Experiment 12: Harmonic Motion

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Abstract

This experiment aims to study the effects of damping on an oscillatory system. The system is a classic mass-spring oscillator, with a rotating wheel to study force driven oscillations. We find the spring constant at $6.56 \pm 0.18\frac{N}{m}$ from a force vs displacement plot, which is consistent with the theoretical value of $6.57 \pm 0.06\frac{N}{m}$ determined from the mass of the system and the observed period $(0.56 \pm 0.1s)$. The damping rod (attached mass) is released away from equilibrium to create oscillations, tracking the amplitudes to determine the damping constant.The damping constants of air and water were 0.092 and 0.49, respectively. Water's damping constant measures much larger than air's, which is expected and explains why water's system returns to equilibrium much faster. Lastly, the phase dependence of the system during force driven oscillations showed a 180° shift occur once the resonant frequency is passed. Water damped oscillations had a more gradual rate of "flipping" due to the higher damping constant.

I Introduction

Hook's Law describes the force required to stretch or compress a spring is linearly proportional to the distance stretched/compressed from equilibrium. This law can be written as

$$F = Kx \tag{1}$$

where K is called the spring constant, and x is the distance moved from equilibrium. By Newton's second law,

$$F = Ma = -Kx \tag{2}$$

with the negative sign indicating the force's direction is opposite to the direction of displacement. This is the condition required for harmonic motion: force and displacement are proportional in magnitude, but negative in direction. Rewriting the above equation in differential form

$$-Kx = Ma = M\frac{d^2x}{dt^2} \tag{3}$$

or

$$M\frac{d^2x}{dt^2} + Kx = 0 \tag{4}$$

Adding a mass to the spring creates more momentum in the system, allowing more accentuated harmonic motion. If a mass attached to a spring is released from some starting displacement A_0 at time $t = t_0 = 0$, the solution to equation (4) is

$$x(t) = A_0cos(2\pi f_0t) \tag{5}$$

where

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{K}{M}} \tag{6}$$

is the natural frequency of the system. In a frictionless system with no external forces, the oscillation will occur forever at amplitude A_0.

Damped free oscillations: In all real world systems, there exists some form of damping. Immersed in fluid, the system will experienced a frictional damping force proportional to velocity, converting equation (4) into

$$M\frac{d^2x}{dt^2} + b\frac{dx}{dt} + Kx = 0 \tag{7}$$

with a new solution

$$x(t) = A_0e^{-\frac{b}{2M}t}cos(2\pi f_0t) \tag{8}$$

which is a cosine wave that exponentially loses amplitude in time, eventually tapering off to zero movement. The new frequency of oscillation is given by

$$f = \frac{1}{2\pi}\sqrt{\frac{K}{M} - (\frac{b}{2M})^2} \tag{9}$$

The damping constant $\frac{b}{2M}$ can be experimentally determined from the measurement of sequential oscillations from

$$\frac{b}{2M}T = \ln\frac{x1}{x2} \tag{10}$$

where T is the period, x_1 is the amplitude of an oscillation, and x_2 is the amplitude of the following oscillation.

Force damped oscillations: Add a sinusoidal driving force to the system converts equation (7) into

$$M\frac{d^2x}{dt^2} + b\frac{dx}{dt} + Kx = F_0cos2\pi f_dt \tag{11}$$

where F_0 is the driving force amplitude at a frequency f_d. The solution becomes

$$x(t) = A_0 cos(2\pi f_d t - \delta) \tag{12}$$

where δ is the phase angle: the phase difference between the driving force oscillations and the spring's. Depending on whether the spring's oscillations are ahead or behind of the driving motor's, the phase angle can be described as "leading" or "lagging." The amplitude of the oscillation depends on the frequency, given by

$$A(f) = \frac{F_0 \pi^2 M}{\sqrt{(f_0^2 - f_d^2)^2 + (bf_d/2\pi M)^2}} \tag{13}$$

The phase angle depends on the driving frequency given by

$$tan\delta = \frac{bf_d/2\pi M}{f_0^2 - f_d^2} \tag{14}$$

II Apparatus

The apparatus (figure 1): DHMA Oscillation Unit, string, spring, mass bar, phase measuring guide/LED, damping rod, damping liquid.

