Goal : Show that in site percolation, $p_c(\triangle) = \frac{1}{2}$.

$$\boxed{\text{Partial order on } \Omega : \omega \leq \omega' \Leftrightarrow \omega(e) \leq \omega'(e),\ \forall e \in \Lambda}$$

We first introduce some definition :

Definition

$*$) A function $f : \Omega \to \mathbb{R}$ is said to be **increasing**(**decreasing**) if

$$\omega \leq (\geq)\omega' \Rightarrow f(\omega) \leq f(\omega')$$

$*$) An event A is said to be **increasing**(**decreasing**) if $\mathbb{1}_A$ is increasing(decreasing).

Harris-FKG inequality

Proposition (Harris inequality)

(a) If A and B are increasing event, then

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\mathbb{P}(B).$$

(b) If f and g are increasing functions, then

$$\mathbb{E}[fg] \geq \mathbb{E}[f]\mathbb{E}[g].$$

Exercise 1 (The square-root trick) Given n increasing events $A_1, \cdots, A_n$, show that

$$\max_{1\leq i\leq n}\{\mathbb{P}(A_i)\} \geq 1 - \Big[1 - \mathbb{P}(A_1 \cup \cdots \cup A_n)\Big]^{\frac{1}{n}}$$

$\boxed{\texttt{SoL:}}$ Note that $A_1^{\complement}, \cdots, A_n^{\complement}$ are decreasing events, and thus also has FKG inequality.

$$\begin{aligned}
1 - \Big[1 - \mathbb{P}(A_1 \cup \cdots \cup A_n)\Big]^{\frac{1}{n}} &= 1 - \Big[\mathbb{P}(A_1^{\complement} \cap \cdots \cap A_n^{\complement})\Big]^{\frac{1}{n}} \\
\text{(FKG inequality)} &\leq 1 - \Big(\prod_{k=1}^{n} \mathbb{P}(A_k^{\complement})\Big)^{\frac{1}{n}} \\
&\leq 1 - \Big(\prod_{k=1}^{n} \min_{1\leq i\leq n}\{\mathbb{P}(A_i^{\complement})\}\Big)^{\frac{1}{n}} \\
&= 1 - \min_{1\leq i\leq n}\{\mathbb{P}(A_i^{\complement})\} \\
&= \max_{1\leq i\leq n}\{1 - \mathbb{P}(A_i^{\complement})\} \\
&= \max_{1\leq i\leq n}\{\mathbb{P}(A_i)\}
\end{aligned}$$

Application Show that $\theta(\frac{1}{2}) = 0$ on $\mathbb{Z}^2$, which implies that $p_c(\mathbb{Z}^2) \geq \frac{1}{2}$.

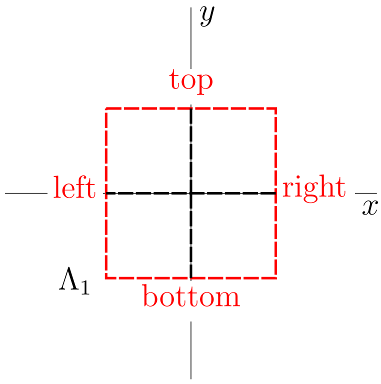
Proof. Assume that $\theta(\frac{1}{2}) > 0$, then we can know that $\mathbb{P}_{\frac{1}{2}}(\exists \text{ an infinite cluster }) = 1$. Define

$$B_n := [\partial\Lambda_n \leftrightarrow \infty] = \bigcup_{x\in\partial\Lambda_n} [x \leftrightarrow \infty],\ n \in \mathbb{N},$$

because of

$$\begin{aligned}[\exists \text{ an infinite cluster}] &= \bigcup_{x\in\mathbb{Z}^2} [x \leftrightarrow \infty] \\ &= \bigcup_{n\in\mathbb{N}} \bigcup_{x\in\partial\Lambda_n} [x \leftrightarrow \infty] = \lim_{n\to\infty} [\partial\Lambda_n \leftrightarrow \infty],\end{aligned}$$

Thus $\lim\limits_{n\to\infty} \mathbb{P}_{\frac{1}{2}}[\partial\Lambda_n \leftrightarrow \infty] = 1$.



