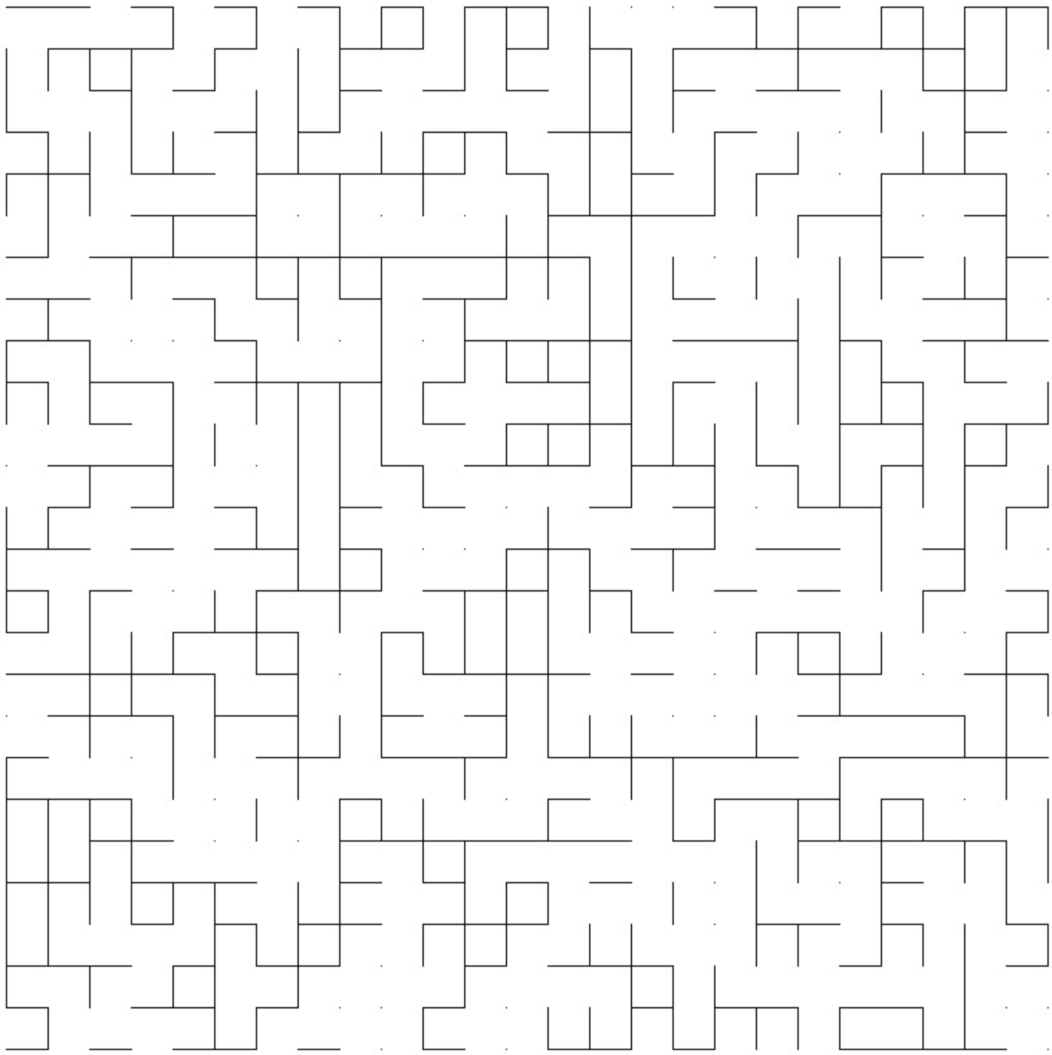
Group 7 Note of Planar Statistical Physics and Bernoulli Percolation



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0. (Planer) Statistical Mechanics

0.1) Motivation

- We want to describe the (macroscopic) behavior of a system composed of huge number of (microscopic) particles. Theoretically, if the behavior of each particle can be described, then one should be able to say something about the system. However, the number of particles are astronomical (1 mole $\approx 10^{23}$) and it is costly (even impossible) to solve the system.
- Boltzmann (end 19^{th} century) gives a probabilistic formalism

  (a) $\Omega =$ sample space (discrete, countable, possibly infinite)

  (b) $H : \Omega \to \mathbb{R}$ energy function (Hamiltonian)

  (c) $\beta = \frac{1}{T}$ inverse temperature.

  $\Rightarrow$ **Boltzmann distribution**

$$\mathbb{P}_\beta(\omega) = \frac{1}{z_\beta} \exp(-\beta H(\omega))$$

  Where $z_\beta = \sum\limits_{\omega \in \Omega} \exp(-\beta H(\omega))$ is called **partition function**

- Interpretations

  $*$) At a fixed temperature, an exponential weight is associated to each configuration

  $*$) At high temperature ($\beta \to 0^+$) the weights are almost a constant function of configurations, so each state equiprobable.

  $*$) At low temperature, ($\beta \to +\infty$), the weight concentrate on minimizers of the energy function H.

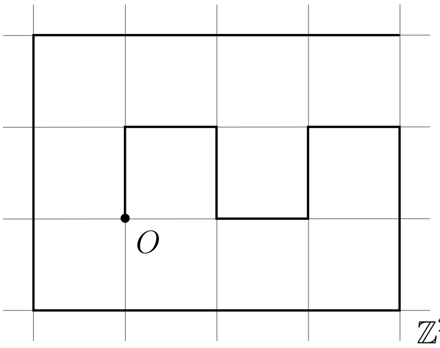
  $\Rightarrow$ this corresponds to the intuitions we have in physics (thermodynamics?)

Remark (a) $z_\beta < \infty$ if Ω is finite, so $\mathbb{P}_\beta$ is well defined.

(b) If Ω is infinite, then z_β is not always finite. The finiteness of z_β may depend on β.

0.2) First Model (Polymer Model)

- Regular lattices, e.g., $\mathbb{Z}^d$
- Self-avoiding walk (SAW) (a polymer)



Example A self-avoiding walk

- Consider $\Omega = \{$ all SAWs $\}$, define $H(\omega) = |\omega|$ where $\omega \in \Omega$
  Let $n \in \mathbb{N}$, define $\lambda_n = \#$ SAWs of length n.

Observe : Given $n, m \geq 1$
Any SAW of length $n+m$ can be decomposed into 2 SAWs of length n and m.
$\Rightarrow \lambda_{n+m} \leq \lambda_n \cdot \lambda_m$.

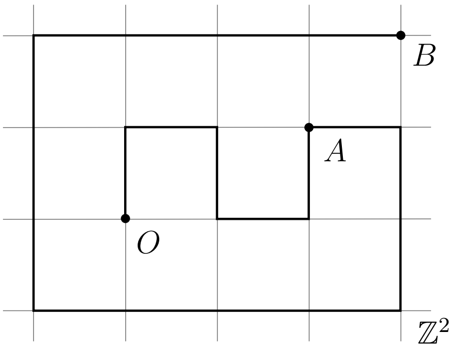


Figure $\lambda_{19} \leq \lambda_5 \cdot \lambda_{14}$

Exercise 1 $\mu \equiv \lim\limits_{n\to\infty} (\lambda_n)^{\frac{1}{n}}$ exists and $\lambda_N \geq \mu^N$ for all $N \geq 1$.

