Homework 2

Due by 3:05 PM 9/12/2024

1. (16 points) Show that the following probability density functions belong to the exponential family. Define $a(\cdot)$, $b(\cdot)$, $c(\cdot)$, $d(\cdot)$ components.

   (a) Pareto distribution $f(y;\theta) = \theta y^{-\theta-1}$

   (b) Exponential distribution $f(y;\theta) = \theta e^{-y\theta}$

   (c) Negative Binomial distribution

   $$f(y;\theta) = \binom{y+r-1}{r-1} \theta^r (1-\theta)^y,$$

   where r is known.

   (d) Extreme value (Gumbel) distribution

   $$f(y;\theta) = \frac{1}{\phi} \exp\left\{ \frac{(y-\theta)}{\phi} - exp\left[\frac{(y-\theta)}{\phi} \right] \right\},$$

   where $\phi > 0$ is regarded as a nuisance parameter.

2. (10 points) [Textbook Exercise 3.2 on page 61] Consider a random variable Y with the following Gamma distribution with a scale parameter β, the parameter of interest, and a known shape parameter α,

   $$f(y;\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-y\beta}, \text{ where } y > 0, \alpha > 0, \beta > 0.$$

   (a) Does this distribution belong to the exponential family?

   (b) Derive expectation of Y.

   (c) Derive variation of Y.

   (d) Derive variance of the score statistic.

3. (5 points) Derive expression for information (consider θ the parameter of interest) for a single observation from the Weibull distribution

   $$f(y;\lambda,\theta) = \frac{\lambda y^{\lambda-1}}{\theta^\lambda} \exp\left[-\left(\frac{y}{\theta}\right)^\lambda \right]$$

   where $y \geq 0$, $\lambda > 0$, and $\theta > 0$. Show your work.

4. (10 points) [Textbook Exercise 3.9 on page 63] Suppose $Y_1, \dots, Y_N$ are independent random variables each with the Pareto distribution

$$f(y_i;\theta) = \frac{\theta}{y_i^{\theta+1}}, \text{ where } y_i > 1, \theta > 0 \text{ for all } i = 1, \dots, N,$$

and

$$E(Y_i) = (\beta_0 + \beta_1 x_i)^2.$$

(a) Does this distribution have the canonical form?

(b) Are the distributions of all the Y_i's of the same form?

(c) What is the link function? Is it monotone and differentiable?

(d) Given your answers in (a) to (c), is this a generalized linear model? Give reasons for your answer.