# Comprehensive Exam

## Injection and Measurement of Self-Consistent Beams

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# Outline

1. Background
    - Linear motion
    - Space charge
    - Self-consistent beams
    - Spallation Neutron Source (SNS)
2. Research plan
    - Objectives
    - Timeline

# Background

# What is accelerator physics?

“Accelerator physics is the field that deals with all the physics issues associated with the production of energetic particle beams.”

— Jeff Holmes, ORNL

**Applications**

- Particle physics
- Material science
- Nuclear physics
- Cancer treatment
- Isotope production
- Heavy ion fusion

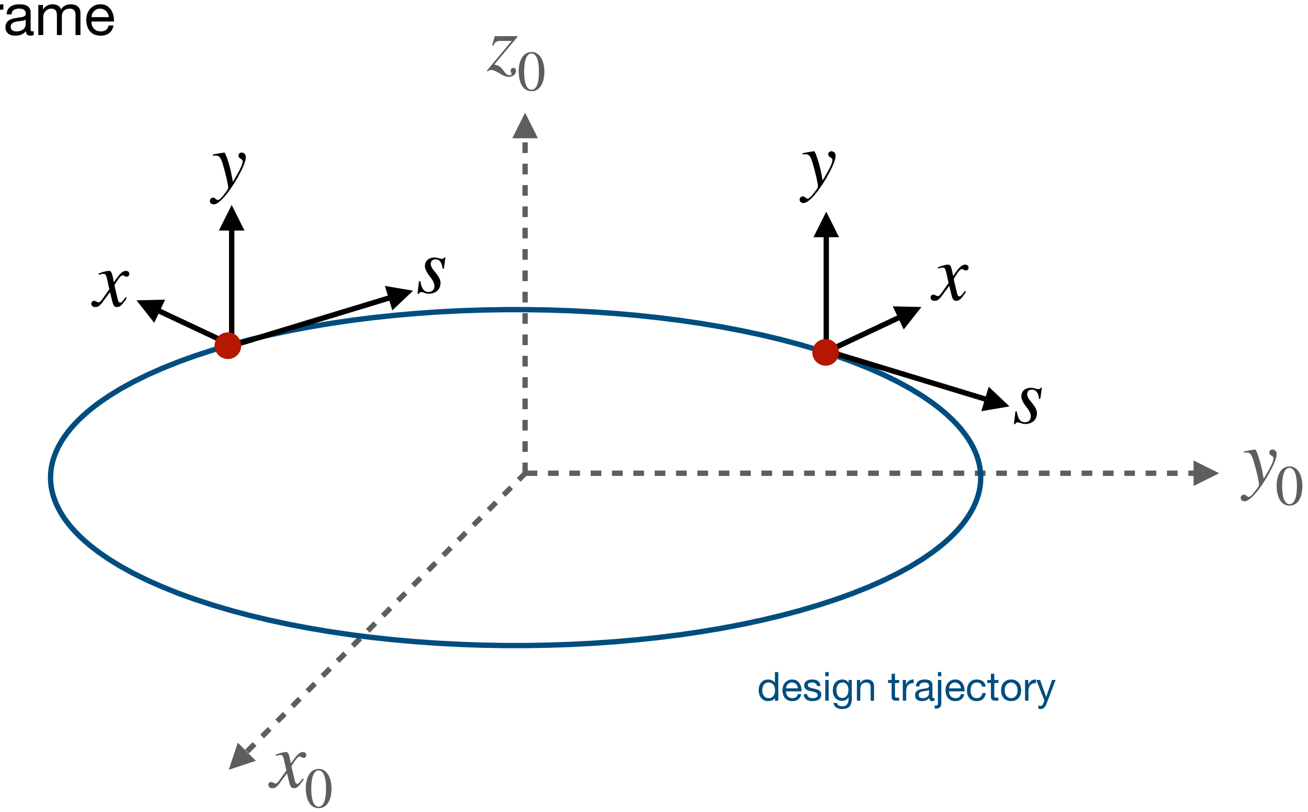
→

**Requirements**

- Energy
- Intensity
- Beam size
- Time structure
- Particle species
- Lifetime

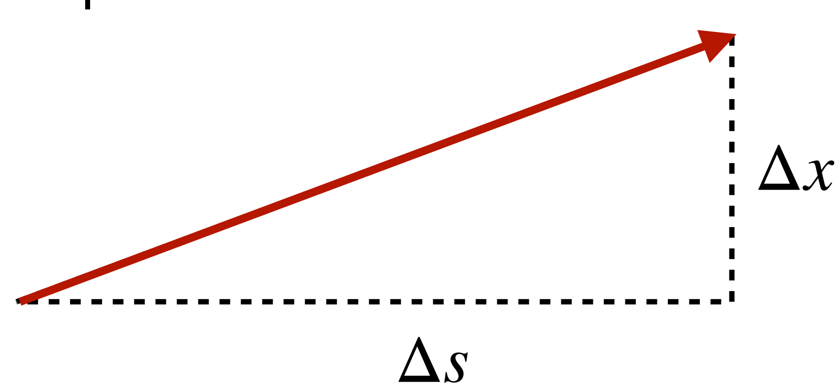
# Coordinate system

- Defined in particle's rest frame




- Assume no acceleration ( $v_s = constant$ )

- Replace time $t$ with position $s$

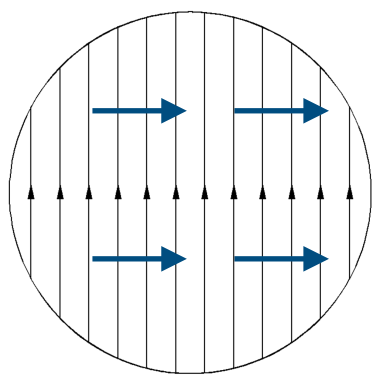



$$x' = \frac{dx}{ds} = \frac{1}{\beta c}\frac{dx}{dt} = slope$$

# Alternate gradient focusing

- **Lattice**: sequence of elements which act on beam

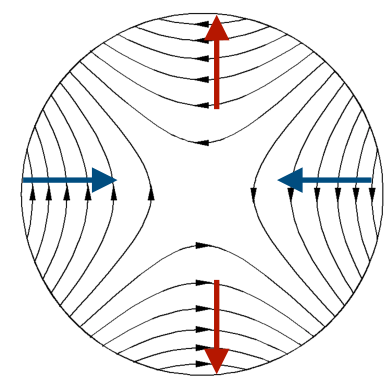
**Dipole** - steering



$$\vec{B} \propto \hat{y}$$
$$\vec{F} \propto \hat{x}$$

**Quadrupole** - focusing

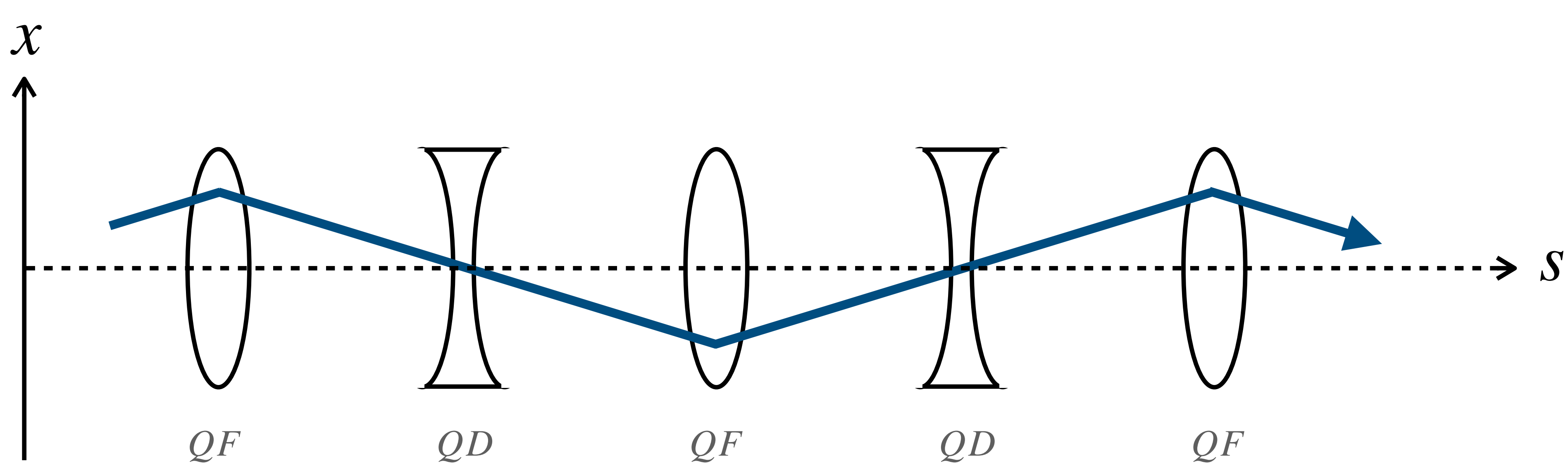
$\otimes \; \vec{v} = v\hat{s}$



$$\vec{B} \propto y\hat{x} + x\hat{y}$$
$$\vec{F} \propto -x\hat{x} + y\hat{y}$$

focusing defocusing

- Series of focusing/defocusing elements gives stable motion



# Hill's equation

- Single particle EOM in transverse plane ($u = x$ or $y$)

$$u'' + k_u(s)u = 0$$

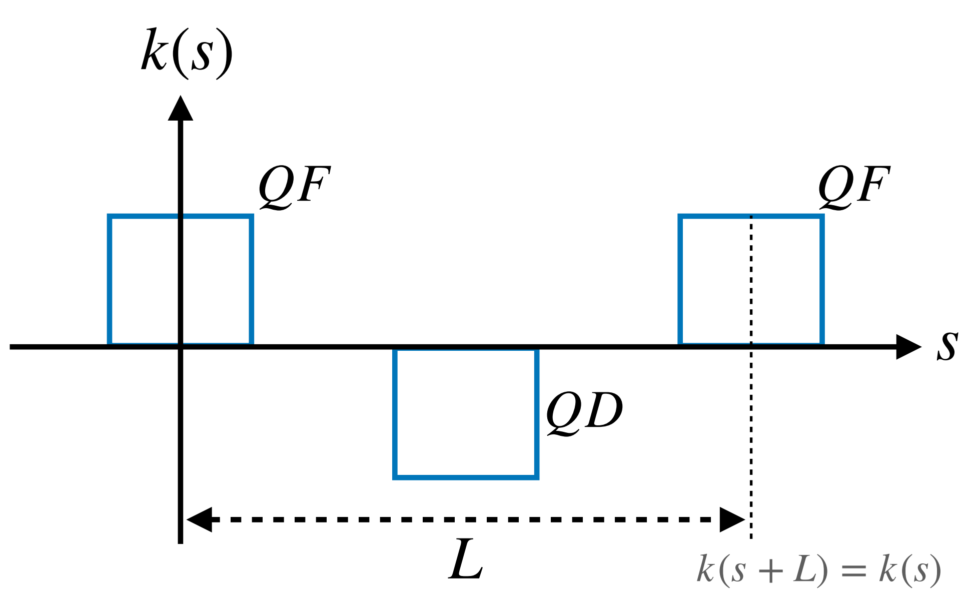
- Linear, periodic lattice: pseudo-harmonic motion

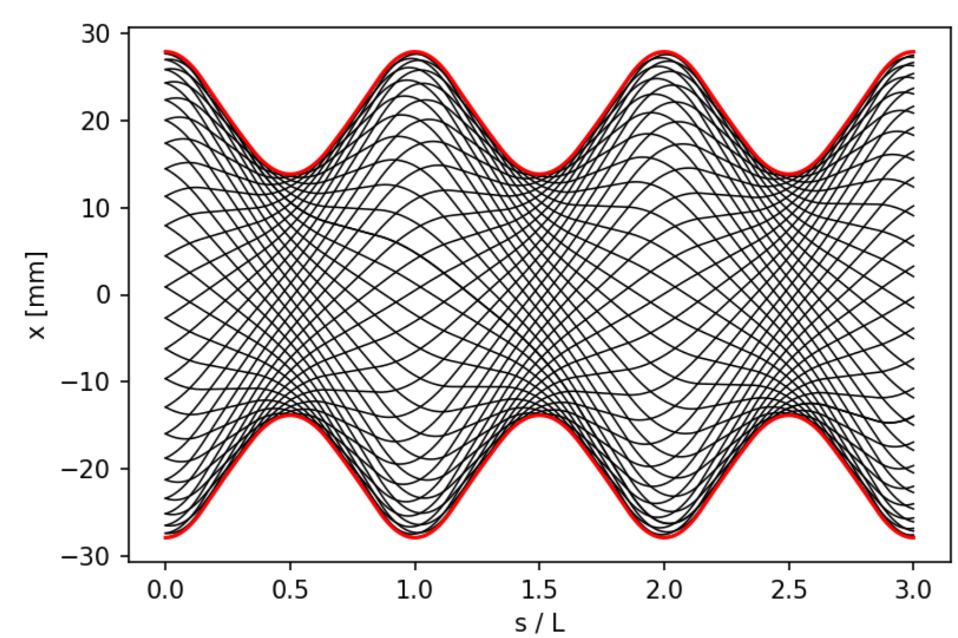
$$u(s) = \sqrt{I_u}\,w(s)\,\cos\left(\mu(s) + \delta\right)$$

Envelope: $w\,(s + L) = w\,(s)$

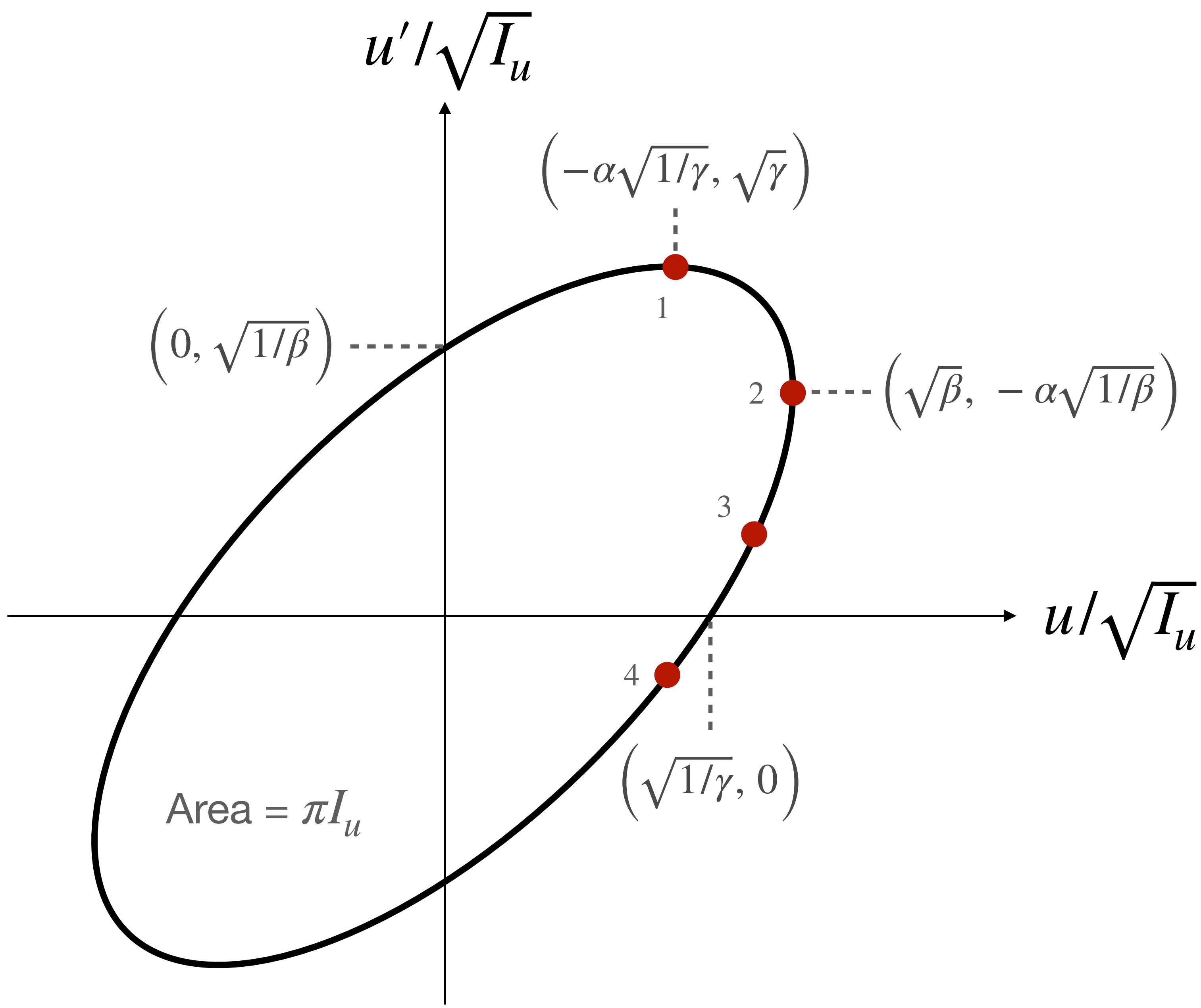
Phase: $\mu(s) = \int_{s_0}^{s} \frac{ds}{w^2}$

- Example: FODO cell



# Courant-Snyder parameterization




**Lattice parameters**

$$\beta = w^2$$

$$\alpha = -\frac{1}{2}\frac{d\beta}{ds}$$

$$\gamma = (1+\alpha^2)/\beta$$

**CS Invariant**

$$\gamma u^2 + 2\alpha u u' + \beta u'^2 = I_u$$

**Tune**

$$\nu = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}$$

# Statistical beam description

- **Covariance matrix** (measurable)

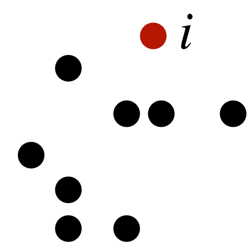
$$\Sigma = \begin{bmatrix} \langle x^2 \rangle & \langle xx' \rangle & \langle xy \rangle & \langle xy' \rangle \\ \langle xx' \rangle & \langle x'^2 \rangle & \langle x'y \rangle & \langle x'y' \rangle \\ \langle xy \rangle & \langle x'y \rangle & \langle y^2 \rangle & \langle yy' \rangle \\ \langle xy' \rangle & \langle x'y' \rangle & \langle yy' \rangle & \langle y'^2 \rangle \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy}^T & \sigma_{yy} \end{bmatrix}$$

- **Emittance** gives volume in phase space (up to a factor)

$$\varepsilon_{4D} = |\Sigma|^{1/2}$$ 4D emittance (invariant under linear forces)

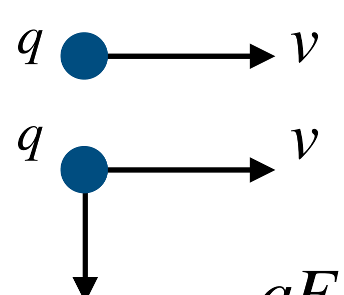
$$\varepsilon_u = |\sigma_{uu}|^{1/2}$$ 2D emittance (invariant under linear, uncoupled forces)