$\boxed{\text{SOL}}$ Note that $\forall n \in \mathbb{N}$, we have $(\lambda_n)^{\frac{1}{n}} \geq 0$, thus 0 is a lower bound of $\{(\lambda_n)^{\frac{1}{n}}\}_{n=0}^{\infty}$, therefore

$$\inf_{n\in\mathbb{N}_0} (\lambda_n)^{\frac{1}{n}} = K \in \mathbb{R}$$

Now, given $\varepsilon > 0$, 1° $\exists N_1 \in \mathbb{N}$ s.t.

$$K + \frac{\varepsilon}{2} > (\lambda_{N_1})^{\frac{1}{N_1}}$$

2° By division algorithm, $\forall n \geq N_1$, $\exists m, \ell \in \mathbb{Z}$ with $0 \leq \ell \leq N_1$ such that $n = mN_1 + \ell$, thus

$$\lambda_n = \lambda_{mN_1+\ell} \leq (\lambda_{N_1})^m \cdot \lambda_\ell$$

therefore,

$$(\lambda_n)^{\frac{1}{n}} \le (\lambda_{N_1})^{\frac{m}{n}} \cdot (\lambda_\ell)^{\frac{1}{n}} = (\lambda_{N_1})^{\frac{1}{N_1 + \frac{\ell}{m}}} \cdot (\lambda_\ell)^{\frac{1}{n}} \le (\lambda_{N_1})^{\frac{1}{N_1}} \cdot (\lambda_\ell)^{\frac{1}{n}}$$
$$\le (\lambda_{N_1})^{\frac{1}{N_1}} \cdot (\lambda_1)^{\frac{\ell}{n}} \le (\lambda_{N_1})^{\frac{1}{N_1}} \cdot (\lambda_1)^{\frac{N_1}{n}}$$

Because of $\lim\limits_{n\to\infty} (\lambda_{N_1})^{\frac{1}{N_1}} \cdot (\lambda_1)^{\frac{N_1}{n}} = (\lambda_{N_1})^{\frac{1}{N_1}}$, $\exists N \in \mathbb{N}$ such that $n \ge N \Rightarrow$

$$(\lambda_{N_1})^{\frac{1}{N_1}} + \frac{\varepsilon}{2} > (\lambda_{N_1})^{\frac{1}{N_1}} \cdot (\lambda_1)^{\frac{N_1}{n}} \ge (\lambda_n)^{\frac{1}{n}}$$

Therefore

$$K + \varepsilon = K + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} > \lambda_{N_1}^{\frac{1}{N_1}} + \frac{\varepsilon}{2} > (\lambda_n)^{\frac{1}{n}} \ge K.$$

Thus, $\mu = \lim\limits_{n\to\infty} (\lambda_n)^{\frac{1}{n}} = K$ exists.

Next, for $N > 1$, we have $\forall n \in \mathbb{N}, \lambda_{nN} \le (\lambda_N)^n$, thus $(\lambda_{nN})^{\frac{1}{nN}} \le (\lambda_N)^{\frac{1}{N}}$. Note that $\{(\lambda_{nN})^{\frac{1}{nN}}\}_{n=1}^{\infty}$ is a subsequence of $\{(\lambda_n)^{\frac{1}{n}}\}_{n=1}^{\infty}$, therefore $\mu = \lim\limits_{n\to\infty} (\lambda_{nN})^{\frac{1}{nN}} \le (\lambda_N)^{\frac{1}{N}}$ ■

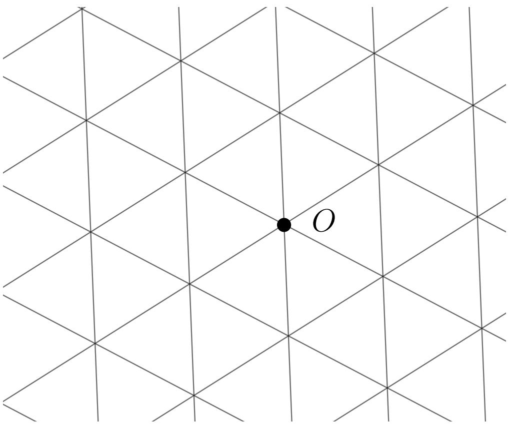
Exercise 2 (a) In $\mathbb{Z}^2$, $\mu \in (2,3)$

(b) In $\mathbb{Z}^3$, $\mu > 3$

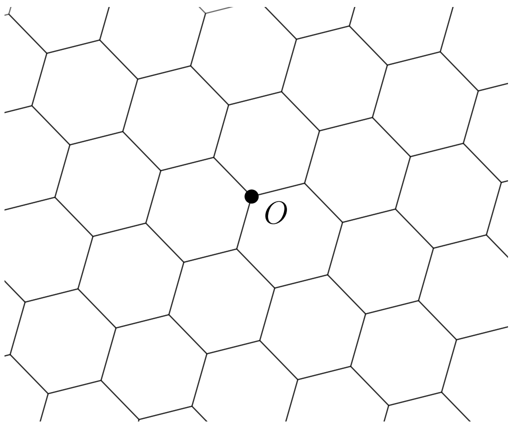
(c) In the triangular mesh, $\mu > 3$;

(d) In the hexagonal mesh, $\mu < 2$

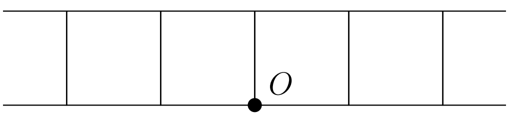
(e) In $\mathbb{Z} \times \{0,1\}$ (i.e. a ladder), $\mu = \frac{1+\sqrt{5}}{2}$.



The triangular mesh



The hexagonal mesh



$\mathbb{Z} \times \{0,1\}$

$\boxed{\texttt{SoL:}}$ We first show a fact :

Fact(ratio test and root test) : Let $\{a_n\}_{n=1}^{\infty}$ be a sequence that take value in $(0,\infty)$. Then

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=K\in\mathbb{R}\Rightarrow\lim_{n\to\infty}\sqrt[n]{a_n}=K.$$

Proof. If $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=K\in\mathbb{R}$, then given $\varepsilon>0$, there exists $N_1>0$ such that

$$n\geq N_1\Rightarrow\left|\frac{a_{n+1}}{a_n}-K\right|<\frac{\varepsilon}{2}\Rightarrow K-\frac{\varepsilon}{2}<\frac{a_{n+1}}{a_n}<K+\frac{\varepsilon}{2}$$

