Group 7 Note of Planar Statistical Physics and Bernoulli Percolation
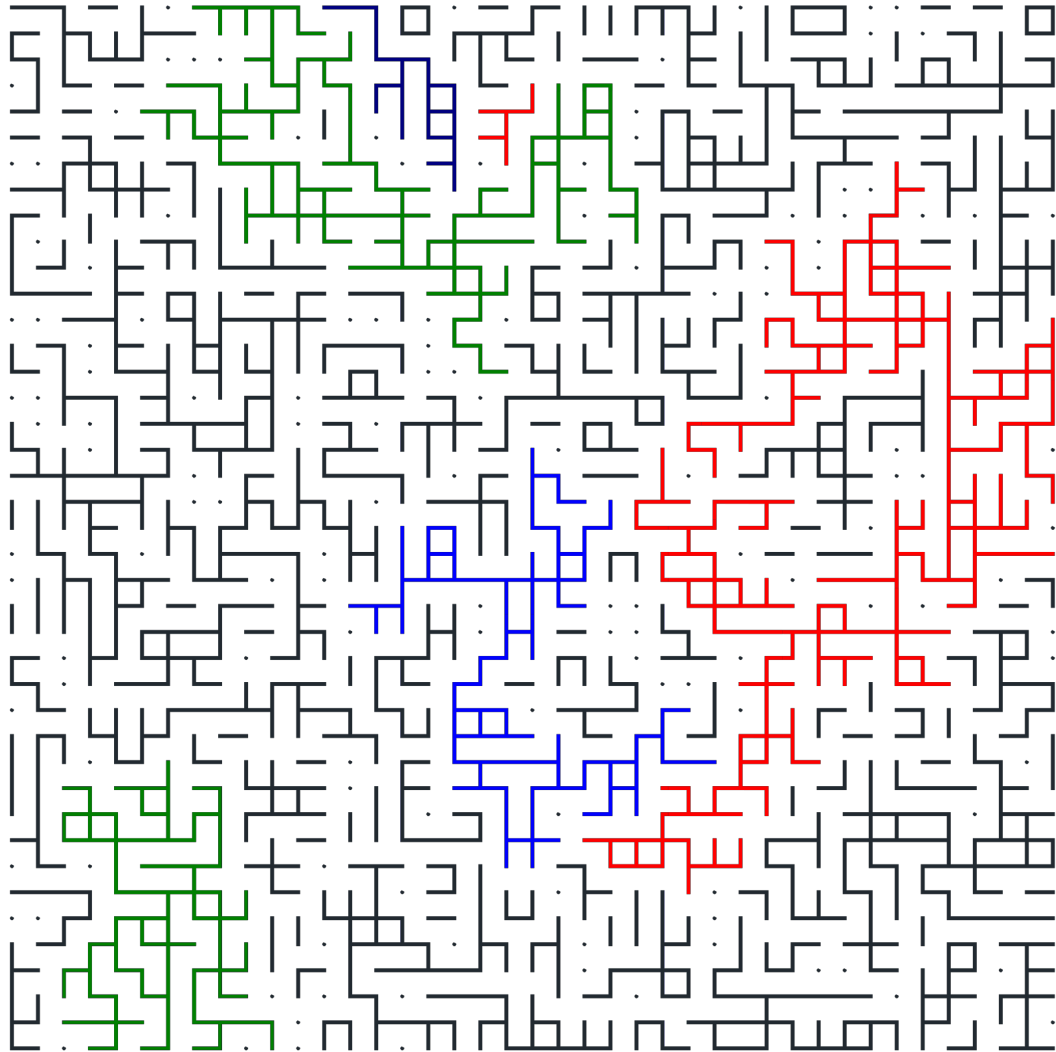
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Chapter 0

(Planer) Statistical Mechanics

0.1 Motivation

- We want to describe the (macroscopic) behavior of a system composed of huge number of (microscopic) particles. Theoretically, if the behavior of each particle can be described, then one should be able to say something about the system. However, the number of particles are astronomical (1 mole $\approx 10^{23}$) and it is costly (even impossible) to solve the system.

- Boltzmann (end 19th century) gives a probabilistic formalism

  (a) Ω = sample space (discrete, countable, possibly infinite)

  (b) $H : \Omega \to \mathbb{R}$ energy function (Hamiltonian)

  (c) $\beta = \frac{1}{T}$ inverse temperature.

  $\Rightarrow$ **Boltzmann distribution**

  $$\mathbb{P}_\beta(\omega) = \frac{1}{z_\beta} \exp(-\beta H(\omega))$$

  Where $z_\beta = \sum_{\omega \in \Omega} \exp(-\beta H(\omega))$ is called **partition function**

- Interpretations

  *) At a fixed temperature, an exponential weight is associated to each configuration

  *) At high temperature ($\beta \to 0^+$) the weights are almost a constant function of configurations, so each state equiprobable.

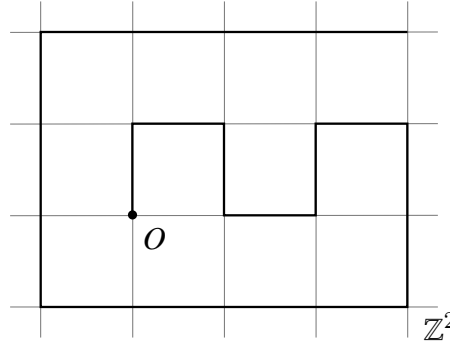 *) At low temperature, ($\beta \to +\infty$), the weight concentrate on minimizers of the energy function H.

  $\Rightarrow$ this corresponds to the intuitions we have in physics (thermodynamics?)

Remark   (a) $z_\beta < \infty$ if $\boldsymbol{\Omega}$ is finite, so $\mathbb{P}_\beta$ is well defined.

   (b) If $\boldsymbol{\Omega}$ is infinite, then z_β is not always finite. The finiteness of z_β may depend on β.

0.2   First Model (Polymer Model)

- Regular lattices, e.g., $\mathbb{Z}^d$

- Self-avoiding walk (SAW) (a polymer)

Example A self-avoiding walk

- Consider $\Omega = \{$ all SAWs $\}$, define $H(\omega) = |\omega|$ where $\omega \in \Omega$
  Let $n \in \mathbb{N}$, define $\lambda_n = \#$ SAWs of length n.

<u>Observe</u> :   Given $n, m \geq 1$
  Any SAW of length $n + m$ can be decomposed into 2 SAWs of length n and m.
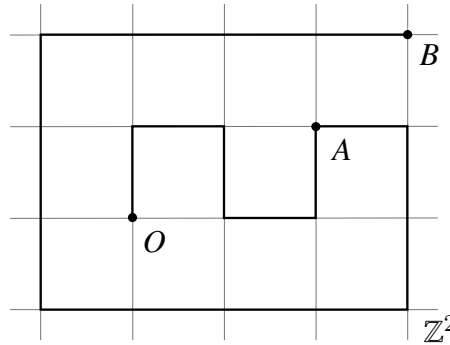 $\Rightarrow \lambda_{n+m} \leq \lambda_n \cdot \lambda_m$.

Figure $\lambda_{19} \leq \lambda_5 \cdot \lambda_{14}$

Exercise 1 $\mu \equiv \lim\limits_{n \to \infty} (\lambda_n)^{\frac{1}{n}}$ exists and $\lambda_N \geq \mu^N$ for all $N \geq 1$. μ is called the **connective constant**
  (of the graph)

$\boxed{\texttt{soL:}}$ Note that $\forall n \in \mathbb{N}$, we have $(\lambda_n)^{\frac{1}{n}} \geq 0$, thus 0 is a lower bound of $\{(\lambda_n)^{\frac{1}{n}}\}_{n=0}^{\infty}$, therefore

$$\inf_{n \in \mathbb{N}_0} (\lambda_n)^{\frac{1}{n}} = K \in \mathbb{R}$$

Now, given $\varepsilon > 0$, $1°$ $\exists N_1 \in \mathbb{N}$ s.t.

$$K + \frac{\varepsilon}{2} > (\lambda_{N_1})^{\frac{1}{N_1}}$$

$2°$ By division algorithm, $\forall n \geq N_1$, $\exists m, \ell \in \mathbb{Z}$ with $0 \leq \ell \leq N_1$ such that $n = mN_1 + \ell$,
  thus

$$\lambda_n = \lambda_{mN_1+\ell} \leq (\lambda_{N_1})^m \cdot \lambda_\ell$$

therefore,

$$(\lambda_n)^{\frac{1}{n}} \le (\lambda_{N_1})^{\frac{m}{n}} \cdot (\lambda_\ell)^{\frac{1}{n}} = (\lambda_{N_1})^{\frac{1}{N_1 + \frac{\ell}{m}}} \cdot (\lambda_\ell)^{\frac{1}{n}} \le (\lambda_{N_1})^{\frac{1}{N_1}} \cdot (\lambda_\ell)^{\frac{1}{n}}$$

$$\le (\lambda_{N_1})^{\frac{1}{N_1}} \cdot (\lambda_1)^{\frac{\ell}{n}} \le (\lambda_{N_1})^{\frac{1}{N_1}} \cdot (\lambda_1)^{\frac{N_1}{n}}$$

Because of $\lim\limits_{n\to\infty} (\lambda_{N_1})^{\frac{1}{N_1}} \cdot (\lambda_1)^{\frac{N_1}{n}} = (\lambda_{N_1})^{\frac{1}{N_1}}$, $\exists N \in \mathbb{N}$ such that $n \ge N \Rightarrow$

$$(\lambda_{N_1})^{\frac{1}{N_1}} + \frac{\varepsilon}{2} > (\lambda_{N_1})^{\frac{1}{N_1}} \cdot (\lambda_1)^{\frac{N_1}{n}} \ge (\lambda_n)^{\frac{1}{n}}$$
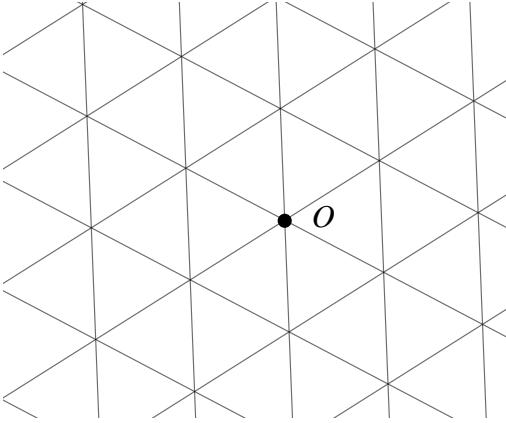
Therefore

$$K + \varepsilon = K + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} > \lambda_{N_1}^{\frac{1}{N_1}} + \frac{\varepsilon}{2} > (\lambda_n)^{\frac{1}{n}} \ge K.$$
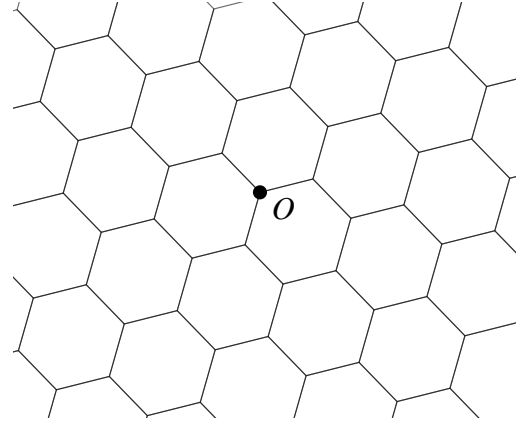
Thus, $\mu = \lim\limits_{n\to\infty} (\lambda_n)^{\frac{1}{n}} = K$ exists.

Next, for $N > 1$, we have $\forall n \in \mathbb{N}, \lambda_{nN} \le (\lambda_N)^n$, thus $(\lambda_{nN})^{\frac{1}{nN}} \le (\lambda_N)^{\frac{1}{N}}$. Note that $\{(\lambda_{nN})^{\frac{1}{nN}}\}_{n=1}^{\infty}$ is a subsequence of $\{(\lambda_n)^{\frac{1}{n}}\}_{n=1}^{\infty}$, therefore $\mu = \lim\limits_{n\to\infty} (\lambda_{nN})^{\frac{1}{nN}} \le (\lambda_N)^{\frac{1}{N}}$ ∎

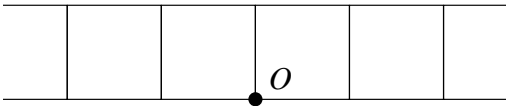**Exercise 2** (a) In $\mathbb{Z}^2$, $\mu \in (2,3)$

(b) In $\mathbb{Z}^3$, $\mu > 3$

(c) In the triangular mesh, $\mu > 3$;

(d) In the hexagonal mesh, $\mu < 2$

(e) In $\mathbb{Z} \times \{0,1\}$ (i.e. a ladder), $\mu = \frac{1+\sqrt{5}}{2}$.

The triangular mesh

The hexagonal mesh

$\mathbb{Z} \times \{0,1\}$

$\boxed{\text{s}_\text{o}\text{L:}}$ We first show a fact :

Fact(ratio test and root test) : Let $\{a_n\}_{n=1}^{\infty}$ be a sequence that take value in $(0, \infty)$. Then

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = K \in \mathbb{R} \implies \lim_{n \to \infty} \sqrt[n]{a_n} = K.$$

Proof. If $\displaystyle\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = K \in \mathbb{R}$, then given $\varepsilon > 0$, there exists $N_1 > 0$ such that

$$n \geq N_1 \implies \left| \frac{a_{n+1}}{a_n} - K \right| < \frac{\varepsilon}{2} \implies K - \frac{\varepsilon}{2} < \frac{a_{n+1}}{a_n} < K + \frac{\varepsilon}{2}$$

Note that $\displaystyle a_n = a_1 \times \frac{a_2}{a_1} \times \frac{a_3}{a_2} \times \cdots \times \frac{a_n}{a_{n-1}} = a_1 \times \prod_{k=1}^{n-1} \frac{a_{k+1}}{a_k} = a_1 \times \prod_{k=1}^{N_1-1} \frac{a_{k+1}}{a_k} \times \prod_{k=N_1}^{n-1} \frac{a_{k+1}}{a_k}$.

