Parameterization of Linear Coupled Motion

Austin Hoover

AP Meeting - 01.15.2021

Outline

- Motivation
- Equations of motion and symplecticity condition
- Transfer matrix eigenvectors
- 4D Twiss parameters
- Physical meaning of Twiss parameters
- Other approaches

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Betatron motion with coupling of horizontal and vertical degrees of freedom[1]

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Abstract: Presntly, there are two most frequently used parameterezations of linear $x-y$ coupled motion used in the accelerator physics. They are the Edwards-Teng and Mais-Ripken parameterizations. The article is devoted to an analysis of close relationship between the two representations, thus adding a clarity to their physical meaning. It also discusses the relationship between the eigen-vectors, the beta-functions, second order moments and the bilinear form representing the particle ellipsoid in the 4D phase space. Then, it consideres a further development of Mais-Ripken parameteresation where the particle motion is descrabed by 10 parameters: four beta-functions, four alpha-functions and two betatron phase advances. In comparison with Edwards-Teng parameterization the chosen parametrization has an advantage that it works equally well for analysis of coupled betatron motion in circular accelerators and in transfer lines. Considered relationship between second order moments, eigen-vectors and beta-functions can be useful in interpreting tracking results and experimental data. As an example, the developed formalizm is applied to the FNAL electron cooler and Derbenev's vertex-to-plane adapter.

Keywords: Accelerator modelling and simulations (multi-particle dynamics; single-particle dynamics); Beam-line instrumentation (beam position and profile monitors; beam-intensity monitors; bunch length monitors)

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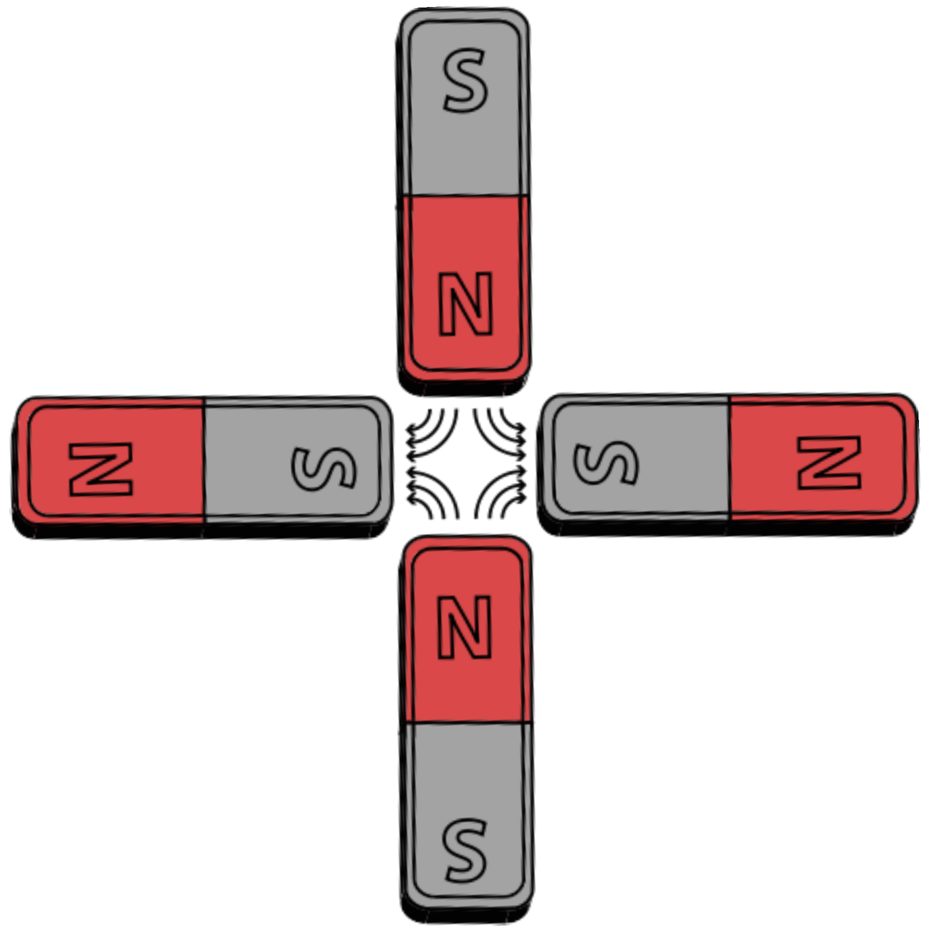
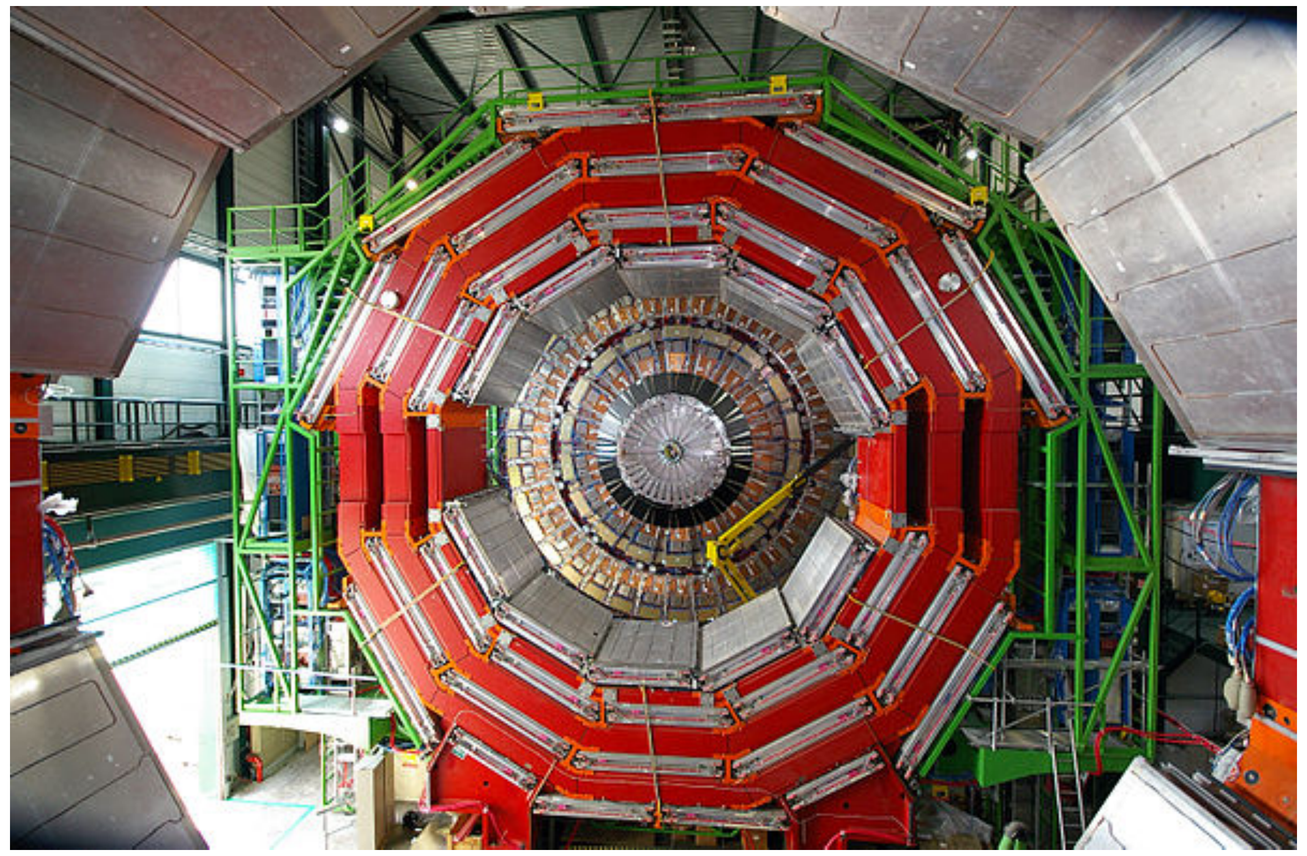
Everything in this presentation (other than images/animations) can be found in this paper.

Motivation

- Usually coupling is undesirable (motion is more complicated)
- Sometimes coupling is introduced intentionally
  - Collider interaction points
  - Round to flat beam conversion
  - Self-consistent beam studies @ SNS
  - Present in every machine due to small/random misalignments
- Accelerator community has yet to adopt a single method to handle coupled motion
  - Traditional methods: Edwards-Teng, Mais-Ripken/**Lebedev-Bogacz**
  - More recent methods: Wolski, Qin-Davidson