# Space charge




$$\vec{F}_i = \frac{1}{4\pi\varepsilon_0}\sum_{j\neq i}\frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|^2}$$

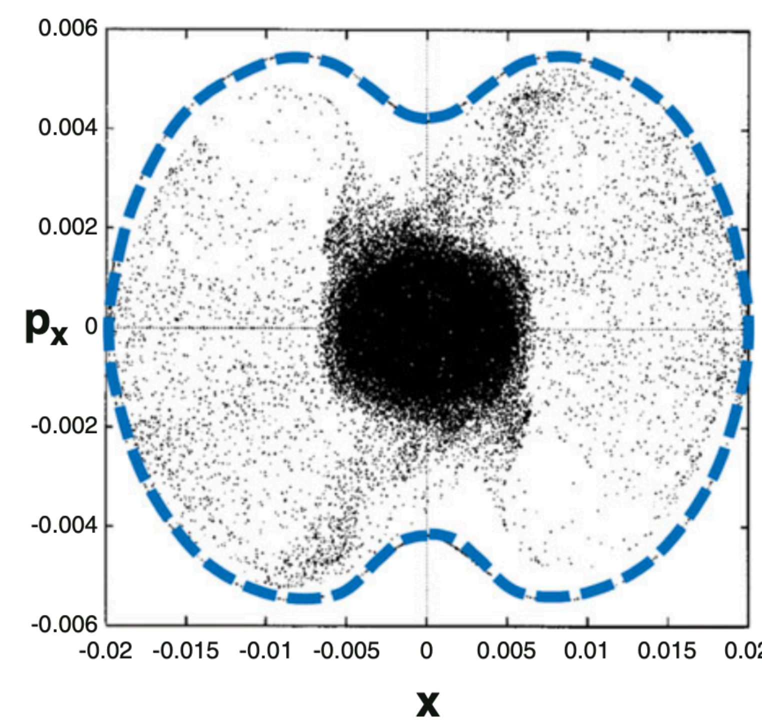
- Important for high intensity, low energy




$$F_{lab} = \frac{qE_{rest}}{\gamma}$$

perveance: $Q = \frac{2\lambda r_0}{\beta^2\gamma^3}$

- Can drive halo formation → beam loss

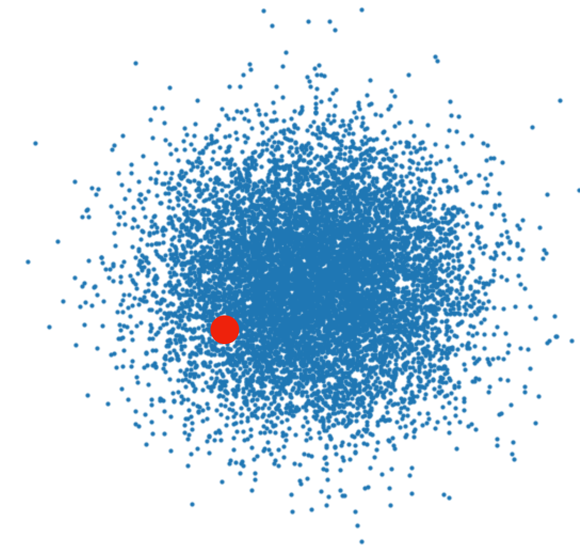



Hofmann, *Space Charge Physics for Particle Accelerators*

# Space charge: frozen beam

- Hill's equation modified by beam's electric field

$$\boxed{u'' + k_{0u}(s)\,u - \frac{e}{m\gamma^3 v_s^2}E_u = 0}$$



$$E_u = \sum_{n,m} c_{n,m}\, x^n y^m$$

- If fields are linear, focusing remains linear but reduced in strength

$$u'' + \underbrace{\left[k_{0u}(s) - h(s)\right]}_{k_{1u}(s)}\, u = 0$$

Linear space charge acts as defocusing quadrupole

- Single particle tune is decreased (**space charge tune shift**)

$$\nu_{1u} < \nu_{0u}$$

In general: different for each particle

# Space charge: full evolution

- *Self-consistent* motion

Particles move $\qquad$ New field generated

$\Delta \vec{x} \qquad \Delta \vec{E}$

- Distribution function

$$f\left(\vec{x}, s\right) dx\, dx'\, dy\, dy'$$

Number of particles per unit axial length at position $s$

- Vlasov equation

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial t} + \{H, f\} = 0}$$

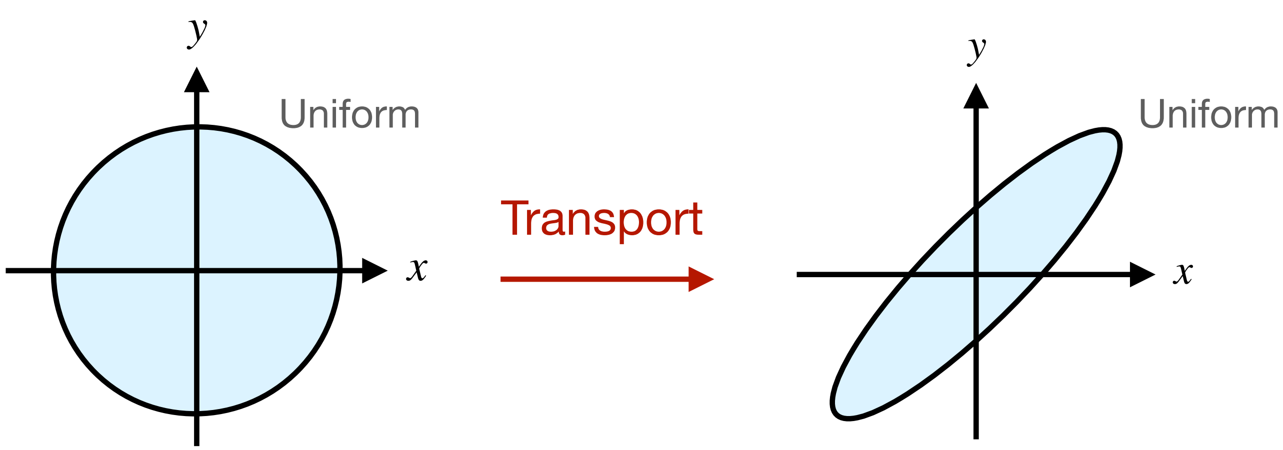
$$\{f, H\} = \frac{\partial f}{\partial \vec{x}} \cdot \frac{\partial H}{\partial \vec{p}} - \frac{\partial f}{\partial \vec{p}} \cdot \frac{\partial H}{\partial \vec{x}}$$

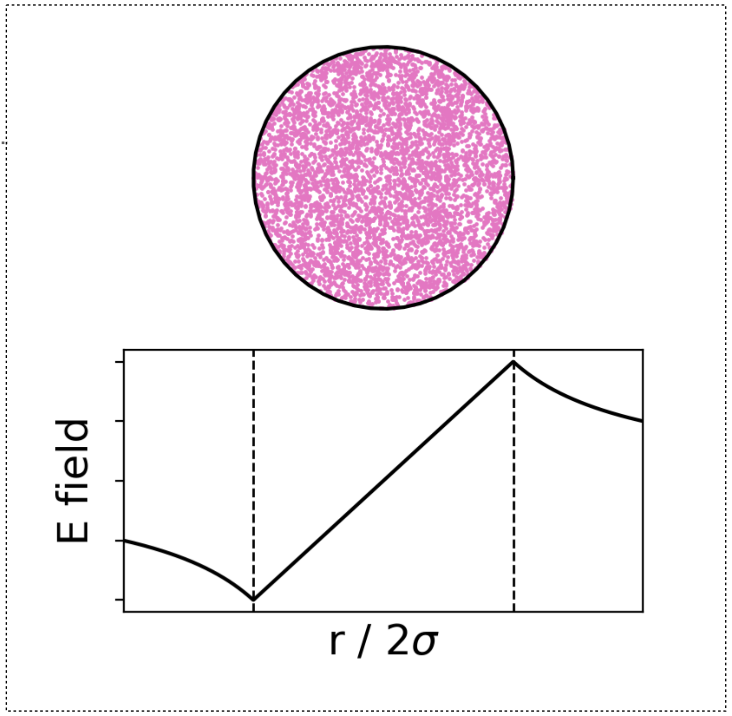
*Hamiltonian:* $H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}k_{0x}x^2 + \frac{1}{2}k_{0y}y^2 + \frac{q}{m\gamma^3 v_s^2}\phi$

*Space charge potential:* $\nabla^2 \phi = -\frac{q}{\epsilon_o}\int\int f\, dx' dy'$

# Self-consistent beams

- Definition
  - Uniform density ← → linear electric fields
  - Above point remains true under any linear transport (including self-fields)



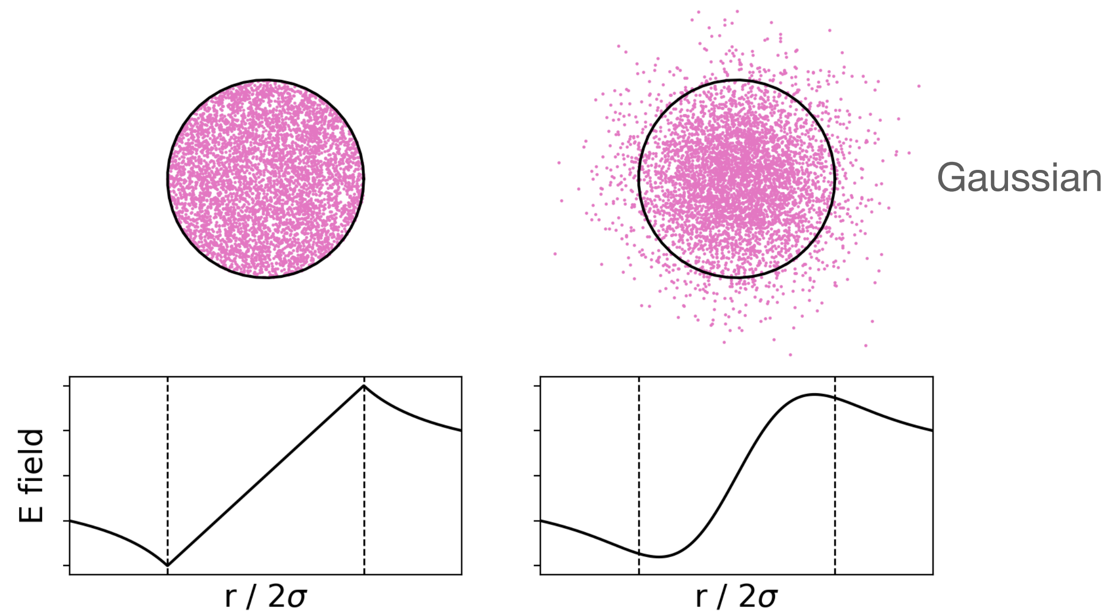

- Properties
  - Equilibrium solution of Vlasov equation
  - Function of invariants (Hamiltonian, angular momentum)

equivalent

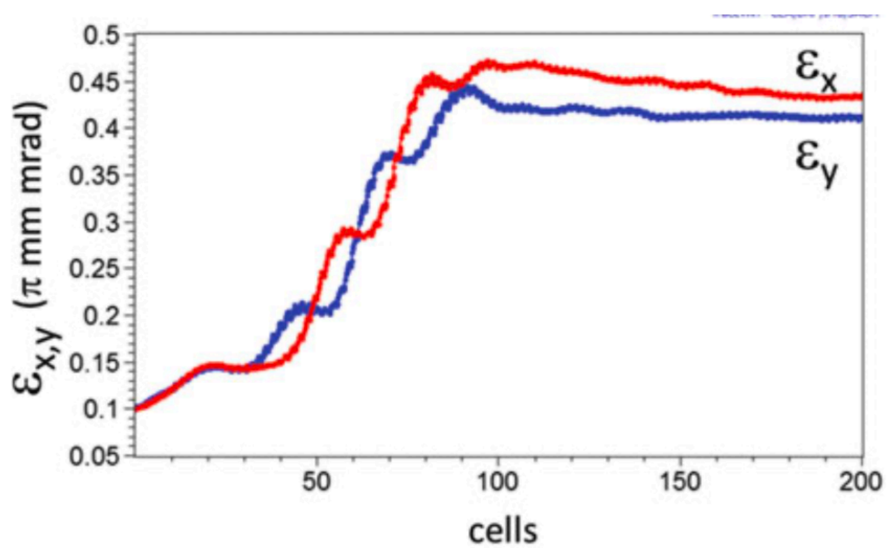
# Attractive properties of SC beams

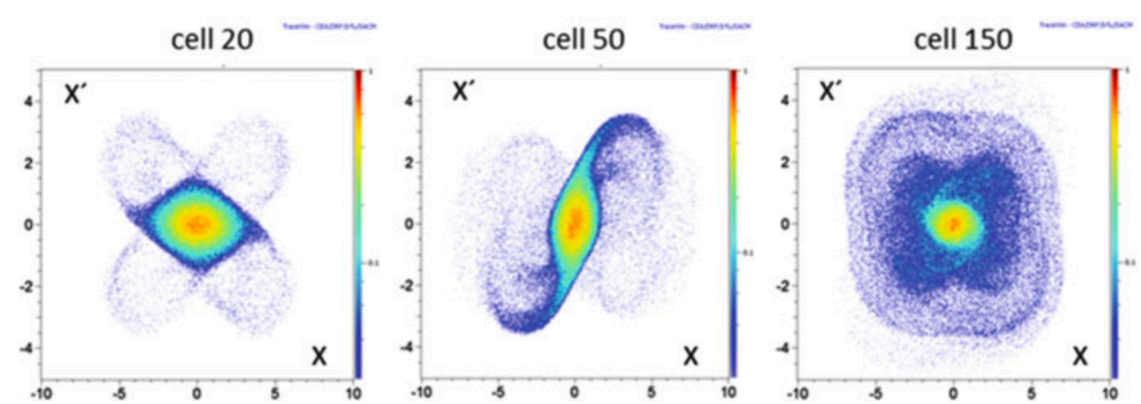
- Linear forces conserve emittance

$$\frac{d\varepsilon}{ds} = 0$$




- Nonlinear space charge forces lead to emittance growth

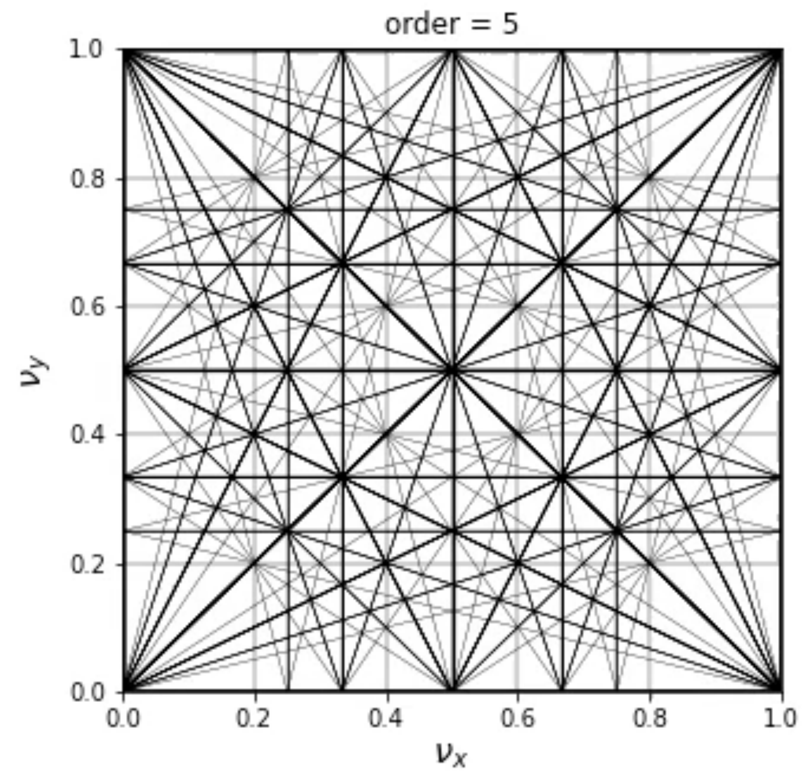



Hofmann, *Space Charge Physics for Particle*

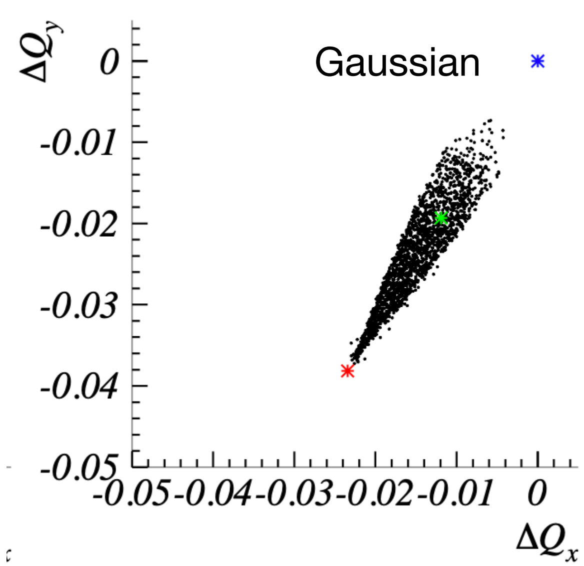
# Attractive properties of SC beams

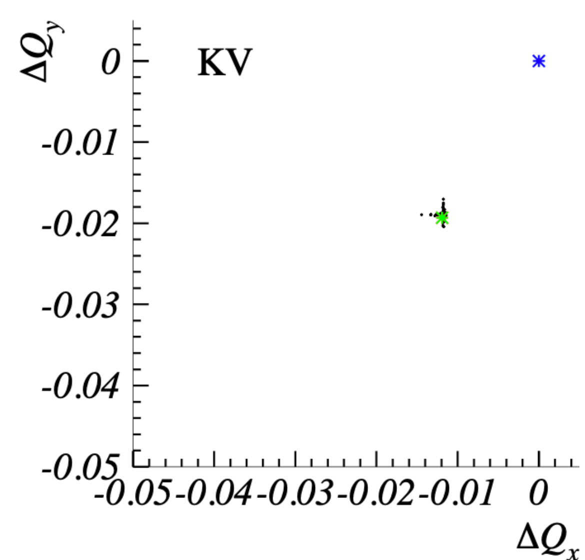
- Important to avoid resonance lines
  - Resonance condition: $m\nu_x + n\nu_y = p$
  - Order of resonance: $|m| + |n|$