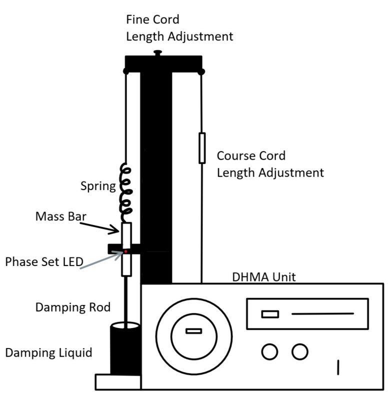

Figure 1. **Harmonic Motion Analyzer**

III Procedure

Initial steps include measuring all relevant apparatus pieces (values located in section IV). The base of the system was leveled so the mass bar dropped through the mass guide without touching the sides, which would have adding their own damping due to friction.

Spring constant: Various masses were attached to the spring, with its displacement recorded from equilibrium. Refer to figure 2 for the plot of displacement versus mass. The slope of the curve represents the spring constant, K, from Hook's law.

Period determination: The mass bar was adjusted (using course and fine controls) so the phase LED was half-on, or flickering dimly, finding the zero point for phase measurements. With the DHMA unit set to display the oscillation period, the bar was raised approximately 1cm above equilibrium and released.

Extraneous Damping: Attaching the damping rod and re-centering the system, the function switch was set to measure peak-to-peak amplitude of oscillations. The bar was raised 2-3cm above equilibrium and dropped. On every 10th oscillation, the amplitude was recorded, until the system returned to equilibrium. This process was repeated 10 times.

Damped Oscillations: The driver was turned on to about 0.1Hz. Once the system stabilized, the string was adjusted so the phase LED flashed at precisely 0°. The driver frequency was increased by about 0.2Hz to a maximum of 3.0Hz, with recordings of the phase and amplitude taken once the system stabilized. Due to the resonance point violently tossing the mass bar, those nearby frequencies were skipped to ensure all lab members' safety.

Liquid Damped: A plastic cylinder was filled with water, and the damping rod was inserted directly down the middle, ensuring no contact between the rod and the plastic. Both extraneous damping and force damped oscillation techniques were repeated for the new system; however, for extraneous damping, every single oscillation had its amplitude recorded (opposed to every 10th oscillation), due to the water causing the system to reach equilibrium much quicker.

IV Results

For determining the force constant of our spring, we us the data presented in table 1 plotted in figure 2. All full sized plots are included in appendix A.

Mass (g)	Force (N)	Δx(cm)
0	0	0
10	0.0981	1.3
20	0.196	2.7
50	0.491	7.4

TABLE I. Mass/Displacement data for calculating the force constant of the spring.

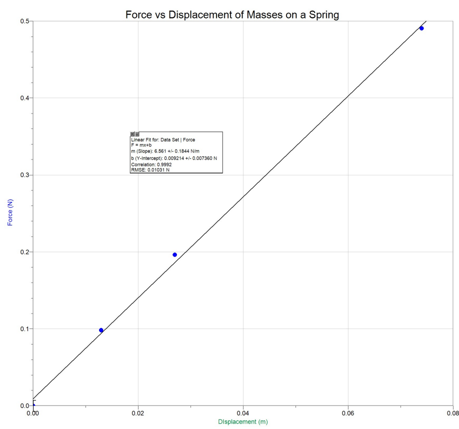

Figure 2. **Harmonic Motion Analyzer**

Using equation 2, the force constant of the spring is the slope in figure 2, $K = (6.56 \pm 0.18)\frac{N}{m}$. We can also use equation 6 to determine the force constant, using the measured period of $T = (0.56 \pm 0.01)s$ and $f_0 = \frac{1}{T} = \frac{1}{0.56s \pm 2\%} = 1.78s^{-1} \pm 1.8\%$.

