Computation of the Matched Envelope of the Danilov Distribution

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AP Meeting - 01.29.2021

Outline

- Danilov distribution
  - Definition, properties, and envelope equations
- Problem & motivation
- Methods
- Example applications
  - Coupled lattice
  - Uncoupled lattice

Danilov distribution

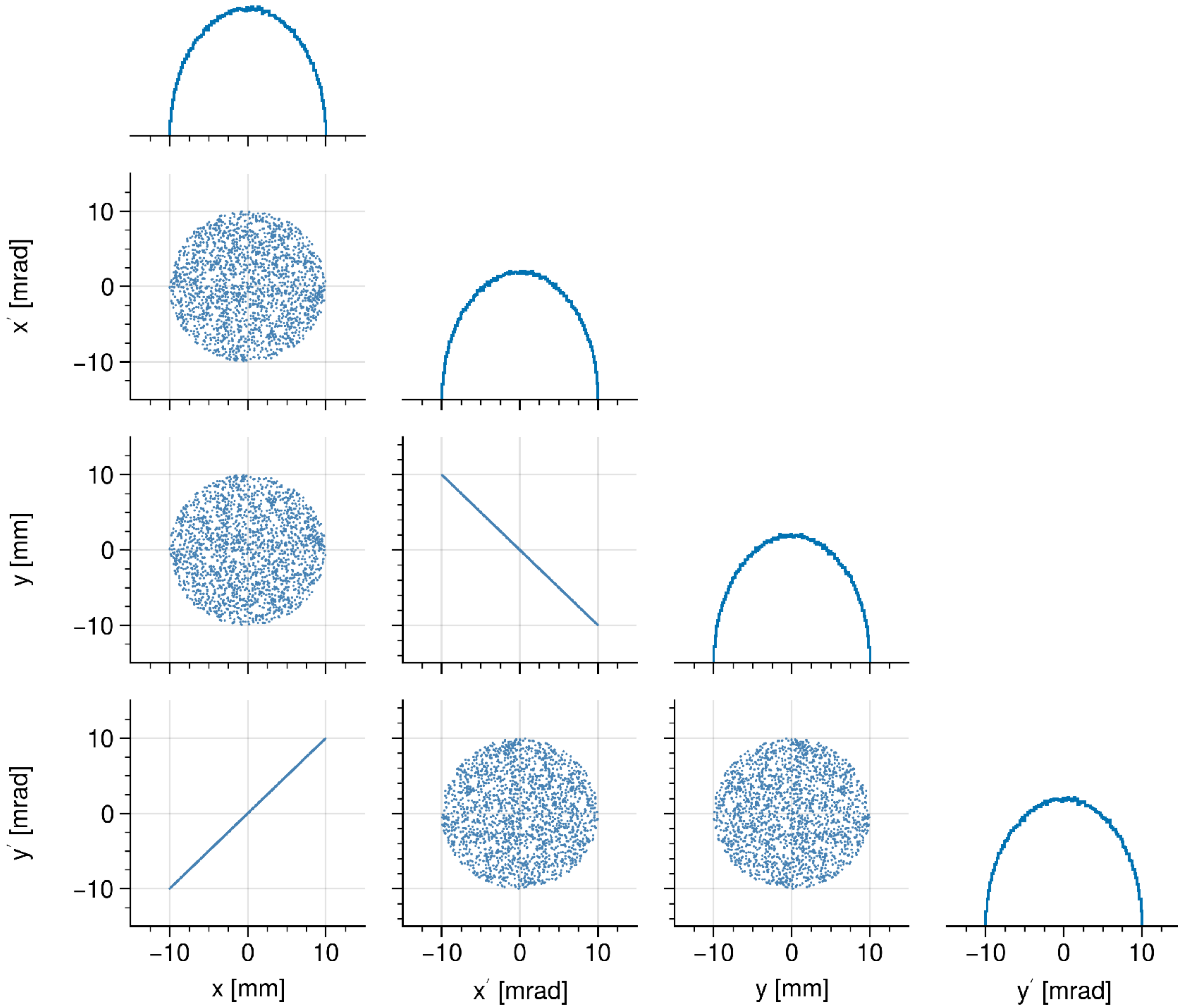
- Self-consistent
- 2D coasting beam (infinite or zero length)
- Uniform density in every 2D projection
- Linear relationships between coordinates

$$f(\mathbf{x}) \propto \delta\left(x' + e_{11}x + e_{12}y\right)\delta\left(y' + e_{21}x + e_{22}y\right)$$

- Zero emittance in 4D: $\varepsilon_{4D} = \varepsilon_1\varepsilon_2 = 0$

$$\varepsilon_1 = 0 \qquad \varepsilon_2 = \varepsilon_x + \varepsilon_y$$

$$\varepsilon_1 = \varepsilon_x + \varepsilon_y \qquad \varepsilon_2 = 0$$



Phase space projections of the Danilov distribution

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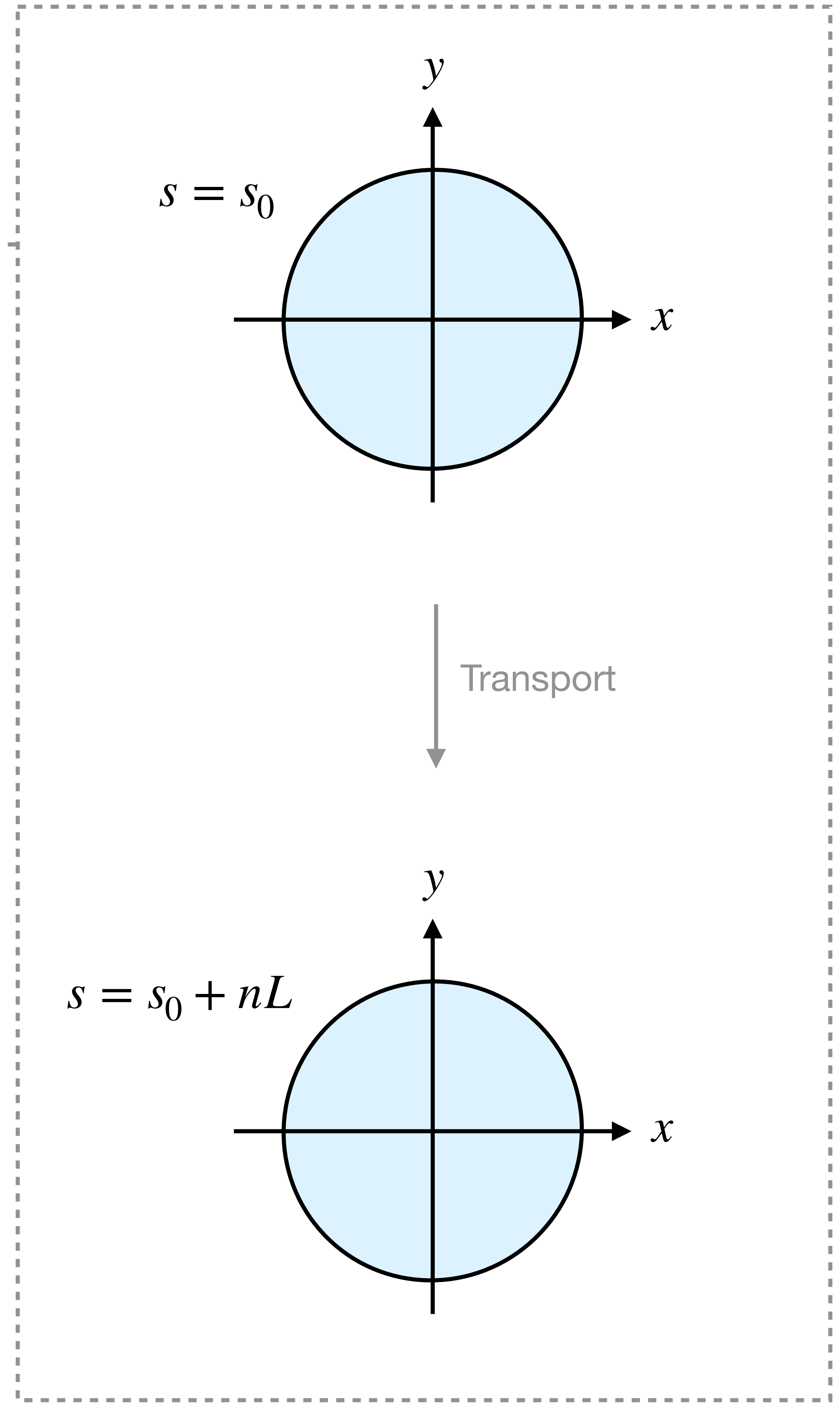
Problem and motivation

- Matched beam has same periodicity as lattice:

$$\mathbf{\Sigma}(s + L) = \mathbf{\Sigma}(s) \text{ for all } s$$

- Importance (general)
  - Minimum energy solution (free energy in mismatched beam → emittance growth)
  - Maximizes possible beam intensity within a given aperture
  - Starting point for stability analysis
- Importance (SNS)
  - Lattice + space charge introduce coupling… effect on beam not yet studied
  - Final painted beam is non-uniform if mismatched during injection
- Goal: solve problem for simple lattices, study their properties, then extend to SNS

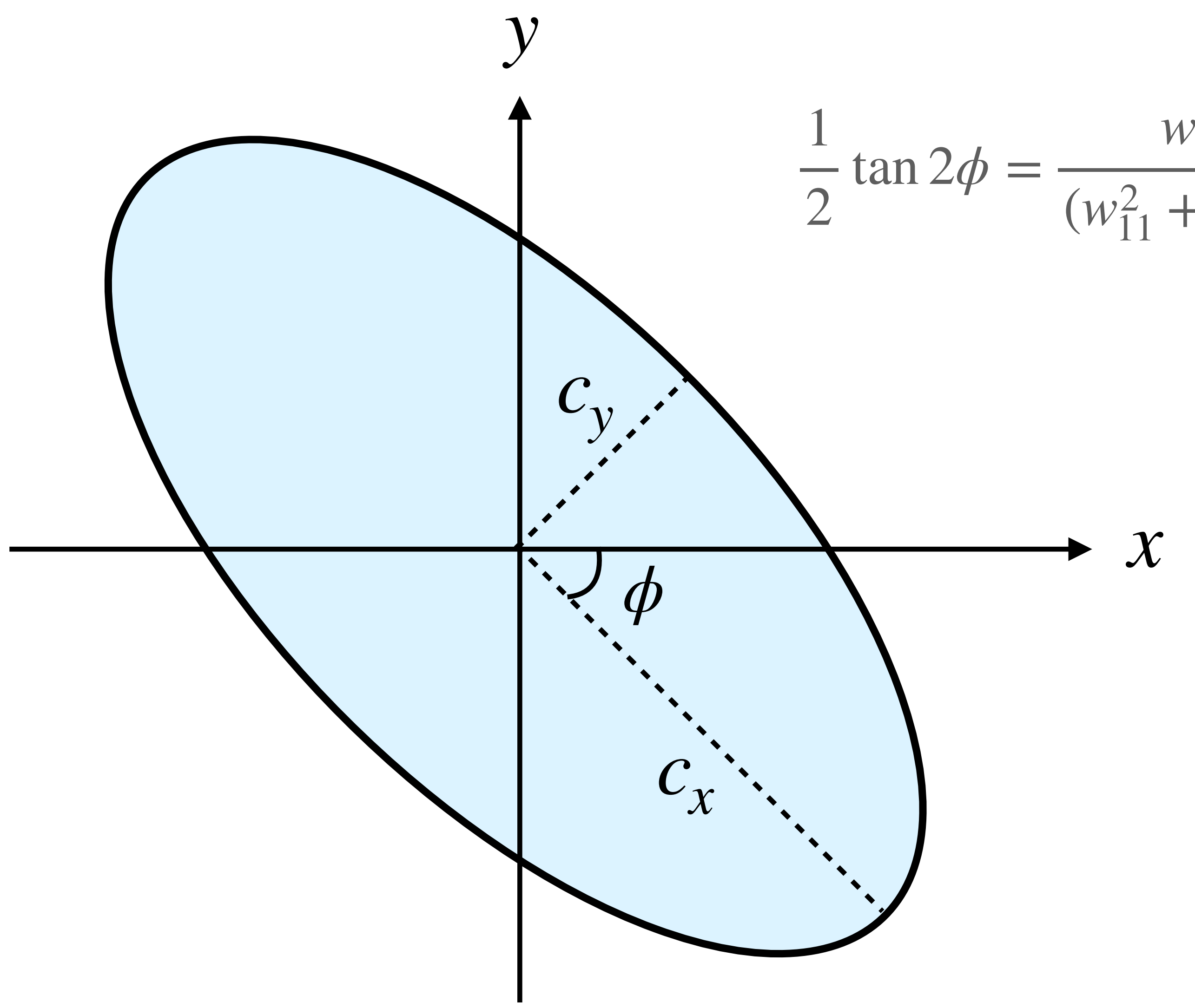


Envelope equations

- Parameterize coordinates on beam envelope

$$x = w_{11}\cos\psi + w_{12}\sin\psi$$
$$y = w_{21}\cos\psi + w_{22}\sin\psi$$

$$w = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$$



$$\frac{1}{2}\tan 2\phi = \frac{w_{11}w_{21} + w_{12}w_{22}}{(w_{11}^2 + w_{12}^2) - (w_{12}^2 + w_{22}^2)}$$

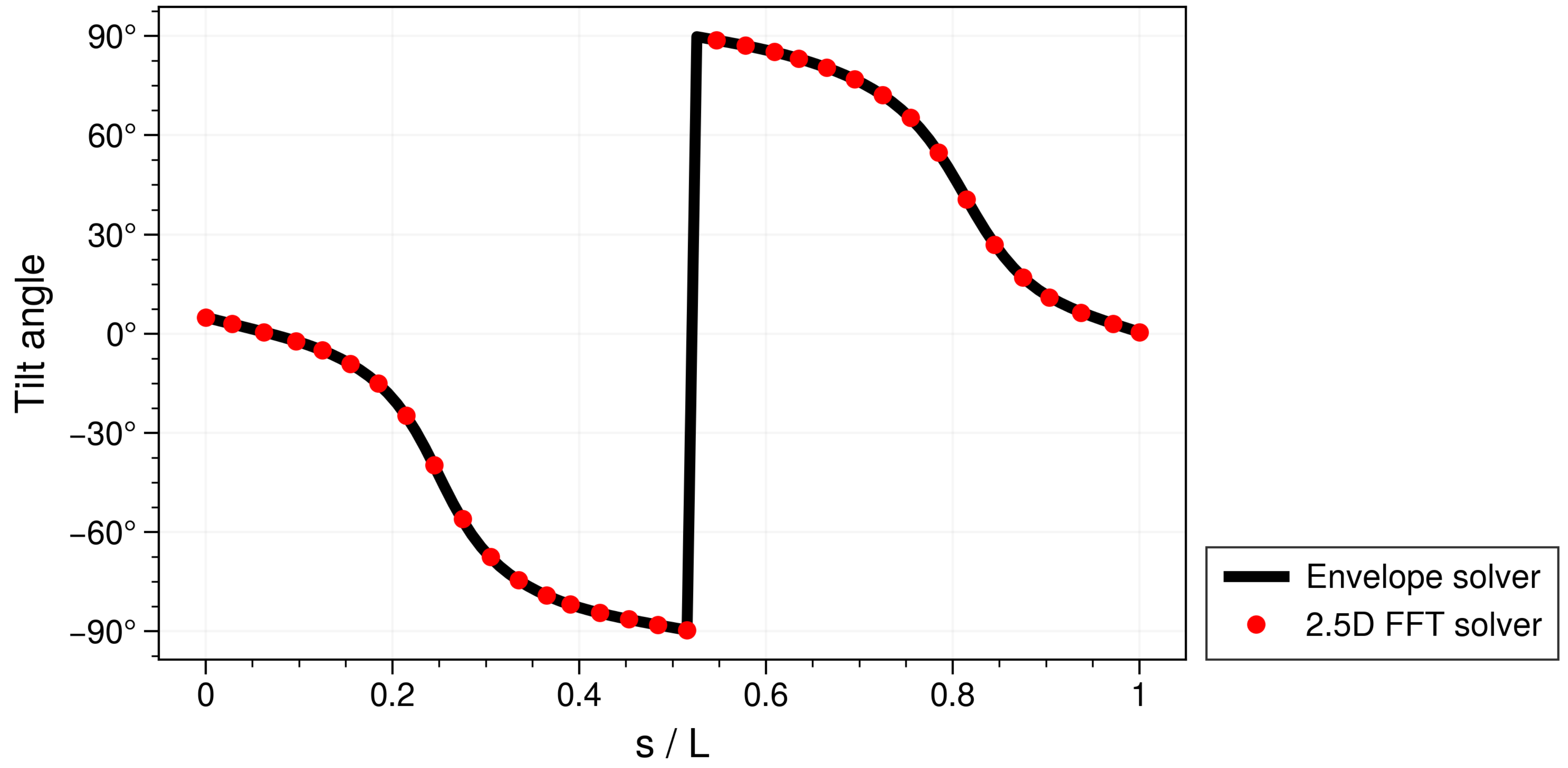
- Envelope equations: $\boxed{w'' + \left(\kappa_0 + \kappa_{sc}\right) w + \kappa_1 w' = 0}$