- Uniform density beam minimizes tune spread

Tune spread limits beam intensity



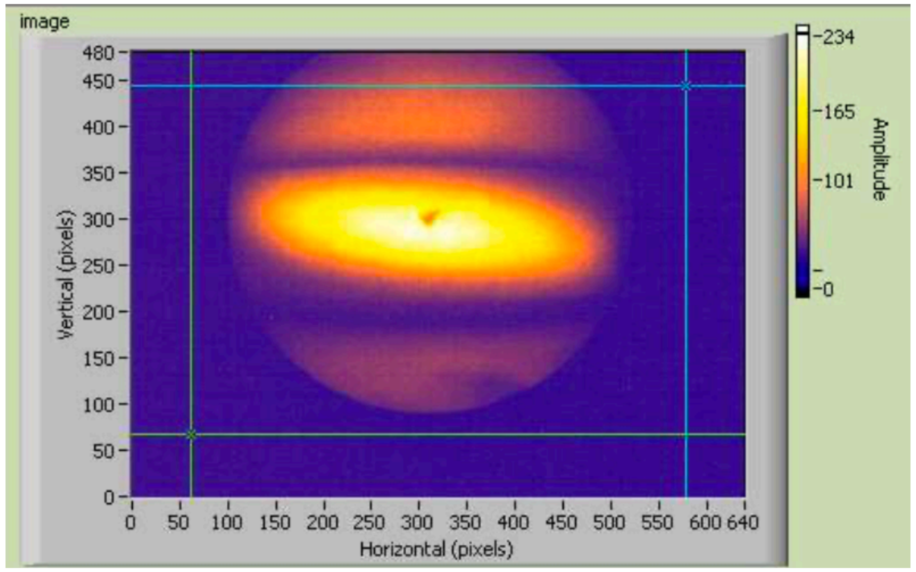

Maximum tune shift is minimized for SC beam

Higher possible beam intensity

Franchetti, *Space Charge in Circular Machines*

# Attractive properties of SC beams

- Uniformity of energy density on fixed target



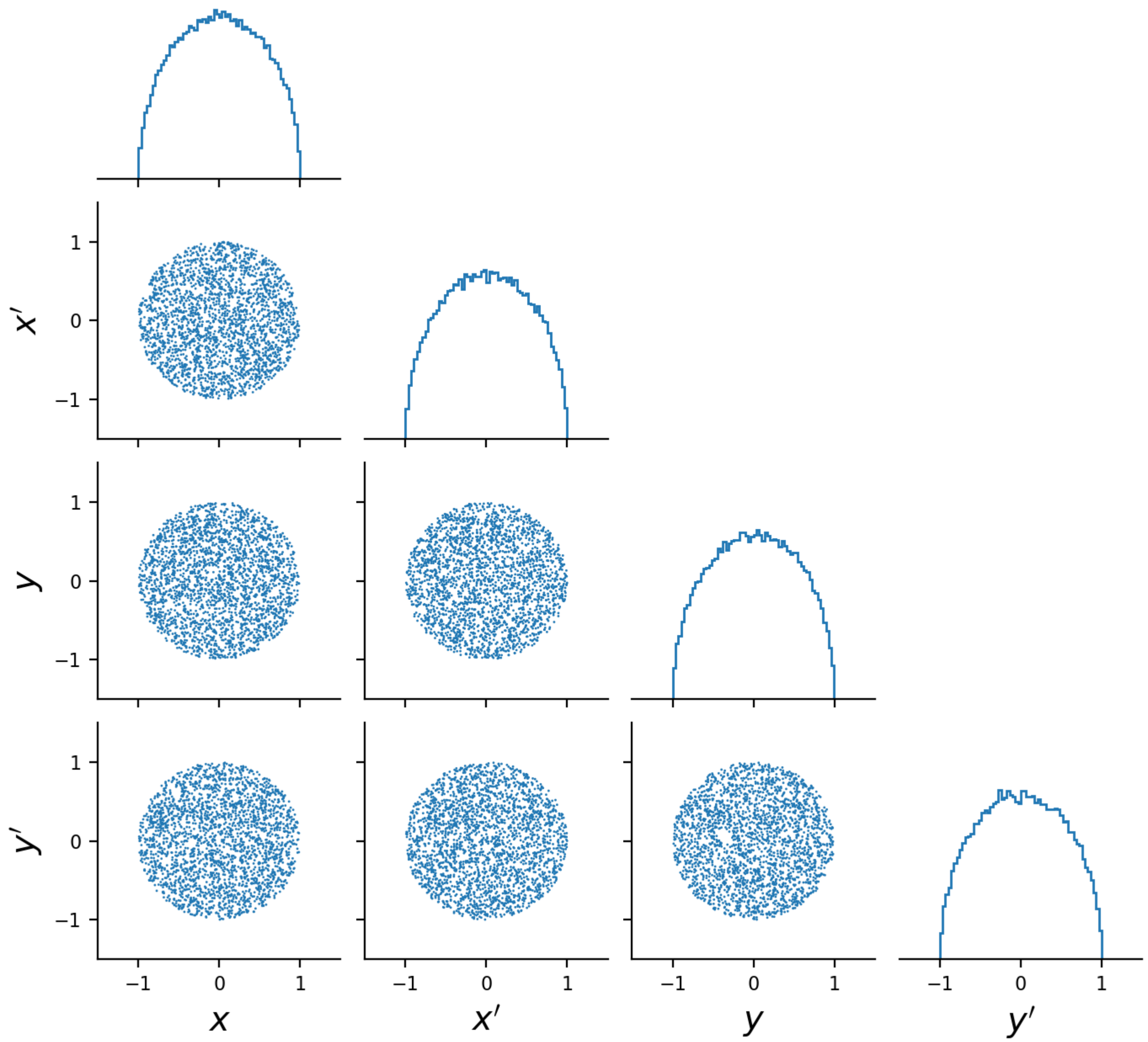

Beam distribution on target in Spallation Neutron Source (SNS)

- Next we discuss an example of a SC beam: the well-known Kapchinskij-Vladimirskij (KV) distribution

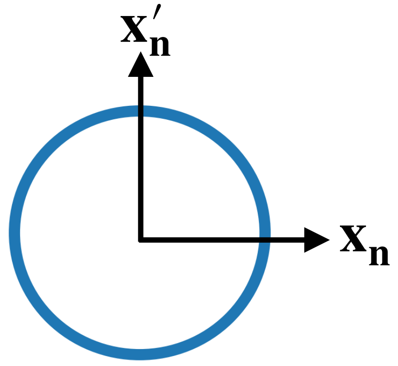
# KV distribution

$$\hat{f}\left(\overrightarrow{x}\right) = \delta\left(1 - \frac{I_x}{\varepsilon_x} - \frac{I_y}{\varepsilon_y}\right)$$

*2D beam: infinite length, uniform density along $s$




- Discovered in 1959
- Only known SC beam* for $s$-dependent focusing (until recently)
- Particles fill boundary of 4D ellipsoid

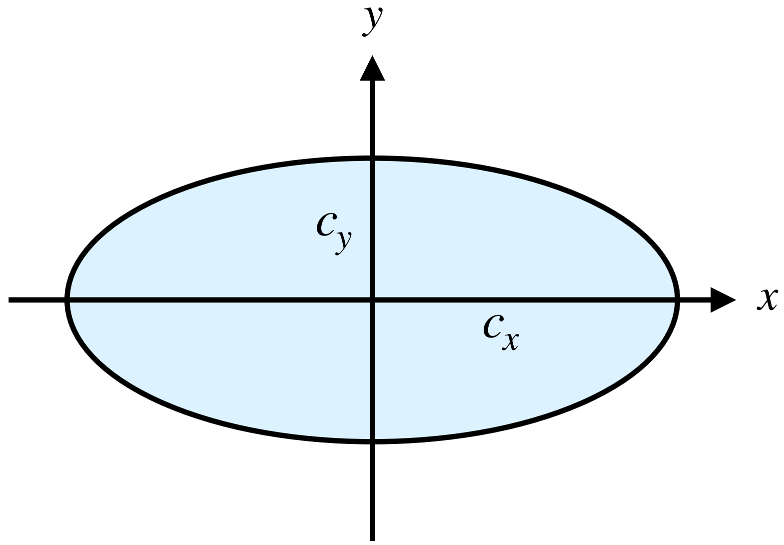



- Every 2D projection is uniform density ellipse
- Does not exist in 3D

# KV envelope equations

- $E$ field in uniform density, upright ellipse

$$\vec{E}(x,y) = \frac{\lambda}{\pi\epsilon_0}\left[\frac{x}{c_x(c_x+c_y)}\hat{x} + \frac{y}{c_y(c_x+c_y)}\hat{y}\right]$$




- 2 equations needed to track envelope

$$\tilde{x}'' + k_{0x}(s)\tilde{x} - \frac{\varepsilon_x^2}{\tilde{x}^3} - \frac{Q}{2\left(\tilde{x}+\tilde{y}\right)} = 0$$

$$\tilde{y}'' + k_{0y}(s)\tilde{y} - \frac{\varepsilon_y^2}{\tilde{y}^3} - \frac{Q}{2\left(\tilde{x}+\tilde{y}\right)} = 0$$

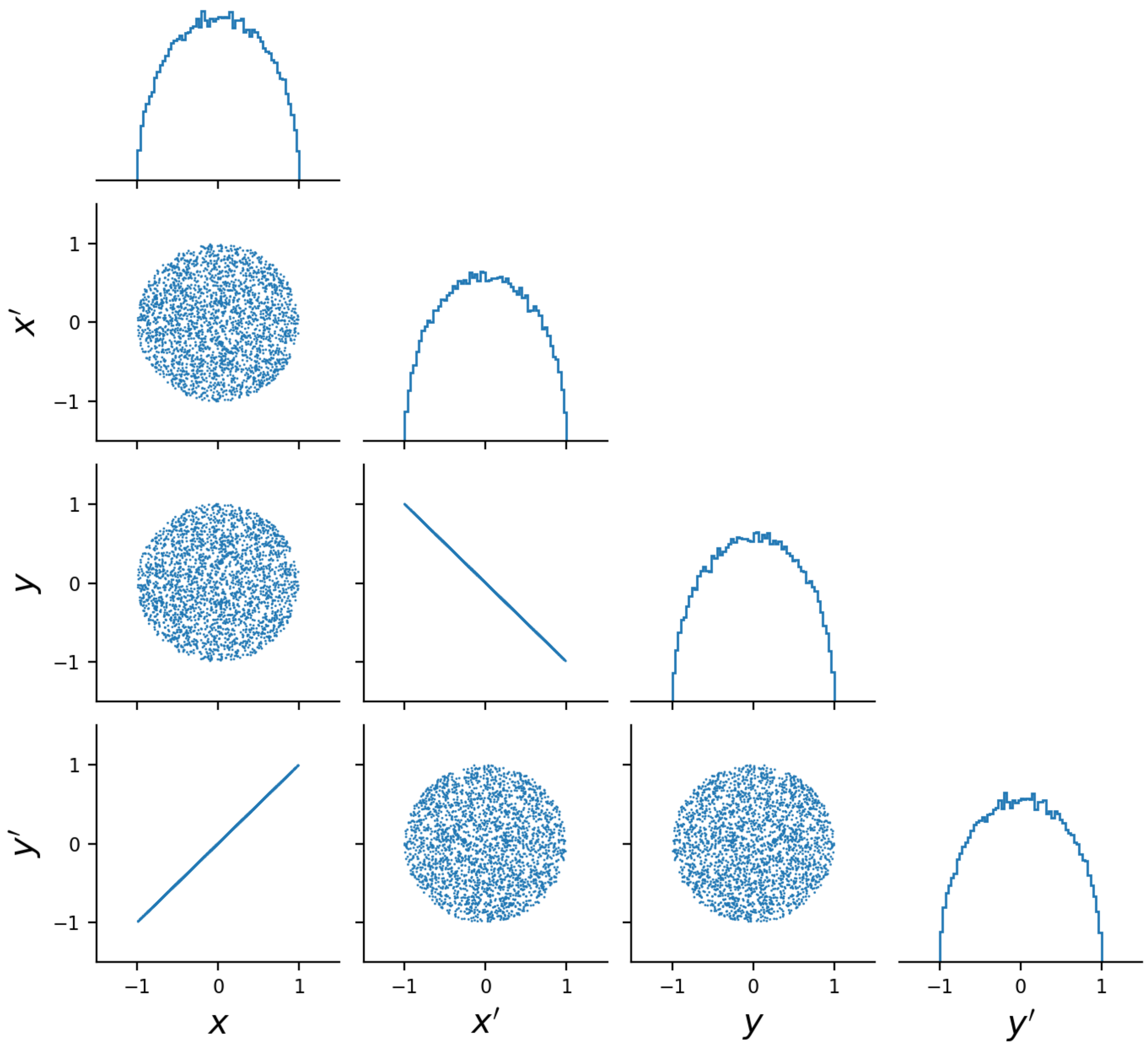
r.m.s. size: $\tilde{u} = \sqrt{\langle u^2 \rangle} = \frac{c_u}{2}$

perveance: $Q = \frac{2\lambda r_0}{\beta_s^2\gamma_s^3}$

- Extensively used/studied to understand low-order space charge effects

# Rotating distribution

$$\hat{f}\left(\overrightarrow{x}\right) = \delta\left(x' - e_{11}x - e_{12}y\right)\delta\left(y' - e_{21}x - e_{22}y\right)$$




$e_{21} = -e_{12} = 1, \quad e_{11} = e_{22} = 0$

- Every 2D projection is uniform density ellipse
- Can be tilted in real space
- Zero 4D emittance

$$\varepsilon_{4D} = \varepsilon_1\varepsilon_2 = 0$$

$$\varepsilon_1 = \varepsilon_x + \varepsilon_y$$

$$\varepsilon_2 = 0$$

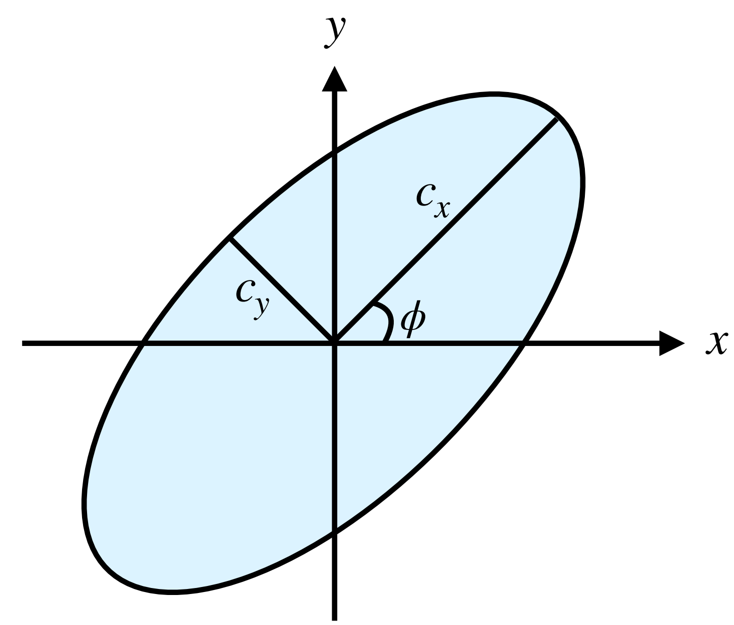
- Possibility to paint in the SNS

# Rotating envelope equations

- $E$ field in uniform density, tilted ellipse

$$E_x \propto \left[\frac{\cos^2\phi}{c_x(c_x+c_y)} + \frac{\sin^2\phi}{c_y(c_x+c_y)}\right] x + \sin\phi\cos\phi\left[\frac{1}{c_y(c_x+c_y)} - \frac{1}{c_x(c_x+c_y)}\right] y$$

$$E_y \propto \left[\frac{\cos^2\phi}{c_y(c_x+c_y)} + \frac{\sin^2\phi}{c_x(c_x+c_y)}\right] y + \sin\phi\cos\phi\left[\frac{1}{c_y(c_x+c_y)} - \frac{1}{c_x(c_x+c_y)}\right] x$$




- 4 equations needed to track envelope

$$\boxed{w'' + (\kappa_0 + \kappa_{sc})w = 0}$$

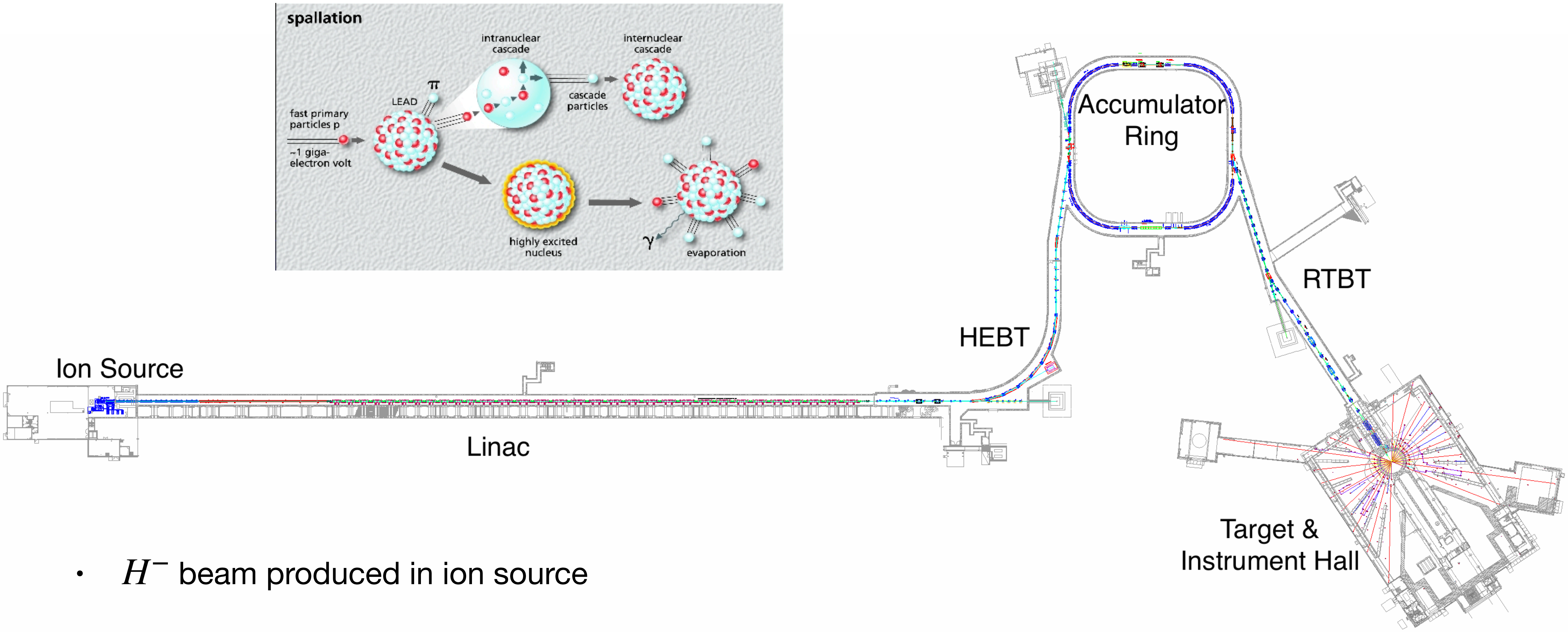
$$w = \begin{bmatrix} w_1 & w_2 \\ w_3 & w_4 \end{bmatrix}$$

$\kappa_0 \rightarrow$ lattice

$\kappa_{sc} \rightarrow$ space charge

$$x = w_1\cos\psi + w_2\sin\psi$$
$$y = w_3\cos\psi + w_4\sin\psi \qquad \cdots \qquad 0 \le \psi \le 2\pi$$