Note that $a_n=a_1\times\frac{a_2}{a_1}\times\frac{a_3}{a_2}\times\cdots\times\frac{a_n}{a_{n-1}}=a_1\times\prod_{k=1}^{n-1}\frac{a_{k+1}}{a_k}=a_1\times\prod_{k=1}^{N_1-1}\frac{a_{k+1}}{a_k}\times\prod_{k=N_1}^{n-1}\frac{a_{k+1}}{a_k}$.
Now define $\mathcal{Q}=a_1\times\prod_{k=1}^{N_1-1}\frac{a_{k+1}}{a_k}$, then $\sqrt[n]{a_n}=\sqrt[n]{\mathcal{Q}}\times\prod_{k=N_1}^{n-1}\left(\frac{a_{k+1}}{a_k}\right)^{\frac{1}{n}}$, thus

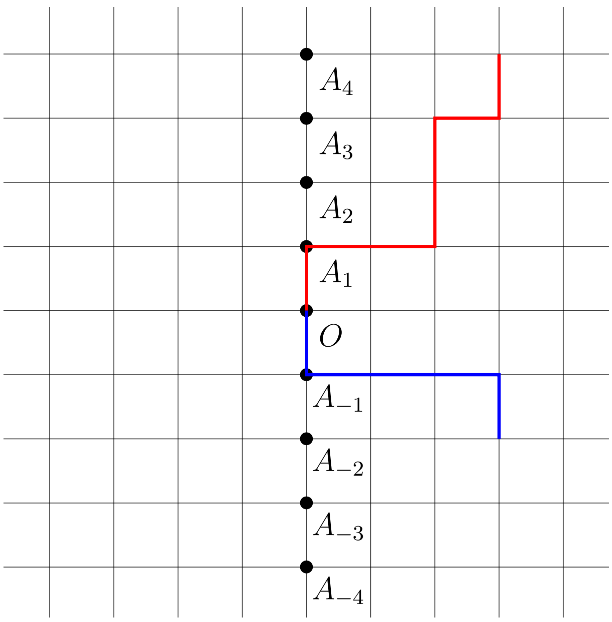
$$\begin{aligned}\sqrt[n]{\mathcal{Q}}\times\left(K-\frac{\varepsilon}{2}\right)^{\frac{n-N_1-1}{n}}&=\sqrt[n]{\mathcal{Q}}\times\prod_{k=N_1}^{n-1}\left(K-\frac{\varepsilon}{2}\right)^{\frac{1}{n}}<\sqrt[n]{a_n}=\sqrt[n]{\mathcal{Q}}\times\prod_{k=N_1}^{n-1}\left(\frac{a_{k+1}}{a_k}\right)^{\frac{1}{n}}\\&<\sqrt[n]{\mathcal{Q}}\times\prod_{k=N_1}^{n-1}\left(K+\frac{\varepsilon}{2}\right)^{\frac{1}{n}}=\sqrt[n]{\mathcal{Q}}\times\left(K+\frac{\varepsilon}{2}\right)^{\frac{n-N_1-1}{n}}.\end{aligned}$$

By the fact that
$A(n)=\sqrt[n]{\mathcal{Q}}\times\left(K-\frac{\varepsilon}{2}\right)^{\frac{n-N_1-1}{n}}\to K-\frac{\varepsilon}{2}$, $B(n)=\sqrt[n]{\mathcal{Q}}\times\left(K+\frac{\varepsilon}{2}\right)^{\frac{n-N_1-1}{n}}\to K+\frac{\varepsilon}{2}$ as $n\to\infty$, there is $N>N_1$ such that $n\geq N\Rightarrow A(n)>K-\varepsilon$ and $B(n)<K+\varepsilon$, thus

$$K-\varepsilon<A(n)<\sqrt[n]{a_n}<B(n)<K+\varepsilon.$$

Therefore, $\lim_{n\to\infty}\sqrt[n]{a_n}=K$. □

(a) We first show that $\mu(\mathbb{Z}^2)>2$:



If we only consider the 3 direction $\uparrow, \rightarrow, \downarrow$ to choice for next step, ex : red line and blue line, we define $S(n) = \#$ SAWs with length n that under this restriction, $n \geq 1$ and $S(0) = 1$. Now considering the samples SAWs are of length n, then

$$\begin{aligned}
S(n) =& \# \text{ SAWs passing } O \text{ with } \rightarrow \\
&+ \sum_{k=1}^{n} \# \text{ SAWs passing } A_i \text{ with } \rightarrow + \sum_{k=1}^{n} \# \text{ SAWs passing } A_{-i} \text{ with } \rightarrow \\
&+ \# \text{ SAWs lying in } y\text{- axis} \\
&= S(n-1) + 2S(n-2) + \cdots + 2S(1) + 2S(0) + 2 \\
&= S(n-1) + 2 + 2\sum_{k=0}^{n-2} S(k), \quad n \geq 2
\end{aligned}$$

Similarly, $S(n+1) = S(n) + 2 + 2\sum_{k=0}^{n-1} S(k)$, by subtraction of the two equation, we get a recursive formula :

$$S(n+1) = 2S(n) + S(n-1), \quad n \geq 1, \ \cdots\cdots (*)$$

We get $\{S(n)\}_{n=0}^{\infty} : 1, 3, 7, 17, 31, \cdots$, we use matrix representation of $(*)$:

$$\begin{bmatrix} S(n+1) \\ S(n) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} S(n) \\ S(n-1) \end{bmatrix}.$$

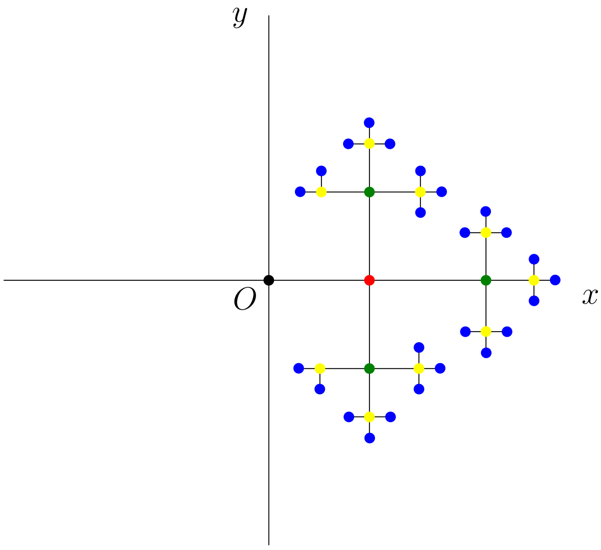
And note that by calculating eigenvalue and eigenvector, we have

$$\begin{aligned}
\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} &= P \begin{bmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{bmatrix} P^{-1}, \quad P = \begin{bmatrix} 1 & -1 \\ -1+\sqrt{2} & 1+\sqrt{2} \end{bmatrix} \\
\Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n &= P \begin{bmatrix} (1+\sqrt{2})^n & 0 \\ 0 & (1-\sqrt{2})^n \end{bmatrix} P^{-1} \\
\Rightarrow \begin{bmatrix} S(n+1) \\ S(n) \end{bmatrix} &= \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} S(1) \\ S(0) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\
&= \frac{1}{2} P \begin{bmatrix} (1+\sqrt{2})^n & 0 \\ 0 & (1-\sqrt{2})^n \end{bmatrix} \begin{bmatrix} 4+3\sqrt{2} \\ 4-3\sqrt{2} \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1+\sqrt{2} & 1+\sqrt{2} \end{bmatrix} \begin{bmatrix} (1+\sqrt{2})^n(4+3\sqrt{2}) \\ (1-\sqrt{2})^n(4-3\sqrt{2}) \end{bmatrix} \\
\Rightarrow S(n+1) &= \frac{1}{2}\Big((1+\sqrt{2})^n(4+3\sqrt{2}) - (1-\sqrt{2})^n(4-3\sqrt{2})\Big)
\end{aligned}$$

Thus, $\mu(\mathbb{Z}^2) = \lim\limits_{n\to\infty} \sqrt[n+1]{\lambda_{n+1}} \geq \lim\limits_{n\to\infty} \sqrt[n+1]{S(n+1)} = 1 + \sqrt{2} > 2$.