Lattice focusing: $\kappa_{0,1}$ (2×2 matrices)

Space charge defocusing: $\kappa_{sc} = -\frac{2Q}{c_x + c_y} R(\phi) \begin{bmatrix} 1/c_x & 0 \\ 0 & 1/c_y \end{bmatrix} R(\phi)^T$

Beam perveance: $Q = \frac{qI}{2\pi\epsilon_0 mc^3\gamma^3\beta^3}$

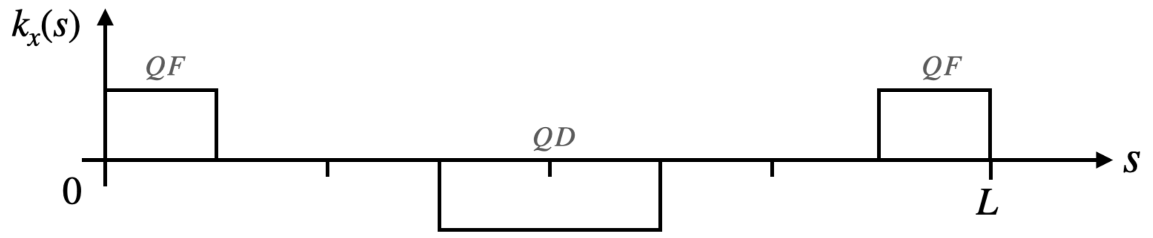
Rotation matrix: $R(\phi)$



- Quick to implement in PyORBIT

- Possible to track test particles inside our outside beam

Beam tilt without space charge



- Every particle has same x-y phase relationship

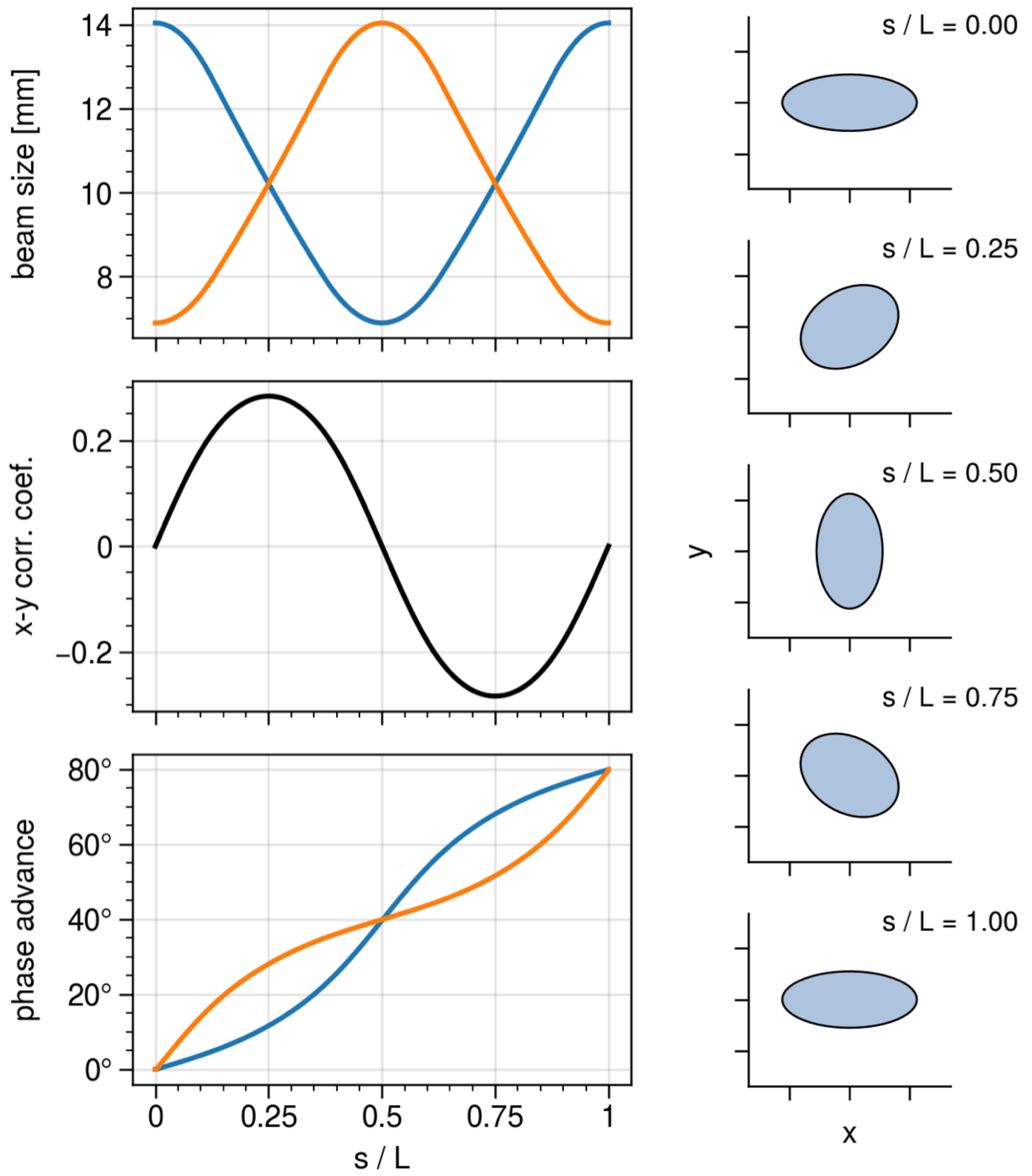
$$\frac{d\mu_x}{ds} = \frac{\varepsilon_x(s)}{\tilde{x}(s)^2} \qquad \frac{d\mu_y}{ds} = \frac{\varepsilon_y(s)}{\tilde{y}(s)^2}$$

$$\tilde{u}^2 = \langle u^2 \rangle$$

- Any change in relationship leads to tilting

$$\frac{\langle xy \rangle}{\sqrt{\langle x^2 \rangle \langle y^2 \rangle}} \propto \mu_y(s) - \mu_x(s)$$

- True without any coupled forces



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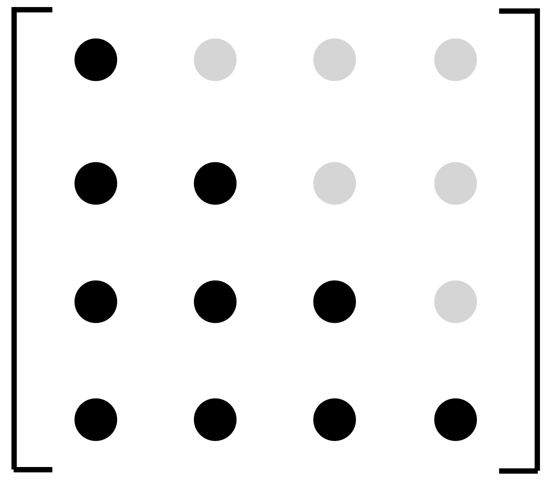
Problem formulation

- Minimize $C(\sigma) = \left| \mathbf{T}\sigma - \sigma \right|^2$ subject to $\left| \mathbf{\Sigma} \right| = 0$
  - $\sigma \equiv$ 10 element vector of unique elements of $\mathbf{\Sigma}$
  - $\mathbf{T}$: map connecting initial/final envelope for one lattice period
  - Constraint present so that $\varepsilon_{4D} = 0$
  - Would also like to hold intrinsic emittance $\varepsilon_{1,2}$ fixed:
- Difficulty: $\mathbf{T} = \mathbf{T}(\sigma)$
  - Potentially complicated dependence on initial beam
  - Dependence unknown before tracking (iterative method needed)

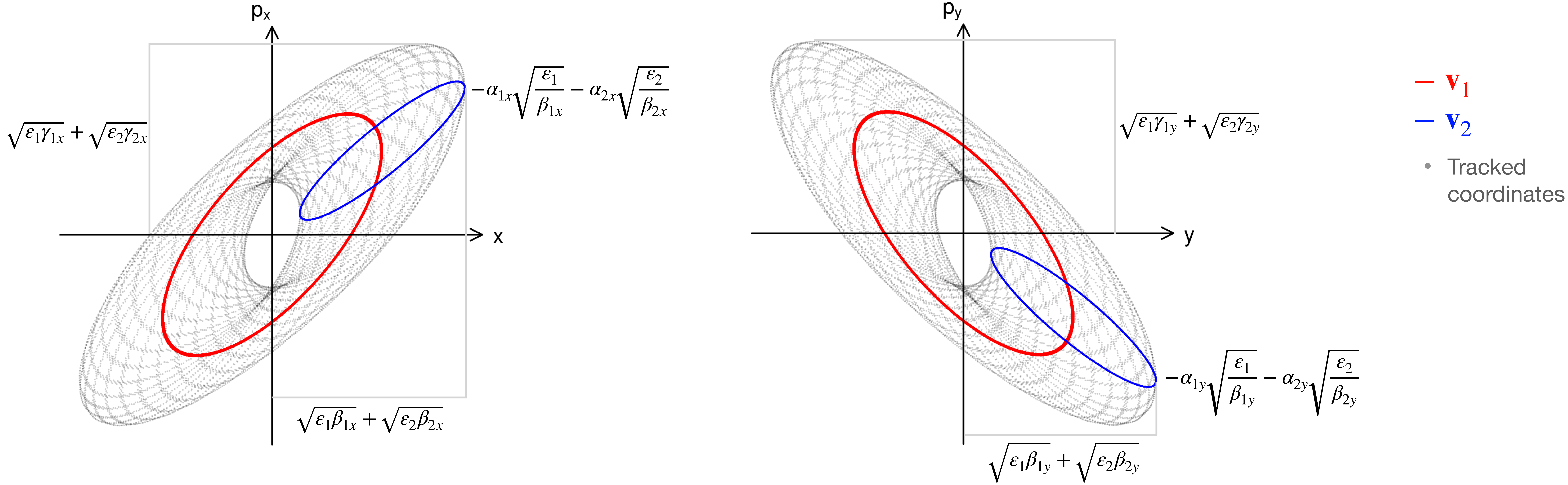
$\mathbf{\Sigma} =$ 

$$\varepsilon_1 = \frac{1}{2}\sqrt{-tr\left[(\Sigma U)^2\right] + \sqrt{tr\left[(\Sigma U)^2\right] - 16|\Sigma|}} = \varepsilon_x + \varepsilon_y$$

$$\varepsilon_2 = \frac{1}{2}\sqrt{-tr\left[(\Sigma U)^2\right] - \sqrt{tr\left[(\Sigma U)^2\right] - 16|\Sigma|}} = 0$$

Matching to known transfer matrix

$$\mathbf{Mv} = e^{-i\mu}\mathbf{v}$$



- Turn-by-turn coordinates form closed surface in 4D phase space
- Particles are distributed uniformly over this surface → matched
- Matched distribution = collection of these surfaces with different volumes

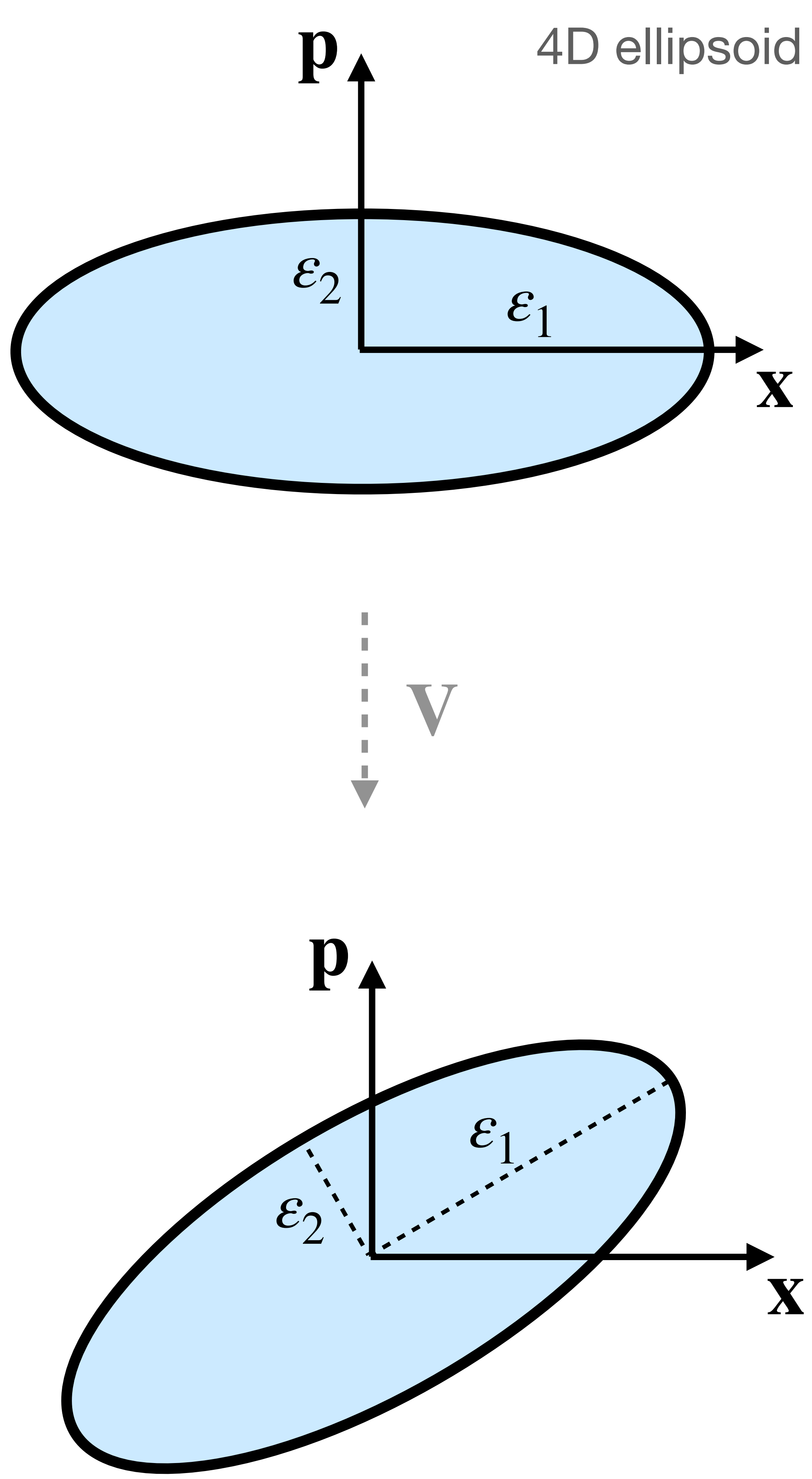
$$\gamma_{1x}\,\beta_{1x} = (1-u)^2 + \alpha_{1x}^2$$
$$\gamma_{2x}\,\beta_{2x} = u^2 + \alpha_{2x}^2$$
$$\gamma_{1y}\,\beta_{1y} = u^2 + \alpha_{1y}^2$$
$$\gamma_{2y}\,\beta_{2y} = (1-u)^2 + \alpha_{2y}^2$$

