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NUMERICAL ANALYSIS WITH LAGRANGIAN AND THE HAMILTONIAN EQUATIONS OF MOTIONS

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ABSTRACT

A planar double pendulum is a simple mechanical system that has two simple pendula attached end to end that exhibits chaotic behavior. The aim of this research will be to numerically analyze the dynamics of the double pendulum system. First, the physical system is introduced and a system of coordinates is fixed, and then the Lagrangian and the Hamiltonian equations of motions are derived.We will find that the system is governed by a set of coupled non-linear ordinary differential equations and using these, the system can be simulated.

Finally we analyze Poincare sections, the largest lyapunov exponent, progression of trajectories, and change of angular velocities with time for certain system parameters at varying initial conditions.

All numerical analysis was done using MATLAB, specifically ode45, to solve the system of 4 first-order Hamilton's Equations of Motion.

Keywords: Hamilton's Equations, numerical analysis, Pendulum

1. INTRODUCTION

The planar double pendulum serves as a paradigm for chaotic dynamics in classical mechanics. The character of its motion changes dramatically as the energy is increased from zero to infinity. At low energies, the system represents a typical case of coupled harmonic oscillators, and therefore, can be treated as an integrable system. At very high energies, the system is again integrable because the total angular momentum is a second conserved quantity besides energy. At intermediate energies, however, it exhibits more or less chaotic features. In this region of intermediate energies, we can observe a transition to global chaos via the decay of a last surviving Kolmogorov-Arnold-Moser (KAM) torus [1].

This paper shows our attempts to numerically analyze the double pendulum system to show the passing of the system from an integrable one to a chaotic one as energy increases to intermediate values.

1.1 DERIVATION OF HAMILTON'S EQUATIONS OF MOTION:

Consider a double bob pendulum with masses m_1 and m_2 and attached by rigid, massless wires of lengths l_1 and l_2. Also, let the angles the wires make with the vertical be denoted as q_1 and q_2. Finally, let the acceleration due to gravity be g.

The positions of the bobs are given by the following equations:

$$x_1 = l_1 \sin(q_1)\ ;\ y_1 = -l_1 \cos(q_1) \tag{1}$$

$$x_2 = l_1 \sin(q_1) + l_2 \sin(q_2)\ ;\ y_2 = -l_1 \cos(q_1) - l_2 \cos(q_2) \tag{2}$$

The potential energy (V) of the system is then given by:

$$\mathrm{V} = m_1 g y_1 + m_2 g y_2 = -(m_1 + m_2) g l_1 \cos(q_1) - m_2 g l_2 \cos(q_2) \tag{3}$$

The kinetic energy (K) is given by:

$$K = \frac{1}{2} m_1 v_1{}^2 + \frac{1}{2} m_2 v_2{}^2 = \frac{1}{2} m_1 l_1{}^2 \dot{q_1}{}^2 + \frac{1}{2} m_2 [l_1{}^2 \dot{q_1}{}^2 + l_2{}^2 \dot{q_2}{}^2 + 2 l_1 l_2 \dot{q_1} \dot{q_2} \cos(q_1 - q_2)] \tag{4}$$

The Lagrangian (L) is then:

$$\mathrm{L} \equiv T - V = \frac{1}{2}(m_1 + m_2) l_1{}^2 \dot{q_1}{}^2 + \frac{1}{2} m_2 l_2{}^2 \dot{q_2}{}^2 + m_2 l_1 l_2 \dot{q_1} \dot{q_2} \cos(q_1 - q_2) + (m_1 + m_2) g l_1 \cos(q_1) + m_2 g l_2 \cos(q_2) \tag{5}$$

Now we can compute the generalized momenta,

$$p_{q_1} = \frac{\partial L}{\partial \dot{q_1}} = (m_1 + m_2) l_1{}^2 \dot{q_1} + m_2 l_1 l_2 \dot{q_2} \cos(q_1 - q_2) \tag{6}$$

$$p_{q_2} = \frac{\partial L}{\partial \dot{q_2}} = m_2 l_2{}^2 \dot{q_2}{}^2 + m_2 l_1 l_2 \dot{q_1} \cos(q_1 - q_2) \tag{7}$$

The Hamiltonian (H) is then given by:

$$H = \dot{q_i} p_i - L = \frac{1}{2}(m_1 + m_2) l_1{}^2 \dot{q_1}{}^2 + \frac{1}{2} m_2 l_2 \dot{q_2}{}^2 + m_2 l_1 l_2 \dot{q_1} \dot{q_2} \cos(q_1 - q_2) - (m_1 + m_2) g l_1 \cos(q_1) - m_2 g l_2 \cos(q_2) \tag{8}$$

Solving the generalized momenta equations for q_1 and q_2 and plugging back into the Hamiltonian equation:

$$H = \frac{m_2 l_2{}^2 p_{q_1}{}^2 + (m_1 + m_2) l_1{}^2 p_{q_2}{}^2 - 2 m_2 l_1 l_2 p_{q_1} p_{q_2} \cos(q_1 - q_2)}{2 l_1{}^2 l_2{}^2 m_2 [m_1 + \sin^2(q_1 - q_2) m_2]} - (m_1 + m_2) g l_1 \cos(q_1) - m_2 g l_2 \cos(q_2) \tag{9}$$

