

Goal : Show that in site percolation, $p_c(\triangle) = \frac{1}{2}$.

Partial order on Ω : $\omega \leq \omega' \Leftrightarrow \omega(e) \leq \omega'(e), \forall e \in \Lambda$

We first introduce some definition :

Definition

*) A function $f : \Omega \rightarrow \mathbb{R}$ is said to be **increasing**(**decreasing**) if

$$\omega \leq (\geq) \omega' \Rightarrow f(\omega) \leq f(\omega')$$

*) An event A is said to be **increasing**(**decreasing**) if $\mathbb{1}_A$ is increasing(decreasing).

Harris-FKG inequality

Proposition (Harris inequality)

(a) If A and B are increasing event, then

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\mathbb{P}(B).$$

(b) If f and g are increasing functions, then

$$\mathbb{E}[fg] \geq \mathbb{E}[f]\mathbb{E}[g].$$

Exercise 1 (The square-root trick) Given n increasing events A_1, \dots, A_n , show that

$$\max_{1 \leq i \leq n} \{\mathbb{P}(A_i)\} \geq 1 - \left[1 - \mathbb{P}(A_1 \cup \dots \cup A_n)\right]^{\frac{1}{n}}$$

sOL: Note that A_1^c, \dots, A_n^c are decreasing events, and thus also has FKG inequality.

$$\begin{aligned} 1 - \left[1 - \mathbb{P}(A_1 \cup \dots \cup A_n)\right]^{\frac{1}{n}} &= 1 - \left[\mathbb{P}(A_1^c \cap \dots \cap A_n^c)\right]^{\frac{1}{n}} \\ (\text{FKG inequality}) &\leq 1 - \left(\prod_{k=1}^n \mathbb{P}(A_k^c)\right)^{\frac{1}{n}} \\ &\leq 1 - \left(\prod_{k=1}^n \min_{1 \leq i \leq n} \{\mathbb{P}(A_i^c)\}\right)^{\frac{1}{n}} \\ &= 1 - \min_{1 \leq i \leq n} \{\mathbb{P}(A_i^c)\} \\ &= \max_{1 \leq i \leq n} \{1 - \mathbb{P}(A_i^c)\} \\ &= \max_{1 \leq i \leq n} \{\mathbb{P}(A_i)\} \end{aligned}$$

Application Show that $\theta(\frac{1}{2}) = 0$ on \mathbb{Z}^2 , which implies that $p_c(\mathbb{Z}^2) \geq \frac{1}{2}$.

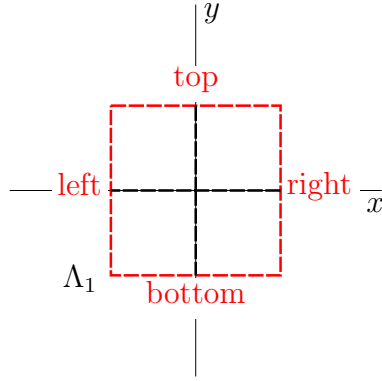
Proof. Assume that $\theta(\frac{1}{2}) > 0$, then we can know that $\mathbb{P}_{\frac{1}{2}}(\exists \text{ an infinite cluster}) = 1$. Define

$$B_n := [\partial\Lambda_n \leftrightarrow \infty] = \bigcup_{x \in \partial\Lambda_n} [x \leftrightarrow \infty], \quad n \in \mathbb{N},$$

because of

$$\begin{aligned} [\exists \text{ an infinite cluster}] &= \bigcup_{x \in \mathbb{Z}^2} [x \leftrightarrow \infty] \\ &= \bigcup_{n \in \mathbb{N}} \bigcup_{x \in \partial\Lambda_n} [x \leftrightarrow \infty] = \lim_{n \rightarrow \infty} [\partial\Lambda_n \leftrightarrow \infty], \end{aligned}$$

Thus $\lim_{n \rightarrow \infty} \mathbb{P}_{\frac{1}{2}}[\partial\Lambda_n \leftrightarrow \infty] = 1$.