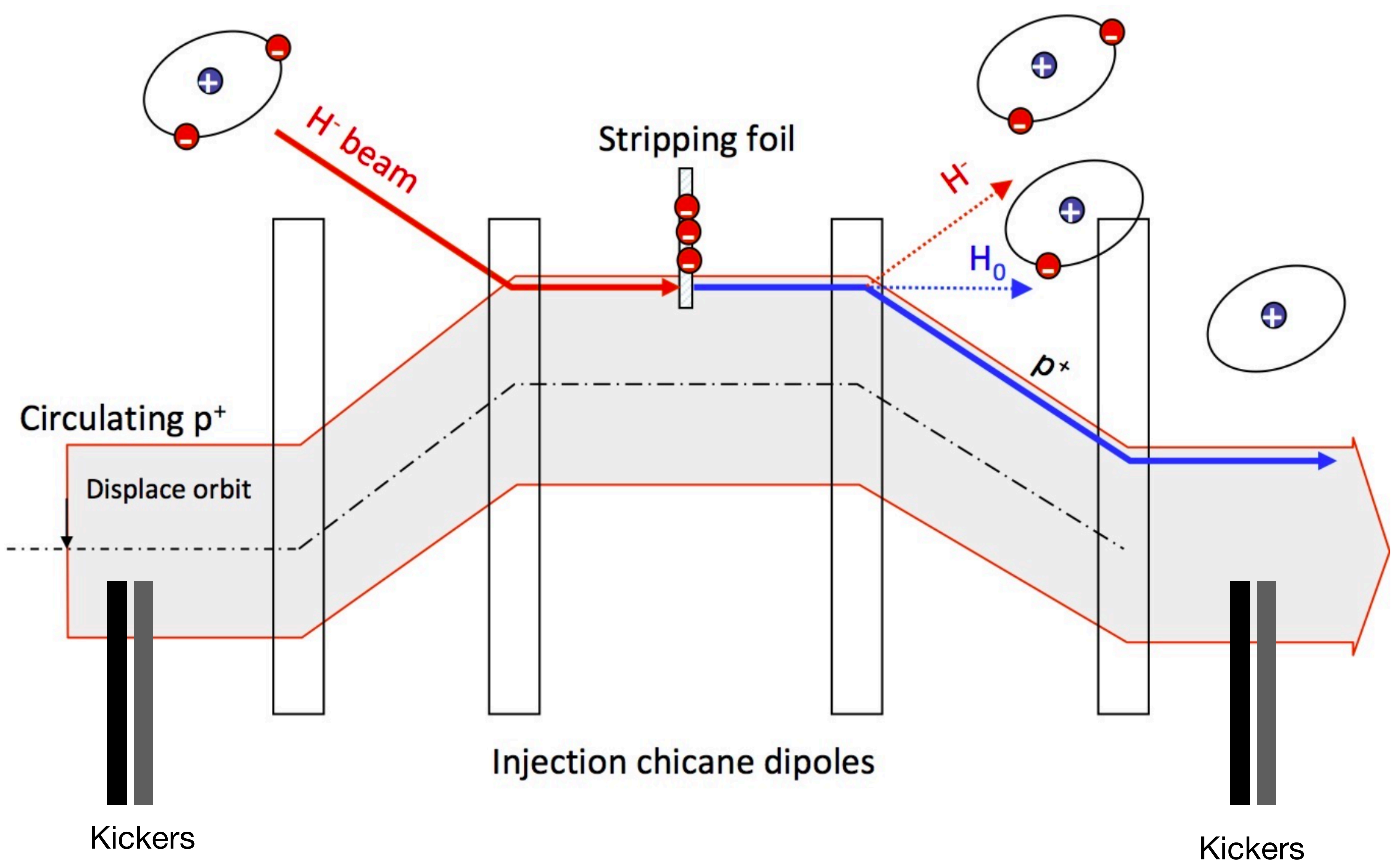
# SNS overview

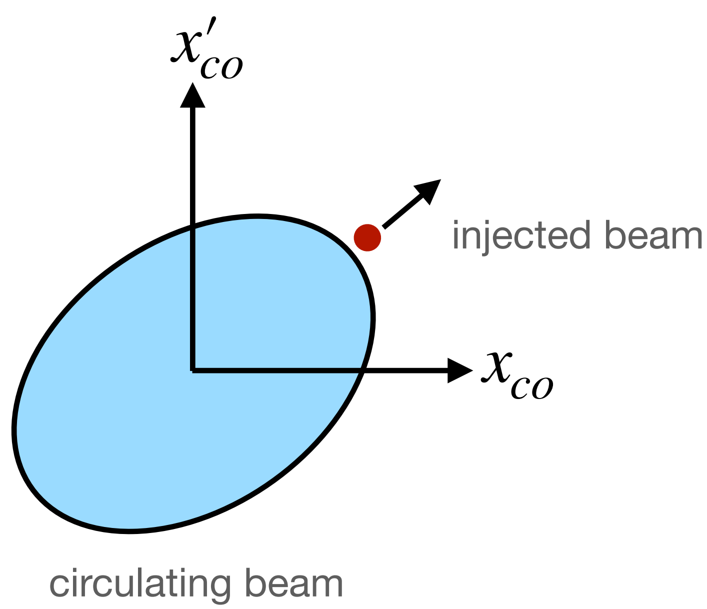



- $H^-$ beam produced in ion source
- Linac accelerates to 1 GeV
- $\approx$ 1000 pulses accumulated in ring ($10^{14}$ particles in final beam… high space charge)
- Collision with Hg target releases neutrons through spallation

# SNS injection

- Closed orbit modified to align with $H^-$ beam
- Electrons in $H^-$ beam are stripped
- Circulating beam moved relative to injected beam in phase space (8 kicker magnets)
- Slowly fill in or "paint" the transverse phase space




Injected beam is stationary in lab frame

# Research Plan

# Motivation

**Injection of the rotating distribution in the SNS**

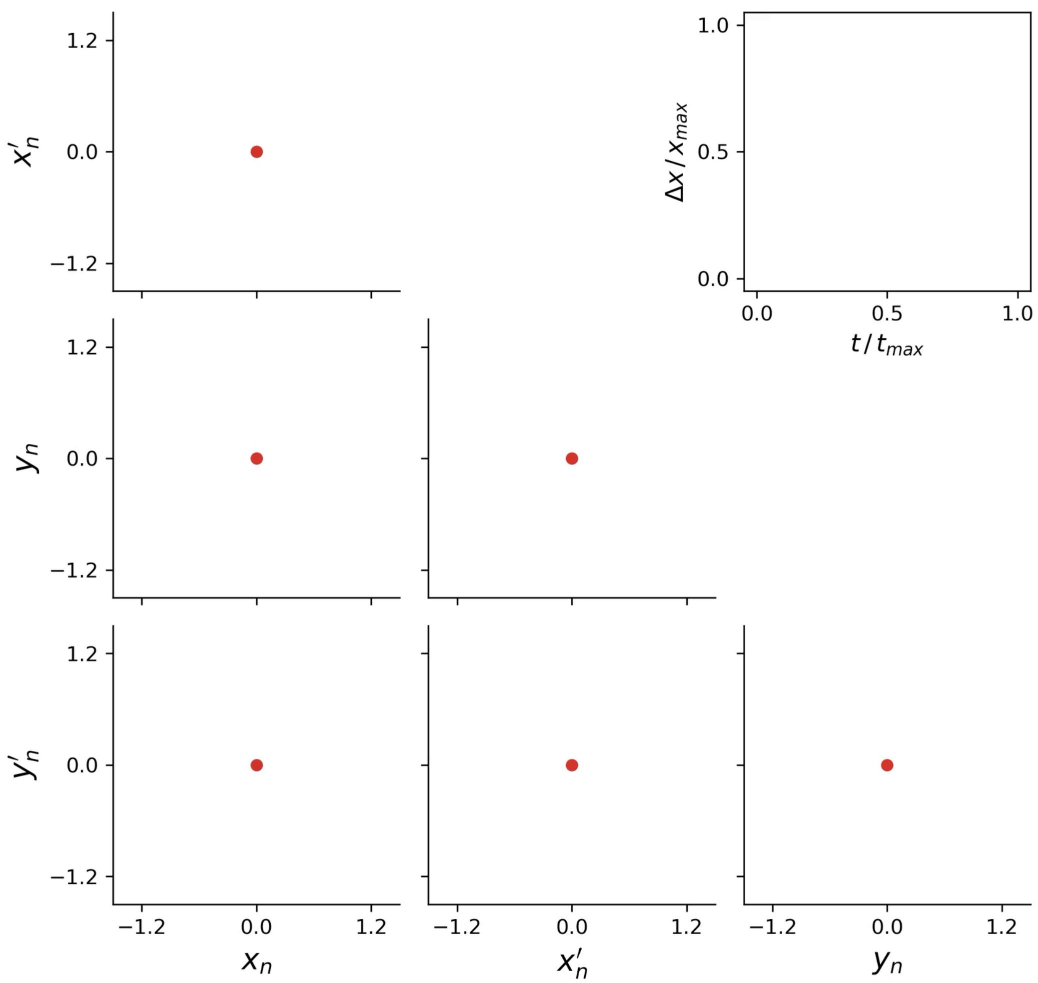
- Initial plan suggested — it seems to work okay in simulation
- Can it be improved upon by considering the *matched beam*?

**Measurement of the rotating distribution**

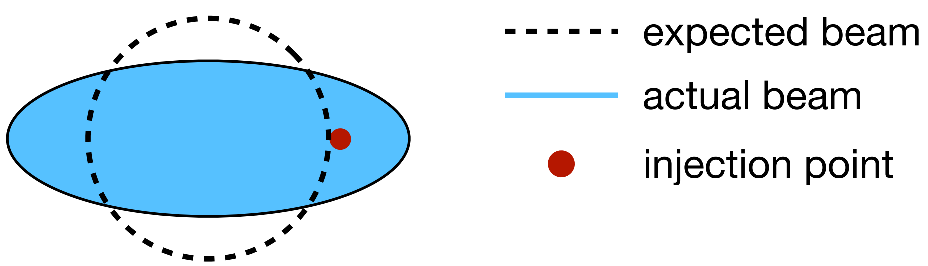
- Requires measurement of cross-plane moments — unusual for SNS
- Sensitivity and error analysis needed to optimize method

# Injection of the rotating distribution

$$x_{co} - x_{inj} = x_{max}\sqrt{t/t_{max}}$$
$$x'_{co} - x'_{inj} = x'_{max}\sqrt{t/t_{max}}$$
$$y_{co} - y_{inj} = y_{max}\sqrt{t/t_{max}}$$
$$y'_{co} - y'_{inj} = y'_{max}\sqrt{t/t_{max}}$$




- Promising approach in SNS:
  - Set $x'_{max} = y_{max} = 0$
  - Paints upright ellipse
- Assumes circulating beam is matched
  - Does it matter in SNS?
  - How to calculate the matched beam?

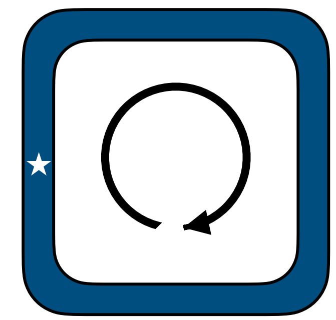



Mismatched circulating beam results in non-uniform density

# Matched beam

- Matched beam: same periodicity as lattice

$$\Sigma(s+L) = \Sigma(s)$$

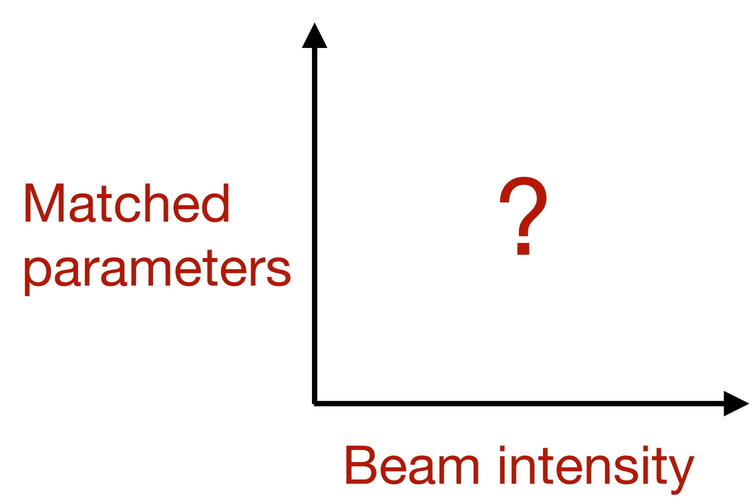


- Minimize the cost function: $C(\sigma) = |M\sigma - \sigma|^2$

$$\sigma = (\sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{22}, \sigma_{23}, \sigma_{24}, \sigma_{33}, \sigma_{34}, \sigma_{44})^T$$ moment vector

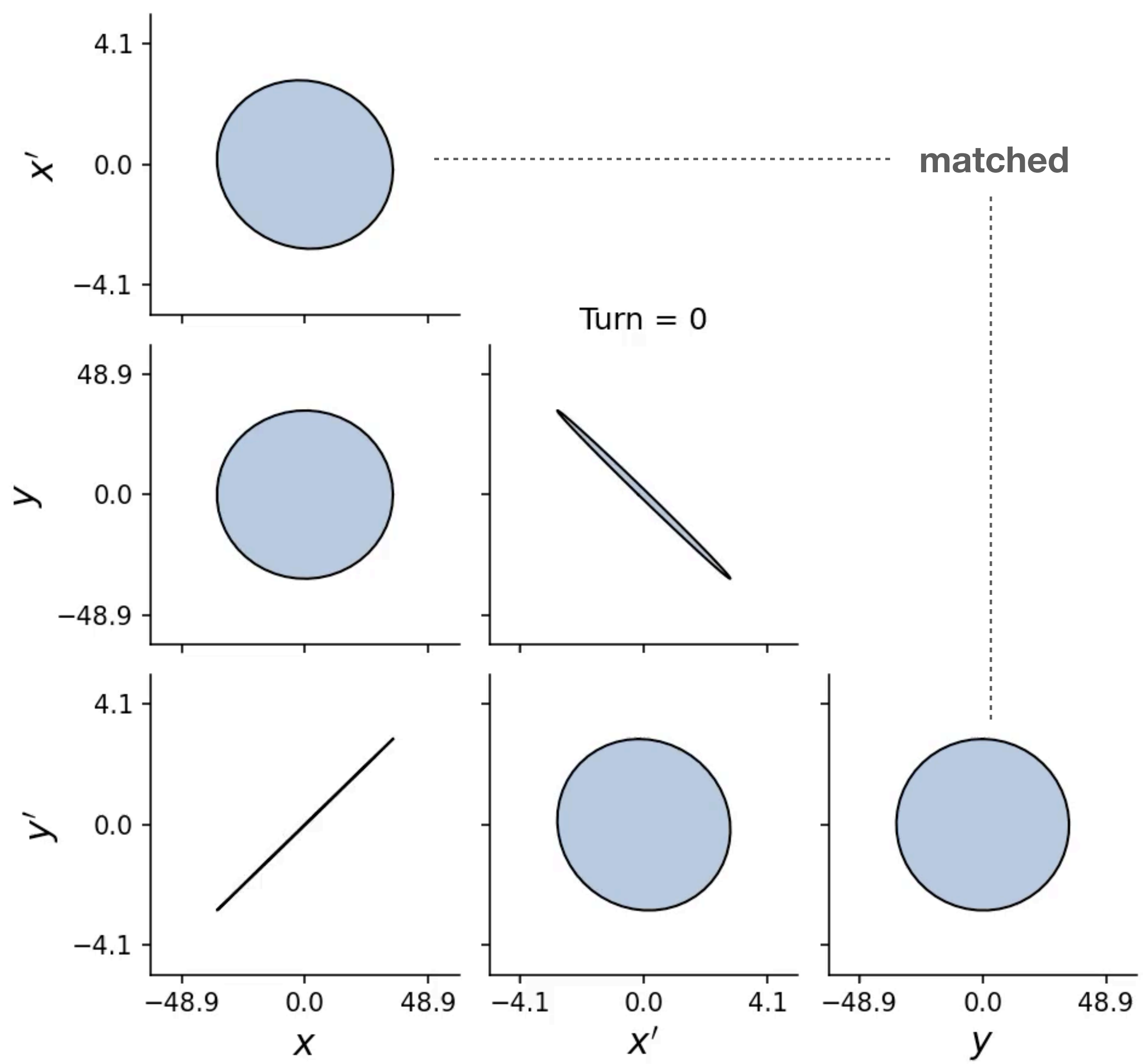
$$M = M(\sigma)$$ nonlinear one-turn map

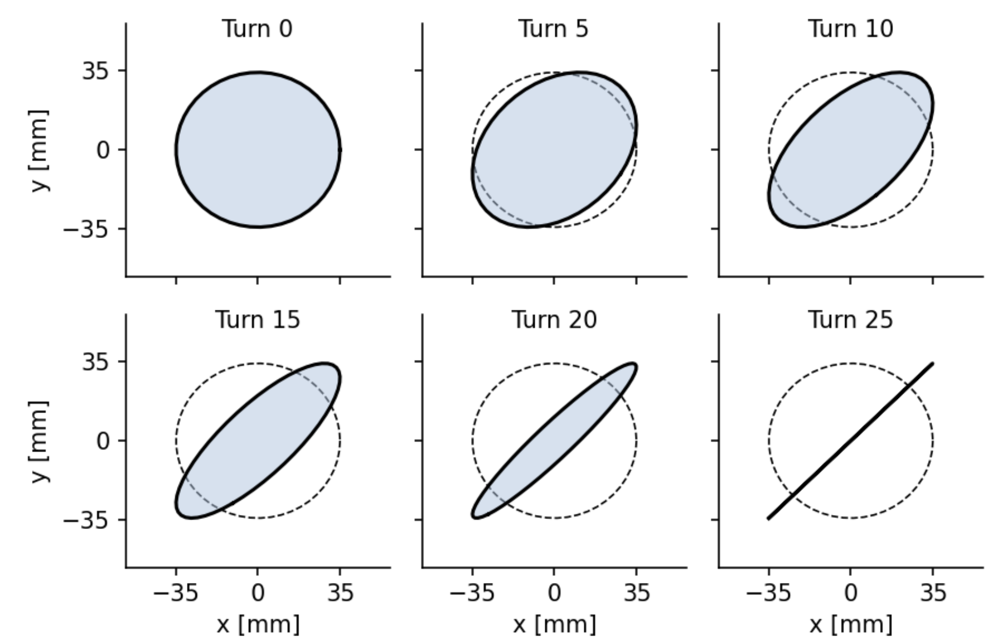
- How does $\sigma_{matched}$ change as a function of intensity?
  - Relevance to SNS experiments
  - Scaling law

