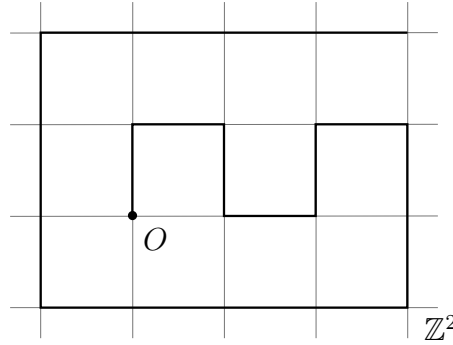


0.2) First Model (Polymer Model)

- Regular lattices, e.g., \mathbb{Z}^d
- Self-avoiding walk (SAW) (a polymer)



Example A self-avoiding walk

- Consider $\Omega = \{ \text{all SAWs} \}$, define $H(\omega) = |\omega|$ where $\omega \in \Omega$
Let $n \in \mathbb{N}$, define $\lambda_n = \# \text{ SAWs of length } n$.

Observe : Given $n, m \geq 1$

Any SAW of length $n + m$ can be decomposed into 2 SAWs of length n and m .

$$\Rightarrow \lambda_{n+m} \leq \lambda_n \cdot \lambda_m.$$

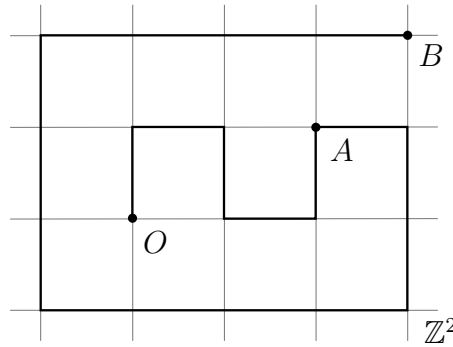


Figure $\lambda_{19} \leq \lambda_5 \cdot \lambda_{14}$

Exercise 1 $\mu \equiv \lim_{n \rightarrow \infty} (\lambda_n)^{\frac{1}{n}}$ exists and $\lambda_N \geq \mu^N$ for all $N \geq 1$.

SOL Note that $\forall n \in \mathbb{N}$, we have $(\lambda_n)^{\frac{1}{n}} \geq 0$, thus 0 is a lower bound of $\{(\lambda_n)^{\frac{1}{n}}\}_{n=0}^{\infty}$, therefore

$$\inf_{n \in \mathbb{N}_0} (\lambda_n)^{\frac{1}{n}} = K \in \mathbb{R}$$

Now, given $\varepsilon > 0$, 1° $\exists N_1 \in \mathbb{N}$ s.t.

$$K + \frac{\varepsilon}{2} > (\lambda_{N_1})^{\frac{1}{N_1}}$$

2° By division algorithm, $\forall n \geq N_1$, $\exists m, \ell \in \mathbb{Z}$ with $0 \leq \ell \leq N_1$ such that $n = mN_1 + \ell$, thus

$$\lambda_n = \lambda_{mN_1 + \ell} \leq (\lambda_{N_1})^m \cdot \lambda_{\ell}$$

therefore,

$$\begin{aligned} (\lambda_n)^{\frac{1}{n}} &\leq (\lambda_{N_1})^{\frac{m}{n}} \cdot (\lambda_\ell)^{\frac{1}{n}} = (\lambda_{N_1})^{\frac{1}{N_1} + \frac{\ell}{m}} \cdot (\lambda_\ell)^{\frac{1}{n}} \leq (\lambda_{N_1})^{\frac{1}{N_1}} \cdot (\lambda_\ell)^{\frac{1}{n}} \\ &\leq (\lambda_{N_1})^{\frac{1}{N_1}} \cdot (\lambda_1)^{\frac{\ell}{n}} \leq (\lambda_{N_1})^{\frac{1}{N_1}} \cdot (\lambda_1)^{\frac{N_1}{n}} \end{aligned}$$

