PEP336: Simulation of the orbital motion.

Due Wednesday February 21.

Part 2. [Runge-Kutta.]

At this time you should have a working simulation of orbital motion using Euler's methods, and now you can easily upgrade it to RK2 and RK4 methods. RK stands for Runge-Kutta, amd '2' and '4' are orders of approximation. At the beginning of each time interval Δt we know the initial position vector $\vec{r} = (x, y)$ and velocity vector $\vec{v} = (v_x, v_y)$.

I. RK2

In RK2 we calculate auxiliary quantities k_1 and k_2 for each of these variables, and use them to find the updated variable at the end of the time interval. The procedure is (initial variables) $\longrightarrow k_1 \longrightarrow k_2 \longrightarrow$ (updated variables):

RK2 Step 1: Calculate the first set of k's:

$$k_{1x} = v_x \Delta t \tag{1}$$

$$k_{1y} = v_y \Delta t \tag{2}$$

$$k_{1vx} = -\frac{GM}{x^2 + y^2} \frac{x}{\sqrt{x^2 + y^2}} \Delta t \tag{3}$$

$$k_{1vy} = -\frac{GM}{x^2 + y^2} \frac{y}{\sqrt{x^2 + y^2}} \Delta t \tag{4}$$

RK2 Step 2: Calculate the second set of k's:

$$k_{2x} = (v_x + k_{1vx}) \Delta t \tag{5}$$

$$k_{2y} = (v_y + k_{1vy}) \Delta t \tag{6}$$

$$k_{2vx} = -\frac{GM}{(x + k_{1x})^2 + (y + k_{1y})^2} \frac{x + k_{1x}}{\sqrt{(x + k_{1x})^2 + (y + k_{1y})^2}} \Delta t \tag{7}$$

$$k_{2vy} = -\frac{GM}{(x + k_{1x})^2 + (y + k_{1y})^2} \frac{y + k_{1y}}{\sqrt{(x + k_{1x})^2 + (y + k_{1y})^2}} \Delta t \tag{8}$$

RK2 Step 3: Calculate the new values of the variables:

$$
\begin{aligned}
x &= x + \frac{1}{2}\left(k_{1x} + k_{2x}\right) && (9)\\
y &= y + \frac{1}{2}\left(k_{1y} + k_{2y}\right) && (10)\\
v_x &= v_x + \frac{1}{2}\left(k_{1vx} + k_{2vx}\right) && (11)\\
v_y &= v_y + \frac{1}{2}\left(k_{1vy} + k_{2vy}\right) && (12)
\end{aligned}
$$

II. RK4

In RK4 we use the same idea with more steps:
(initial variables) $\longrightarrow k_1 \longrightarrow k_2 \longrightarrow k_3 \longrightarrow k_4 \longrightarrow$ (updated variables):

RK4 Step 1: Calculate the first set of k's:

$$
\begin{aligned}
k_{1x} &= v_x \Delta t && (13)\\
k_{1y} &= v_y \Delta t && (14)\\
k_{1vx} &= -\frac{GM}{x^2+y^2}\frac{x}{\sqrt{x^2+y^2}}\Delta t && (15)\\
k_{1vy} &= -\frac{GM}{x^2+y^2}\frac{y}{\sqrt{x^2+y^2}}\Delta t && (16)
\end{aligned}
$$

RK4 Step 2: Calculate the second set of k's (note the factors of 1/2):

$$
\begin{aligned}
k_{2x} &= (v_x + k_{1vx}/2)\Delta t && (17)\\
k_{2y} &= (v_y + k_{1vy}/2)\Delta t && (18)\\
k_{2vx} &= -\frac{GM}{(x+k_{1x}/2)^2+(y+k_{1y}/2)^2}\frac{x+k_{1x}/2}{\sqrt{(x+k_{1x}/2)^2+(y+k_{1y}/2)^2}}\Delta t && (19)\\
k_{2vy} &= -\frac{GM}{(x+k_{1x}/2)^2+(y+k_{1y}/2)^2}\frac{y+k_{1y}/2}{\sqrt{(x+k_{1x}/2)^2+(y+k_{1y}/2)^2}}\Delta t && (20)
\end{aligned}
$$