Note that $B_n = [\exists \text{ an infinite SAW in } \mathbb{Z}^2 \setminus \Lambda_n \text{ starting in } \partial\Lambda_n]$, define

$$\begin{aligned} B_n^{\text{top}} &= [\exists \text{ an infinite SAW in } \mathbb{Z}^2 \setminus \Lambda_n \text{ starting in } \partial\Lambda_n^{\text{top}}] \\ B_n^{\text{bottom}} &= [\exists \text{ an infinite SAW in } \mathbb{Z}^2 \setminus \Lambda_n \text{ starting in } \partial\Lambda_n^{\text{bottom}}] \\ B_n^{\text{left}} &= [\exists \text{ an infinite SAW in } \mathbb{Z}^2 \setminus \Lambda_n \text{ starting in } \partial\Lambda_n^{\text{left}}] \\ B_n^{\text{right}} &= [\exists \text{ an infinite SAW in } \mathbb{Z}^2 \setminus \Lambda_n \text{ starting in } \partial\Lambda_n^{\text{right}}]. \end{aligned}$$

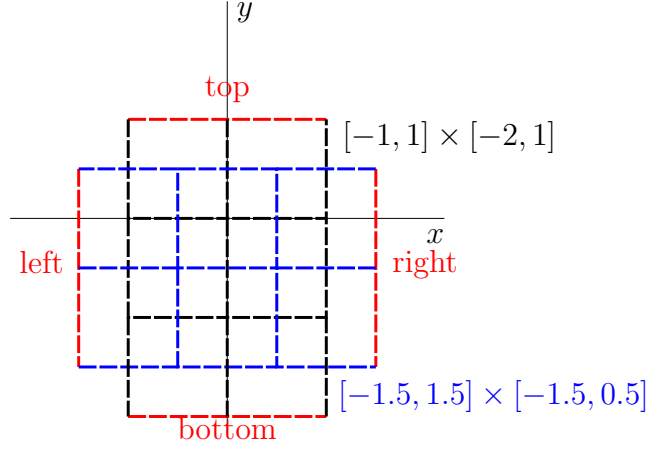
Then

$$B_n = B_n^{\text{top}} \cup B_n^{\text{bottom}} \cup B_n^{\text{left}} \cup B_n^{\text{right}}$$

It's no doubt that for all $n \in \mathbb{N}$, B_n^{top} , B_n^{bottom} , B_n^{left} , B_n^{right} are increasing events, thus by square-root trick and the fact that $\lim_{n \rightarrow \infty} \mathbb{P}_{\frac{1}{2}} B_n = 1$, we got that

$$\lim_{n \rightarrow \infty} \mathbb{P}_{\frac{1}{2}}(B_n^{\text{top}}) \geq \lim_{n \rightarrow \infty} 1 - \left[1 - \mathbb{P}_{\frac{1}{2}}(B_n)\right]^{\frac{1}{4}} = 1$$

Note that the probability of B_n^{top} remains the same for any "shift" of this event.



Now, we consider

$$\begin{aligned}
C_n^{\text{top}} &= B_n^{\text{top}} \\
C_n^{\text{bottom}} &= \{\omega \subseteq \mathbb{Z}^2 \mid \omega + (0, 1) \in B_n^{\text{bottom}}\} \\
C_n^{\text{left}} &= \{\omega \subseteq \mathbb{Z}^2 \mid \omega^* + (0.5, 0.5) \in B_n^{\text{left}}\} \\
C_n^{\text{right}} &= \{\omega \subseteq \mathbb{Z}^2 \mid \omega^* + (-0.5, 0.5) \in B_n^{\text{right}}\}
\end{aligned}$$

then

$$\lim_{n \rightarrow \infty} \mathbb{P}_{\frac{1}{2}}(C_n^{\text{top}}) = \lim_{n \rightarrow \infty} \mathbb{P}_{\frac{1}{2}}(C_n^{\text{bottom}}) = \lim_{n \rightarrow \infty} \mathbb{P}_{\frac{1}{2}}(C_n^{\text{left}}) = \lim_{n \rightarrow \infty} \mathbb{P}_{\frac{1}{2}}(C_n^{\text{right}}) = 1,$$

define $C_n = C_n^{\text{top}} \cap C_n^{\text{bottom}} \cap C_n^{\text{left}} \cap C_n^{\text{right}}$, then

