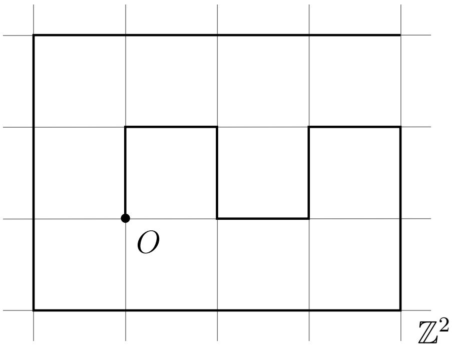
0.2) First Model (Polymer Model)

- Regular lattices, e.g., $\mathbb{Z}^d$
- Self-avoiding walk (SAW) (a polymer)



Example A self-avoiding walk

- Consider $\Omega = \{$ all SAWs $\}$, define $H(\omega) = |\omega|$ where $\omega \in \Omega$
  Let $n \in \mathbb{N}$, define $\lambda_n = \#$ SAWs of length n.

Observe : Given $n, m \geq 1$
Any SAW of length $n+m$ can be decomposed into 2 SAWs of length n and m.
$\Rightarrow \lambda_{n+m} \leq \lambda_n \cdot \lambda_m$.

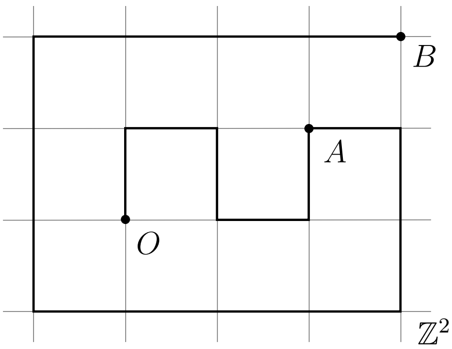


Figure $\lambda_{19} \leq \lambda_5 \cdot \lambda_{14}$

Exercise 1 $\mu \equiv \lim_{n\to\infty} (\lambda_n)^{\frac{1}{n}}$ exists and $\lambda_N \geq \mu^N$ for all $N \geq 1$.

$\boxed{\text{SOL}}$ Note that $\forall n \in \mathbb{N}$, we have $(\lambda_n)^{\frac{1}{n}} \geq 0$, thus 0 is a lower bound of $\{(\lambda_n)^{\frac{1}{n}}\}_{n=0}^{\infty}$, therefore

$$\inf_{n\in\mathbb{N}_0} (\lambda_n)^{\frac{1}{n}} = K \in \mathbb{R}$$

Now, given $\varepsilon > 0$, 1° $\exists N_1 \in \mathbb{N}$ s.t.

$$K + \frac{\varepsilon}{2} > (\lambda_{N_1})^{\frac{1}{N_1}}$$

2° By division algorithm, $\forall n \geq N_1$, $\exists m, \ell \in \mathbb{Z}$ with $0 \leq \ell \leq N_1$ such that $n = mN_1 + \ell$, thus

$$\lambda_n = \lambda_{mN_1+\ell} \leq (\lambda_{N_1})^m \cdot \lambda_\ell$$

therefore,

$$
\begin{aligned}
(\lambda_n)^{\frac{1}{n}} &\leq (\lambda_{N_1})^{\frac{m}{n}} \cdot (\lambda_\ell)^{\frac{1}{n}} = (\lambda_{N_1})^{\frac{1}{N_1 + \frac{\ell}{m}}} \cdot (\lambda_\ell)^{\frac{1}{n}} \leq (\lambda_{N_1})^{\frac{1}{N_1}} \cdot (\lambda_\ell)^{\frac{1}{n}} \\
&\leq (\lambda_{N_1})^{\frac{1}{N_1}} \cdot (\lambda_1)^{\frac{\ell}{n}} \leq (\lambda_{N_1})^{\frac{1}{N_1}} \cdot (\lambda_1)^{\frac{N_1}{n}}
\end{aligned}
$$