We take the mass on the string plus 1/3 of the spring's mass for M,

$$M = m_b + m_r + m_s$$

where m_b, m_r, and m_s are the masses of the bar, damping rod, and spring, respectively.

$$M = (19.29 \pm 0.01)g + (32.35 \pm 0.01)g + \frac{1}{3}(2.74 \pm 0.01)g$$

$$= 52.55g \pm 0.14\% =$$

Rearranging equation 6,

$$K = (f_0 2\pi)^2 M$$

$$K = ((1.78s^{-1} \pm 1.8\%)2\pi)^2(.05255kg \pm 0.14\%)$$

$$= 6.57\frac{kg}{s^2} \pm 0.9\%$$

Performing a consistency check

$$|K_1 - K_2| \overset{?}{\leq} \Delta K_1 + \Delta K_2$$

$$|6.56 - 6.57| \overset{?}{\leq} 0.18 + 0.06$$

$$0.01 \leq 0.24$$

therefore, the values are consistent.

Using equation 10 we can find the damping constant for air and water. For air damping, because we counted every 10th oscillation, the value for T in the calculations is $T_{10} = 5.6s$. The code for the full calculations is included at the end of the report, with a sample calculations here:

$$\frac{b}{2M} = \frac{1}{T}\ln\frac{x_1}{x_2}$$

$$= \frac{1}{5.6s}\ln\frac{27}{21} = 0.045s^{-1}$$

where this process for x_1, x_2 is repeated for all x_1, x_2 across a trial and averaged. All oscillation data points are stored as the lists in the calculation code, with single-trial tables in appendix A. The damping constants for air and water were calculated as 0.092 and 0.49, respectively. We see that water has a much larger damping constant, which is suggested by the system's quicker return to equilibrium.

We plot the amplitude and phase versus frequency for air and water damped cases.

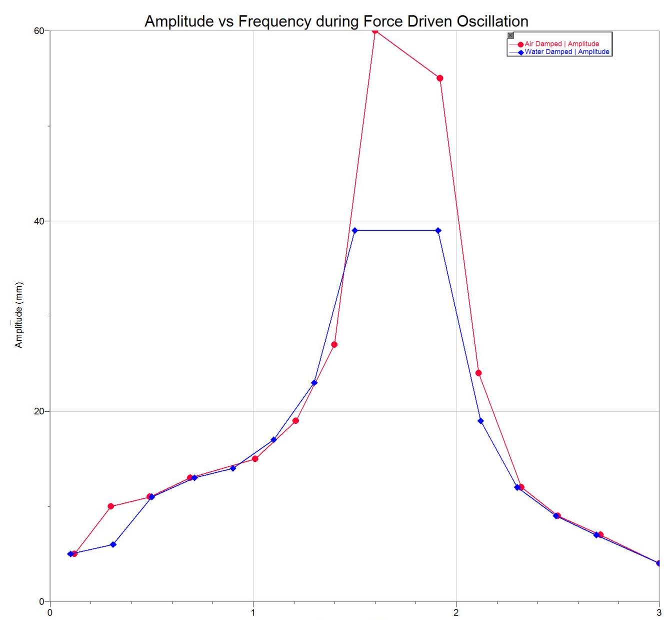

Figure 3. **Amplitude vs. Frequency** Air (red) and Water (blue)