Skew quadrupole

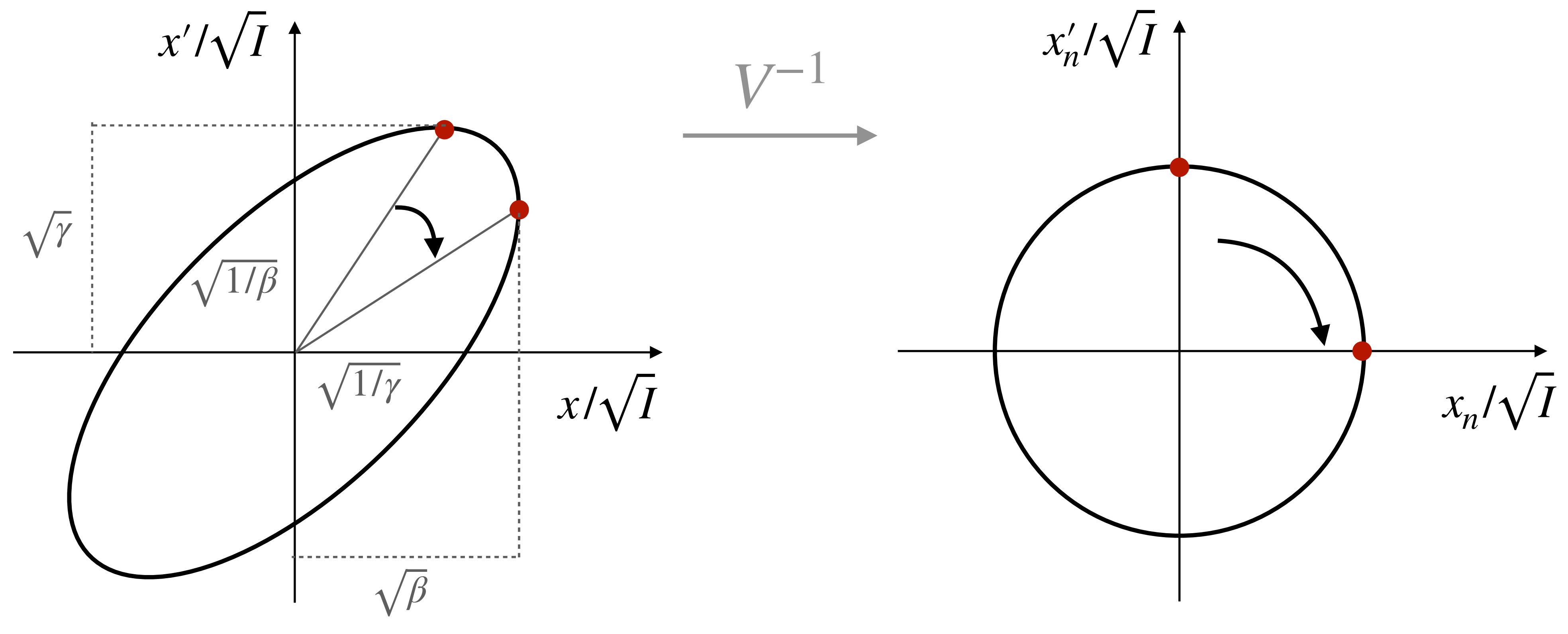


Compact Muon Solenoid (CMS) detector at LHC [≈ 4 T]

Review: Courant-Snyder (CS) theory

- Parameterize symplectic transfer matrix $\mathbf{M}$ as

$$\mathbf{M}(0,s) = \underbrace{\begin{bmatrix} \sqrt{\beta(s)} & 0 \\ \frac{-\alpha(s)}{\sqrt{\beta(s)}} & \frac{1}{\sqrt{\beta(s)}} \end{bmatrix}}_{V(\alpha,\beta)} \underbrace{\begin{bmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{bmatrix}}_{P(\mu)} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{\beta(s)}} & 0 \\ \frac{\alpha(s)}{\sqrt{\beta(s)}} & \sqrt{\beta(s)} \end{bmatrix}}_{V^{-1}(\alpha,\beta)}$$



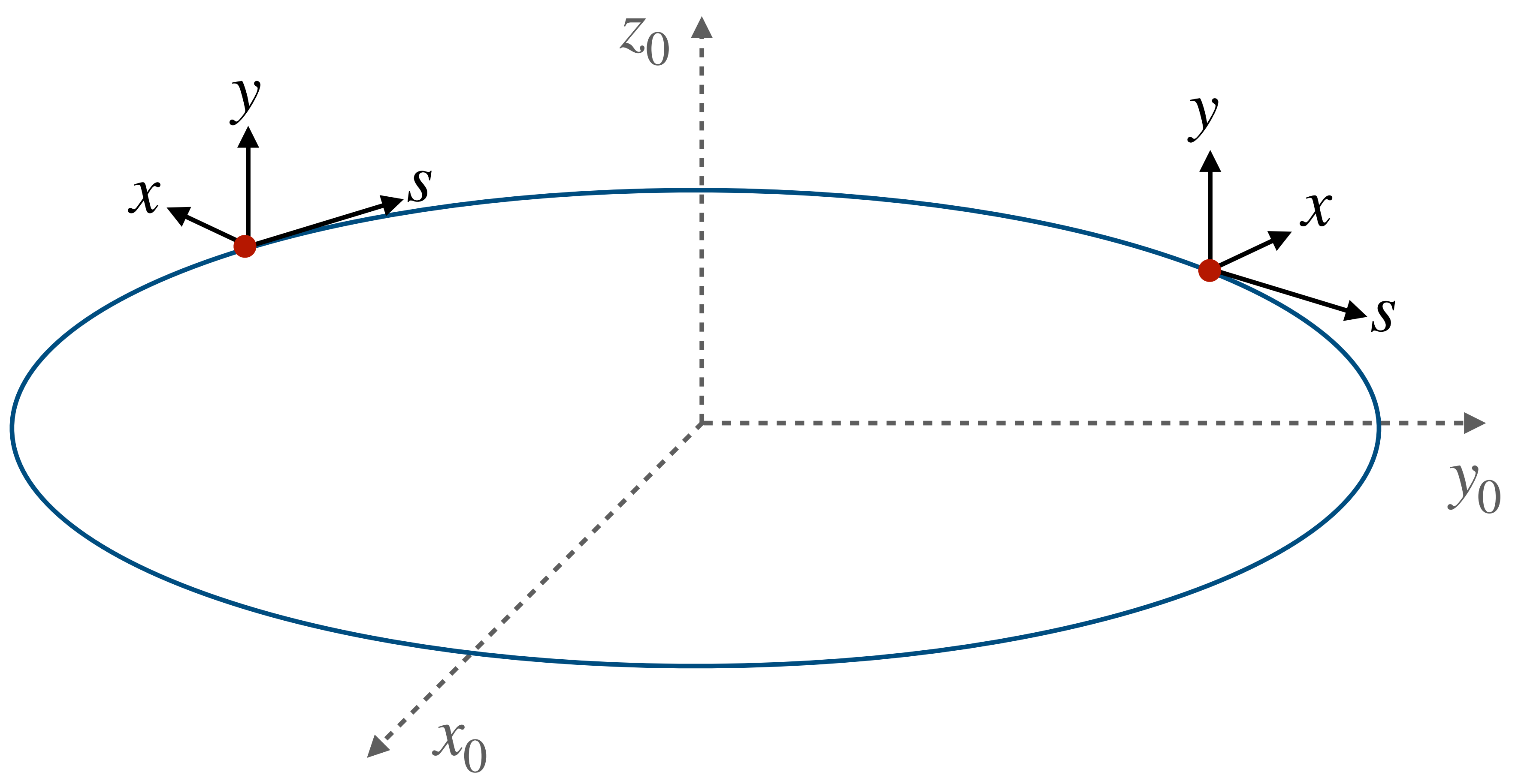
I = Area / π = $\gamma x^2 + \beta x'^2 + 2\alpha x x'$

- Time-dependent transformation V removes variance in focusing strength
- Parameters connect particle coordinates to an invariant
- Would like to do something similar for 2D motion

Equations of motion

$$x'' + \left(K_x^{\ 2} + k\right) x + \left(N - \frac{1}{2} R'\right) y - R y' = 0$$

$$y'' + \left(K_y^{\ 2} - k\right) y + \left(N + \frac{1}{2} R'\right) x + R x' = 0$$



Dipole $K_{x,y} = \left(e/pc\right) B_{y,x}$

Quadrupole coefficient $k = \left(e/pc\right) \dfrac{\partial B_y}{\partial x}$

Skew quadrupole coefficient $N = \left(e/pc\right) \dfrac{1}{2} \left(\dfrac{\partial B_x}{\partial x} - \dfrac{\partial B_y}{\partial y}\right)$

Axial magnetic field $R = \left(e/pc\right) B_s$

$$\vec{B} = \left(B_x, B_y, B_s\right)^T$$

Equations of motion

$$\boxed{\mathbf{x}' = \mathbf{U}\mathbf{H}\mathbf{x}}$$

Canonical coordinate vector

$$\mathbf{x} = \begin{bmatrix} x \\ p_x \\ y \\ p_y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -R/2 & 0 \\ 0 & 0 & 1 & 0 \\ R/2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}$$

Unit symplectic matrix

$$\mathbf{U} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Hamiltonian

$$H = \frac{1}{2}\mathbf{x}^T\mathbf{H}\mathbf{x}$$

$$\mathbf{H} = \begin{bmatrix} {K_x}^2 + k + \frac{R^2}{4} & 0 & N & -R/2 \\ 0 & 1 & R/2 & 0 \\ N & R/2 & {K_y}^2 - k + \frac{R^2}{4} & 0 \\ -R/2 & 0 & 0 & 1 \end{bmatrix}$$

Symplecticity condition

- Lagrange invariant:

$$\frac{d}{ds}\left(\mathbf{x}_1{}^T\,\mathbf{U}\,\mathbf{x}_2\right) = 0$$

- Transfer matrix is symplectic:

$$\mathbf{x}^T\,\mathbf{U}\,\mathbf{x} = (\mathbf{M}\mathbf{x})^T\;\mathbf{U}\;(\mathbf{M}\mathbf{x}) = \mathbf{x}^T\mathbf{M}^T\mathbf{U}\mathbf{M}\mathbf{x}$$

$$\longrightarrow \boxed{\mathbf{M}^T\mathbf{U}\mathbf{M} = \mathbf{U}}$$

Antisymmetric

- Leaves 10 independent elements

Transfer matrix

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{xx} & \mathbf{M}_{xy} \\ \mathbf{M}_{yx} & \mathbf{M}_{yy} \end{bmatrix}$$

Unit symplectic matrix

$$\mathbf{U} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Eigenvalues

$$\mathbf{M}\mathbf{v} = \lambda \mathbf{v}$$

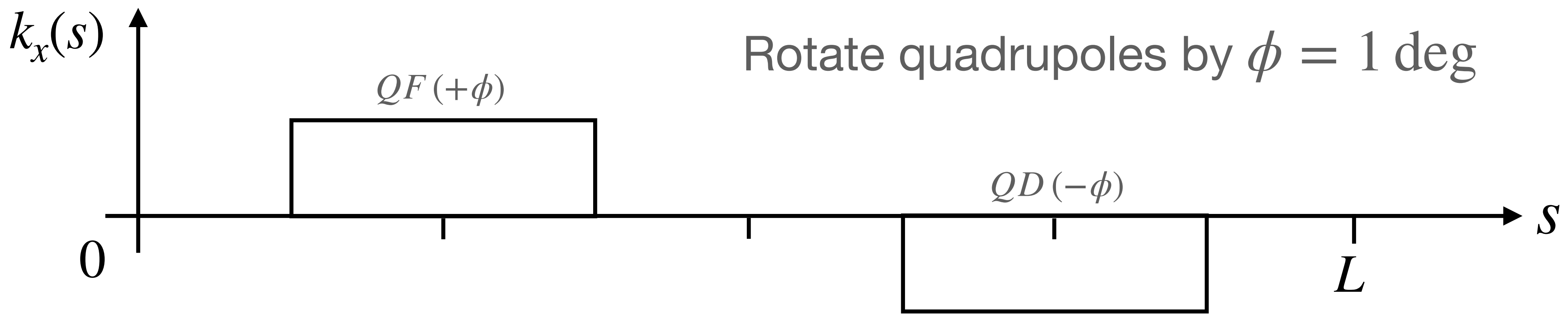
- Two complex conjugate pairs $(l = 1, 2)$