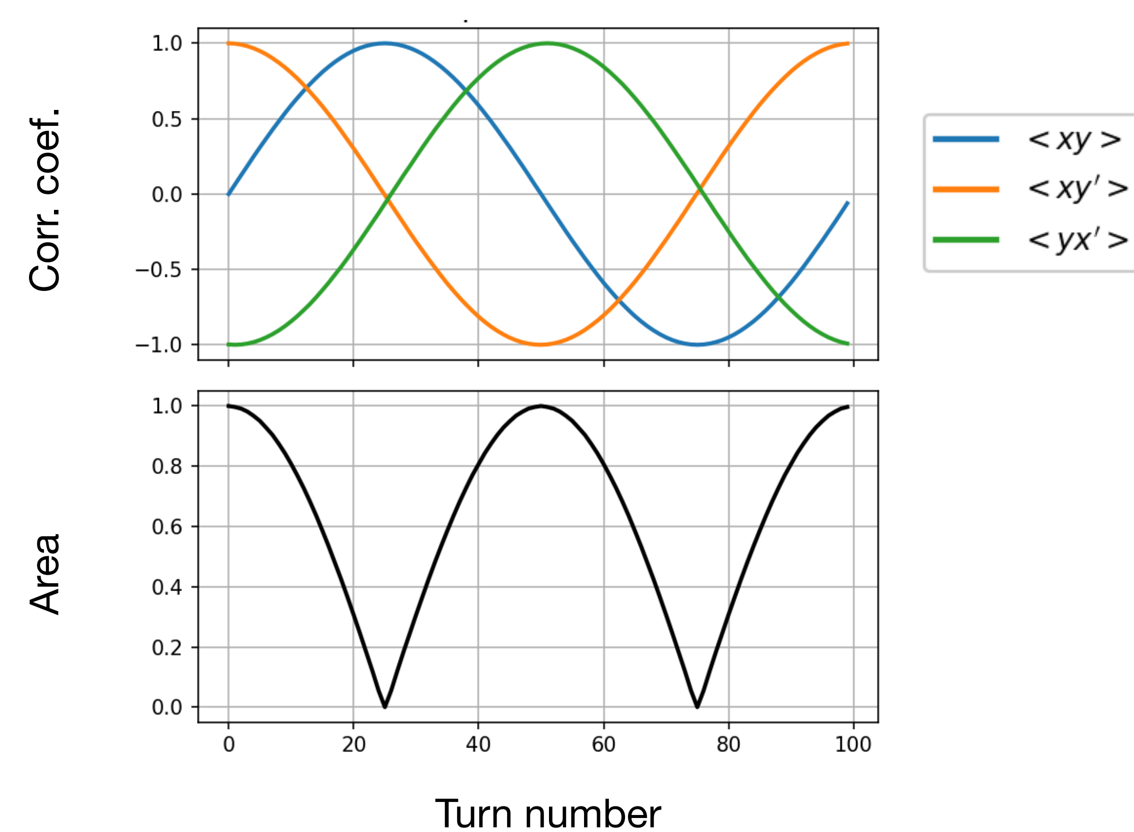


# Unequal tunes: no space charge

- Matching is impossible under these conditions

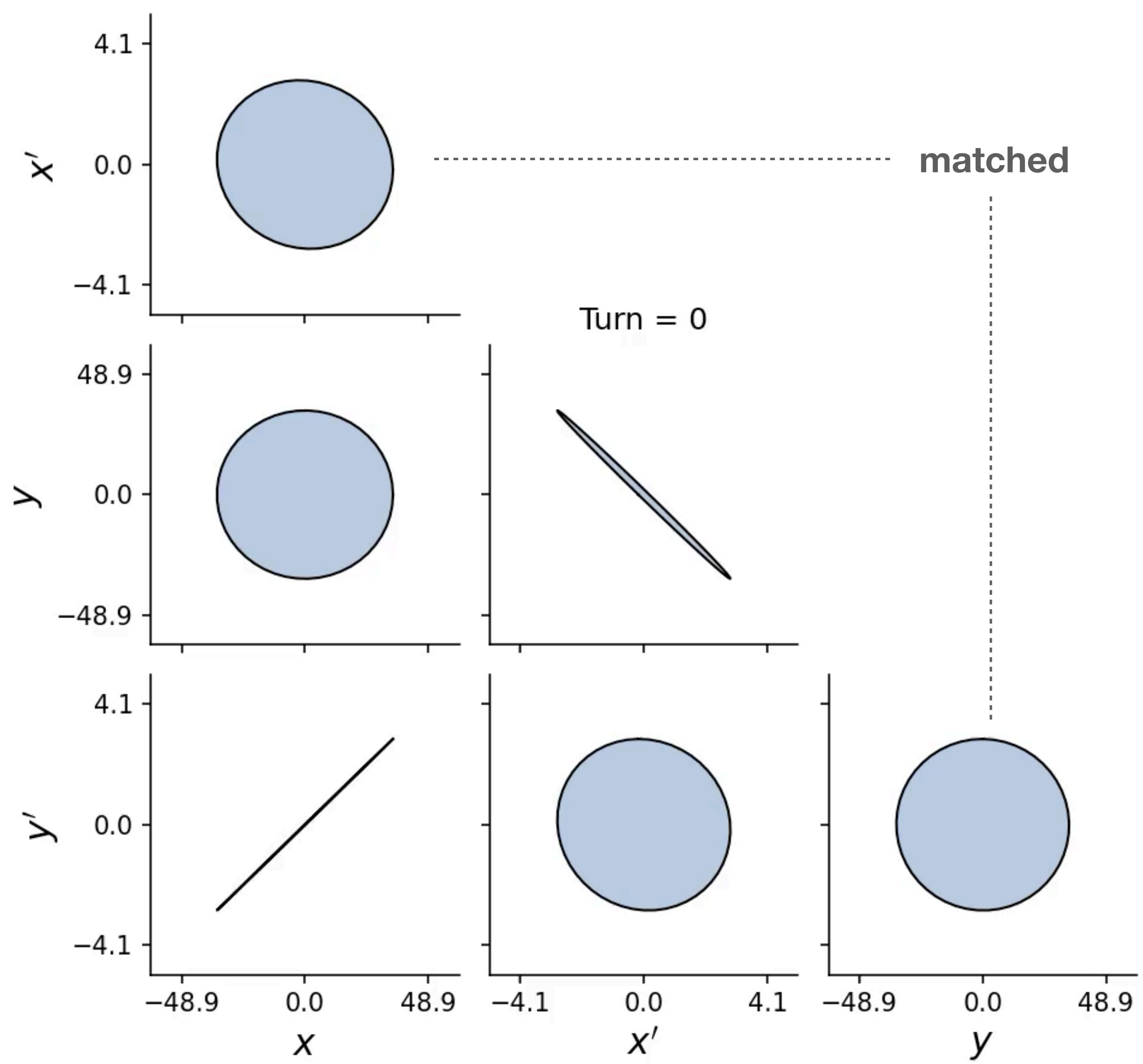


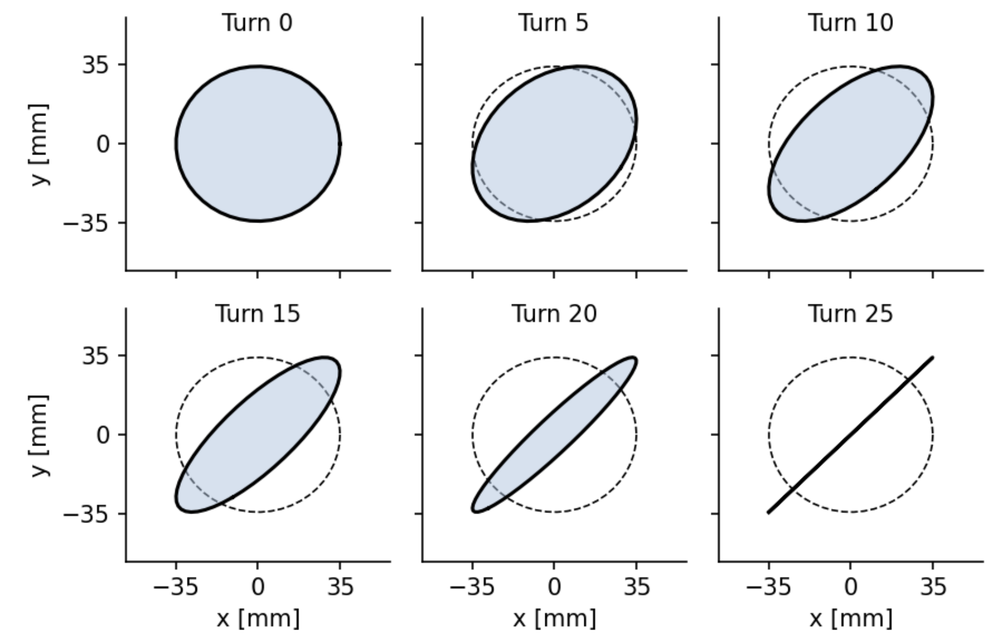



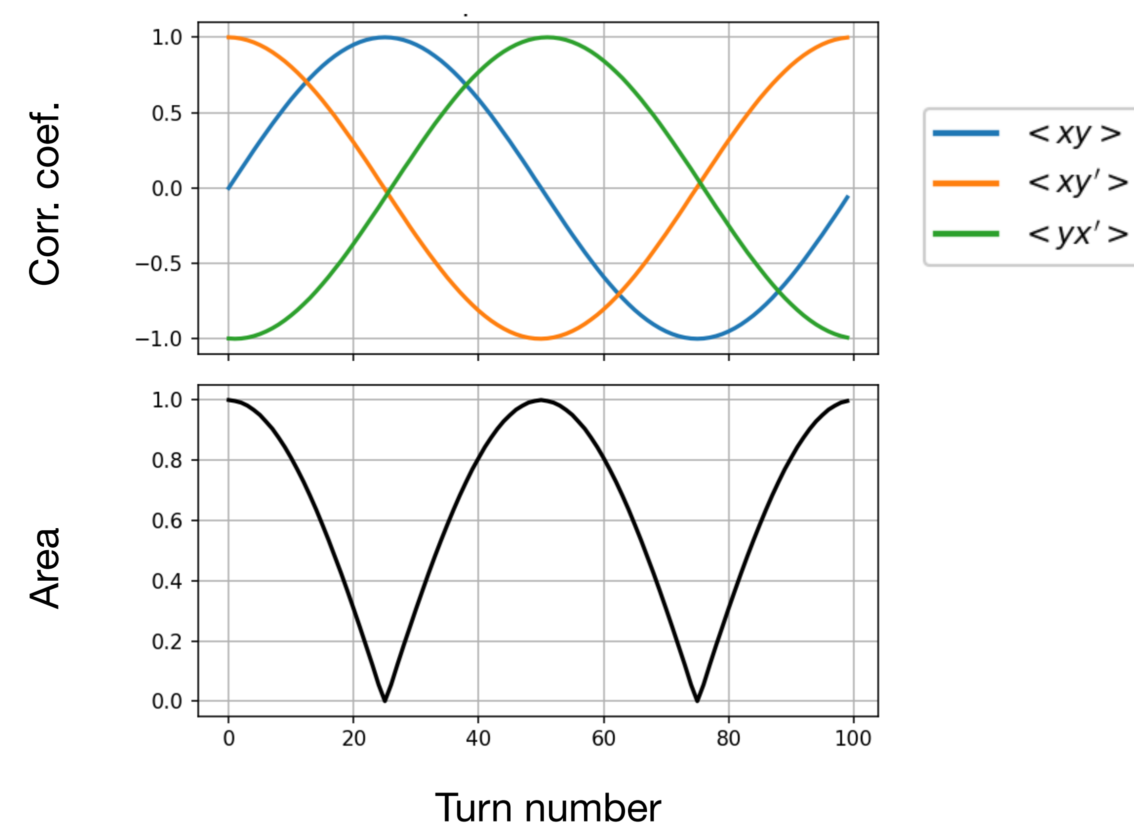



# Unequal tunes: no space charge

- Painting: end up with square beam



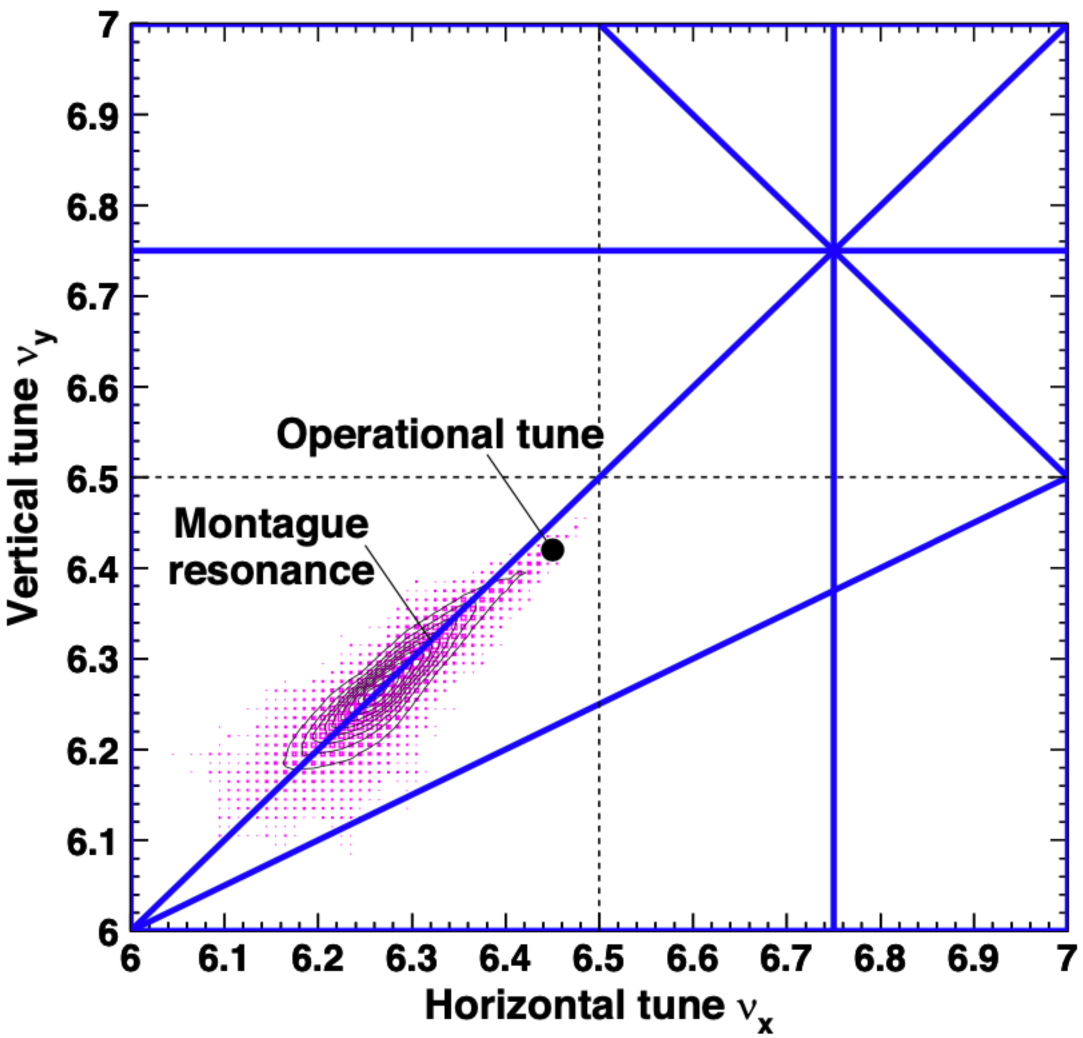



# Matching with unequal tunes

- Does space charge relax the condition $\nu_x = \nu_y$?

- Space charge driven resonance known to affect beam at $\nu_x \approx \nu_y$



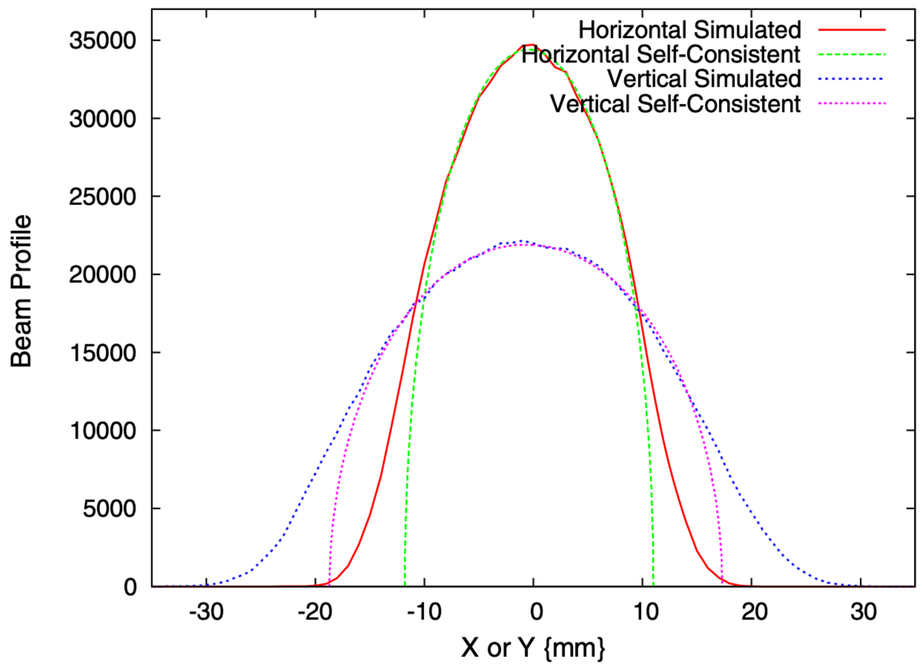

Simulated tune spread at J-PARC (Hotchi 2020)

# Objectives: matching

1. Develop method to find matched envelope
2. Study matched beam parameters as function of intensity
3. Investigate matching with unequal tunes
4. Use results to optimize painting scheme in SNS

# Signatures of the rotating distribution

Uniform density profile




Nonzero cross-plane moments

$$\Sigma = \begin{bmatrix} \langle x^2 \rangle & \langle xx' \rangle & \boxed{\langle xy \rangle \quad \langle xy' \rangle} \\ \langle xx' \rangle & \langle x'^2 \rangle & \boxed{\langle x'y \rangle \quad \langle x'y' \rangle} \\ \langle xy \rangle & \langle x'y \rangle & \langle y^2 \rangle \quad \langle yy' \rangle \\ \langle xy' \rangle & \langle x'y' \rangle & \langle yy' \rangle \quad \langle y'^2 \rangle \end{bmatrix}$$