Because of $\lim_{n \rightarrow \infty} (\lambda_{N_1})^{\frac{1}{N_1}} \cdot (\lambda_1)^{\frac{N_1}{n}} = (\lambda_{N_1})^{\frac{1}{N_1}}$, $\exists N \in \mathbb{N}$ such that $n \geq N \Rightarrow$

$$(\lambda_{N_1})^{\frac{1}{N_1}} + \frac{\varepsilon}{2} > (\lambda_{N_1})^{\frac{1}{N_1}} \cdot (\lambda_1)^{\frac{N_1}{n}} \geq (\lambda_n)^{\frac{1}{n}}$$

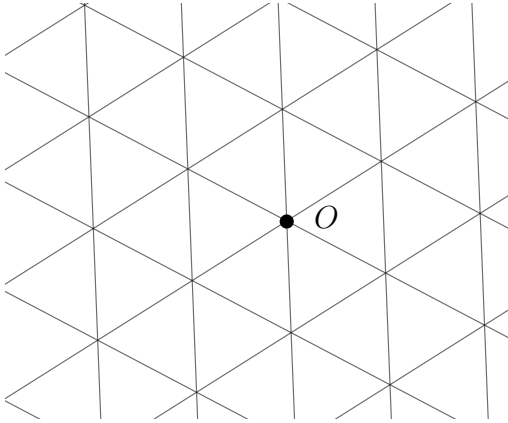
Therefore

$$K + \varepsilon = K + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} > \lambda_{N_1}^{\frac{1}{N_1}} + \frac{\varepsilon}{2} > (\lambda_n)^{\frac{1}{n}} \geq K.$$

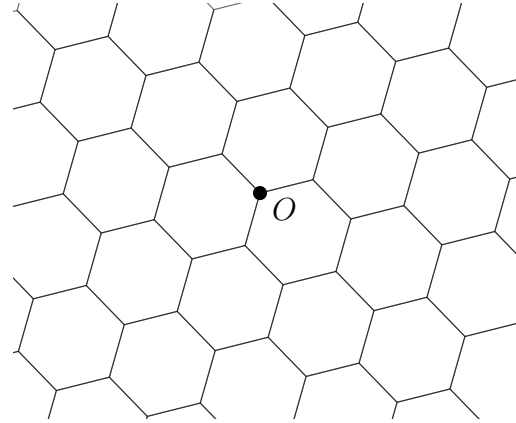
Thus, $\mu = \lim_{n \rightarrow \infty} (\lambda_n)^{\frac{1}{n}} = K$ exists.

Next, for $N > 1$, we have $\forall n \in \mathbb{N}, \lambda_{nN} \leq (\lambda_N)^n$, thus $(\lambda_{nN})^{\frac{1}{nN}} \leq (\lambda_N)^{\frac{1}{N}}$. Note that $\{(\lambda_{nN})^{\frac{1}{nN}}\}_{n=1}^\infty$ is a subsequence of $\{(\lambda_n)^{\frac{1}{n}}\}_{n=1}^\infty$, therefore $\mu = \lim_{n \rightarrow \infty} (\lambda_{nN})^{\frac{1}{nN}} \leq (\lambda_N)^{\frac{1}{N}}$ ■

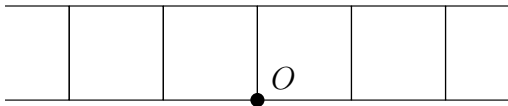
- Exercise 2**
- (a) In \mathbb{Z}^2 , $\mu \in (2, 3)$
 - (b) In \mathbb{Z}^3 , $\mu > 3$
 - (c) In the triangular mesh, $\mu > 3$;
 - (d) In the hexagonal mesh, $\mu < 2$
 - (e) In $\mathbb{Z} \times \{0, 1\}$ (i.e. a ladder), $\mu = \frac{1+\sqrt{5}}{2}$.



