Cheat Sheet (Exam 1)

PROOFS

Direct-Proofs *Directly, i.e. working forward: Assume A is true, and then show B is true.* A direct proof is an argument to establish an implication $p \to q$ by assuming p and constructing a sequence of valid inferences that establish the statement q.

Indirect-Proof An indirect proof also establishes an implication $p \to q$, but does so by way of the contrapositive: one assumes $\neg q$ and constructs an argument that $\neg p$ must follow.

Proof by Contradiciton: A proof by contradiction establishes a statement p by proving $\neg p$ must be false. In particular, one assumes $\neg p$ and shows that a contradiction arises from this assumption

Proof by Cases: A proof by cases establishes a statement q by considering a number of cases $p_1, p_2, ..., p_n$, one of which must hold, and showing that in each of these cases, $p_i \to q$.

Existence Proof: An existence proof establishes a statement of the form $\exists x P(x)$. There are two kinds. A **constructive proof** is given by providing an explicit element c and establishing $P(c)$. A **nonconstructive** proof is any other proof of $\exists x P(x)$ that does not provide an explicit element c such that P(c). For example, one might proceed by contradiction and show that $\forall x \neg P(x)$ is impossible.

Uniqueness Proof: Mathematical theorems often assert the existence of a unique element satisfying a given predicate. For such theorems, a uniqueness proof must be supplied. That is, one must first prove the existence, as described above, but then one must also argue the uniqueness of the element under discussion. In other words, after proving $\exists x P(x)$, a uniqueness proof also requires that you prove $\forall x \forall y ((P(x)P(y)) \to (x = y))$.

Note: The converse of an implication $p \to q$ is the implication $q \to p$.

equivalences involving quantifiers	name
$\sim \forall x\, P(x) = \exists x \sim P(x)$ $\sim \exists x\, P(x) = \forall x \sim P(x)$	generalized De Morgan's laws
$\forall x \forall y\, P(x,y) = \forall y \forall x P(x,y)$ $\exists x \exists y\, P(x,y) = \exists y \exists x P(x,y)$	commutative laws

Logical Equivalences

1.	$\neg(\neg p) \iff p$	Double Negation
2.a.	$(p \vee q) \iff (q \vee p)$	Commutative Laws
2.b.	$(p \wedge q) \iff (q \wedge p)$	
2.c.	$(p \leftrightarrow q) \iff (q \leftrightarrow p)$	
3.a.	$[(p \vee q) \vee r] \iff [p \vee (q \vee r)]$	Associative Laws
3.b.	$[(p \wedge q) \wedge r] \iff [p \wedge (q \wedge r)]$	
4.a.	$[p \vee (q \wedge r)] \iff [(p \vee q) \wedge (p \vee r)]$	Distributive Laws
4.b.	$[p \wedge (q \vee r)] \iff [(p \wedge q) \vee (p \wedge r)]$	
5.a.	$(p \vee p) \iff p$	Idempotent Laws
5.b.	$(p \wedge p) \iff p$	
6.a.	$(p \vee 0) \iff p$	Identity Laws
6.b.	$(p \wedge 1) \iff p$	
7.a.	$(p \vee 1) \iff 1$	Domination Law
7.b.	$(p \wedge 0) \iff 0$	
8.a.	$\neg(p \vee q) \iff (\neg p \wedge \neg q)$	DeMorgan's Laws
8.b.	$\neg(p \wedge q) \iff (\neg p \vee \neg q)$	
8.c.	$(p \vee q) \iff \neg(\neg p \wedge \neg q)$	
8.d.	$(p \wedge q) \iff \neg(\neg p \vee \neg q)$	
9.	$(p \to q) \iff (\neg q \to \neg p)$	Contrapositive
10.a.	$(p \to q) \iff (\neg p \vee q)$	Implication
10.b.	$(p \to q) \iff \neg(p \wedge \neg q)$	
11.a.	$(p \vee q) \iff (\neg p \to q)$	
11.b.	$(p \wedge q) \iff \neg(p \to \neg q)$	
12.a.	$[(p \to r) \wedge (q \to r)] \iff [(p \vee q) \to r]$	
12.b.	$[(p \to q) \wedge (p \to r)] \iff [p \to (q \wedge r)]$	
13.	$(p \leftrightarrow q) \iff [(p \to q) \wedge (q \to p)]$	Equivalence
14.	$[(p \wedge q) \to r] \iff [p \to (q \to r)]$	Exportation Law
15.	$(p \to q) \iff [(p \wedge \neg q) \to c]$	Reductio ad Absurdum
16.	$(p \oplus q) \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$	Exclusive Or
17.a.	$p \vee (p \wedge q) \equiv p$	Absorption Laws
17.b.	$p \wedge (p \vee q) \equiv p$	
18.a.	$(p \vee \neg p) \iff 1$	Negation Laws
18.b.	$(p \wedge \neg p) \iff 0$	

