HW 2

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6/8/2021

Homework 2

4-4

Consider a process $X_n, n = 0, 1, ...$, which takes on the vales 0, 1, or 2. Suppose

$$P\{X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, ..., X_0 = i_0\}$$

$$= \begin{cases} P^I_{ij} & \text{when n is even} \\ P^{II}_{ij} & \text{when n is odd} \end{cases}$$

where $\sum_{j=0}^{2} P^I_{ij} = \sum_{ij}^{II} = 1, i = 0, 1, 2$. Is $\{X_n, n \geq 0\}$ a Markov chain? If not, then show how, by enlarging the state space, we may transform it into a Markov chain.

The set $X_n, n = 0, 1, ...$ is not a Markov chain because there are infinite values of n. It can be transformed into a Markov chain by assuming the state space is $S = \{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$, where $i(\bar{i})$ signifies that the present value is i and whether the present day is even or odd.

4-20

A transition probability matrix P is said to be doubly stochastic if the sum over each column equals one; that is,

$$\sum_i P_{ij} = 1, \text{for all j}$$

If such a chain is irreducible and aperiodic and consists of $M+1$ states $0, 1, ..., M$, show that the long-run proportions are given by

If $\sum_{i=0}^{m} P_{ij} = 1$ for all j, then $r_j = \frac{1}{(M+1)}$ satisfies $r_j = \sum_{i=0}^{m} r_i P_{ij}, \sum_{0}^{m} r_j = 1$.

Therefore, by uniqueness, these are the limiting probabilities.

5-9

Machine 1 is currently working. Machine 2 will be put in use at a time t from now. If the lifetime of the machine i is exponential with rate λ_i, $i = 1, 2$, what is the probability that machine 1 is the first machine to fail?

$$\begin{aligned}
P(\text{Machine 1 Fails First}) &= [P(\text{Machine 1 Fails First}|T_1 < t)P(T_1 < t) + P(\text{Machine 1 fails first}|T_1 > t)P(T_1 > t)] \\
&= P(T_1 < t) + P(T_1 < T_x|T_1 > t)P(T_1 > t)P(T_1 > t) \\
&= 1 - e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-\lambda_1 t}
\end{aligned}$$

Therefore, the probability that machine 1 fails first is

$$1 - e^{-\lambda_1 t} + e^{-\lambda_1 t}\left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)$$

5-20

Consider a two-server system in which a customer is served first by server 1, then by server 2, and then departs. The service times at server i are exponential random variables with rates $\mu_i, i = 1, 2$. When you arrive, you find server 1 free and two customers at server 2-customer A in service and customer B waiting in line.

Service time of server 1 is T_1 $Exp(\mu_1)$.

Service time of server 2 is T_2 $Exp(\mu_2)$

(a) Find P_A, the probability that A is still in service when you move over to server 2.

$$\begin{aligned} P_A &= P(\text{A Is Still In Service When Moving To Server 2}) \\ &= P(T_2 > T_1) \\ &= P(T_1 < T_2) \\ &= \frac{\mu_1}{\mu_1 +_m u_2} \end{aligned}$$

(b) Find P_B, the probability that B is still in the system when you move over to server 2.

$$\begin{aligned} P_B &= P(\text{B Is Still In Service When Moving To Server 2}) \\ &= P(\text{A Is Completed and B Is In Service When Moving To Server 2}) \\ &= P(T_2 < T_1)P(\text{B Is In Service When Moving To Server 2}) \\ &= [P(T_2 < T_1)] * [P(T_2 > T_1)] \\ &= \frac{\mu_2}{\mu_1 + \mu_2} * \frac{\mu_1}{\mu_1 + \mu_2} \\ &= \frac{\mu_1 \mu_2}{(\mu_1 + \mu_2)^2} \end{aligned}$$

Therefor, the probability that B is still in the system when moving over to server 2 is

$$P_B = \frac{\mu_1}{\mu_1 + \mu_2} + \frac{\mu_1 \mu_2}{(\mu_1 + \mu_2)^2}$$

(c) Find $E[T]$, where T is the time that you spend in the system. Hint: Write $T = S_1 + S_2 + W_A + W_B$ where S_i is your service time at server i, W_A is the amount of time you wait in queue while A is being served, and W_B is the amount of time you wait in queue while B is being served.