Matching to known transfer matrix

- Matched beam covariance matrix: $\mathbf{\Sigma} = \mathbf{V}\,\mathbf{\Sigma}_n \mathbf{V}^T$

$$\mathbf{\Sigma}_n = \begin{bmatrix} \varepsilon_1 & 0 & 0 & 0 \\ 0 & \varepsilon_1 & 0 & 0 \\ 0 & 0 & \varepsilon_2 & 0 \\ 0 & 0 & 0 & \varepsilon_2 \end{bmatrix}$$

$$\mathbf{V} = \begin{array}{c} \begin{bmatrix} \sqrt{\beta_{1x}} & 0 & \sqrt{\beta_{2x}}\cos\nu_2 & -\sqrt{\beta_{2x}}\sin\nu_2 \\ -\frac{\alpha_{1x}}{\sqrt{\beta_{1x}}} & \frac{(1-u)}{\sqrt{\beta_{1x}}} & \frac{u\sin\nu_2 - \alpha_{2x}\cos\nu_2}{\sqrt{\beta_{2x}}} & \frac{u\cos\nu_2 + \alpha_{2x}\sin\nu_2}{\sqrt{\beta_{2x}}} \\ \sqrt{\beta_{1y}}\cos\nu_1 & -\sqrt{\beta_{1y}}\sin\nu_1 & \sqrt{\beta_{2y}} & 0 \\ \frac{u\sin\nu_1 - \alpha_{1y}\cos\nu_1}{\sqrt{\beta_{1y}}} & \frac{u\cos\nu_1 + \alpha_{1y}\sin\nu_1}{\sqrt{\beta_{1y}}} & -\frac{\alpha_{2y}}{\sqrt{\beta_{2y}}} & \frac{(1-u)}{\sqrt{\beta_{2y}}} \end{bmatrix} \\ \begin{array}{cccc} Re[\mathbf{v}_1] & -Im[\mathbf{v}_1] & Re[\mathbf{v}_2] & -Im[\mathbf{v}_2] \end{array} \end{array}$$



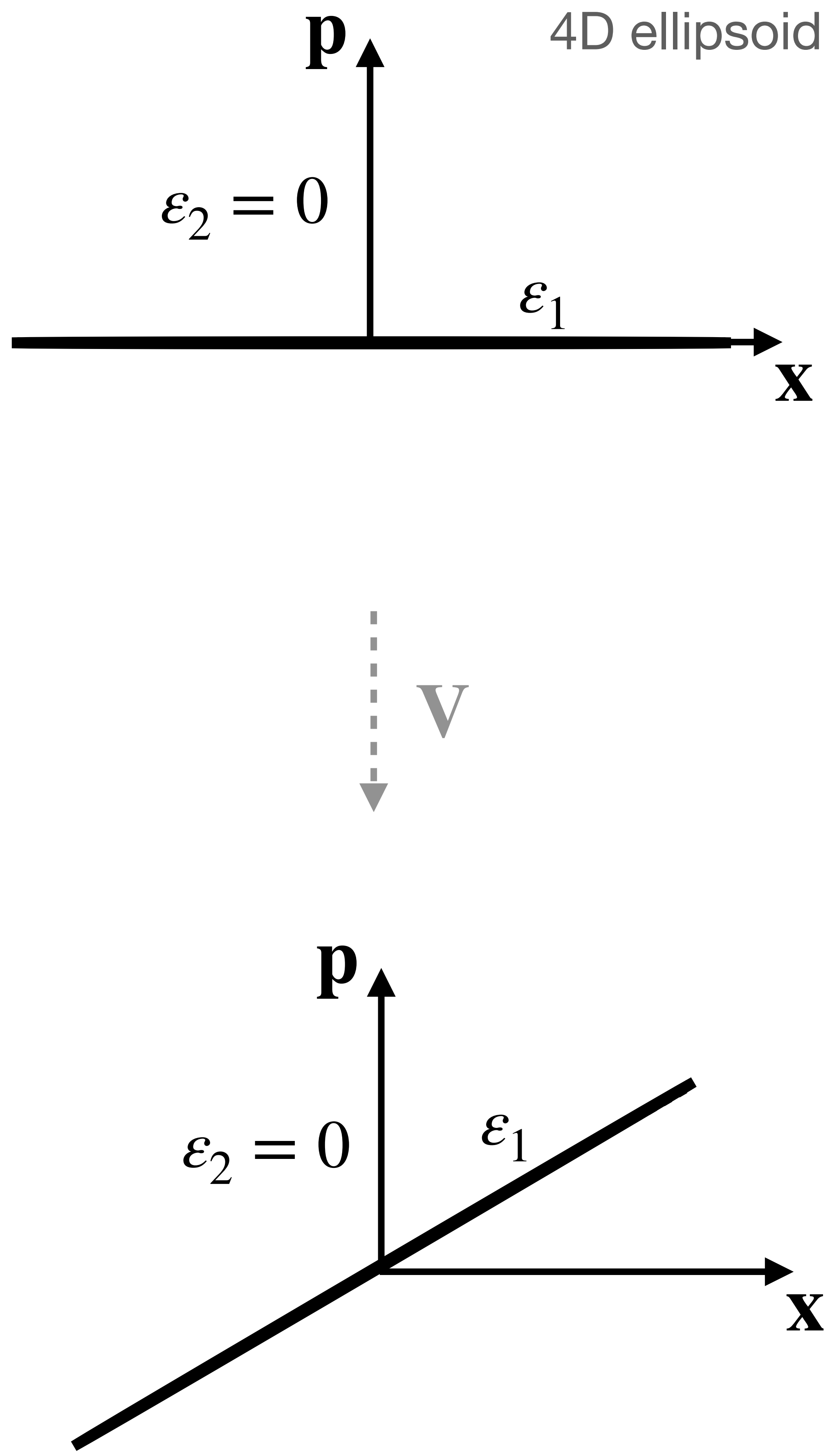
Matching to known transfer matrix

- Matched beam covariance matrix: $\boldsymbol{\Sigma} = \mathbf{V}\,\boldsymbol{\Sigma}_n\mathbf{V}^T$

$$\boldsymbol{\Sigma}_n = \begin{bmatrix} \varepsilon_1 & 0 & 0 & 0 \\ 0 & \varepsilon_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Matched Danilov distribution: function of single eigenvector

$$\mathbf{V} = \begin{array}{c} \begin{bmatrix} \sqrt{\beta_{1x}} & 0 & \sqrt{\beta_{2x}}\cos\nu_2 & -\sqrt{\beta_{2x}}\sin\nu_2 \\ -\dfrac{\alpha_{1x}}{\sqrt{\beta_{1x}}} & \dfrac{(1-u)}{\sqrt{\beta_{1x}}} & \dfrac{u\sin\nu_2 - \alpha_{2x}\cos\nu_2}{\sqrt{\beta_{2x}}} & \dfrac{u\cos\nu_2 + \alpha_{2x}\sin\nu_2}{\sqrt{\beta_{2x}}} \\ \sqrt{\beta_{1y}}\cos\nu_1 & -\sqrt{\beta_{1y}}\sin\nu_1 & \sqrt{\beta_{2y}} & 0 \\ \dfrac{u\sin\nu_1 - \alpha_{1y}\cos\nu_1}{\sqrt{\beta_{1y}}} & \dfrac{u\cos\nu_1 + \alpha_{1y}\sin\nu_1}{\sqrt{\beta_{1y}}} & -\dfrac{\alpha_{2y}}{\sqrt{\beta_{2y}}} & \dfrac{(1-u)}{\sqrt{\beta_{2y}}} \end{bmatrix} \\ \begin{matrix} Re[\mathbf{v}_1] & -Im[\mathbf{v}_1] & Re[\mathbf{v}_2] & -Im[\mathbf{v}_2] \end{matrix} \end{array}$$



Matching to known transfer matrix

- Matched beam is function of single eigenvector → 4D Twiss parameters simplify
- 6 independent parameters given by $\mathbf{p}$

$$\mathbf{p} = \left(\beta_{lx}, \beta_{ly}, \alpha_{lx}, \alpha_{ly}, u, \nu\right) \qquad l = 1, 2$$

  - Size in either plane (β)
  - Divergence in either plane (α)
  - Ratio between apparent emittances (u)
  - Phase difference between x and y (ν)
- Two choices of eigenvector

Solution 1 $(\varepsilon_2 = 0)$

$$\beta_{1x} = \frac{\langle x^2 \rangle}{\varepsilon_1}$$

$$\beta_{1y} = \frac{\langle y^2 \rangle}{\varepsilon_1}$$

$$\alpha_{1x} = -\frac{\langle x p_x \rangle}{\varepsilon_1}$$

$$\alpha_{1y} = -\frac{\langle y p_y \rangle}{\varepsilon_1}$$

$$\cos\nu = \frac{\langle xy \rangle}{\sqrt{\langle x^2 \rangle \langle y^2 \rangle}}$$

$$1 - u = \varepsilon_x / \varepsilon_1$$

$$u = \varepsilon_y / \varepsilon_1$$

Solution 2 $(\varepsilon_1 = 0)$

$$\beta_{2x} = \frac{\langle x^2 \rangle}{\varepsilon_2}$$

$$\beta_{2y} = \frac{\langle y^2 \rangle}{\varepsilon_2}$$

$$\alpha_{2x} = -\frac{\langle x p_x \rangle}{\varepsilon_2}$$

$$\alpha_{2y} = -\frac{\langle y p_y \rangle}{\varepsilon_2}$$

$$\cos\nu = \frac{\langle xy \rangle}{\sqrt{\langle x^2 \rangle \langle y^2 \rangle}}$$