Now define $\displaystyle Q = a_1 \times \prod_{k=1}^{N_1-1} \frac{a_{k+1}}{a_k}$, then $\displaystyle \sqrt[n]{a_n} = \sqrt[n]{Q} \times \prod_{k=N_1}^{n-1} \left(\frac{a_{k+1}}{a_k} \right)^{\frac{1}{n}}$, thus

$$\sqrt[n]{Q} \times \left(K - \frac{\varepsilon}{2} \right)^{\frac{n-N_1-1}{n}} = \sqrt[n]{Q} \times \prod_{k=N_1}^{n-1} \left(K - \frac{\varepsilon}{2} \right)^{\frac{1}{n}} < \sqrt[n]{a_n} = \sqrt[n]{Q} \times \prod_{k=N_1}^{n-1} \left(\frac{a_{k+1}}{a_k} \right)^{\frac{1}{n}}$$

$$< \sqrt[n]{Q} \times \prod_{k=N_1}^{n-1} \left(K + \frac{\varepsilon}{2} \right)^{\frac{1}{n}} = \sqrt[n]{Q} \times \left(K + \frac{\varepsilon}{2} \right)^{\frac{n-N_1-1}{n}}.$$
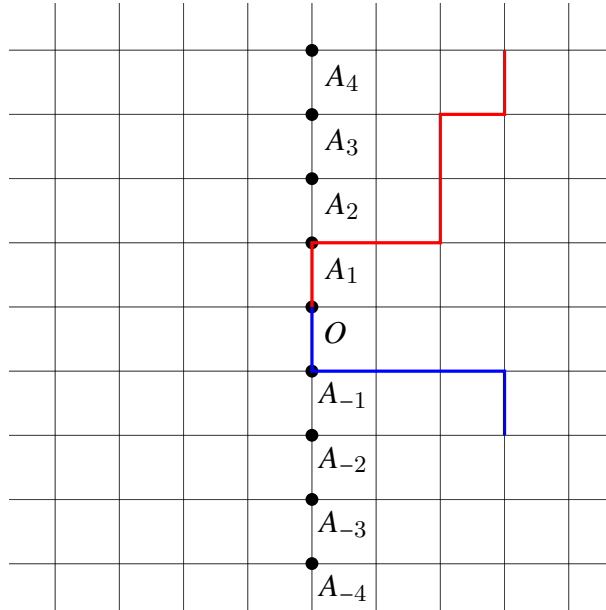
By the fact that

$$A(n) = \sqrt[n]{Q} \times \left(K - \frac{\varepsilon}{2} \right)^{\frac{n-N_1-1}{n}} \to K - \frac{\varepsilon}{2}, \quad B(n) = \sqrt[n]{Q} \times \left(K + \frac{\varepsilon}{2} \right)^{\frac{n-N_1-1}{n}} \to K + \frac{\varepsilon}{2} \text{ as}$$

$n \to \infty$, there is $N > N_1$ such that $n \geq N \implies A(n) > K - \varepsilon$ and $B(n) < K + \varepsilon$, thus

$$K - \varepsilon < A(n) < \sqrt[n]{a_n} < B(n) < K + \varepsilon.$$

Therefore, $\displaystyle\lim_{n \to \infty} \sqrt[n]{a_n} = K$. $\qquad\square$

(a) We first show that $\mu(\mathbb{Z}^2) > 2$:

If we only consider the 3 direction $\uparrow, \rightarrow, \downarrow$ to choose for next step, ex : red line and blue line, we define $S(n) = \#$ SAWs with length n that under this restriction, $n \geq 1$ and $S(0) = 1$. Now considering the samples SAWs are of length n, then

$$S(n) = \# \text{ SAWs passing } O \text{ with } \rightarrow$$

$$+ \sum_{k=1}^{n} \# \text{ SAWs passing } A_i \text{ with } \rightarrow + \sum_{k=1}^{n} \# \text{ SAWs passing } A_{-i} \text{ with } \rightarrow$$

$$+ \# \text{ SAWs lying in } y\text{- axis}$$

$$= S(n-1) + 2S(n-2) + \cdots + 2S(1) + 2S(0) + 2$$

$$= S(n-1) + 2 + 2 \sum_{k=0}^{n-2} S(k), \quad n \geq 2$$

Similarly, $S(n+1) = S(n) + 2 + 2 \sum\limits_{k=0}^{n-1} S(k)$, by subtraction of the two equation, we get a recursive formula :

$$S(n+1) = 2S(n) + S(n-1), \quad n \geq 1, \quad \cdots \cdots (*)$$

We get $\{S(n)\}_{n=0}^{\infty} : 1, 3, 7, 17, 31, \cdots$, we use matrix representation of $(*)$:

$$\begin{bmatrix} S(n+1) \\ S(n) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} S(n) \\ S(n-1) \end{bmatrix}.$$

And note that by calculating eigenvalue and eigenvector, we have

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = P \begin{bmatrix} 1 + \sqrt{2} & 0 \\ 0 & 1 - \sqrt{2} \end{bmatrix} P^{-1}, \quad P = \begin{bmatrix} 1 & -1 \\ -1 + \sqrt{2} & 1 + \sqrt{2} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n = P \begin{bmatrix} (1 + \sqrt{2})^n & 0 \\ 0 & (1 - \sqrt{2})^n \end{bmatrix} P^{-1}$$

$$\Rightarrow \begin{bmatrix} S(n+1) \\ S(n) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} S(1) \\ S(0) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2} P \begin{bmatrix} (1 + \sqrt{2})^n & 0 \\ 0 & (1 - \sqrt{2})^n \end{bmatrix} \begin{bmatrix} 4 + 3\sqrt{2} \\ 4 - 3\sqrt{2} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 + \sqrt{2} & 1 + \sqrt{2} \end{bmatrix} \begin{bmatrix} (1 + \sqrt{2})^n (4 + 3\sqrt{2}) \\ (1 - \sqrt{2})^n (4 - 3\sqrt{2}) \end{bmatrix}$$

$$\Rightarrow S(n+1) = \frac{1}{2} \left((1 + \sqrt{2})^n (4 + 3\sqrt{2}) - (1 - \sqrt{2})^n (4 - 3\sqrt{2}) \right)$$

Thus, $\mu(\mathbb{Z}^2) = \lim\limits_{n \to \infty} \sqrt[n+1]{\lambda_{n+1}} \geq \lim\limits_{n \to \infty} \sqrt[n+1]{S(n+1)} = 1 + \sqrt{2} > 2.$

Next, we show that $\mu < 3$:

First, we calculate the number of SAWs of length 4 with the first move is " $\rightarrow$ "
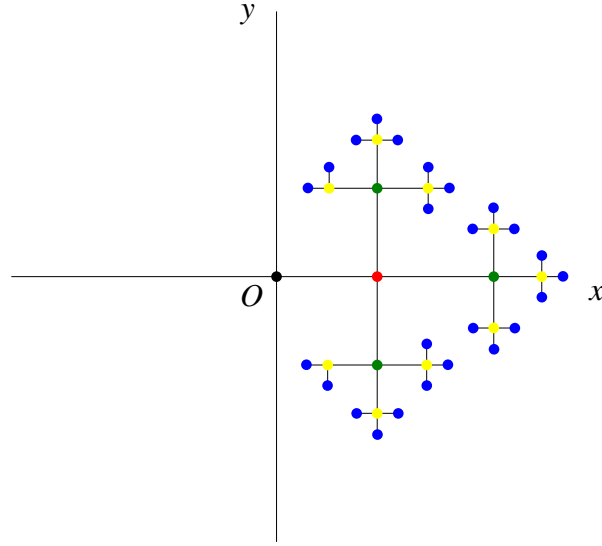
We got that number is 25 (blue points),this figure concentrate about the direction of next move of SAWs instead of the distance between two connected points (it is always 1). we get $\lambda_4 = 4 \times 25 = 100$ Now we replace blue point by red point, and then we got $\lambda_7 \le \lambda_4 \times 25 = 2500$, continue this process, we got $\lambda_{1+3n} \le 4 \times 25^n$. Thus

$$\mu(\mathbb{Z}^2) = \lim_{n\to\infty} (\lambda_{3n+1})^{\frac{1}{3n+1}} \le \lim_{n\to\infty} 4^{\frac{1}{3n+1}} \times 25^{\frac{n}{3n+1}} = \sqrt[3]{25} < 3.$$

(b) Because of $\mathbb{Z}^3 = \mathbb{Z}^2 \times \mathbb{Z}$, thus we define β_n be the number of SAWs of length n in $\mathbb{Z}^2$, by (a), we have $\lim_{n\to\infty} \sqrt[n]{\beta_n} = \mu(\mathbb{Z}^2) \in (2, 3)$. Note that $\lambda_{n+1} \ge \beta_n \times 2^{n+1}$, therefore

$$\mu(\mathbb{Z}^3) = \lim_{n\to\infty} \sqrt[n+1]{\lambda_{n+1}} \ge \lim_{n\to\infty} \sqrt[n+1]{\beta^n} \times 2^{\frac{n+1}{n+1}} \ge \lim_{n\to\infty} \left(\frac{\beta_{n+1}}{4}\right)^{\frac{1}{n+1}} \times 2 > 2 \times 2 > 3$$

(c) If we look more seriously, we can find that there's a hidden $\mathbb{Z}^2$ in the triangular mesh :
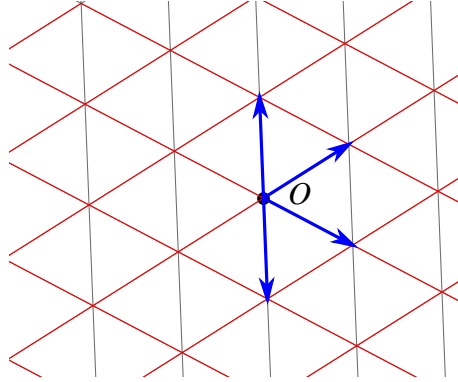
Figure 1 The triangular mesh

But, we are not going to use this fact. Instead, we'll prove $\mu(\triangle) > 3$ by using the similar way to prove $\mu(\mathbb{Z}^2) > 2$. Define $S(n)$ be # of SAWs with length n that every move only choose the four blue directions (**Figure 1**), if we define $S(0) = 1$, it is easy to see that

$$S(n) = 2S(n-1) + 4S(n-2) + \cdots + 4S(0) + 2 = 2 + 2S(n-1) + 4\sum_{k=0}^{n-2} S(k)$$

$$S(n+1) = 2S(n) + 4S(n-1) + \cdots + 4S(0) + 2 = 2 + 2S(n) + 4\sum_{k=0}^{n-1} S(k) \quad n \ge 2.$$

Thus, $S(n+1) - S(n) = 2S(n) + 2S(n-1)$

$\Rightarrow S(n+1) = 3S(n) + 2S(n-1), \quad n \geq 2, \; S(0) = 1, \; S(1) = 4, \; S(2) = 14$, thus

$$\begin{bmatrix} S(n+1) \\ S(n) \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} S(n) \\ S(n-1) \end{bmatrix}$$

We note that

$$\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = P \begin{bmatrix} \frac{3+\sqrt{17}}{2} & 0 \\ 0 & \frac{3-\sqrt{17}}{2} \end{bmatrix} P^{-1}, \quad \text{where } P = \begin{bmatrix} 3 + \sqrt{17} & 3 - \sqrt{17} \\ 2 & 2 \end{bmatrix}$$

And

$$P^{-1} = \frac{1}{4\sqrt{17}} \begin{bmatrix} 2 & -3 + \sqrt{17} \\ -2 & 3 + \sqrt{17} \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} S(n+1) \\ S(n) \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} S(1) \\ S(0) \end{bmatrix} = P \begin{bmatrix} \frac{3+\sqrt{17}}{2} & 0 \\ 0 & \frac{3-\sqrt{17}}{2} \end{bmatrix}^n P^{-1} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$= \frac{1}{4\sqrt{17}} P \begin{bmatrix} \left(\frac{3+\sqrt{17}}{2}\right)^n & 0 \\ 0 & \left(\frac{3-\sqrt{17}}{2}\right)^n \end{bmatrix} \begin{bmatrix} 5 + \sqrt{17} \\ -5 + \sqrt{17} \end{bmatrix}$$

$$= \frac{1}{4\sqrt{17}} \begin{bmatrix} 3 + \sqrt{17} & 3 - \sqrt{17} \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \left(\frac{3+\sqrt{17}}{2}\right)^n (5 + \sqrt{17}) \\ \left(\frac{3-\sqrt{17}}{2}\right)^n (-5 + \sqrt{17}) \end{bmatrix}$$

Thus implies that

$$S(n) = \frac{2}{4\sqrt{17}} \left(\left(\frac{3 + \sqrt{17}}{2}\right)^n (5 + \sqrt{17}) + \left(\frac{3 - \sqrt{17}}{2}\right)^n (-5 + \sqrt{17}) \right)$$

And therefore

$$\mu(\triangle) = \lim_{n \to \infty} \sqrt[n]{\lambda_n} \geq \lim_{n \to \infty} \sqrt[n]{S(n)} = \frac{3 + \sqrt{17}}{2} > 3.$$

(d) Similar way to prove $\mu(\mathbb{Z}^2) < 3$, we use a simple calculate, deduce that

$$\lambda_{1+5} = (3 \times 2^5 - 2) = \lambda_1 \times 30$$

Thus,

$$\lambda_{1+5n} \leq \lambda_1 \times 30^n$$

Therefore,

$$\mu(\Large\text{⬡}\normalsize) = \lim_{n \to \infty} \sqrt[1+5n]{\lambda_{1+5n}} \leq \lim_{n \to \infty} \sqrt[1+5n]{\lambda_1 \times 30^n} = \sqrt[5]{30} < 2$$

(e) *Claim* " $\mu(\mathbb{Z} \times \{0, 1\}) \geq \frac{1+\sqrt{5}}{2}$ " :

Let $S(n)$ be # of SAWs with length n that only move $\uparrow$, $\downarrow$, $\to$ in every step, and it is easy to see that $S(n) = S(n-1) + S(n_2), \quad n \geq 3$, therefore

$$\mu(\mathbb{Z} \times \{0, 1\}) = \lim_{n \to \infty} \sqrt[n]{\lambda_n} \geq \lim_{n \to \infty} \sqrt[n]{S(n)} = \frac{1 + \sqrt{5}}{2}$$

Claim " $\mu(\mathbb{Z} \times \{0,1\}) \leq \frac{1+\sqrt{5}}{2}$ " :