This leads to the **Hamilton's Equations of Motion**:

$$\dot{q_1} = \frac{\partial H}{\partial p_{q_1}} = \frac{l_2 p_{q_1} - l_1 p_{q_2} \cos(q_1 - q_2)}{l^2{}_1 l_2 (m_1 + m_2 sin^2(q_1 - q_2))} \tag{10}$$

$$\dot{q_2} = \frac{\partial H}{\partial p_{q_2}} = \frac{l_1 (m_1 + m_2) p_{q_2} - l_2 m_2 p_{q_2} \cos(q_1 - q_2)}{l^2{}_2 l_1 (m_1 + m_2 sin^2(q_1 - q_2))} \tag{11}$$

$$\dot{p_{q_1}} = -\frac{\partial H}{\partial q_1} = -(m_1 + m_2)gl_1 \sin(q_1) - C_1 + C_2 \quad (12)$$

$$\dot{p_{q_2}} = -\frac{\partial H}{\partial q_2} = -m_2 g l_2 \sin(q_2) + C_1 - C_2 \quad (13)$$

$$C_1 = \frac{p_{q_1} p_{q_2} m_2\, sin(q_1 - q_2)}{l_1 l_2 (m_1 + m_2\, sin^2(q_1 - q_2)} \quad (14)$$

$$C_2 = \frac{[(l^2{}_1 p^2{}_{q_2}(m_1+m_2) + \left(l^2{}_2 p^2{}_{q_1} m_2\right) - (l_1 l_2 m_2\, p_{q_1} p_{q_2} \cos(q_1 - q_2))] \sin(2(q_1 - q_2))}{2 l^2{}_1 l^2{}_2 (m_1 + m_2\, sin^2(q_1 - q_2))^2} \quad (15)$$

1.2 Some Theory

We introduce a new concept of quasi periodic motion. Roughly speaking, they refer to "almost periodic" motion. More mathematically, it can be thought of as the type of motion executed by a dynamical system containing a finite number of incommensurable frequencies [2]. From equations 10 to13, we see that the dynamics of a double pendulum can be described with 4 variables, the two angles and their corresponding (angular) velocities, which span the four dimensional phase space of the system [3]. Since the double pendulum is a Hamiltonian system, total energy is conserved (when m=l=1, equation 9) and this reduces the four-dimensional phase space to a three dimensional manifold. Further, The KAM theorem states that if a Hamiltonian system is subjected to a weak nonlinear perturbation, some of the invariant tori that have "sufficiently irrational" frequencies survive. In other words, the motion continues to be quasi periodic[3]. KAM tells us that at lower energies, the function is integrable (it has as many conserved quantities as there are degrees of freedom in the system). At high energy the pendulum behaves like a simple rotor, with the system rotating rapidly in a stretched cofiguration ($q_1 = \pi$, $q_2 = 0$). In this case the kinetic energy terms in the Lagrangian dominate the potential energy terms and may be described by setting $g = 0$ in the equations of motion. The total angular momentum is conserved, because in the absence of gravity, there is no torque on the pendulum. The resulting motion of the system is regular (non-chaotic), because a system with two degrees of freedom and two constraints (conservation of total energy and total angular momentum) cannot exhibit chaos. It follows, for example, that the double square pendulum would not exhibit chaos if installed on the space station [4]. Lower energies and higher energies = periodic motion. From this theoretical evidence, we hypothesize that the behavior of a double pendulum varies from regular motion at low energies, to chaos at intermediate energies, and back to regular motion at high energies.

1.3 Largest Lyapunov Exponent

Sensitive dependence on initial conditions – smallvseparations between arbitrarily close initial conditions are amplified exponentially in time – is the hallmark of chaos. The underlying cause of this behavior, namely the exponential growth, can be numerically and analytically evaluated using lyapunov exponents.Largest lyapunov exponents [5], as it effectively gives us the information on the divergence of two close trajectories. We can use the same first order equations used in the MATLAB simulation to evaluate the exponent. The method to calculate the lyapunov exponent is to first plot the natural logarithm of the separation between the two closely launched trajectories against time and then finds the slope of the region where it is increasing. As usual, positive lyapunov exponents are indicative of chaotic behavior.[7]

1.4 Point care

Poincare allows fast and informative insight into the dynamics of the double pendulum. The different types of motion appear as finite number of points for periodic orbits, curve filling points ('invariant curves') for quasi periodic motion and area filling points for chaotic trajectories. We can construct a two-dimensional Poincare section by looking at the trajectory only at those points when the outer pendulum passes the vertical position, that is $q_2 = 0$ [6].

1.5 NUMERICAL EXPERIMENTS

1.5.1 System Parameters:

For simplicity, we decided to use unit lengths and unit masses, i.e. $m_1 = m_2 = l_1 = l_2 = 1$. The acceleration due to gravity, g = 9.81 m/s2.

We start with low energy conditions and using the Hamiltonian, the energy can easily be calculated in Joules. When we have initial conditions = y0 = [$\dot{q}_1$, $\dot{q}_2$, $\dot{p}_{q_1}$, $\dot{p}_{q_2}$]= [0.2,0.2828,0,0], the energy = 0.7809 J. At this low energy we expected periodic behavior. (Fig2, 3, 4, 5, 6) The periodic trajectory of the outer bob is clear from fig 2a, the Poincare sections are presented in fig 2b and show a finite number of points that grow outwards with time but form a general pear shape. The plot in 3-D forms a pear shape when rotated about the x_1 axis. When a second trajectory is launched at a distance of 10^{-9} from this initial condition, we see from fig 2d that they move together indicating that there is no chaos. When the lyapunov graph is plotted, it is clear that the lyapunov exponent is negative (= -3.426) and hence the system is not chaotic. The angular velocities of the inner and outer bobs are in phase and periodic further confirming that these initial conditions at low energy are indicative of the non-chaotic regime.

As we increase the energy to 1.2807J with the intial conditions y0 = [0.7, 0.3825, 0, 0] we enter the quasi periodic regime. The periodic trajectory of the outer bob is in fig 3a and we see that the inner and the outer bob are out of phase (also exemplified in the angular velocities graph in 3d.