$T = S_1 + S_2 + W_A + W_B$, where S_1, W_A, and W_B are the time waiting in queue A and queue B for service.

$$E(T) = E(S_1) + E(S_2) + E(W_A) + E(W_B)$$

The service times at server i are exponential random variables with parameters μ_1. Therefore,

$$\begin{aligned} E(S_1) &= \frac{1}{\mu_1} \\ E(S_2) &= \frac{1}{\mu_2} \\ E(W_A) &= \frac{1}{\mu_2}\left(\frac{\mu_1}{\mu_1 + \mu_2}\right) \end{aligned}$$

Similarly,

$$E(W_B) = \frac{1}{\mu_2}\left(\frac{\mu_1}{\mu_1 + \mu_2} + \frac{\mu_1}{(\mu_1 + \mu_2)^2}\right)$$

5-33

Let X and Y be independent exponential random variables with respective rates λ and μ.

(a) Argue that, conditional on $X > Y$, the random variables min(x, Y) and $X - Y$ are independent.

Let U =Min(X, Y).

$$\begin{aligned}
F_U(u) &= P[U \leq u] \\
&= P[min(X, Y) \leq u] \\
&= 1 - P[min(X, Y) \leq u] \\
&= 1 - P[X > u, Y > u] \\
&= 1 - P[X > u] * P[Y > u] \\
&= 1 - p(X > u) * P[Y > u] \\
&= 1 - e^{-\lambda u} e^{-\mu u} \\
f(u) &= (\lambda + \mu) e^{-(\lambda+\mu)u} I_{(0,\infty)}(\mu)\beta
\end{aligned}$$

$$V = X - Y$$

$$\begin{aligned}
F_V(u) &= P[X - Y \leq u] \\
&= P[X \leq u + y] \\
&= \int_0^\infty P[X \leq u + Y | Y = y] f_y(y) dy \\
&= \int_{y=0}^\infty \int_{x=0}^{u+y} f_x(x) dx f_y(y) dy \\
&= 1 - \frac{\mu}{\lambda + \mu} e^{-\lambda u} \\
f_u(u) &= \frac{\lambda\mu}{\lambda + \mu} e^{-\lambda u} \\
&X \sim Exp(\lambda); Y \sim Exp(\mu) \\
&u = min(xy) | X > y = Y
\end{aligned}$$

$V = X - \frac{Y}{X} > Y$; for joint distribution.

$$|J| = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1$$

$$f_{uv}(u, v) = \lambda\mu e^{-(\lambda+\mu)u} e^{-\lambda v}$$

$U = min(X, Y) | X > Y$ and $u = X - Y | X > Y$ $f_{uv}(u, v) = f_u(u) f_v(u)$ therefore, both are independent.

(b) Use part (a) to conclude that for any positive constant c

$$\begin{aligned}
&E[min(X, Y | X > Y + c] = E[min(X, Y) | X > Y] \\
&= E[min(X, Y)] \\
&= \int_0^\infty u(\lambda + \mu)^{-(\lambda+\mu)u} du \\
&= \frac{1}{\lambda + \mu} \qquad \text{(from equation 1)}
\end{aligned}$$

(c) Give a verbal explanation of why min(X, Y) and $X - Y$ are (unconditionally) independent.

$\min(x, y)$ and $X - Y$ are conditionally independent because in the case of an exponential distribution and x and y are independent, the minimum of those two random variables are independent of any operations between these two random variables.