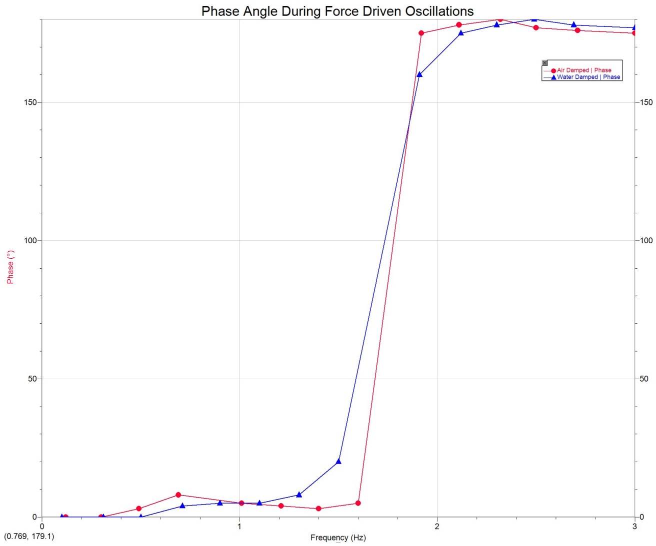

Figure 4. **Phase vs. Frequency:** Air (red) and Water (blue)

Maximum amplitude: As mentioned previously, frequencies around 1.8Hz were skipped due to resonant frequencies causing amplitudes larger than the system can handle. This can be seen from equation 13, where the largest amplitude occurs when the denominator is at a minimum: when $f_0^2 - f_d^2 = 0$. The degree of damping appears inversely proportional to the amplitudes recorded: a larger damping constant results in lower amplitudes. For the frequency graph, the phase makes a 180° flip once passed the resonant frequency; however, the air damped system makes that flip quicker, as the water damped system has a more gradual curve before flipping.

Limiting phase angle: Equation 14 suggests that maximum and minimum phase angles are $\pm\frac{\pi}{2}$, coming from the limits of $tan^{-1}(x)$. At low values, $tan^{-1}(x) = -\frac{\pi}{2}$ and at high values $tan^{-1}(x) = \frac{\pi}{2}$, with a steep shift around zero. Our results confirm these values, however on a difference scale of $[0, \pi]$, which is an offset of $+\frac{\pi}{2}$ from our apparatus zeroing. The phase angle flip occurs around the natural frequency when the denominator approaches zero.

V Discussion

Our results prove consistent with expected values. The amplitude of a force driven oscillation clearly depends on the damping constant of the medium. A larger damping constant exerts a larger drag force on the system, causing smaller oscillation amplitudes. Water damping caused the system to return to equilibrium much quicker than in air, which is expected as the more viscous water has a higher damping constant.

Although consistent results, there are ways to improve the experiment. Firstly, the spring constant was measured on a different day then the damping experiments were performed, potentially being used and stretched by another research group. Stretching the string too far causes permanent deformation, changing the spring constant. It appears the spring was not overly stretched, but if conducted again, the spring constant would be measured immediately preceding the experiment. Another potential source of error comes from counting every n^{th} oscillation during the first portion of the experiment. A single researcher had to simultaneously count oscillations, read the amplitude off the DHMA unit, and record them on the computer. During the water damped oscillations, a video was taken of the entire system, making playback and measurement recording easier. Although more time consuming, a missed reading was much less likely using a video recording. It is recommended to either record and playback each trial, or operate with multiple researchers, each tracking an element of the experiment (oscillation counting, amplitude readings/recordings).

VI Conclusion

We calculate the force constant of a spring from a force vs displacement graph and from a theoretical equation. With a value of $6.56 \pm 0.18\frac{N}{m}$ from the graph and $6.57 \pm 0.06\frac{N}{m}$ from the equation, these results are consistent and suggest a correct measurement for the spring force constant. From the oscillation amplitudes to equilibrium, we calculate the damping constants of air and water as 0.092 and 0.49, respectively. The damping constant of water is much larger than for air, as expected. During force driven oscillations, the resonant frequency occurred around 1.8Hz, which caused the oscillation amplitude to become very large. At frequencies below resonant frequency, the phase stayed around 0°. For frequencies above the resonant frequency, the phase shifted by a factor of π to become 180°.

Appendix A: Data Plots

10^{th} oscillation (Air)	Peak-to-Peak Amplitude (mm)
1	33
2	27
3	21
4	18
5	15
6	10
7	8
8	5
9	2

TABLE II. Every 10th oscillation for extraneous damping in air with the mass bar raised about 3cm and released.