Figure 3e shows that two closely launched trajectories do not diverge from each other. Using this distance separation, when the lyapunov graph was generated in 3e, we see that the lyapunov exponent does not remain entirely negative and starts to begin increasing to positive values with time. The average lyapunov exponent is -1.203. which is higher than for the periodic condition. When the energy is further increased to be 29.4 J with initial conditions y0 = [π, π, 0.5,0]. We repeat the same process and find an average positive lyapunov exponent of 1.906 (fig 4e). This average number was obtained by calculating the average over a range of initial conditions around this y0 (fig 4f). We see in fig 4a that the path of the outer bob is random and unpredictable. The Poincare section is clearly area filling and the two closely launched trajectories (with an initial separation of 10^{-9}) start off together but move away from each other. The angular velocities which are qualitative indicators also show that there is no regular behavior. All of these clearly indicate that at these energies, the system is chaotic. So far we found that as we increase energy, the system moves from being periodic to quasi-periodic to chaos. Now, if we increase the energy even more to reach 104.25 J with initial conditions y0 = [π, 0, 0.5, 0.5], we see the quasi-periodic state again! Clearly, from 5d the trajectories have not diverged from each other. Also, there is a decrease in the lyapunov exponent to 0.4320. The angular velocities (fig 5c) are out of phase together. The trajectory in fig 5a is a certain indication of quasi-periodic motion. Although the Poincare sections are a little difficult to understand, the other graphs are indicative of quasi-periodic behavior at these energies.With look at figure 6, which has other informative plots of the variables plotted against each other. There is qualitative trend indicating the system starts out to be periodic, in the sense that the shape of the curves can be predicted for later times, to chaotic unpredictable behavior and finally back to shapes resembling periodicity. In each of them, the top left is the angles of the inner and outer bob, top right has the angular velocities of the inner and the outer bob. Bottom and left and right show the angle versus angular velocity of the inner bob and the outer bob respectively.

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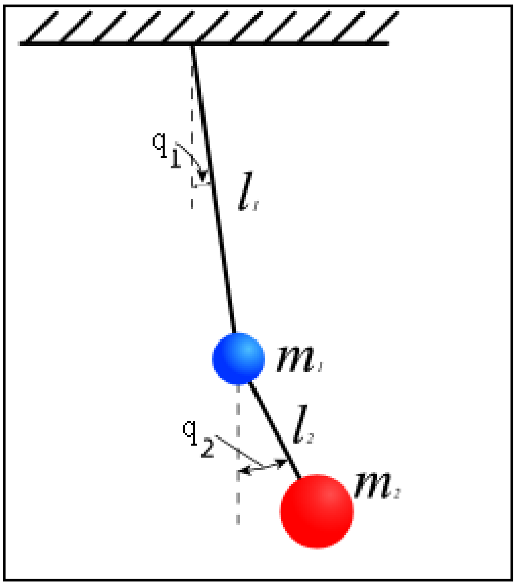
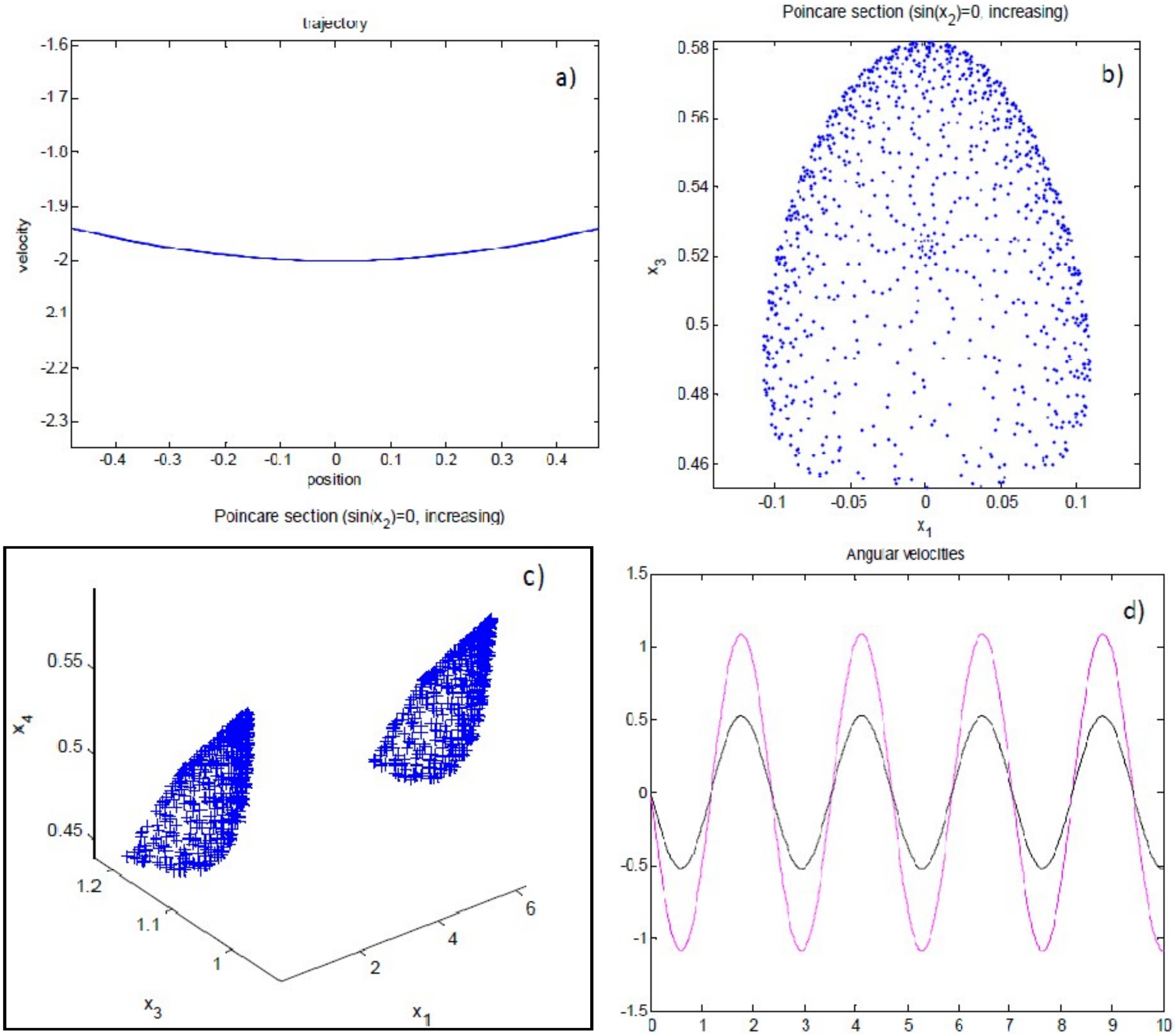