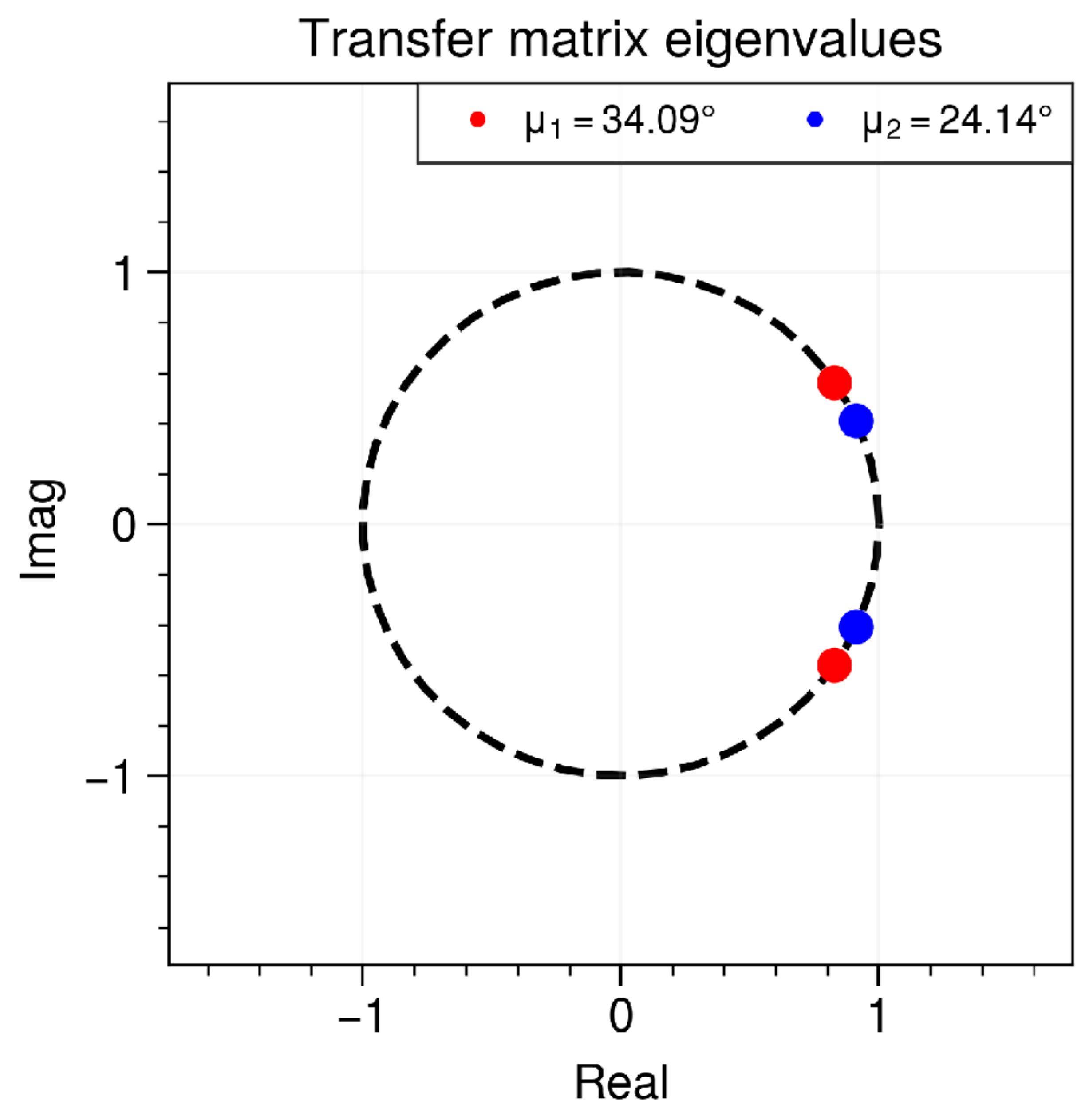
$$\lambda_l\,, \lambda_l^*$$

$$\mathbf{v}_l\,, \mathbf{v}_l^*$$

$$\lambda_1 \lambda_2 = 1$$

- Write eigenvalues as $\lambda_l = e^{-i\mu_l}$
  - μ_l is phase advance imparted to $\mathbf{v}_l$
  - For stable motion $|\lambda_l| = 1$ (phase advance is real)





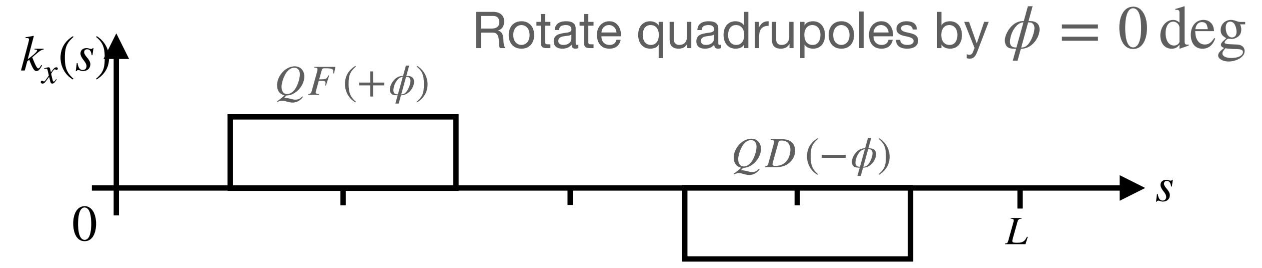
Eigenvectors: normalization conditions

- Symplecticity condition leads to:

$$\left(1 - \lambda_i \lambda_j\right) \mathbf{v}_i{}^{\dagger} \, \mathbf{U} \, \mathbf{v}_j = 0 \longrightarrow \begin{aligned} \mathbf{v}_1{}^{\dagger}\mathbf{U}\mathbf{v}_1 &\neq 0 \\ \mathbf{v}_2{}^{\dagger}\mathbf{U}\mathbf{v}_2 &\neq 0 \\ \mathbf{v}_i{}^{\dagger}\mathbf{U}\mathbf{v}_j &= 0 \\ \mathbf{v}_i{}^{T}\mathbf{U}\mathbf{v}_j &= 0 \end{aligned} \quad i \neq j$$

- Gives 6 equations
- Each eigenvector has 8 components (4 real, 4 imag)
- 16 - 6 = 10 numbers needed to parameterize eigenvectors
  - 2 of these will be phases

Eigenvectors: physical meaning



- Coordinate vector at lattice entrance:

$$\mathbf{x} = Re\left(\sqrt{\varepsilon_1}\,\mathbf{v}_1\,e^{-i\psi_1} + \sqrt{\varepsilon_2}\,\mathbf{v}_2\,e^{-i\psi_2}\right)$$

$$\mathbf{Mx} = Re\left(\sqrt{\varepsilon_1}\,\mathbf{v}_1\,e^{-i(\psi_1+\mu_1)} + \sqrt{\varepsilon_2}\,\mathbf{v}_2\,e^{-i(\psi_2+\mu_2)}\right)$$

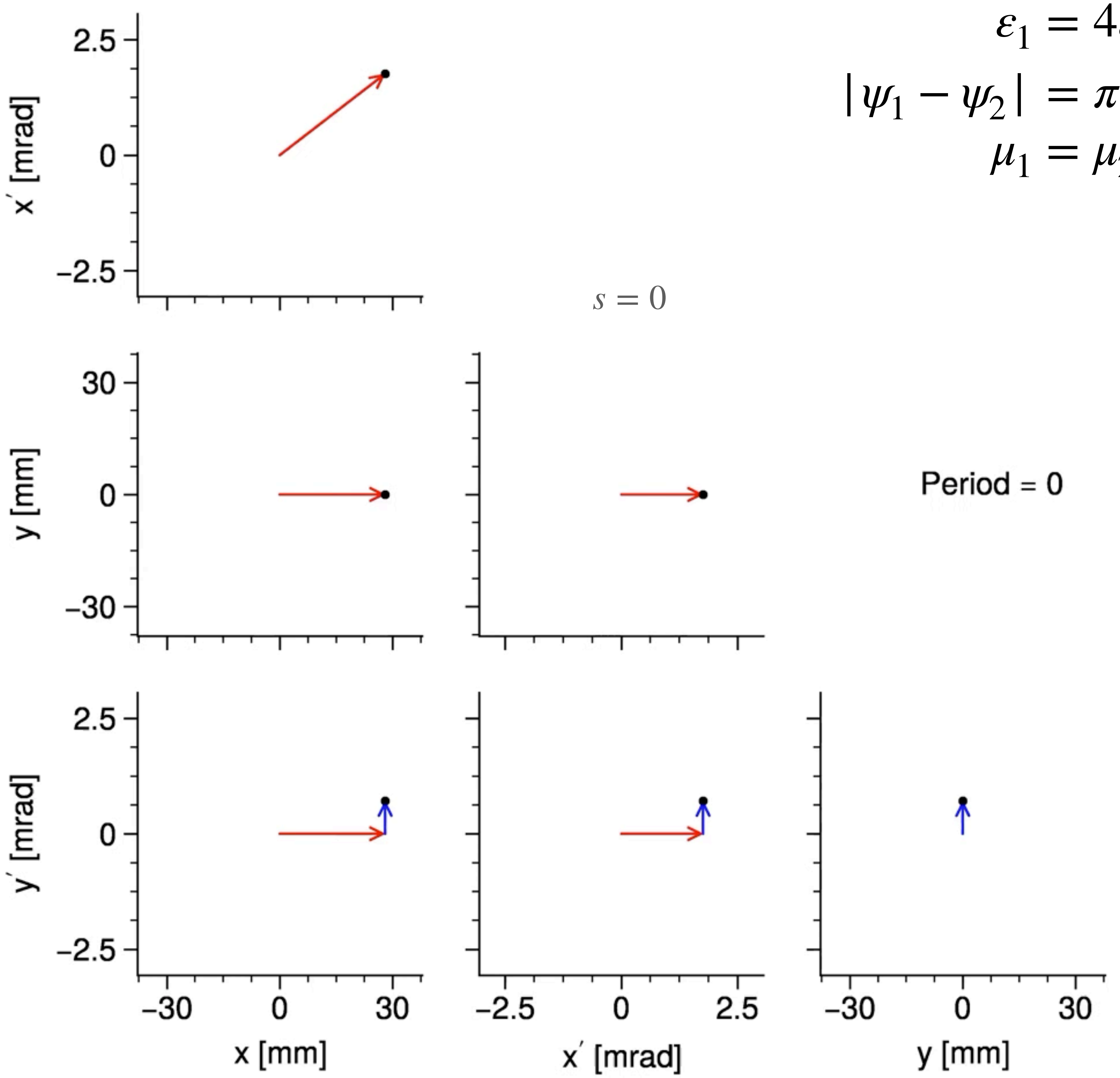
- Eigenvectors for uncoupled lattice:

