
2022 NCTS USRP Group 7
Planar Statistical Physics and Bernoulli Percolation

FINAL REPORT

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Abstract

We study the percolation phenomenon in planar statistical physics using probability theory tools. We especially focus on *Bernoulli percolation model*. In the first three weeks, we received classes covering four main topics. Those classes introduce methods to discuss the connecting property and phase transition behavior on some regular lattice such as \mathbb{Z}^2 , \mathbb{T}_d , triangular lattice or hexagon lattice. After the classes were over, we went on individual research to explore topics such as exponential decay near critical probability and scaling invariant property of the crossing events.

Course Progress

(Planer) Statistical Mechanics

Planar Statistical Physics and Bernoulli Percolation

Useful Identities & Applications

Exponential Decay

Russo-Seymour-Welsh Theorem

In this section, we discuss the scale invariant property of some connecting events when $p = p_c$ on \mathbb{Z}^2 lattice. In particular, we presents the invariant behaviour of the *horizontal crossing event in a rectangle of size* $[0, pn] \times [0, n]$, which is denoted by $\mathcal{H}(\rho n, n)$.

Theorem (Russo-Seymour-Welsh). *Let $\rho > 0$. There exists $c = c(\rho) > 0$ such that for all $n \geq 1$, we have*

$$c \leq \mathbb{P}_{\frac{1}{2}}[\mathcal{H}(\rho n, n)] \leq 1 - c,$$

where $\mathcal{H}(\rho n, n)$ denotes the horizontal crossing event in a rectangle with size $[0, \rho n] \times [0, n]$.

They proved this theorem by proving a special case:

Theorem. *For all $n \geq 1$,*

$$\mathbb{P}_{\frac{1}{2}}[\mathcal{H}(3n, 2n)] \geq \frac{1}{128}.$$

Once one get this result, i.e. when one find $c(\rho)$ for some $\rho > 1$ (e.g. $c(3/2)$), then one can get $c(\rho')$ for arbitrary $\rho' > 1$ by construct the crossing events $\mathcal{H}(\rho n, n)$ that assures $\mathcal{H}(\rho' n, n)$ to occur, and hence we can prove the first theorem. For example, to get a lower bound for $\mathbb{P}_{p_c}[\mathcal{H}(4n, n)]$, we can place five $(2n, n)$ boxes as follows: Let $R_1 = [0, 2n] \times [0, n]$, $R_2 = [n, 2n] \times [-n, n]$, $R_3 = [n, 3n] \times [-n, 0]$, $R_4 = [2n, 3n] \times [-n, n]$ and $R_5 = [2n, 4n] \times [0, n]$. Then we have

$$\mathbb{P}_{p_c}[\mathcal{H}(R_1) \cap \mathcal{V}(R_2) \cap \mathcal{H}(R_3) \cap \mathcal{V}(R_4) \cap \mathcal{H}(R_5)] \leq \mathbb{P}_{p_c}[\mathcal{H}([0, 4n] \times [0, n])].$$

Now by Harris-FKG inequality and translation invariant property on \mathbb{Z}^2 lattice, we immediately have

$$c(2)^5 \leq \mathbb{P}_{p_c}[\mathcal{H}(4n, n)].$$

With Russo-Seymour-Welsh's theory, we're able to give a scale invariant property for more general crossing events. Consider a simply connected domain with a smooth boundary Ω with distinct boundary points a, b, c, d . For $\delta > 0$, we define a finite graph $\Omega^\delta = \delta\mathbb{Z}^2 \cap \Omega$. And let $a^\delta, b^\delta, c^\delta, d^\delta \in \Omega^\delta$ to be the closest points to $a, b, c, d \in \partial\Omega$. Also define $(a^\delta b^\delta), (c^\delta d^\delta)$ as the paths on $\partial\Omega^\delta$ from a^δ to b^δ , from c^δ to d^δ counterclockwise.

Theorem. *There exists $c = c(\Omega, a, b, c, d) > 0$ such that for any $\delta > 0$,*

$$\mathbb{P}_{\frac{1}{2}}[(a^\delta b^\delta) \xleftrightarrow{\Omega^\delta} (c^\delta d^\delta)] \geq c.$$

Individual Research

Hao: Exponential Decay

Future Work

There are several directions in the future. For example, on the scaling invariant property, RSW theory gave us a way to obtain the uniform probability bounds of crossing events, but we can also ask the question about the scaling limit of a crossing event, not only the uniform bounds. We'll try to apply the discrete analytic ideas developed by Smirnov to extend the scaling problem on different types of lattice.

Reference

1. S. Smirnov, *Critical percolation in the plane: Conformal invariance, Cardy's formula, scaling limits*, *C. R. Acad. Sci. Paris* (2001).
2. Hugo Duminil-Copin, *Introduction to Bernoulli percolation*, (2018).
3. Geoffrey R. Grimmett, Ioan Manolescu, *Universality for bond percolation in two dimensions*, *Ann. Probab.* (2013).
4. R. Lyons, Y. Peres, *Probability on Trees and Networks*. (2016)