Fig.1. Double pendulum



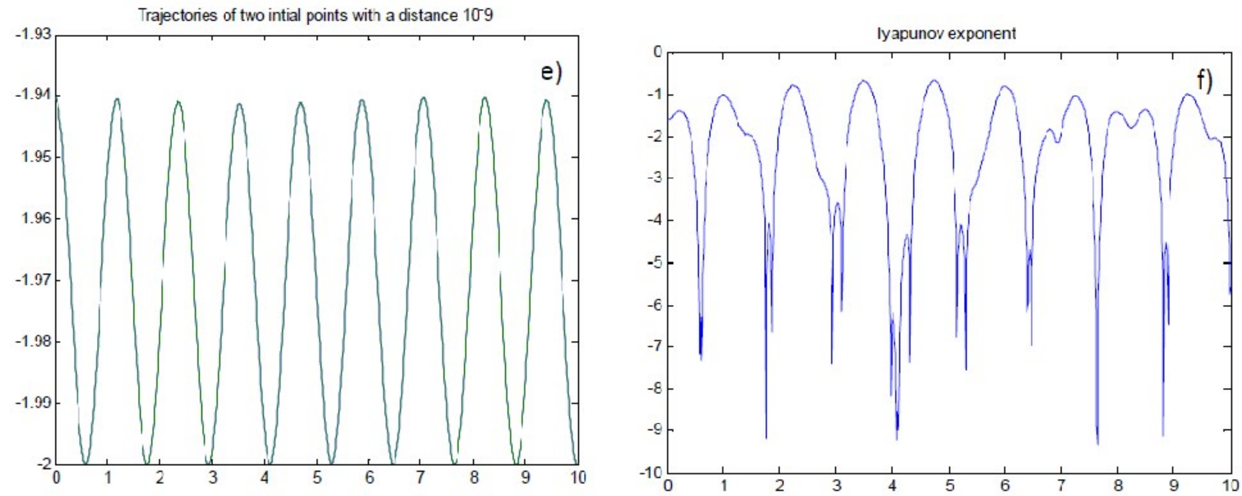
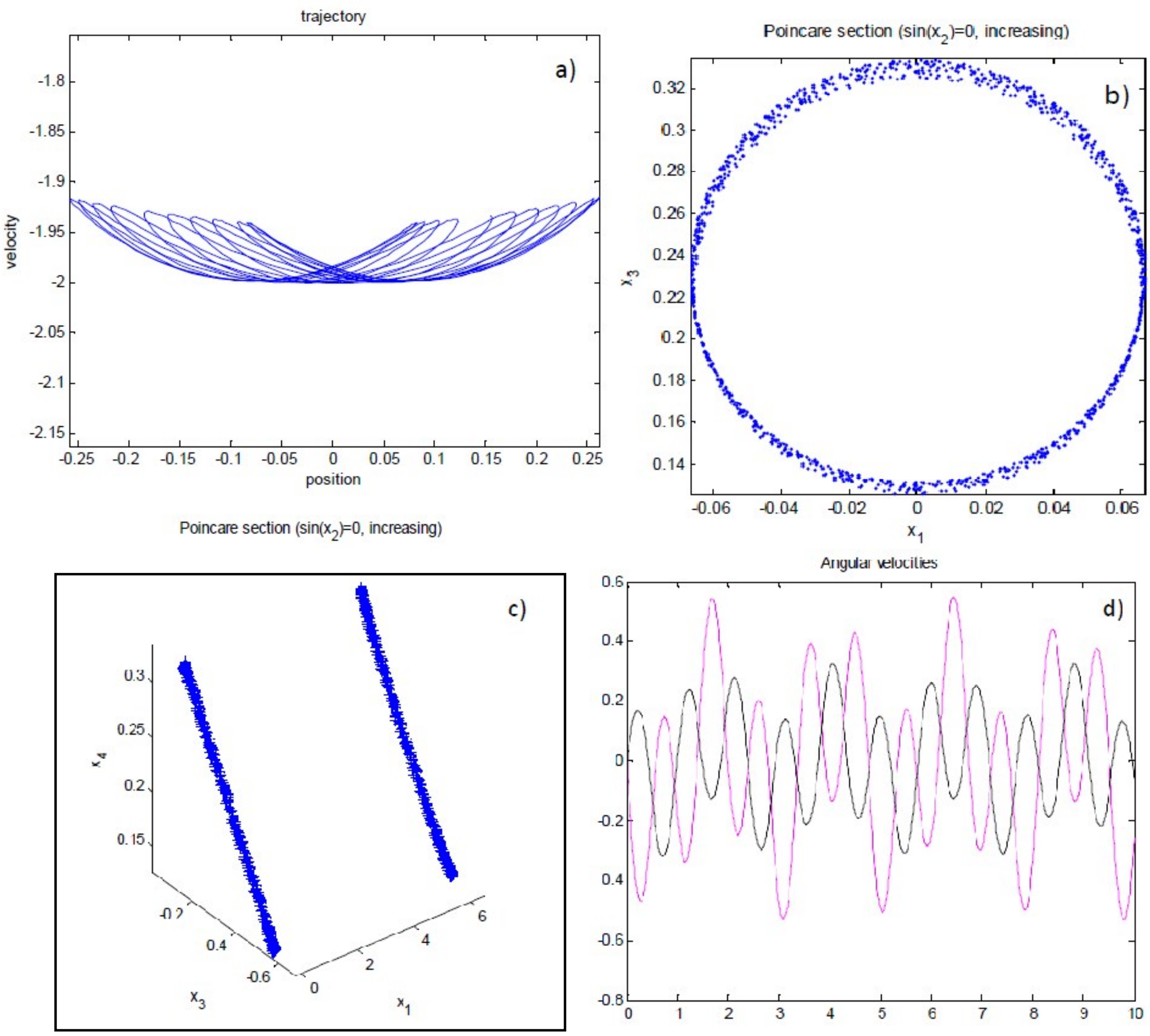