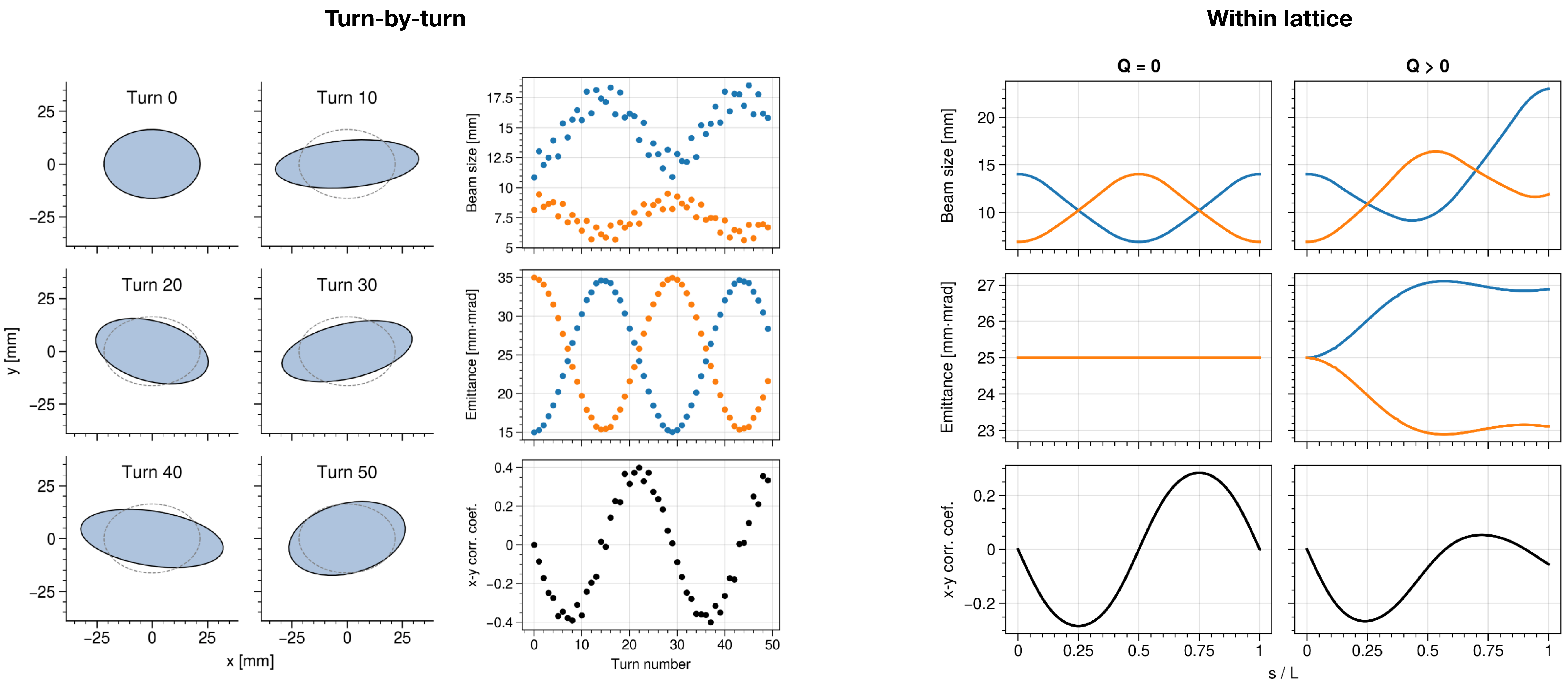
$$u = \varepsilon_x / \varepsilon_1$$

$$1 - u = \varepsilon_y / \varepsilon_1$$

$$w'' + \left(\kappa_0 + \kappa_{sc}\right) w + \kappa_1 w' = 0$$

Effect of space charge

- Linear coupling evident from exchange of apparent emittances

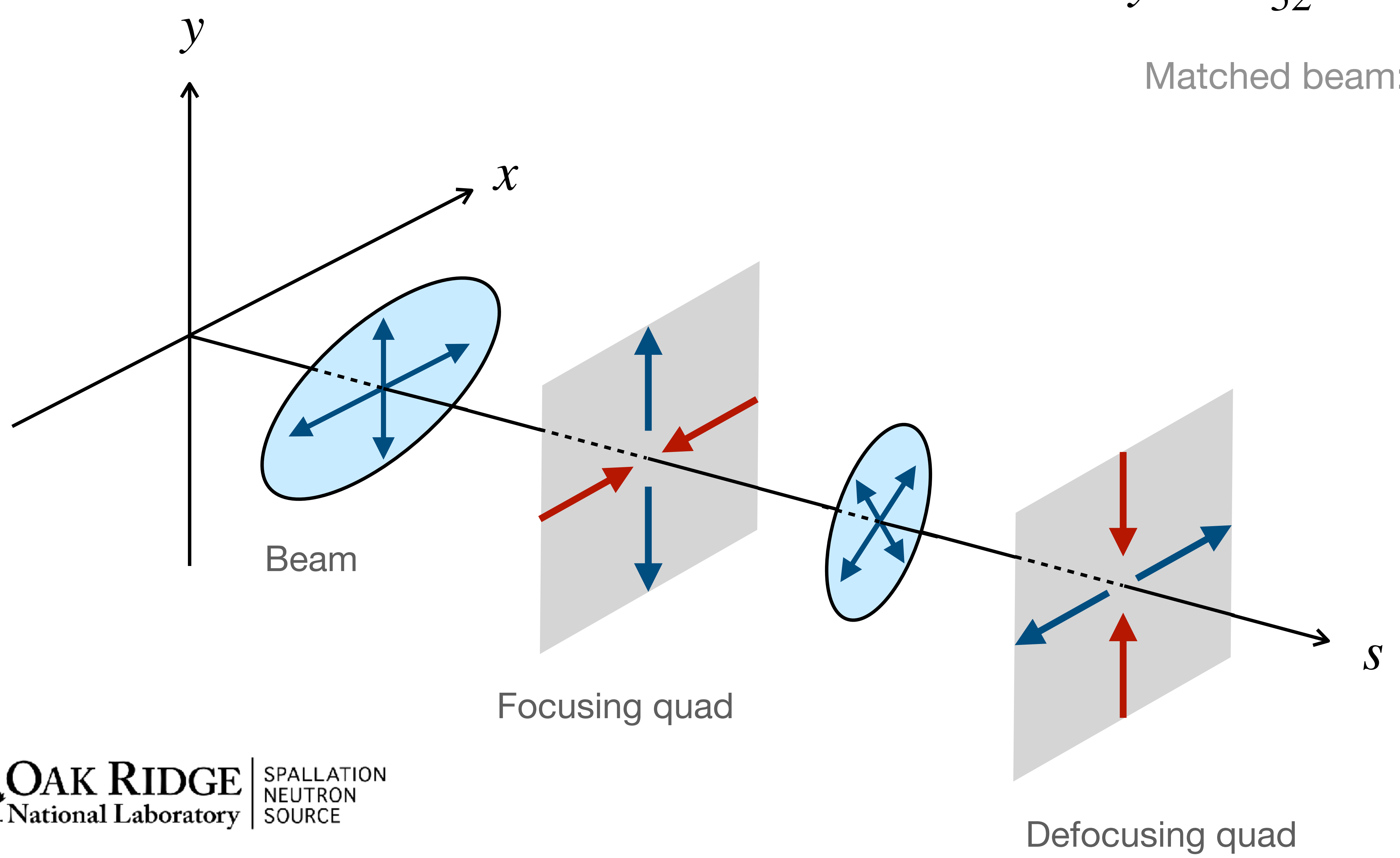


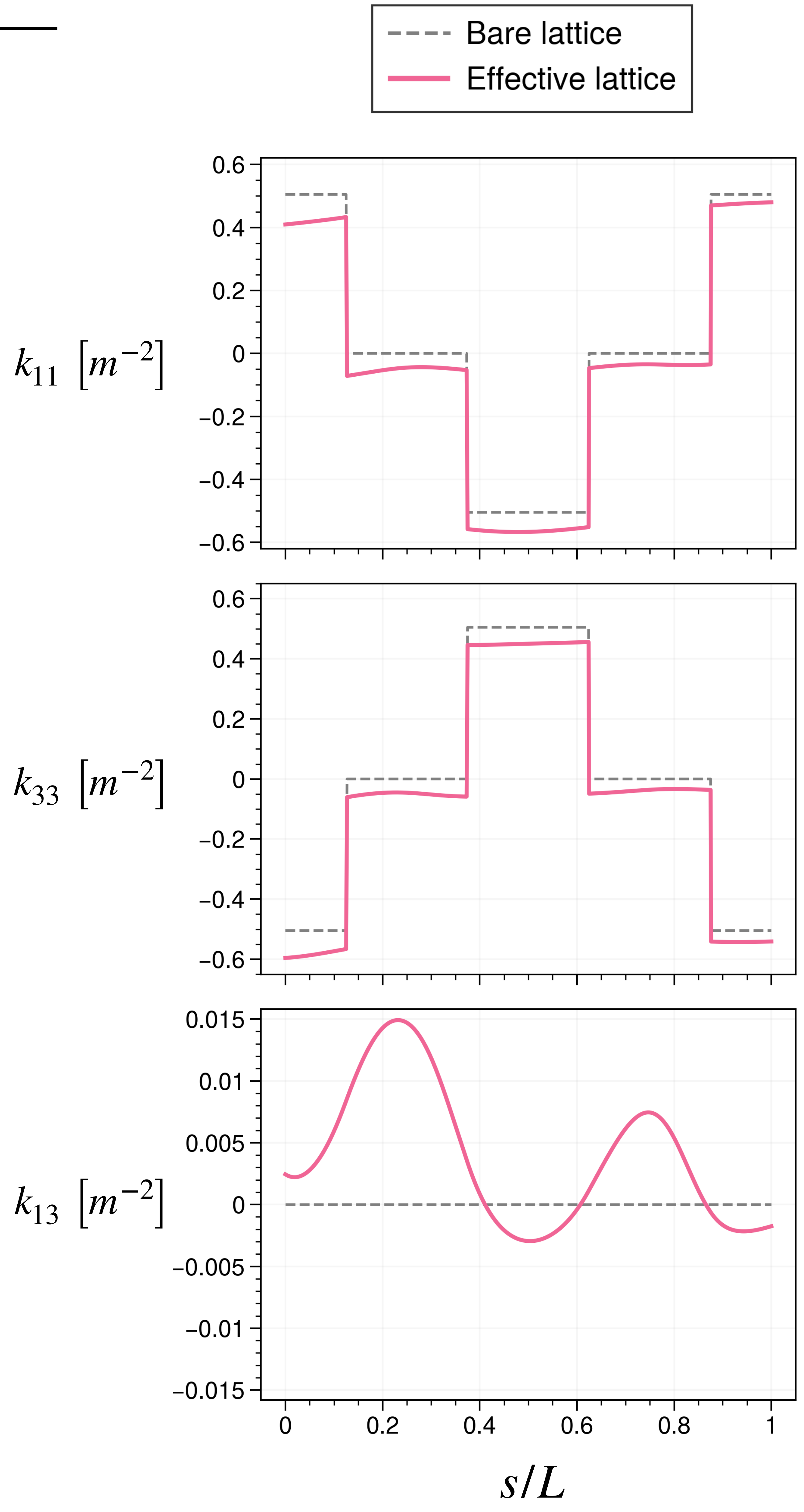
Matching with space charge

- Space charge creates *effective lattice* by modifying focusing strength
- Effective lattice is linear and coupled
- Effective lattice is periodic if produced by matched beam

$$x'' + k_{11}x + k_{13}y + k_{14}y' = 0$$
$$y'' + k_{32}x + k_{33}y + k_{23}x' = 0$$

Matched beam: $k_{ij}(s+L) = k_{ij}(s)$



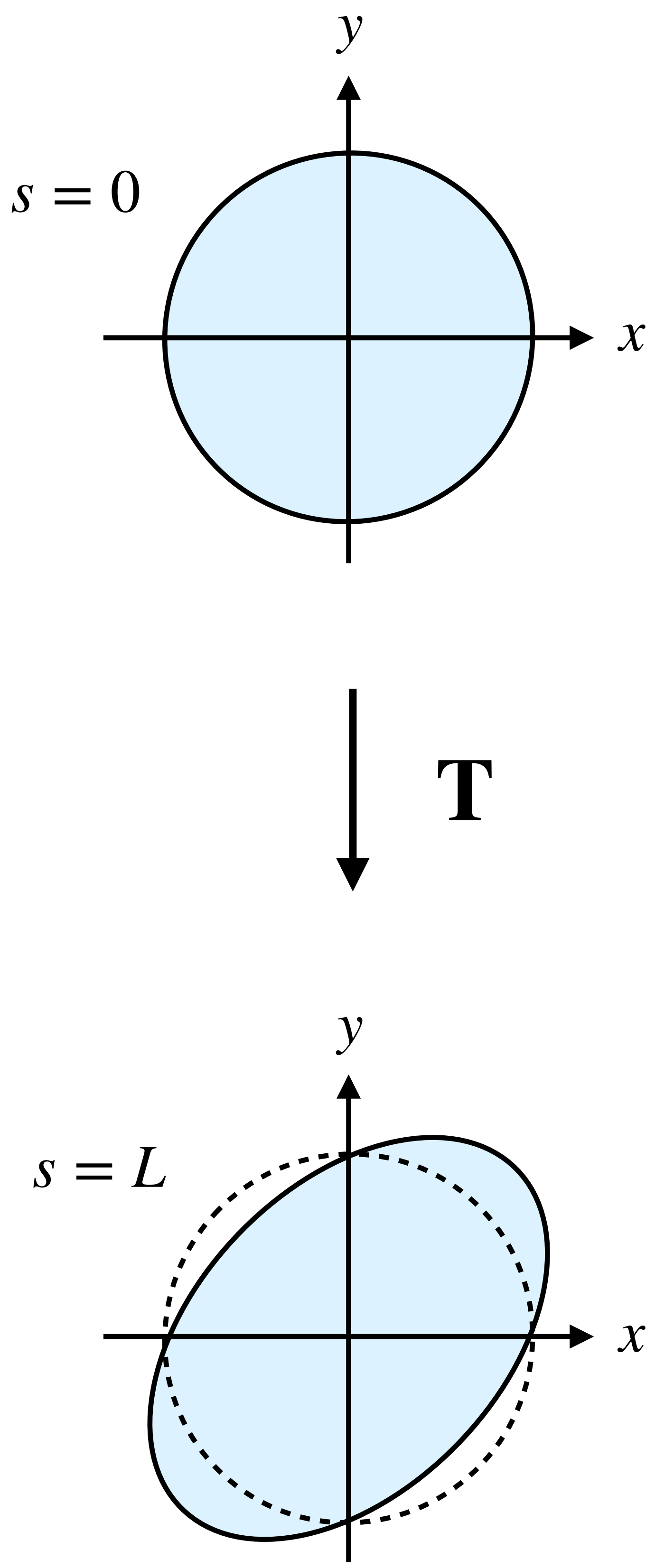


Matching with space charge

- In other words… $\mathbf{\Sigma U}$ and $\mathbf{M}_{eff}$ share an eigenvector
- Just have to find correct effective lattice

Algorithm

- Initialize $\mathbf{p}$ from bare lattice parameters
- While not matched:
  1. Generate envelope σ from $\mathbf{p}$
  2. Track one turn, calculate $C(\sigma)$
  3. Update $\mathbf{p}$
  4. Stop if relative change in C or $|\mathbf{p}|$ is below tolerance



$$C(\sigma) = \left| \mathbf{T}\sigma - \sigma \right|^2$$

Updating the parameter vector

- Trust region minimization (packaged)

SciPy.org

SciPy.org | Docs | SciPy v1.6.0 Reference Guide | Optimization and root finding (scipy.optimize)

scipy.optimize.least_squares

scipy.optimize.least_squares(*fun, x0, jac='2-point', bounds=- inf, inf, method='trf', ftol=1e-08, xtol=1e-08, gtol=1e-08, x_scale=1.0, loss='linear', f_scale=1.0, diff_step=None, tr_solver=None, tr_options={}, jac_sparsity=None, max_nfev=None, verbose=0, args=(), kwargs={}*) [source]

Solve a nonlinear least-squares problem with bounds on the variables.

SIAM J. Sci. Comput.
Vol. 21, No. 1, pp. 1–23

A SUBSPACE, INTERIOR, AND CONJUGATE GRADIENT METHOD FOR LARGE-SCALE BOUND-CONSTRAINED MINIMIZATION PROBLEMS*

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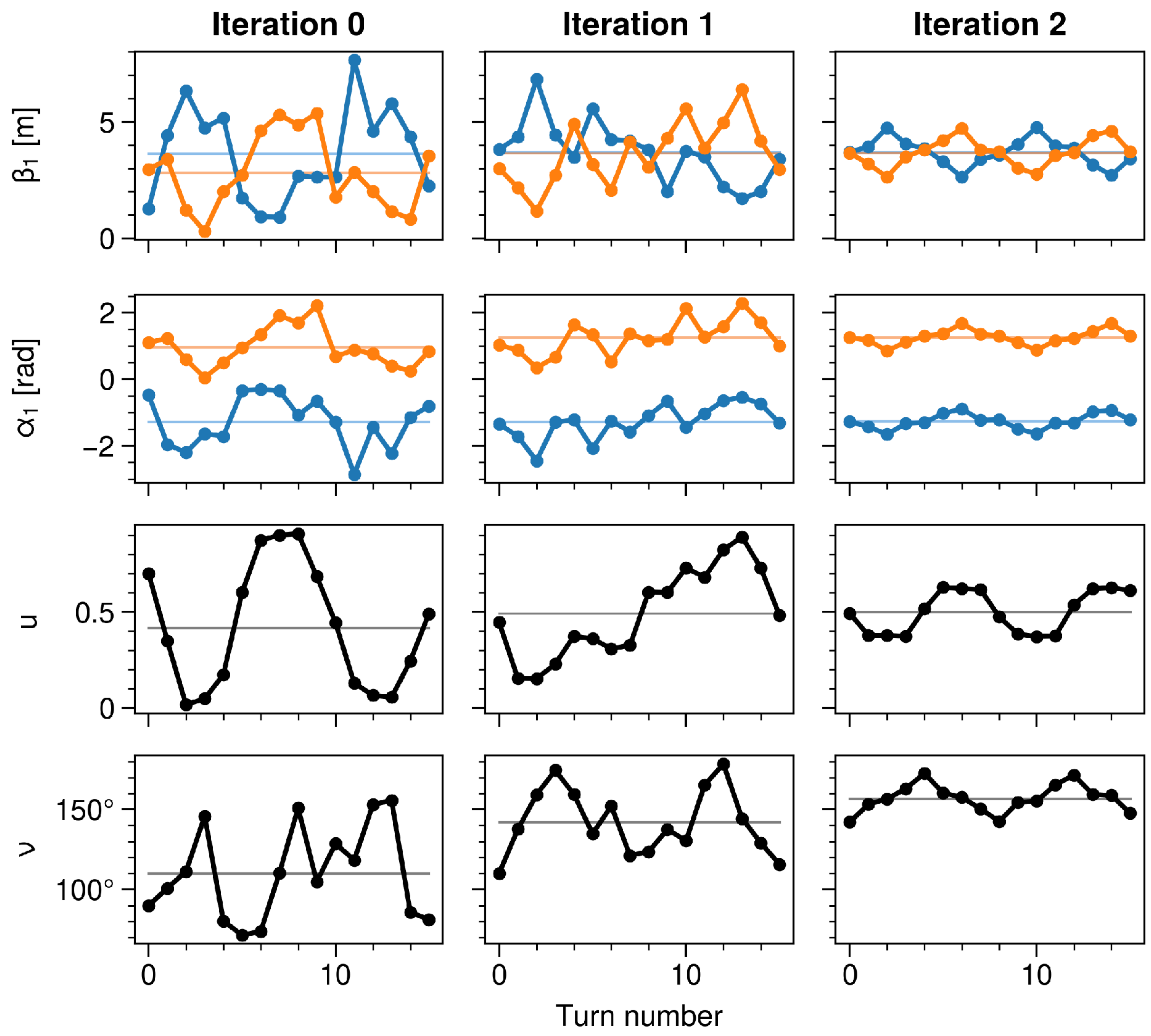
Abstract. A subspace adaptation of the Coleman–Li trust region and interior method is proposed for solving large-scale bound-constrained minimization problems. This method can be implemented with either sparse Cholesky factorization or conjugate gradient computation. Under reasonable conditions the convergence properties of this subspace trust region method are as strong as those of its full-space version.