RK4 Step 3: Calculate the third set of k's (note the factors of $1/2$):

$$
\begin{aligned}
k_{3x} &= (v_x + k_{2vx}/2)\Delta t && (21)\\
k_{3y} &= (v_y + k_{2vy}/2)\Delta t && (22)\\
k_{3vx} &= -\frac{GM}{(x+k_{2x}/2)^2+(y+k_{2y}/2)^2}\frac{x+k_{2x}/2}{\sqrt{(x+k_{2x}/2)^2+(y+k_{2y}/2)^2}}\Delta t && (23)\\
k_{3vy} &= -\frac{GM}{(x+k_{2x}/2)^2+(y+k_{2y}/2)^2}\frac{y+k_{2y}/2}{\sqrt{(x+k_{2x}/2)^2+(y+k_{2y}/2)^2}}\Delta t && (24)
\end{aligned}
$$

RK4 Step 4: Calculate the fourth set of k's:

$$
\begin{aligned}
k_{4x} &= (v_x + k_{3vx})\Delta t && (25)\\
k_{4y} &= (v_y + k_{3vy})\Delta t && (26)\\
k_{4vx} &= -\frac{GM}{(x+k_{3x})^2+(y+k_{3y})^2}\frac{x+k_{3x}}{\sqrt{(x+k_{3x})^2+(y+k_{3y})^2}}\Delta t && (27)\\
k_{4vy} &= -\frac{GM}{(x+k_{3x})^2+(y+k_{3y})^2}\frac{y+k_{3y}}{\sqrt{(x+k_{3x})^2+(y+k_{3y})^2}}\Delta t && (28)
\end{aligned}
$$

RK4 Step 5: Calculate the new values of the variables:

$$
\begin{aligned}
x &= x + \frac{1}{6}\left(k_{1x} + 2k_{2x} + 2k_{3x} + k_{4x}\right) && (29)\\
y &= y + \frac{1}{6}\left(k_{1y} + 2k_{2y} + 2k_{3y} + k_{4y}\right) && (30)\\
v_x &= v_x + \frac{1}{6}\left(k_{1vx} + 2k_{2vx} + 2k_{3vx} + k_{4vx}\right) && (31)\\
v_y &= v_y + \frac{1}{6}\left(k_{1vy} + 2k_{2vy} + 2k_{3vy} + k_{4vy}\right) && (32)
\end{aligned}
$$

III. ASSIGNMENT

Let's make sure it works:

(a) Give your test particle a tangential velocity of 29.8 km/s at a distance of 1 AU from the Sun. These are exactly the orbital elements of a circular orbit at a distance of 1 AU. Is your orbit circular?

(b) At a same distance of 1 *AU* give your particle a velocity in an arbitrary direction just a bit larger than the escape speed at that distance (42.1 km/s). Your particle should slowly escape the solar system. Is that what happens?

(c) Now for the elliptical orbits. If you were to calculate the (elliptical) Hohmann transfer orbit between the Earth and Mars (take PEP 351 Planetary Science for details), you would find that such an orbit has the orbital speed at the Earth's orbit equal to 32.7 km/s. Give your particle that initial speed in the tangential direction, and check whether the maximal orbital distance is equal to about 1.52 AU (orbit of Mars). Did it work?

(d) Now repeat steps (a), (b) and (c) for all three methods you developed (Euler, RK2 and RK4) using the *same* time steps, and let the system evolve for several (e.g 5, 10, or whatever you want) orbital periods. Print out the results of the three methods, and submit with your comments on accuracy. Don't use too small time steps (like one second), as the differences will not be apparent. Play with your simulation a little, and choose the right time steps to demonstrate the differences in accuracy.

That's all. Submit your code by email. On paper (or by email in pdf) submit the printouts of the orbits and a brief (a page or so) report on what have you done, how do you interpret the results, and what further use you can envisage for your code.