2023-01-23

Monday, January 23, 2023

1:20 PM

Linear Algebra

From Linear Equations

$$\underline{a_1} x_1 + \underline{\underline{a_2}} x_2 + \dots + a_n x_n = b$$

'scale'

a's = constants

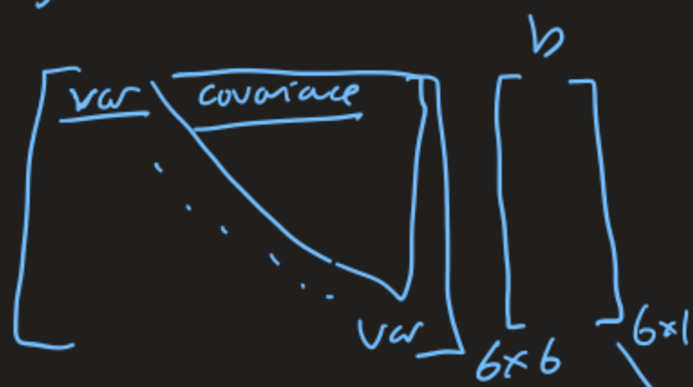
x's = random variables

$$y = \underset{\text{design}}{X}\,\underset{\text{coefs}}{\underline{b}} + e$$

Where:

Selection Index:

1. info on relatives to calculate genetic values



Replaced w/ BLUP

solutions = weights

2: Economic Index

- combine EBVs (estimated breeding values) of diff traits into 1 'index' value

Inbreeding

- Optimal Contribution Selection OCS
- Matings

Economics

Genomics

Machine Learning

Intro, Notation & Terminology

Matrix: rectangular array of numbers, symbols, expressions, arranged in rows and columns

$$A = \begin{bmatrix} 1 & 3 \\ 5 & 9 \end{bmatrix}$$

represent

$$a_{ij} \Rightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

i = row
j = column

a, b, c = constants or coefficients
x, y, z = typically random variables

$A = \{a_{ij}\}$ $\quad i = 1, 2, \dots m$
$\quad j = 1, 2, \dots, n$

$A_{m \times n}$ order is $m \times n$
"m by n"

Square Matrix

where $m = n$
or $n \times n$

diagonal $i = j$ $\quad a_{11}$ or a_{55}
off-diag $i \neq j$

Trace: sum of diagonal elements

$$a_{11} + a_{22} \cdots + a_{n \times n}$$

$$tr(\ )$$

Diagonal Matrix

square matrix w/ all off-diag $= 0$

with

$= 0$

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -17 & 0 \\ 0 & 0 & 99 \end{bmatrix}$$

a_3

$$a_{11} = a_1$$

$$a_{22} = a_2$$

Identity Matrix

all diagonal elements $= 1$

off-diag $= 0$

$$e \sim N(0, I\sigma_e)$$

Lower / Upper Triangular

$$B = \begin{bmatrix} 1 & 5 & 13 \\ 0 & -2 & 9 \\ 0 & 0 & 7 \end{bmatrix}$$ Upper below diag = 0

$$C = \begin{bmatrix} 2 & 0 & 0 \\ 8 & 3 & 0 \\ 1 & -1 & 2 \end{bmatrix}$$ Lower above diag = 0

Vectors

special case of matrix w/
1 row or column

$x' = \begin{bmatrix} 1 & 3 & 5 & 9 \end{bmatrix}$ row vector

$y = \begin{bmatrix} 2 \\ 5 \\ 3 \\ 8 \end{bmatrix}$ column vector

$\underset{\sim}{y}$ $\underset{\sim}{A}$ signifies matrix or vector

in text **BOLD**

$y = 2$

$y = \begin{bmatrix} \\ \\ \end{bmatrix}$

Scalar

1×1 matrix or just 1 number

often w/ greek letter

$\lambda = 5$

λA multiply λ by each element of A

Dot Notation

$$A = \begin{array}{cl} \begin{bmatrix} 5 & 8 & -1 \\ 2 & -1 & 6 \\ 1 & 3 & 4 \end{bmatrix} & \begin{array}{l} \text{— sum} \\ \\ \\ \end{array} \\ \quad\quad\quad\ \text{sum} & \end{array}$$

a_{ij}

$$a_{1\cdot} = a_{11} + a_{12} + a_{13}$$

$$= \sum_{j=1}^{3} a_{1j} = a_{11} + a_{12} + a_{13}$$

$$a_{\cdot 2} = a_{12} + a_{22} + a_{32}$$

$$i = 1, 2, 3$$

Transpose

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$A' = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

