Bryce McLaughlin
VRDI
Overlapping Existence Proof
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> Let G be a graph. If P_1 and P_2 are two partitionings of the vertices of G into k groups such that there does not exist $S \subseteq P_1, T \subseteq P_2$ where $|S| = |T|$ and $\cup_{s\in S} s \subset \cup_{t \in T} t$ (total equality being allowed), then there exists a bijective map $\chi : P_1 \to P_2$ such that for every $p \in P_1$, $p \cap \chi(p) \neq \emptyset$

Suppose to the contrary that no such mapping χ exists. Let $\phi : P_1 \to P_2$ be the mapping such that for $f : P_1 \to P_2$, if $I(f) = \{p \in P_1 | p \cap f(p) \neq \emptyset\}$, then $I(\phi)$ is largest. Let $j = |I(\phi)|$. Let $N(\phi) = P_1 \setminus I(\phi)$ If there exists $a, b \in N(\phi)$ such that $a \cap \phi(b) \neq \emptyset$, then we can construct a function $\psi : P_1 \to P_2$ such that

$$\psi(p) = \begin{cases} \phi(b) & \text{when } p = a \\ \phi(a) & \text{when } p = b \\ \phi(p) & \text{otherwise} \end{cases}.$$

This reaches a contradiction as $I(\psi) = j + 1$, so we only need to consider the case in which $N(\phi) \cap \cup_{n \in N(\phi)} \phi(n) = \emptyset$.

In this case we know that $N(\phi) \cap \cup i \in I(\phi)\phi(i) \neq \emptyset$ otherwise we would contradict that $N(\phi) \cap \cup_{n\in N(\phi)} \phi(n) = \emptyset$. Therefore, there exists $c \in N(\phi)$ such that $c \cap \cup i \in I(\phi)\phi(i) \neq \emptyset$. Let $d \in I(\phi)$ be such that $c \cap \phi(d) \neq \emptyset$. Construct the mapping $\theta : P_1 \to P_2$ as follows. Let $\theta(c) = \phi(d)$. If $d \cap \phi(e) \neq \emptyset$, for some $e \in N(\phi)$, then let $\theta(d) = \phi(e)$, $\theta(e) = \phi(c)$ and map everything else as it would by ϕ. This creates a contradiction as $|I(\theta)| = j + 1$. If no such e exists, created an empty set U and add d to it. There must be some $f \in I(\phi) \setminus U$, such that $d \cap \phi(f) \neq \emptyset$, otherwise $d \subset \phi(d)$ which contradicts the idea that there does not exist $|S| = |T|$ and $\cup_{s\in S} s \subset \cup_{t\in T} t$ (here $S = \{d\}$ and $T = \{f\}$). Therefore we let $\theta(d) = \phi(f)$ and perform the same search procedure for f that we did for d. If we cannot find an element in $N(\phi)$ whose image intersects f, we add f to S, and let $g \in I(\phi) \setminus U$ be such that $f \cap \phi(g) \neq \emptyset$, which must exists, otherwise U and $\cup_{u\in U} \phi(u)$ act as two sets which meet the criteria $S \subseteq P_1, T \subseteq P_2$ where $|S| = |T|$ and $\cup_{s\in S} s \subset \cup_{t\in T} t$, which is a contradiction. We then set $\theta(f) = \phi(g)$ If we continue this process, we must eventually come across some element in $I(\phi)$ which intersects $\cup_{n\in N(\phi)} \phi(n)$ as the number of elements in $I(\phi) \setminus U$ becomes 0 after j iterations. If at any point we $h \in I(\phi)$ which does not intersect $\cup_{n\in N(\phi)} \phi(n)$ and we cannot find some $i \in I(\phi) \setminus U$ such that $g \cap \phi(i) \neq \emptyset$, then U and $\cup_{u\in U} \phi(u)$ act as two sets which meet the criteria $S \subseteq P_1, T \subseteq P_2$ where $|S| = |T|$ and $\cup_{s\in S} s \subset \cup_{t\in T} t$, which is a contradiction. Therefore we are able to construct θ such that $|I(\theta)| = j + 1$, a contradiction which ends the proof.