Define $T(n)$ be # of SAWs in $\mathbb{N}_0 \times \{0,1\}$ of length n, $n \in \mathbb{N}$, and define $T(0) = 1$. You well find that recursive formula is a great way to slove these problem.

$$T(n) = T(n-2) + T(n-3) + \cdots + T(0) + \gamma(n),$$

where $\gamma(n) = \begin{cases} \frac{n+3}{2}, & \text{if } n \text{ is odd.} \\ \frac{n+2}{2}, & \text{if } n \text{ is even.} \end{cases}$ We find that
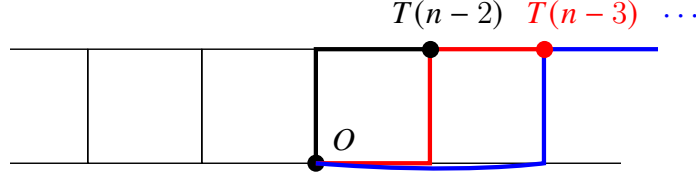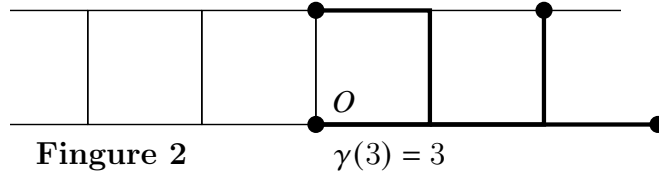

$T(n-2)$ $T(n-3)$ $\cdots$

Finqure 1

Finqure 2 $\gamma(3) = 3$

$$T(n+1) - T(n) = T(n-1) + \gamma(n+1) - \gamma(n) = T(n-1) + \mathbb{1}_{even}(n),$$

where $\mathbb{1}_{even}(n) = \begin{cases} 0, & \text{if } n \text{ is odd.} \\ 1, & \text{if } n \text{ is even.} \end{cases}$

Moreover, $\lambda_n \leq 2T(n) + 2(T(n-4) + T(n-6) + \cdots)$, $n \in \mathbb{N}$, thus

$$\begin{aligned} \lambda_n \leq \lambda_n + \lambda_{n+1} &\leq 2(T(n+1) + T(n)) + 2(T(n-3) + \cdots + T(0)) \\ &= 2(T(n+1) + T(n) + T(n-1) - \gamma(n-1)) \\ &= 2(2T(n+1) + \mathbb{1}_{even}(n) - \gamma(n)) \leq 4T(n+1), \ \forall n \in \mathbb{N} \end{aligned}$$

Because of $T(0) = 1$, $T(1) = 2$, and

$$T(n+1) \geq T(n) + T(n_1), \ T(n+1) + 1 \leq (T(n) + 1) + (T(n-1) + 1)$$

Define
$\{a_n\}_{n=0}^{\infty}$ as $a_0 = 1$, $a_1 = 2$, $a_{n+1} = a_n + a_{n-1}$, $n \geq 1$
$\{b_n\}_{n=0}^{\infty}$ as $b_0 = 2$, $b_1 = 3$, $b_{n+1} = b_n + b_{n-1}$, $n \geq 1$, then

$$a_n \leq T(n) < T(n) + 1 \leq b_n, \quad \forall n \in \mathbb{N}.$$

We note that both a_n and b_n are Fibonacci sequence, thus

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{b_n} = \frac{1+\sqrt{5}}{2} \Rightarrow \lim_{n\to\infty} \sqrt[n]{T(n)} = \frac{1+\sqrt{5}}{2}$$

Therefore,

$$\mu(\mathbb{Z} \times \{0,1\}) = \lim_{n\to\infty} \sqrt[n]{\lambda_n} \leq \lim_{n\to\infty} \sqrt[n]{4T(n+1)} = \frac{1+\sqrt{5}}{2} \quad \blacksquare$$

Remark. (a) Highly non-trivial to compute μ.

(b) On the hexagonal lattice, it is shown that $\mu = \sqrt{2 + \sqrt{2}}$ (2010, Copin at el.)

(c) By computer simulation,
   Conjecture : $Z_\beta = (\beta - \beta_c)^{-\gamma + o(1)}$ for $\beta \to \beta_c^+$, where γ only depends on that dimension of the lattice. In 2D, $\gamma = \frac{43}{32}$ (conjectured)

(d) If we have $\lambda_N \sim \mu^N N^\alpha$, then Z_β can be computed to satisfy $Z_\beta \sim (\beta - \beta_c)^{-1+\alpha}$, $\alpha = \frac{11}{32}$ in 2D (conjectured)

Exercise 3 How to sample SAWs with computer. (Think of Markov process whose invariant measure is $\mathbb{P}_\beta$)

0.3   Bernoulli Percolation

We first consider $\mathbb{Z}^d$ lattices,

- **Bond percolation** Give $p \in [0, 1]$, consider $\omega = (\omega(e))_{e \in \mathbb{Z}^d}$ such that $\{\omega(e)\}_{e \in \mathbb{Z}^d}$ is i.i.d with $\omega(e) \sim$ Brenoulli(p).
  If $\omega(e) = 1$, we say that e is open. If $\omega(e) = 0$, we say that e is closed.
  Remark We got a model of random subgraph. By above of notation, we may also write ω for the random subgraph (consisting of that open edges).

- **Site percolation** Same thing with i.i.d Bernoulli(p),
  open means that the node can pass, closed means that the node can not pass.

- Notation :
  $\mathbb{P}_p$ = The Bernoulli percolation of parameter p.
  ω_p a sample of $\mathbb{P}_p$

A bond percolation   A site percolation
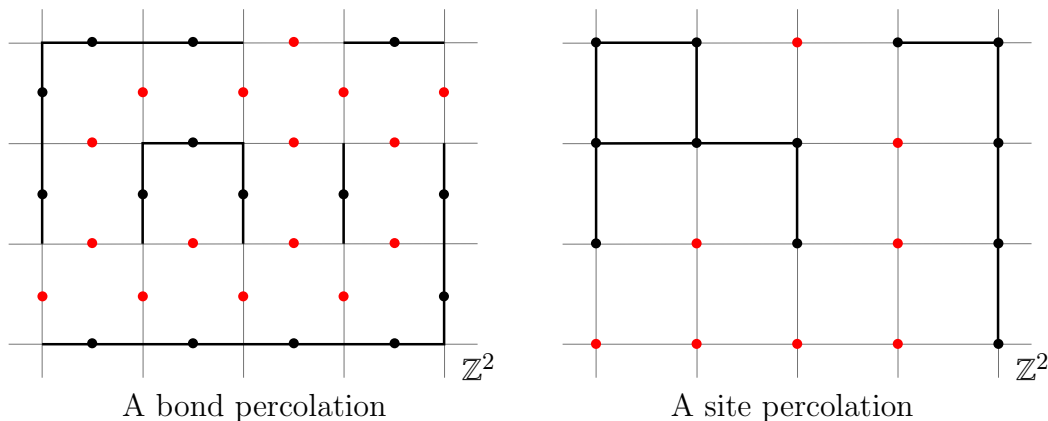
Remark The terminology "Bernoulli percolation" stands for **i.i.d**, on the other hand, without independence, we simply say that we have a "percolation model", e.g. random cluster model.
For the following classes we use "percolation" to refer to Bernoulli percolation.

Exercise 1 Show that a bond percolation is equivalent to a site percolation. How about the other way? Construct an example.

Question : What are the interesting behavior when p varies? e.g. # component, size of component etc.
$p = 0$ is an empty graph, $p = 1$ is a full graph.

- **Connected component (cluster)**
  Let a, b be two vertex of $\mathbb{Z}^d$, we say that $a \sim b$ if exists an path in ω_p from a to b. It is clearly that $\sim$ is an equivalence relation.
  A **connected component**(cluster) is an element in equivalence classes of $\sim$

- **Infinite cluster**
  A infinite cluster is a cluster of ω_p that has infinite edges and infinite vertex.
  Let $[O \leftrightarrow \infty]$ be the event in $\mathbb{P}_p$ that O belongs to a infinite cluster.
  $\theta(p) = \mathbb{P}_p[O \leftrightarrow \infty]$.

Exercise 2 Let $\mathbb{T}_d$ be a regular tree, i.e., an infinite connected graph with countably many vertices without cycles where all the vertices is of degree d. <u>Ex</u> : $\mathbb{T}_2 = \mathbb{Z}$,

$$\mathbb{T}_3$$
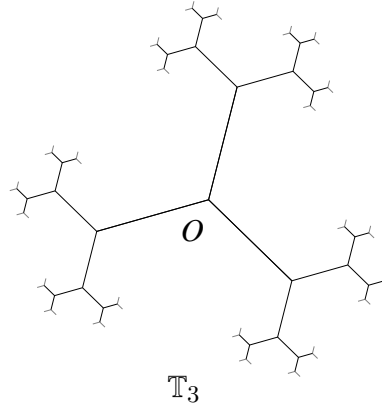
We study p_c of $\mathbb{T}_d$, $d \geq 2$.

If $d = 2$, i.e., $\mathbb{T}_2 = \mathbb{Z}$, consider

$$\beta(p) := \mathbb{P}_p[O \leftrightarrow \infty \text{ in } \mathbb{N}_0] = 1 - (1-p) - p(1-p) - p^2(1-p) - \cdots$$

$$= 1 - \frac{1-p}{1-p} = 0, \quad \text{if } p \neq 1.$$

$$= 1, \quad \text{if } p = 1.$$

Then, $\theta(p) = 2\beta(p) - \beta(p)^2 = \beta(p)$. Thus, $p_c(\mathbb{T}_2) = \sup\{p \mid \theta(p) = 0\} = 1$.

If $d = 3$, consider $\mathbb{I}_3 \subseteq \mathbb{T}_3$ and define $\beta(p) = \mathbb{P}_p[O \leftrightarrow \infty \text{ in } \mathbb{I}_3]$.
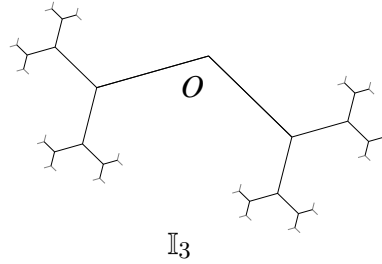

$$\mathbb{I}_3$$

Then,

$$\beta(p) = 2p\beta(p) - p^2\beta(p)^2 \Rightarrow \beta(p) = 0 \text{ or } \frac{2p-1}{p^2}.$$

Because of $\beta(p) \geq 0$ for all $p \in [0, 1]$, we get $\beta(p) = 0$ if $p \in [0, \frac{1}{2}]$.

On the other hand, we use a similar way of $d = 2$ to calculate $\beta(p)$:

$$\beta(p) = 1 - \sum_{n=0}^{\infty} a_n p^n (1-p)^{n+2} = 1 - (1-p)^2 \sum_{n=0}^{\infty} a_n p^n (1-p)^n$$

where $a_n = \#$ of n-edges clusters in $\mathbb{I}_2$ that contains O.

If $p \in [0, \frac{1}{2}]$, then $\beta(p) = 0$, thus

$$\sum_{n=0}^{\infty} a_n p^n (1-p)^n = \frac{1}{(1-p)^2}$$

Therefore, if $p \in [\frac{1}{2}, 1]$, we get $1 - p \in [0, \frac{1}{2}]$, and thus

$$\sum_{n=0}^{\infty} a_n p^n (1-p)^n = \sum_{n=0}^{\infty} a_n (1-p)^n (1-(1-p))^n = \frac{1}{p^2}$$

Thus,

$$\beta(p) = 1 - \frac{(1-p)^2}{p^2} = \frac{2p-1}{p^2}$$

And we can calculate $\theta(p) = \beta(p) + p\beta(p) - p\beta(p)^2$, and find that

$$p_c(\mathbb{T}_3) = \sup\{p \mid \theta(p) = 0\} = \sup\{p \mid \beta(p) = 0\} = \frac{1}{2}$$

For a general $d \geq 3$, and still studying $p_c(\mathbb{T}_d)$, still consider $\mathbb{I}_d \subseteq \mathbb{T}_d$ be a connected graph that the degree of O is $d-1$ and is d in other vertices.

1° Consider

$$\beta(p) = \mathbb{P}_p[O \leftrightarrow \infty \text{ in } \mathbb{I}_d],$$

then $1 - \beta(p) = (1 - p\beta(p))^{d-1}$. Now, consider the case that $p = \frac{1}{d-1}$, then

$$1 - \beta\left(\frac{1}{d-1}\right) = \left(1 - \frac{1}{d-1}\beta\left(\frac{1}{d-1}\right)\right)^{d-1}, \quad \text{let } u = 1 - \frac{1}{d-1}\beta\left(\frac{1}{d-1}\right),$$

we get

$$-d + 2 + (d-1)u = u^{d-1} \cdots\cdots (*)$$

Because of $0 \leq \beta\left(\frac{1}{d-1}\right) \leq 1$, we have $1 - \frac{1}{d-1} \leq u \leq 1$, we know that if $0 < u < 1$, then $u^{d-1} - (d-1)u = u(u^{d-2} - (d-1)) > u^{d-2} - d + 1 > -d + 2$, thus the solution of $(*)$ is $u = 1$, and thus $\beta\left(\frac{1}{d-1}\right) = 0$. Remark that

$$\theta(p) = \beta(p) + p\beta(p) - p\beta(p)^2,$$

thus $\theta\left(\frac{1}{d-1}\right) = 0$, thus $p_c \geq \frac{1}{d-1}$.

2° Note that

$$\beta(p) = 1 - \sum_{n=0}^{\infty} a_n p^n (1-p)^{(d-2)n+d-1} = 1 - (1-p)^{d-1} \sum_{n=0}^{\infty} a_n p^n (1-p)^{(d-2)n}$$

where $a_n = \#$ of n-edges clusters in $\mathbb{I}_d$ that contain O.