Because of $\lim_{n\to\infty} (\lambda_{N_1})^{\frac{1}{N_1}} \cdot (\lambda_1)^{\frac{N_1}{n}} = (\lambda_{N_1})^{\frac{1}{N_1}}$, $\exists N \in \mathbb{N}$ such that $n \geq N \Rightarrow$

$$
(\lambda_{N_1})^{\frac{1}{N_1}} + \frac{\varepsilon}{2} > (\lambda_{N_1})^{\frac{1}{N_1}} \cdot (\lambda_1)^{\frac{N_1}{n}} \geq (\lambda_n)^{\frac{1}{n}}
$$

Therefore

$$
K + \varepsilon = K + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} > \lambda_{N_1}^{\frac{1}{N_1}} + \frac{\varepsilon}{2} > (\lambda_n)^{\frac{1}{n}} \geq K.
$$

Thus, $\mu = \lim_{n\to\infty} (\lambda_n)^{\frac{1}{n}} = K$ exists.

Next, for $N > 1$, we have $\forall n \in \mathbb{N}, \lambda_{nN} \leq (\lambda_N)^n$, thus $(\lambda_{nN})^{\frac{1}{nN}} \leq (\lambda_N)^{\frac{1}{N}}$. Note that $\{(\lambda_{nN})^{\frac{1}{nN}}\}_{n=1}^{\infty}$ is a subsequence of $\{(\lambda_n)^{\frac{1}{n}}\}_{n=1}^{\infty}$, therefore $\mu = \lim_{n\to\infty} (\lambda_{nN})^{\frac{1}{nN}} \leq (\lambda_N)\frac{1}{N}$ ■

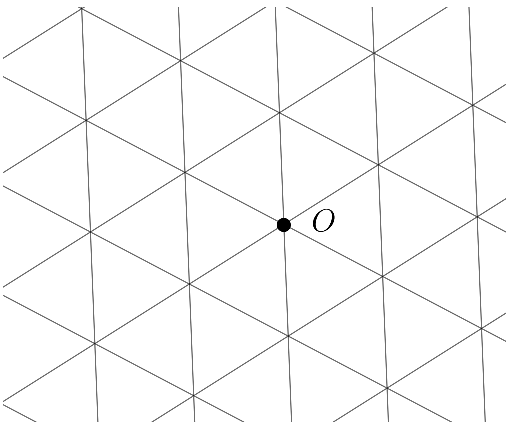
Exercise 2 (a) In $\mathbb{Z}^2$, $\mu \in (2,3)$

(b) In $\mathbb{Z}^3$, $\mu > 3$

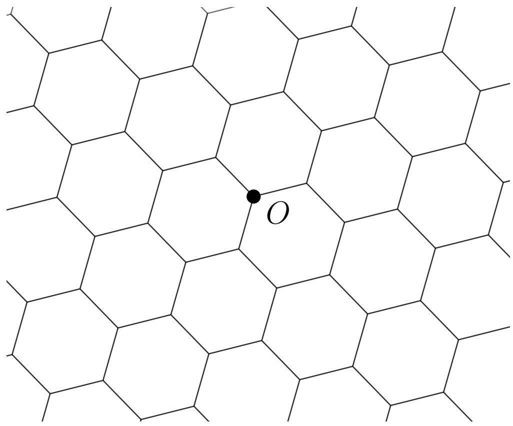
(c) In the triangular mesh, $\mu > 3$;

(d) In the hexagonal mesh, $\mu < 2$

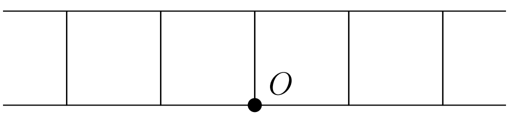
(e) In $\mathbb{Z} \times \{0,1\}$ (i.e. a ladder), $\mu = \frac{1+\sqrt{5}}{2}$.