FLIP along diagonal

$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad x' = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

t( ) in R,
A' in Julia

$a_{ij} \Rightarrow a_{ji}$

$a_{12} \Rightarrow a_{21}$

A^T for transpose

Basic Matrix Operations

- addition
- subtraction
- multiplication
- inverse => division

Addition

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

Same for vectors

ONLY Condition

must be CONFORMABLE

= have same dimensions

$$[m \times n] + [m \times n]$$

Subtraction

element-by-element subtraction

$$c_{ij} = a_{ij} - b_{ij}$$

Multiplication

$\boxed{\otimes}$ → direct element by element

$$P = AB$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}_{2\times 2}$$

$$(1\times 5)+(2\times 7)$$

$$\begin{bmatrix} (1\times5)+(2\times7) & (1\times6)+(2\times8) \\ =19 & =22 \\ (3\times5)+(4\times7) & (3\times6)+(4\times8) \\ =43 & =50 \end{bmatrix}_{2\times2}$$

Must be $\boxed{\text{Conformable}}$

\# of colums in A

$=$ \# rows in B

$$A_{r\times c}\ B_{c\times s}$$

must be =

$$P_{r\times s}$$

$X_{100 \times \cancel{3}} \beta_{\cancel{3} \times 1}$

100×1

w/ vectors

$y = \begin{bmatrix} 9 \\ 2 \\ 6 \end{bmatrix}_{3 \times \underline{1}} \qquad x = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}_{\underline{3} \times 1}$

$y'_{1 \times 3} \, x_{3 \times 1} = \text{scaler}_{1 \times 1}$

$\begin{bmatrix} 9 & 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$

$$(9\times 3) + (2\times -2) + \cancel{(6\times 0)}$$

$$27 + -4$$

$$\boxed{23}$$

<u>inner</u> product

or <u>dot</u> product

if inner product $= 0$

vectors are <u>orthogonal</u>

$$(X'X)$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ & & 2 & 1 \\ & & & 3 \\ & & & \end{bmatrix}$$

orthogonal

2×3 inner prod

orthogonal => uncorrelated

Outer Product

$$y \; x'$$

$$\begin{bmatrix} 9 \\ 2 \\ 6 \end{bmatrix}_{3\times 1} \begin{bmatrix} 3 & -2 & 0 \end{bmatrix}_{1\times 3}$$

$$\begin{bmatrix} 27 & -18 & 0 \\ 6 & -4 & 0 \\ 18 & -12 & 0 \end{bmatrix}_{3\times 3}$$

Generalize

$$P_{r\times s} = A_{r\times c} \, B_{c\times s}$$

$$ab_{ij} = a_{i1} b_{1j} + \dots + a_{ic} b_{cj}$$

$$= \sum_{k=1}^{c} a_{ik} b_{kj}$$

Inversion

matrix equivalent to <u>division</u>

$$\left.\frac{\cancel{2}x}{\cancel{2}} = \frac{10}{2}\right\} = 5$$

$$x = 5$$

$$\left(\frac{1}{2}\right)2x = 10\left(\frac{1}{2}\right)$$

$$\left[\frac{1}{2} + \frac{2}{1}\right] = \frac{2}{2} = 1$$

$$x = 5$$

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$\cancel{A^{-1}A}\,x = A^{-1}y$$

cancel

$$x = A^{-1}y$$

$$AA^{-1} = I \quad \text{(identity matrix)}$$

$$A^{-1}A = I$$

$$Ix = x$$

Illustration

allele codes

$$m + a = GG = 14$$

$$m + d = Gg = 12$$

$$m - a = gg = 6$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} m \\ a \\ d \end{bmatrix} = \begin{bmatrix} 14 \\ 12 \\ 6 \end{bmatrix}$$