We consider the polynomial $f(p) = p(1-p)^{d-2}$, we have $f(0) = f(1) = 0$, and $f(p) \geq 0$ if $p \in [0,1]$. Note that

$$f'(p) = (1-p)^{d-2} - (d-2)p(1-p)^{d-3} = (1-p)^{d-3}\left[1 - (d-1)p\right]$$

Thus, critical points of f on $[0,1]$ is 0, 1, $\frac{1}{d-1}$, thus $f\left(\frac{1}{d-1}\right)$ is the maximal value in $[0,1]$. Thus, if $p \in \left(\frac{1}{n-1}, 1\right]$

$$\beta(p) = 1 - (1-p)^{d-1} \sum_{n=0}^{\infty} a_n f(p)^n$$

$$\geq 1 - (1-p)^{d-1} \sum_{n=0}^{\infty} a_n f\left(\frac{1}{n-1}\right)^n$$

$$> 1 - \left(1 - \frac{1}{d-1}\right)^{d-1} \sum_{n=0}^{\infty} a_n f\left(\frac{1}{n-1}\right)^n$$

$$= \beta\left(\frac{1}{n-1}\right) = 0.$$

Thus, $p_c(\mathbb{T}_d) = \frac{1}{1-d}$, $d \geq 2$ ∎

Exercise 3 Define the random-cluster model and discuss the following :

(a) Why it can only be defined on a finite graph?

(b) The heuristic behavior of the model when q varies.
($q \to 0^+$ UST,
$q = 1$ Bernoulli percolation,
$q > 1$ configurations with more clusters are preferred
$q < 1$ configurations with less clusters are preferred)

(c) The definition of the model is "global", i.e., one needs to count the number of clusters so needs to know where a configuration looks like in the <u>whole</u> graph.

(d) For a planer random-cluster model, the FK-loop representation makes the definition "local". (This may be hard to discuss at the moment)

Chapter 1

Basic Properties of the Bernoulli Percolation

Consider $G = \mathbb{Z}^d$ or some "nice" graph.

1.1 Coupling (耦合)

- Given $p \leq p'$, how to compute $X \sim \text{Bernoulli}(p)$, $X' \sim \text{Bernoulli}(p')$?
  Consider $\mathcal{U} \sim \text{Uniform}([0,1])$, define $Y = \mathbf{1}_{\mathcal{U} \leq p}$, $Y' = \mathbf{1}_{\mathcal{U} \leq p'}$, we get

$$X \stackrel{(id)}{=} Y, \quad X' \stackrel{(id)}{=} Y', \quad Y \leq Y'. \quad a.s.(almost\ sure)$$

  This is called a coupling.

Remark. In coupling, usually we do not want independence, so that we can compute values between random variables.

Exercise 1 Construct a coupling between $\omega \sim \mathbb{P}_p$, $\omega' \sim \mathbb{P}_{p'}$ with $p \leq p'$, so that values between edges can be computed.
Wanted : $p \leq p' \Rightarrow \omega_p \leq \omega'_p (\Leftrightarrow \omega_p(e) \leq \omega_{p'}(e),\ \forall e \in E)$

$\boxed{\texttt{soL:}}$ Let $\omega = (\omega(e))_{e \in G}$ such that $\{\omega(e)\}_{e \in G}$ is i.i.d. and $\omega(e) \sim \text{Uniform}([0,1])$.
Define $\omega_p \sim \mathbb{P}_p$, $\omega_{p'} \sim \mathbb{P}_{p'}$ as $\forall e \in E$, $\omega_p(e) = \mathbf{1}_{\omega(e) \leq p}$, $\omega_{p'}(e) = \mathbf{1}_{\omega(e) \leq p'}$, thus, $p \leq p' \Rightarrow \forall e \in E$, $\omega_p(e) \leq \omega_{p'}(e)$.

Exercise 2 Given $O \in V(G)$, define $\theta : \begin{array}{ccc} [0,1] & \to & \mathbb{R} \\ p & \mapsto & \mathbb{P}_p([O \leftrightarrow \infty]) \end{array}$ (percolation function).
Show that θ is increasing. In more general case, at must how many different θ function can be obtain?

$\boxed{\texttt{soL:}}$ If $p \leq p'$, let $\omega_1 \sim \mathbb{P}_p$, $\omega_2 \sim \mathbb{P}_{p'}$, we use the definition of **Exercise 1**, we have
$\omega_1 \stackrel{(id)}{=} \omega_p$, $\omega_2 \stackrel{(id)}{=} \omega_{p'}$, thus $\mathbb{P}_p([O \leftrightarrow \infty]) = \mathbb{P}_p([O \leftrightarrow \infty]$ in $\omega_1) = \mathbb{P}_p([O \leftrightarrow \infty]$ in $\omega_p)$,

similarly, $\mathbb{P}_{p'}([O \leftrightarrow \infty]) = \mathbb{P}_{p'}([O \leftrightarrow \infty]$ in $\omega_1) = \mathbb{P}_{p'}([O \leftrightarrow \infty]$ in $\omega_{p'})$. We note that ω_p is always a subgraph of $\omega_{p'}$ (by **Exercise 1**), thus $\{[O \leftrightarrow \infty]$ in $\omega_p\} \subseteq \{[O \leftrightarrow \infty]$ in $\omega_{p'}\}$, we have $\mathbb{P}_p([O \leftrightarrow \infty]) = \mathbb{P}_p([O \leftrightarrow \infty]$ in $\omega_p) \le \mathbb{P}_{p'}([O \leftrightarrow \infty]$ in $\omega_{p'}) = \mathbb{P}_{p'}([O \leftrightarrow \infty])$.

And, because of $\omega_1 \overset{(id)}{=} \omega_p$, where ω_1 is a arbitrary random variable with $\omega_1 \sim \mathbb{P}_p$ thus there are only one choice of θ. i.e., θ is well-defined. ∎

- Define $p_c = \sup\{p \in [0, 1] \mid \theta(p) = 0\}$

Exercise 3 Check the following properties :

    (a) The function $p \mapsto \theta(p)$ is right-continuous on $[0, 1]$.

    (b) The function $p \mapsto \theta(p)$ is left-continuous on $(p_c, 1)$. (Hint : use the uniqueness)

    (c) Show that $p \mapsto \theta(p)$ is strictly increasing in $(p_c, 1]$.
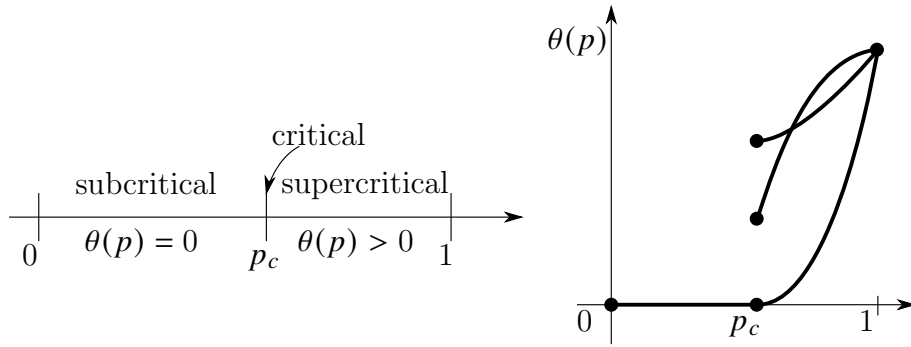
SOL: (a) Given $n \in \mathbb{N}$, define $\theta_n : [0, 1] \to [0, 1]$ as $\theta_n(p) = \mathbb{P}_p[O \leftrightarrow \partial \Lambda_n]$. Because $\{[O \leftrightarrow \partial \Lambda_n]\}_{n \in \mathbb{N}}$ is a decreasing sequence of events and $\bigcup_{n=1}^{\infty}[O \leftrightarrow \partial \Lambda_n] = [0 \leftrightarrow \infty]$, we get
$$\lim_{n \to \infty} \theta_n(p) = \theta(p), \quad \forall p \in [0, 1].$$

And because of $\theta_n(p)$ is just a polynomial in p, we have θ_n is continuous and non-decreasing on $[0, 1]$, thus given $\varepsilon > 0$, $p \in [0, 1]$, there exist $\delta := \delta(n) > 0$ such that
$$p < q < p + \delta \Rightarrow \theta_n(q) - \theta_n(p) < \frac{\varepsilon}{2}.$$

And there exists $N \in \mathbb{N}$ such that
$$n \ge N \Rightarrow \theta_n(p) - \theta(p) < \frac{\varepsilon}{2}$$

Thus,
$$p < q < p + \delta \Rightarrow \theta(q) - \theta(p) \le \theta_N(q) - \theta(p)$$
$$= \theta_N(q) - \theta_N(p) + \theta_N(p) - \theta(p) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

which implies that θ is right-continuous on $[0, 1]$.

  (b)

  (c) Suppose that θ is not strictly increasing on $(p_c, 1]$, then there exist p, q, $p_c < p < q \le 1$ such that $\theta(p) = \theta(q)$.
Now, given $x \in V(G)$, we say that x is a **escape point** (esc-point) of a subgraph ω of G if $x \leftrightarrow \infty$. Given $e \in E(G)$, we say that e is an **isolated edge** (iso-edge) of ω

if e connects an esc-point and a non-esc-point. Therefore, if e is an iso-edge of ω, then $\omega(e) = 0$.

Given $s \in [0, 1]$ define $X_{n,s} = \#$ esc-points of Λ_n in ω_s, $Y_{n,s} = \#$ iso-edges of $E(\Lambda_n)$ in ω_s. We find that

$$\mathbb{E}[X_{n,p}] = \mathbb{E}[X_{n,q}] = |\Lambda_n|\theta(p).$$

Let $Z_n = \#$ iso-edges of Λ_n in ω_p which are open in ω_q (under the coupling). Then

$$X_{n,q} \geq X_{n,p} + \frac{1}{2d}Z_n, \quad \text{if } G = \mathbb{Z}^d$$

Thus, $\mathbb{E}[Z_n] = 0$, and this further implies that

$$\begin{aligned}
&\mathbb{P}(\{e \in \mathrm{iso}(\omega_p)\} \cap \{\omega_q(e) = 1\}) \\
&= \mathbb{P}\{e \in \mathrm{iso}(\omega_p)\} \times \mathbb{P}\{\omega_q(e) = 1 \mid \omega_p(e) = 0\} = 0 \text{ for all } e \in \mathbb{Z}^d \\
&\Rightarrow \mathbb{P}\{e \in \mathrm{iso}(\omega_p)\} = 0 \text{ for all } e \in \mathbb{Z}^d.
\end{aligned}$$

Claim :

(i) $\theta(p) = \mathbb{P}[\exists \text{ an infinite cluster in } \omega_p] = 1$.

(ii) $\theta(p) = 1 \Rightarrow p = 1$

proof of claim :

(i) Note that given $O \in G$, then for $X \in V(G)$, we have a sequence of edges $\{e_k\}_{k=1}^n \subseteq G$ such that it connects from O to X. Thus

$$\begin{aligned}
\mathbb{P}[O \notin \mathrm{esc}(\omega_p)] &= \mathbb{P}[O \notin \mathrm{esc}(\omega_p) \text{ and } \{e_k\}_{k=1}^n \subseteq \mathrm{iso}(\omega_p)^{\complement}] \\
&= \mathbb{P}[O \notin \mathrm{esc}(\omega_p) \text{ and } X \notin \mathrm{esc}(\omega_p)].
\end{aligned}$$

Let $V(G) = \{V_n\}_{n=1}^\infty$, $E(G) = \{e_k\}_{k=1}^\infty$, use the fact that $\mathbb{P}\left[\bigcup_{e \in \mathbb{N}}\{e_k \in \mathrm{iso}(\omega_p)\}\right] \leq \sum_{n \in \mathbb{N}} \mathbb{P}\{e_k \in \mathrm{iso}(\omega_p)\} = 0$, we get

$$\begin{aligned}
\mathbb{P}[V_1 \notin \mathrm{esc}(\omega_p)] &= \mathbb{P}[V_1 \notin \mathrm{esc}(\omega_p) \text{ and } \{e_k\}_{k=1}^\infty \subseteq \mathrm{iso}(\omega_p)^{\complement}] \\
&= \mathbb{P}[\{V_n\}_{n=1}^\infty \subseteq \mathrm{esc}(\omega_p)^{\complement}]
\end{aligned}$$

Therefore,

$$\begin{aligned}
1 &= \mathbb{P}[\exists \text{ an infinite cluster in } \omega_p] \\
&= \mathbb{P}[\exists \text{ an infinite cluster in } \omega_p \text{ and } O \in \mathrm{esc}(\omega_p)] \\
&\quad + \mathbb{P}[\exists \text{ an infinite cluster in } \omega_p \text{ and } O \notin \mathrm{esc}(\omega_p)] \\
&= \mathbb{P}[O \leftrightarrow \infty] + \mathbb{P}(\varnothing) = \theta(p).
\end{aligned}$$

(ii) Suppose that $p < 1$, then

$$\theta(p) = \mathbb{P}[O \leftrightarrow \infty] \leq 1 - (1 - p)^{2d} < 1.$$

Thus, $p = 1$ (a contradiction). ∎

- The proof of (c) comes from some ideas :

    i. Use $\mathbb{E}[X_{n,p}]$ to represent $\theta(p)$. Thus induces $\mathbb{E}[Z_n] = 0$.

    ii. Use $\mathbb{E}[Z_n] = 0$ to show that $\theta(p) = 1$, thus $p = 1$ and make a contradiction.

iii. It is worth to mention that $p > p_c$ is a sufficient condition because
$\mathbb{P}[\exists$ an infinite cluster in $\omega_p] = \theta(p)$, if p is given arbitrary, then $\theta(p)$ can be 0 or 1.

Questions :   (a)   Where is p_c? or bounds? i.e., $p_c > 0$? $p_c < 1$?

(b)   continuity of θ? $\theta(p_c) = 0$ or > 0?

We well see that $p_c = 0$ in $\mathbb{Z}^2$.

Conjecture $\theta(p_c) = 0$ in $\mathbb{Z}^d$.
Current result :

- '90, Hora-Slade, $d \geq 19$
- '15, Fitzner-van der Rofstod, $d \geq 11$

$\rightarrow$ the method should be hold for $d \geq 6$, how about $d = 3, 4, 5$?

Exercise 4   (a)   Show that on $\mathbb{Z}^d$, $p_c \geq \frac{1}{2d}$.

(b)   Show that on $\mathbb{Z}^d$, $p_c < 1$ if $d \geq 2$ (Hint : look at $\mathbb{Z}^2$+ dual graph)

(c)   For a general graph, shoe that $p_c \geq \frac{1}{\mu}$. (connective constant)

1.2 Uniqueness of the Infinite Cluster (on $\mathbb{Z}^d$)

Theorem. For a fixed $p \in [0, 1]$, we have

*) Either a.s. (almost surely) there is no infinite cluster.