$$\lim_{n \rightarrow \infty} \mathbb{P}_{\frac{1}{2}}(C_n) = 1.$$

Note that C_n does not depending on edges on $[-n, n] \times [-n-1, n]$, we take n be large enough such that $\mathbb{P}_{\frac{1}{2}}(C_n) > \frac{1}{2}$, let $\mathcal{C}([-n, n] \times [-n-1, n])$ be the event that all edge in $[-n, n] \times [-n-1, n]$ are closed, then

$$\begin{aligned}
\mathbb{P}_{\frac{1}{2}}(N \geq 2) &\geq \mathbb{P}_{\frac{1}{2}}(C_n \cap \mathcal{C}([-n, n] \times [-n-1, n])) \\
&\geq \frac{1}{2} \times \frac{1}{2^{8n^2+8n+1}} > 0.
\end{aligned}$$

Which contradicts that $\mathbb{P}_{\frac{1}{2}}(N \geq 2) = 0$ on \mathbb{Z}^2 . □

Exercise 2 Show that $p_c \geq \frac{1}{2}$ for the site percolation on the triangular lattice.

sOL: We show that $\theta(\frac{1}{2}) = 0$ to conclude this.

Note that we can use similar way of proving that $\mathbb{P}_p(N = 1) = 1$, $\forall p > p_c$ in \mathbb{Z}^d to say that it is also true in triangular lattice.

We suppose that $\theta(\frac{1}{2}) > 0$, then $\frac{1}{2} > p_c(\Delta)$, thus $\mathbb{P}_{\frac{1}{2}}[\exists \text{ an infinite cluster}] = 1$, note that

$$\begin{aligned}
[\exists \text{ an infinite cluster}] &= \bigcup_{x \in \Delta} [x \leftrightarrow \infty] \\
&= \bigcup_{n \in \mathbb{N}} \bigcup_{x \in \partial \Lambda_n} [x \leftrightarrow \infty] = \lim_{n \rightarrow \infty} [\partial \Lambda_n \leftrightarrow \infty],
\end{aligned}$$

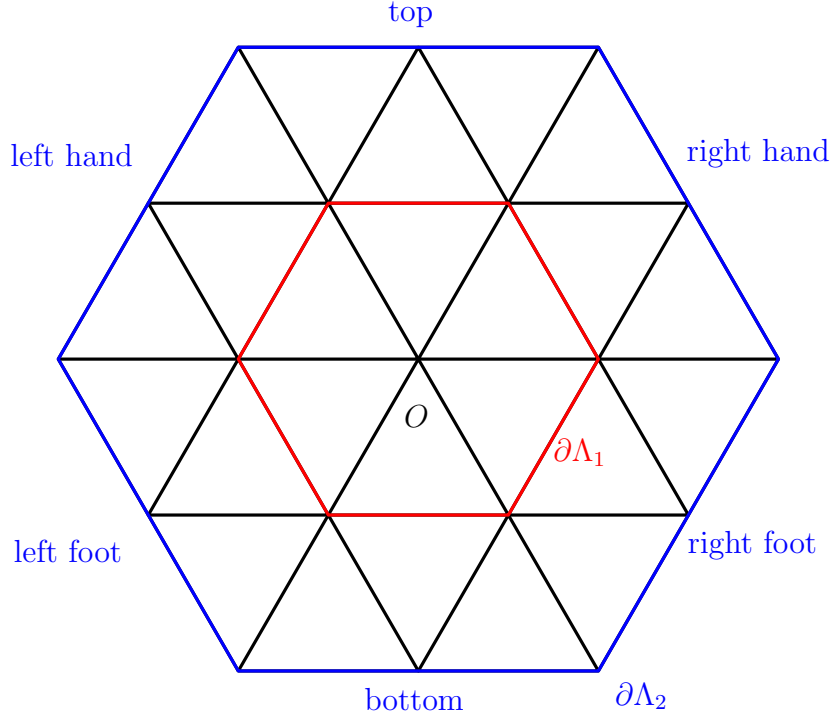
In this case, $\Lambda_n = \{x \in G \mid \exists \text{ SAW of length } \leq n \text{ that connected from } 0 \text{ to } x\}$, where the length of SAWs is allowed to be 0. And, $\partial\Lambda_n = \Lambda_n \setminus \Lambda_{n-1}$, where $\Lambda_{-1} = \emptyset$. We still define

$$B_n := [\partial\Lambda_n \leftrightarrow \infty] = [\exists \text{ an infinite SAW in } \Delta \setminus \Lambda_{n-1} \text{ starting in } \partial\Lambda_n]$$