5-46

Let $N(t), t \geq 0$ be a Poisson process with rate λ that is independent of the sequence $X_1, X_2, ...$ of independent and identically distributed random variables with mean $/mu$ and variance σ^2. Find

$$Cov\left(N(t), \sum_{i=1}^{N(t)} X_i\right)$$

A poisson process with rate λ is independent of the sequence $X_1, X_2, ..., X_n$ of independent and identically distributed random variables with mean μ and variance σ^2

$$Cov\left(N(t), \sum_{i=1}^{N(t)} X_i\right) = E\left(N(t) * \sum_{i-1}^{N(t)} X_i\right) - E(N(t)) * E\left(\sum_{i=1}^{N(t)} X_i\right)$$

$$E\left(\sum_{i-1}^{N(t)} \frac{X_i}{N(t)} = k\right) = E\left(\sum_{i-1}^{N(t)} X_i\right) P(N(t) = K)$$

$$= k\mu \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

$$= \sum_{p=0}^{\infty} k\mu * \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$
$$= \mu\lambda t$$
$$E\left(N(t) \sum_{i=1}^{N(t)} \frac{X_i}{N(t)} = K\right) = E\left(K \sum_{i=1}^{N(t)} X_i\right) P(N(t) = K)$$
$$= kk\mu P(NH) = k$$
$$= k^2 \mu \frac{e^{-\lambda t}}{k!} (\lambda t)^k$$
$$E\left(N(t) \sum_{i=1}^{N(t)} X_i\right) = E\left(E\left(N(t) \sum_{i=1}^{N(t)} \frac{X_i}{N(t)} = k\right)\right)$$
$$= \sum_{k=0}^{\infty} k^2 \mu \frac{e^{-\lambda t}}{k!} (\lambda t)^k$$
$$= \mu(\lambda^2 t^2 + \lambda t)$$

Therefore

$$Cov\left(N(t), \sum_{i=1}^{N(t)} X_i\right) = E\left(N(t) * \sum_{i-1}^{N(t)} X_i\right) - E(N(t)) * E\left(\sum_{i=1}^{N(t)} X_i\right)$$
$$= \mu(\lambda^2 t^2 + \lambda t) - \lambda t \mu \lambda t$$
$$= \mu\lambda t$$

Therefore $Cov\left(N(t), \sum_{i=1}^{N(t)} X_i\right) = \mu\lambda t$

5-73

Shocks occur according to a Poisson process with rate λ, and each shock independently causes a certain system to fail with probability p. Let T denote the time at which the system fails and let N denote the number of shocks that it takes.

(a) Find the conditional distribution of T given that $N = n$.

$$P(T = t|N = n) = \lambda e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}, t \geq 0$$

(b) Calculate the conditional distribution of N, give that $T = t$, and notice it is distributed as 1 plus a Poisson random variable with mean $\lambda(1-p)t$.

$$\begin{aligned} P(N(t) = n|T = t) &= \frac{P(T = t|N = n)P(N = n)}{P(T = t)} \\ &= C_1 \lambda e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}p(1-p)^{n-1} \\ &= C_2 \frac{(\lambda(1-p)t)^{n-1}}{(n-1)!} \end{aligned}$$

The distribution looks like a Poisson process with the parameter $\lambda(1-p)$. For the probabilities to add up to 1, $C_2 = e^{-\lambda(1-p)t}$. Therefore,

$$P(N(t) = n|T = t) = e^{-\lambda(1-p)t}\frac{(\lambda(1-p)t)^{n-1}}{(n-1)!}$$

(c) Explain how the result in part (b) could have been obtained without any calculations.

Because the events are broken into two independent classes, they each have the following Poisson distribution rate λp and $\lambda(1-p)$:

$$\begin{aligned} P_{N|T}(n, t) &= P(N(t) = n|T = t) \\ &= P(N_1(t) + N_2(t) = n|T = t) \\ &= P(1 + N_2(t) = n) \\ &= P(N_2(t) = n - 1) \\ &= \frac{e^{-\lambda(1-p)t}((1-p)\lambda t)^{n-1}}{(n-1)!} \end{aligned}$$