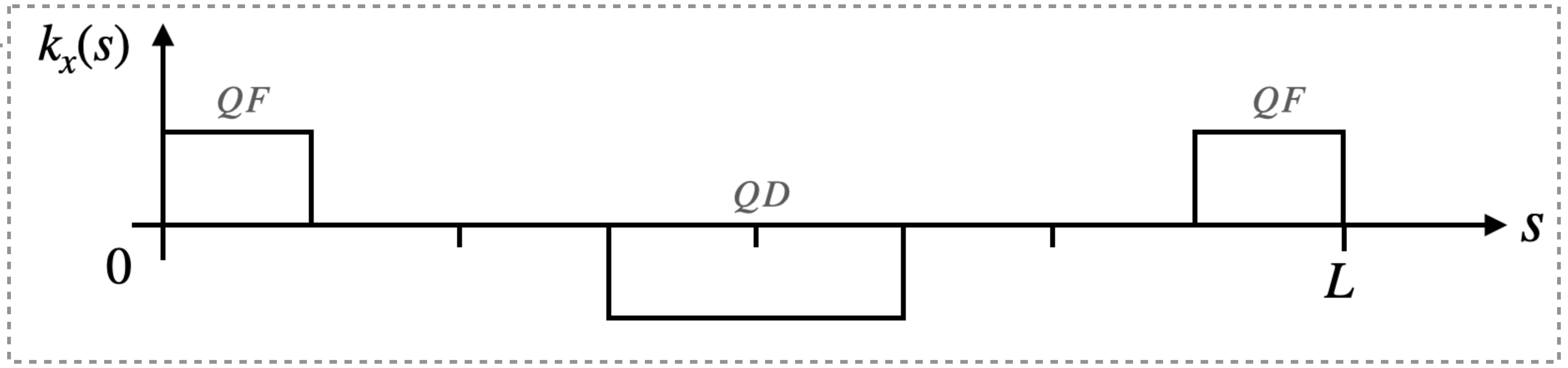
Computational performance on various large test problems is reported; advantages of our approach are demonstrated. Our experience indicates that our proposed method represents an efficient way to solve large bound-constrained minimization problems.

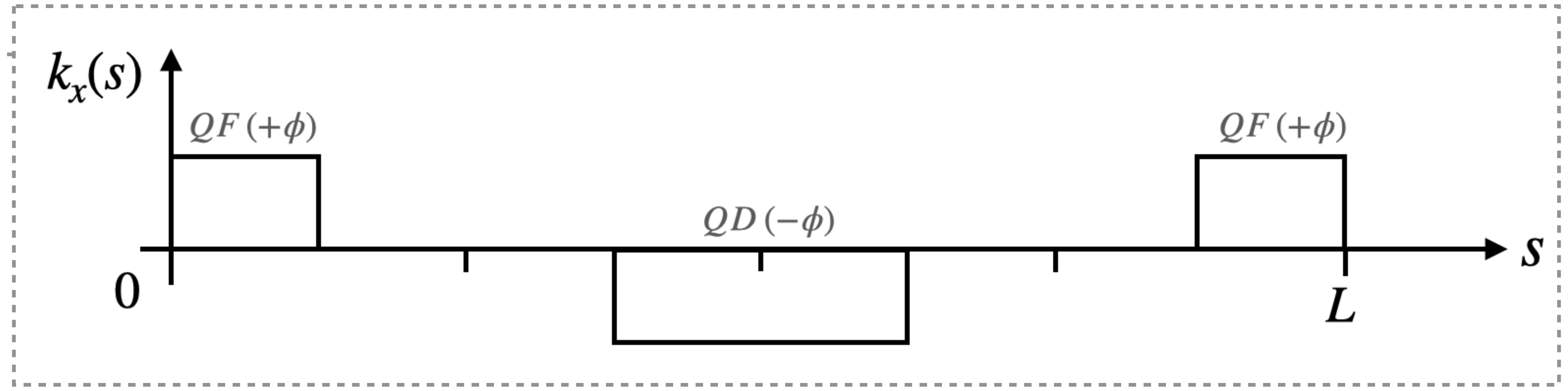
- Replace by average* (custom)

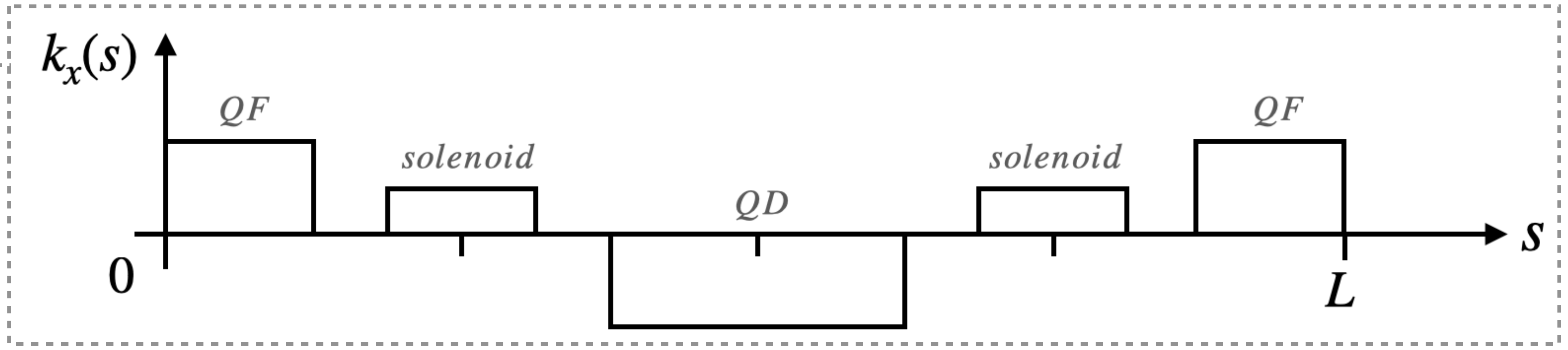


Example applications

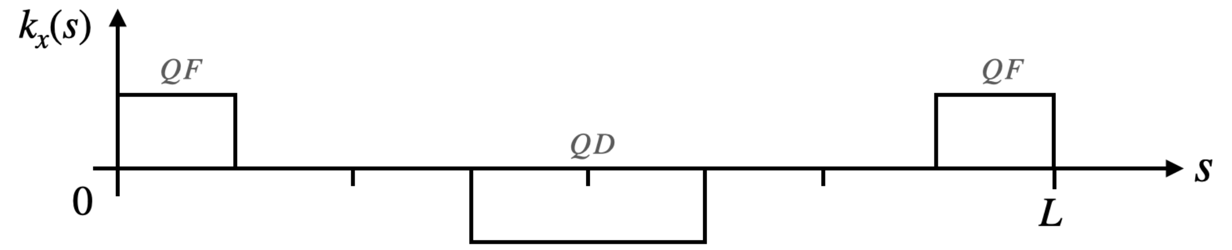
- FODO
  - Simplest case of time-dependent focusing
  - Can vary horizontal/vertical tunes
- FODO with skew quadrupoles
  - QF and QD rotated in opposite directions
  - Introduces linear coupling
- FODO with solenoid inserts
  - Solenoids fill half the drift space
  - Introduces linear coupling

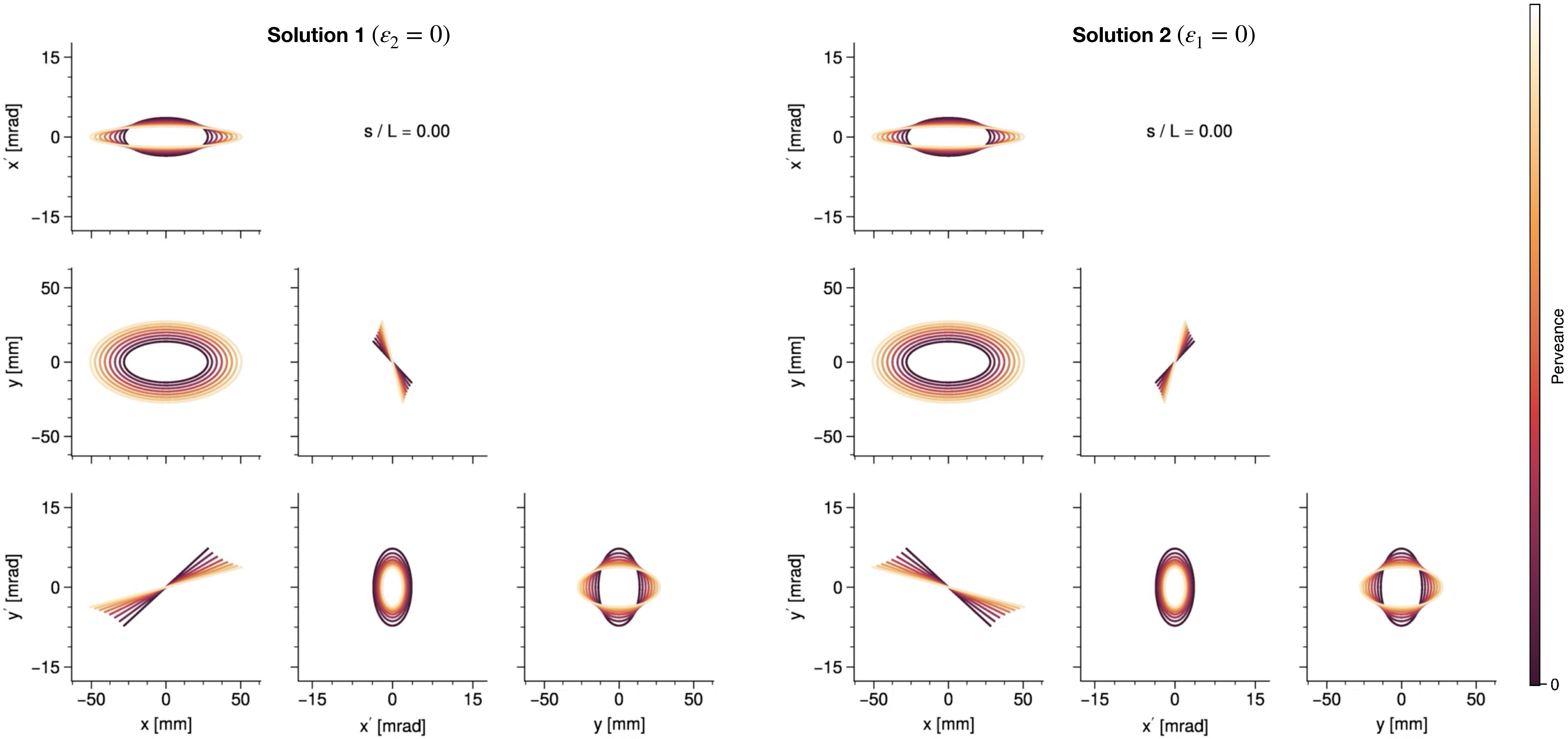




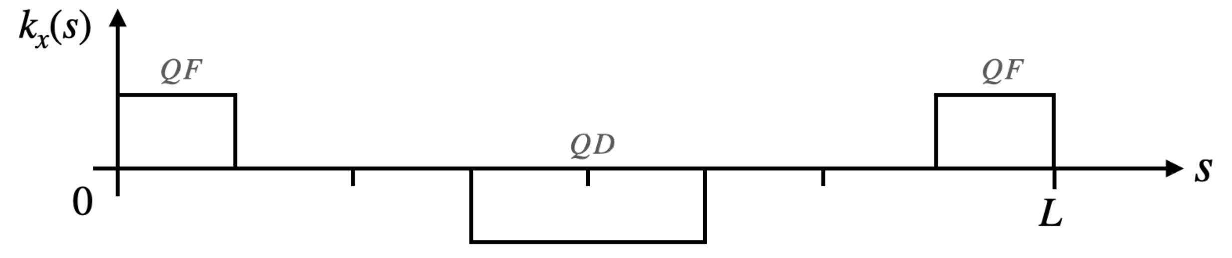


Matched beam (FODO)