Oscillation (Water)	Peak-to-Peak Amplitude (mm)
1	46
2	33
3	24
4	18
5	16
6	13
7	10
8	9
9	8
10	6
11	4
12	4
13	2

TABLE III. Every oscillation for extraneous damping in water with the mass bar raised about 3cm and released.

Frequency (Hz)	Phase (°)	Amplitude (mm)
0.12	0	5
0.30	0	10
0.49	3	11
0.69	8	13
1.01	5	15
1.21	4	19
1.40	3	27
1.60	5	60
1.92	175	55
2.11	178	24
2.32	180	12
2.50	177	9
2.71	176	7
3.00	175	4

TABLE IV. Frequency, phase, and amplitude of force driven oscillations in air.

Frequency (Hz)	**Phase (°)**	**Amplitude (mm)**
0.10	0	5
0.31	0	10
0.50	0	11
0.71	4	13
0.90	5	15
1.10	5	19
1.30	8	27
1.50	20	60
1.91	160	55
2.12	175	24
2.30	178	12
2.49	180	9
2.69	178	7
3.00	177	4

TABLE V. Frequency, phase, and amplitude of force driven oscillations in water.

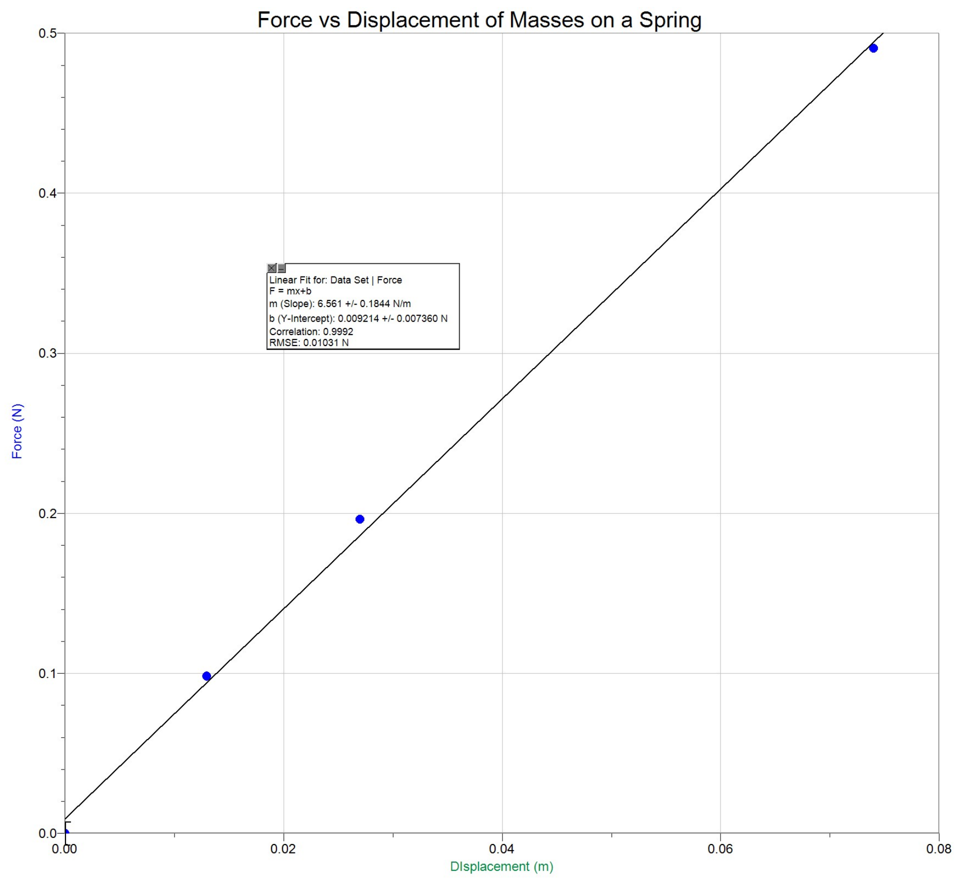

Figure 5. **Harmonic Motion Analyzer**

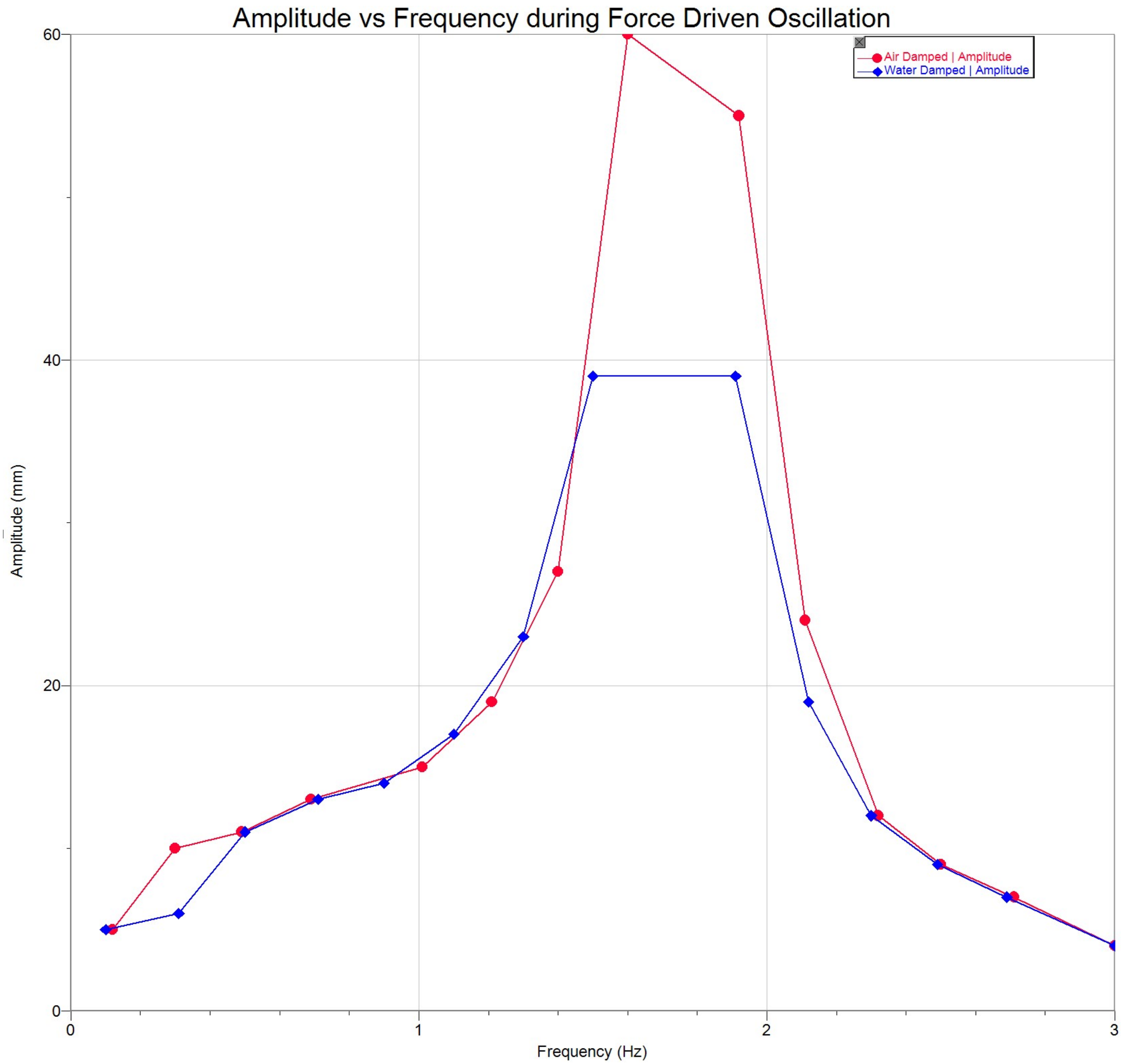

Figure 6. **Amplitude vs. Frequency** Air (red) and Water (blue)