Fig.2. y0 = [0.2,0.2828,0,0] produces periodic motion. a) Trajectory of the outer bob; b) 2-D Poincare map for the section when the outer bob is hanging vertically i.e q_2 = 0; c) 3-D Poincare map when the position of the outer bob is at zero i.e sinq_2=0; d) angular velocities of the outer and the inner bob against time; e) Trajectories of two curves at a distance of 10^{-9} between them; f) Illustrates the negative lyapunov exponent.



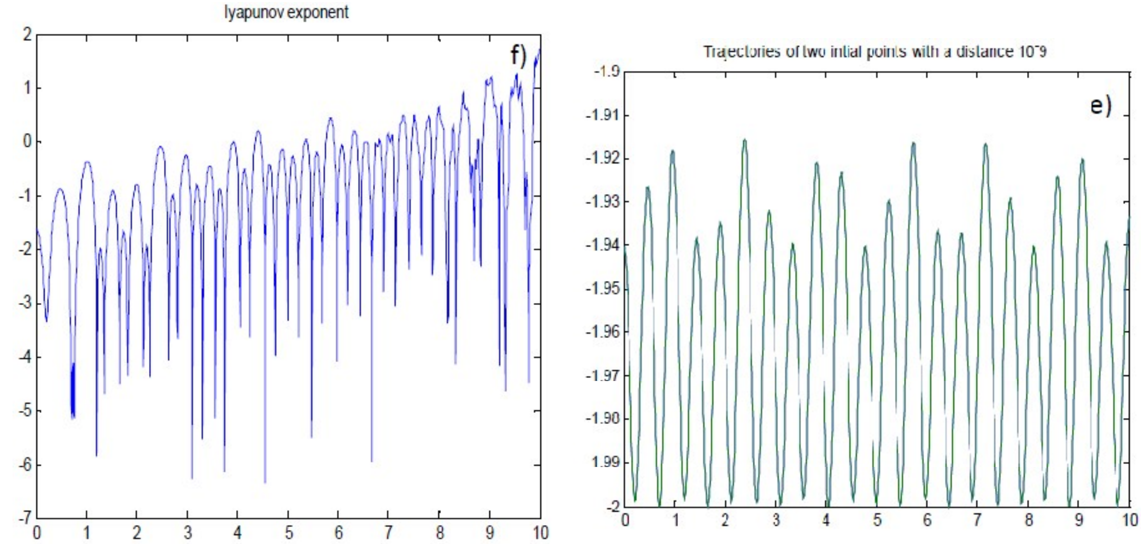
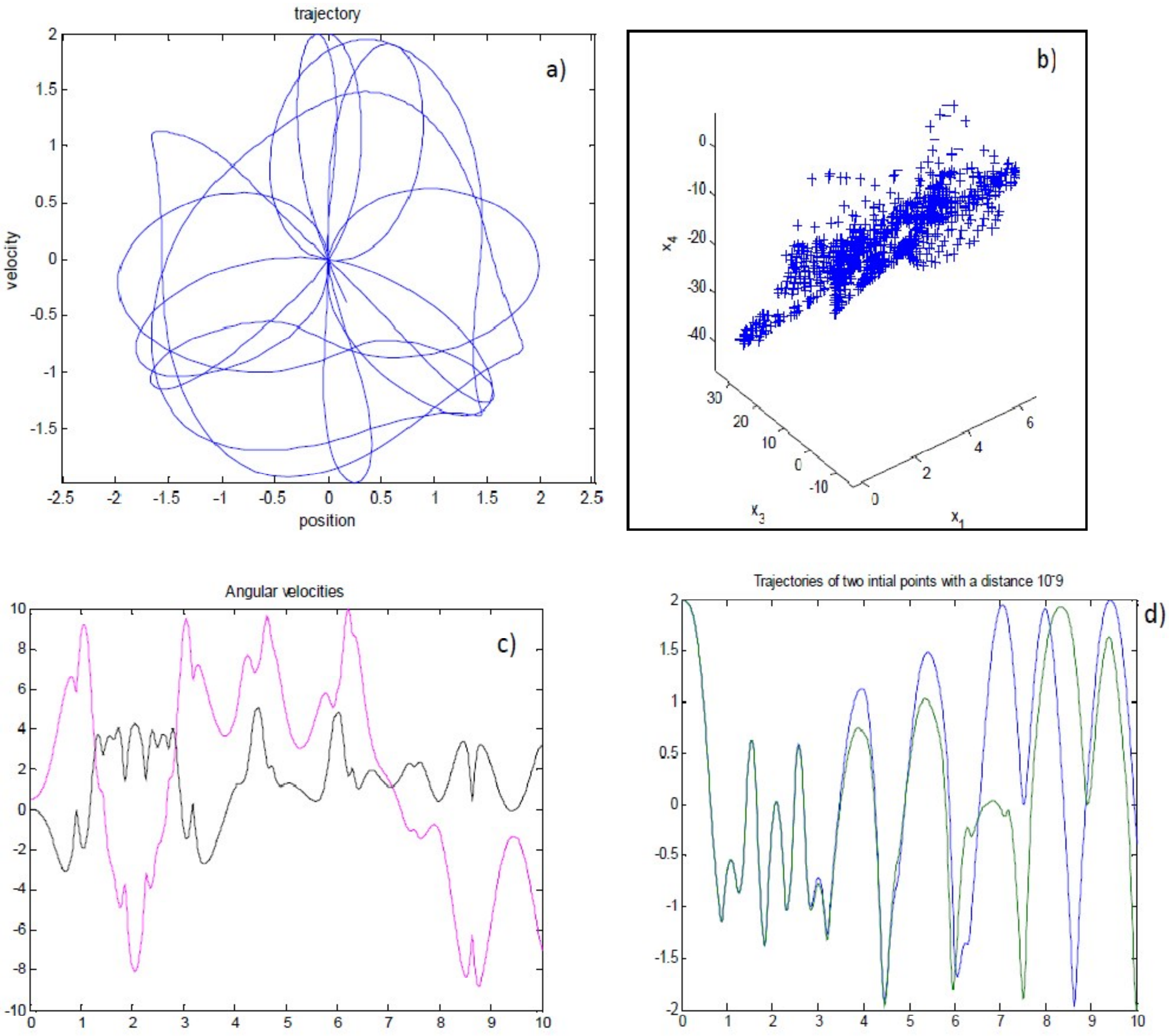