Next, we show that $\mu < 3$:

First, we calculate the number of SAWs of length 4 with the first move is " $\rightarrow$ "



We got that number is 25 (blue points),this figure concentrate about the direction of next move of SAWs instead of the distance between two connected points (it is always 1). we get $\lambda_4 = 4 \times 25 = 100$ Now we replace blue point by red point, and then we got $\lambda_7 \le \lambda_4 \times 25 = 2500$, continue this process, we got $\lambda_{1+3n} \le 4 \times 25^n$. Thus

$$\mu(\mathbb{Z}^2) = \lim_{n\to\infty} (\lambda_{3n+1})^{\frac{1}{3n+1}} \le \lim_{n\to\infty} 4^{\frac{1}{3n+1}} \times 25^{\frac{n}{3n+1}} = \sqrt[3]{25} < 3.$$

(b) Because of $\mathbb{Z}^3 = \mathbb{Z}^2 \times \mathbb{Z}$, thus we define β_n be the number of SAWs of length n in $\mathbb{Z}^2$, by (a), we have $\lim_{n\to\infty} \sqrt[n]{\beta_n} = \mu(\mathbb{Z}^2) \in (2,3)$. Note that $\lambda_{n+1} \ge \beta_n \times 2^{n+1}$, therefore

$$\mu(\mathbb{Z}^3) = \lim_{n\to\infty} \sqrt[n+1]{\lambda_{n+1}} \ge \lim_{n\to\infty} \sqrt[n+1]{\beta^n} \times 2^{\frac{n+1}{n+1}} \ge \lim_{n\to\infty} \left(\frac{\beta_{n+1}}{4}\right)^{\frac{1}{n+1}} \times 2 > 2 \times 2 > 3$$

(c) If we look more seriously, we can find that there's a hidden $\mathbb{Z}^2$ in the triangular mesh :

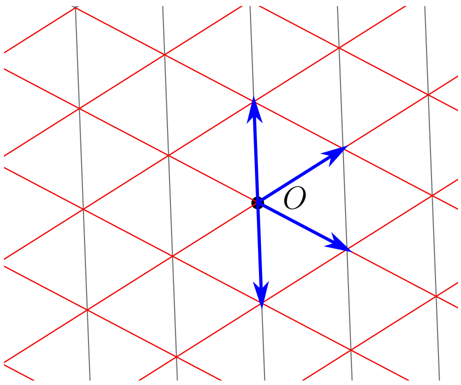


Figure 1 The triangular mesh

But, we are not going to use this fact. Instead, we'll prove $\mu(\triangle) > 3$ by using the similar way to prove $\mu(\mathbb{Z}^2) > 2$. Define $S(n)$ be # of SAWs with length n that every move only choose the four blue directions (**Figure 1**), if we define $S(0) = 1$, it is easy to see that

$$S(n) = 2S(n-1) + 4S(n-2) + \cdots + 4S(0) + 2 = 2 + 2S(n-1) + 4\sum_{k=0}^{n-2} S(k)$$

$$S(n+1) = 2S(n) + 4S(n-1) + \cdots + 4S(0) + 2 = 2 + 2S(n) + 4\sum_{k=0}^{n-1} S(k) \quad n \ge 2.$$

Thus, $S(n+1)-S(n)=2S(n)+2S(n-1) \Rightarrow S(n+1)=3S(n)+2S(n-1), \quad n \geq 2,\ S(0)=1,\ S(1)=4,\ S(2)=14$, thus

$$\begin{bmatrix} S(n+1) \\ S(n) \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} S(n) \\ S(n-1) \end{bmatrix}$$

We note that

$$\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = P \begin{bmatrix} \frac{3+\sqrt{17}}{2} & 0 \\ 0 & \frac{3-\sqrt{17}}{2} \end{bmatrix} P^{-1}, \quad \text{where } P = \begin{bmatrix} 3+\sqrt{17} & 3-\sqrt{17} \\ 2 & 2 \end{bmatrix}$$

And

$$P^{-1} = \frac{1}{4\sqrt{17}} \begin{bmatrix} 2 & -3+\sqrt{17} \\ -2 & 3+\sqrt{17} \end{bmatrix}$$

Therefore,

$$\begin{aligned} \begin{bmatrix} S(n+1) \\ S(n) \end{bmatrix} &= \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} S(1) \\ S(0) \end{bmatrix} = P \begin{bmatrix} \frac{3+\sqrt{17}}{2} & 0 \\ 0 & \frac{3-\sqrt{17}}{2} \end{bmatrix}^n P^{-1} \begin{bmatrix} 4 \\ 1 \end{bmatrix} \\ &= \frac{1}{4\sqrt{17}} P \begin{bmatrix} \left(\frac{3+\sqrt{17}}{2}\right)^n & 0 \\ 0 & \left(\frac{3-\sqrt{17}}{2}\right)^n \end{bmatrix} \begin{bmatrix} 5+\sqrt{17} \\ -5+\sqrt{17} \end{bmatrix} \\ &= \frac{1}{4\sqrt{17}} \begin{bmatrix} 3+\sqrt{17} & 3-\sqrt{17} \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \left(\frac{3+\sqrt{17}}{2}\right)^n (5+\sqrt{17}) \\ \left(\frac{3-\sqrt{17}}{2}\right)^n (-5+\sqrt{17}) \end{bmatrix} \end{aligned}$$

Thus implies that

$$S(n) = \frac{2}{4\sqrt{17}} \left(\left(\frac{3+\sqrt{17}}{2}\right)^n (5+\sqrt{17}) + \left(\frac{3-\sqrt{17}}{2}\right)^n (-5+\sqrt{17}) \right)$$

And therefore

$$\mu(\triangle) = \lim_{n\to\infty} \sqrt[n]{\lambda_n} \geq \lim_{n\to\infty} \sqrt[n]{S(n)} = \frac{3+\sqrt{17}}{2} > 3.$$

(d) Similar way to prove $\mu(\mathbb{Z}^2) < 3$, we use a simple calculate, deduce that

$$\lambda_{1+5} = (3 \times 2^5 - 2) = \lambda_1 \times 30$$

Thus,

$$\lambda_{1+5n} \leq \lambda_1 \times 30^n$$

Therefore,

$$\mu(\hexagon) = \lim_{n\to\infty} \sqrt[1+5n]{\lambda_{1+5n}} \leq \lim_{n\to\infty} \sqrt[1+5n]{\lambda_1 \times 30^n} = \sqrt[5]{30} < 2$$

