Russo-Seymour-Welsh Theory

In this section, we discuss the scale invariant property of some connecting events when $p = p_c$ on $\mathbb{Z}^2$ lattice. In particular, we presents the invariant behaviour of the *horizontal crossing event in a rectangle of size* $[0, \rho n] \times [0, n]$, which is denoted by $\mathcal{H}(\rho n, n)$.

Theorem (Russo-Seymour-Welsh). *Let $\rho > 0$. There exists $c = c(\rho) > 0$ such that for all $n \geq 1$, we have*

$$c \leq \mathbb{P}_{\frac{1}{2}}\left[\mathcal{H}(\rho n, n)\right] \leq 1 - c,$$

where $\mathcal{H}(\rho n, n)$ denotes the horizontal crossing event in a rectangle with size $[0, \rho n] \times [0, n]$.

They proved this theorem by proving a special case:
Theorem. *For all $n \geq 1$,*

$$\mathbb{P}_{\frac{1}{2}}\left[\mathcal{H}(3n, 2n)\right] \geq \frac{1}{128}.$$

Once one get this result, i.e. when one find $c(\rho)$ for some $\rho > 1$ (e.g. $c(3/2)$), then one can get $c(\rho')$ for arbitrary $\rho' > 1$ by construct the crossing events $\mathcal{H}(\rho n, n)$ that assures $\mathcal{H}(\rho' n, n)$ to occur, and hence we can prove the first theorem. For example, to get a lower bound for $\mathbb{P}_{p_c}[\mathcal{H}(4n, n)]$, we can place five $(2n, n)$ boxes as follows: Let $R_1 = [0, 2n] \times [0, n]$, $R_2 = [n, 2n] \times [-n, n]$, $R_3 = [n, 3n] \times [-n, 0]$, $R_4 = [2n, 3n] \times [-n, n]$ and $R_5 = [2n, 4n] \times [0, n]$. Then we have

$$\mathbb{P}_{p_c}[\mathcal{H}(R_1) \cap \mathcal{V}(R_2) \cap \mathcal{H}(R_3) \cap \mathcal{V}(R_4) \cap \mathcal{H}(R_5)] \leq \mathbb{P}_{p_c}[\mathcal{H}([0, 4n] \times [0, n])].$$

Now by Harris-FKG inequality and translation invariant property on $\mathbb{Z}^2$ lattice, we immediately have

$$c(2)^5 \leq \mathbb{P}_{p_c}[\mathcal{H}(4n, n)].$$

With Russo-Seymour-Welsh's theory, we're able to give a scale invariant property for more general crossing events. Consider a simply connected domain with a smooth boundary $\mathbf{\Omega}$ with distinct boundary points a, b, c, d. For $\delta > 0$, we define a finite graph $\mathbf{\Omega}^\delta = \delta\mathbb{Z}^2 \cap \mathbf{\Omega}$. And let $a^\delta, b^\delta, c^\delta, d^\delta \in \mathbf{\Omega}^\delta$ to be the closest points to $a, b, c, d \in \partial\mathbf{\Omega}$. Also define $(a^\delta b^\delta)$, $(c^\delta d^\delta)$ as the paths on $\partial\mathbf{\Omega}^\delta$ from a^δ to b^δ, from c^δ to d^δ counterclockwise.

Theorem. *There exists $c = c(\mathbf{\Omega}, a, b, c, d) > 0$ such that for any $\delta > 0$,*

$$\mathbb{P}_{\frac{1}{2}}\left[(a^\delta b^\delta) \xleftrightarrow{\Omega^\delta} (c^\delta d^\delta)\right] \geq c.$$