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12:59 PM

Linear Algebra #2

Basic Laws of Algebra

Associative Law – Addition
If A, B, & C are conformable for addition

$$(A+B)+C = A+B+C$$
$$= A+(B+C)$$

$$1+2+3$$

Associative Law – Multiplication

If A, B, C are conformable for multiplication

$$A_{p\times r};\ B_{r\times c},\ C_{c\times q}$$
$$(AB)C = ABC = A(BC)$$

Distributive Law

$A(B+C) = \underline{AB} + \underline{AC}$

$2(a+b)$

$2a + 2b$

(algebra)

Commutative Law - Addition

$A + B = B + A$

Commutative Law - Multiplication

Generally NOT commutative

$\underbrace{A^{-1}}_{\text{'pre'}} A x = \underbrace{A^{-1}}_{\text{'pre'}} y$

$AB \neq BA$

Exception: Identity Matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{I}A = A\underline{I} = \underline{A}$$

Transpose of Products

$$(AB)' = B'A'$$

Partitioning Matrices

$$A_{r \times c} = \left[\begin{array}{c|c} K_{p \times q} & L_{p \times (c-q)} \\ \hline M_{(r-p) \times q} & N_{(r-p) \times (c-q)} \end{array}\right]$$

submatrices

$$\underset{r\times c}{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

among non-genotyped

relationship matrix among genotyped animals

Quadratic Form ($x' A x$)

Sum of Squares

Modeling, F-test

System of Equation

Elimination

$$3\underline{x} + 4\underline{y} = 10$$
$$2\underline{x} + 3\underline{y} = 7$$

(elimination method)

find common multiple

solve for y

$$3(3x + 4y = 10)$$
$$-4(2x + 3y = 7)$$

$$9x \cancel{+12y} = 30$$
$$-8x \cancel{+-12y} = -28$$

$$1x + 0 = 2$$

$$\boxed{x = 2}$$

plug in $x = 2$ into either

$$2(2) + 3y = 7$$

$$3y = 3$$

$$\boxed{y = 1}$$

$$\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \end{bmatrix}$$

"design matrix"

"coefficient matrix"

"incidence matrix"

"model matrix"

solve for this vector

Determinant

- polynomial of the elements of a <u>square</u> matrix (undefined if NOT square)
- scalar
- denoted w/ $|A|$ NOT abs value
  = determinant

$$A = \begin{bmatrix} 7 & 3 \\ 4 & 6 \end{bmatrix}$$

$$(7 \times 6) - (4 \times 3) = \boxed{30}$$

Singularity –

- matrix is singular if det = 0
- implication: NOT invertible

Non-singular (✓)

- matrix det $\neq 0$
- invertible (can solve the system of equations)

related to amount of
redundancy

Computational cost is $n!$
HUGE

VERY intensive

Properties of Traspose

- $|A'| = |A|$

- If 2 rows of A are the same
  $|A| = 0$

  Due to matrix rank (revisit today)

- $|AB| = |A|\,|B|$

  scalar = scalar   when A & B are square & same order

- If one row/colum is a multiple of another
  det = 0
  NOT Full Rank

Direct Sums & Products

Direct Sums

matrix of 0's

rows

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

colums

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \oplus \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$$

$$
\left[\begin{array}{ccc|cc}
1 & 2 & 3 & 0 & 0 \\
\hline
0 & 0 & 0 & 6 & 7 \\
0 & 0 & 0 & 8 & 9
\end{array}\right]
$$

Direct Product

$$A_{\underline{p}\times \underline{q}} \otimes B_{\underline{m}\times \underline{n}}$$

$$
= \begin{bmatrix}
a_{11} B & \cdots & a_{1q} B \\
 & & \\
 & & a_{pq} B
\end{bmatrix}
$$

$$\underset{1\times 3}{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}} \otimes \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}_{2\times 2}$$

$\boxed{2\times 6}$

$$= \left[\begin{array}{cc:cc:cc} 6 & 7 & 12 & 14 & 18 & 21 \\ 8 & 9 & 16 & 18 & 24 & 27 \end{array}\right]$$

order of product =

$pm \times qn$

product of rows

product of colums

Kronecker Product

Properties

1) $(A \otimes B)' = A' \otimes B'$

2) x & y are vectors

$$x' \otimes y = y x' = y \otimes x'$$

3) $\lambda \otimes A = \underline{\lambda} A = A \otimes \lambda = A \underline{\lambda}$

4) for partitioned Matrices

$$[A_1 \quad A_2] \otimes B = [A_1 \otimes B \quad A_2 \otimes B]$$

5) $tr(A \otimes B) = tr(A)\,tr(B)$

sum diag

Inverse

$A^{-1} = A$ inverse

$A^{-1} A = I$

anything $\times$ I = itself

Linear Dependency

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 4 & 5 & 6 \end{bmatrix}$$ row 2 = row 3