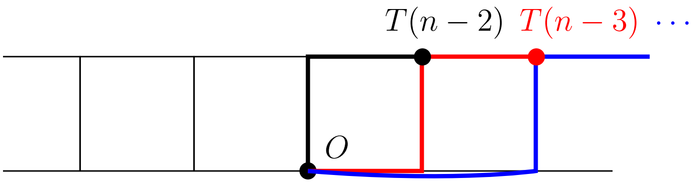
(e) *Claim* " $\mu(\mathbb{Z} \times \{0,1\}) \geq \frac{1+\sqrt{5}}{2}$ " :
Let $S(n)$ be # of SAWs with length n that only move $\uparrow$, $\downarrow$, $\rightarrow$ in every step, and it is easy to see that $S(n) = S(n-1) + S(n_2), \quad n \geq 3$, therefore

$$\mu(\mathbb{Z} \times \{0,1\}) = \lim_{n\to\infty} \sqrt[n]{\lambda_n} \geq \lim_{n\to\infty} \sqrt[n]{S(n)} = \frac{1+\sqrt{5}}{2}$$

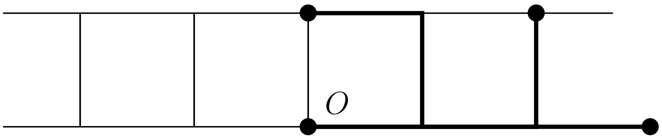
Claim " $\mu(\mathbb{Z}\times\{0,1\}) \le \frac{1+\sqrt{5}}{2}$ " :
Define $T(n)$ be # of SAWs in $\mathbb{N}_0 \times \{0,1\}$ of length n, $n\in\mathbb{N}$, and define $T(0)=1$. You well find that recursive formula is a great way to slove these problem.

$$T(n) = T(n-2) + T(n-3) + \cdots + T(0) + \gamma(n),$$

where $\gamma(n) = \begin{cases} \frac{n+3}{2}, & \text{if } n \text{ is odd.} \\ \frac{n+2}{2}, & \text{if } n \text{ is even.} \end{cases}$ We find that



Fingure 1



Fingure 2 $\gamma(3)=3$

$$T(n+1) - T(n) = T(n-1) + \gamma(n+1) - \gamma(n) = T(n-1) + \mathbb{1}_{even}(n),$$

where $\mathbb{1}_{even}(n) = \begin{cases} 0, & \text{if } n \text{ is odd.} \\ 1, & \text{if } n \text{ is even.} \end{cases}$
Moreover, $\lambda_n \le 2T(n) + 2(T(n-4)+T(n-6)+\cdots), \quad n\in\mathbb{N}$, thus

$$\begin{aligned} \lambda_n \le \lambda_n + \lambda_{n+1} &\le 2(T(n+1)+T(n)) + 2(T(n-3)+\cdots+T(0)) \\ &= 2(T(n+1)+T(n)+T(n-1)-\gamma(n-1)) \\ &= 2(2T(n+1)+\mathbb{1}_{even}(n)-\gamma(n)) \le 4T(n+1), \ \forall n\in\mathbb{N} \end{aligned}$$

Because of $T(0)=1,\ T(1)=2$, and

$$T(n+1) \ge T(n) + T(n_1),\ T(n+1)+1 \le (T(n)+1) + (T(n-1)+1)$$

Define
$\{a_n\}_{n=0}^{\infty}$ as $a_0=1,\ a_1=2,\ a_{n+1}=a_n+a_{n-1}, \quad n\ge 1$
$\{b_n\}_{n=0}^{\infty}$ as $b_0=2,\ b_1=3,\ b_{n+1}=b_n+b_{n-1}, \quad n\ge 1$, then

$$a_n \le T(n) < T(n)+1 \le b_n, \quad \forall n\in\mathbb{N}.$$

We note that both a_n and b_n are Fibonacci sequence, thus

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{b_n} = \frac{1+\sqrt{5}}{2} \Rightarrow \lim_{n\to\infty} \sqrt[n]{T(n)} = \frac{1+\sqrt{5}}{2}$$

Therefore,

$$\mu(\mathbb{Z}\times\{0,1\}) = \lim_{n\to\infty} \sqrt[n]{\lambda_n} \le \lim_{n\to\infty} \sqrt[n]{4T(n+1)} = \frac{1+\sqrt{5}}{2} \quad \blacksquare$$

Remark. (a) Highly non-trivial to compute μ.

(b) On the hexagonal lattice, it is shown that $\mu = \sqrt{2+\sqrt{2}}$ (2010, Copin at el.)

(c) By computer simulation,
Conjecture : $Z_\beta = (\beta - \beta_c)^{-\gamma + o(1)}$ for $\beta \to \beta_c^+$, where γ only depends on that dimension of the lattice. In 2D, $\gamma = \frac{43}{32}$ (conjectured)

(d) If we have $\lambda_N \sim \mu^N N^\alpha$, then Z_β can be computed to satisfy $Z_\beta \sim (\beta - \beta_c)^{-1+\alpha}$, $\alpha = \frac{11}{32}$ in 2D (conjectured)

Exercise 3 How to sample SAWs with computer.

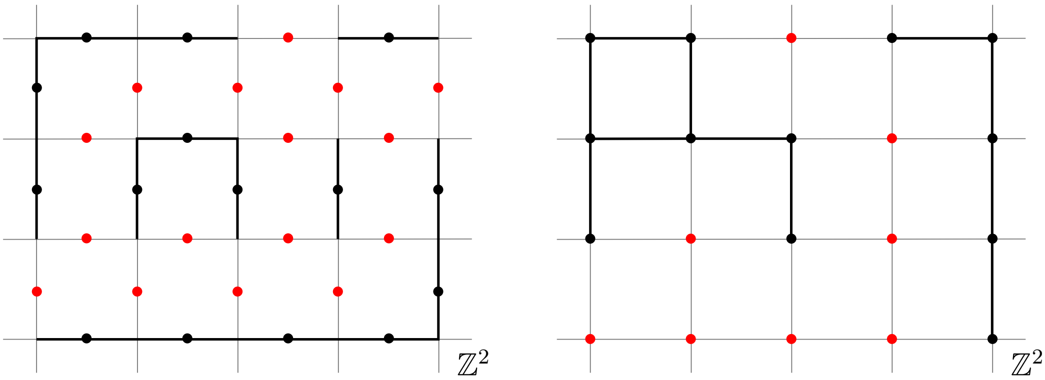
0.3) Bernoulli Percolation

We first consider $\mathbb{Z}^d$ lattices,

- **Bond percolation** Give $p \in [0,1]$, consider $\omega = (\omega(e))_{e\in\mathbb{Z}^d}$ such that $\{\omega(e)\}_{e\in\mathbb{Z}^d}$ is i.i.d with $\omega(e) \sim$ Brenoulli(p).
  If $\omega(e) = 1$, we say that e is open. If $\omega(e) = 0$, we say that e is closed.
  Remark We got a model of random subgraph. By above of notation, we may also write ω for the random subgraph (consisting of that open edges).

