = x_2 - x_1$ and present a formula to recover the rotation angle θ from the corresponding unit vectors.

The secret here is to rewrite the seen in equation 11 to get the translation out of the way, and instead integrate it using the definitions of the vectors v and v′. Thus:

$$\begin{pmatrix} x'_2 - x'_1 \\ y'_2 - y'_1 \end{pmatrix} = s \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} \tag{12}$$

Now, to recover the rotation angle θ , it is necessary to first convert the vectors v and v′ into unit vectors, as this way the angle between these two unit vectors can be computed using the dot product formula:

$$\theta = \cos^{-1}\left(\hat{v'} \cdot \hat{v}\right) \tag{13}$$

where $\hat{v}$ and $\hat{v'}$ are the unit form of the vectors v and v′ respectively.

2.2 Compute the scale factor s as the ratio of the norms of vectors v′ and v.

Now that θ has been computed, the scale factor s becomes quite trivial, as it can be recovered by a direct comparison of the magnitude of the vectors v′ and v:

$$s = \frac{||v'||}{||v||} \tag{14}$$

2.3 After solving s and θ compute t using equation 11 and either one of the two point correspondences.

Thus, one can simply play around with equation 11 to solve for t:

$$t = x'_1 - sRx_1 \tag{15}$$

2.4 Use the procedure to compute the transformation from the following point correspondences: $\{(\frac{1}{2}, 0) \to (0, 0)\}$, $\{(0, \frac{1}{2}) \to (-1, -1)\}$.

Hint: Drawing the point correspondences on a grid paper may help you to check your answer.
Let us start by giving value to the vectors v and v′:

$$\begin{cases} v' = (0,0) - (-1,-1) = (1,1) \\ v = \left(0, \frac{1}{2}\right) - \left(\frac{1}{2}, 0\right) = \left(-\frac{1}{2}, \frac{1}{2}\right) \end{cases} \tag{16}$$

then, one can compute their corresponding unit vectors and thus, the angle θ:

$$\begin{cases} \hat{v'} = \frac{v'}{\sqrt{1^2+1^2}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \\ \hat{v} = \frac{v}{\sqrt{\frac{1}{\sqrt{2}}^2 + \frac{1}{\sqrt{2}}^2}} = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \end{cases} \tag{17}$$

$$\theta = \cos^{-1}\left(\frac{\frac{-1}{2} + \frac{1}{2}}{2}\right) = \frac{\pi}{2} \tag{18}$$

Now for the scale factor s:

$$s = \frac{\sqrt{2}}{\sqrt{1/2}} = 2 \tag{19}$$

And finally, the translation t:

$$\mathsf{t} = \mathsf{x}_1' - \mathsf{s}\mathsf{R}\mathsf{x}_1 = (0,0)^\mathsf{T} - 2\mathsf{R}\cdot(\frac{1}{2},0)^\mathsf{T} \tag{20}$$

where

$$\mathsf{R} = \begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{21}$$

thus,

$$\mathsf{t} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tag{22}$$