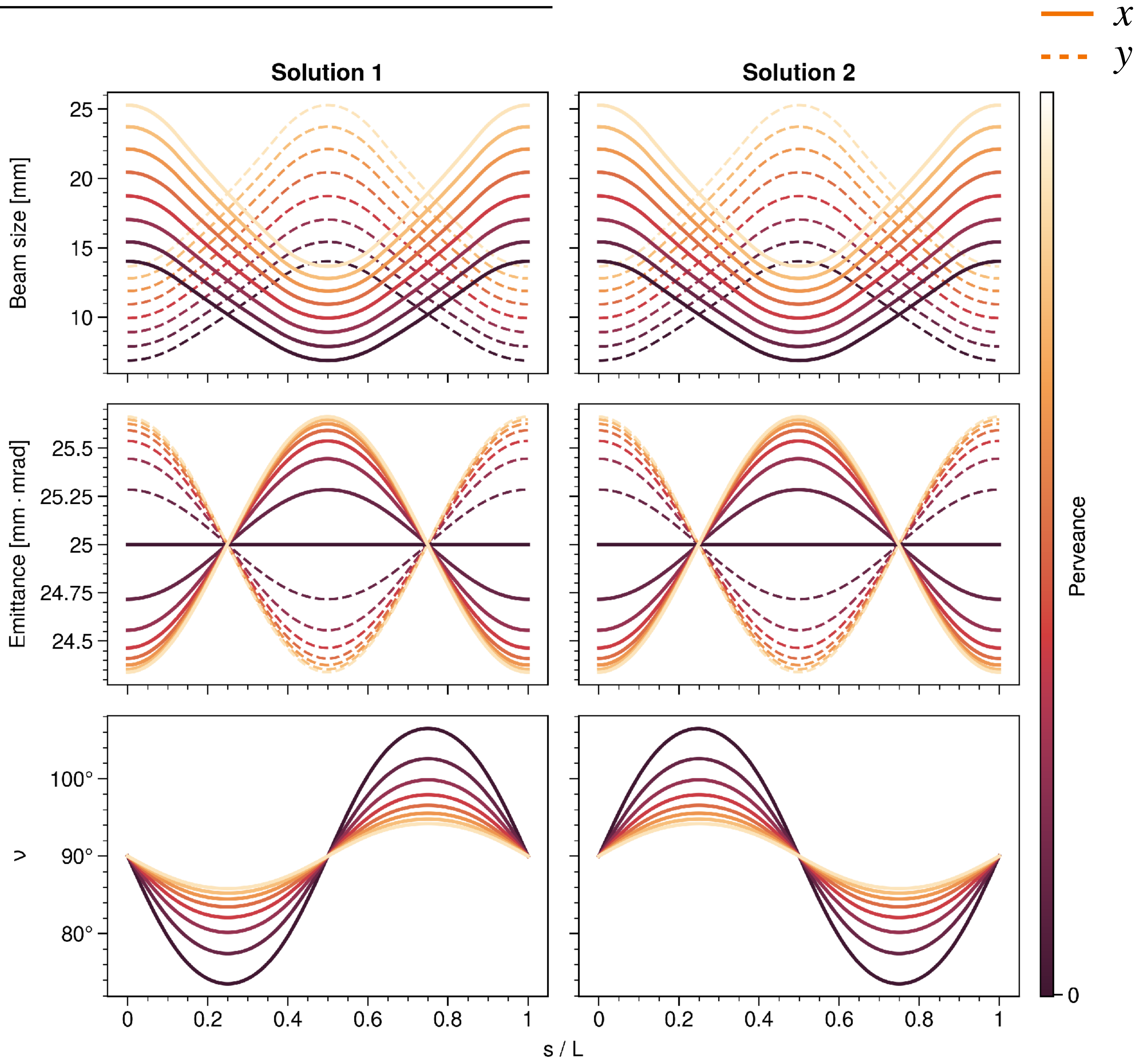




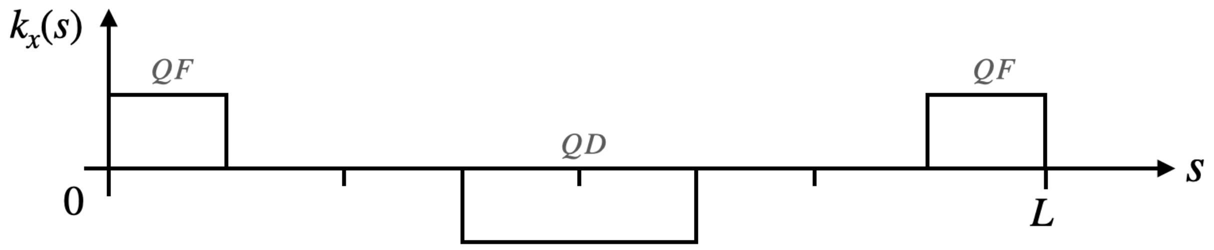
Matched beam (FODO)

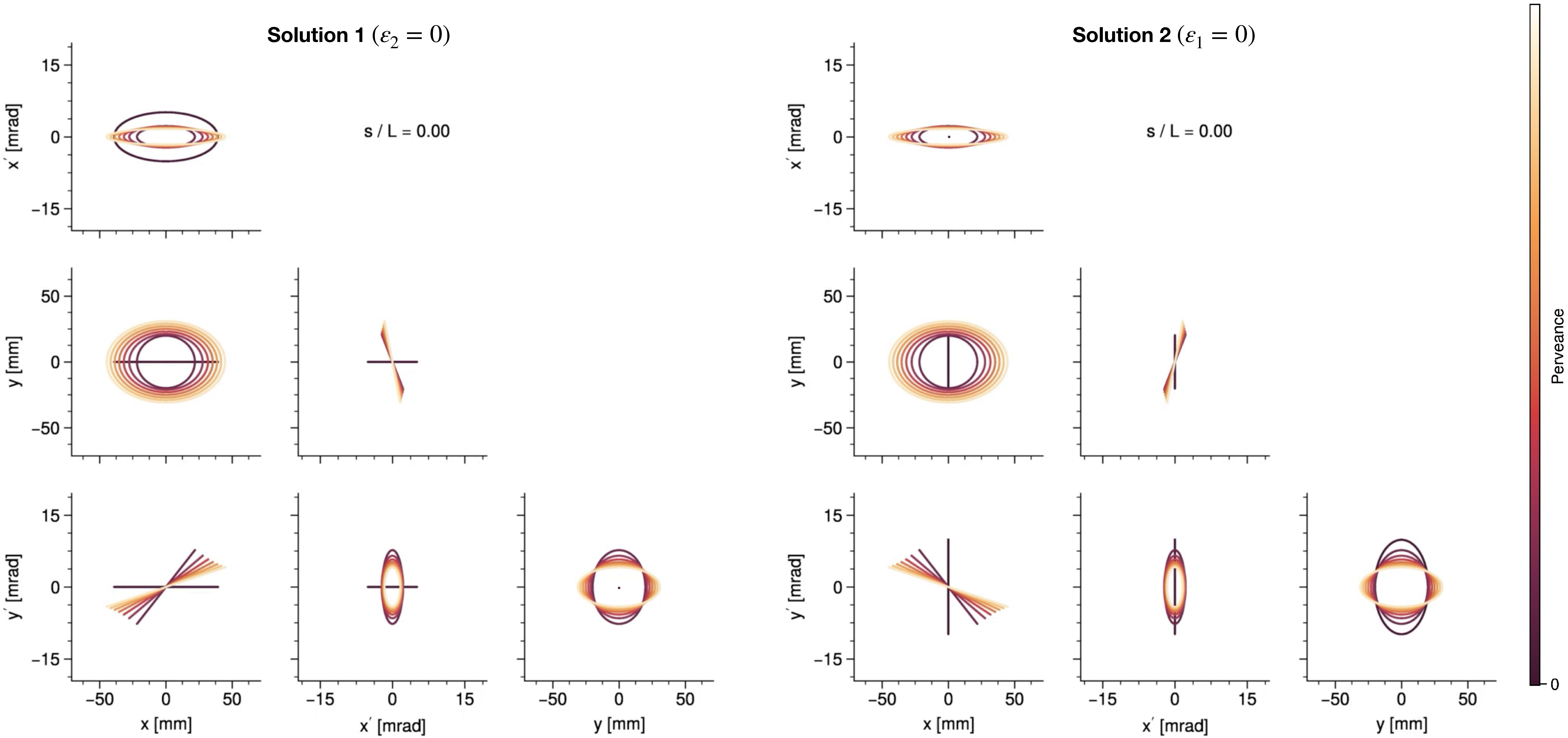


- Beam size scales ~ linearly with space charge
- x and y emittances are not constant
- Two solutions differ in sign of angular momentum (eigenvectors rotate opposite directions)
- Relevance to painting:
    - Space charge doesn't dramatically change matched beam

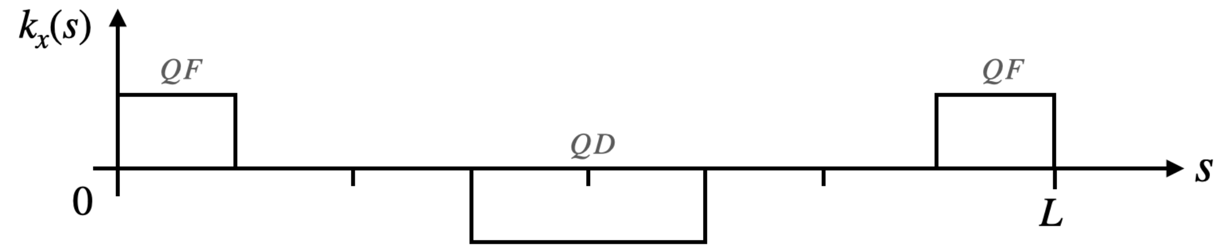


Matched beam (FODO — split tunes)

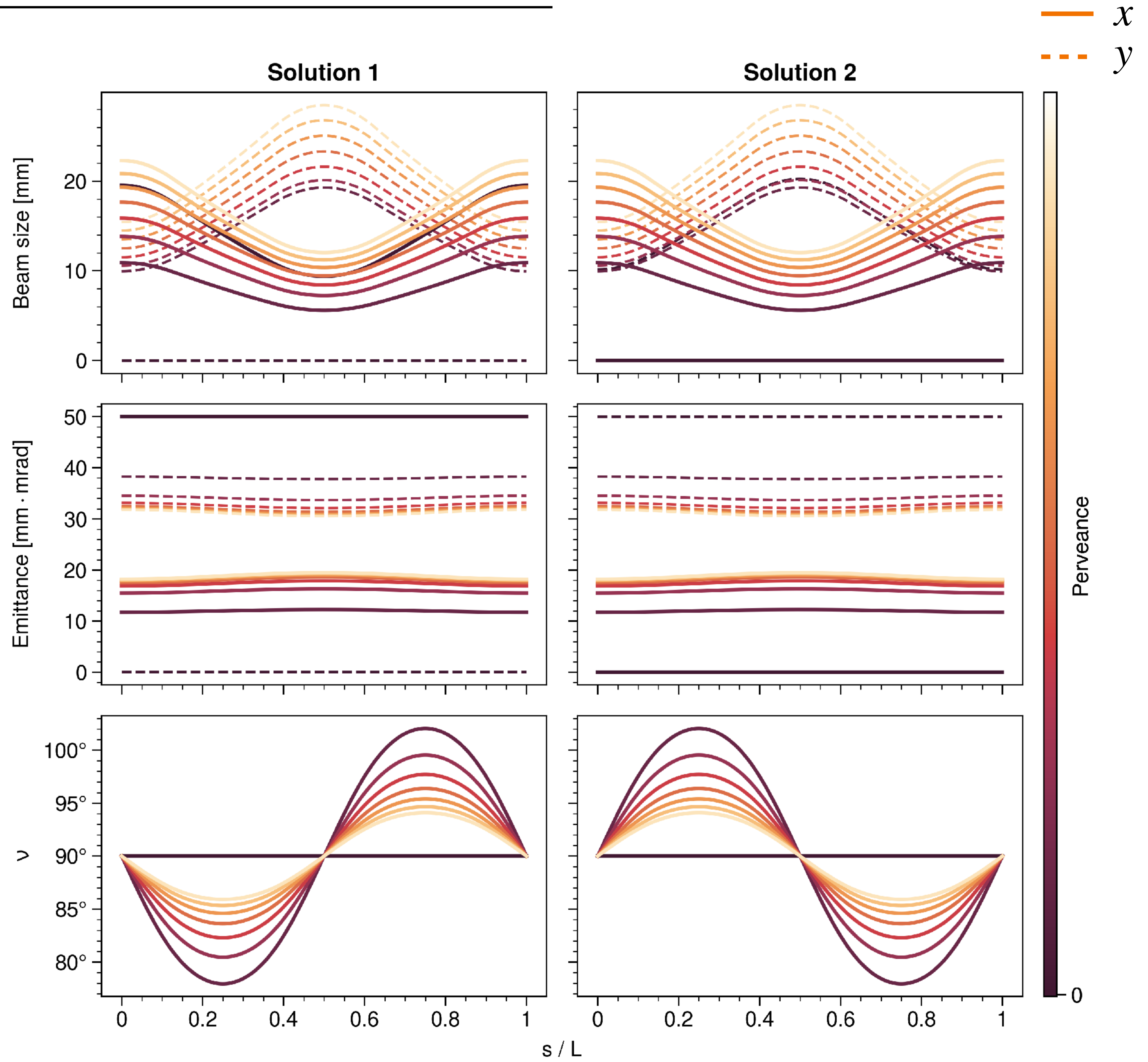




Matched beam (FODO — split tunes)



- Space charge reduces phases by unequal amounts → in the end they are equal
- Emittances initially maximally split, space charge brings together
- Otherwise, evolution similar to equal tunes case
- Relevance for painting
    - Montague resonance



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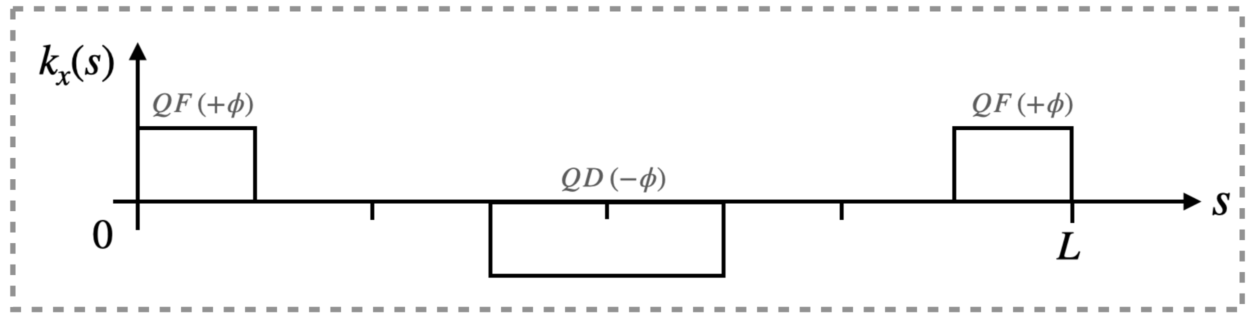
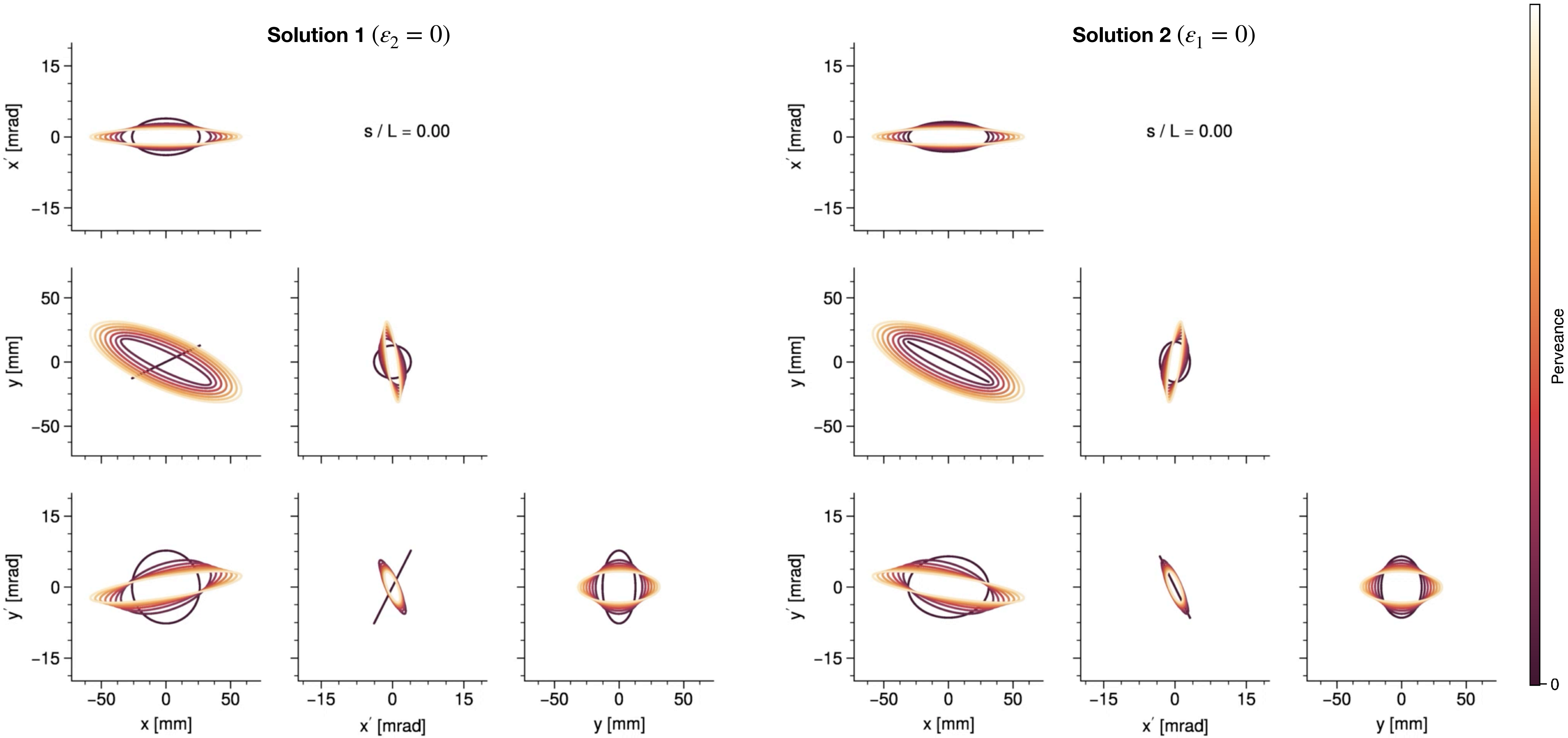
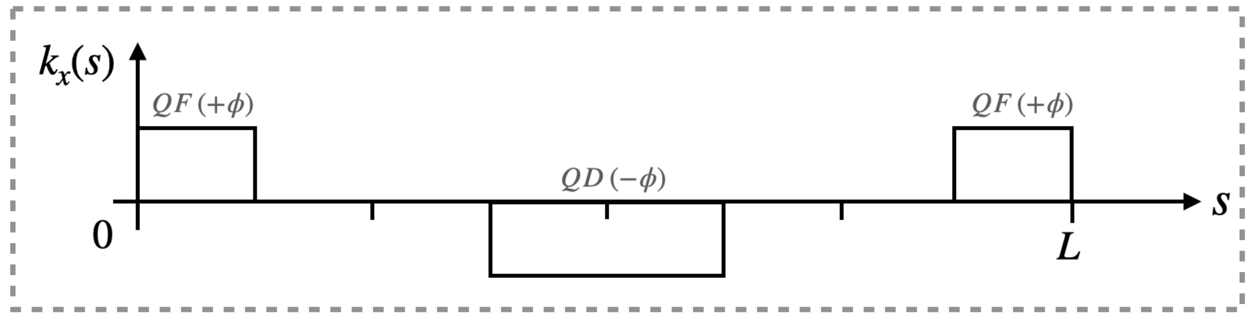
Matched beam (FODO — skew quads)