Fig.3. y0 = [0.2, - 0.2828,0,0] produces quasi-periodic motion. a) Trajectory of the outer bob; b) 2-D Poincare map for the section when the outer bob is hanging vertically i.e $q_2 = 0$; c) 3-D Poincare map when the position of the outer bob is at zero i.e $\sin q_2=0$; d) angular velocities of the outer and the inner bob against time; e) Trajectories of two curves at a distance of 10^{-9} between them; f) slowly increasing lyapunov exponent.



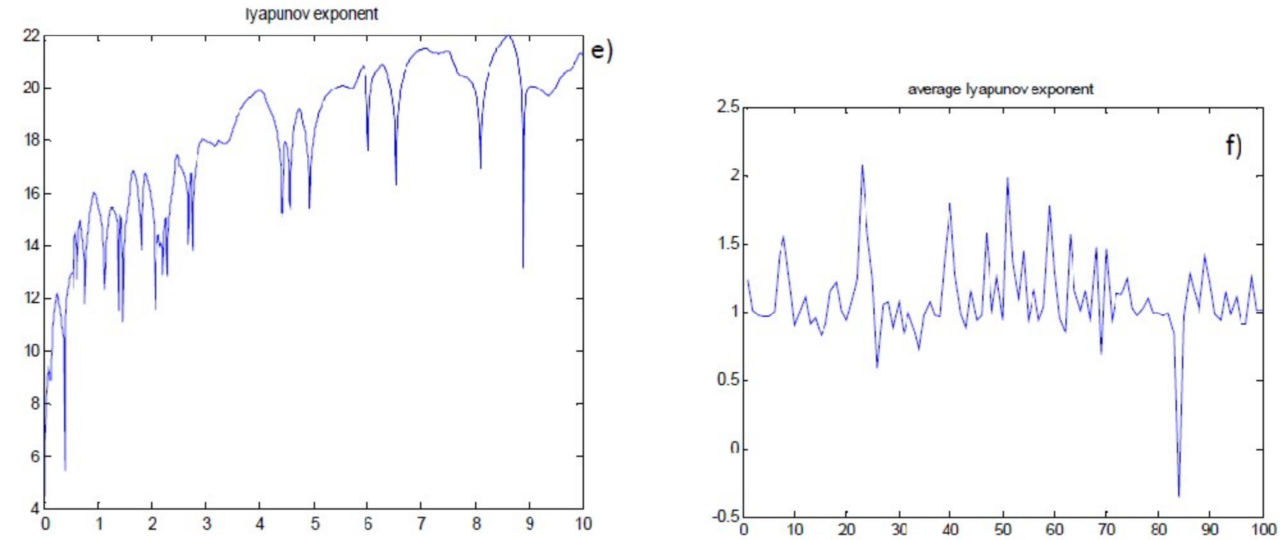
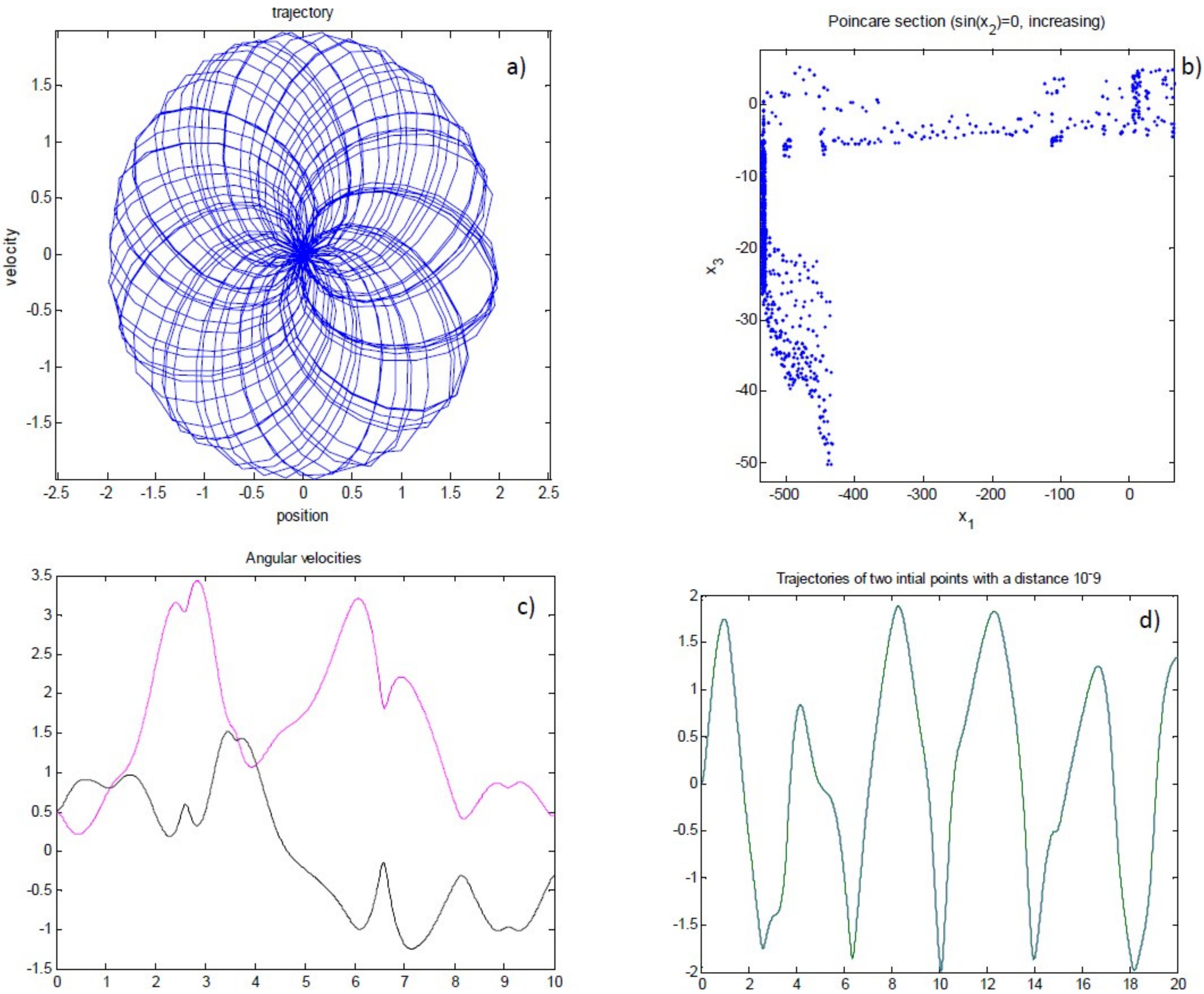


