# Homework 3

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### 6-4

**Potential customers arrive at single-server station in accordance with a Poisson process with rate $\lambda$. However, if the arrival finds $n$ customers already in the station, then he will enter the system with probability $\alpha_n$. Assuming an exponential service rate $\mu$, set this up as a birth and death process and determine the birth and death rates.**

Let $N(t)$ be the number of customers in the station at time $t$.

$\{N(t), t \geq 0\}$ is a birth and death process with birth rate $\lambda_n = \lambda\alpha_n$ and death rate $\mu_n = \mu$.

## 6-9

**The birth and death process with parameters $\lambda_n = 0$ and $\mu_n = \mu$, $n > 0$ is called a pure death process.Find$P_{ij}(t)$.**

Birth rate: $\lambda_n = 0$

Death rate: $\mu_n = \mu$, where $n > 0$ is a pure death process.

First we find the transition probability, $P_{ij}(t)$ for the continuous Markov chain. $P_{ij}(t)$ is the death rate, $\mu_n = \mu$. The number of deaths in any interval of length $t$ follows a Poisson process with mean $\mu t$ until the system empties.

Therefore, $P_{ij}(t)$ is the probability of a Poisson distribution that assumes the value $(i - j)$ for $i > j$. This gives $(i - j)$ events for the interval of length $t$. The PMF of a Poisson distribution is as follows:

$$P_{ij}(t) = \frac{e^{-\mu t}(\mu t)^{i-j}}{(i-j)!}; 0 < j \leq i$$

When $j = 0$,

$$\sum_{j=0}^{i} P_{i,j}(t) = 1$$
$$P_{i,0}(t) + \sum_{j=1}^{i} P_{i,j}(t) = 1$$

$P_{i,0}(t) + \sum_{j=1}^{i} P_{i,j}(t) = 1$ can be re-written as:

$$\begin{aligned}
P_{i,0}(t) &= 1 - \sum_{j=1}^{i} P_{i,j}(t) \\
&= 1 - \sum_{j=1}^{i} \frac{e^{-\mu t}(\mu t)^{i-j}}{(i-j)!} \\
&= 1 - \sum_{j=1}^{i} \frac{e^{-\mu t}(\mu t)^{k}}{k!}; k = i - j \\
&= \sum_{k=1}^{\infty} \frac{e^{-\mu t}(\mu t)^{k}}{k!}
\end{aligned}$$

## 6-18

*After being repaired, a machine function for an exponential time with rate $\lambda$ and then fails. Upon failure, a repair process begins. The repair process proceeds sequentially through $k$ distinct phases. First a phase 1 repair must be performed, then a phase 2, and so on. The times to complete these phases are independent with phase $i$ taking an exponential time with rate $\mu_i, i = 1, ..., k$.*

Let the process have $k+1$ states.

State 0: THe machine is working.

State 1: The machien is undergoing repair phase 1.

...

State $k$: The Machine is undergoing repair phase $k$.

This is a birth and death process with parameters:

$$\begin{aligned} v_0 &= \lambda & q_{01} &= \lambda \\ v_i &= \mu_i, i > 0 & q_{i,j+1} &= \mu_i, 1 < i < k \\ & & q_{k0} &= \mu_k \end{aligned}$$

We can stae the equations of balance of the process as

$$v_j P_j = \sum_{k \neq j} q_{kq} P_k$$

| State | Rate At Which Leave | = | Rate At Which Enter |
|---|---|---|---|
| 0 | $\lambda P_0$ | = | $\mu_k P_k$ |
| 1 | $\mu_1 P_1$ | = | $\lambda P_0$ |
| $i$ | $\mu_i P_i$ | = | $\mu_{i-1} P_{i-1}, \quad 1 < i < k$ |