The triangular mesh



The hexagonal mesh



$\mathbb{Z} \times \{0, 1\}$

sOL: We first show a fact :

Fact(ratio test and root test) : Let $\{a_n\}_{n=1}^{\infty}$ be a sequence that take value in $(0, \infty)$. Then

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = K \in \mathbb{R} \Rightarrow \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = K.$$

Proof. If $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = K \in \mathbb{R}$, then given $\varepsilon > 0$, there exists $N_1 > 0$ such that

$$n \geq N_1 \Rightarrow \left| \frac{a_{n+1}}{a_n} - K \right| < \frac{\varepsilon}{2} \Rightarrow K - \frac{\varepsilon}{2} < \frac{a_{n+1}}{a_n} < K + \frac{\varepsilon}{2}$$

Note that $a_n = a_1 \times \frac{a_2}{a_1} \times \frac{a_3}{a_2} \times \cdots \times \frac{a_n}{a_{n-1}} = a_1 \times \prod_{k=1}^{n-1} \frac{a_{k+1}}{a_k} = a_1 \times \prod_{k=1}^{N_1-1} \frac{a_{k+1}}{a_k} \times \prod_{k=N_1}^{n-1} \frac{a_{k+1}}{a_k}$.

Now define $\mathcal{Q} = a_1 \times \prod_{k=1}^{N_1-1} \frac{a_{k+1}}{a_k}$, then $\sqrt[n]{a_n} = \sqrt[n]{\mathcal{Q}} \times \prod_{k=N_1}^{n-1} \left(\frac{a_{k+1}}{a_k} \right)^{\frac{1}{n}}$, thus

$$\begin{aligned} \sqrt[n]{\mathcal{Q}} \times \left(K - \frac{\varepsilon}{2} \right)^{\frac{n-N_1-1}{n}} &= \sqrt[n]{\mathcal{Q}} \times \prod_{k=N_1}^{n-1} \left(K - \frac{\varepsilon}{2} \right)^{\frac{1}{n}} < \sqrt[n]{a_n} = \sqrt[n]{\mathcal{Q}} \times \prod_{k=N_1}^{n-1} \left(\frac{a_{k+1}}{a_k} \right)^{\frac{1}{n}} \\ &< \sqrt[n]{\mathcal{Q}} \times \prod_{k=N_1}^{n-1} \left(K + \frac{\varepsilon}{2} \right)^{\frac{1}{n}} = \sqrt[n]{\mathcal{Q}} \times \left(K + \frac{\varepsilon}{2} \right)^{\frac{n-N_1-1}{n}}. \end{aligned}$$

By the fact that

$A(n) = \sqrt[n]{\mathcal{Q}} \times \left(K - \frac{\varepsilon}{2} \right)^{\frac{n-N_1-1}{n}} \rightarrow K - \frac{\varepsilon}{2}$, $B(n) = \sqrt[n]{\mathcal{Q}} \times \left(K + \frac{\varepsilon}{2} \right)^{\frac{n-N_1-1}{n}} \rightarrow K + \frac{\varepsilon}{2}$ as $n \rightarrow \infty$, there is $N > N_1$ such that $n \geq N \Rightarrow A(n) > K - \varepsilon$ and $B(n) < K + \varepsilon$, thus

$$K - \varepsilon < A(n) < \sqrt[n]{a_n} < B(n) < K + \varepsilon.$$

Therefore, $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = K$. □

(a)

Remark. (a) Highly non-trivial to compute μ .

(b) On the hexagonal lattice, it is shown that $\mu = \sqrt{2 + \sqrt{2}}$ (2010, Copin et al.)