It is needed to mention that we consider a configuration ω to be a subset of $V(\Delta)$ instead of a subset of $E(\Delta)$. Define

$$\begin{aligned} B_n^t &= [\exists \text{ an infinite SAW in } \mathbb{Z}^2 \setminus \Lambda_{n-1} \text{ starting in } \partial\Lambda_n^{\text{top}}] \\ B_n^b &= [\exists \text{ an infinite SAW in } \mathbb{Z}^2 \setminus \Lambda_{n-1} \text{ starting in } \partial\Lambda_n^{\text{bottom}}] \\ B_n^{\text{lh}} &= [\exists \text{ an infinite SAW in } \mathbb{Z}^2 \setminus \Lambda_n \text{ starting in } \partial\Lambda_n^{\text{left hand}}] \\ B_n^{\text{lf}} &= [\exists \text{ an infinite SAW in } \mathbb{Z}^2 \setminus \Lambda_{n-1} \text{ starting in } \partial\Lambda_n^{\text{left foot}}] \\ B_n^{\text{rh}} &= [\exists \text{ an infinite SAW in } \mathbb{Z}^2 \setminus \Lambda_{n-1} \text{ starting in } \partial\Lambda_n^{\text{right hand}}] \\ B_n^{\text{rf}} &= [\exists \text{ an infinite SAW in } \mathbb{Z}^2 \setminus \Lambda_{n-1} \text{ starting in } \partial\Lambda_n^{\text{right foot}}]. \end{aligned}$$

Then, $B_n = B_n^t \cup B_n^b \cup B_n^{\text{lh}} \cup B_n^{\text{lf}} \cup B_n^{\text{rh}} \cup B_n^{\text{rf}}$, because of all these events are increasing



events, by square-root trick and $\lim_{n \rightarrow \infty} \mathbb{P}_{\frac{1}{2}}(B_n) = 1$, we got that

$$\lim_{n \rightarrow \infty} \mathbb{P}_{\frac{1}{2}}(B_n^t) \geq \lim_{n \rightarrow \infty} 1 - \left[1 - \mathbb{P}_{\frac{1}{2}}(B_n)\right]^{\frac{1}{6}} = 1.$$

Given a configuration $\omega \subseteq \Delta$, we define $\omega^c = V(\Delta) \setminus \omega$, which is also a (site) configuration of Δ . Thus for any given event A , we have $\mathbb{P}_p\{\omega \mid \omega \in A\} = \mathbb{P}_{1-p}\{\omega \mid \omega^c \in A\}$. Because of $p = 1 - p$ when $p = \frac{1}{2}$, for convenience, we say $[\omega^c \in A] := \{\omega \mid \omega^c \in A\}$ for A be a event, then

$$\lim_{n \rightarrow \infty} \mathbb{P}_{\frac{1}{2}}([\omega \in B_n^t] \cap [\omega^c \in B_n^b] \cap [\omega^c \in B_n^{\text{lh}}] \cap [\omega \in B_n^{\text{lf}}] \cap [\omega^c \in B_n^{\text{rh}}] \cap [\omega \in B_n^{\text{rf}}]) = 1.$$

We define $C_n := [\omega \in B_n^t] \cap \dots \cap [\omega \in B_n^{\text{rf}}]$ in the previous step, we can take large n such that $\mathbb{P}_{\frac{1}{2}}(C_n) \geq \frac{1}{2}$, we use \blacksquare_n to denote the event that all sites in Λ_n are closed, note that

C_n dose not depend on sites in Λ_{n-1} , thus

$$\mathbb{P}(N \geq 3) \geq \mathbb{P}(C_n \cap \blacklozenge_{n-1}) = \frac{1}{2} \times \frac{1}{2^{1+3n(n-1)}} > 0,$$

which is a contradiction because of $\mathbb{P}_{\frac{1}{2}}(N = 1) = 1$. ■