Fig.4. y0 = [π,π,0.5,0] produces chaos! a) Trajectory of the outer bob; b) 2-D Poincare map for the section when the outer bob is hanging vertically i.e q_2 = 0; c) angular velocities of the outer and the inner bob against time; d) Trajectories of two curves at a distance of 10^{-9} between them; e) positive lyapunov exponent; f) average lyapunov exponent for various initial conditions.



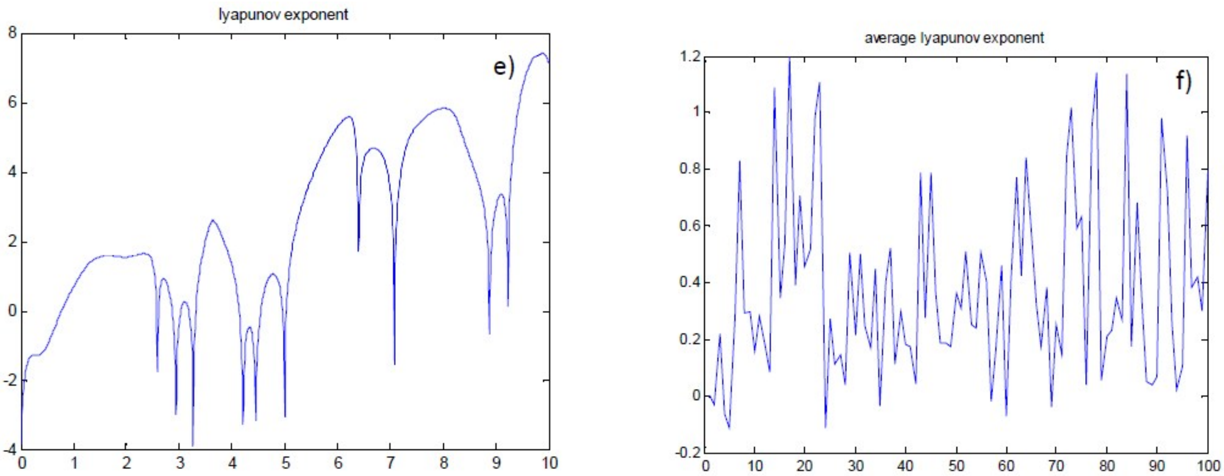


Fig.5. y0 = [π,0,0.5,0.5] produces quasi-periodic motion. a) Trajectory of the outer bob; b) 2-D Poincare map for the section when the outer bob is hanging vertically i.e $q_2 = 0$; c) angular velocities of the outer and the inner bob against time; d) Trajectories of two curves at a distance of 10^{-9} between them; e) slightly positive lyapunov exponent; f) average lyapunov exponent for various initial conditions.