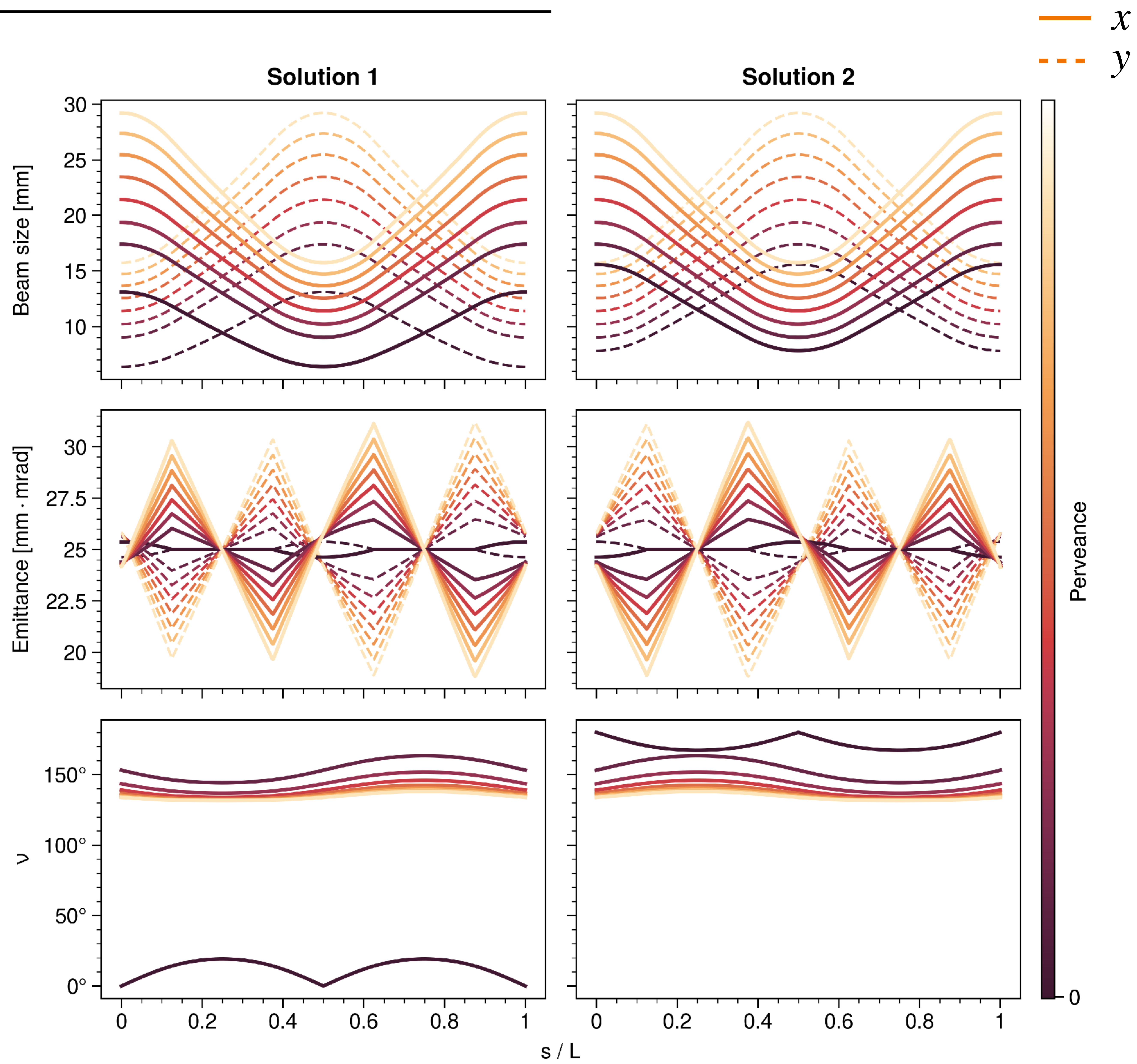




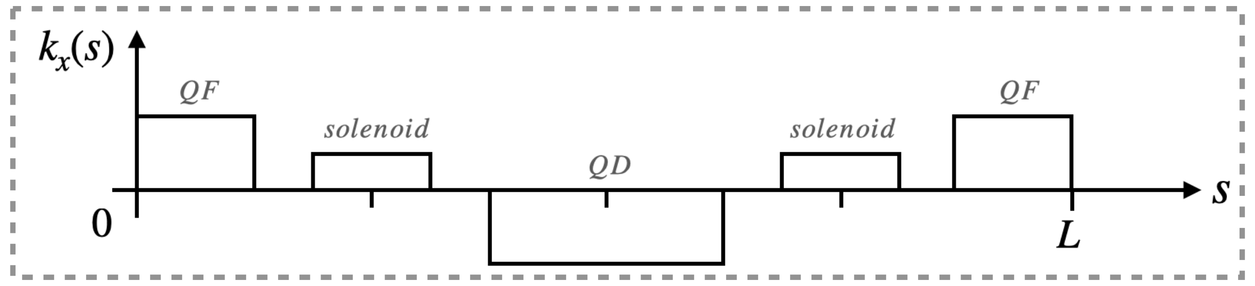
Matched beam (FODO — skew quads)

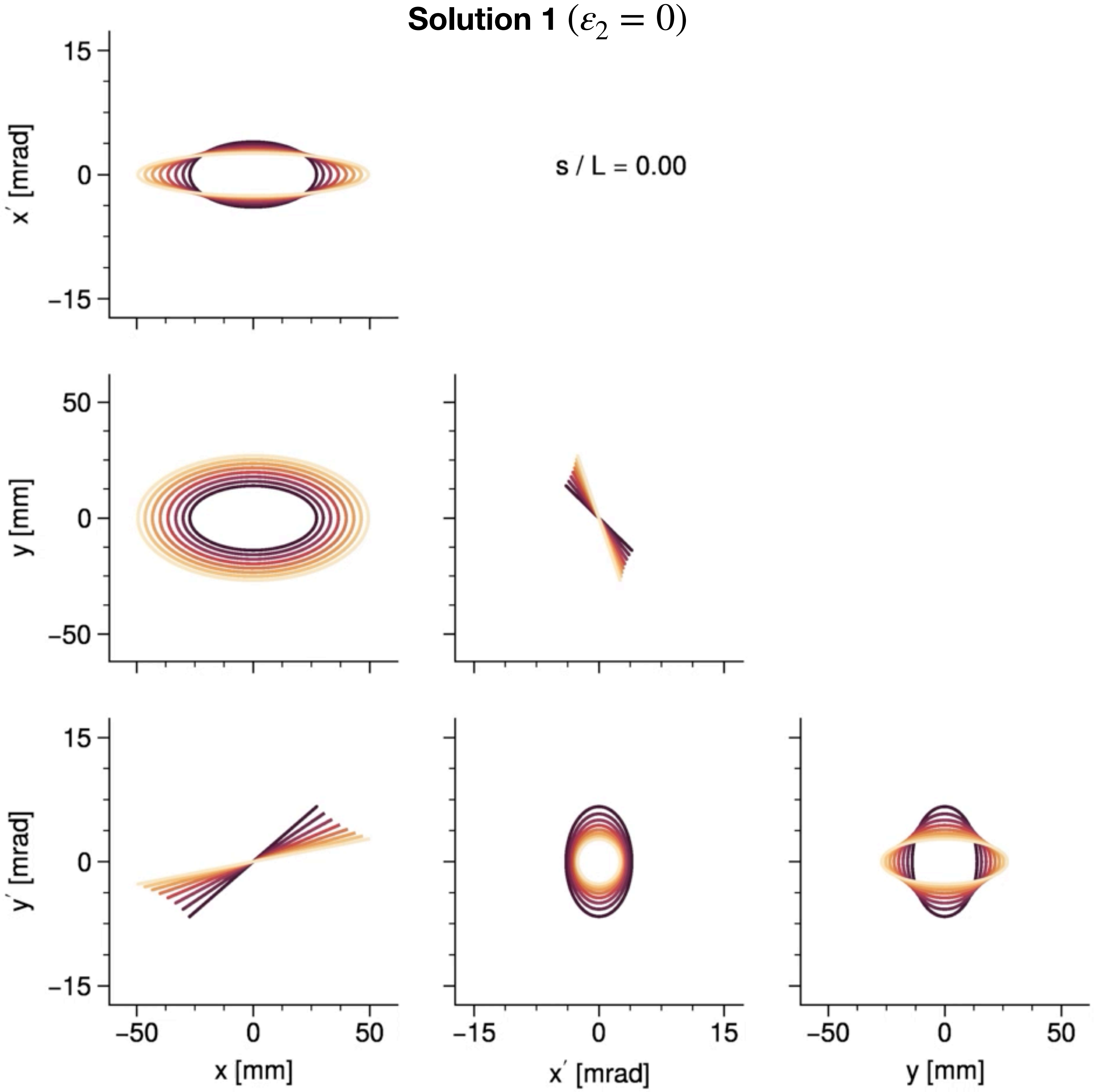


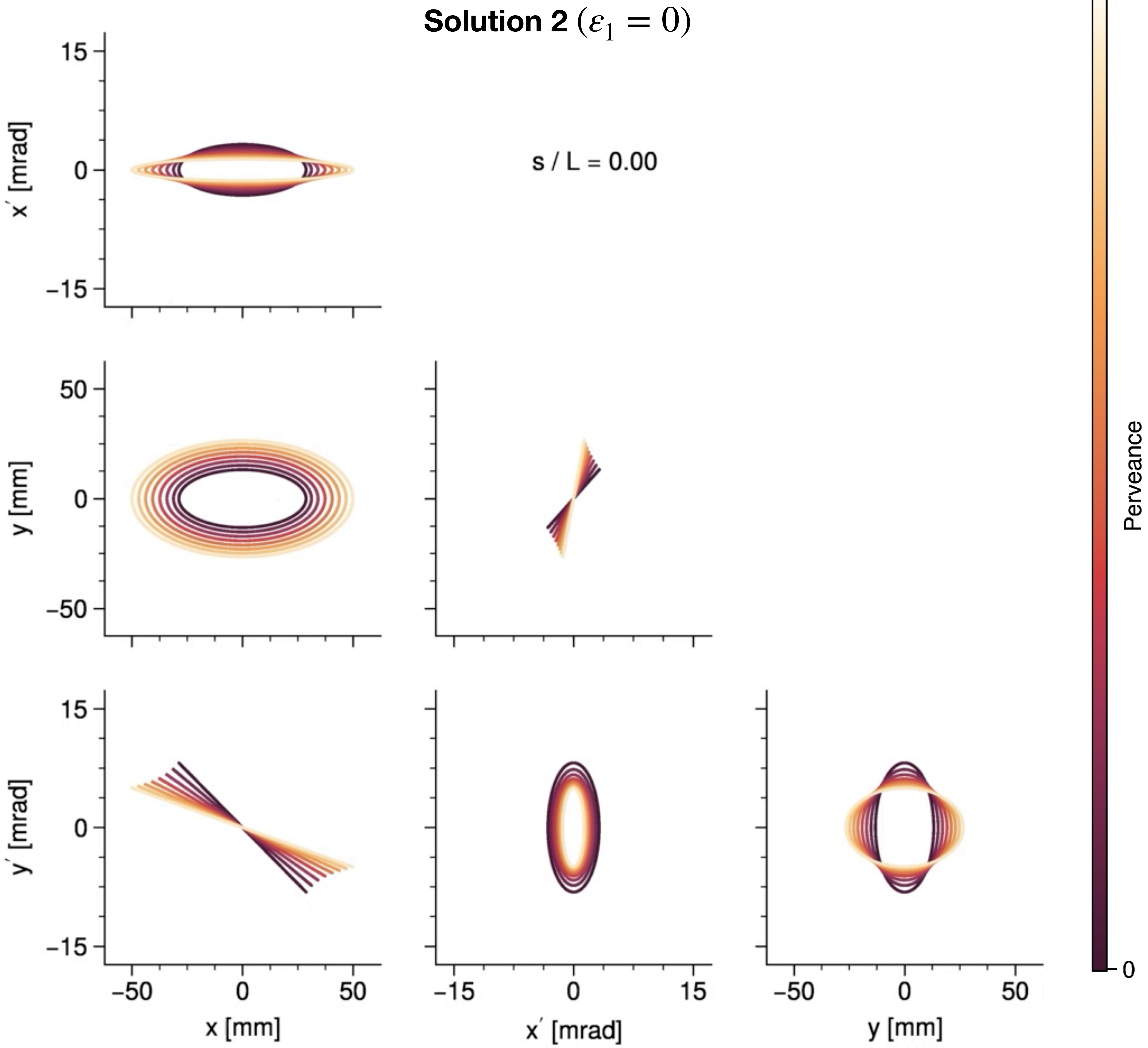
- Space charge enhances emittance exchange
- Lack of symmetry between solutions
- Relevance to painting
    - Non-rotating beam = easier on kickers
    - Unsure if flat beam adds other complications