Logical Implications

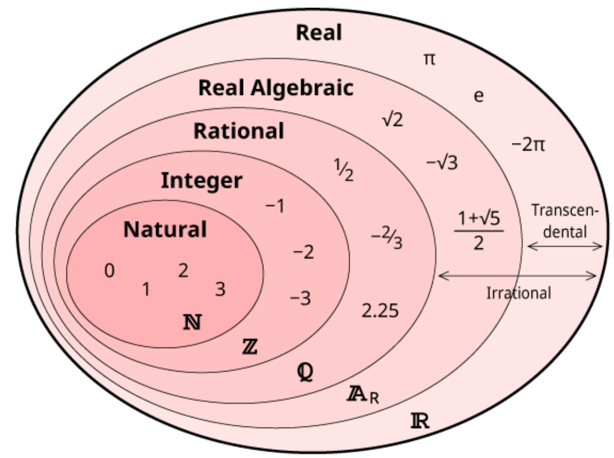
1.	$p \Rightarrow (p \vee q)$	Addition
2.	$(p \wedge q) \Rightarrow p$	Simplification
3.	$(p \to c) \Rightarrow \neg p$	Absurdity
4.	$[p \wedge (p \to q)] \Rightarrow q$	Modus Ponens
5.	$[(p \to q) \wedge \neg q] \Rightarrow \neg p$	Modus Tollens
6.	$[(p \vee q) \wedge \neg p] \Rightarrow q$	Disjunctive Syllogism
7.	$p \Rightarrow [q \to (p \wedge q)]$	
8.a.	$[p \leftrightarrow q) \wedge (q \leftrightarrow r)] \Rightarrow (p \leftrightarrow r)$	Transitivity
8.b.	$[p \to q) \wedge (q \to r)] \Rightarrow (p \to r)$	
9.a.	$(p \to q) \Rightarrow [(p \vee r) \to (q \vee r)]$	
9.b.	$(p \to q) \Rightarrow [(p \wedge r) \to (q \wedge r)]$	
9.c.	$(p \to q) \Rightarrow [(p \to r) \to (q \to r)]$	
10.a.	$[(p \to q) \wedge (r \to s)] \Rightarrow [(p \vee r) \to (q \vee s)]$	Constructive Dilemmas
10.b.	$[(p \to q) \wedge (r \to s)] \Rightarrow [(p \wedge r) \to (q \wedge s)]$	
11.a.	$[(p \to q) \wedge (r \to s)] \Rightarrow [(\neg q \vee \neg s) \to (\neg p \vee \neg r)]$	Destructive Dilemmas
11.b.	$[(p \to q) \wedge (r \to s)] \Rightarrow [(\neg q \wedge \neg s) \to (\neg p \wedge \neg r)]$	

Negations

1. The contrapositive of the implication $p \to q$ is $\neg q \to \neg p$.
2. The negation of "for every/for all" ($\forall$) is "there exits" ($\exists$).
3. The negation of "there exists" ($\exists$) is "for every/for all" ($\forall$).
4. The negation of "if p, then q" is "p and ¬ q" (note: the negation of an "if, then" statement is NOT an "if, then" statement).
5. The negation of "p or q" is "¬p and ¬q".
6. The negation of "p and q" is "¬p or ¬q".

SET THEORY

Random Notes: The empty set is a subset of all sets.
The set $R \setminus S = x \in R | x \notin S$ is called the complement of S in R. Sometimes $R \setminus S$ is called the relative complement of S in R.
If A and B are nonempty finite sets, then $|A \times B| = |A| * |B|$



$\emptyset$ empty set
$\mathbb{N}$ natural numbers
$\mathbb{Z}$ integers (from Zahl, German for number).
$\mathbb{Q}$ rational numbers (from quotient)
$\mathbb{R}$ real numbers
$\mathbb{C}$ complex numbers

Set Identities

$A \cup \emptyset = A$ $A \cap U = A$	Identity Laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination Laws
$A \cup A = A$ $A \cap A = A$	Idempotent Laws
$\overline{(\overline{A})} = A$	Complementation Law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative Laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative Laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive Laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's Laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption Laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement Laws

