Double Phase Transition

In this section, we introduce the interesting property, double phase transition of $\mathbb{T}_d \times \mathbb{Z}$, which is different from integers lattices $\mathbb{Z}^d$, triangle lattices and regular trees. The number of infinite clusters changes on $[0,1]$; there's no infinite cluster on $(0, p_c)$, infinitely many on (p_c, p_u) and one on $(p_u, 1)$, where p_u is defined as

$$p_u = \{p \in [0,1] \mid \text{there is a.s. a unique infinite cluster}\}.$$

The property is given by the theorem:

Theorem. *For $d \geq 6$,*

$$0 < p_c(\mathbb{T}_d \times \mathbb{Z}) < p_u(\mathbb{T}_d \times \mathbb{Z}) < 1.$$

For the lower bound of $p_c > 0$, we can use the following lemma:

Lemma. *For an infinite connected graph G, $p_c \geq \frac{1}{\mu}$, where μ is the connective constant of G.*

The main idea is through bounding the $\theta(p)$ by the number of self-avoiding walks of length n with probability p^n.
To make sure $p_c < 1$, we can use

Theorem. *If G is Cayley graph of a group with exponential growth, then $p_c < 1$.*

where we bound the p_c of G by the p_c of its subgraph, lexicographically minimal spanning tree. As for the inequality $p_u < 1$, we apply the theorem showed by Babson and Benjamini

Theorem. *If G is the Cayley graph of a nonamenable finitely presented group with one end, then $p_u < 1$.*

where we consider special graphs and a combinatorial fact to obtain the desired result. The most difficult part is to show the inequality $p_c < p_u$. The essential ingredients are the following:

Theorem. *If G is a $d-$regular connected multigraph, then*

$$\mathit{cogr}(G) > \sqrt{d-1} \textit{ iff } \rho(G) > \frac{2\sqrt{d-1}}{d}$$

in which case

$$d\rho(G) = \frac{d-1}{\mathit{cogr}(G)} + \mathit{cogr}(G).$$

Corollary. *For all $b \geq 1$, we have*

$$\rho(\mathbb{T}_d \times \mathbb{Z}) = \frac{2\sqrt{b}+2}{b+3}.$$

Of course, the proofs are nontrivial but they give the interesting property that the number of infinite clusters varies on the interval in p.

reference: R. Lyons, Y. Peres, *Probability on Trees and Networks.* (2016)