$$A^{-1} \; A \; b = A^{-1} y$$

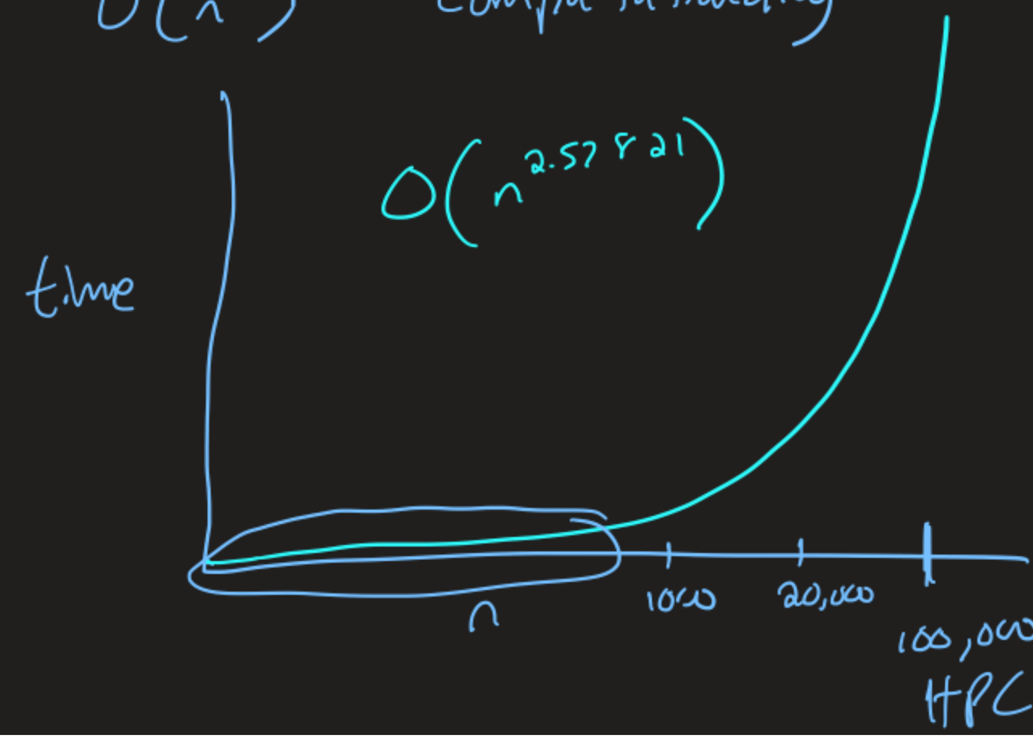
$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix}$$

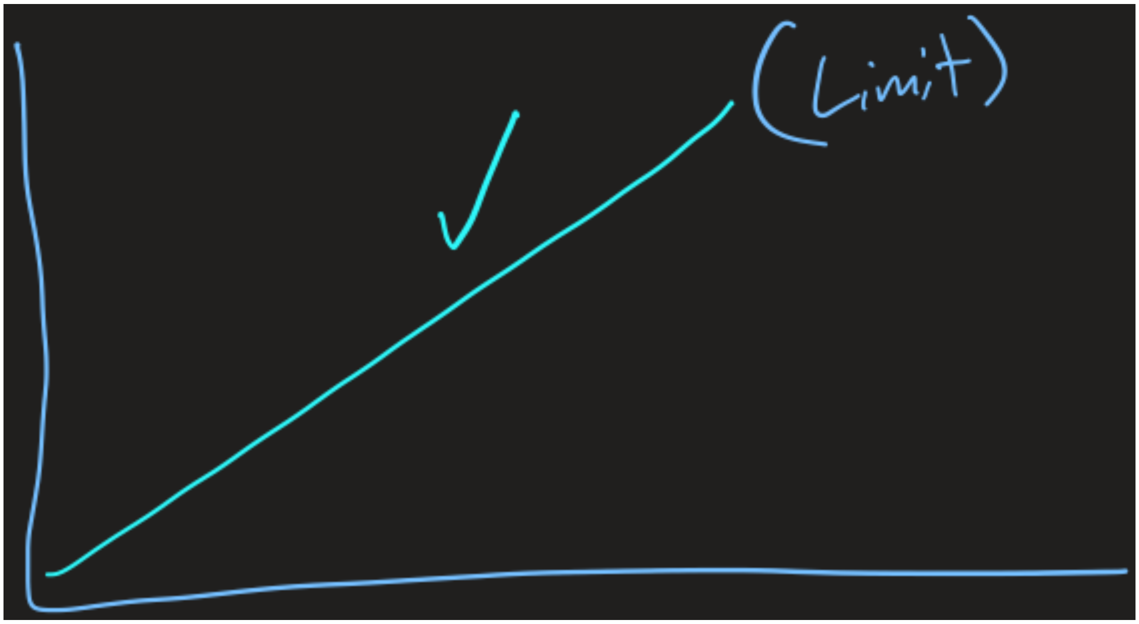
$$A^{-1} A = I \quad \text{(verify)}$$

$$b = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix} \begin{array}{c} y \\ \begin{bmatrix} 14 \\ 12 \\ 5 \end{bmatrix} \end{array}$$

$$b = \begin{bmatrix} 10 \\ 4 \\ 2 \end{bmatrix}$$

* only exists for square matrices
* A^{-1} is unique for given A
* only exists when $\det(A) \neq 0$
* $O(n^3)$ computationally





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