Solving in terms of $P_0$ we get:

$$\begin{aligned} P_k &= \frac{\lambda}{\mu_k} P_0 & P_i &= \frac{\mu_{i-1}}{\mu_i} P_{i-1} \\ P_1 &= \frac{\lambda}{\mu_1} P_0 & &= \frac{\mu_{i-1}}{\mu_i} \frac{\mu_{i-2}}{\mu_{i-1}} P_{i-2} \\ P_2 &= \frac{\lambda}{\mu_2} P_1 & &= \frac{\mu_{i-1}}{\mu_i} \frac{\mu_{i-2}}{\mu_{i-1}} ... \frac{\lambda}{\mu_1} P_0 \\ &= \frac{\lambda}{\mu_2} P_0 & &= \frac{\lambda}{\mu_1} P_0, \qquad 1 < i < k \end{aligned}$$

Because $P_i = \frac{\lambda}{\mu_1} P_0, 1 \leq k$ and using $\sum_{i=0}^{k} P_i = 1$, we have:

$$1 = P_0 + \left( \frac{\lambda}{\mu_1} + ... + \frac{\lambda}{\mu_k} \right) P_0 P_0 = \frac{1}{\left(1 + \lambda \sum_{i=1}^{k} \frac{1}{\mu_i}\right)}$$

### (a) What proportion of time is the machine undergoing a phase $i$ repair?

From the above,

$$P_i = \frac{\lambda}{\mu_i} \frac{1}{\left(1 + \lambda \sum_{i=1}^{k} \frac{1}{\mu_i}\right)}$$

**(b) What proportion of time is the machine working?**

$$P_0 = \frac{1}{\left(1 + \lambda \sum_{i=1}^{k} \frac{1}{\mu_i}\right)}$$

## 7-12

*Events occur according to a Poisson process with rate $\lambda$. Any event that occurs within a time d of the event that immediately proceed it is called a d-event. FOr instance, if $d = 1$ and events occur at times $2, 2, 8, 4, 6, 6, 6...$, then the events at times 2.8 and 6.6 would be d-events.*

Let $X$ be the time between successive $d$-events. Conditioning on $T$, the time until the next event following a $d$-event gives:

$$\begin{aligned}
E[X] &= \int_0^d x\lambda e^{-\lambda x} dx + \int_d^\infty x + E[X]\lambda e^{-\lambda x} dx \\
&= \frac{1}{\lambda} + E[X]e^{-\lambda d} \\
E[X] &= \frac{1}{\lambda(1 - e^{-\lambda d})}
\end{aligned}$$

### (a) At what rate do $d$-events occur?

$$\frac{1}{E[X]} = \lambda\left(1 - e^{-\lambda d}\right)$$

### (b) What proportion of all events are $d$-events?

$$1 - e^{-\lambda d}$$

## 7-23

In a serve and rally competition involving players A and B, each rally that begins with a serve by player A is won by player A with probability $p_a$ and is won by player B with probability $q_a = 1 - p_a$, whereas each rally that begins with a serve by player B is won by player A with the probability $p_b$ and is won by player B with probability $q_b = 1 - p_b$. The winner of the rally earns a point and becomes the server of the next rally.

**(a) In the long run, what proportion of points are won by A?**

If a new cycle begins each time that A wins, with N equal to the number of points in a cycle,

$$E[N] = 1 + \frac{q_a}{p_b}$$

When starting with B serving, the number of the points played until A wins is a geometric with parameter $p_b$, therefore the proportion of points won by A is:

$$\frac{1}{E[N]} = \frac{p_b}{p_b + q_a}$$

**(b) What proportion of points are won by A if the protocol is that the players alternate service? That is, if the service protocol is that A serves for the first point, then B for the second, then A for the third point, and so on.**

$$\frac{(p_a + p_b)}{2}$$

**(c) GIve the condition under which A wins a higher percentage of points under the winner serves protocol than under the alternating service protocol.**

$$\frac{p_b}{p_b + q_a} > \frac{(p_a + p_b)}{2}$$

This is equivalent to:

$$p_a q_a > p_b q_b$$

$$\begin{aligned}
\frac{A(t)}{t} &= \frac{t - S_{N(t)}}{t} \\
&= \frac{1 - S_{N(t)}}{t} \\
&= \frac{1 - S_{N(t)}}{N(t)} \frac{N(t)}{t}
\end{aligned}$$