HW_5

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9-2

Show that

(a) if $\phi(0, 0, ..., 0) = 0$ **and** $\phi(1, 1, ..., 1) = 1$ **then min** $x_i \leq \phi(\vec{x}) \leq$ **max** x_i

$\phi(\vec{x}) \leq \max\ \vec{x}_i$ with state vector $\vec{x} = (x_1, ..., x_n)$
$x_i = 0$ for some i , $i = 1, ..., n,$ or $x_i = 1$ for all i.
If $x_i = 0$ for some i, min $x_i = 0 \therefore \min\ x_i \leq \phi(\vec{x})$
If $x_i = 1$ for all i, $i = 1, ..., n$, that is $\vec{x} = (1, ..., 1)$ then min $x_i = 1$ and $\phi(\vec{x} = 1) \therefore$ min $x_i \leq \phi(\vec{x})$

$\phi\vec{x} \leq \max\ x_i$ with state vector $\vec{x} = (x_1, ..., x_n)$
$x_i = 1$ for some i , $i = 1, ..., n,$ or $x_i = 0$ for all i, $i = 1, ..., n.$
If $x_i = 1$ for some i, max $x_i = 0$ and $\phi(\vec{x}) = 0 \therefore \phi(\vec{x}) \leq$ max x_i

Thus, min $x_i \leq \phi\vec{x} \leq$ max x_i

(b) $\phi($**max** $(\vec{x}, \vec{y})) \geq$ **max** $(\phi(\vec{x}), \phi(\vec{y}))$

max $(\vec{x}, \vec{y}) \geq \vec{x}$ and max $(\vec{x}, \vec{y}) \geq \vec{y}$
Because $\phi(\vec{x})$ is an increasing function,
$\phi(\max\ (\vec{x}, \vec{y})) \geq \phi(\vec{x})$ and $\phi(\max\ (\vec{x}, \vec{y})) \geq \phi(\vec{y})$
$\therefore \phi(\max(\vec{x}, \vec{y})) \geq \max(\phi(\vec{x}), \phi(\vec{y}))$

(c) $\phi($**min** $(\vec{x}, \vec{y})) \leq$ **min** $(\phi(\vec{x}), \phi(\vec{y}))$

min $(\vec{x}, \vec{y}) \leq \vec{x}$ and min $(\vec{x}, \vec{y}) \leq \vec{y}$
Because $\phi(\vec{x})$ is an increasing function,
$\phi(\min\ (\vec{x}, \vec{y})) \leq \phi(\vec{x})$ and $\phi(\min\ (\vec{x}, \vec{y})) \leq \phi(\vec{y})$
$\therefore \phi(\min(\vec{x}, \vec{y}) \leq \min(\phi(\vec{x}), \phi(\vec{y}))$

9-13

Let $r(\vec{p})$ be the reliability function.

Show that

$$r(\vec{p}) = p_i r(1_i, \vec{p}) + (1 - p_i) r(0_i, \vec{p})$$

$$\begin{aligned}
r(p) &= E[\phi(X)] \\
&= p_i \{E\phi(X)|X_i = 1\} + (1 - p_i) E(\phi(X)|X_i = 0) \\
&= p_i E[\phi(1_i, X)] + (1 - p_i) E[\phi(o_i, X)] \\
r(p) &= p_i [r(1_i, p) + (1 - p_i \{(o_i, p)\})] \\
\therefore r(p) &= p_i [r(1_i, p)] + (1 - p_i) \{o_i, p\}
\end{aligned}$$

9-23

Show that if each (independent) component of a series system has an IFR distribution, then the system lifetime is itself IFR by

(a) showing that

$$\lambda_F(t) = \sum_i \lambda_i(t)$$

where $\lambda_F(t)$ is the failure rate function of the system; and $\lambda_i(t)$ the failure rate function fo the lifetime of component i.

$$\begin{aligned}\bar{F}(t) &= \Pi_i \bar{F}_i(t)\\ F(t) &= 1 - \Pi_i(1 - F_i(t))\\ &= 1 - (1 - F_1(t))(1 - F_2(t))...(1 - F_n(t))\end{aligned}$$

where F_i is the lifetime distribution of the ith component

Taking the derivatives of boths sides, we get

$$\begin{aligned}F'(t) &= F_1'(t)(1 - F_2(t))...(1 - F_n(t))\\ &+ F_2'(t)(1 - F_1(t))...(1 - F_n(t)) +\\ &... + F_n'(t)(1 - F_1(t))(1 - F_2(t))...(1 - F_{n-1}(t))\\ &= \sum_i F_i'(t)\Pi_{j\neq i}(1 - F_j(t))\end{aligned}$$

The system failure rate function is given by:

$$\begin{aligned}\lambda_F(t) &= \frac{F'(t)}{1 - F(t)}\\ &= \frac{\sum_i F_i'(t)\Pi_{j\neq 1}(1 - F_j(t))}{\Pi_i(1 - F_i(t))}\\ &= \frac{F_1'(t)}{1 - F_1(t)} + \frac{F_2'(t)}{1 - F_2(t)} + \frac{F_3'(t)}{1 - F_3(t)} + ... + \frac{F_n'(t)}{1 - F_n(t)}\\ &= \sum_i \frac{F_i'(t)}{1 - F_i(t)}\\ &= \sum_i \lambda_i(t)\end{aligned}$$

(b) Using the definition of IFR given in Exercise 22.

If each F_i is an IFR, then $\bar{F}_{it}(a)$ is a decreasing function in t. $\bar{F}_t(a)$ is the probability that a t-year old system surives additional time t.

$$\begin{aligned}\bar{F}_t(a) &= \frac{\bar{F}(t + a)}{\bar{F}(t)}\\ &= \frac{\Pi_i \bar{F}_i(t + a)}{\Pi_i \bar{F}_i(t)}\\ &= \Pi_i \frac{\bar{F}_i(t + a)}{\bar{F}_i(t)}\\ &= \Pi_i \bar{F}_{it}(a)\end{aligned}$$