- **Site percolation** Same thing with i.i.d Bernoulli(p),
  open means that the node can pass, closed means that the node can not pass.

- Notation :
  $\mathbb{P}_p =$ The Bernoulli percolation of parameter p.
  ω_p a sample of $\mathbb{P}_p$



A bond percolation　　　　A site percolation

Remark The terminology "Bernoulli percolation" stands for **i.i.d**, on the other hand, without independence, we simply say that we have a "percolation model", e.g. random cluster model.
For the following classes we use "percolation" to refer to Bernoulli percolation.

Exercise 1 Show that a bond percolation is equivalent to a site percolation. How about the other way? Construct an example.

Question : What are the interesting behavior when p varies? e.g. # component, size of component etc.
$p = 0$ is an empty graph, $p = 1$ is a full graph.

- **Connected component (cluster)**
  Let a, b be two vertex of $\mathbb{Z}^d$, we say that $a \sim b$ if exists an path in ω_p from a to b. It is clearly that $\sim$ is an equivalence relation.
  A **connected component**(cluster) is an element in equivalence classes of $\sim$

- **Infinite cluster**
  A infinite cluster is a cluster of ω_p that has infinite edges and infinite vertex.
  Let $[O \leftrightarrow \infty]$ be the event in $\mathbb{P}_p$ that O belongs to a infinite cluster.
  $\theta(p) = \mathbb{P}_p[O \leftrightarrow \infty]$.

1. Basic Properties of the Bernoulli Percolation

Consider $G = \mathbb{Z}^d$ or some "nice" graph.

1.1) Coupling (耦合)

- Given $p \le p'$, how to compute $X \sim \text{Bernoulli}(p),\ X' \sim \text{Bernoulli}(p')$?
  Consider $\mathcal{U} \sim \text{Uniform}([0,1])$, define $Y = \mathbf{1}_{\mathcal{U} \le p}$, $Y' = \mathbf{1}_{\mathcal{U} \le p'}$, we get

  $$X \overset{(id)}{=} Y, \quad X' \overset{(id)}{=} Y', \quad Y \le Y'. \quad a.s.(almost\ sure)$$

  This is called a coupling.

Remark. In coupling, usually we do not want independence, so that we can compute values between random variables.

Exercise 1 Construct a coupling between $\omega \sim \mathbb{P}_p,\ \omega' \sim \mathbb{P}_{p'}$ with $p \le p'$, so that values between edges can be computed.
Wanted : $p \le p' \Rightarrow \omega_p \le \omega'_p (\Leftrightarrow \omega_p(e) \le \omega_{p'}(e),\ \forall e \in E)$

$\boxed{\texttt{SoL:}}$ Let $\omega = (\omega(e))_{e \in G}$ such that $\{\omega(e)\}_{e \in G}$ is i.i.d. and $\omega(e) \sim \text{Uniform}([0,1])$.
Define $\omega_p \sim \mathbb{P}_p,\ \omega_{p'} \sim \mathbb{P}_{p'}$ as $\forall e \in E,\ \omega_p(e) = \mathbf{1}_{\omega(e) \le p},\ \omega_{p'}(e) = \mathbf{1}_{\omega(e) \le p'}$, thus, $p \le p' \Rightarrow \forall e \in E,\ \omega_p(e) \le \omega_{p'}(e)$.

Exercise 2 Given $O \in V(G)$, define $\theta : \begin{array}{ccl} [0,1] & \to & \mathbb{R} \\ p & \mapsto & \mathbb{P}_p([O \leftrightarrow \infty]) \end{array}$ (percolation function).
Show that θ is increasing. In more general case, at must how many different θ function can be obtain?

$\boxed{\texttt{SoL:}}$ If $p \le p'$, let $\omega_1 \sim \mathbb{P}_p,\ \omega_2 \sim \mathbb{P}_{p'}$, we use the definition of **Exercise 1**, we have
$\omega_1 \overset{(id)}{=} \omega_p,\ \omega_2 \overset{(id)}{=} \omega_{p'}$, thus $\mathbb{P}_p([O \leftrightarrow \infty]) = \mathbb{P}_p([O \leftrightarrow \infty] \text{ in } \omega_1) = \mathbb{P}_p([O \leftrightarrow \infty] \text{ in } \omega_p)$,

similarly, $\mathbb{P}_{p'}([O \leftrightarrow \infty]) = \mathbb{P}_{p'}([O \leftrightarrow \infty] \text{ in } \omega_1) = \mathbb{P}_{p'}([O \leftrightarrow \infty] \text{ in } \omega_{p'})$. We note that ω_p is always a subgraph of $\omega_{p'}$ (by **Exercise 1**), thus $\{[O \leftrightarrow \infty] \text{ in } \omega_p\} \subseteq \{[O \leftrightarrow \infty] \text{ in } \omega_{p'}\}$, we have $\mathbb{P}_p([O \leftrightarrow \infty]) = \mathbb{P}_p([O \leftrightarrow \infty] \text{ in } \omega_p) \leq \mathbb{P}_{p'}([O \leftrightarrow \infty] \text{ in } \omega_{p'}) = \mathbb{P}_{p'}([O \leftrightarrow \infty])$.
And, because of $\omega_1 \overset{(id)}{=} \omega_p$, where ω_1 is a arbitrary random variable with $\omega_1 \sim \mathbb{P}_p$ thus there are only one choice of θ. i.e., θ is well-defined. ■

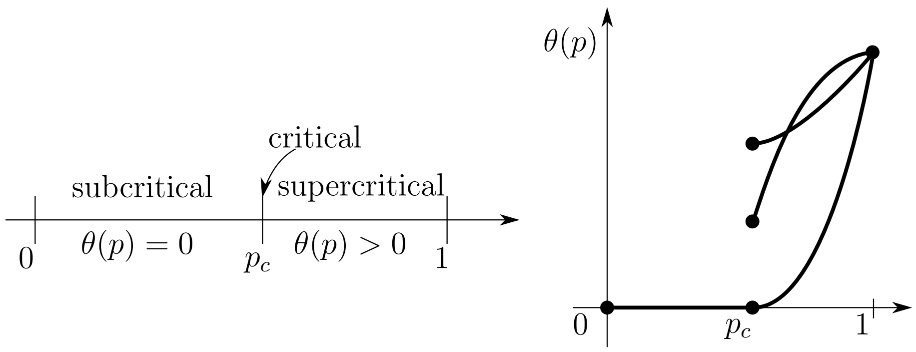
- Define $p_c = \sup\{p \in [0,1] \,|\, \theta(p) = 0\}$

Exercise 3 Check the following properties :

(a) The function $p \mapsto \theta(p)$ is right-continuous on $[0, 1]$.

(b) The function $p \mapsto \theta(p)$ is left-continuous on $(p_c, 1)$.