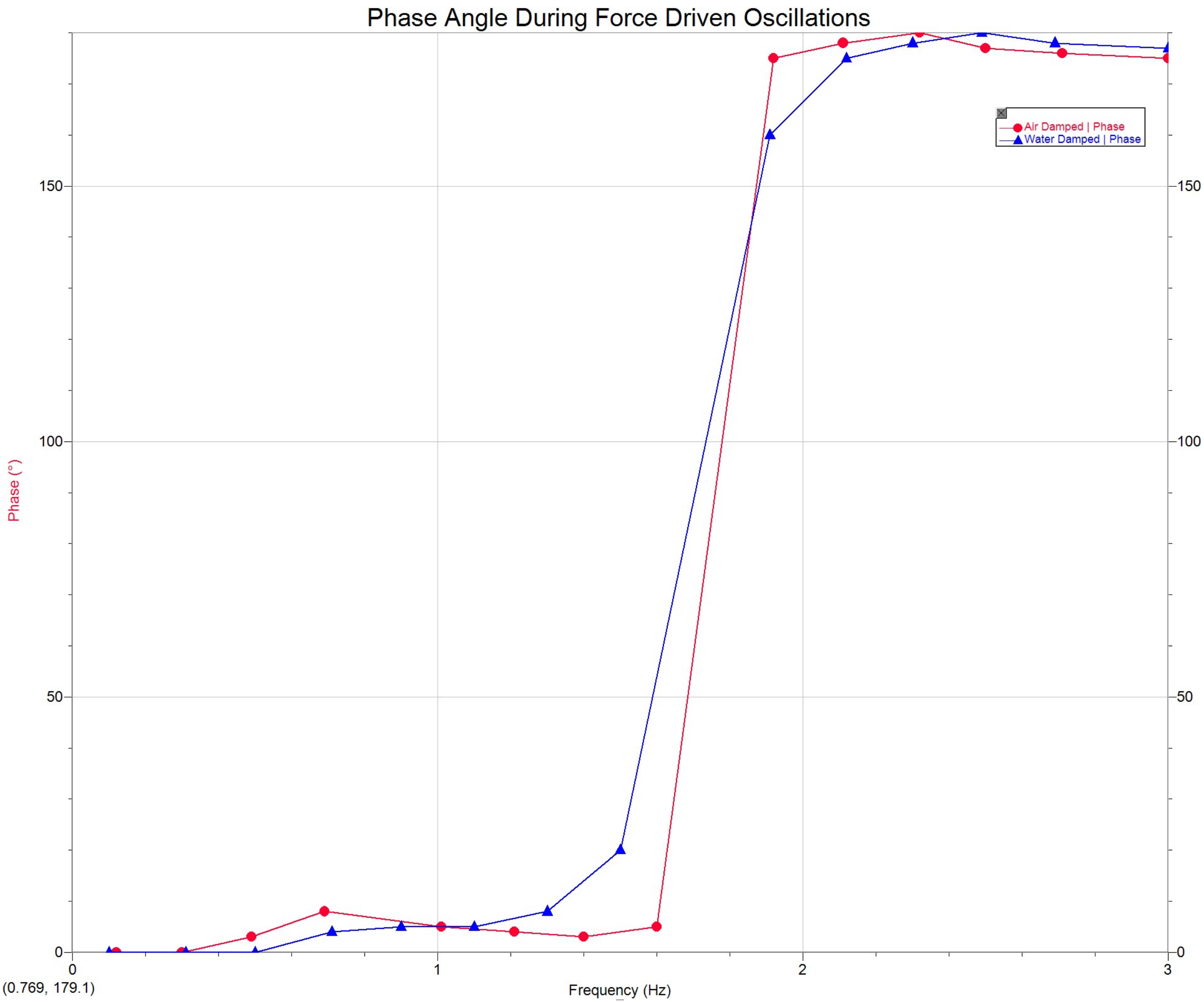

Figure 7. **Phase vs. Frequency:** Air (red) and Water (blue)