$$\mathbf{v}_1 = \begin{bmatrix} v_{1x} \\ v_{1x'} \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ v_{2y} \\ v_{2y'} \end{bmatrix}$$

- Same phase advance

$$M\left(\mathbf{v}_1 + \mathbf{v}_2\right) = e^{-i\mu_1}\left(\mathbf{v}_1 + \mathbf{v}_2\right)$$

$$\varepsilon_1 = 4\varepsilon_2$$
$$|\psi_1 - \psi_2| = \pi/2$$
$$\mu_1 = \mu_2$$



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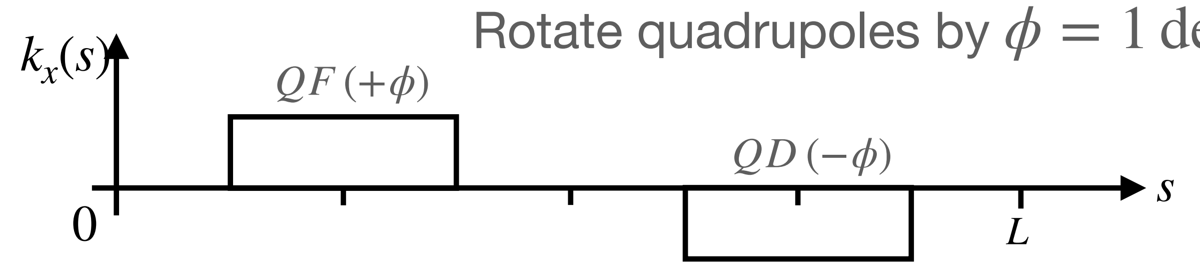
Rotate quadrupoles by $\phi = 1$ deg



Eigenvectors: physical meaning

- Coordinate vector at lattice entrance:

$$\mathbf{x} = Re\left(\sqrt{\varepsilon_1}\,\mathbf{v}_1\,e^{-i\psi_1} + \sqrt{\varepsilon_2}\,\mathbf{v}_2\,e^{-i\psi_2}\right)$$

$$\mathbf{M}\mathbf{x} = Re\left(\sqrt{\varepsilon_1}\,\mathbf{v}_1\,e^{-i(\psi_1+\mu_1)} + \sqrt{\varepsilon_2}\,\mathbf{v}_2\,e^{-i(\psi_2+\mu_2)}\right)$$

- Eigenvectors are coupled:

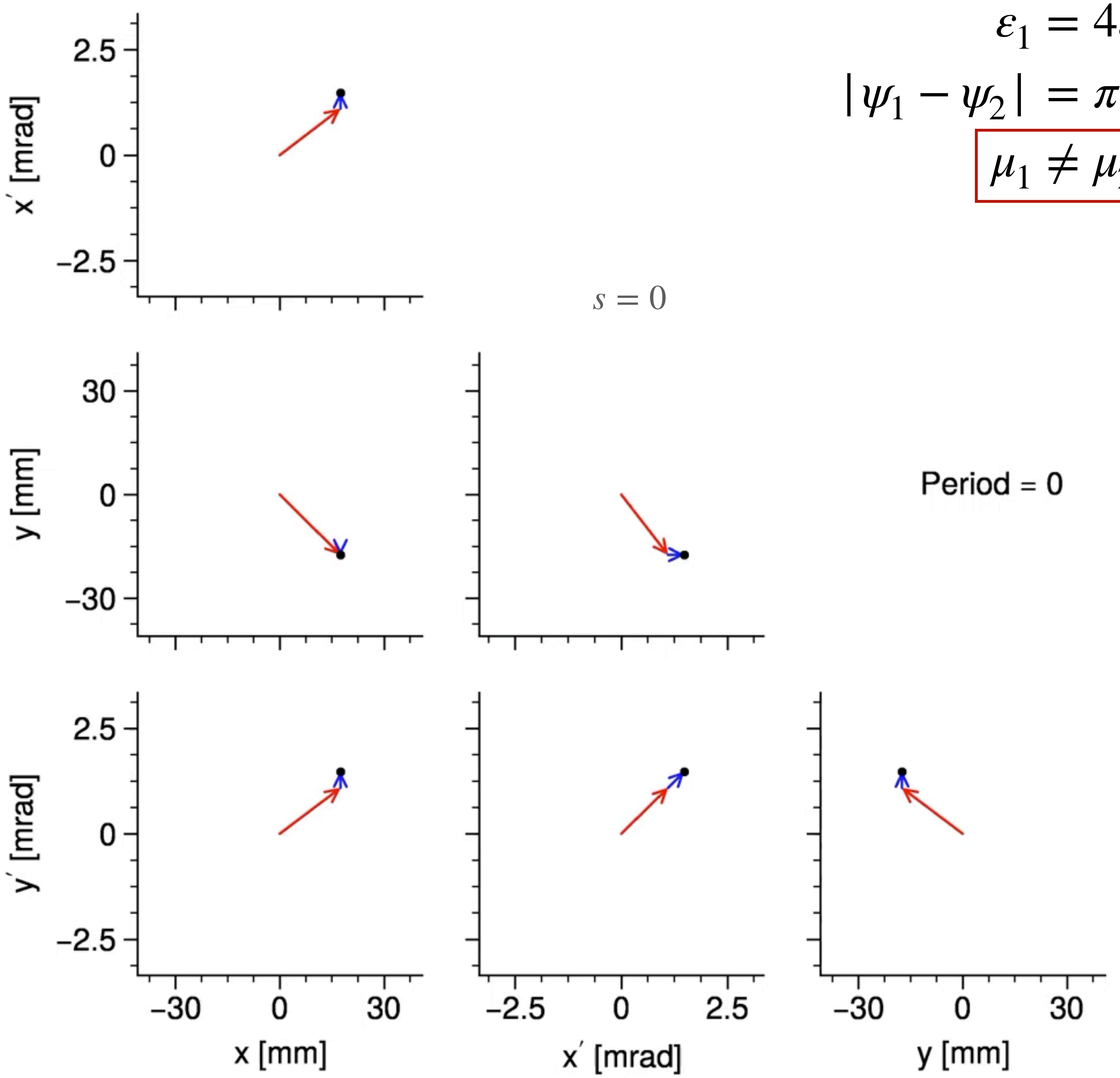
$$\mathbf{v}_1 = \begin{bmatrix} v_{1x} \\ v_{1x'} \\ v_{1y} \\ v_{1y'} \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} v_{1x} \\ v_{1x'} \\ v_{2y} \\ v_{2y'} \end{bmatrix}$$

