TOWARDS DENSITY OF SOSX IN SMSX

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I record some attempted solutions and possible avenues for proving density $SOSX \longrightarrow SMSX$.

Convolution Approach - One idea was to use the original idea from the Weierstrass Approximation Theorem (convolution with a polynomial kernel) to guarantee convexity. Convolution (over $\mathbb{R}^p$) with a positive kernel preserves convexity, while convolution with a polynomial kernel (over a compact set) yields a polynomial. The issue is that we cannot guarantee both of these simultaneously.

Our convex function f is defined on a compact set K. Shift and extend f to $K_1 \supset K$ such that $f = 0$ on K_1^c. Let $g(x)$ be a standard normal density in $\mathbb{R}^p$. It follows from a theorem in Prolla (1988) that the cone of psd polynomials is dense (in the uniform norm) in the cone of continuous psd functions on K. Then let $p_m \to g$ be such a sequence of psd polynomials.

The idea is to notice that

$$\int_{\mathbb{R}^p} f(x-t)p_m(t)dt = \int_{-K+x} f(x-t)p_m(t)dt = \int_K f(t)p_m(x-t)$$

Notice that the last expression is clearly a polynomial in x. We want to say that the first integral is convex (as a convolution of a convex function with a psd kernel p_m. We would actually want to build $p_m^\epsilon \to \frac{1}{\epsilon}g(x/\epsilon)$ uniformly on some set containing K, following the standard convolution argument.

In the context of the truncated convolution above, we get uniform convergence on any compact set contained in K. Extending to $K \subset\subset K_1$ gives uniform convergence on K.

The issue is that we need to do something to make the integral converge. Making $f = 0$ on K_1^c destroys convexity of the first expression, while if we trucate the polynomial outside of K, the last expression will no longer be a polynomial. An argument of this type seemed promising at first, but it is not clear if we can actually get something like this to work.

Gradient Approximation Approach - last time, we took an $SMSX$ function f with Hessian $H_f = L_f L_f^T$ and approximated with $L_f^m \to L_f$ in $\|\cdot\|_\infty$. This was supposed to give an approximation $p_m \to f$ such that all p_m are $SOSX$ polynomials. The issue is that our approximation H_f^m is not necessarily a Hessian. I can show that given a gradient $g = \nabla h$, under some regularity conditions there exists a polynomial gradient $\nabla p_m \to g$ uniformly on a compact set in the case $p = 2$.

The proof takes advantage of the fact that the "curl-free" conditions are very nice in 2-dimensions. It is not obvious how to extend this argument to the general case.