(c) By computer simulation,

Conjecture : $Z_\beta = (\beta - \beta_c)^{-\gamma+o(1)}$ for $\beta \rightarrow \beta_c^+$, where γ only depends on that dimension of the lattice. In 2D, $\gamma = \frac{43}{32}$ (conjectured)

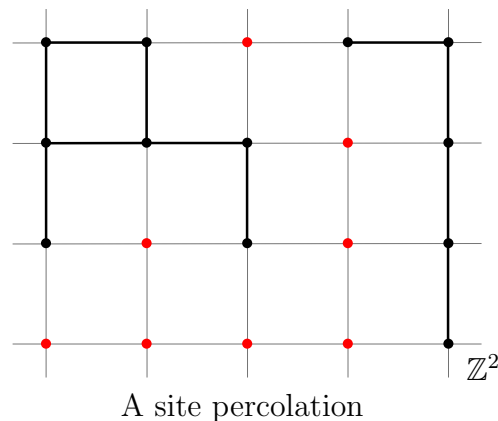
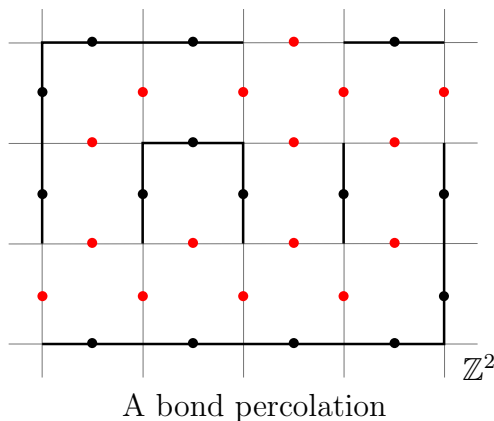
(d) If we have $\lambda_N \sim \mu^N N^\alpha$, then Z_β can be computed to satisfy $Z_\beta \sim (\beta - \beta_c)^{-1+\alpha}$, $\alpha = \frac{11}{32}$ in 2D (conjectured)

Exercise 3 How to sample SAWs with computer.

0.3) Bernoulli Percolation

We first consider \mathbb{Z}^d lattices,

- **Bond percolation** Give $p \in [0, 1]$, consider $\omega = (\omega(e))_{e \in \mathbb{Z}^d}$ such that $\{\omega(e)\}_{e \in \mathbb{Z}^d}$ is i.i.d with $\omega(e) \sim \text{Bernoulli}(p)$.
If $\omega(e) = 1$, we say that e is open. If $\omega(e) = 0$, we say that e is closed.
- **Remark** We got a model of random subgraph. By above of notation, we may also write ω for the random subgraph (consisting of that open edges).
- **Site percolation** Same thing with i.i.d $\text{Bernoulli}(p)$,
open means that the node can pass, closed means that the node can not pass.
- **Notation :**
 \mathbb{P}_p = The Bernoulli percolation of parameter p .
 ω_p a sample of \mathbb{P}_p



Remark The terminology “Bernoulli percolation” stands for **i.i.d**, on the other hand, without independence, we simply say that we have a “percolation model”, e.g. random cluster model.

For the following classes we use “percolation” to refer to Bernoulli percolation.

Exercise 1 Show that a bond percolation is equivalent to a site percolation. How about the other way? Construct an example.

Question : What are the interesting behavior when p varies? e.g. # component, size of component etc.

$p = 0$ is an empty graph, $p = 1$ is a full graph.

- **Connected component (cluster)**

Let a, b be two vertex of \mathbb{Z}^d , we say that $a \sim b$ if exists an path in ω_p from a to b . It is clearly that \sim is an equivalence relation.

A **connected component**(cluster) is an element in equivalence classes of \sim

- **Infinite cluster**

A infinite cluster is a cluster of ω_p that has infinite edges and infinite vertex.

Let $[O \leftrightarrow \infty]$ be the event in \mathbb{P}_p that O belongs to a infinite cluster.

$\theta(p) = \mathbb{P}_p[O \leftrightarrow \infty]$.

1. Basic Properties of the Bernoulli Percolation

Consider $G = \mathbb{Z}^d$ or some “nice” graph.