(c) Show that $p \mapsto \theta(p)$ is strictly increasing in $(p_c, 1]$.



1.2) Uniqueness of the Infinite Cluster (on $\mathbb{Z}^d$)

Theorem. For a fixed $p \in [0,1]$, we have

- $*)$ Either a.s. (almost surely) there is no infinite cluster.
- $*)$ or a.s. there is a unique infinite cluster.

Remark $N = \#$ infinite clusters. (a random variable)
We have $\mathbb{P}(N=0)=1$ or $\mathbb{P}(N=1)=1$.

Proof. We decompose the proof in 3 steps :

Step 1 Show that if an event A is invariant under some vector e (平移不變性), then

$$\mathbb{P}(A) = 0 \text{ or } \mathbb{P}(A) = 1.$$

$\to$ Kolmogorov's 0-1 law / "approximation" method.
$\Rightarrow$ For $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, the event $\{N = k\}$ satisfy the property. Thus, there exist $k = k(p)$ s.t. $\mathbb{P}_p(N = k) = 1$.

Step 2 Exclude the case $N = k$ a.s. for $k \geq 2,\ k \neq \infty$ using the "finite energy property".

Step 3 Exclude the case $N = \infty$ by looking at "**trifurcation points**".

Step 1 *Claim* : (**Exercise 1**)
Any event A can be "approximated" by a sequence of events that depend only on a finitely many edges. In other words, we can find a sequence of events $(A_n)_{n\geq 0}$ such that A_n only depends on edges in $[-m_n, m_n]^d$ for all $n \geq 0$ and

$$\lim_{n\to\infty} \mathbb{P}(A \triangle A_n) = 0, \quad \text{where } A \triangle A_n = (A \backslash A_n) \cup (A_n \backslash A).$$

$\boxed{\texttt{sol:}}$ We use monotone class theorem to show. "The theorem says that the smallest monotone class containing an algebra of sets $\mathcal{G}$ is precisely the smallest σ-algebra containing $\mathcal{G}$"- wiki
First, we note that it is true the collection of finite dimensional cylinder set is an algebra that generate the σ-algebra $\mathcal{F}$. We define the desired class $\mathcal{M}$ to be

$$\mathcal{M} = \{A \in \mathcal{F} | \forall \epsilon > 0, \exists A_\epsilon \text{ depend only on finitely many edges s.t } \mathbb{P}(A \triangle A_\epsilon) < \epsilon\}.$$

Then, by definition the cylinder set is a subset of $\mathcal{M}$. We proceed to check that $\mathcal{M}$ is a monotone class. For countable monotone unions, consider the union $A = \bigcup_{i=1}^\infty A_i$, where $A_1, A_2, \cdots \in \mathcal{M}$ and $A_1 \subseteq A_2 \subseteq \ldots$, since $\mathbb{P}(A) \leq 1$, for any given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\mathbb{P}(A \setminus A_N) < \epsilon/2$. On the other hand, the set A_N is in $\mathcal{M}$, so there is a set $\tilde{A_N}$ defined on finitely many edges that satisfies $\mathbb{P}(A_N \triangle \tilde{A_N}) < \epsilon/2$. Therefore,

$$\mathbb{P}(A \triangle \tilde{A_N}) < \mathbb{P}(A \setminus A_N) + \mathbb{P}(A_N \triangle \tilde{A_N}) < \epsilon.$$

ϵ is arbitrary, so $A \in \mathcal{M}$ The countable monotone intersection is shown in a similar fasion. Let $B = \bigcap_{i=1}^\infty B_i$, where $B_1, B_2, \cdots \in \mathcal{M}$ and $B_1 \subseteq B_2 \subseteq \ldots$. WLOG we may assume $B \neq \emptyset$, then given any $\epsilon > 0$,there exists $N \in \mathbb{N}$ such that $\mathbb{P}(B_N \setminus B) < \epsilon/2$. Also, there is $\tilde{B_N}$ that defined on finitely many edges and $\mathbb{P}(B_N \triangle \tilde{B_N}) < \epsilon/2$. Hence,

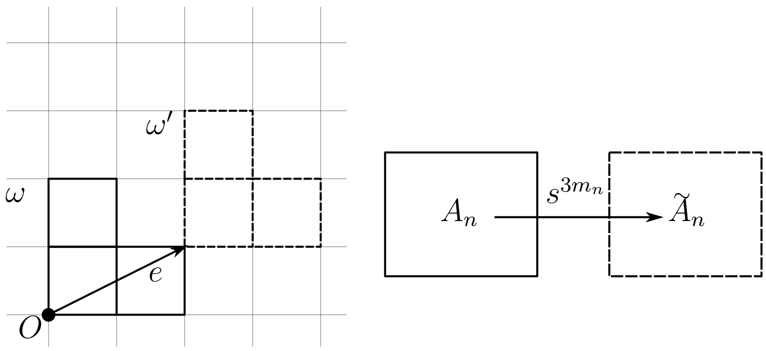
$$\mathbb{P}(B \triangle \tilde{B_N}) < \mathbb{P}(B_N \setminus B) + \mathbb{P}(B_N \triangle \tilde{B_N}) < \epsilon$$

this implies $B \in \mathcal{M}$. Finally, $\mathcal{M}$ is a monotone class containing the generating algebra (i.e cylinder set) of the σ-algebra $\mathcal{F}$, so by monotone class theorem, $\mathcal{F} \subseteq \mathcal{M}$. $\square$

Given an event A that is invariant under a vector $e \neq 0$, we approximate A by $(A_n)_{n\geq 1}$, i.e., $\mathbb{P}(A \triangle A_n) \to 0$ as $n \to \infty$. Where A_n depends only on edges in $[-m_n, m_n]^d$.
Define $\widetilde{A}_n := s^{3m_n}(A_n)$ where

$$s : \begin{array}{ccc} \Omega & \to & \Omega \\ \omega(\cdot) & \mapsto & \omega(\cdot - e) \end{array} \text{ (translation operator)}$$

Moreover, because of A is invariant under vector e, we have $s(A) = A$. Now, using



the independence between A_n and $\widetilde{A}_n$,

$$\mathbb{P}(A_n \cap \widetilde{A}_n) = \mathbb{P}(A_n)\mathbb{P}(\widetilde{A}_n) = \mathbb{P}(A_n)^2 \underset{n\to\infty}{\longrightarrow} \mathbb{P}(A)^2$$

Therefore,

$$\mathbb{P}(A \triangle (A_n \cap \widetilde{A}_n)) \leq \mathbb{P}(A \triangle A_n) + \mathbb{P}(A \triangle \widetilde{A}_n) \underset{n\to\infty}{\longrightarrow} 0$$
$$\Rightarrow \mathbb{P}(A) = \mathbb{P}(A)^2 \Rightarrow \mathbb{P}(A) = 0 \text{ or } 1$$

Remark If $\lim\limits_{n\to\infty} (A \triangle (A_n \cap \widetilde{A}_n)) = 0$, then