**Zero 4D emittance** : strict requirements on moments

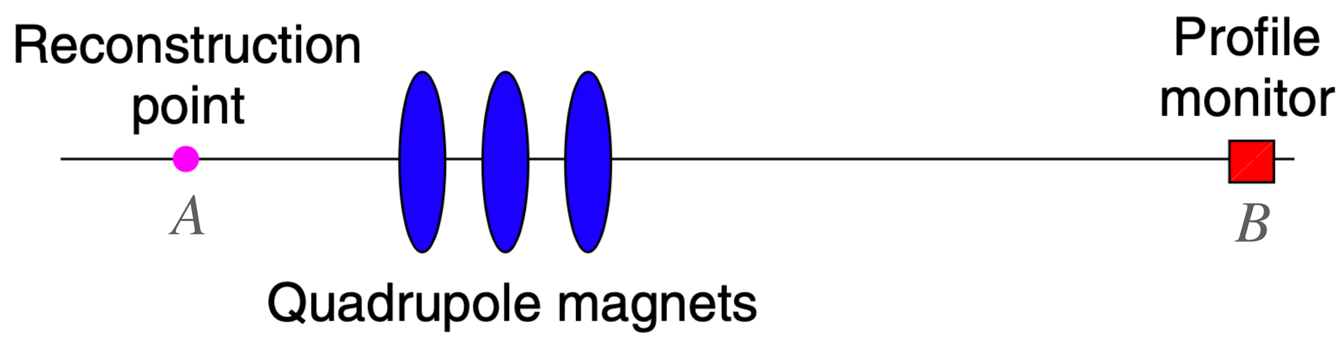
$$\varepsilon_{4D} = \varepsilon_1 \varepsilon_2 = \sqrt{\det \Sigma} = 0$$

$$\varepsilon_1 = \frac{1}{2}\sqrt{-tr\left[(\Sigma U)^2\right] + \sqrt{tr\left[(\Sigma U)^2\right] - 16 \det \Sigma}} = \varepsilon_x + \varepsilon_y$$

$$\varepsilon_2 = \frac{1}{2}\sqrt{-tr\left[(\Sigma U)^2\right] - \sqrt{tr\left[(\Sigma U)^2\right] - 16 \det \Sigma}} = 0$$

# Measuring cross-plane moments

- Reconstruction point lies upstream of measurement point "single location, multiple optics"

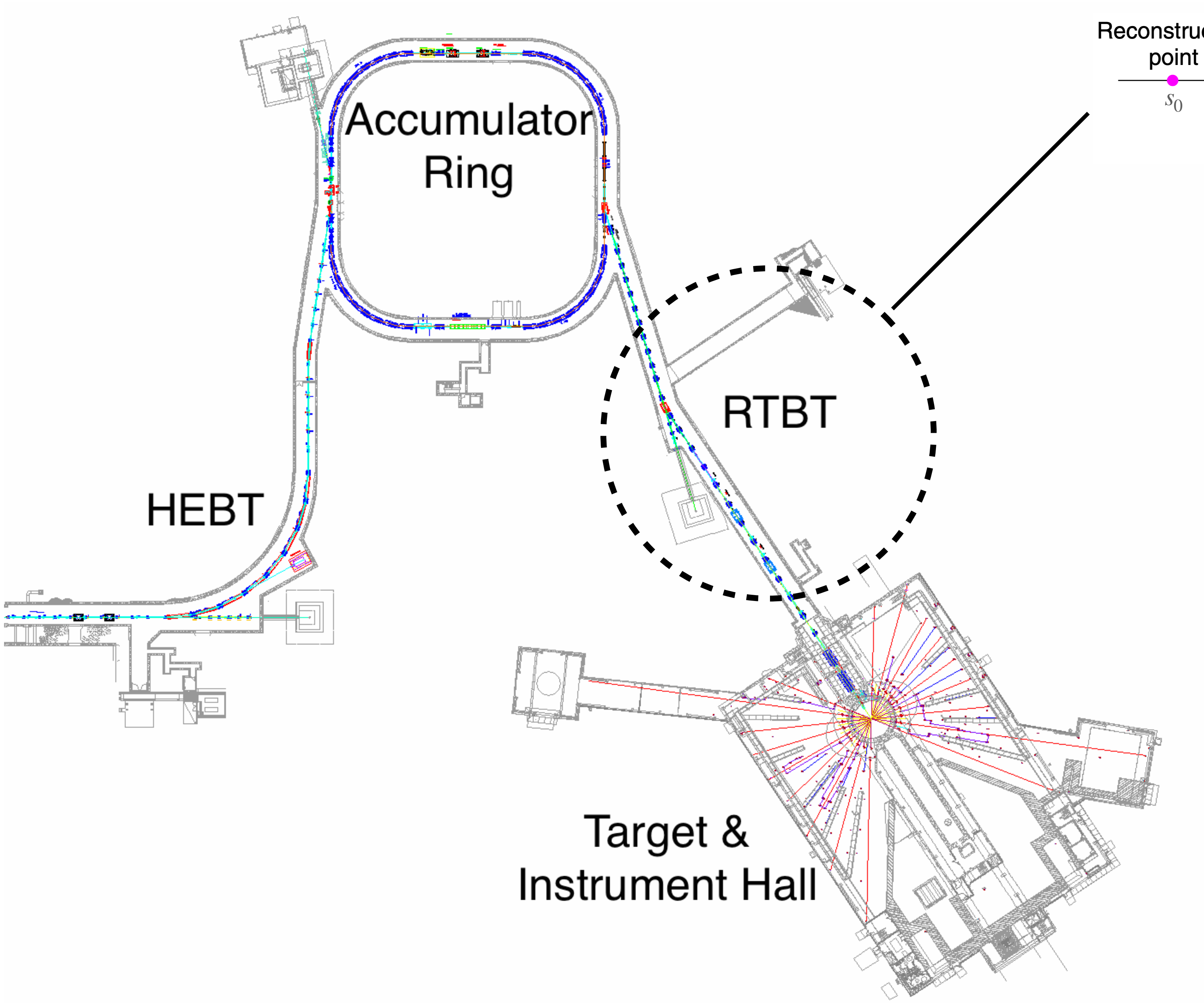


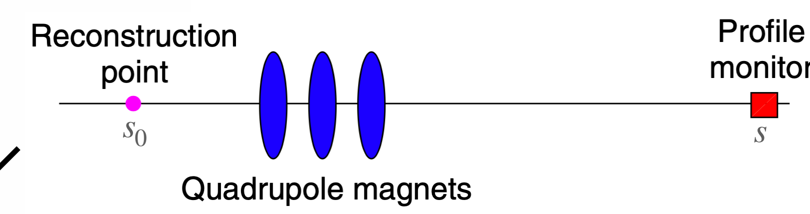

- Neglect space charge: $\boxed{\Sigma_B = R\,\Sigma_A\,R^T}$ $\quad \vec{x}_B = R\vec{x}_A$

- At least 4 measurements needed to solve for $\Sigma_A$

$$\langle x^2\rangle_B = R_{11}^2\langle x^2\rangle_A + R_{12}^2\langle x'^2\rangle_A + 2R_{11}R_{12}\langle xx'\rangle_A,$$
$$\langle y^2\rangle_B = R_{33}^2\langle y^2\rangle_A + R_{34}^2\langle y'^2\rangle_A + 2R_{33}R_{34}\langle yy'\rangle_A,$$
$$\langle xy\rangle_B = R_{11}R_{33}\langle xy\rangle_A + R_{12}R_{33}\langle yx'\rangle_A + R_{11}R_{34}\langle xy'\rangle_A + R_{12}R_{34}\langle x'y'\rangle_A,$$

# Measuring cross-plane moments

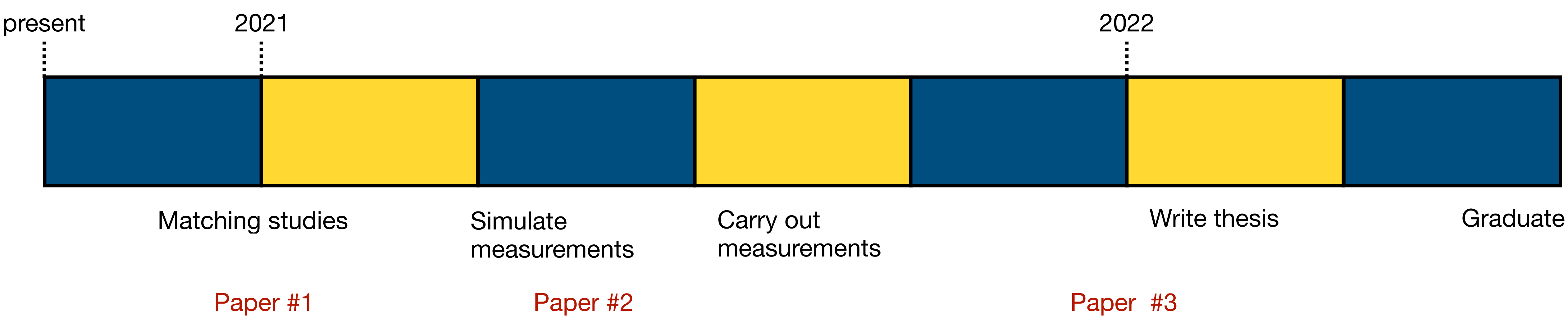


# Objectives: measurement

1. Simulation of the measurement (error + sensitivity analysis)
   - Investigate measurement precision with available SNS optics
   - Discriminate between rotating distribution and production type beams in SNS
   - Optimize measurement for the available optics
   - Is “single location, multiple optics” the best method?
2. The actual measurement
   - Write and test scripts to perform measurement in RTBT
   - Analyze data from experiments
   - Create “4D measurement” application for future use at SNS

# Proposed Timeline



# Backup slides

# Software tools

python™ SciPy — Integration, optimization, data analysis, data visualization

PyORBIT-Collaboration — Beam physics simulation

European Laboratory for Particle Physics

CERN **MAD - Methodical Accelerator Design**

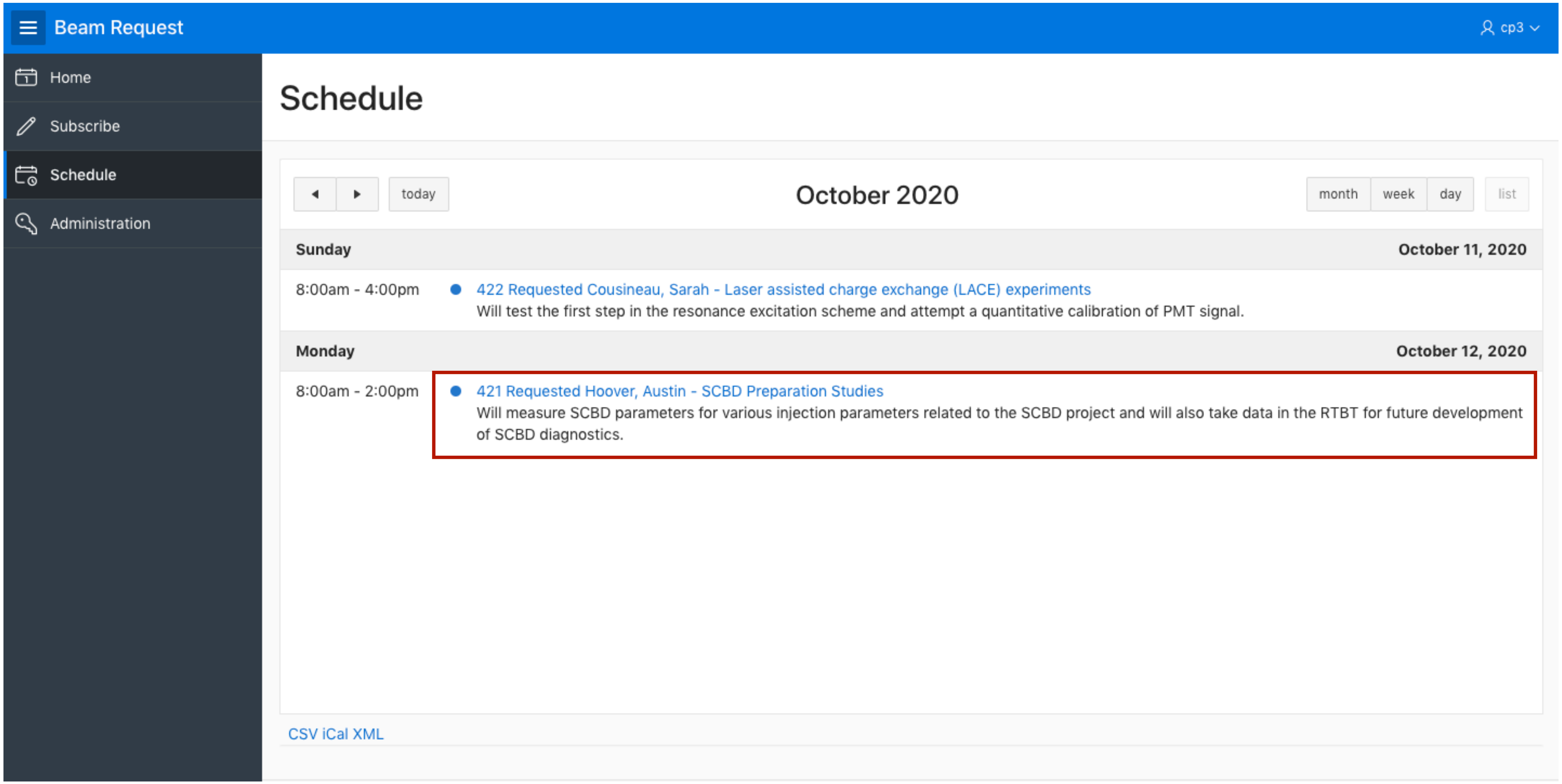
CERN - BE/ABP Accelerator Beam Physics Group — Lattice design

Open XAL

Accelerator Physics software platform

https://openxal.github.io — Interface with SNS

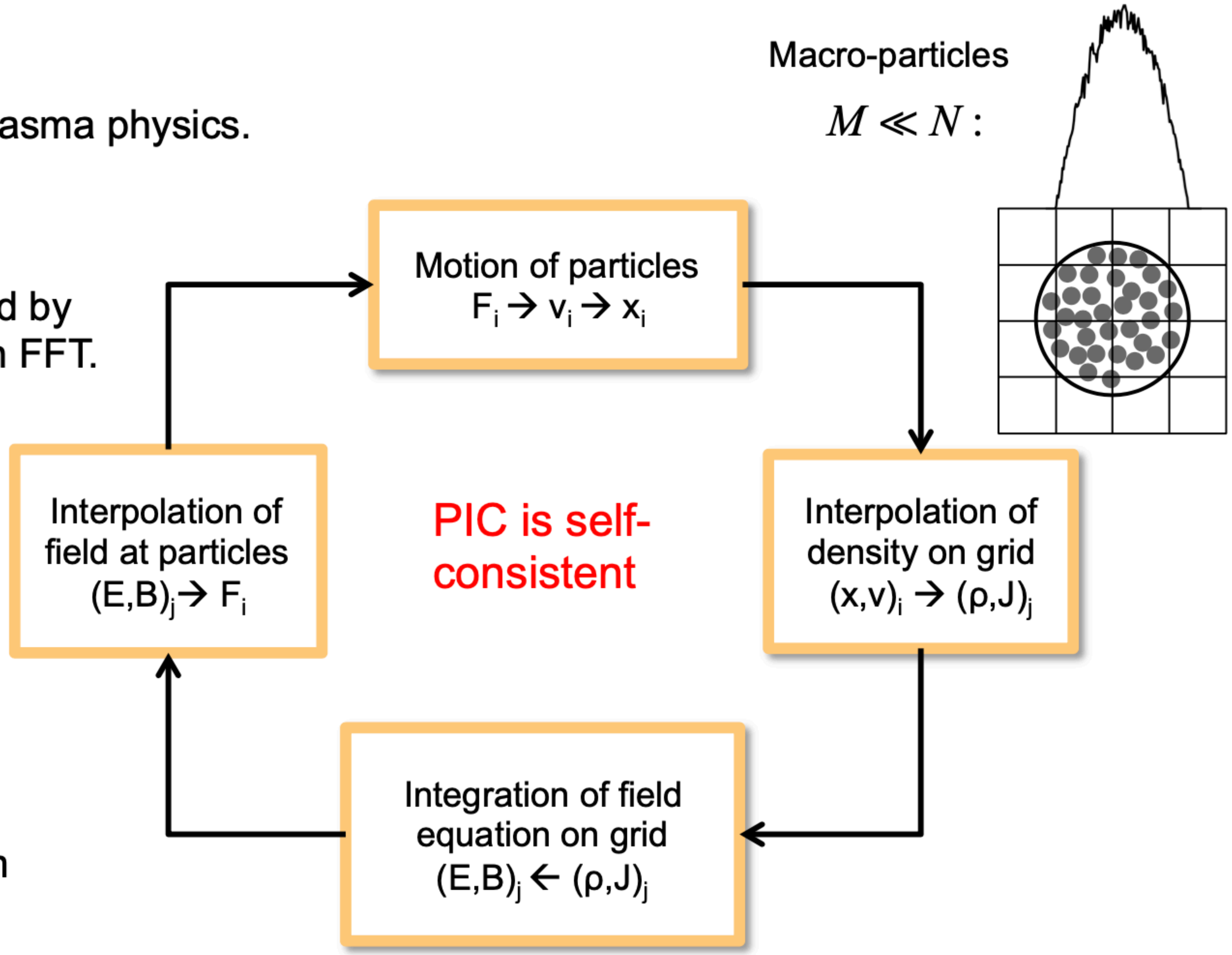
# Scheduled beam time



# PIC (Particle In Cell) algorithms

**Particle In Cell**

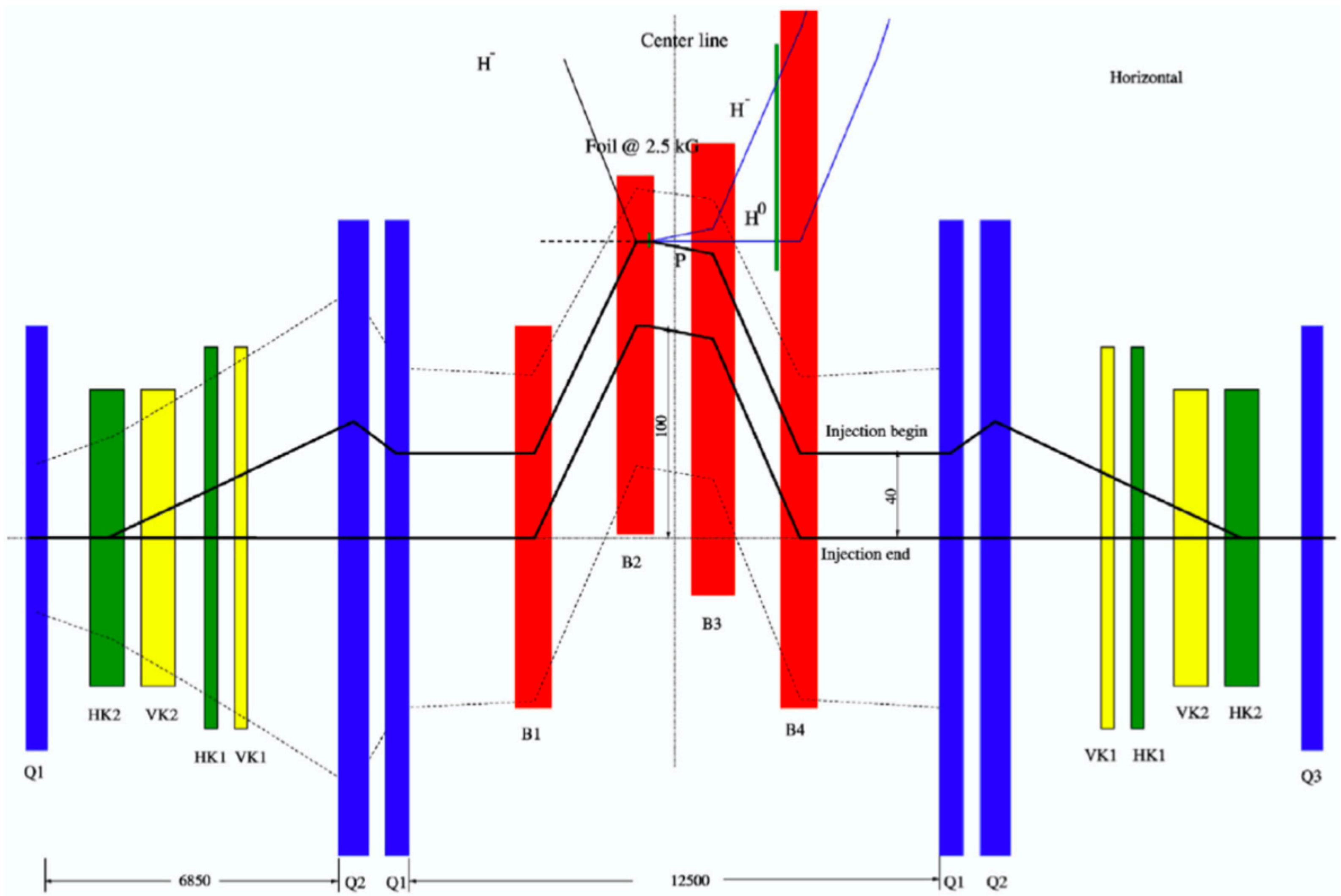
- Applied also in Astrophysics & Plasma physics.
- Space charge forces are obtained by solving the Poisson equation with FFT.
- Between the evaluation of SC forces, also other external forces can act on the beam.
- Between the “SC kicks” the beam oscillations have to be resolved