*) or a.s. there is a unique infinite cluster.

Remark $N = \#$ infinite clusters. (a random variable)
We have $\mathbb{P}(N = 0) = 1$ or $\mathbb{P}(N = 1) = 1$.

Proof. We decompose the proof in 3 steps :

Step 1 Show that if an event A is invariant under some vector e (平移不變性), then

$$\mathbb{P}(A) = 0 \text{ or } \mathbb{P}(A) = 1.$$

$\rightarrow$ Kolmogorov's 0-1 law / "approximation" method.
$\Rightarrow$ For $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, the event $\{N = k\}$ satisfy the property. Thus, there exist $k = k(p)$ s.t. $\mathbb{P}_p(N = k) = 1$.

Step 2 Exclude the case $N = k$ a.s. for $k \geq 2$, $k \neq \infty$ using the "finite energy property".

Step 3 Exclude the case $N = \infty$ by looking at "**trifurcation points**".

Step 1 *Claim* : (**Exercise 1**)
Any event A can be "approximated" by a sequence of events that depend only on a finitely many edges. In other words, we can find a sequence of events $(A_n)_{n \geq 0}$ such that A_n only depends on edges in $[-m_n, m_n]^d$ for all $n \geq 0$ and

$$\lim_{n \to \infty} \mathbb{P}(A \triangle A_n) = 0, \quad \text{where } A \triangle A_n = (A \backslash A_n) \cup (A_n \backslash A).$$

$\boxed{\text{s}_\text{o}\text{L:}}$ We use monotone class theorem to show. "The theorem says that the smallest monotone class containing an algebra of sets $\mathcal{G}$ is precisely the smallest σ-algebra containing $\mathcal{G}$"- wiki

First, we note that it is true the collection of finite dimensional cylinder set is an algebra that generate the σ-algebra $\mathcal{F}$. We define the desired class $\mathcal{M}$ to be

$$\mathcal{M} = \{A \in \mathcal{F} | \forall \epsilon > 0, \exists A_\epsilon \text{ depend only on finitely many edges s.t } \mathbb{P}(A \triangle A_\epsilon) < \epsilon\}.$$

Then, by definition the cylinder set is a subset of $\mathcal{M}$. We proceed to check that $\mathcal{M}$ is a monotone class. For countable monotone unions, consider the union $A = \bigcup_{i=1}^{\infty} A_i$, where $A_1, A_2, \cdots \in \mathcal{M}$ and $A_1 \subseteq A_2 \subseteq \ldots$, since $\mathbb{P}(A) \leq 1$, for any given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\mathbb{P}(A \backslash A_N) < \epsilon/2$. On the other hand, the set A_N is in $\mathcal{M}$, so there is a set $\tilde{A_N}$ defined on finitely many edges that satisfies $\mathbb{P}(A_N \triangle \tilde{A_N}) < \epsilon/2$. Therefore,

$$\mathbb{P}(A \triangle \tilde{A_N}) < \mathbb{P}(A \backslash A_N) + \mathbb{P}(A_N \triangle \tilde{A_N}) < \epsilon.$$

ϵ is arbitrary, so $A \in \mathcal{M}$ The countable monotone intersection is shown in a similar fasion. Let $B = \bigcap_{i=1}^{\infty} B_i$, where $B_1, B_2, \cdots \in \mathcal{M}$ and $B_1 \subseteq B_2 \subseteq \ldots$. WLOG we may assume $B \neq \emptyset$, then given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\mathbb{P}(B_N \backslash B) < \epsilon/2$. Also, there is $\tilde{B_N}$ that defined on finitely many edges and $\mathbb{P}(B_N \triangle \tilde{B_N}) < \epsilon/2$. Hence,
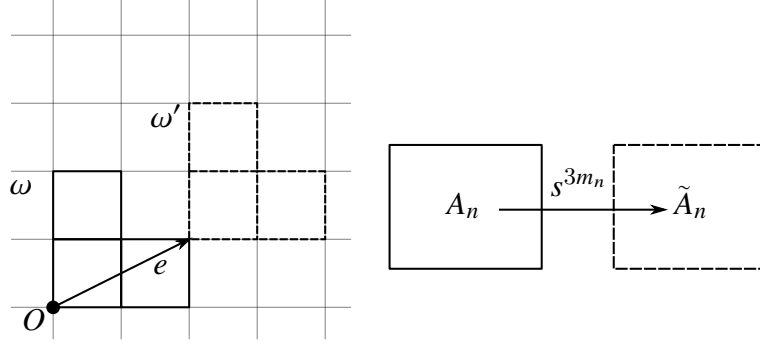
$$\mathbb{P}(B \triangle \tilde{B_N}) < \mathbb{P}(B_N \backslash B) + \mathbb{P}(B_N \triangle \tilde{B_N}) < \epsilon$$

this implies $B \in \mathcal{M}$. Finally, $\mathcal{M}$ is a monotone class containing the generating algebra (i.e cylinder set) of the σ-algebra $\mathcal{F}$, so by monotone class theorem, $\mathcal{F} \subseteq \mathcal{M}$. $\square$

Given an event A that is invariant under a vector $e \neq 0$, we approximate A by $(A_n)_{n\geq 1}$, i.e., $\mathbb{P}(A \triangle A_n) \to 0$ as $n \to \infty$. Where A_n depends only on edges in $[-m_n, m_n]^d$.

Define $\tilde{A}_n := s^{3m_n}(A_n)$ where

$$s : \begin{array}{ccc} \Omega & \to & \Omega \\ \omega(\cdot) & \mapsto & \omega(\cdot - e) \end{array} \quad \text{(translation operator)}$$

Moreover, because of A is invariant under vector e, we have $s(A) = A$. Now, using the independence between A_n and $\tilde{A}_n$,

$$\mathbb{P}(A_n \cap \tilde{A}_n) = \mathbb{P}(A_n)\mathbb{P}(\tilde{A}_n) = \mathbb{P}(A_n)^2 \underset{n\to\infty}{\longrightarrow} \mathbb{P}(A)^2$$

Therefore,

$$\mathbb{P}(A \triangle (A_n \cap \tilde{A}_n)) \leq \mathbb{P}(A \triangle A_n) + \mathbb{P}(A \triangle \tilde{A}_n) \underset{n\to\infty}{\longrightarrow} 0$$

$$\Rightarrow \mathbb{P}(A) = \mathbb{P}(A)^2 \Rightarrow \mathbb{P}(A) = 0 \text{ or } 1$$

Remark 1 It is needed to say that $\{N = k\} \in \sigma(\mathcal{F})$ for all $k \in \mathbb{N}$

Remark 2 If $\lim_{n\to\infty}(A \triangle (A_n \cap \tilde{A}_n)) = 0$, then

$$\lim_{n\to\infty} \mathbb{P}(A \triangle (A_n \cap \tilde{A}_n)) = \lim_{n\to\infty} \mathbb{P}\big((A \cap (A_n \cap \tilde{A}_n)^{\complement}) \cup (A^{\complement} \cap (A_n \cap \tilde{A}_n))\big)$$

$$= \lim_{n\to\infty} \mathbb{P}(A \cap (A_n \cap \tilde{A}_n)^{\complement}) + \mathbb{P}(A^{\complement} \cap (A_n \cap \tilde{A}_n)) = 0$$

$$\Rightarrow \lim_{n\to\infty} \mathbb{P}(A \cap (A_n \cap \tilde{A}_n)^{\complement}) = 0 \text{ and}$$

$$\lim_{n\to\infty} \mathbb{P}(A^{\complement} \cap (A_n \cap \tilde{A}_n)) = 0.$$

$$\Rightarrow \lim_{n\to\infty} \mathbb{P}(A) - \mathbb{P}(A_n \cap \tilde{A}_n)$$

$$= \lim_{n\to\infty} \mathbb{P}(A \cap (A_n \cap \tilde{A}_n)) - \mathbb{P}(A \cap (A_n \cap \tilde{A}_n)) = 0.$$

Step 2 We want to show that it is impossible to have $N = k$ as for $k \geq 2, \ k \neq \infty$. We assume that $\mathbb{P}(N = k) = 1$ for some $k \geq 2$, consider
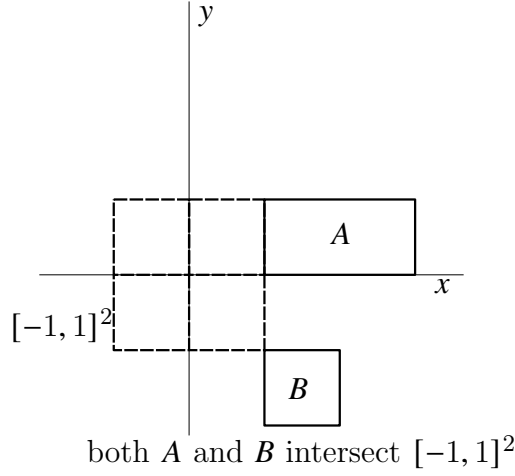
$$B_n = \{N = k \text{ and } k \text{ infinite clusters intersect } [-n, n]^d\}$$

Note that there exist $n \in \mathbb{N}$ such that $\mathbb{P}(B_n) > 0$. Define

$$C_n = \{ \text{ all infinite clusters intersect } [-n, n]^d\}$$

Note that

i. $B_n \subseteq C_n$.

ii. B_n depends on the edges in $[-n, n]^d$, but C_n does not.

both A and B intersect $[-1, 1]^2$

We use $\blacksquare_n$ to be the event that all edges in $[-n, n]^d$ are open, then

$$\mathbb{P}(C_n \cap \blacksquare_n) = \mathbb{P}(C_n)\mathbb{P}(\blacksquare_n) > 0.$$

Moreover, $C_n \cap \blacksquare_n \subseteq \{N = 1\}$, we find that $\mathbb{P}(N = 1) > 0 \Rightarrow \mathbb{P}(N = 1) = 1$ (contradicts).

Remark The fact that we can modify the state of the edges in $[-n, n]$ while keeping the probability positive is called **finite-energy property**

Step 3 Now, we want to show that it is impossible to have $\mathbb{P}(N = \infty) = 1$.

Assume that $0 < p < 1$, A **trifurcation point** (三叉點) x in ω_p is defined by
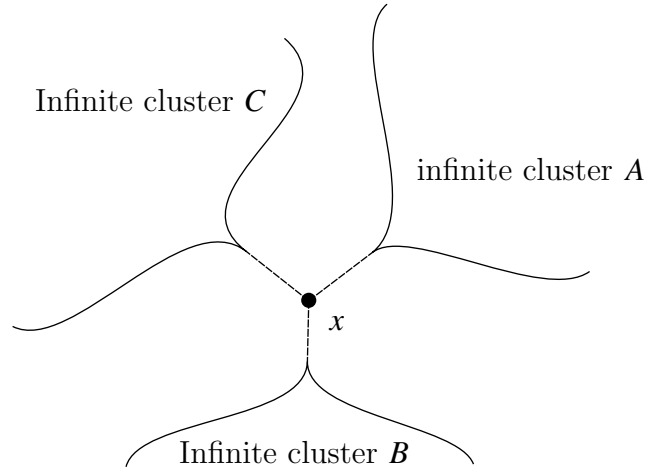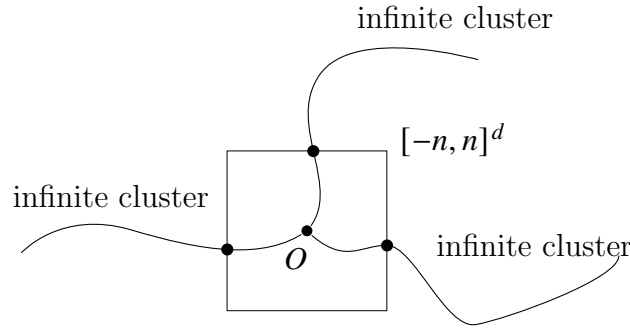

∗) x is in a infinite cluster of ω.

∗) Exactly 3 adjacent edge of x are open.

∗) If all the adjacent edge of x are closed, we have 3 infinite clusters.

Assume that $\mathbb{P}(N = \infty) = 1$, define

$$D_n = \{ \text{ at least 3 infinite clusters intersect} [-n, n]^d \}$$

We can find $n \in \mathbb{N}$ such that $\mathbb{P}(D_n) > 0$.

(a) The box $\Lambda_n = [-n, n]^d$ is finite, thus any edge configuration has at lest probability $\min\{p, 1-p\}^{|E(\Lambda_n)|} > 0$ to be happen.

(b) For each configuration $\omega \in D_n$, we can find a nice configuration in Λ_n such that O is a trifurcation point.
$\Rightarrow c := \mathbb{P}(O \text{ is a trifurcation }) \geq \mathbb{P}(D_n) \cdot \min\{p, 1-p\}^{|E(\Lambda_n)|} > 0$

For each trifurcation point x, we can define at least three self-avoiding path form x to ∞, each path goes from 3 distinct infinite clusters, respectively. A intersection of a SA paths form trifurcation point and $\partial\Lambda_n$ is called a leaf. We deduce that

$$\# \text{ of leaves in } \partial\Lambda_n \geq 2\# \text{ of trifurcations in } \Lambda_n$$

Define the following random variables :

$$X_n = \# \text{ trifurcations in } \Lambda_n.$$
$$= \sum_{x \in \Lambda_n} \mathbb{1}_{\{x \text{ is a trifurcation point}\}}$$
$$Y_n = \# \text{ leaves in } \partial\Lambda_n.$$

Then we have
$$\mathbb{E}[Y_n] \geq 2\mathbb{E}[X_n], \quad \forall n \in \mathbb{N}.$$

Note that $Y_n \leq |\partial\Lambda_n| \leq 2 \times (2n-1)^{d-1} \times d$, thus

$$2 \times (2n-1)^{d-1} \times d \geq \mathbb{E}[Y_n] \geq 2\mathbb{E}[X_n] = 2c \times (2n-1)^d, \quad \forall n \in \mathbb{N}$$
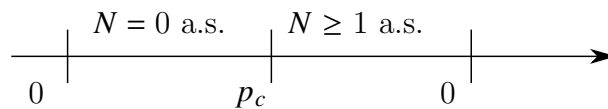
This implies $c = 0$ (contradiction)

$\square$

Exercise 2 (a) For which general graphs does this theorem hold?

