Exam 1

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1. Two in a Row

Two players take turns shooting at a target, with each shot by player i hitting the target with probability $p_i, i = 1, 2$. Shooting ends when two consecutive shots hit the target. Let μ_i denote the mean number of shots taken when player i shoots first, $i = 1, 2$.

Let $q_i = 1 - p_i, i = 1, 2$.

$h =$ hit

$m =$ miss

(a) Find μ_1 and μ_2.

$$\begin{aligned}
\mu_1 &= E[N|h]p_1 + E[N|m]q_1 \\
&= p_i(E[N|h,h]p_2 + E[N|h,m]q_2) + (1 + \mu 2)q_1 \\
&= 2p_1p_2 + (2 + \mu_1)p_1q_2 + (1 + \mu_2)q_1 \\
&= \mu_1(1 - p_1q_2) = 1 + p_1 + \mu_2q_1
\end{aligned}$$

Similarly,

$$\mu_2(1 - p_2q_1) = 1 + p_2 + \mu_1q_2$$

(b) Let h_i denote the mean number of times that the target is hit when player i shoots first, $i = 1, 2$. Find h_1 and h_2.

$$\begin{aligned}
h_1 &= E[H|h]p_1 + E[H|m]q_1 \\
&= p_1(E[H|h,h]p_2 + E[H|h,m]q_2) + h_2q_1 \\
&= 2p_1p_2 + (1 + h_1)p_1q_2 + h_2q_1
\end{aligned}$$

Similarly,

$$h_2 = 2p_1p_2 + 1(1 + h_2)p_2q_1 + h_1q_2$$

By solving these equations, we find h_1 and h_2

2. Varying Claims

The number of accidents in a month follows a Poisson distribution with mean 12. Each accident generates 1,2, or 3 claimants with probabilities $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{6}$, respectively.

Calculate the variance in the total number of claimants.

```
n1 <- (12*1/2)
n2 <- (12*1/3)
n3 <- (12*1/6)

dist <- c(n1, n2, n3)
dist
```

```
## [1] 6 4 2
```

$$\mathrm{Var}(n_1 + 2n_2 + 3n_3 = \mathrm{Var}(n_1) + 4\mathrm{Var}(n_2) + 9\mathrm{Var}(n_3)$$

```
dist[1]+4*dist[2]+9*dist[3]
```

```
## [1] 40
```

3. Heavy Training

A coach can give two types of training, "light" or "heavy" to his sports team before a game. If the team wins the prior game, the next training is equally likely to be light or heavy. But, if the team loses the prior game, the next training is always heavy. The probability that the team will win the game is 0.4 after light training and 0.8 after heavy training.

Calculate the long run proportion of time that the coach will give heavy training to the team.

```
library(markovchain)
```

```
## Warning: package 'markovchain' was built under R version 4.0.5

## Package:  markovchain
## Version:  0.8.6
## Date:     2021-05-17
## BugReport: https://github.com/spedygiorgio/markovchain/issues
```

```
# Setting up transitional matrix

state <- c("win", "lose")
transmat <- matrix(c(0.4, 0.6, 0.8, 0.2), nrow = 2, byrow = T)
row.names(transmat) <- c("light", "heavy")
colnames(transmat) <- state
transmat
```

```
##       win lose
## light 0.4  0.6
## heavy 0.8  0.2
```

```
train_mc <- new("markovchain", transitionMatrix=transmat, states=row.names(transmat), name="Training Ma
train_mc
```

```
## Training Markov Chain
##  A  2 - dimensional discrete Markov Chain defined by the following states:
##  light, heavy
##  The transition matrix  (by rows)  is defined as follows:
##       light heavy
## light   0.4   0.6
## heavy   0.8   0.2
```

```
P <- transmat
P2 <- P%*%P; P2
```

```
##        win lose
## light 0.64 0.36
## heavy 0.48 0.52
```

```
P3 <- P2%*%P2; P3
```

```
##          win   lose
## light 0.5824 0.4176
## heavy 0.5568 0.4432
```

```r
P4 <- P3%*%P3; P4
```

```
##             win      lose
## light 0.5717094 0.4282906
## heavy 0.5710541 0.4289459
```

```r
P5 <- P4%*%P4; P5
```

```
##             win      lose
## light 0.5714288 0.4285712
## heavy 0.5714283 0.4285717
```

As we can see,the probability of a win converges to 0.5714288 or $\frac{4}{7}$, meaning the probability of a loss is 0.4285717 or $\frac{3}{7}$. With heavy training guaranteed after a loss, the probability of heavy training is 42.9%.