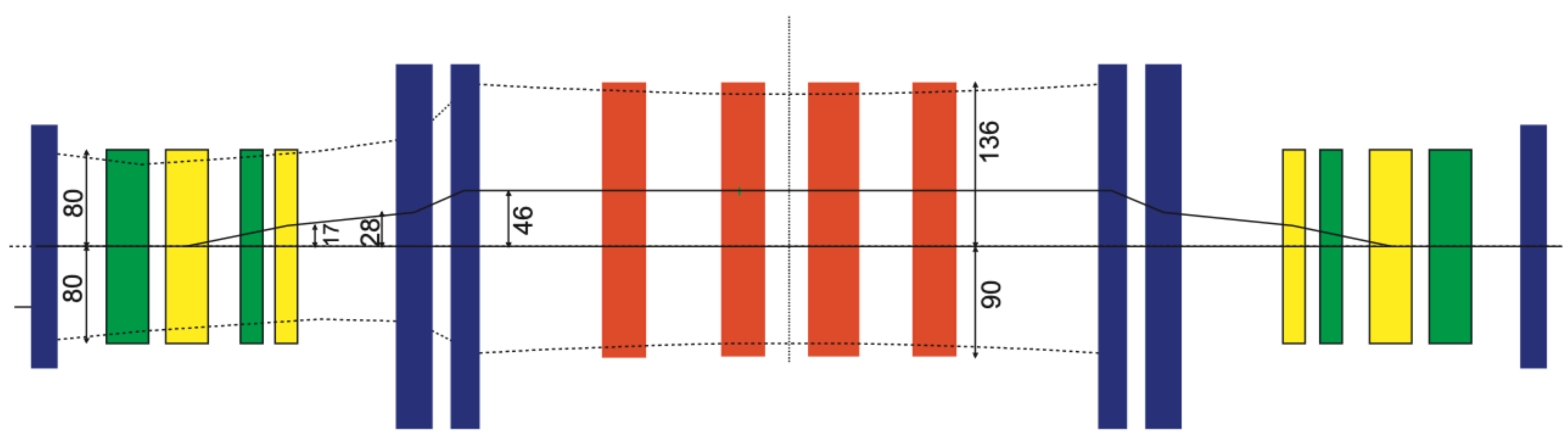

*C. Bridsall & A B Langdon: Plasma Physics via computer simulation; R W Hockney & J W Eastwood: Computer simulation using particles*

# SNS injection region

Horizontal



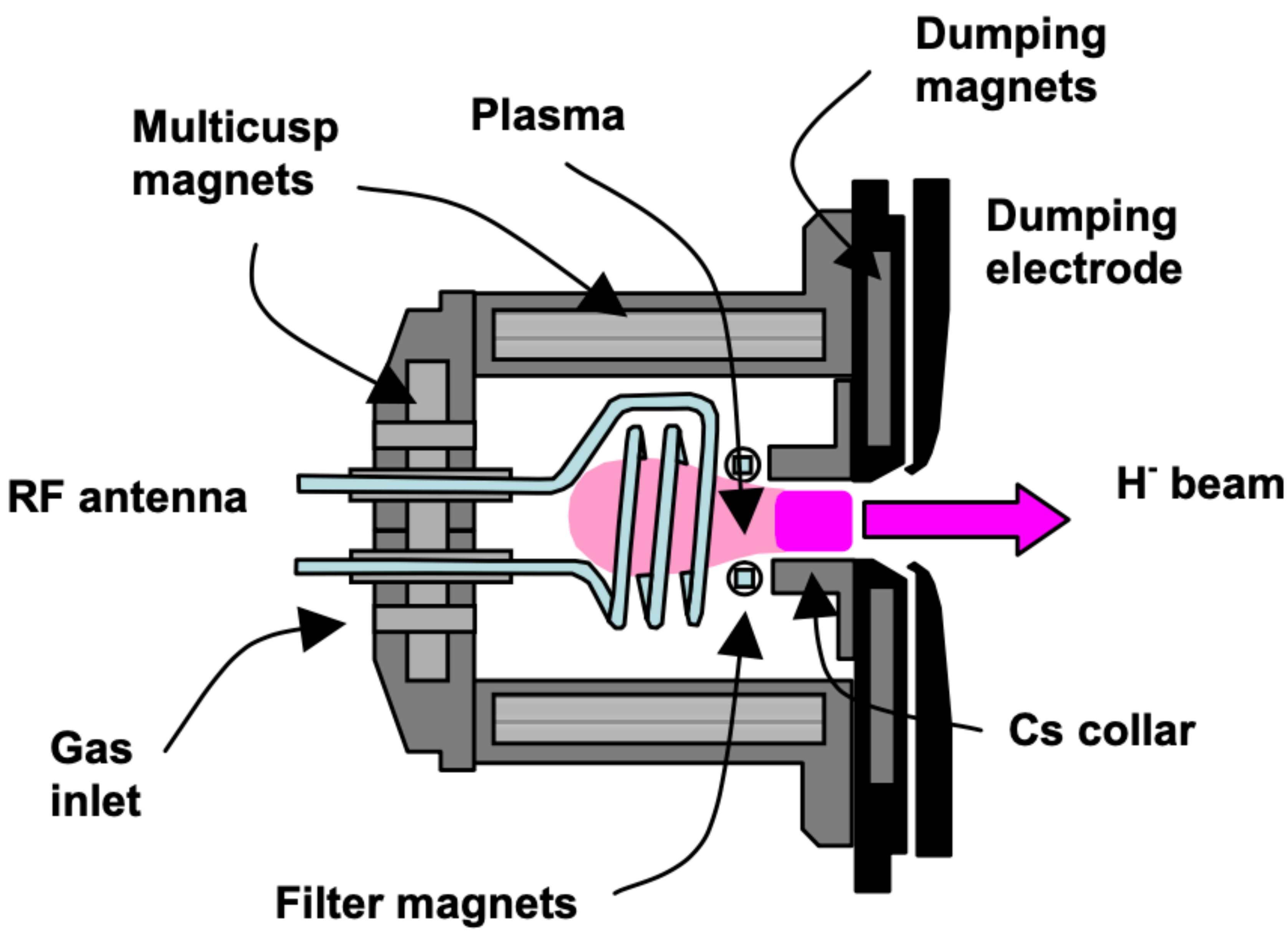

Vertical




credit: Henderson

# SNS ion source

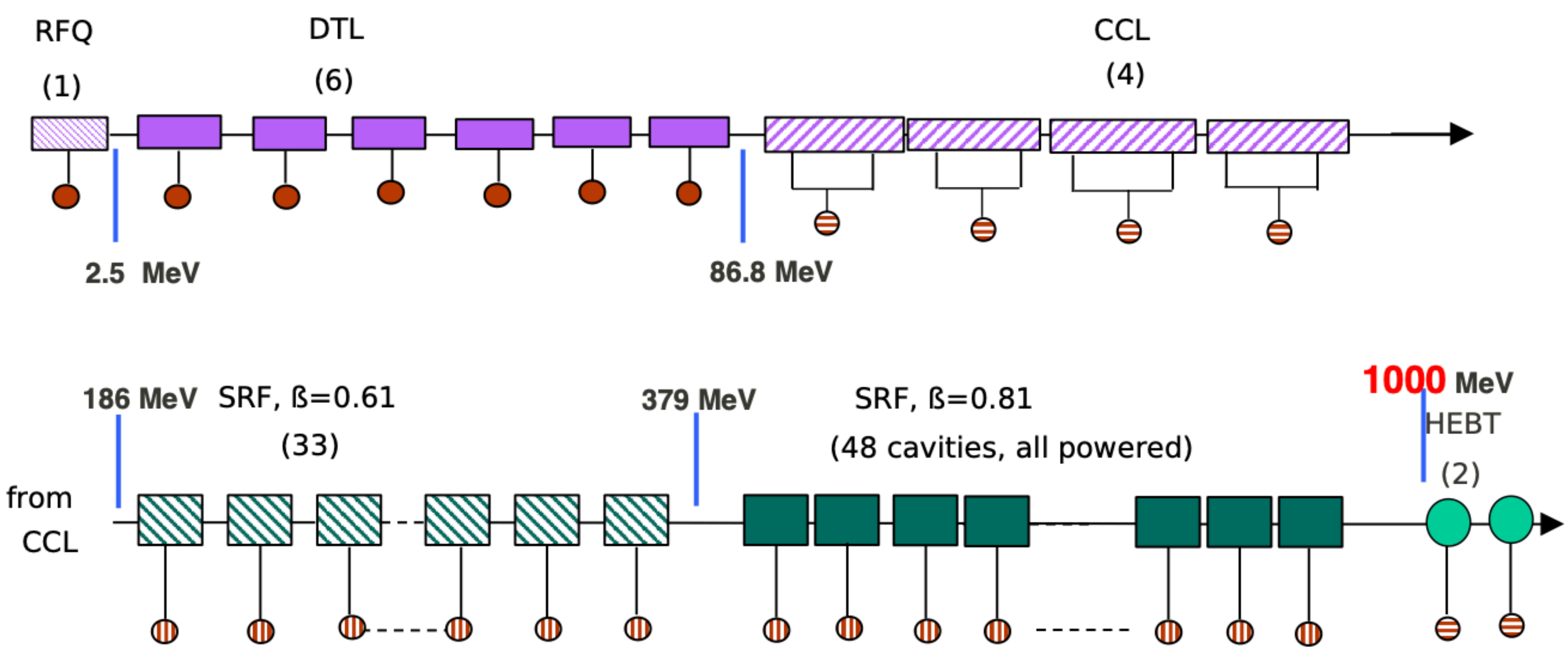
- $\approx$ 45 mA $H^{-}$ pulse




credit: Welton

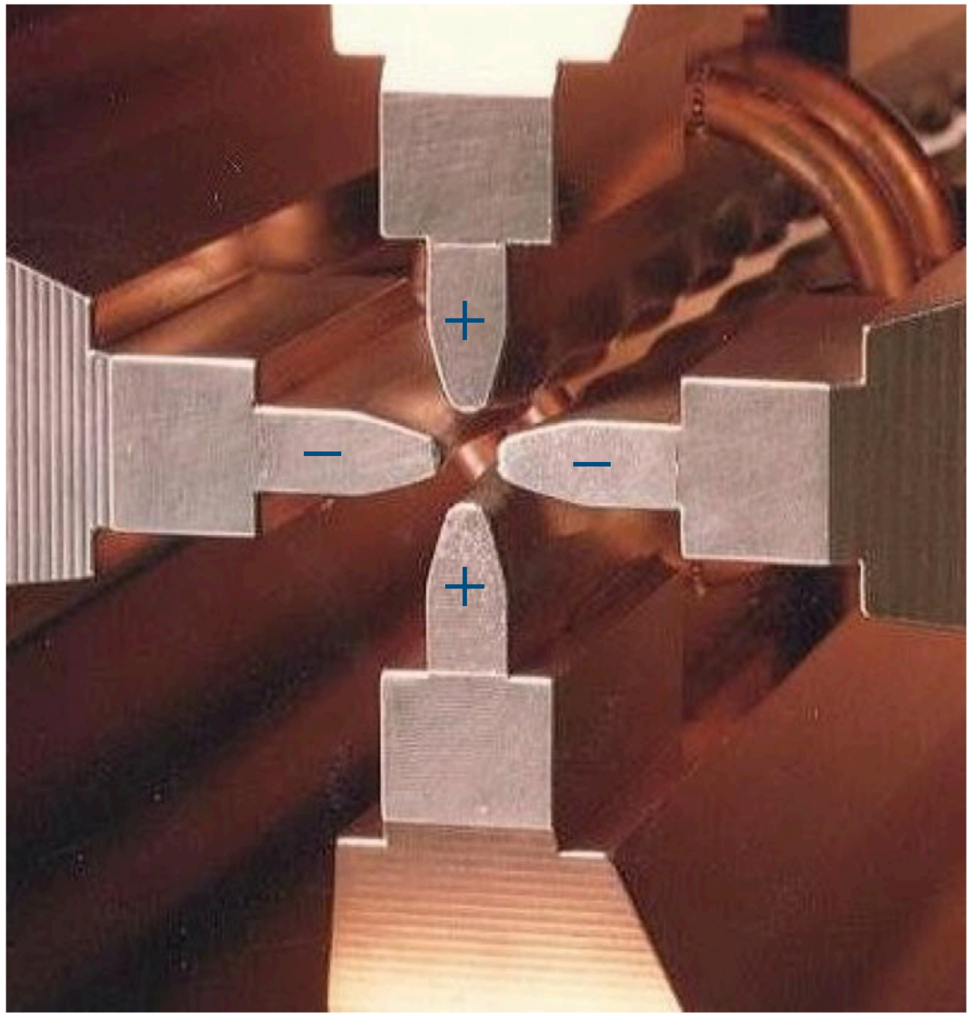
# SNS linac

- 2.5 MeV RFQ
- 86 MeV drift-tube inac
- 186 MeV side-coupled linac
- 1 GeV superconducting linac



# RFQ

- Electric quadrupole field
- Flip polarity → transverse focusing
- Ripples → longitudinal $\overrightarrow{E}$ for acceleration
- Low beam current



# Transfer matrix formulation

- Transfer matrices connect initial and final phase space coordinates

$$\begin{bmatrix} u \\ u' \end{bmatrix}_{s+L} = M \begin{bmatrix} u \\ u' \end{bmatrix}_s$$

$$M_{k=0} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \qquad M_{k>0} = \begin{bmatrix} \cos(\omega s) & \sin(\omega s)/\omega \\ -\omega \sin(\omega s) & \cos(\omega s) \end{bmatrix} \qquad M_{k<0} = \begin{bmatrix} \cosh(\omega s) & \sinh(\omega s)/\omega \\ \omega \sinh(\omega s) & \cosh(\omega s) \end{bmatrix}$$

drift focusing quad defocusing quad

- $M$ is symplectic: $MUM^T = U$

$$U = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

- Total transfer matrix given by product

$$\mathbf{M} = M_n M_{n-1} \ldots M_2 M_1$$

- Motion is stable if $tr\,\mathbf{M} < 2$

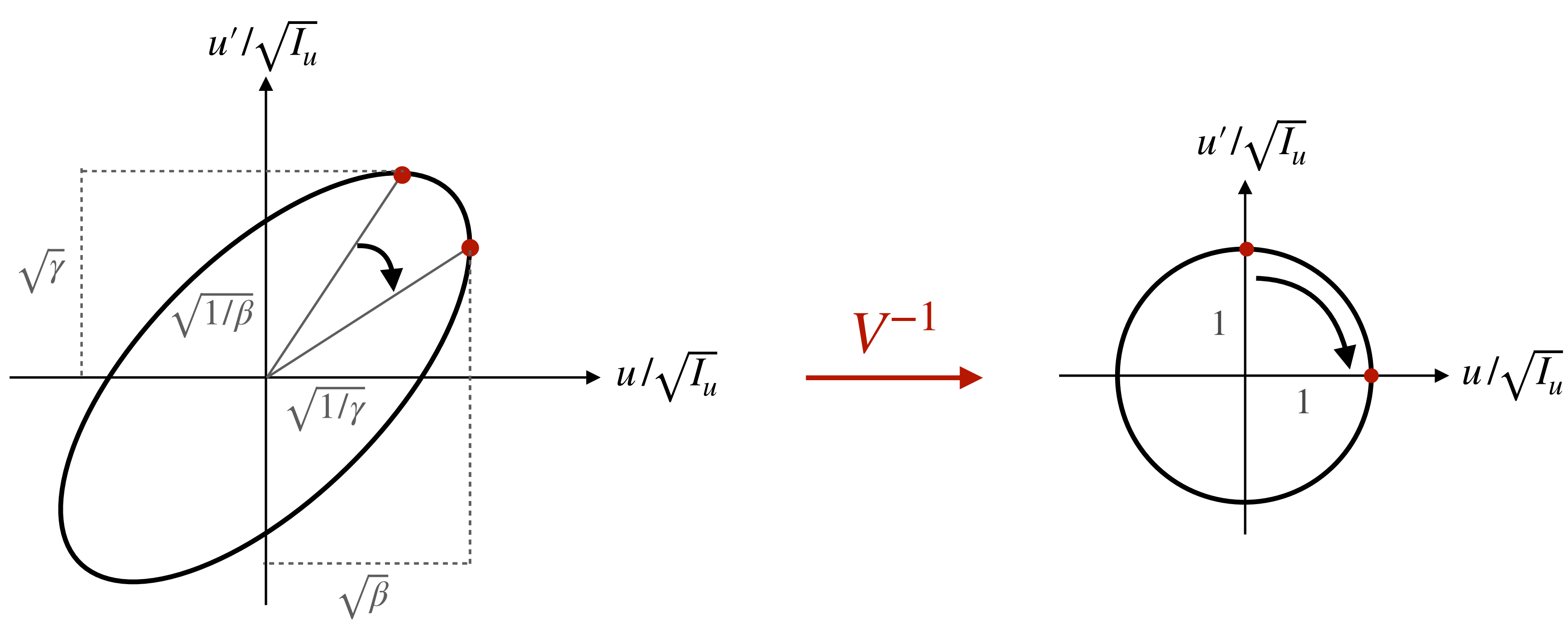
# Normalized coordinates

- $\mathbf{M}$ can be nicely factored

$$\mathbf{M} = \begin{bmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{bmatrix} = \underset{V(\alpha,\beta)}{\begin{bmatrix} \sqrt{\beta} & 0 \\ \frac{-\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{bmatrix}} \underset{P(\mu)}{\begin{bmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{bmatrix}} \underset{V^{-1}(\alpha,\beta)}{\begin{bmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{bmatrix}}$$