(b) If we do not require uniqueness, to which graph can this theorem be extended?

Exercise 3 What happens if we look at the tree $\mathbb{T}_d$?

Exercise 4 Check that
$$\theta(p) > 0 \Leftrightarrow \mathbb{P}_p(N \geq 1) = 1$$

$\boxed{\text{s\underline{o}L:}}$ Note that $[N \geq 1] = [\exists$ an infinite cluster $]$.

" $\Rightarrow$ " : If $\theta(p) > 0$, then $\mathbb{P}_p[\exists$ an infinite cluster$] \geq \theta(p) > 0$, because of $[\exists$ an infinite cluster$]$ is invariant under any vector $e \in G$, we have

$$\mathbb{P}_p[\exists \text{ an infinite cluster}] = 0 \text{ or } 1.$$

Therefore, $\mathbb{P}_p(N \geq 1) = \mathbb{P}_p[\exists$ an infinite cluster$] = 1$.

" $\Leftarrow$ " : If $\mathbb{P}_p(N \geq 1) = \mathbb{P}_p[\exists$ an infinite cluster$] = 1$, suppose that $\theta(p) = 0$, we say $V(G) = \{V_n\}_{n=1}^{\infty}$, then
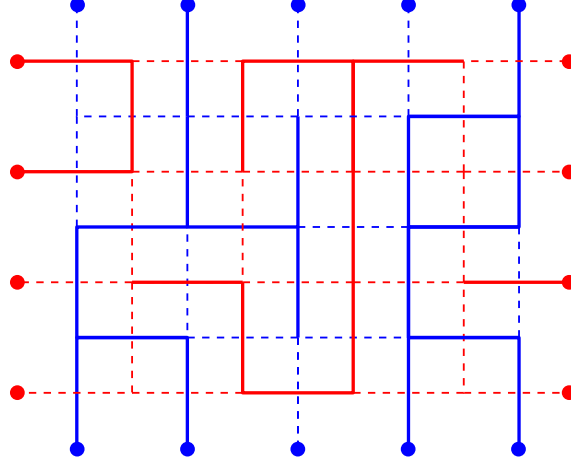
$$[\exists \text{ an infinite cluster}] = \bigcup_{n \in \mathbb{N}} [V_n \leftrightarrow \infty], \text{ and } \mathbb{P}_p[V_n \leftrightarrow \infty] = 0, \ \forall n \in \mathbb{N}.$$

And,

$$\mathbb{P}_p\left[\bigcup_{n \in \mathbb{N}} [V_n \leftrightarrow \infty] \right] \leq \sum_{n \in \mathbb{N}} \mathbb{P}_p[V_n \leftrightarrow \infty] = 0,$$

a contradiction. $\blacksquare$

1.3   Ideas Forward $p_c(\mathbb{Z}^2) = \frac{1}{2}$

A dual graph $\bullet$ $\omega_p \sim \mathbb{P}_{p,\mathbb{Z}^2}$ with goal of horizental crossing
$\bullet$ $\omega^*_{1-p} \sim \mathbb{P}_{1-p,(\mathbb{Z}+\frac{1}{2})^2}$ with goal of vertical crossing

We well show that for $\mathbb{Z}^2$, $p_c = \frac{1}{2}$ and $\theta(p_c) = 0$.

<u>Heuristics</u> : Define the dual graph of $\mathbb{Z}^2$ as follows : $G = \mathbb{Z}^2$, G^* = dual graph of G such that

$$V(G^*) = F(G), \ E(G^*) \cong E(G)$$

where two adjacent faces of G become two adjacent vertices of G^*.
To a configuration ω on $E(G)$ we obtain its dual configuration ω^* on $E(G^*)$ as follows :

$$\omega^*(e^*) = 1 - \omega(e) \implies \text{ if } \omega \sim \mathbb{P}_{p,G} \text{ then } \omega^* \sim \mathbb{P}_{1-p,G^*}$$

- The dual of $\mathbb{Z}^2$ is $(\mathbb{Z} + \frac{1}{2})^2 \cong \mathbb{Z}^2$. It is **self-dual**.

- Heuristically, if under $\mathbb{P}_p$, there is a huge/infinite cluster, then under $\mathbb{P}_{1-p}$, the clusters need to be small, so p_c must be $\frac{1}{2}$.

- More precisely, if we look at $p = \frac{1}{2}$ and a finite rectangle $\mathcal{R}_n = [0, n+1] \times [0, n]$, the "dual" of $\mathcal{R}_n$ is also $\mathcal{R}_n^* \cong \mathcal{R}_n$. Moreover, there's a horizontal crossing in $\mathcal{R}_n$ if and only if there's no vertical crossing in $\mathcal{R}_n^*$. We define

  $\mathcal{H}(\mathcal{R}_n)$ (or $\mathcal{H}(n+1, n)$) be the event that there's a walk in ω that cross from left edge of $\mathcal{R}_n$ to right edge of $\mathcal{R}_n$,

  $\mathcal{V}(\mathcal{R}_n)$ (or $\mathcal{V}(n+1, n)$) be the event that there's a walk in ω that cross from top edge of $\mathcal{R}_n$ to bottom edge of $\mathcal{R}_n$. Then,

$$\mathbb{P}(\mathcal{H}(\mathcal{R}_n)) = \mathbb{P}(\mathcal{H}(n+1, n)) = \mathbb{P}(\mathcal{V}(n, n+1)) = \mathbb{P}(\mathcal{V}(\mathcal{R}_n^*)),$$

and,

$$\mathbb{P}(\mathcal{H}(\mathcal{R}_n)) + \mathbb{P}(\mathcal{V}(\mathcal{R}_n^*)) = 1$$

This implies that $\mathbb{P}(\mathcal{H}(n+1, n)) = \frac{1}{2}$.

Chapter 2

Useful Identities & Applications

Probability space $\Omega = \{0, 1\}^{\Lambda}$ where Λ is finite or countable.

$\underline{\text{Ex}}$: $\Lambda = E(\mathbb{Z}^d)$, $\Lambda = E([-n, n]^d) =: \Lambda_n$

$$\boxed{\text{Partial order on } \Omega : \omega \leq \omega' \Leftrightarrow \omega(e) \leq \omega'(e), \ \forall e \in \Lambda}$$

$*$) A function $f : \Omega \to \mathbb{R}$ is said to be increasing if

$$\omega \leq \omega' \Rightarrow f(\omega) \leq f(\omega')$$

$*$) An event A is said to be increasing if $\mathbb{1}_A$ is increasing.

Exercise 1 Check that the increasing event generates all the event (σ-algebra).

$\underline{\text{Ex}}$: (a) Let $x, y \in V(\Lambda)$, $\{x \leftrightarrow y\} := \{\exists$ an open path connects from x to $y\}$

(b) $\mathcal{U}, \mathcal{V} \subseteq V(\Lambda)$, $\{\mathcal{U} \leftrightarrow \mathcal{V}\} := \bigcup_{\substack{x \in \mathcal{U} \\ y \in \mathcal{V}}} \{x \leftrightarrow y\}$

(c) Let $x \in V(\Lambda)$, $\{x \leftrightarrow \infty\} := \{x$ belongs to an infinite cluster$\}$.

$\boxed{\text{sol:}}$ Let $\mathcal{F}$ be the collection of events that only depend on finite edges, and let $\mathcal{I}$ be all increasing event in $\sigma(\mathcal{F})$. Our goal is to check that $\sigma(\mathcal{F}) = \sigma(\mathcal{I})$.

$1°$ Obviously, because $\mathcal{I} \subseteq \sigma(\mathcal{F})$, thus $\sigma(\mathcal{I}) \subseteq \sigma(\mathcal{F})$

$2°$ Observe that for all $e \in E(G)$, the event $\{\omega(e) = 1\}$ is an increasing event, and because $\{\omega(e) = 0\} = \{\omega(e) = 1\}^{C} \Rightarrow \{\omega(e) = 1\}$, $\{\omega(e) = 0\} \in \sigma(\mathcal{I})$. Now, we take an event $A \in \mathcal{F}$, say A only depending on finite edges $\{e_1, \cdots, e_n\} \subseteq E(G)$, thus

$$A \in \sigma(\{e_1 = 1\}, \{e_2 = 1\}, \cdots, \{e_n = 1\}) \subseteq \sigma(\mathcal{I}).$$

Thus, $\mathcal{F} \subseteq \sigma(\mathcal{I}) \Rightarrow \sigma(\mathcal{F}) \subseteq \sigma(\mathcal{I})$. $\blacksquare$

2.1 Harris–FKG inequality

<u>Proposition</u> (Harris inequality)

    (a) If A and B are increasing event, then

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\mathbb{P}(B).$$

    (b) If f and g are increasing functions, then

$$\mathbb{E}[fg] \geq \mathbb{E}[f]\mathbb{E}[g].$$

Remark (i) The two statement are equivalent. (why?) We will show (b).

    (ii) By "approximation", it is sufficient to show (b) for functions depending on finitely many edges.

    (iii) In the i.i.d. case (Bernoulli percolation), it was first proved by Harris ($'$60), before being extended to a more general setting by Fortuin, Kasteleyn, and Ginibre ($'$71), so sometimes it is also called FKG inequality.

    (iv) (a) if and only if $\mathbb{P}(B \mid A) \geq \mathbb{P}(B)$, which can be interperted as "When an increasing event hold, it becomes easier for another increasing event to holds." OR "A and B are positive correlated."

    (v) Define decreasing functions / events in a similar way, what are the inequalities one can obtain?

  $\boxed{\text{SoL:}}$ (i) "$\Rightarrow$" : For $f, g : \Omega \to [\infty)$ be increasing functions, we have $\forall t > 0, \mathbb{1}_{\{f>t\}}, \mathbb{1}_{\{g>t\}}$ are two increasing functions, and

$$f = \int_0^\infty \mathbb{1}_{\{f>t\}}\, dt, \quad g = \int_0^\infty \mathbb{1}_{\{g>t\}}\, dt$$

Thus

$$
\begin{aligned}
\mathbb{E}[fg] &= \mathbb{E}\left[\left(\int_0^\infty \mathbb{1}_{\{f>t\}}\, dt\right)\left(\int_0^\infty \mathbb{1}_{\{g>s\}}\, ds\right)\right] \\
&= \mathbb{E}\left[\int_0^\infty\int_0^\infty \mathbb{1}_{\{f>t\}\cap\{g>s\}}\, dt\, ds\right] \\
(\text{Fubini's theorem}) &= \int_0^\infty\int_0^\infty \mathbb{E}[\mathbb{1}_{\{f>t\}\cap\{g>s\}}]\, dt\, ds \\
&= \int_0^\infty\int_0^\infty \mathbb{P}(\{f>t\}\cap\{g>s\})\, dt\, ds \\
(\text{assumption}) &\geq \int_0^\infty\int_0^\infty \mathbb{P}(\{f>t\})\mathbb{P}(\{g>s\})\, dt\, ds \\
&= \mathbb{E}[f]\mathbb{E}[g].
\end{aligned}
$$

If we consider more general increasing functions $f, g : \Omega \to \mathbb{R}$, we define $\omega_\varnothing \in \Omega$ be graph that has no edge and vertices, then $f \geq f(\omega_\varnothing) =: c,\ g \geq g(\omega_\varnothing) := d$, then $f + c, g + d \in [0, \infty)$, thus

$$\mathbb{E}[(f+c)(g+d)] \geq \mathbb{E}[f+c]\mathbb{E}[g+d] \Rightarrow \mathbb{E}[fg] \geq \mathbb{E}[f]\mathbb{E}[g].$$

"$\Leftarrow$" : For A, B be two increasing events, we have $\mathbb{1}_A$, $\mathbb{1}_B$ are two increasing functions, thus

$$\mathbb{P}(A \cap B) = \mathbb{E}[\mathbb{1}_{A\cap B}] = \mathbb{E}[\mathbb{1}_A \mathbb{1}_B] \geq \mathbb{E}[\mathbb{1}_A]\mathbb{E}[\mathbb{1}_B] = \mathbb{P}(A)\mathbb{P}(B).$$

(v) A **function** $f : \Omega \to \mathbb{R}$ is said to be **decreasing** if

$$\omega \leq \omega' \Rightarrow f(\omega) \geq f(\omega')$$

An **event** A is said to be **decreasing** if $\mathbb{1}_A$ is decreasing.
Suppose that FKG inequality is hold, observe that if A and B are decreasing event, then

$$\mathbb{1}_A, \ \mathbb{1}_B \text{ are decreasing } \Rightarrow \mathbb{1}_{A^\complement} = 1 - \mathbb{1}_A, \ \mathbb{1}_{B^\complement} = 1 - \mathbb{1}_B \text{ are increasing.}$$

Thus $A^\complement$, $B^\complement$ are increasing event. Therefore

$$\mathbb{P}(A \cap B) = 1 - \mathbb{P}(A^\complement) - \mathbb{P}(B^\complement) + \mathbb{P}(A^\complement \cap B^\complement)$$
$$\geq 1 - \mathbb{P}(A^\complement) - \mathbb{P}(B^\complement) + \mathbb{P}(A^\complement)\mathbb{P}(B^\complement) = \mathbb{P}(A)\mathbb{P}(B)$$

Similarly, if f, g are decreasing function, then $-f, -g$ are increasing function. Thus

$$\mathbb{E}[(-f)(-g)] \geq \mathbb{E}[-f]\mathbb{E}[-g] \Rightarrow \mathbb{E}[fg] \geq \mathbb{E}[f]\mathbb{E}[g]. \quad \blacksquare$$

Proof of $\boxed{\text{Harris inequality}}$

We show the inequality for increasing functions depending on finitely many edge by induction.