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}(A \triangle (A_n \cap \widetilde{A}_n)) &= \lim_{n\to\infty} \mathbb{P}\big((A \cap (A_n \cap \widetilde{A}_n)^\complement) \cup (A^\complement \cap (A_n \cap \widetilde{A}_n))\big) \\
&= \lim_{n\to\infty} \mathbb{P}(A \cap (A_n \cap \widetilde{A}_n)^\complement) + \mathbb{P}(A^\complement \cap (A_n \cap \widetilde{A}_n)) = 0 \\
&\Rightarrow \lim_{n\to\infty} \mathbb{P}(A \cap (A_n \cap \widetilde{A}_n)^\complement) = 0 \text{ and} \\
&\quad \lim_{n\to\infty} \mathbb{P}(A^\complement \cap (A_n \cap \widetilde{A}_n)) = 0. \\
&\Rightarrow \lim_{n\to\infty} \mathbb{P}(A) - \mathbb{P}(A_n \cap \widetilde{A}_n) \\
&= \lim_{n\to\infty} \mathbb{P}(A \cap (A_n \cap \widetilde{A}_n)) - \mathbb{P}(A \cap (A_n \cap \widetilde{A}_n)) = 0.
\end{aligned}$$

Step 2 We want to show that it is impossible to have $N = k$ as for $k \geq 2,\ k \neq \infty$. We assume that $\mathbb{P}(N = k) = 1$ for some $k \geq 2$, consider

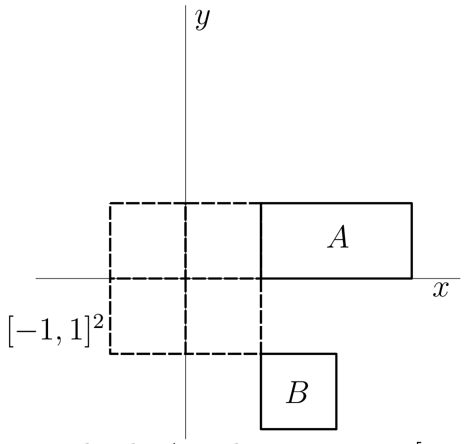
$$B_n = \{N = k \text{ and } k \text{ infinite clusters intersect } [-n, n]^d\}$$

Note that there exist $n \in \mathbb{N}$ such that $\mathbb{P}(B_n) > 0$. Define

$$C_n = \{ \text{ all infinite clusters intersect } [-n, n]^d\}$$

Note that

i. $B_n \subseteq C_n$.
ii. B_n depends on the edges in $[-n, n]^d$, but C_n does not.



both A and B intersect $[-1,1]^2$

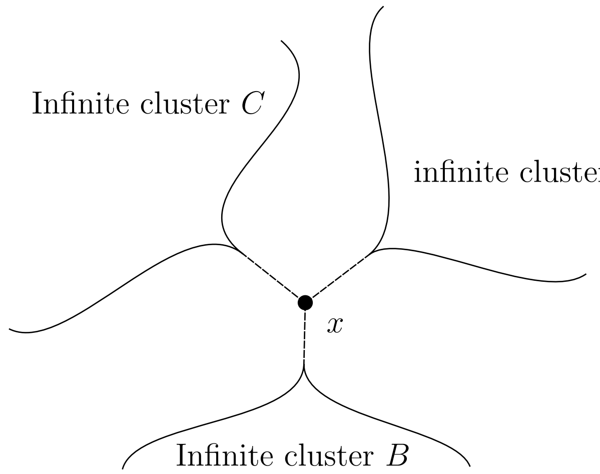
We use $\blacksquare_n$ to be the event that all edges in $[-n,n]^d$ are open, then

$$\mathbb{P}(C_n \cap \blacksquare_n) = \mathbb{P}(C_n)\mathbb{P}(\blacksquare_n) > 0.$$

Moreover, $C_n \cap \blacksquare_n \subseteq \{N = 1\}$, we find that $\mathbb{P}(N = 1) > 0 \Rightarrow \mathbb{P}(N = 1) = 1$ (contradicts).

Remark The fact that we can modify the state of the edges in $[-n,n]$ while keeping the probability positive is called **finite-energy property**

Step 3 Now, we want to show that it is impossible to have $\mathbb{P}(N = \infty) = 1$.
Assume that $0 < p < 1$, A **trifurcation point** (三叉點) x in ω_p is defined by

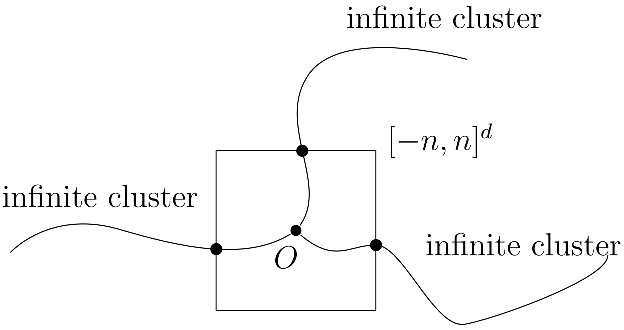


*) x is in a infinite cluster of ω.
*) Exactly 3 adjacent edge of x are open.
*) If all the adjacent edge of x are closed, we have 3 infinite clusters.

Assume that $\mathbb{P}(N = \infty) = 1$, define

$$D_n = \{ \text{ at least 3 infinite clusters intersect}[-n,n]^d \ \}$$

We can find $n \in \mathbb{N}$ such that $\mathbb{P}(D_n) > 0$.



(a) The box $\Lambda_n = [-n,n]^d$ is finite, thus any edge configuration has at lest probability $\min\{p, 1-p\}^{|E(\Lambda_n)|} > 0$ to be happen.

(b) For each configuration $\omega \in D_n$, we can find a nice configuration in Λ_n such that O is a trifurcation point.
$\Rightarrow c := \mathbb{P}(O \text{ is a trifurcation }) \geq \mathbb{P}(D_n) \cdot \min\{p, 1-p\}^{|E(\Lambda_n)|} > 0$

For each trifurcation point x, we can define at least three self-avoiding path form x to ∞, each path goes from 3 distinct infinite clusters, respectively. A intersection of a SA paths form trifurcation point and $\partial\Lambda_n$ is called a leaf. We deduce that

$$\#\text{ of leaves in } \partial\Lambda_n \geq 2\#\text{ of trifurcations in } \Lambda_n$$

Define the following random variables :

$$\begin{aligned} X_n &= \#\text{ trifurcations in } \Lambda_n. \\ &= \sum_{x \in \Lambda_n} \mathbb{1}_{\{x \text{ is a trifurcation point}\}} \\ Y_n &= \#\text{ leaves in } \partial\Lambda_n. \end{aligned}$$

Then we have

$$\mathbb{E}[Y_n] \geq 2\mathbb{E}[X_n], \quad \forall n \in \mathbb{N}.$$

Note that $Y_n \leq |\partial\Lambda_n| = 8n$, thus

$$8n \geq \mathbb{E}[Y_n] \geq 2\mathbb{E}[X_n] = 2c \times (2n+1)^2, \quad \forall n \in \mathbb{N}$$

This implies $c = 0$ (contradiction)

□