Note that $B_n = [\exists \text{ an infinite SAW in } \mathbb{Z}^2\backslash\Lambda_n \text{ starting in } \partial\Lambda_n]$, define

$$\begin{aligned}B_n^{\text{top}} &= [\exists \text{ an infinite SAW in } \mathbb{Z}^2\backslash\Lambda_n \text{ starting in } \partial\Lambda_n^{\text{top}}] \\ B_n^{\text{bottom}} &= [\exists \text{ an infinite SAW in } \mathbb{Z}^2\backslash\Lambda_n \text{ starting in } \partial\Lambda_n^{\text{bottom}}] \\ B_n^{\text{left}} &= [\exists \text{ an infinite SAW in } \mathbb{Z}^2\backslash\Lambda_n \text{ starting in } \partial\Lambda_n^{\text{left}}] \\ B_n^{\text{right}} &= [\exists \text{ an infinite SAW in } \mathbb{Z}^2\backslash\Lambda_n \text{ starting in } \partial\Lambda_n^{\text{right}}].\end{aligned}$$

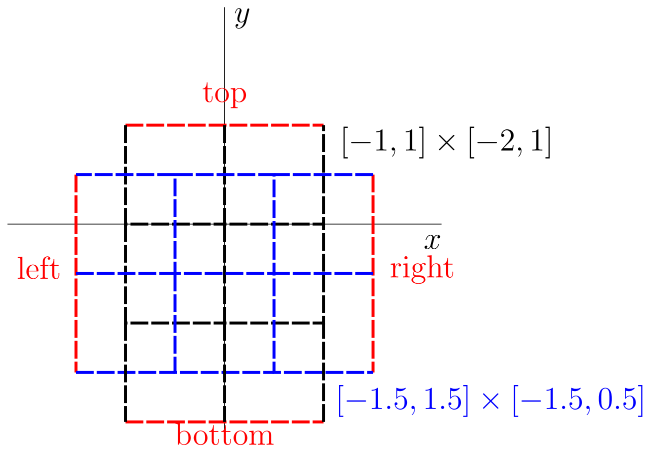
Then

$$B_n = B_n^{\text{top}} \cup B_n^{\text{bottom}} \cup B_n^{\text{left}} \cup B_n^{\text{right}}$$

It's no doubt that for all $n \in \mathbb{N}$, B_n^{top}, B_n^{bottom}, B_n^{left}, B_n^{right} are increasing events, thus by square-root trick and the fact that $\lim\limits_{n\to\infty} \mathbb{P}_{\frac{1}{2}} B_n = 1$, we got that

$$\lim_{n\to\infty} \mathbb{P}_{\frac{1}{2}}(B_n^{\text{top}}) \geq \lim_{n\to\infty} 1 - \left[1 - \mathbb{P}_{\frac{1}{2}}(B_n)\right]^{\frac{1}{4}} = 1$$

Note that the probability of B_n^{top} remains the same for any "shift" of this event.



Now, we consider

$$
\begin{aligned}
&C_n^{\text{top}} = B_n^{\text{top}}\\
&C_n^{\text{bottom}} = \{\omega \subseteq \mathbb{Z}^2 \mid \omega + (0,1) \in B_n^{\text{bottom}}\}\\
&C_n^{\text{left}} = \{\omega \subseteq \mathbb{Z}^2 \mid \omega^* + (0.5, 0.5) \in B_n^{\text{left}}\}\\
&C_n^{\text{right}} = \{\omega \subseteq \mathbb{Z}^2 \mid \omega^* + (-0.5, 0.5) \in B_n^{\text{right}}\}
\end{aligned}
$$

then

$$\lim_{n\to\infty} \mathbb{P}_{\frac{1}{2}}(C_n^{\text{top}}) = \lim_{n\to\infty} \mathbb{P}_{\frac{1}{2}}(C_n^{\text{bottom}}) = \lim_{n\to\infty} \mathbb{P}_{\frac{1}{2}}(C_n^{\text{left}}) = \lim_{n\to\infty} \mathbb{P}_{\frac{1}{2}}(C_n^{\text{right}}) = 1,$$

define $C_n = C_n^{\text{top}} \cap C_n^{\text{bottom}} \cap C_n^{\text{left}} \cap C_n^{\text{right}}$, then

$$\lim_{n\to\infty} \mathbb{P}_{\frac{1}{2}}(C_n) = 1.$$

Note that C_n does not depending on edges on $[-n,n]\times[-n-1,n]$, we take n be large enough such that $\mathbb{P}_{\frac{1}{2}}(C_n) > \frac{1}{2}$, let $\mathcal{C}([-n,n]\times[-n-1,n])$ be the event that all edge in $[-n,n]\times[-n-1,n]$ are closed, then

$$
\begin{aligned}
\mathbb{P}_{\frac{1}{2}}(N \geq 2) &\geq \mathbb{P}_{\frac{1}{2}}\big(C_n \cap \mathcal{C}([-n,n]\times[-n-1,n])\big)\\
&\geq \frac{1}{2} \times \frac{1}{2^{8n^2+8n+1}} > 0.
\end{aligned}
$$