Step 1 If f, g are increasing functions depending only on $\omega(e_1)$ for $\omega \in \Omega$ and for some edge e_1, observe that
if $\omega(e_1) = 1$, then $f(\omega) = f(G)$, $g(\omega) = g(G)$;
if $\omega(e_1) = 0$, then $f(\omega) = f(\omega_\varnothing)$, $g(\omega) = g(\omega_\varnothing)$.
Because $f(G) > f(\omega_\varnothing)$, $g(G) > g(\omega_\varnothing)$, we have

$$[f(\omega) - f(\omega')][g(\omega) - g(\omega')] \geq 0 \quad \text{for all } \omega, \ \omega' \in \Omega.$$

We can calculate $\mathbb{E}[f]$, $\mathbb{E}[g]$, $\mathbb{E}[fg]$ as follows :

$$\mathbb{E}[f] = f(G) \cdot \mathbb{P}\{\omega(e_1) = 1\} + f(\omega_\varnothing) \cdot \mathbb{P}\{\omega(e_1) = 0\}$$
$$\mathbb{E}[g] = g(G) \cdot \mathbb{P}\{\omega(e_1) = 1\} + g(\omega_\varnothing) \cdot \mathbb{P}\{\omega(e_1) = 0\}$$
$$\mathbb{E}[fg] = f(G)g(G) \cdot \mathbb{P}\{\omega(e_1) = 1, \omega(e_1) = 1\}$$
$$+ f(\omega_\varnothing)g(G) \cdot \mathbb{P}\{\omega(e_1) = 0, \omega(e_1) = 1\}$$
$$+ f(G)g(\omega_\varnothing) \cdot \mathbb{P}\{\omega(e_1) = 1, \omega(e_1) = 0\}$$
$$+ f(\omega_\varnothing)g(\omega_\varnothing) \cdot \mathbb{P}\{\omega(e_1) = 0, \omega(e_1) = 0\}.$$

Thus

$$\mathbb{E}[fg] - \mathbb{E}[f]\mathbb{E}[g] = [f(G) - f(\omega_\varnothing)][g(G) - g(\omega_\varnothing)]\mathbb{P}\{\omega(e_1) = 1\}\mathbb{P}\{\omega(e_1) = 0\} \geq 0.$$

Step 2 Suppose that $k \geq 2$, FKG inequality is hold for any f, g be increasing functions depending only on $\{e_1, \cdots e_{k-1}\} \subseteq E(G)$, $\{e_1, \cdots, e_{k-1}\}$ is arbitrary chosen with respect to f, g. Now, consider f, g be increasing functions that depend on $\{e_1, \cdots, e_k\} \subseteq E(G)$, we use the idea of conditional expectation to compute the following :

$$
\begin{aligned}
\mathbb{E}[fg] &= \mathbb{E}[fg \mid e_k = 0]\mathbb{P}[e_k = 0] + \mathbb{E}[fg \mid e_k = 1]\mathbb{P}[e_k = 1] \\
(\text{assumption}) &\geq \mathbb{E}[f \mid e_k = 0]\mathbb{E}[g \mid e_k = 0]\mathbb{P}[e_k = 0] \\
&\quad + \mathbb{E}[f \mid e_k = 1]\mathbb{E}[g \mid e_k = 1]\mathbb{P}[e_k = 1], \\
\mathbb{E}[f] &= \mathbb{E}[f \mid e_k = 0]\mathbb{P}[e_k = 0] + \mathbb{E}[f \mid e_k = 1]\mathbb{P}[e_k = 1], \\
\mathbb{E}[g] &= \mathbb{E}[g \mid e_k = 0]\mathbb{P}[e_k = 0] + \mathbb{E}[g \mid e_k = 1]\mathbb{P}[e_k = 1].
\end{aligned}
$$

Note that $\mathbb{E}[f \mid \omega(e_k)]$, $\mathbb{E}[g \mid \omega(e_k)] : \{0, 1\} \to \mathbb{R}$ are two increasing functions (why?), thus

$$
\begin{aligned}
&\mathbb{E}[fg] - \mathbb{E}[f]\mathbb{E}[g] \\
&\geq \big(\mathbb{E}[f \mid e_k = 1] - \mathbb{E}[f \mid e_k = 0]\big)\big(\mathbb{E}[g \mid e_k = 1] - \mathbb{E}[g \mid e_k = 0]\big)\mathbb{P}[e_1 = 1]\mathbb{P}[e_1 = 0] \\
&\geq 0.
\end{aligned}
$$

Thus by Math Induction, we have FKG inequality holds for any $f, g : \Omega \to \mathbb{R}$ be increasing functions depending on finitely many edges.

Exercise 1 (The square-root trick) Given n increasing events $A_1, \cdots, A_n$, show that

$$
\max_{1 \leq i \leq n} \{\mathbb{P}(A_i)\} \geq 1 - \left[1 - \mathbb{P}(A_1 \cup \cdots \cup A_n)\right]^{\frac{1}{n}}
$$

$\boxed{\text{soL:}}$ Note that $A_1^{\mathsf{C}}, \cdots, A_n^{\mathsf{C}}$ are decreasing events, and thus also has FKG inequality.

$$
\begin{aligned}
1 - \left[1 - \mathbb{P}(A_1 \cup \cdots \cup A_n)\right]^{\frac{1}{n}} &= 1 - \left[\mathbb{P}(A_1^{\mathsf{C}} \cap \cdots \cap A_n^{\mathsf{C}})\right]^{\frac{1}{n}} \\
(\text{FKG inequality}) &\leq 1 - \left(\prod_{k=1}^{n} \mathbb{P}(A_k^{\mathsf{C}})\right)^{\frac{1}{n}} \\
&\leq 1 - \left(\prod_{k=1}^{n} \min_{1 \leq i \leq n}\{\mathbb{P}(A_i^{\mathsf{C}})\}\right)^{\frac{1}{n}} \\
&= 1 - \min_{1 \leq i \leq n}\{\mathbb{P}(A_i^{\mathsf{C}})\} \\
&= \max_{1 \leq i \leq n}\{1 - \mathbb{P}(A_i^{\mathsf{C}})\} \\
&= \max_{1 \leq i \leq n}\{\mathbb{P}(A_i)\}
\end{aligned}
$$

Application Show that $\theta(\frac{1}{2}) = 0$ on $\mathbb{Z}^2$, which implies that $p_c(\mathbb{Z}^2) \geq \frac{1}{2}$.
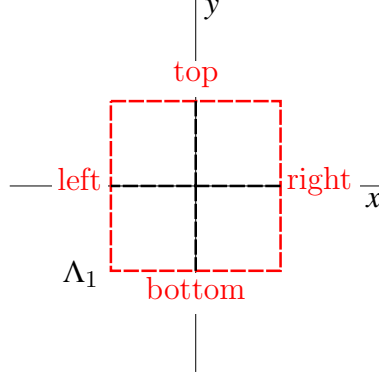
Proof. Assume that $\theta(\frac{1}{2}) > 0$, then we can know that $\mathbb{P}_{\frac{1}{2}}(\exists \text{ an infinite cluster }) = 1$. Define

$$
B_n := [\partial \Lambda_n \leftrightarrow \infty] = \bigcup_{x \in \partial \Lambda_n} [x \leftrightarrow \infty], \ n \in \mathbb{N},
$$

because of

$$[\exists \text{ an infinite cluster}] = \bigcup_{x \in \mathbb{Z}^2} [x \leftrightarrow \infty]$$

$$= \bigcup_{n \in \mathbb{N}} \bigcup_{x \in \partial \Lambda_n} [x \leftrightarrow \infty] = \lim_{n \to \infty} [\partial \Lambda_n \leftrightarrow \infty],$$

Thus $\lim_{n \to \infty} \mathbb{P}_{\frac{1}{2}} [\partial \Lambda_n \leftrightarrow \infty] = 1$.

Note that $B_n = [\exists \text{ an infinite SAW in } \mathbb{Z}^2 \backslash \Lambda_n \text{ starting in } \partial \Lambda_n]$, define

$$B_n^{\text{top}} = [\exists \text{ an infinite SAW in } \mathbb{Z}^2 \backslash \Lambda_n \text{ starting in } \partial \Lambda_n^{\text{top}}]$$
$$B_n^{\text{bottom}} = [\exists \text{ an infinite SAW in } \mathbb{Z}^2 \backslash \Lambda_n \text{ starting in } \partial \Lambda_n^{\text{bottom}}]$$
$$B_n^{\text{left}} = [\exists \text{ an infinite SAW in } \mathbb{Z}^2 \backslash \Lambda_n \text{ starting in } \partial \Lambda_n^{\text{left}}]$$
$$B_n^{\text{right}} = [\exists \text{ an infinite SAW in } \mathbb{Z}^2 \backslash \Lambda_n \text{ starting in } \partial \Lambda_n^{\text{right}}].$$
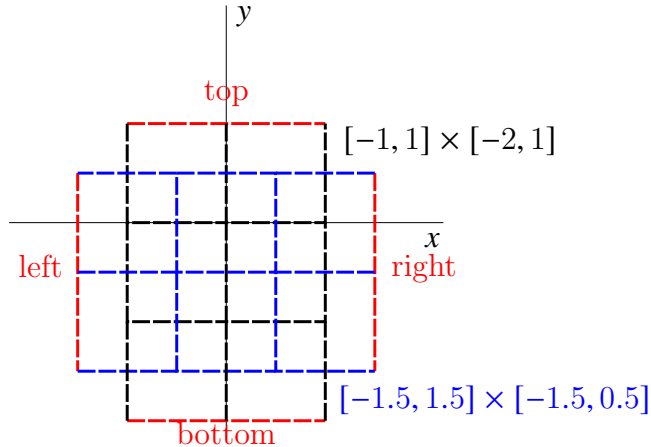
Then

$$B_n = B_n^{\text{top}} \cup B_n^{\text{bottom}} \cup B_n^{\text{left}} \cup B_n^{\text{right}}$$

It's no doubt that for all $n \in \mathbb{N}$, B_n^{top}, B_n^{bottom}, B_n^{left}, B_n^{right} are increasing events, thus by square-root trick and the fact that $\lim_{n \to \infty} \mathbb{P}_{\frac{1}{2}} B_n = 1$, we got that

$$\lim_{n \to \infty} \mathbb{P}_{\frac{1}{2}}(B_n^{\text{top}}) \geq \lim_{n \to \infty} 1 - \left[1 - \mathbb{P}_{\frac{1}{2}}(B_n)\right]^{\frac{1}{4}} = 1$$

Note that the probability of B_n^{top} remains the same for any "shift" of this event.

Now, we consider

$$C_n^{\text{top}} = B_n^{\text{top}}$$
$$C_n^{\text{bottom}} = \{\omega \subseteq \mathbb{Z}^2 \mid \omega + (0,1) \in B_n^{\text{bottom}}\}$$
$$C_n^{\text{left}} = \{\omega \subseteq \mathbb{Z}^2 \mid \omega^* + (0.5, 0.5) \in B_n^{\text{left}}\}$$
$$C_n^{\text{right}} = \{\omega \subseteq \mathbb{Z}^2 \mid \omega^* + (-0.5, 0.5) \in B_n^{\text{right}}\}$$

then

$$\lim_{n \to \infty} \mathbb{P}_{\frac{1}{2}}(C_n^{\text{top}}) = \lim_{n \to \infty} \mathbb{P}_{\frac{1}{2}}(C_n^{\text{bottom}}) = \lim_{n \to \infty} \mathbb{P}_{\frac{1}{2}}(C_n^{\text{left}}) = \lim_{n \to \infty} \mathbb{P}_{\frac{1}{2}}(C_n^{\text{right}}) = 1,$$

define $C_n = C_n^{\text{top}} \cap C_n^{\text{bottom}} \cap C_n^{\text{left}} \cap C_n^{\text{right}}$, then

$$\lim_{n \to \infty} \mathbb{P}_{\frac{1}{2}}(C_n) = 1.$$

Note that C_n does not depending on edges on $[-n, n] \times [-n-1, n]$, we take n be large enough such that $\mathbb{P}_{\frac{1}{2}}(C_n) > \frac{1}{2}$, let $C([-n, n] \times [-n-1, n])$ be the event that all edge in $[-n, n] \times [-n-1, n]$ are closed, then

$$\mathbb{P}_{\frac{1}{2}}(N \geq 2) \geq \mathbb{P}_{\frac{1}{2}}\big(C_n \cap C([-n, n] \times [-n-1, n])\big)$$
$$\geq \frac{1}{2} \times \frac{1}{2^{8n^2 + 8n + 1}} > 0.$$

Which contradicts that $\mathbb{P}_{\frac{1}{2}}(N \geq 2) = 0$ on $\mathbb{Z}^2$. $\qquad\qquad\square$

Exercise 2 Show that $p_c \geq \frac{1}{2}$ for the site percolation on the triangular lattice.

$\boxed{\text{sol:}}$ We show that $\theta(\frac{1}{2}) = 0$ to conclude this.

Note that we can use similar way of proving that $\mathbb{P}_p(N = 1) = 1$, $\forall p > p_c$ in $\mathbb{Z}^d$ to say that it is also true in triangular lattice.

We suppose that $\theta(\frac{1}{2}) > 0$, then $\frac{1}{2} > p_c(\triangle)$, thus $\mathbb{P}_{\frac{1}{2}}[\exists \text{ an infinite cluster}] = 1$, note that

$$[\exists \text{ an infinite cluster}] = \bigcup_{x \in \triangle}[x \leftrightarrow \infty]$$
$$= \bigcup_{n \in \mathbb{N}} \bigcup_{x \in \partial\Lambda_n}[x \leftrightarrow \infty] = \lim_{n \to \infty}[\partial\Lambda_n \leftrightarrow \infty],$$

In this case, $\Lambda_n = \{x \in G \mid \exists \text{ SAW of length} \leq n \text{ that connected form } 0 \text{ to } x\}$, where the length of SAWs is allowed to be 0. And, $\partial\Lambda_n = \Lambda_n \backslash \Lambda_{n-1}$, where $\Lambda_{-1} = \varnothing$.