The triangular mesh



The hexagonal mesh



$\mathbb{Z} \times \{0,1\}$

SOL: We first show a fact :

Fact(ratio test and root test) : Let $\{a_n\}_{n=1}^{\infty}$ be a sequence that take value in $(0,\infty)$. Then

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=K\in\mathbb{R}\Rightarrow\lim_{n\to\infty}\sqrt[n]{a_n}=K.$$

Proof. If $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=K\in\mathbb{R}$, then given $\varepsilon>0$, there exists $N_1>0$ such that

$$n\geq N_1\Rightarrow\left|\frac{a_{n+1}}{a_n}-K\right|<\frac{\varepsilon}{2}\Rightarrow K-\frac{\varepsilon}{2}<\frac{a_{n+1}}{a_n}<K+\frac{\varepsilon}{2}$$

Note that $a_n=a_1\times\frac{a_2}{a_1}\times\frac{a_3}{a_2}\times\cdots\times\frac{a_n}{a_{n-1}}=a_1\times\prod_{k=1}^{n-1}\frac{a_{k+1}}{a_k}=a_1\times\prod_{k=1}^{N_1-1}\frac{a_{k+1}}{a_k}\times\prod_{k=N_1}^{n-1}\frac{a_{k+1}}{a_k}$.
Now define $\mathcal{Q}=a_1\times\prod_{k=1}^{N_1-1}\frac{a_{k+1}}{a_k}$, then $\sqrt[n]{a_n}=\sqrt[n]{\mathcal{Q}}\times\prod_{k=N_1}^{n-1}\left(\frac{a_{k+1}}{a_k}\right)^{\frac{1}{n}}$, thus

$$\begin{aligned}\sqrt[n]{\mathcal{Q}}\times\left(K-\frac{\varepsilon}{2}\right)^{\frac{n-N_1-1}{n}}&=\sqrt[n]{\mathcal{Q}}\times\prod_{k=N_1}^{n-1}\left(K-\frac{\varepsilon}{2}\right)^{\frac{1}{n}}<\sqrt[n]{a_n}=\sqrt[n]{\mathcal{Q}}\times\prod_{k=N_1}^{n-1}\left(\frac{a_{k+1}}{a_k}\right)^{\frac{1}{n}}\\&<\sqrt[n]{\mathcal{Q}}\times\prod_{k=N_1}^{n-1}\left(K+\frac{\varepsilon}{2}\right)^{\frac{1}{n}}=\sqrt[n]{\mathcal{Q}}\times\left(K+\frac{\varepsilon}{2}\right)^{\frac{n-N_1-1}{n}}.\end{aligned}$$

By the fact that
$A(n)=\sqrt[n]{\mathcal{Q}}\times\left(K-\frac{\varepsilon}{2}\right)^{\frac{n-N_1-1}{n}}\to K-\frac{\varepsilon}{2}$, $B(n)=\sqrt[n]{\mathcal{Q}}\times\left(K+\frac{\varepsilon}{2}\right)^{\frac{n-N_1-1}{n}}\to K+\frac{\varepsilon}{2}$ as $n\to\infty$, there is $N>N_1$ such that $n\geq N\Rightarrow A(n)>K-\varepsilon$ and $B(n)<K+\varepsilon$, thus

$$K-\varepsilon<A(n)<\sqrt[n]{a_n}<B(n)<K+\varepsilon.$$

Therefore, $\lim_{n\to\infty}\sqrt[n]{a_n}=K$. □

(a)

Remark.
(a) Highly non-trivial to compute μ.
(b) On the hexagonal lattice, it is shown that $\mu=\sqrt{2+\sqrt{2}}$ (2010, Copin at el.)
(c) By computer simulation,
Conjecture : $Z_\beta=(\beta-\beta_c)^{-\gamma+o(1)}$ for $\beta\to\beta_c^+$, where γ only depends on that dimension of the lattice. In 2D, $\gamma=\frac{43}{32}$ (conjectured)
(d) If we have $\lambda_N\sim\mu^N N^\alpha$, then Z_β can be computed to satisfy $Z_\beta\sim(\beta-\beta_c)^{-1+\alpha}$, $\alpha=\frac{11}{32}$ in 2D (conjectured)

Exercise 3 How to sample SAWs with computer.