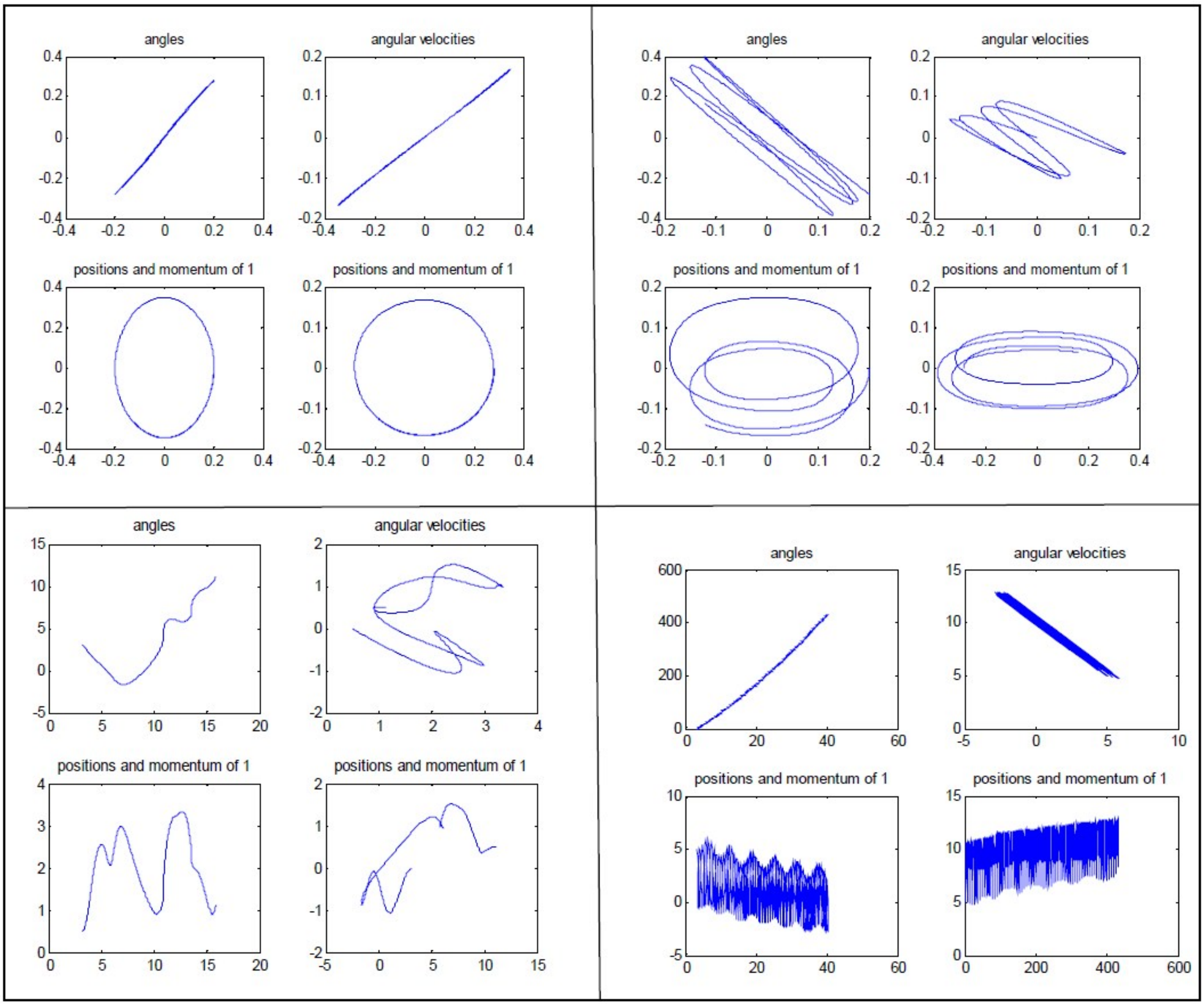


Fig.6. in each square, the top left is the angles of the inner and outer bob, top right has the angular velocities of the inner and the outer bob. Bottom and left and right show the angle versus angular velocity of the inner bob and the outer bob respectively. The top left square shows periodic behavior, the top right square indicates quasi-periodic aka almost periodic behavior, the bottom left square has the chaotic system depicted and the bottom right square starts to having resemblances to periodic behavior.

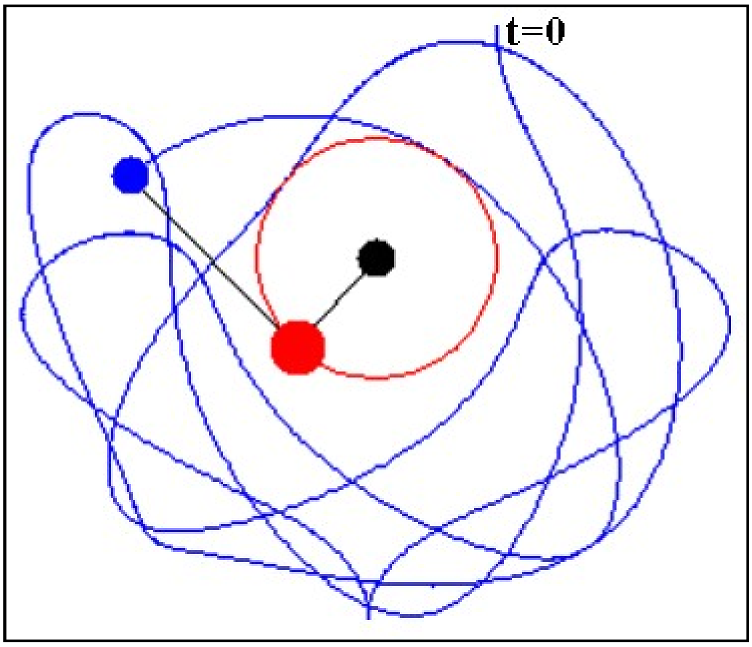


Fig.7. Double pendulum motion