- *Emittance exchange* due to unequal phase advances

$$\varepsilon_1 = 4\varepsilon_2$$

$$|\psi_1 - \psi_2| = \pi/2$$

$$\boxed{\mu_1 \neq \mu_2}$$

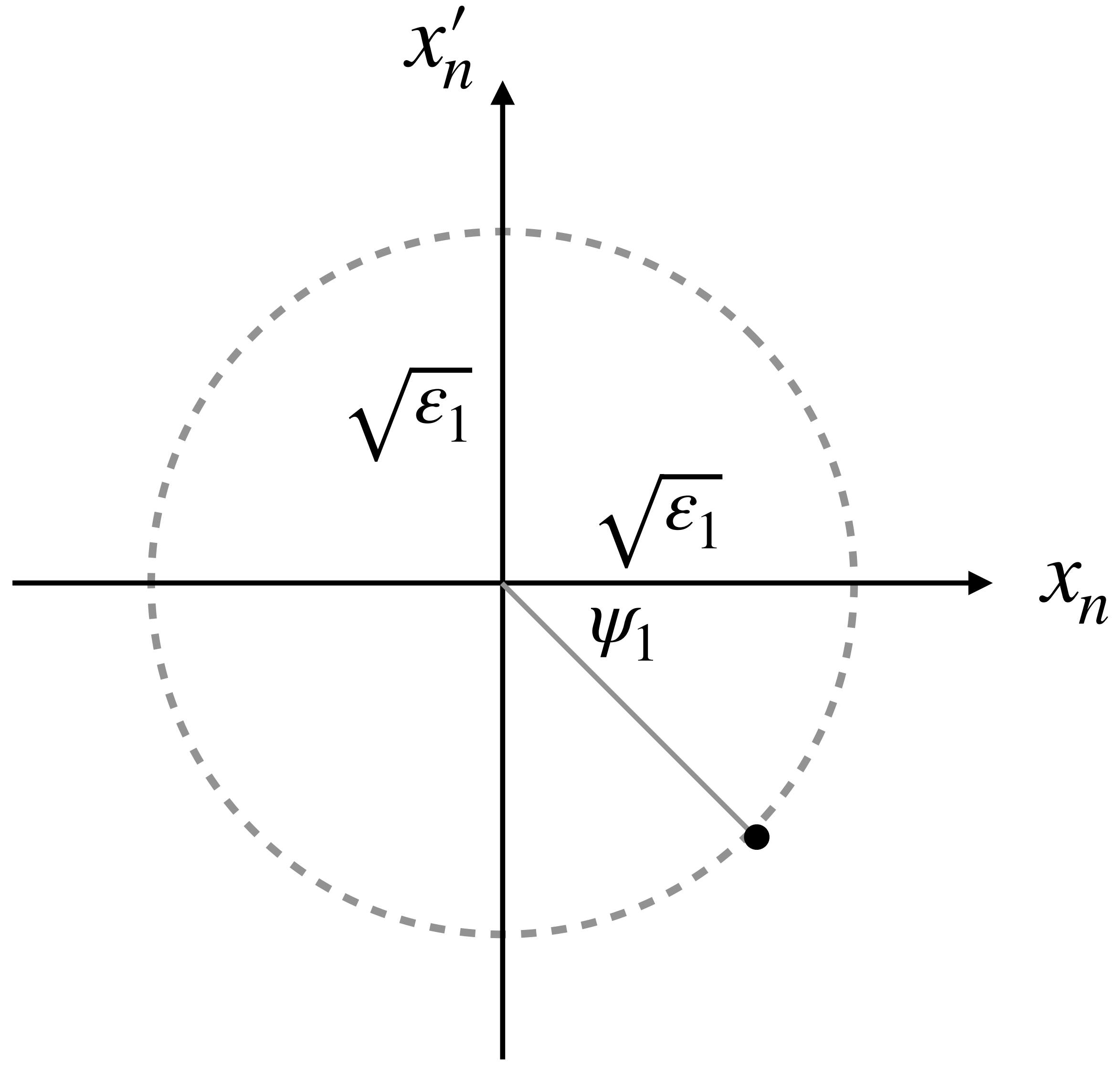


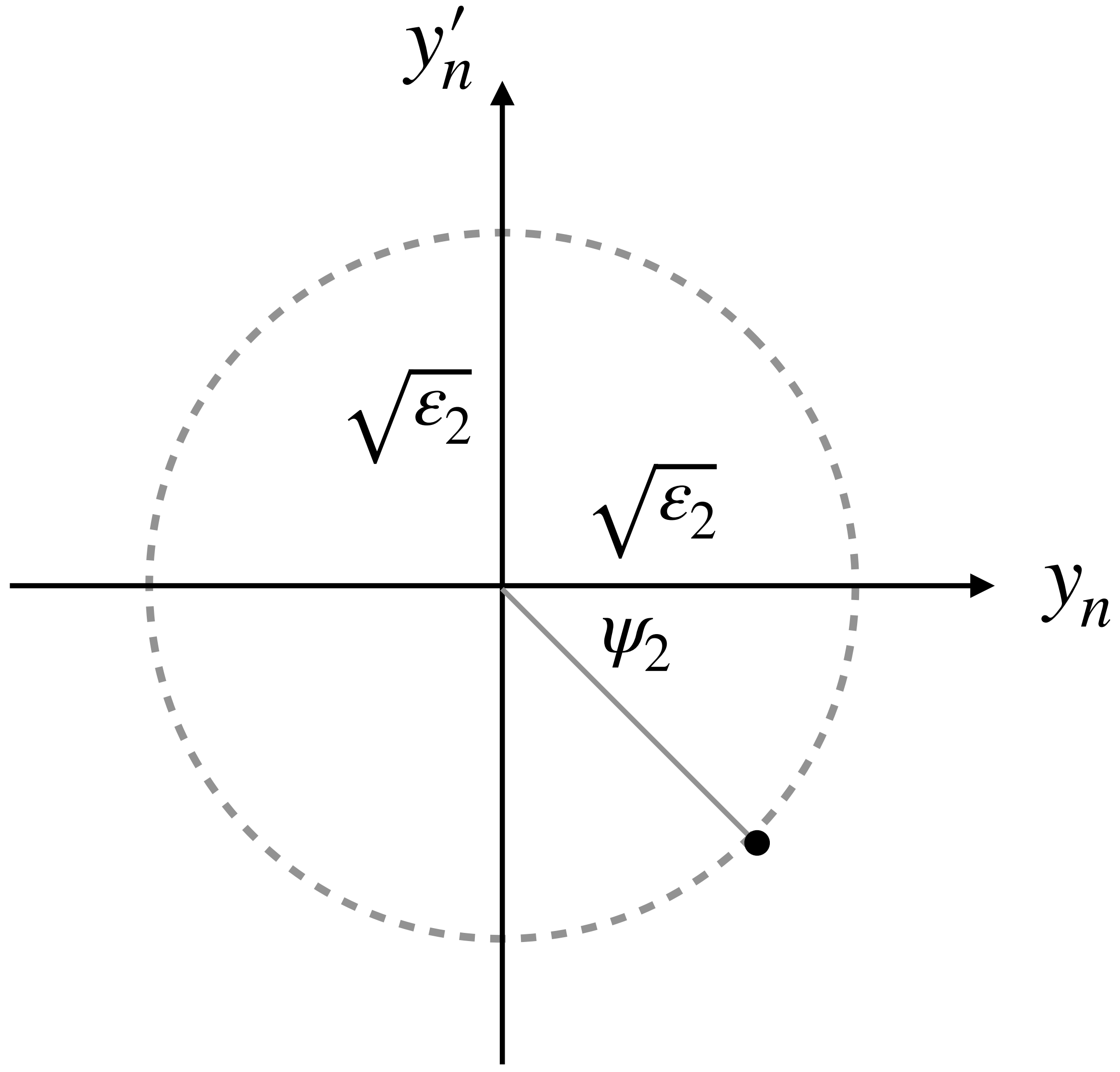
Normalized coordinates

$$\mathbf{x} = Re\left(\sqrt{\varepsilon_1}\,\mathbf{v}_1\,e^{-i\psi_1} + \sqrt{\varepsilon_2}\,\mathbf{v}_2\,e^{-i\psi_2}\right) = \mathbf{V}\mathbf{x}_n$$

Normalized coordinate vector

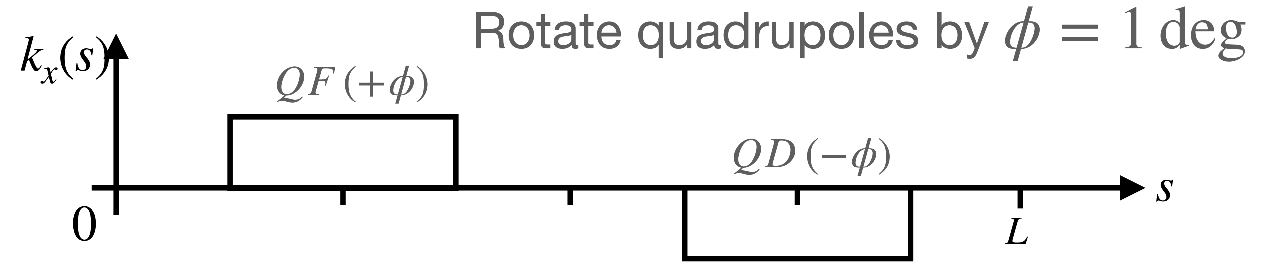
$$\mathbf{x}_n = \begin{bmatrix} +\sqrt{\varepsilon_1}\cos\psi_1 \\ -\sqrt{\varepsilon_1}\sin\psi_1 \\ +\sqrt{\varepsilon_2}\cos\psi_2 \\ -\sqrt{\varepsilon_2}\sin\psi_2 \end{bmatrix}$$





Symplectic transformation

$$\mathbf{V} = \Big[Re\left[\mathbf{v_1}\right],\ -Im\left[\mathbf{v_1}\right], Re\left[\mathbf{v_2}\right],\ -Im\left[\mathbf{v_2}\right],\Big]$$