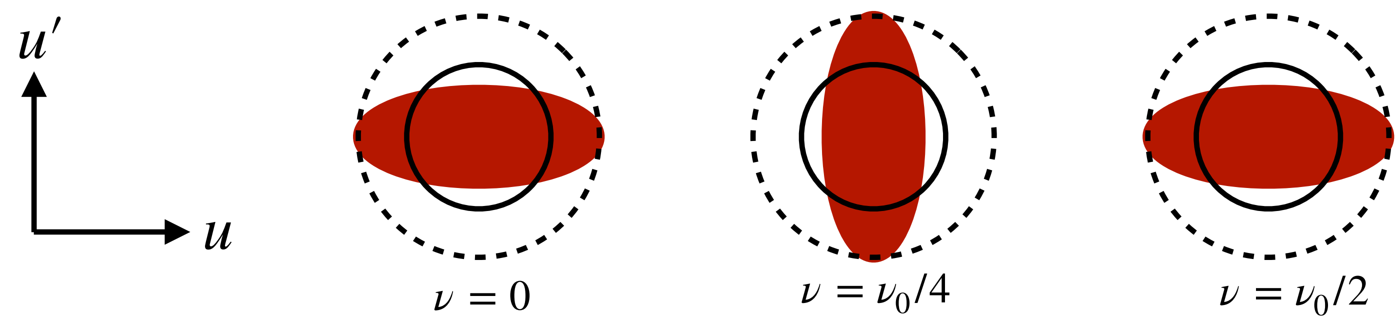
- Normalized coordinates $\overrightarrow{x}_n = V^{-1}\overrightarrow{x}$ behave as harmonic oscillator

$$\overrightarrow{x} \to M\overrightarrow{x}$$
$$\overrightarrow{x}_n \to V^{-1}MV\overrightarrow{x}_n$$
$$\overrightarrow{x}_n \to P\overrightarrow{x}_n$$

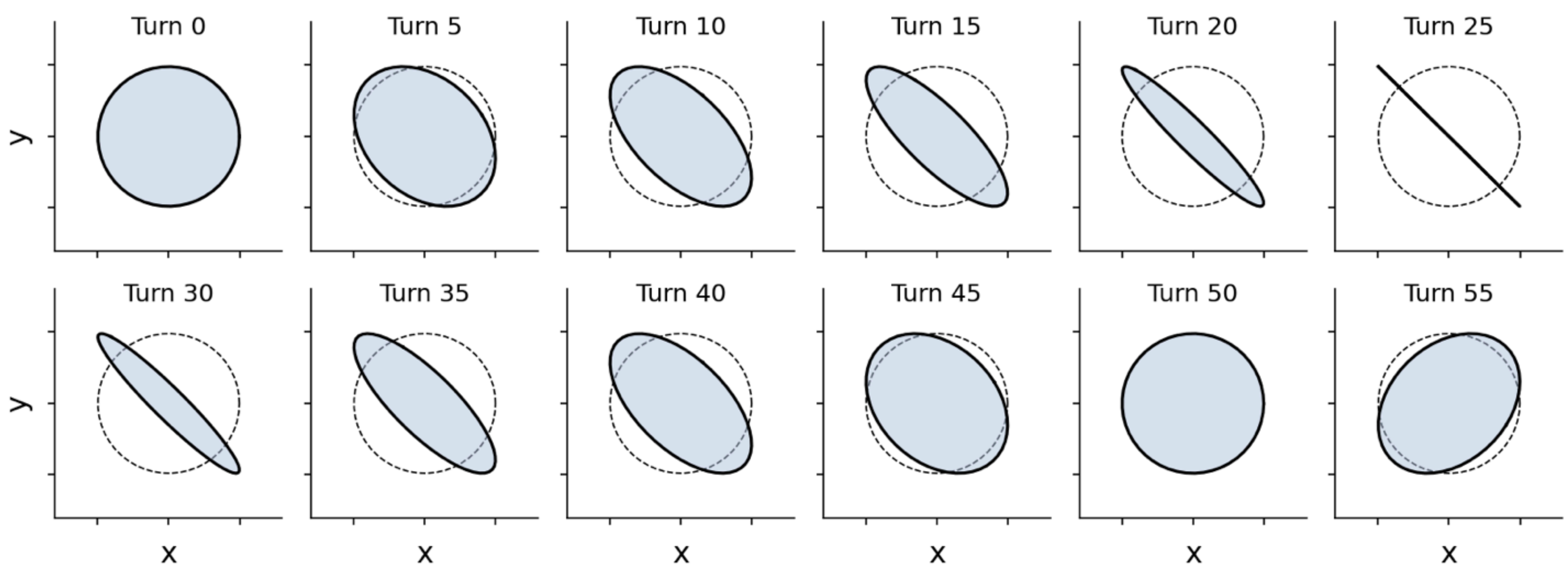


# Mismatch without space charge

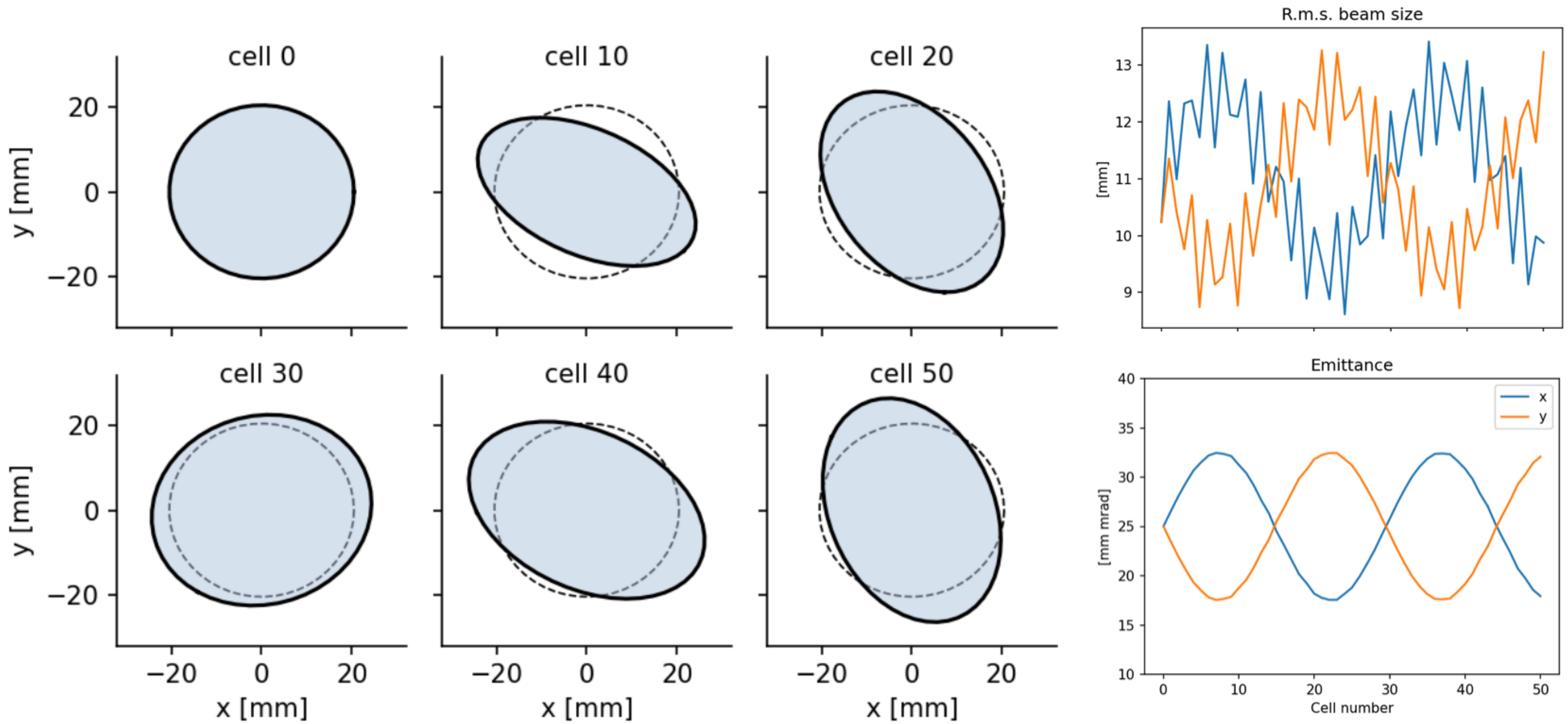
- The $x$-$x'$ and $y$-$y'$ ellipses do not have same parameters as lattice $\{\alpha_x, \alpha_y, \beta_x, \beta_y\}$
- Beam size breathes at twice the lattice tune




- Rotating distribution: $\nu_x \neq \nu_y$



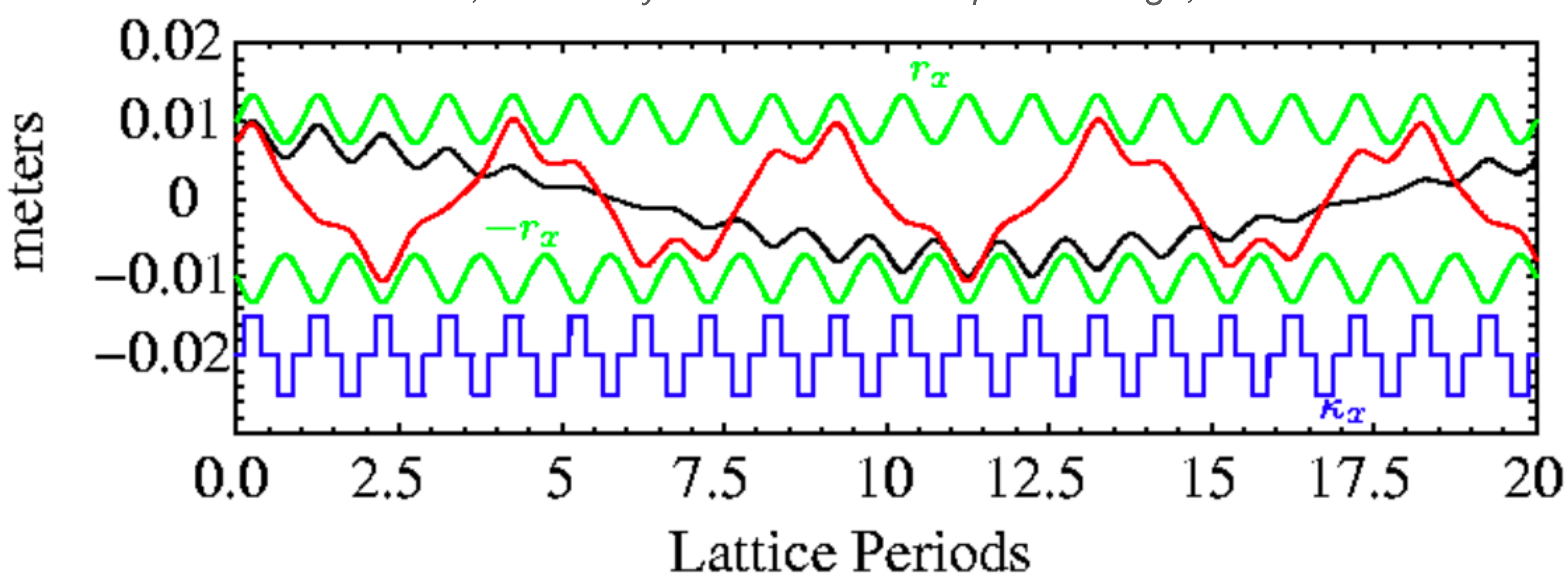
# Mismatch with space charge




- Phase advance is no longer equal in the two planes → tilted modes
- In addition to breathing: $\varepsilon_x$ and $\varepsilon_y$ oscillate from coupled space charge force
- This makes matching an interesting problem for the rotating distribution

# Space charge in FODO lattice

Lund, *Beam Physics with Intense Space Charge,* USPAS



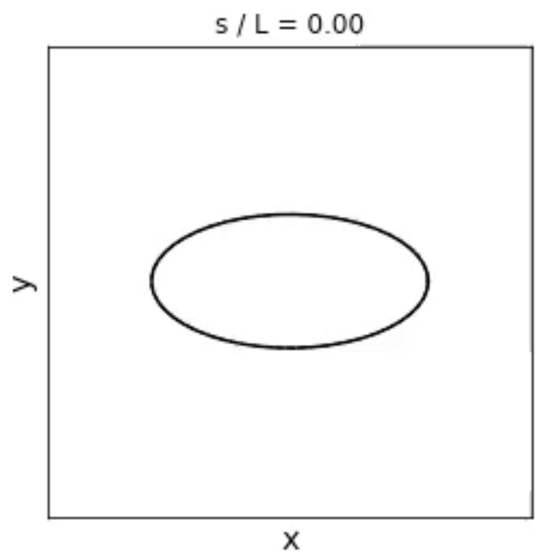

— envelope

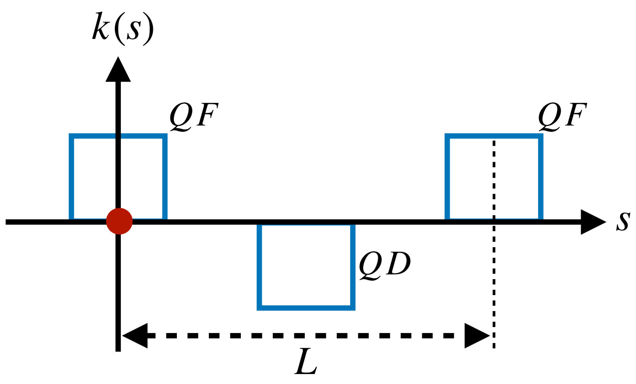
— focusing strength

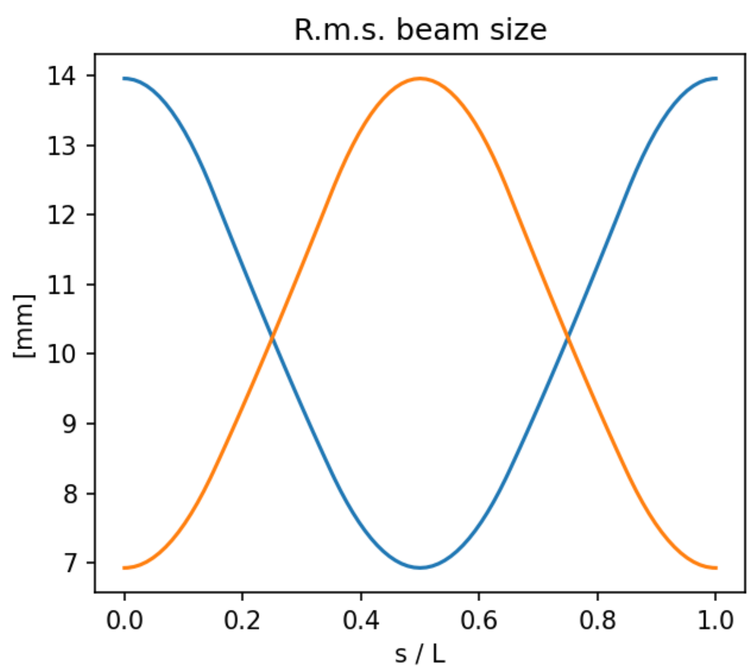
— depressed motion

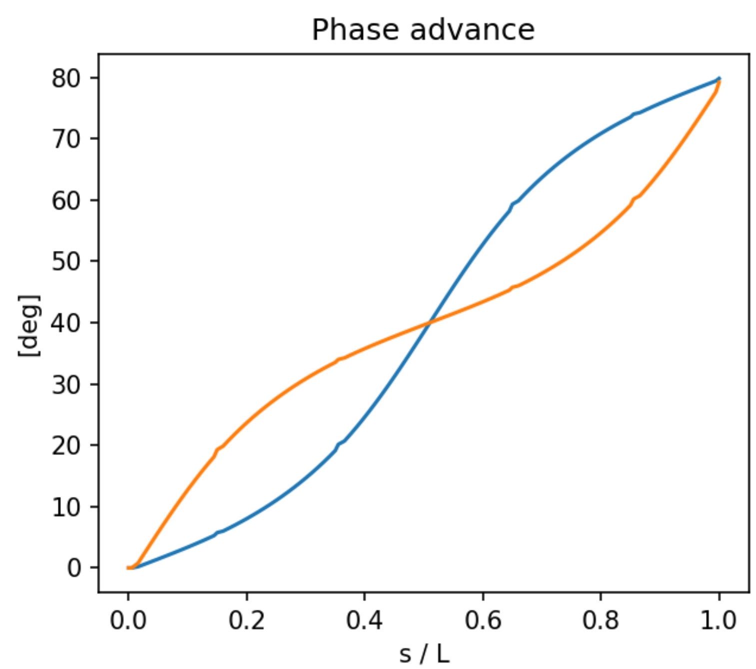
— undepressed motion

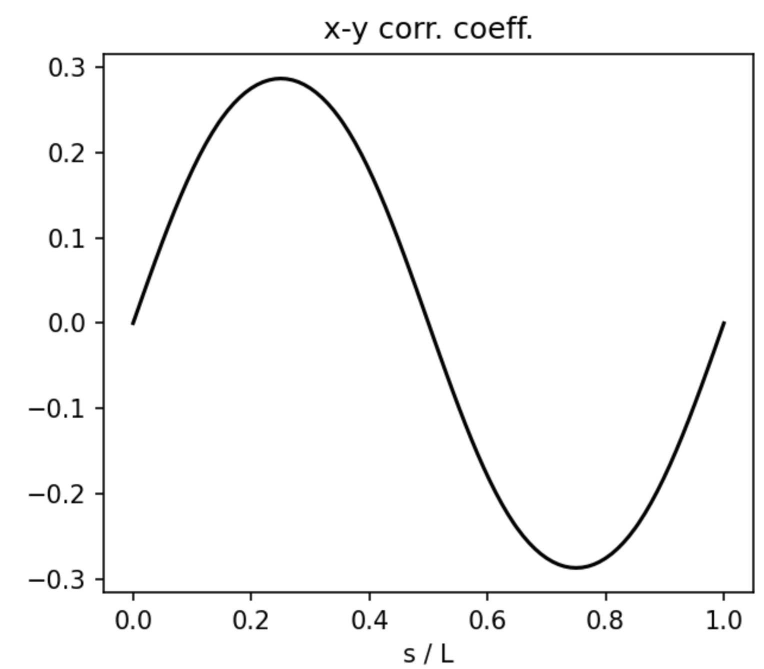
# Rotating distribution in FODO cell




Tilt is function of $\mu_y(s) - \mu_x(s)$