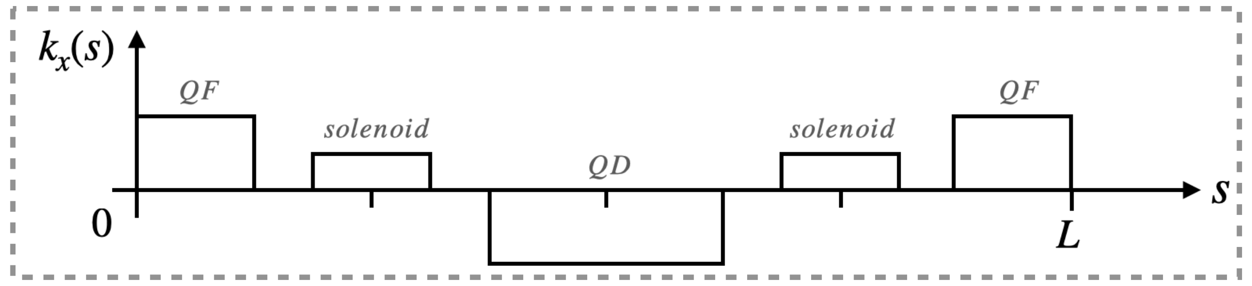
Matched beam (FODO — solenoid inserts)



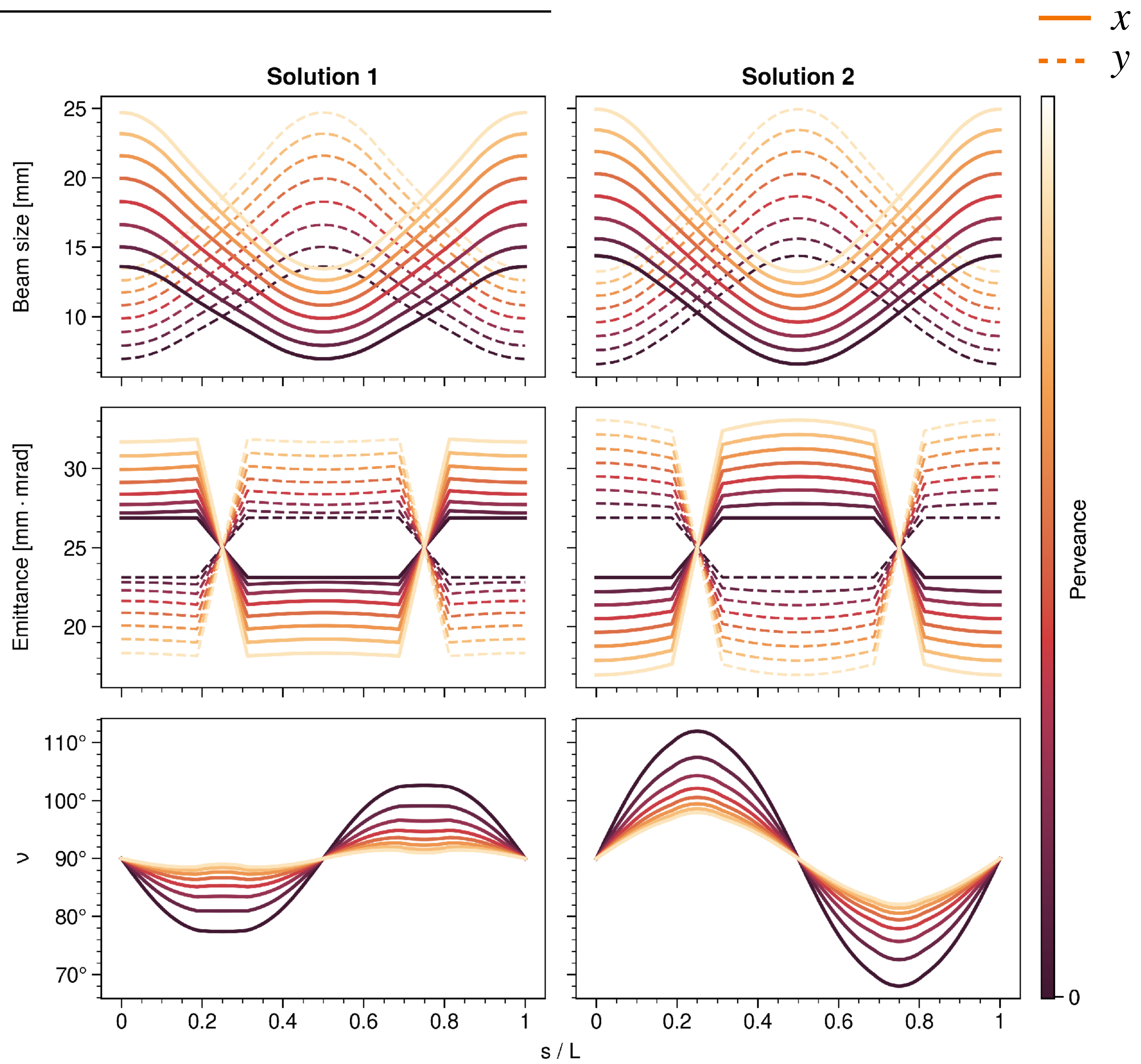




Matched beam (FODO — solenoid inserts)



- Large emittance exchange within solenoids
- Oscillations grow with space charge
- Other than emittance curves, evolution is similar to regular FODO lattice



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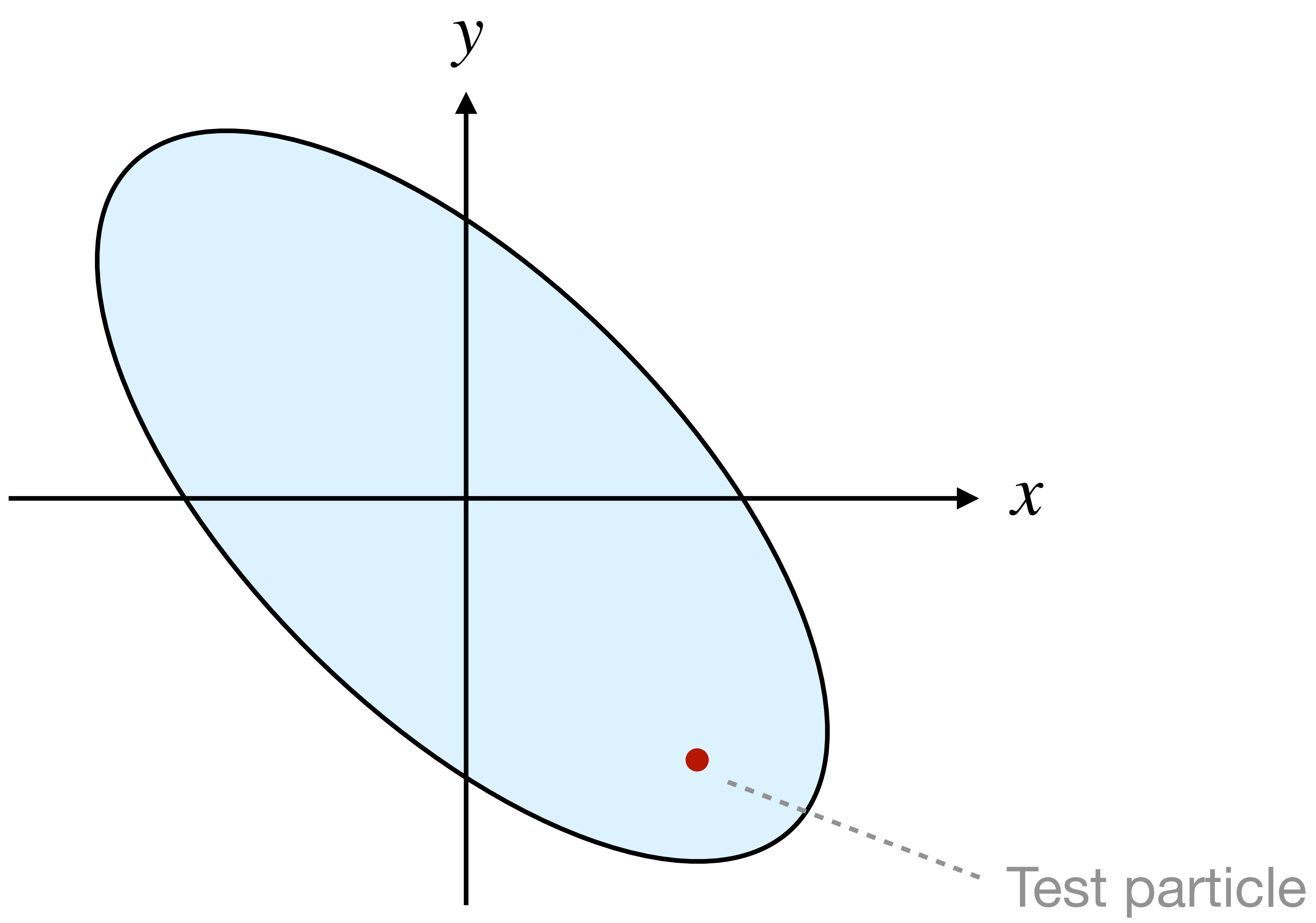
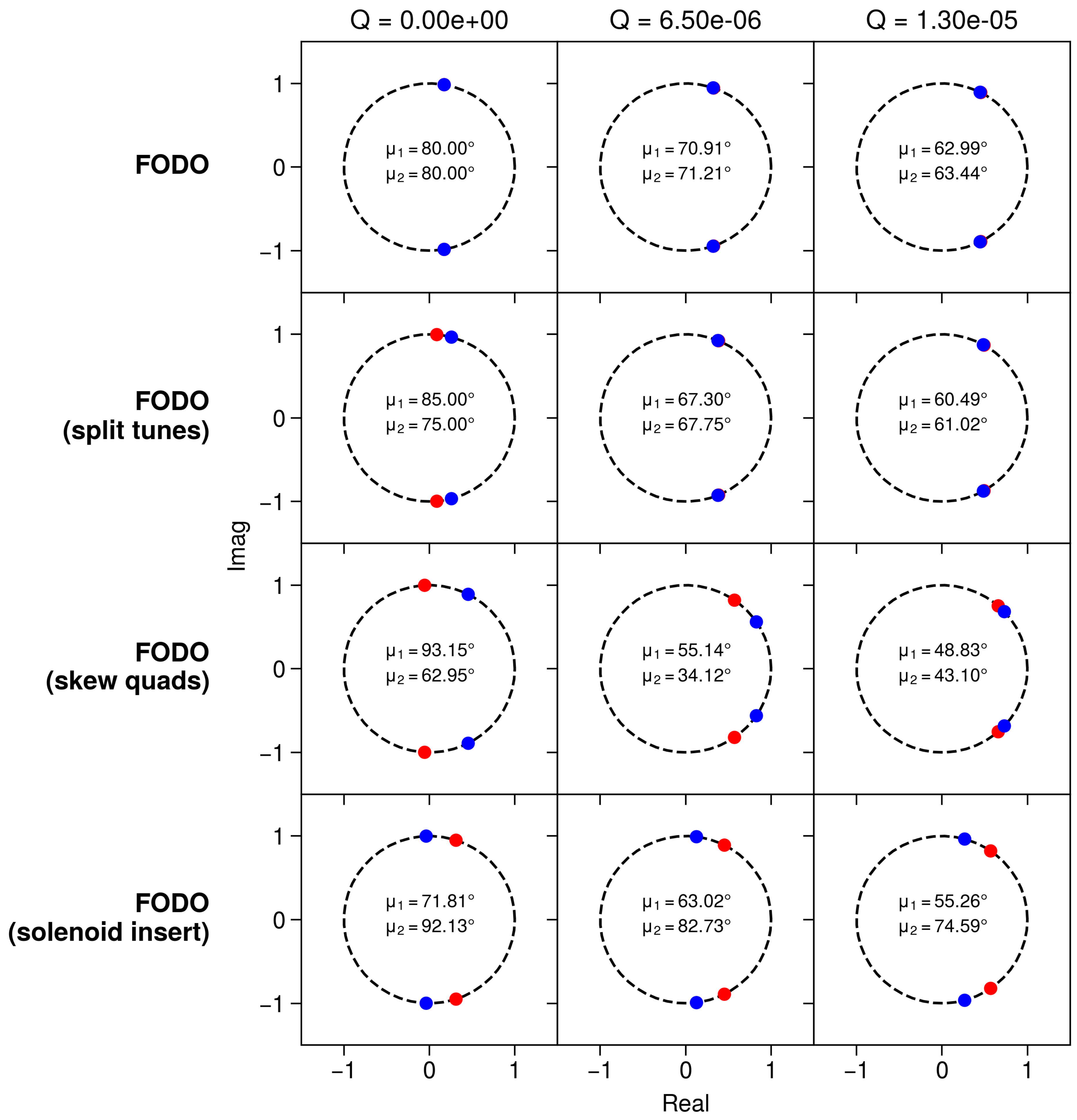
Effective transfer matrix

- Track test particles in field of matched beam to find $\mathbf{M}_{eff}$
- Calculate tunes as normal
- $\Delta = |\mu_1 - \mu_2|$ is measure of coupling





Δ small but nonzero (small emittance

Δ decreases with space charge (tunes forced back together)

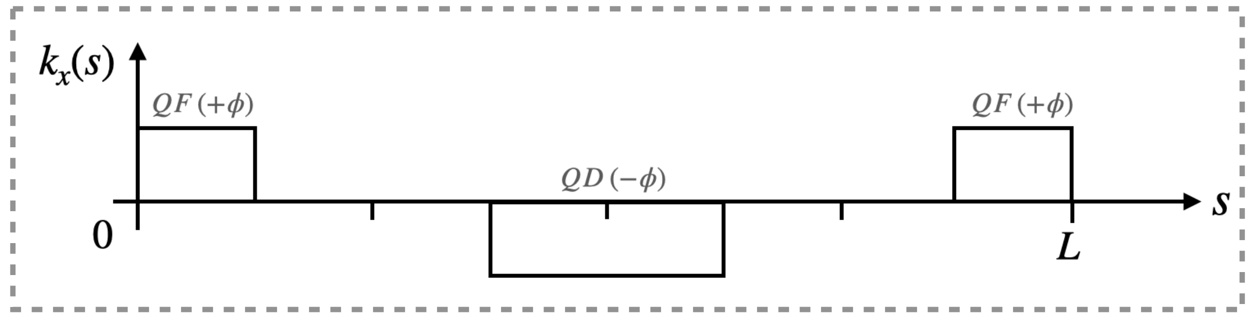
Δ decreases with space charge

$\Delta \approx$ constant

Conclusions

- Developed successful matching algorithm
  - Matched beam parameters identified using existing parameterization of coupled motion
  - Works in uncoupled and coupled lattices
- Gained knowledge of Danilov distribution properties, effect of space charge, and coupled motion
- Found some new things to try in painting method
- Future work
  - Stability — Are the matched solutions stable? (perturbation analysis + PIC simulation)
  - Painting — Will this information influence the way we paint the distribution in SNS?
  - Measurement — What is the best way to measure the 4D covariance matrix?

Matched beam (FODO — skew quads)



- Optimizer struggles with small space charge in this case ($\nu : 0 \to \pi$)
- Important to smooth over a number of turns using the custom method