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CS-E4850
Computer Vision

Homework 1

Homogeneous coordinates and transformations

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Exercice 1: Homogeneous coordinates

Hint: In tasks b, c, d [...], you can utilize the fact that for three-element vectors a, b, c *the scalar triple product,* $(\mathrm{a} \times \mathrm{b})^{\mathrm{T}}\mathrm{c}$ *is zero if any two of the vectors are parallel.*

1.1 Show using homogeneous coordinates

The equation of a line in the plane is

$$ax + by + c = 0 \tag{1}$$

Show that by using homogeneous coordinates this can be written as

$$X^T l = 0 \tag{2}$$

$$\text{where: } l = (a\ b\ c)^T$$

Understanding that in homogeneous coordinates,

$$\mathrm{X} = \begin{bmatrix} \mathrm{x} \\ \mathrm{y} \\ 1 \end{bmatrix} \tag{3}$$

one only needs to input the definitions of X and l into equation 2. By computing the multiplication:

$$[\mathrm{x}\ \mathrm{y}\ 1] \begin{bmatrix} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \end{bmatrix} = \mathrm{ax} + \mathrm{by} + \mathrm{c} = 0 \tag{4}$$

1.2 Show that the intersection of two lines l and l' is in the point $\mathrm{x} = \mathrm{l} \times \mathrm{l}'$

As hinted, one can use the fact the scalar triple product is zero for two parallel vectors. Thus,

$$(l \times l')^T l = (l' \times l)^T l' = 0 \tag{5}$$

which can be reused with the definition of x:

$$l^T x = l'^T x = 0 \tag{6}$$

This means that x needs to necessarily be a part of the two lines l and l′, thus the intersection of both.

1.3 Show that the line through two points x and x' is $\mathrm{l} = \mathrm{x} \times \mathrm{x}'$

By taking the equation of the line in homogeneous coordinates $\mathrm{Y}^{\mathrm{T}}\mathrm{l} = 0$, one can substitute l by its definition of line $\mathrm{x} \times \mathrm{x}'$:

$$Y^T (x \times x') = 0 \tag{7}$$

By using again the fact that a scalar triple product necessarily means two of the vectors are parallel for it to be zero; one infers that any point y on the line formed by x and x' satisfies the equation.

1.4 Show that for all $\alpha \in \mathbb{R}$ the point $\mathsf{y} = \alpha\mathsf{x} + (1-\alpha)\mathsf{x}'$ lies on the line through points x an x'.

Any point y $\forall\alpha \in$ **R** neds to be on the line defined by points x and x', thus:

$$ly = (x \times x')\, y \tag{8}$$

knowing the definition of y and operating a bit:

$$\begin{aligned} \mathsf{l}\mathsf{y} &= (\mathsf{x} \times \mathsf{x}')\,(\alpha\mathsf{x} + (1-\alpha)\mathsf{x}') && (9)\\ &= \alpha\,(\mathsf{x} \times \mathsf{x}')\,\mathsf{x} + (1-\alpha)\,(\mathsf{x} \times \mathsf{x}')\,\mathsf{x}' && (10) \end{aligned}$$

of course, the cross product of oneself is zero, thus:

$$ly = 0\alpha + 0(1-\alpha) = 0 \tag{11}$$

which means that, indeed, $\forall\alpha \in \mathbb{R}$ every y point belongs to the line defined by x and x'.

Exercice 2: Transformations in 2D

Hint: The answers to the first two sub-tasks are directly given in Table 2.1 in Hartley & Zisserman.

2.1 Use homogeneous coordinates and give the matrix representations of the following transformation groups

2.1.1 Translation

In Euclidean coordinates:

$$\mathsf{x}'_{2\times1} = \mathsf{x}_{2\times1} + \mathsf{t}_{2\times1} \tag{12}$$

Moving to homogeneous coordinates by adding the value 1 to the fourth dimension of the x and x' points:

$$\begin{bmatrix} \mathsf{x}'_{2\times1} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathsf{x}_{2\times1} \\ 1 \end{bmatrix} + \mathsf{t}_{3\times1} \tag{13}$$

which leaves the mystery third component of t to only possibly be zero to maintain the equality. This, in turn, means that it is possible to express this transformation as a matrix multiplication instead:

$$\begin{bmatrix} \mathsf{x}'_{2\times1} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathsf{I}_{2\times2} & \mathsf{t}_{2\times1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathsf{x}_{2\times1} \\ 1 \end{bmatrix} \tag{14}$$

where I is the identity matrix.

2.1.2 Euclidean transformation (rotation+translation)

Again, starting in Euclidean coordinates with a simple rotation:

$$\mathsf{x}'_{2\times1} = \mathsf{R}_{2\times2}\mathsf{x}_{2\times1} \tag{15}$$

one can move to homogeneous coordinates while maintaining the equality:

$$\begin{bmatrix} x'_{2\times1} \\ 1 \end{bmatrix} = \begin{bmatrix} R_{2\times2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{2\times1} \\ 1 \end{bmatrix} \tag{16}$$

Thanks to the results achieved in equation 14 one can simply "add" the translation for a full Euclidean transformation by multiplying the matrices accordingly:

$$\begin{bmatrix} x'_{2\times1} \\ 1 \end{bmatrix} = \begin{bmatrix} I_{2\times2} & t_{2\times1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{2\times2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{2\times1} \\ 1 \end{bmatrix} \tag{17}$$