We still define

$$B_n := [\partial\Lambda_n \leftrightarrow \infty] = [\exists \text{ an infinite SAW in } \triangle\backslash\Lambda_n \text{ starting in } \partial\Lambda_n]$$

It is needed to mention that we consider a configuration ω to be a subset of $V(\triangle)$ instead of a subset of $E(\triangle)$. Define

$$B_n^{\text{t}} = [\exists \text{ an infinite SAW in } \mathbb{Z}^2\backslash\Lambda_n \text{ starting in } \partial\Lambda_n^{\text{top}}]$$
$$B_n^{\text{b}} = [\exists \text{ an infinite SAW in } \mathbb{Z}^2\backslash\Lambda_n \text{ starting in } \partial\Lambda_n^{\text{bottom}}]$$
$$B_n^{\text{lh}} = [\exists \text{ an infinite SAW in } \mathbb{Z}^2\backslash\Lambda_n \text{ starting in } \partial\Lambda_n^{\text{left hand}}]$$
$$B_n^{\text{lf}} = [\exists \text{ an infinite SAW in } \mathbb{Z}^2\backslash\Lambda_n \text{ starting in } \partial\Lambda_n^{\text{left foot}}]$$
$$B_n^{\text{rh}} = [\exists \text{ an infinite SAW in } \mathbb{Z}^2\backslash\Lambda_n \text{ starting in } \partial\Lambda_n^{\text{right hand}}]$$
$$B_n^{\text{rf}} = [\exists \text{ an infinite SAW in } \mathbb{Z}^2\backslash\Lambda_n \text{ starting in } \partial\Lambda_n^{\text{right foot}}].$$
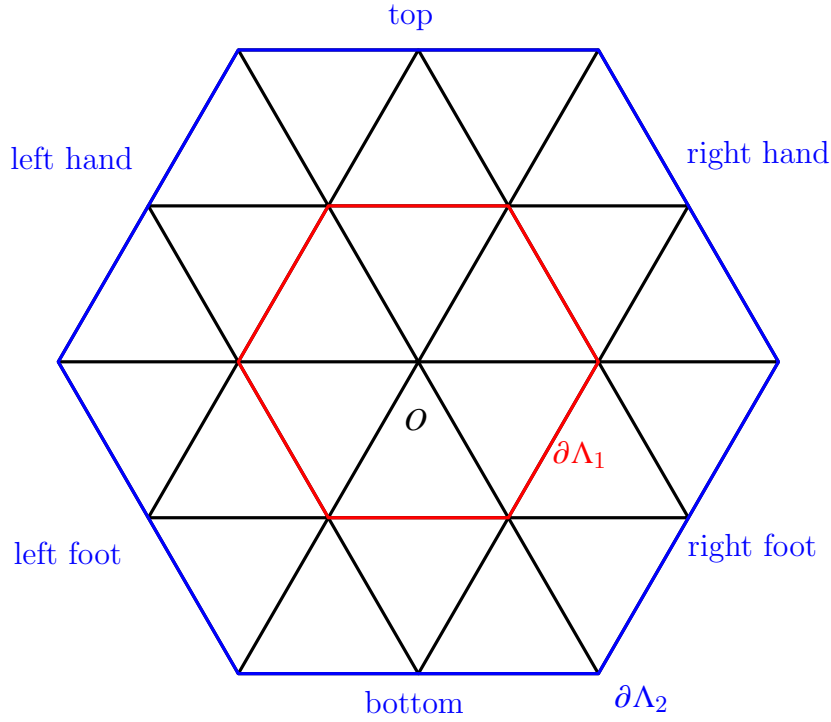
Then, $B_n = B_n^{\mathrm{t}} \cup B_n^{\mathrm{b}} \cup B_n^{\mathrm{lh}} \cup B_n^{\mathrm{lf}} \cup B_n^{\mathrm{rh}} \cup B_n^{\mathrm{rf}}$, because of all these events are increasing events, by square-root trick and $\lim_{n \to \infty} \mathbb{P}_{\frac{1}{2}}(B_n) = 1$, we got that

$$\lim_{n \to \infty} \mathbb{P}_{\frac{1}{2}}(B_n^{\mathrm{t}}) \geq \lim_{n \to \infty} 1 - \left[1 - \mathbb{P}_{\frac{1}{2}}(B_n)\right]^{\frac{1}{6}} = 1.$$

Given a configuration $\omega \subseteq \triangle$, we define $\omega^{\complement} = V(\triangle)\backslash\omega$, which is also a (site) configuration of $\triangle$. Thus for any given event A, we have $\mathbb{P}_p\{\omega \mid \omega \in A\} = \mathbb{P}_{1-p}\{\omega \mid \omega^{\complement} \in A\}$.

Because of $p = 1 - p$ when $p = \frac{1}{2}$, for convenience, we say $[\omega^{\complement} \in A] := \{\omega \mid \omega^{\complement} \in A\}$ for A be a event, then

$$\lim_{n \to \infty} \mathbb{P}_{\frac{1}{2}}([\omega \in B_n^{\mathrm{t}}] \cap [\omega^{\complement} \in B_n^{\mathrm{b}}] \cap [\omega^{\complement} \in B_n^{\mathrm{lh}}] \cap [\omega \in B_n^{\mathrm{lf}}] \cap [\omega^{\complement} \in B_n^{\mathrm{rh}}] \cap [\omega \in B_n^{\mathrm{rf}}]) = 1.$$

We define $C_n := [\omega \in B_n^{\mathrm{t}}] \cap \cdots \cap [\omega \in B_n^{\mathrm{rf}}]$ in the previous step, we can take large n such that $\mathbb{P}_{\frac{1}{2}}(C_n) \geq \frac{1}{2}$, we use $\ominus_n$ to denote the event that all sites in Λ_n are closed, note that C_n dose not depend on sites in Λ_{n-1}, thus

$$\mathbb{P}(N \geq 3) \geq \mathbb{P}(C_n \cap \ominus_{n-1}) = \frac{1}{2} \times \frac{1}{2^{1+3n(n-1)}} > 0,$$

which is a contradiction because of $\mathbb{P}_{\frac{1}{2}}(N = 1) = 1.$ ∎

Exercise 3 Assume that we have a Bernoulli percolation on a graph G with $N = 1$ a.s. or $N = \infty$ a.s. at some parameter p, show that

$$\inf_{x,y \in G} \mathbb{P}_p[x \leftrightarrow y] > 0 \Leftrightarrow N = 1 \text{ a.s.}$$

2.2 van den Berg-Kesten Inequality

2.3 Russo's Identity

We want to show study the behavior of $\mathbb{P}_p(A)$ as p varies (One can readily assume that A is increasing).

Definition An edge $e \in E$ is said to be pivotal for A and ω if $\omega_e \notin A$, $\quad omega^e \in A$, where ω_e / ω^e stand for the configuration with $\omega(e) = 1 / \omega(e) = 0$.

Remark $\omega(e)$ decides ω in A or not. The value of $\omega(e)$ **does not** is independent to the event "e is **pivotal** for A". We write $N(A) := N(A, \omega)$ for the number of pivotal edges for A and ω.

Proposition Russo's Identity

> If A is an increasing event depending on finitely many edges, then
>
> $$\frac{d}{dp} \mathbb{P}_p(A) = \mathbb{E}_p(N(A))$$

Exercise (Margulis-Russo formula) For a finite edge set E and **any** function $F : \{0, 1\}^E \to \{0, 1\}$, use a direct computation to show that

$$\frac{d}{dp} f(p) = \frac{1}{p(1-p)} \sum_{e \in E} \mathrm{Cov}(F, \omega_e), \qquad f(p) := \mathbb{E}_p[F]$$

Deduce Russo's identity from this.

sol: First, we deduce the given formula. Notice that $f(p) = \mathbb{E}_p[F] = \sum_\omega F(\omega) p^{|\omega_E|}(1 - p)^{|E| - |\omega_E|}$, where $|\omega_E| = \sum_{e \in E} \omega_e$.

Therefore,

$$f(p)' = \sum_\omega F(\omega) \left[|\omega_e| / p - (|E| - |\omega_e|)/(1 - p) \right]$$

$$= \frac{1}{p(1-p)} \sum_{e \in E} \mathbb{E}_p[F(\omega)(\omega_e - p)] \qquad \begin{array}{l} \text{Combine the terms and use} \\ |\omega_E| = \sum_{e \in E} \omega_e \text{ and } |E| = \\ \sum_{e \in E} 1 \end{array}$$

$$= \frac{1}{p(1-p)} \sum_{e \in E} \mathbb{E}_p[(F - \mathbb{E}_p[F])(\omega_e - p)] \qquad \text{We can add terms since } \mathbb{E}_p[c(\omega_e - p)] = 0$$

$$= \frac{1}{p(1-p)} \sum_{e \in E} \mathrm{Cov}(F, \omega_e)$$

Then, let's consider $F = \mathbb{1}_A$, where A is depending on edge set E and $|E| < \infty$. We can see that e must be **pivotal** for A to have influence. To be more specific,

$$\mathbb{E}_p[\mathbb{1}_{\mathrm{Piv}_e(A)^c}(\omega_e - p)F(\omega)] = \mathbb{E}_p[\omega_e - p]\mathbb{E}_p[F(\omega)\mathbb{1}_{\mathrm{Piv}_e(A)^c}] = 0$$

Thus, $\mathbb{E}_p[F(\omega)(\omega_e - p)] = \mathbb{E}_p[F(\omega)(\omega_e - p)\mathbb{1}_{\mathrm{Piv}_e(A)}]$

Now we consdier $f(p)'$

$$f(p)' = \frac{1}{p(1-p)} \sum_{e \in E} \mathbb{E}_p[F(\omega)(\omega_e - p)]$$

$$= \frac{1}{p(1-p)} \sum_{e \in E} \mathbb{E}_p[F(\omega)(\omega_e - p)\mathbb{1}_{\mathrm{Piv}_e(A)}]$$

$$= \frac{1}{p(1-p)} \sum_{e \in E} \mathbb{E}_p[\omega_e = 1]\mathbb{E}_p[F(\omega)(\omega_e - p)\mathbb{1}_{\mathrm{Piv}_e(A)}|\omega_e = 1]$$

$$+ \mathbb{E}_p[\omega_e = 0]\mathbb{E}_p[F(\omega)(\omega_e - p)\mathbb{1}_{\mathrm{Piv}_e(A)}|\omega_e = 0] \qquad$$

e is a pivot of A whether $\omega_e = 1$ or not given $\omega \in \mathrm{Piv}_e(A)$.

$$= \sum_{e \in E} \mathbb{E}_p[\mathbb{1}_{\mathrm{Piv}_e(A)}]$$

$$= \mathbb{E}_p[N(A)] \quad \blacksquare$$

Chapter 3

Exponential Decay

One of the reason we introduce this tool is to show that $p_c \leq \frac{1}{2}$ on $\mathbb{Z}^d$

3.1 Statement and Proof

For $d \geq 2$, there are two facts

1. For $p < p_c$, there exists $c = c(p) > 0$ s.t For all $n \geq 1$,

$$\mathbb{P}_p[0 \longleftrightarrow \partial \Lambda_n] \leq e^{-cn}$$

2. For $p > p_c$,

$$\mathbb{P}[0 \longleftrightarrow \infty] \geq \frac{p - p_c}{p(1 - p_c)}$$

Remark For the case $p < p_c$, we can think of the event $0 \longleftarrow \partial \Lambda_{nL}$. If the previous event happens, it implies that there must exsists a path from 0 to $\partial \Lambda_{(n-1)L}$. This is shown in Fig. 3.1. However, the inequalities may not holds.
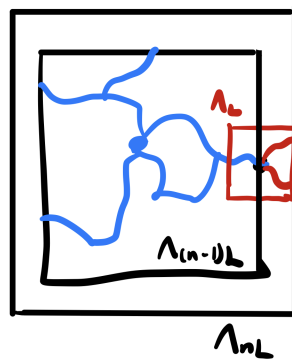

Figure 3.1: Intuitive understanding of $\mathbb{P}_p[0 \longleftrightarrow \Lambda_{nL}] \quad \leq \quad \mathbb{P}_p[0 \longleftrightarrow \Lambda_L]\mathbb{P}_p[0 \longleftrightarrow \Lambda_{(n-1)L}]$.

Exercise Can the decay rate be faster than exponential?

$\boxed{\text{s}_\text{o}\text{L:}}$ No. One direct argument is that the probability from $\Lambda_{(n-1)L}$ to Λ_{nL} will not be smaller than p^L. Hence, the decay rate decay at some exponential rates.

Proof. TBD. Check out the handout "USRP3" $\qquad\qquad \square$

Exercises For a planar periodic graph G and look at the bond percolation on G, show that $p_c(G) + p_c(G*) = 1$.

> $\boxed{\text{s}_\text{o}\text{L:}}$ TBD

Exercise (Adapted from exercise 31 in [1])

> $\boxed{\text{s}_\text{o}\text{L:}}$ TBD

3.2 Size of the Finite Cluster Containing the Origin

It is another events that exhibits exponential decay under subcritical p.

Exercise (Adapted from Exercise 25 in [1])

> $\boxed{\text{s}_\text{o}\text{L:}}$ TBD

3.3 Correlation Length

(Adapted from exercise 26 in [1]) With the help of sub-additivity lemma, we can actually get the rate of decay.

Let us carry on the study of $\mathbb{P}_p[0 \longleftarrow \partial\Lambda_n]$. Note that if $0 \longleftarrow \partial\Lambda_{(n+m)}$, then there exists $x \in \partial\Lambda_n$ s.t $0 \longleftarrow x$ and $x \longleftarrow x + \partial\Lambda_n$ holds disjointly. Therefore,

$$\mathbb{P}_p(0 \longleftarrow \partial\Lambda_{n+m}) \leq |\partial\Lambda_n|\mathbb{P}_p(0 \longleftarrow \partial\Lambda_n)\mathbb{P}_p(0 \longleftarrow \partial\Lambda_m), \qquad |\partial\Lambda_n| \sim 2d(2n+1)^{d-1} \qquad (3.1)$$

Exercise Given $p < p_c$, find a polynomial $f(n)$ s.t $(\ln f(n) + \ln\mathbb{P}_p(0 \longleftarrow \partial\Lambda_n))_{n\geq 1}$ satisfies the subadditive lemma. Then deduce that

$$\mathbb{P}_p[0 \longleftarrow \partial\Lambda_n] \geq \frac{1}{f(n)}\exp\left(-n/\varphi_1(p)\right), \quad \text{for some } \varphi_1(p) > 0$$

> $\boxed{\text{s}_\text{o}\text{L:}}$ W.L.O.G, we may assume that $n \geq m$ in the equation 3.1 Note that
>
> $$|\partial\Lambda_n| \leq C(3n)^d = f(n), \quad C > 0 \qquad (3.2)$$

Chapter 4

Russo-Seymour-Welsh Theorem

Bibliography

[1] Hugo Duminil-Copin. *Introduction to Bernoulli percolation.* 2018. URL: `https://www.ihes.fr/~duminil/publi/2017percolation.pdf`.