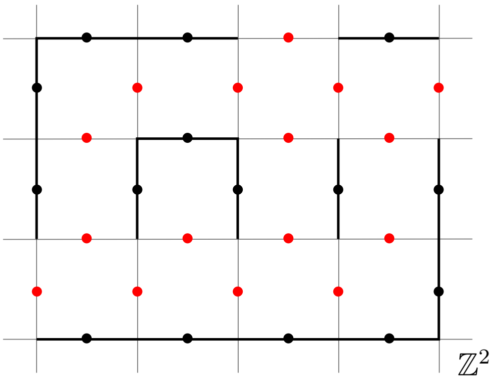
0.3) Bernoulli Percolation

We first consider $\mathbb{Z}^d$ lattices,

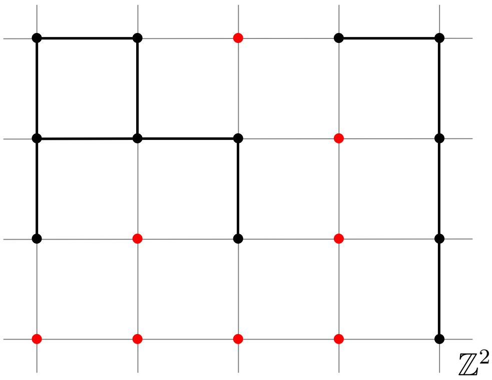
- **Bond percolation** Give $p \in [0,1]$, consider $\omega = (\omega(e))_{e\in\mathbb{Z}^d}$ such that $\{\omega(e)\}_{e\in\mathbb{Z}^d}$ is i.i.d with $\omega(e) \sim$ Brenoulli(p).
  If $\omega(e) = 1$, we say that e is open. If $\omega(e) = 0$, we say that e is closed.
  Remark We got a model of random subgraph. By above of notation, we may also write ω for the random subgraph (consisting of that open edges).

- **Site percolation** Same thing with i.i.d Bernoulli(p),
  open means that the node can pass, closed means that the node can not pass.

- Notation :
  $\mathbb{P}_p =$ The Bernoulli percolation of parameter p.
  ω_p a sample of $\mathbb{P}_p$



A bond percolation



A site percolation

Remark The terminology "Bernoulli percolation" stands for **i.i.d**, on the other hand, without independence, we simply say that we have a "percolation model", e.g. random cluster model.
For the following classes we use "percolation" to refer to Bernoulli percolation.

Exercise 1 Show that a bond percolation is equivalent to a site percolation. How about the other way? Construct an example.

Question : What are the interesting behavior when p varies? e.g. # component, size of component etc.
$p = 0$ is an empty graph, $p = 1$ is a full graph.

- **Connected component (cluster)**
  Let a, b be two vertex of $\mathbb{Z}^d$, we say that $a \sim b$ if exists an path in ω_p from a to b. It is clearly that $\sim$ is an equivalence relation.
  A **connected component**(cluster) is an element in equivalence classes of $\sim$

- **Infinite cluster**
  A infinite cluster is a cluster of ω_p that has infinite edges and infinite vertex.
  Let $[O \leftrightarrow \infty]$ be the event in $\mathbb{P}_p$ that O belongs to a infinite cluster.
  $\theta(p) = \mathbb{P}_p[O \leftrightarrow \infty]$.

1. Basic Properties of the Bernoulli Percolation

Consider $G = \mathbb{Z}^d$ or some "nice" graph.

1.1) Coupling (耦合)

- Given $p \le p'$, how to compute $X \sim \text{Bernoulli}(p),\ X' \sim \text{Bernoulli}(p')$?
  Consider $\mathcal{U} \sim \text{Uniform}([0,1])$, define $Y = \mathbf{1}_{\mathcal{U} \le p}$, $Y' = \mathbf{1}_{\mathcal{U} \le p'}$, we get

$$X \overset{(id)}{=} Y, \quad X' \overset{(id)}{=} Y', \quad Y \le Y'. \quad a.s.(almost\ sure)$$

  This is called a coupling.