Simplifying:

$$\begin{bmatrix} x'_{2\times1} \\ 1 \end{bmatrix} = \begin{bmatrix} R_{2\times2} & t_{2\times1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{2\times1} \\ 1 \end{bmatrix} \tag{18}$$

2.1.3 Similarity transformation (scaling+rotation+translation)

Again, starting from a simple scaling in Euclidean coordinates:

$$x'_{2\times1} = s x_{2\times1} \tag{19}$$

note that in this case, s is a scalar value. In any case this can, as usual, can be scaled up to homogeneous coordinates:

$$\begin{bmatrix} x'_{2\times1} \\ 1 \end{bmatrix} = \begin{bmatrix} sI_{2\times2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{2\times1} \\ 1 \end{bmatrix} \tag{20}$$

where I is again the identity matrix. By multiplying the transformation matrix obtained here correspondingly with the previously obtained equation 20:

$$\begin{bmatrix} x'_{2\times1} \\ 1 \end{bmatrix} = \begin{bmatrix} sR_{2\times2} & t_{2\times1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{2\times1} \\ 1 \end{bmatrix} \tag{21}$$

2.1.4 Affine transformation

Affinity can be defined by:

$$x'_{2\times1} = a_{2\times2} x_{2\times1} + t_{2\times1} \tag{22}$$

in Euclidean coordinates; which results, in turn, in the following for homogeneous coordinates:

$$\begin{bmatrix} x'_{2\times1} \\ 1 \end{bmatrix} = \begin{bmatrix} a_{2\times2} & t_{2\times1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{2\times1} \\ 1 \end{bmatrix} \tag{23}$$

2.1.5 Projective transformation.

Finally, projective transformations cannot be easily defined in Euclidean coordinates, for they result from a variation of the 0 and 1 terms seen in the bottom of the transformation matrix in every previous example.

$$\begin{bmatrix} x'_{2\times1} \\ 1 \end{bmatrix} = \begin{bmatrix} a_{2\times2} & t_{2\times1} \\ v^T_{2\times1} & \nu \end{bmatrix} \begin{bmatrix} x_{2\times1} \\ 1 \end{bmatrix} \tag{24}$$

2.2 What is the number of degrees of freedom in these transformations?

Starting from the very beginning, in 2D, one gets 2 degrees of freedom for translations, 3 for rotations, 4 for similarities, 6 degrees of freedom for affinities, and finally 8 for projective transformations.

2.3 Why is the number of degrees of freedom in a projective transformation less than the number of elements in a 3×3 matrix?

When defining a 2D transformation in homogeneous coordinates, there exists some ambiguity regarding the terms added for a projective transformation. This ambiguity stems from the higher-dimensional nature of homogeneous coordinates when comparing to a 2D space. Bringing back this transformation to an actual 2D space, it is quite direct to see that there is no way to have an extra ninth degree of freedom. At the end of the day, a projective transformation maps 4 points in the source plane to 4 points in the target plane. Meaning that each pair of corresponding points gives 2 constraints (one for x and one for y), so mapping 4 points provides 8 constraints in total.

Exercice 3: Planar projective transformation

Note: Exercise 3 [...] is from Chapter 2 of the book by Hartley and Zisserman and that chapter can be helpful here.

The equation of a line on a plane, $ax + by + c = 0$*, can be written as* $l^T x = 0$*, where* $l = [a\ b\ c]^T$ *and* x *are homogeneous coordinates for lines and points, respectively. Under a planar projective transformation, represented with an invertible* 3×3 *matrix* H*, points transform as*

$$x' = Hx \tag{25}$$

3.1 Given the matrix H for transforming points, as defined above, define the line transformation (i.e. transformation that gives l' which is a transformed version of l).

The defined line $l^T x = 0$ can be transformed in the same way x has been transformed:

$$l'^T x' = l'^T Hx = 0 \tag{26}$$

this needs to be true $\forall x$, meaning:

$$l'^T H = l^T \tag{27}$$

which can be rewritten to define l' as:

$$l'^T = H^{-1} l^T \tag{28}$$

where H^{-1} is the inverse matrix of H.

3.2 Show that two lines and two points not lying on the lines have the following invariant

A projective invariant is a quantity which does not change its value in the transformation. Using the transformation rules for points and lines, show that two lines, l_1, l_2, and two points, x_1, x_2, not lying on the lines have the following invariant under projective transformation:

$$I = \frac{\left(l_1^T x_1\right)\left(l_2^T x_2\right)}{\left(l_1^T x_2\right)\left(l_2^T x_1\right)} \tag{29}$$

Why similar construction does not give projective invariants with fewer number of points or lines? In other words, explain why we do not get an invariant if the expression would have just the first or second term of the product in both nominator and denominator?
(Hint: Projective invariants defined via homogeneous coordinates must be invariant also to arbitrary scaling of the homogeneous coordinate vectors with a non-zero scaling factor.)

From equations 25 and 28, it is possible to define the term $l_i^T x_j$:

$$l_i'^T x_j' = \left(H^{-T} l_i\right)^T \left(H x_j\right) = l_i^T \left(H^{-1} H\right) x_j = l_i^T x_j \tag{30}$$

This is to mean, $l_i^T x_j$ remains invariant under the projective transformation. So, any combination of multiplications or divisions of line and point will also remain invariant under this transformation, proving this way the veracity of equation 29