1.1) Coupling (耦合)

- Given $p \leq p'$, how to compute $X \sim \text{Bernoulli}(p)$, $X' \sim \text{Bernoulli}(p')$?
Consider $\mathcal{U} \sim \text{Uniform}([0, 1])$, define $Y = \mathbf{1}_{\mathcal{U} \leq p}$, $Y' = \mathbf{1}_{\mathcal{U} \leq p'}$, we get

$$X \stackrel{(id)}{=} Y, \quad X' \stackrel{(id)}{=} Y', \quad Y \leq Y'. \quad a.s. (almost sure)$$

This is called a coupling.

Remark. In coupling, usually we do not want independence, so that we can compute values between random variables.

Exercise 1 Construct a coupling between $\omega \sim \mathbb{P}_p$, $\omega' \sim \mathbb{P}_{p'}$ with $p \leq p'$, so that values between edges can be computed.

Wanted : $p \leq p' \Rightarrow \omega_p \leq \omega_{p'} (\Leftrightarrow \omega_p(e) \leq \omega_{p'}(e), \forall e \in E)$

sOL: Let $\omega = (\omega(e))_{e \in G}$ such that $\{\omega(e)\}_{e \in G}$ is i.i.d. and $\omega(e) \sim \text{Uniform}([0, 1])$.
Define $\omega_p \sim \mathbb{P}_p$, $\omega_{p'} \sim \mathbb{P}_{p'}$ as $\forall e \in E$, $\omega_p(e) = \mathbf{1}_{\omega(e) \leq p}$, $\omega_{p'}(e) = \mathbf{1}_{\omega(e) \leq p'}$, thus, $p \leq p' \Rightarrow \forall e \in E$, $\omega_p(e) \leq \omega_{p'}(e)$.

Exercise 2 Given $O \in V(G)$, define $\theta : [0, 1] \rightarrow \mathbb{R}$ (percolation function).
 $p \mapsto \mathbb{P}_p([O \leftrightarrow \infty])$

Show that θ is increasing. In more general case, at most how many different θ function can be obtain?

sOL: If $p \leq p'$, let $\omega_1 \sim \mathbb{P}_p$, $\omega_2 \sim \mathbb{P}_{p'}$, we use the definition of **Exercise 1**, we have
 $\omega_1 \stackrel{(id)}{=} \omega_p$, $\omega_2 \stackrel{(id)}{=} \omega_{p'}$, thus $\mathbb{P}_p([O \leftrightarrow \infty]) = \mathbb{P}_p([O \leftrightarrow \infty] \text{ in } \omega_1) = \mathbb{P}_p([O \leftrightarrow \infty] \text{ in } \omega_p)$,

similarly, $\mathbb{P}_{p'}([O \leftrightarrow \infty]) = \mathbb{P}_{p'}([O \leftrightarrow \infty] \text{ in } \omega_1) = \mathbb{P}_{p'}([O \leftrightarrow \infty] \text{ in } \omega_{p'})$. We note that ω_p is always a subgraph of $\omega_{p'}$ (by **Exercise 1**), thus $\{[O \leftrightarrow \infty] \text{ in } \omega_p\} \subseteq \{[O \leftrightarrow \infty] \text{ in } \omega_{p'}\}$, we have $\mathbb{P}_p([O \leftrightarrow \infty]) = \mathbb{P}_p([O \leftrightarrow \infty] \text{ in } \omega_p) \leq \mathbb{P}_{p'}([O \leftrightarrow \infty] \text{ in } \omega_{p'}) = \mathbb{P}_{p'}([O \leftrightarrow \infty])$.

And, because of $\omega_1 \stackrel{(id)}{=} \omega_p$, where ω_1 is a arbitrary random variable with $\omega_1 \sim \mathbb{P}_p$ thus there are only one choice of θ . i.e., θ is well-defined. ■

- Define $p_c = \sup\{p \in [0, 1] \mid \theta(p) = 0\}$

Exercise 3 Check the following properties :

- The function $p \mapsto \theta(p)$ is right-continuous on $[0, 1]$.
- The function $p \mapsto \theta(p)$ is left-continuous on $(p_c, 1)$.
- Show that $p \mapsto \theta(p)$ is strictly increasing in $(p_c, 1]$.