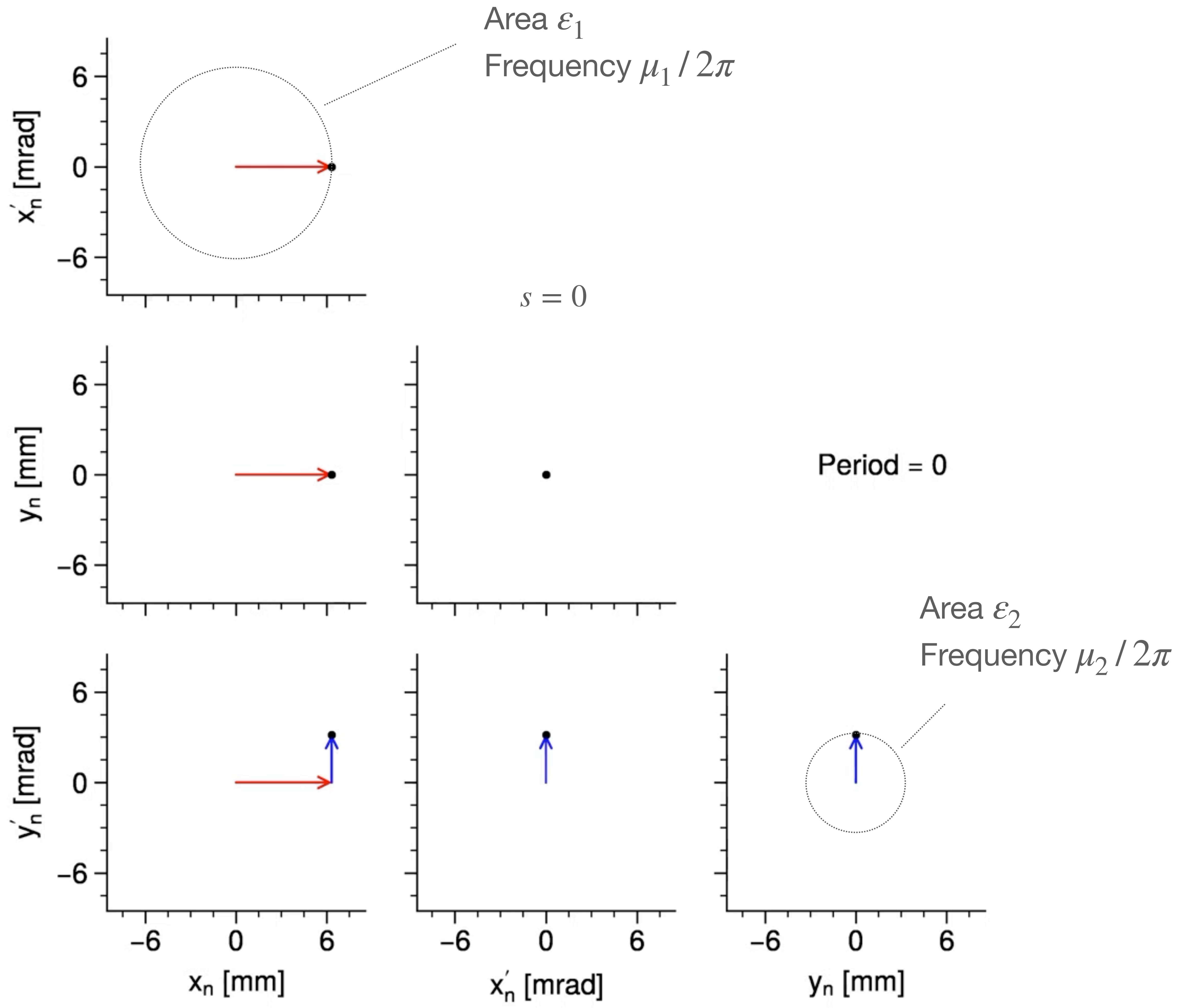
Normalized coordinates

- Motion is uncoupled in normalized coordinates

$$\mathbf{x} \to \mathbf{M}\,\mathbf{x}$$
$$\mathbf{V}^{-1}\mathbf{x} \to \mathbf{V}^{-1}\mathbf{M}\mathbf{V}\mathbf{V}^{-1}\mathbf{x}$$
$$\mathbf{x}_n \to \mathbf{P}\,\mathbf{x}_n$$

$$\mathbf{P} = \mathbf{V}^{-1}\mathbf{M}\mathbf{V} = \begin{bmatrix} \cos\mu_1 & \sin\mu_1 & 0 & 0 \\ -\sin\mu_1 & \cos\mu_1 & 0 & 0 \\ 0 & 0 & \cos\mu_2 & \sin\mu_2 \\ 0 & 0 & -\sin\mu_2 & \cos\mu_2 \end{bmatrix}$$

- Invariants: $\varepsilon_1, \varepsilon_2$



Phase space ellipsoid

- Parameterize maximum amplitude particles: $\mathbf{x}_{max} = \mathbf{V}\mathbf{A}\zeta$

$$\mathbf{A} = \begin{bmatrix} \sqrt{\varepsilon_1} & 0 & 0 & 0 \\ 0 & \sqrt{\varepsilon_1} & 0 & 0 \\ 0 & 0 & \sqrt{\varepsilon_2} & 0 \\ 0 & 0 & 0 & \sqrt{\varepsilon_2} \end{bmatrix} \qquad \zeta = \begin{bmatrix} +\cos\psi_1 \cos\psi_3 \\ -\sin\psi_1 \cos\psi_3 \\ +\cos\psi_2 \sin\psi_3 \\ -\sin\psi_2 \sin\psi_3 \end{bmatrix}$$

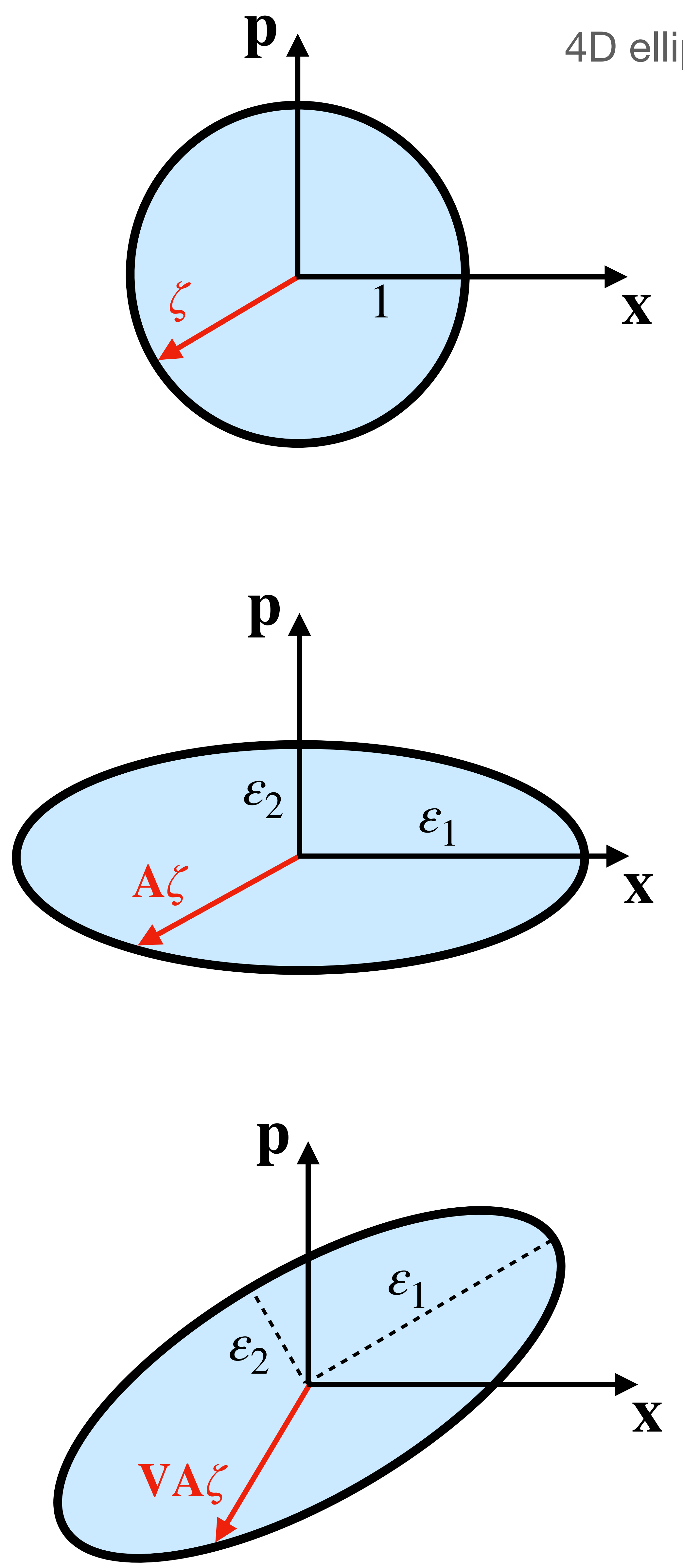
- Matched beam covariance matrix:

$$\zeta^T \zeta = 1$$

$$\mathbf{x}^T \mathbf{\Sigma}^{-1} \mathbf{x} = 1$$

$$\boxed{\mathbf{\Sigma} = \mathbf{V}\,\mathbf{\Sigma}_n \mathbf{V}^T} \qquad \mathbf{\Sigma}_n = \mathbf{A}\mathbf{A} = \begin{bmatrix} \varepsilon_1 & 0 & 0 & 0 \\ 0 & \varepsilon_1 & 0 & 0 \\ 0 & 0 & \varepsilon_2 & 0 \\ 0 & 0 & 0 & \varepsilon_2 \end{bmatrix}$$

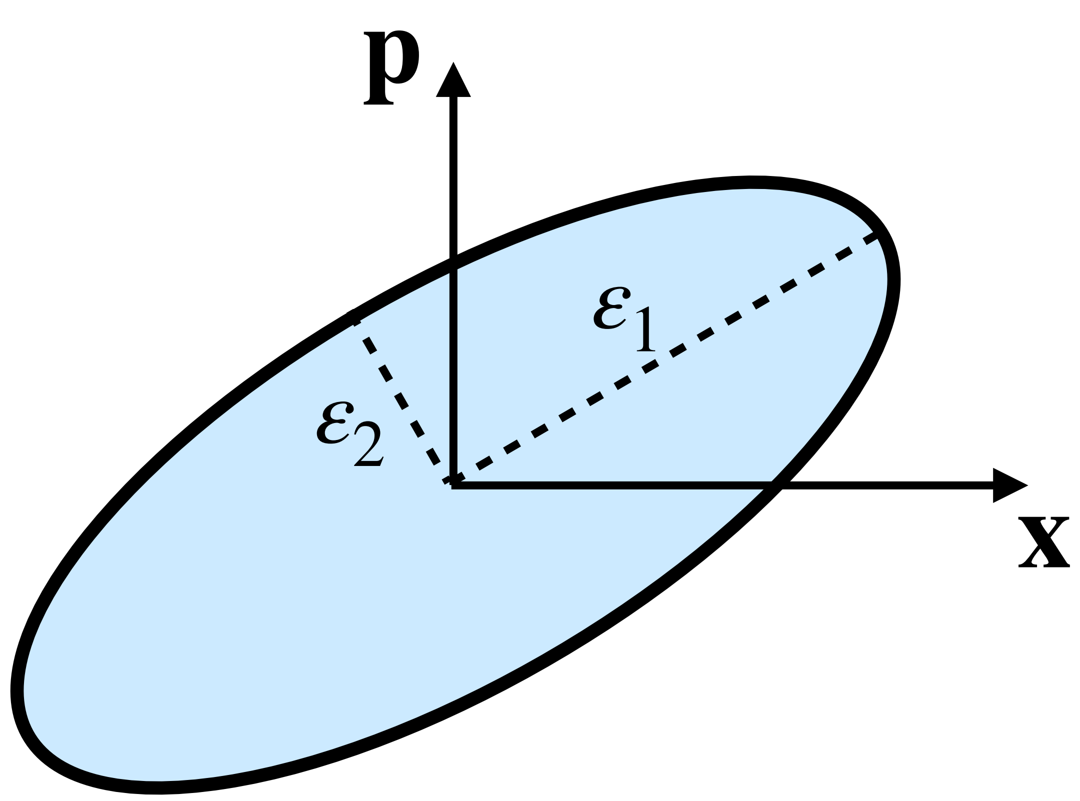
4D ellipsoid



4D emittance

- Determinant is conserved during transformation

$$\left|\mathbf{\Sigma}\right| = \left|\mathbf{V}\,\mathbf{\Sigma}_n\mathbf{V}^T\right| = \left(\varepsilon_1\,\varepsilon_2\right)^2$$