Which contradicts that $\mathbb{P}_{\frac{1}{2}}(N \geq 2) = 0$ on $\mathbb{Z}^2$. $\square$

Exercise 2 Show that $p_c \geq \frac{1}{2}$ for the site percolation on the triangular lattice.

$\boxed{\texttt{SoL:}}$ We show that $\theta(\frac{1}{2}) = 0$ to conclude this.
Note that we can use similar way of proving that $\mathbb{P}_p(N=1) = 1,\ \forall p > p_c$ in $\mathbb{Z}^d$ to say that it is also true in triangular lattice.
We suppose that $\theta(\frac{1}{2}) > 0$, then $\frac{1}{2} > p_c(\triangle)$, thus $\mathbb{P}_{\frac{1}{2}}[\exists$ an infinite cluster$] = 1$, note that

$$
\begin{aligned}
[\exists \text{ an infinite cluster}] &= \bigcup_{x\in\triangle}[x \leftrightarrow \infty]\\
&= \bigcup_{n\in\mathbb{N}} \bigcup_{x\in\partial\Lambda_n}[x \leftrightarrow \infty] = \lim_{n\to\infty}[\partial\Lambda_n \leftrightarrow \infty],
\end{aligned}
$$

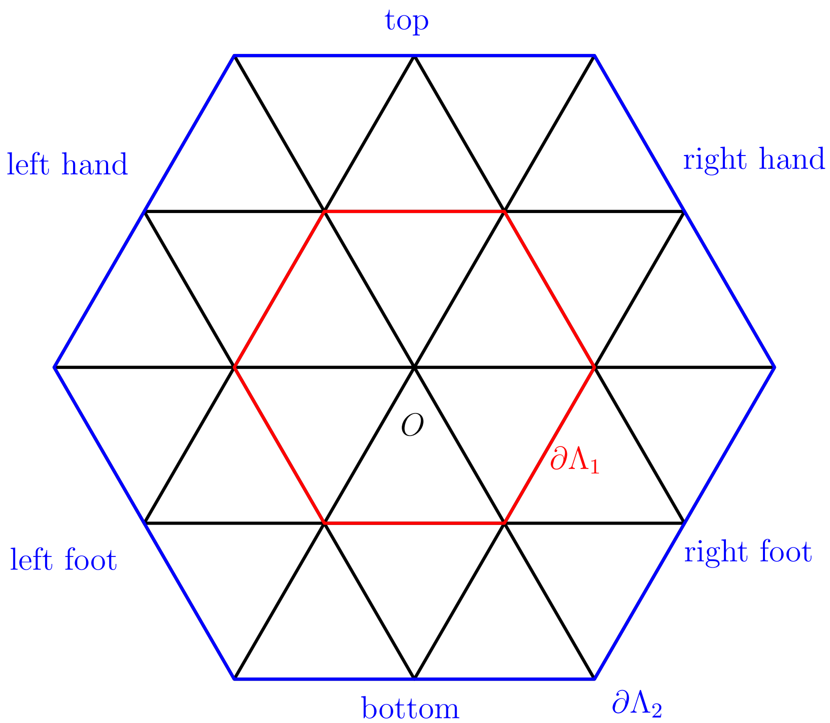
In this case, $\Lambda_n = \{x \in G \mid \exists \text{ SAW of length} \le n \text{ that connected form } 0 \text{ to } x\}$, where the length of SAWs is allowed to be 0. And, $\partial\Lambda_n = \Lambda_n \backslash \Lambda_{n-1}$, where $\Lambda_{-1} = \varnothing$.
We still define

$$B_n := [\partial\Lambda_n \leftrightarrow \infty] = [\exists \text{ an infinite SAW in } \triangle\backslash\Lambda_{n-1} \text{ starting in } \partial\Lambda_n]$$

