HW_1

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Homework 1

3-32

Independent trials, each resulting in success with probability p, are performed.

(a) Find the expected number of trials needed for there to have been both at least n successes and at least m failures.

Let T be the number of trials needed for both at least n successes and m failures. Condition on N, the number of successes in the first $n+m$ trials, to obtain

$E[T]\sum_{i=0}^{n+m} E[T|N=i]\binom{n+1}{i}p^i(1-p)^{n+m-i}$

Now use

$E[T|N=i]=n+m+\frac{n-i}{p}, i\le n$

$E[T|N=i]=n+m+\frac{i-n}{1-p}, i>n$

(b) Find the expected number of trials needed for there to have been either at least n successes or at least m failures.

Let S be the number of trials needed for n successes, and let F be the number needed for m failures. Then $T=$max(S,F). Taking expectations of the identity

min$(S,F)+$ max(S,F)

yields the result

$E[\min(S,F)]=\frac{n}{p}+\frac{m}{1-p}-E[T]$

3-64

A and B roll a pair of dice in turn, with A rolling first. A's objective is to obtain a sum of 6, and B's is to obtain a sum of 7. The game ends when either player reaches their objective, and that player is declared the winner.

(a) Find the probability that A is the winner.

$P(A)=\frac{5}{36}+(\frac{31}{36})(\frac{5}{6})P(A)$

$P(A)=\frac{30}{61}$

(b) Find the expected number of rolls of the dice.

$E[X]=\frac{5}{36}+(\frac{31}{36})[1+\frac{1}{6}+(\frac{5}{6})(1+E[X])]$

$E[X] = \frac{402}{61}$

(c) Find the variance of the number of rolls of the dice.

Let Y equal 1 if A wins on their first attempt, let it equal 2 if B wins on their first attempt, and let it equal 3 otherwise. Then

$Var(X|Y=1) = 0$, $Var(X|Y=2) = 0$, $Var(X|Y=3) = Var(X)$

Hence,

$E[Var(X|Y)] = (\frac{155}{216})Var(X)$

Also,

$E[X|Y=1] = 1$, $E[X|Y=2] = 2$, $E[X|Y=3] = 2 + E[X] = \frac{524}{61}$

Then,

$Var(E[X|Y]) = 1^2(\frac{5}{36}) + 2^2(\frac{31}{216}) + (\frac{524}{61})^2(\frac{155}{216} - (\frac{402}{61})^2$

≈ 10.2345

From the conditional variance formula we see that

$Var(X) \approx z(\frac{155}{216})Var(X) + 10.2345$

≈ 36.24

3-99

Let N be the number of trials until k consecutive successes have occurred, when each trial is independently a success with probability p.

(a) What is $P(N = k)$

(b) Argue that $P(N = k + r) = P(N > r - 1)qp^k, r > 0$

$P(N = k + r) = P(N > r - 1)qp^k$

$\sum_{r=1}^{\infty} P(N = k + r) = \sum_{r=1}^{\infty} P(N > r - 1)qp^k$

Because $N > k$, $\sum_{r=0}^{\infty} P(N = k + r) = 1$

(c) Show that $1 - p^k = qp^k E[N]$

$\sum_{r=1}^{\infty} P(N = k + r) = 1 - P(N = k) = 1 - p^k$ (because $N = k$, all first trials give k successes)

$= \sum_{r=1}^{\infty} P(N > r - 1)qp^k = E(N)qp^k$

For any non-negative discrete random variable X, $E(X) = \sum_{x=0}^{\infty} P(X > x)$

$1 - p^k = E(N)qp^k$