- 4D emittance (∝ volume of 4D ellipsoid)

$$\varepsilon_{4D} = \sqrt{\left|\mathbf{\Sigma}\right|} = \varepsilon_1\,\varepsilon_2 \leq \varepsilon_x\,\varepsilon_y$$

Intrinsic emittance $\varepsilon_1, \varepsilon_2$ (conserved for any linear transfer matrix)

Apparent emittance $\varepsilon_x, \varepsilon_y$ (conserved for uncoupled linear transfer matrix)

- $\varepsilon_1, \varepsilon_2$ found by symplectic diagonalization of $\mathbf{\Sigma}$ (Williamson's theorem)

$$(\mathbf{\Sigma}\mathbf{U} - i\varepsilon\mathbf{I})\,\mathbf{v} = 0$$

(Same eigenvectors as $\mathbf{M}$)

$$\left|\mathbf{\Sigma}\mathbf{U} - i\varepsilon\mathbf{I}\right| = 0$$

$$\varepsilon_{1,2} = \frac{1}{2}\sqrt{-tr\left[(\mathbf{\Sigma}\mathbf{U})^2\right] \pm \sqrt{tr^2\left[(\mathbf{\Sigma}\mathbf{U})^2\right] + 16\left|\mathbf{\Sigma}\right|}}$$

Parameterization of eigenvectors

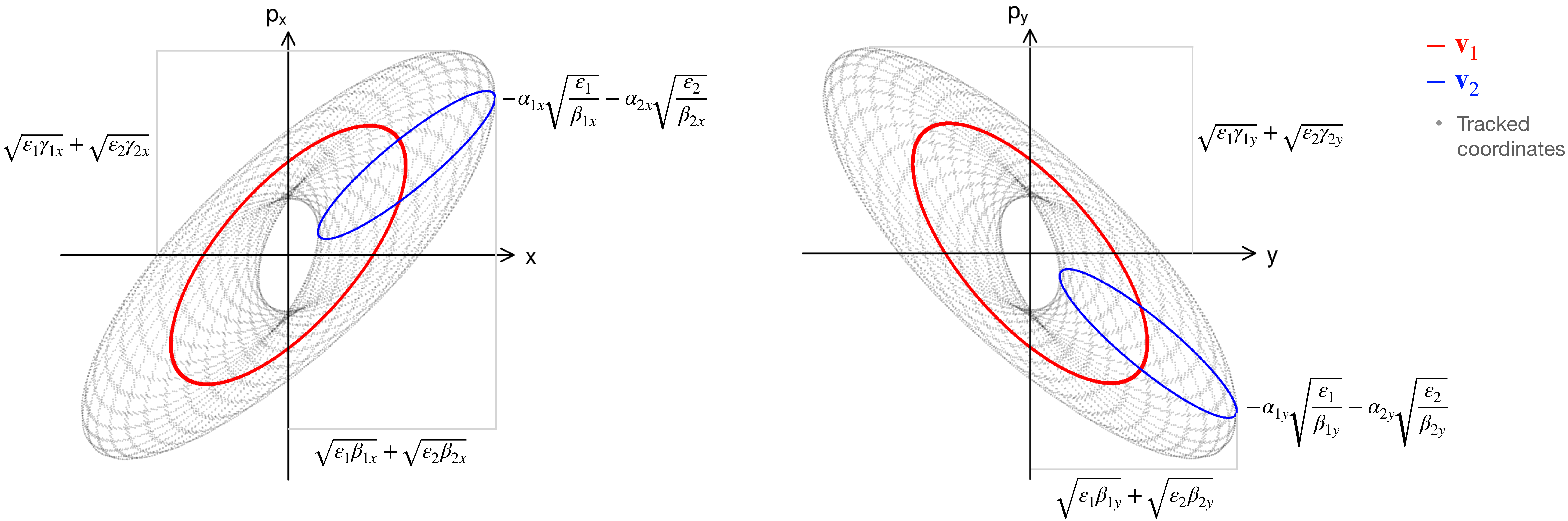
$$\mathbf{v}_1 = \begin{bmatrix} \sqrt{\beta_{1x}} \\ -\dfrac{i(1-u)+\alpha_{1x}}{\sqrt{\beta_{1x}}} \\ \sqrt{\beta_{1y}}\, e^{i\nu_1} \\ -\dfrac{iu+\alpha_{1y}}{\sqrt{\beta_{1y}}}\, e^{i\nu_1} \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} \sqrt{\beta_{2x}}\, e^{i\nu_2} \\ -\dfrac{iu+\alpha_{2x}}{\sqrt{\beta_{2x}}}\, e^{i\nu_2} \\ \sqrt{\beta_{2y}} \\ -\dfrac{i(1-u)+\alpha_{2y}}{\sqrt{\beta_{2y}}} \end{bmatrix}$$

- We've introduced 2 alpha and beta functions in addition to the phase advance for each mode
- Additional parameters ν_1, ν_2, and u (real functions) are determined from the symplecticity condition

$$0 \le u \le 1$$
$$0 \le \nu_{1,2} \le \pi$$

Meaning of parameters

* Pojection of 4D torus: recreated figure from Ripken, *Methods of Beam Optics* (1989)



- α_{lw} and β_{lw} describe ellipses traced by $\mathbf{v}_l$ in the w-p_w plane ($w = x, y$)
- Area of $\mathbf{v}_1$ ellipse is $|1-u|\varepsilon_1$ in horizontal phase space and $|u|\varepsilon_2$ in vertical phase space
- Area of $\mathbf{v}_2$ ellipse is $|u|\varepsilon_1$ in horizontal phase space and $|1-u|\varepsilon_2$ in vertical phase space

$$\gamma_{1x}\beta_{1x} = (1-u)^2 + \alpha_{1x}^2$$
$$\gamma_{2x}\beta_{2x} = u^2 + \alpha_{2x}^2$$
$$\gamma_{1y}\beta_{1y} = u^2 + \alpha_{1y}^2$$
$$\gamma_{2y}\beta_{2y} = (1-u)^2 + \alpha_{2y}^2$$

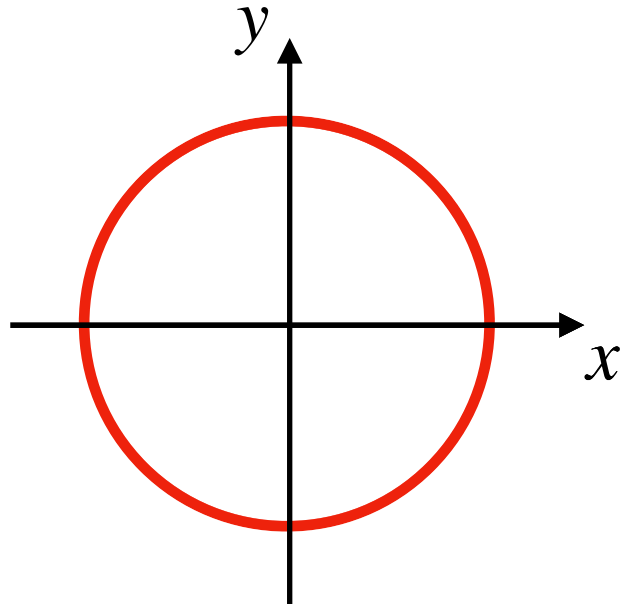
Meaning of parameters

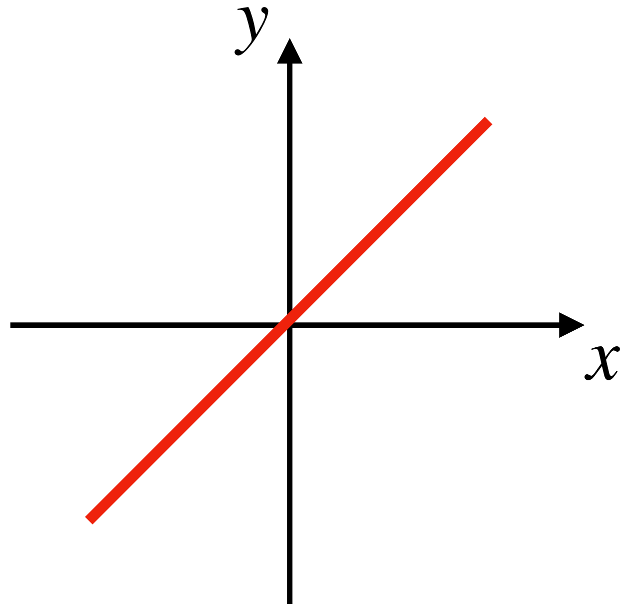
- ν_l is the phase difference between the x and y parts of $\mathbf{v}_l$

$$\mathbf{v}_1 = \begin{bmatrix} \sqrt{\beta_{1x}} \\ -\dfrac{i(1-u)+\alpha_{1x}}{\sqrt{\beta_{1x}}} \\ \sqrt{\beta_{1y}}\, e^{i\nu_1} \\ -\dfrac{iu+\alpha_{1y}}{\sqrt{\beta_{1y}}} e^{i\nu_1} \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} \sqrt{\beta_{2x}}\, e^{i\nu_2} \\ -\dfrac{iu+\alpha_{2x}}{\sqrt{\beta_{2x}}} e^{i\nu_2} \\ \sqrt{\beta_{2y}} \\ -\dfrac{i(1-u)+\alpha_{2y}}{\sqrt{\beta_{2y}}} \end{bmatrix}$$

$$r_{xy} = \frac{\langle xy \rangle}{\sqrt{\langle x^2 \rangle \langle y^2 \rangle}} = \frac{\varepsilon_1 \sqrt{\beta_{1x}\beta_{1y}} \cos\nu_1 + \varepsilon_2 \sqrt{\beta_{2x}\beta_{2y}} \cos\nu_2}{\sqrt{\left(\varepsilon_1\beta_{1x} + \varepsilon_2\beta_{2x}\right)\left(\varepsilon_1\beta_{1y} + \varepsilon_2\beta_{2y}\right)}}$$