Nested / Disjoint

Nested We call a sequence $(A_n)_{n=1}^{\infty}$ of sets a *nested sequence of sets* if the next set is always a subset of its predecessor, i.e.,

$$(\forall n \in \{1, 2, \dots\}) A_{n+1} \subseteq A_n.$$

So the nested sequence looks like this
$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \dots$
A collection δ of sets is said to be *nested* iff for every $A, B \in \delta$, either $A \subseteq B$ or $B \subseteq A$. (Often you'll see $\delta = \{A_0, A_1, A_2, \dots\}$ with either $A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$ or $A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$, but some nested collections have a more complicated structure than those.)
So examples of nested sequences of subsets of $\mathbb{R}$ would be:

1. $A_n = (-\frac{1}{n}, \frac{1}{n})$
2. $A_n = [n, \infty)$
3. $A_n = [0, 1 + \frac{1}{n})$

Disjoint A set (of sets) $\mathcal{A}$ is disjoint if $\bigcap \mathcal{A} = \emptyset$.
The set $\mathcal{A}$ is pairwise disjoint when
$\forall x \in A : \forall y \in A : x \neq y \implies x \cap y = \emptyset$. This implies disjoint if $|\mathcal{A}| \geq 2$.
So $\mathcal{A} = \{x, y\}$ is disjoint iff it is pairwise disjoint.
But in maeasure theory, disjoint is often used as a shorthand for "pairwise disjoint".

Union / Intersection

$\bigcup_{i \in I} A_i$ is the set of all those elements which appear in *some* A_i, and $\bigcap_{i \in I} A_i$ is the set of those elements which appear in *every* A_i.

Example Consider the collection C of intervals in $\mathbb{R}$ given by:
$C = (5 - n, 6 + 1/n] | n \in \mathbb{R}$
It is NOT **nested**, since $7 \in (4, 7] \setminus (3, 6\frac{1}{2}]$ and $4 \in (3, 6\frac{1}{2}] \setminus (4, 7]$
(In this case, work forward from the base)
It is NOT **mutually disjoint**.
$5 \in (4, 7] \cap (3, 6\frac{1}{2}]$
Intersection: (4, 6]
Union:
$(-\infty, 7]$

Partitions

Definition: A *partition* of a nonempty set S is a collection $\mathfrak{C} = B_{i i \in I}$ of nonempty mutually disjoint subsets of S with $\bigcup_{i \in I} B_i = S. The sets B_i \in \mathfrak{C}$ are called *blocks* of the partition.
The collection $\{\{1, 2\}, \{3, 4, 5\}, \{6\}\}$ is a partition of the set $S = \{1, 2, 3, 4, 5, 6\}$ since blocks $B_1 = \{1, 2\}, B_2 = \{3, 4, 5\}, B_3 = 6$ are non-empty, mutually disjoint, and their union $\{1, 2\} \cup \{3, 4, 5\} \cup \{6\}$ is S.
So to show that a collection partitions a space, show that partition is:

1. Non-empty, or: $\forall n \in \mathbb{Z}$ *(or whatever it is)*, that $B_i \neq \emptyset$. Maybe give an interval that's an element of the indexed partition.
2. Disjoint,
3. Spans the space given any (x, y).

Example:
For each $n \in \mathbb{Z}, let$:
$\mathcal{T}_n = \{(x, y) \in \mathbb{R}^2 \mid n \leq x - y < n + 1\}$
Is $\mathcal{T} = \{\mathcal{T}_n \mid n \in \mathbb{Z}\}$ a partition of $\mathbb{R}^2$? Justify your answer using the definition.
Solution:
To start off,
$\mathcal{T}_0 = \{(x, y) \mid 0 \leq x - y < 1\}$
$= \{(x, y) \mid y \leq x \;\&\; y > x - 1\}$.
Therefore $\mathcal{T}_n = \{(x, y) \mid y \leq x - n \;\&\; y > x - n - 1\}$.

1. **Non-Empty**:
   $\forall n \in \mathbb{Z}$, $\mathcal{T}_n \neq \emptyset$ since (0, -n) $\in \mathcal{T}_n$.
2. **Disjoint**:
   $\forall m, n \in \mathbb{Z}$, if $m < n$, then:
   $\mathcal{T}_m \cap \mathcal{T}_n = \{(x, y) \mid m \leq x - y < m + 1 \;\&$
   $n \leq x - y < n + 1\}$
3. **Covers** $\mathbb{R}^2$:
   Given any (x,y) $\in \mathbb{R}^2$, let $m = \lceil x - y \rceil \in \mathbb{Z}$.
   Then $m \leq x - y < m + 1$.
   So (x,y) $\in \mathcal{T}_m$. Therefore, $\mathcal{T}_n$'s partition $\mathbb{R}^2$.