It is needed to mention that we consider a configuration ω to be a subset of $V(\triangle)$ instead of a subset of $E(\triangle)$. Define

$$\begin{aligned}
B_n^{\mathrm{t}} &= [\exists \text{ an infinite SAW in } \mathbb{Z}^2\backslash\Lambda_{n-1} \text{ starting in } \partial\Lambda_n^{\mathrm{top}}] \\
B_n^{\mathrm{b}} &= [\exists \text{ an infinite SAW in } \mathbb{Z}^2\backslash\Lambda_{n-1} \text{ starting in } \partial\Lambda_n^{\mathrm{bottom}}] \\
B_n^{\mathrm{lh}} &= [\exists \text{ an infinite SAW in } \mathbb{Z}^2\backslash\Lambda_{n} \text{ starting in } \partial\Lambda_n^{\mathrm{left\,hand}}] \\
B_n^{\mathrm{lf}} &= [\exists \text{ an infinite SAW in } \mathbb{Z}^2\backslash\Lambda_{n-1} \text{ starting in } \partial\Lambda_n^{\mathrm{left\,foot}}] \\
B_n^{\mathrm{rh}} &= [\exists \text{ an infinite SAW in } \mathbb{Z}^2\backslash\Lambda_{n-1} \text{ starting in } \partial\Lambda_n^{\mathrm{right\,hand}}] \\
B_n^{\mathrm{rf}} &= [\exists \text{ an infinite SAW in } \mathbb{Z}^2\backslash\Lambda_{n-1} \text{ starting in } \partial\Lambda_n^{\mathrm{right\,foot}}].
\end{aligned}$$

Then, $B_n = B_n^{\mathrm{t}} \cup B_n^{\mathrm{b}} \cup B_n^{\mathrm{lh}} \cup B_n^{\mathrm{lf}} \cup B_n^{\mathrm{rh}} \cup B_n^{\mathrm{rf}}$, because of all these events are increasing



events, by square-root trick and $\lim_{n\to\infty} \mathbb{P}_{\frac{1}{2}}(B_n) = 1$, we got that

$$\lim_{n\to\infty} \mathbb{P}_{\frac{1}{2}}(B_n^{\mathrm{t}}) \ge \lim_{n\to\infty} 1 - \left[1 - \mathbb{P}_{\frac{1}{2}}(B_n)\right]^{\frac{1}{6}} = 1.$$

Given a configuration $\omega \subseteq \triangle$, we define $\omega^{\complement} = V(\triangle)\backslash\omega$, which is also a (site) configuration of $\triangle$. Thus for any given event A, we have $\mathbb{P}_p\{\omega \mid \omega \in A\} = \mathbb{P}_{1-p}\{\omega \mid \omega^{\complement} \in A\}$.
Because of $p = 1-p$ when $p = \frac{1}{2}$, for convenience, we say $[\omega^{\complement} \in A] := \{\omega \mid \omega^{\complement} \in A\}$ for A be a event, then

$$\lim_{n\to\infty} \mathbb{P}_{\frac{1}{2}}([\omega \in B_n^{\mathrm{t}}] \cap [\omega^{\complement} \in B_n^{\mathrm{b}}] \cap [\omega^{\complement} \in B_n^{\mathrm{lh}}] \cap [\omega \in B_n^{\mathrm{lf}}] \cap [\omega^{\complement} \in B_n^{\mathrm{rh}}] \cap [\omega \in B_n^{\mathrm{rf}}]) = 1.$$

We define $C_n := [\omega \in B_n^{\mathrm{t}}] \cap \cdots \cap [\omega \in B_n^{\mathrm{rf}}]$ in the previous step, we can take large n such that $\mathbb{P}_{\frac{1}{2}}(C_n) \ge \frac{1}{2}$, we use $\hexagon_n$ to denote the event that all sites in Λ_n are closed, note that

C_n dose not depend on sites in Λ_{n-1}, thus

$$\mathbb{P}(N \geq 3) \geq \mathbb{P}(C_n \cap \varhexagonblack_{n-1}) = \frac{1}{2} \times \frac{1}{2^{1+3n(n-1)}} > 0,$$

which is a contradiction because of $\mathbb{P}_{\frac{1}{2}}(N = 1) = 1$. $\blacksquare$