$\nu_1 = \pi/2$ $\qquad$ $\nu_1 = 0$





Ellipse traced by $\mathbf{v}_1$ in real space

Meaning of parameters

- α is derivative of β (without solenoids)

$$\frac{d\beta_{1x}}{ds} = -2\alpha_{1x} + R\sqrt{\beta_{1x}\beta_{1y}}\cos\nu_1 \qquad \frac{d\beta_{2x}}{ds} = -2\alpha_{2x} + R\sqrt{\beta_{2x}\beta_{2y}}\cos\nu_2$$

$$\frac{d\beta_{1y}}{ds} = -2\alpha_{1y} - R\sqrt{\beta_{1x}\beta_{1y}}\cos\nu_1 \qquad \frac{d\beta_{2y}}{ds} = -2\alpha_{2y} - R\sqrt{\beta_{2x}\beta_{2y}}\cos\nu_2$$

- Phase advance is proportional to β^{-1} (without solenoids)

$$\frac{d\mu_1}{ds} = \frac{1-u}{\beta_{1x}} - \frac{R}{2}\sqrt{\frac{\beta_{1x}}{\beta_{1y}}}\sin\nu_1 \qquad \frac{d\mu_2}{ds} = \frac{1-u}{\beta_{2y}} + \frac{R}{2}\sqrt{\frac{\beta_{2x}}{\beta_{2y}}}\sin\nu_2$$

$$\frac{d\mu_1}{ds} - \frac{d\nu_1}{ds} = \frac{u}{\beta_{1y}} - \frac{R}{2}\sqrt{\frac{\beta_{1x}}{\beta_{1y}}}\sin\nu_1 \qquad \frac{d\mu_2}{ds} - \frac{d\nu_2}{ds} = \frac{u}{\beta_{2x}} - \frac{R}{2}\sqrt{\frac{\beta_{2y}}{\beta_{2x}}}\sin\nu_2$$

Normalization matrix

$$\mathbf{V} = \left[Re\left[\mathbf{v_1}\right], -Im\left[\mathbf{v_1}\right], Re\left[\mathbf{v_2}\right], -Im\left[\mathbf{v_2}\right]\right]$$

$$\mathbf{V} = \begin{bmatrix} \sqrt{\beta_{1x}} & 0 & \sqrt{\beta_{2x}}\cos\nu_2 & -\sqrt{\beta_{2x}}\sin\nu_2 \\ -\dfrac{\alpha_{1x}}{\sqrt{\beta_{1x}}} & \dfrac{(1-u)}{\sqrt{\beta_{1x}}} & \dfrac{u\sin\nu_2 - \alpha_{2x}\cos\nu_2}{\sqrt{\beta_{2x}}} & \dfrac{u\cos\nu_2 + \alpha_{2x}\sin\nu_2}{\sqrt{\beta_{2x}}} \\ \sqrt{\beta_{1y}}\cos\nu_1 & -\sqrt{\beta_{1y}}\sin\nu_1 & \sqrt{\beta_{2y}} & 0 \\ \dfrac{u\sin\nu_1 - \alpha_{1y}\cos\nu_1}{\sqrt{\beta_{1y}}} & \dfrac{u\cos\nu_1 + \alpha_{1y}\sin\nu_1}{\sqrt{\beta_{1y}}} & -\dfrac{\alpha_{2y}}{\sqrt{\beta_{2y}}} & \dfrac{(1-u)}{\sqrt{\beta_{2y}}} \end{bmatrix} = \begin{bmatrix} \mathbf{V}_{xx} & \mathbf{V}_{xy} \\ \mathbf{V}_{yx} & \mathbf{V}_{yy} \end{bmatrix}$$

Matched beam covariance matrix

$$\mathbf{\Sigma} = \mathbf{V}\,\mathbf{\Sigma}_n \mathbf{V}^T$$

$$\langle x^2 \rangle = \varepsilon_1 \beta_{1x} + \varepsilon_2 \beta_{2x} \quad \langle x p_x \rangle = -\,\varepsilon_1 \alpha_{1x} - \varepsilon_2 \alpha_{2x} \quad \langle p_x^2 \rangle = \varepsilon_1 \gamma_{1x} + \varepsilon_2 \gamma_{2x}$$

$$\langle y^2 \rangle = \varepsilon_1 \beta_{1y} + \varepsilon_2 \beta_{2y} \quad \langle y p_y \rangle = -\,\varepsilon_1 \alpha_{1y} - \varepsilon_2 \alpha_{2y} \quad \langle p_y^2 \rangle = \varepsilon_1 \gamma_{1y} + \varepsilon_2 \gamma_{2y}$$

$$\gamma_{1x}\,\beta_{1x} = (1-u)^2 + \alpha_{1x}^2$$
$$\gamma_{2x}\,\beta_{2x} = u^2 + \alpha_{2x}^2$$
$$\gamma_{1y}\,\beta_{1y} = u^2 + \alpha_{1y}^2$$
$$\gamma_{2y}\,\beta_{2y} = (1-u)^2 + \alpha_{2y}^2$$

$$\langle xy \rangle = \varepsilon_1 \sqrt{\beta_{1x}\beta_{1y}} \cos \nu_1 + \varepsilon_2 \sqrt{\beta_{2x}\beta_{2y}} \cos \nu_2$$

$$\langle x p_y \rangle = \varepsilon_1 \sqrt{\frac{\beta_{1x}}{\beta_{1y}}} \left(u \sin \nu_1 - \alpha_{1y} \cos \nu_1 \right) - \varepsilon_2 \sqrt{\frac{\beta_{2x}}{\beta_{2y}}} \left((1-u) \sin \nu_2 + \alpha_{2y} \cos \nu_2 \right)$$

$$\langle y p_x \rangle = -\,\varepsilon_1 \sqrt{\frac{\beta_{1y}}{\beta_{1x}}} \left((1-u) \sin \nu_1 + \alpha_{1x} \cos \nu_1 \right) + \varepsilon_2 \sqrt{\frac{\beta_{2y}}{\beta_{2x}}} \left(u \sin \nu_2 - \alpha_{2x} \cos \nu_2 \right)$$

$$\langle p_x p_y \rangle = \varepsilon_1 \frac{\left(\alpha_{1y}(1-u) - \alpha_{1x} u\right)\, \sin \nu_1 + \left(u(1-u) - \alpha_{1x}\alpha_{1y}\right)\, \cos \nu_1}{\sqrt{\beta_{1x}\beta_{1y}}} + \varepsilon_2 \frac{\left(\alpha_{2x}(1-u) - \alpha_{2y} u\right)\, \sin \nu_2 + \left(u(1-u) - \alpha_{2x}\alpha_{2y}\right)\, \cos \nu_2}{\sqrt{\beta_{2x}\beta_{2y}}}$$

Transfer matrix

$$\mathbf{M}(\mu_1, \mu_2) = \mathbf{VPV^{-1}} = \mathbf{V} \begin{bmatrix} \cos\mu_1 & \sin\mu_1 & 0 & 0 \\ -\sin\mu_1 & \cos\mu_1 & 0 & 0 \\ 0 & 0 & \cos\mu_2 & \sin\mu_2 \\ 0 & 0 & -\sin\mu_2 & \cos\mu_2 \end{bmatrix} \mathbf{V^{-1}}$$

- All 16 elements written explicitly in paper

$$\begin{aligned} \mathbf{M}_{11} &= (1-u)\cos\mu_1 + \alpha_{1x}\sin\mu_1 + u\cos\mu_2 + \alpha_{2x}\sin\mu_2 \\ &\vdots \\ \mathbf{M}_{44} &= \dots \end{aligned}$$

Other methods

- Edwards-Teng (1973)
  - Transform $\mathbf{M}$ to block-diagonal form, then define Twiss parameters as in uncoupled case
  - Used by MADX
  - Does not generalize easily to 3D
  - Transformation is not unique
- Mais-Ripken (1989)
  - Basis for Lebedev-Bogacz parameterization
  - Only valid for circular machines
  - More parameters than necessary
- Wolski (2006)
  - Goal: clear connection between beam size and invariants
  - Write $\mathbf{\Sigma} = \varepsilon_1 \mathbf{B_1} + \varepsilon_2 \mathbf{B_2}$, then define Twiss parameters as elements of $\mathbf{B_1}$ and $\mathbf{B_2}$
  - Not all parameters are independent, but all have clear physical meaning
  - Generalizes easily to 3D (ε_3, $\mathbf{B}_3$)
- Qin-Davidson (2009)
  - Goal: be as elegant as the 1D theory, maintain same form of equations
  - Generalize α and β to 2×2 matrices

Summary

- Particle motion and beam ellipsoid can be expressed in terms of transfer matrix eigenvectors
  - Phase advance: $\mu_x, \mu_y \rightarrow \mu_1, \mu_2$
  - Emittance: $\varepsilon_x, \varepsilon_y \rightarrow \varepsilon_1, \varepsilon_2$
- Parameterization introduced for the eigenvectors
  - $\alpha_{1x}, \alpha_{1y}, \alpha_{2x}, \alpha_{2y}, \beta_{1x}, \beta_{1y}, \beta_{2x}, \beta_{2y}, u, \nu_1, \nu_2$
  - Parameters describe ellipses traced by eigenvectors in phase space
  - Number of parameters is minimized
  - In absence of coupling: $\alpha_{1x}, \beta_{1x}, \alpha_{2y}, \beta_{2y} \rightarrow \alpha_x, \beta_x, \alpha_y, \beta_y$ and $\alpha_{2x}, \beta_{2x}, \alpha_{1y}, \beta_{1y} \rightarrow 0$