4. Tom and Huck

Tom and Huck are painting a fence together. Tom and Huck both start painting at the same time, but lazy Huck stops working after Y hours, where the random variable Y is uniform on $(0, 1)$. Diligent Tom finishes the job at time X, where the conditional distribution of X is uniform on $(Y, 1)$.

```
library(MASS)
mu_y <- 0.5
var <- 1/12
```

$$
\begin{aligned}
\mu_{x|y} &= \frac{y+1}{2} \\
\sigma^2_{x|y} &= \frac{(y-1)^2}{12} = \frac{y^2-2y+1}{12}
\end{aligned}
$$

(a) What is the expected time that Tom works?

$$
\begin{aligned}
E(X) = \mu_x &= E(E(x|y)) \\
&= E\left(\frac{y+1}{2}\right) \\
&= \frac{1}{2}E(Y)+1 \\
&= \frac{1}{2}\mu_y+1
\end{aligned}
$$

```
(1/2)*(mu_y+1)
```

```
## [1] 0.75
```

```
fractions(0.75)
```

```
## [1] 3/4
```

(b) What is the variance of the time that Tom works?

$$
\begin{aligned}
\text{Var}(X) &= E(\text{Var}(X|Y=y)) + \text{Var}(E(X|Y=y)) \\
&= E\left(\frac{y^2-2y+1}{12}\right) + \text{Var}\left(\frac{y+1}{2}\right) \\
&= \frac{1}{12}\left(E(y^2)-2E(y)+1\right) + \frac{1}{4}\text{Var}(y) \\
&= \frac{1}{12}\left(1-2E(y)+E(y^2)\right) + \frac{1}{4}\text{Var}(y) \\
&= \frac{1}{12}\left[1-2*\frac{1}{2}+\left(\frac{1}{12}+\frac{1}{4}\right)\right] + \frac{1}{4}*\frac{1}{12}
\end{aligned}
$$

```
(1/12)*(1-2*(1/2)+((1/12)+1/4))+(1/4)*(1/12)
```

```
## [1] 0.04861111
```

```r
fractions(0.04861111)
```

```
## [1] 7/144
```

(c) If Huck and Tom are paid a total of $10 to do the job, and they split the money according to the length of time each one works, what is the expected amount that Tom gets?

$$\$10 * 0.75$$

```r
library(scales)
```

```
## Warning: package 'scales' was built under R version 4.0.5
```

```r
Tom <- 10*0.75
dollar_format(prefix="$")
```

```
## function (x)
## dollar(x, accuracy = accuracy, scale = scale, prefix = prefix,
##     suffix = suffix, big.mark = big.mark, decimal.mark = decimal.mark,
##     trim = trim, largest_with_cents = largest_with_cents, negative_parens,
##     ...)
## <bytecode: 0x0000000021c8da90>
## <environment: 0x0000000021c8d010>
```

```r
dollar(Tom)
```

```
## [1] "$7.50"
```

5. The Birthday Problem

Consider n people and suppose that each of them has a birthday that is equally likely to be any of the 365 days of the year. Furthermore, assume that their birthdays are independent, and let A be the event that no two of them share the same birthday. Define a "trial" for each of the $\binom{n}{2}$ pairs of people and say that the trial $(i, j), i \neq j$, is a success if persons i and j have the same birthday. Let $S_{i,j}$ be the event that trial (i, j) is a success.

(a) Find $Pr(S_{i,j)}, i \neq j)$.

$$\frac{365 * 364 * 363 * ...(365 - n + 1)}{365!}$$

(b) Are $S_{i,j}$ and $S_{k,r}$ independent when i, j, k, r are all distinct?

Yes, $S_{i,j}$ and $S_{k,r}$ would be independent.

(c) Are $S_{i,j}$ and $S_{k,j}$ independent when i, j, k are all distinct?

No, $S_{i,j}$ and $S_{k,j}$ would not be independent because the events are not pairwise independent.

(d) Are $S_{1,2}, S_{1,3}$, and $S_{2,3}$ independent?

No, $S_{1,2}, S_{1,3}$, and $S_{2,3}$ would not be independent because the events are not pairwise independent.

(e) Employ the Poisson paradigm to approximate $PR(A)$.

$$\begin{aligned}\text{Approx. Pois}(\lambda), \lambda &= \binom{n}{2}\frac{1}{365} \\ &= \frac{n(n-1)}{2} * \frac{1}{365} \\ \lambda &= \frac{n(n-1)}{730}\end{aligned}$$

(f) Show that this approximation yields $Pr(A) \approx 0.5$, when $n = 23$.

```
library(pracma)
```

```
## Warning: package 'pracma' was built under R version 4.0.5
```

```
nchoosek(23,2)
```

```
## [1] 253
```

```
253/365
```

```
## [1] 0.6931507
```

```
1-exp(-0.6931507)
```

```
## [1] 0.5000018
```

(g) Let B be the event that no three people have the same birthday. Approximate the value of n that makes $Pr(B) \approx 0.5$. (Whereas a simple combinatorial argument explicitly determines $Pr(A)$, the exact determination of $Pr(B) is very complicated, so think Poisson.)

$$A = \text{number of triple matches, Approx. Pois}(\lambda), \lambda = \binom{n}{3}\frac{1}{365^2}$$

Probability of at least 1 triple match:

$$\begin{aligned} Pr(A \geq 1) = 1 - Pr(A = 0) &\approx 1 - e^{-\lambda}\frac{\lambda^0}{0!} \\ &= 1 - e^{-\lambda} \end{aligned}$$

```
nchoosek(83,3)
```

```
## [1] 91881
```

```
x <- 91881/(365^2)
1-exp(-x)
```

```
## [1] 0.4982573
```

We approximately reach 50% probability around $n = 83(49.83\%)$.