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 10 & 12 \end{bmatrix}$$ 2x

row 3 = row 2 × 2

dependency exits, det = 0

inverse does **not** exist

$$X = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow$$ add colms "more complex model"

column 1 = col 2 + col 3

redundancy

rank = 2

2 ind colums

$2 < \min(6, \underline{\underline{3}})$

Not full rank $6 = \#$ rows

$3 = \#$ colums

Full Rank

rank $= \min(n, m)$

$n = \#$ rows

$m = \#$ colums

theorem: # of ind rows in a matrix = # ~~a~~ of ind colums (vice versa)

rank cannot exeed

$[6 \times 3]$

$\min(6, 3)$

$= \underline{\underline{3}}$

Full rank = 3

$$M = \begin{array}{c} \begin{array}{cccccc} aa & Aa & AA & & & \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 & 1 & 1 \end{bmatrix}_{\underline{3}\times 6} \end{array}$$

rank = 2

- re genotype same animals
- Identicals twins

BLUPF90 > .9

i, j 6 offdig

$\underline{1001}$, $\underline{2001}$ $\underline{\underline{0.98}}$

delete

$$\underset{0.05}{\underline{A}} + \underset{.95}{G} = G^*$$

unstable

$$(x'x) = \begin{bmatrix} 6.001 & 3 + 3 \\ 3 & 3.001 \; 0 \\ 3 & \boxed{0 + 3\;0.001} \end{bmatrix}$$

(from above)

(try to invert)

cannot invert $(X'X)$

$$MM' = \begin{bmatrix} 6 & 3 & 3 \\ 3 & 8 & 8 \\ 3 & \underbrace{8 \quad 8}_{\text{same}} \end{bmatrix}$$

$(MM')^{-1}$ does <u>not</u> exist

Not-Full Rank Outcomes

- inverse does not exist
- can't solve for all elements in system of equations

$$Ax = y$$

$$\underline{x} = \cancel{A^{-1}y}$$

solve w/ generalized inverse of A (A^{-}) OR make A full rank (typical to drop row(s) or column(s))

square matrix $A_{n \times n}$

rank $A \leq n$

$r(A) < n$ A^{-1} does not exist

det = 0

* Positive Definite Matrix

$x'Ax > 0$ when $x \neq 0$

- all eigenvalues are positive

* Positive Semi-Definite (PSD) Matrix

$x'Ax \geq 0$ all x and
$x'Ax = 0$ for some
$x \neq 0$

- all eigenvalues are positive OR zero for a symmetric matrix

Non-negative Definite (NND) Matrix
PD & PSD

Eigenvalues & Eigenvectors

$A_{n \times n}$ (square)

λ = scalar

$$A\underline{u} = \lambda \underline{u}$$

u = vector valid for $u \neq 0$

$$Au - \lambda u = 0$$

$$(A - \lambda I)u = 0$$

$A - \lambda I$ is singular

$$|A - \lambda I_n| = 0 \quad (\text{rank} < \underline{n})$$

'characteristic equation'

n = roots = eigenvalues
= latent roots

n vectors = eigenvectors
= latent vectors

what do they tell us

- Rank = # non-zero eignvalues for a symmetric matrix

z_A = # of zero eigenvalues of A

$n - z_a$ = # non zero eigen values

$$r_A = r(A) = n - z_A$$

sym => $A = A'$

- eigenvalues represent total amount of variance that can be explained by a given principal component
  - link to PCA

λ_i represents a variance

$\sum \lambda_i$ = total variance for PCA

$\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 \quad \dots > \lambda_n$

- help us understand the condition of a matrix

<u>Condition of a matrix</u>

related to the rank

A is singular (and redundant)

poorly conditioned

well conditioned, basically full rank

'gray scale'

$$M = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 0 \\ 2 & 1 & 0 & 1 & 2 & 1 \\ 2 & 1 & 0 & 1 & 2 & 0 \end{bmatrix}$$

'ill conditioned'

1

vs

1.0001

condition > 87807477

very large

rank = 3

small changes in A (or X or M) result in large changes in solutions if ill-conditioned

→ well conditioned, changes A don't change solutions very much

$$\text{Cond} = \frac{\lambda_{max}}{\lambda_{min}} \quad \begin{array}{l}\text{(largest eigenvalue)} \\ \text{(smallest eigenvalue)}\end{array}$$

kappa() slightly diff in R
(2-norm condition)

leads to poor convergence w/ poorly conditioned matrix
REML or AIREML
or lm()

Generalized Inverse

- rectangular and singular matrices

- analogue to A^{-1} of nonsingular matrix

denote A^{-} for generalized inv

instead of making A full rank we can define A^{-}

invert A^{-1}

$$\underline{(X'X)} = \begin{bmatrix} \boxed{\text{Full rank}} & 0 \\ & 0 \\ 0 \quad 0 & 0 \end{bmatrix} \text{ fix at } 0$$

instead of dropping col 3
we fill in 0's
in $(X'X)$ matrix

R will make X full rank

$$\begin{bmatrix} \cancel{\mu} \\ a_1 \\ \cancel{a_2} \end{bmatrix}$$

$$\begin{bmatrix} \mu + a_2 \\ a_1 \end{bmatrix}$$