Remark. In coupling, usually we do not want independence, so that we can compute values between random variables.

Exercise 1 Construct a coupling between $\omega \sim \mathbb{P}_p,\ \omega' \sim \mathbb{P}_{p'}$ with $p \le p'$, so that values between edges can be computed.
Wanted : $p \le p' \Rightarrow \omega_p \le \omega'_p (\Leftrightarrow \omega_p(e) \le \omega_{p'}(e),\ \forall e \in E)$

$\boxed{\texttt{SoL:}}$ Let $\omega = (\omega(e))_{e \in G}$ such that $\{\omega(e)\}_{e \in G}$ is i.i.d. and $\omega(e) \sim \text{Uniform}([0,1])$.
Define $\omega_p \sim \mathbb{P}_p,\ \omega_{p'} \sim \mathbb{P}_{p'}$ as $\forall e \in E,\ \omega_p(e) = \mathbf{1}_{\omega(e) \le p},\ \omega_{p'}(e) = \mathbf{1}_{\omega(e) \le p'}$, thus, $p \le p' \Rightarrow \forall e \in E,\ \omega_p(e) \le \omega_{p'}(e)$.

Exercise 2 Given $O \in V(G)$, define $\theta : \begin{array}{ccc} [0,1] & \to & \mathbb{R} \\ p & \mapsto & \mathbb{P}_p([O \leftrightarrow \infty]) \end{array}$ (percolation function).
Show that θ is increasing. In more general case, at must how many different θ function can be obtain?

$\boxed{\texttt{SoL:}}$ If $p \le p'$, let $\omega_1 \sim \mathbb{P}_p,\ \omega_2 \sim \mathbb{P}_{p'}$, we use the definition of **Exercise 1**, we have
$\omega_1 \overset{(id)}{=} \omega_p,\ \omega_2 \overset{(id)}{=} \omega_{p'}$, thus $\mathbb{P}_p([O \leftrightarrow \infty]) = \mathbb{P}_p([O \leftrightarrow \infty] \text{ in } \omega_1) = \mathbb{P}_p([O \leftrightarrow \infty] \text{ in } \omega_p)$,

similarly, $\mathbb{P}_{p'}([O \leftrightarrow \infty]) = \mathbb{P}_{p'}([O \leftrightarrow \infty] \text{ in } \omega_1) = \mathbb{P}_{p'}([O \leftrightarrow \infty] \text{ in } \omega_{p'})$. We note that ω_p is always a subgraph of $\omega_{p'}$ (by **Exercise 1**), thus $\{[O \leftrightarrow \infty] \text{ in } \omega_p\} \subseteq \{[O \leftrightarrow \infty] \text{ in } \omega_{p'}\}$, we have $\mathbb{P}_p([O \leftrightarrow \infty]) = \mathbb{P}_p([O \leftrightarrow \infty] \text{ in } \omega_p) \leq \mathbb{P}_{p'}([O \leftrightarrow \infty] \text{ in } \omega_{p'}) = \mathbb{P}_{p'}([O \leftrightarrow \infty])$.

And, because of $\omega_1 \overset{(id)}{=} \omega_p$, where ω_1 is a arbitrary random variable with $\omega_1 \sim \mathbb{P}_p$ thus there are only one choice of θ. i.e., θ is well-defined. ■

- Define $p_c = \sup\{p \in [0,1] \,|\, \theta(p) = 0\}$

Exercise 3 Check the following properties :

(a) The function $p \mapsto \theta(p)$ is right-continuous on $[0, 1]$.

(b) The function $p \mapsto \theta(p)$ is left-continuous on $(p_c, 1)$.

(c) Show that $p \mapsto \theta(p)$ is strictly increasing in $(p_c, 1]$.