

CASELLA AND BERGER NOTES AND PROBLEMS - CHAPTER 5

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Using this problem set: <http://www.stat.ufl.edu/~jhobert/sta6327>

Chapter 5: 3, 5, 6, 11, 12, 15, 16, 17, 19, 21, 22

Chapter 5: 23, 24, 25, 30, 32, 34, 42, 44, 47, 49, 50, 52, 53

Chapter 5: 58 (in part (a), assume $t \in (0, 1)$), 62, 63, 64, 66

CHAPTER 5 EXERCISES

(5.3)

Since each $Y_i = 1_{X_i > \mu}$ is a characteristic function, and (X_i) are iid, the sum is binomial.

(5.5)

I think by convolution $F_{\sum_{1 \leq k \leq n} X_k}$ has a density, but suppose $F_{\sum_{1 \leq k \leq n} X_k}$ exists with density $f_{\sum_{1 \leq k \leq n} X_k}$. Then

$$\begin{aligned} F_{\bar{X}}(x) &= P(y : \bar{X}(y) \leq x) \\ &= P(y : \sum_{1 \leq k \leq n} X_k(y) \leq nx) \\ &= F_{\sum_{1 \leq k \leq n} X_k}(nx) \\ &= \int_{\sum_{1 \leq k \leq n} X_k \leq nx} dF_{\sum_{1 \leq k \leq n} X_k}. \end{aligned}$$

By change of variables and since $F_{\sum_{1 \leq k \leq n} X_k}$ has a density,

$$\begin{aligned} &= \int_{\sum_{1 \leq k \leq n} X_k \leq x} ndF_{\sum_{1 \leq k \leq n} X_k} \\ &= \int_{\sum_{1 \leq k \leq n} X_k \leq x} nf_{\sum_{1 \leq k \leq n} X_k}(x)dx. \end{aligned}$$

By definition of the density being the integrand, we are done.

(5.6)

(a) this is the same as idea as the convolution basically.

(b) TODO This one is a lot of work and I had to use trichotomy, so I will think about it.

(c) Same as (b)

(5.11) Contradiction with the variance identity works for both of these or use Jensons inequality since root is concave.

(5.12)

(a)

Let $(X_k)_{k=1}^n$ be iid $N(0, 1)$. By using the pushforward measure twice,

$$EY_1 = \int_{\Omega} Y_1 dP = \int_{\mathbb{R}} y_1 dP_{Y_1} = \int_{\mathbb{R}} |\bar{x}_n| dP_{(X_k)}.$$

By definition, additivity, and change of variables,

$$= \int_{-\infty}^0 -\bar{x}_n dP_{(X_k)} + \int_0^{\infty} \bar{x}_n dP_{(X_k)} = 2 \int_0^{\infty} \bar{x}_n dP_{(X_k)}.$$

By additivity and Fubini,

$$= \frac{2}{n} \sum_{k=1}^n \int_0^{\infty} x_k dP_{X_k}$$

Since X_k is $N(0, 1)$, it has density, so by change of variables,

$$= \frac{2}{n} \sum_{k=1}^n \int_0^{\infty} x_k f_{X_k}(x_k) dx_k = \frac{2}{n} \frac{1}{\sqrt{2\pi}} \sum_{k=1}^n \int_0^{\infty} x_k \exp(-1/2x_k^2) dx_k.$$

Let $\varphi(x) = -1/2x^2$. Then $D_1\varphi(x) = -x$, and by the Inverse Function Theorem, φ is a diffeomorphism on $(0, \infty)$.

Since the boundary of $\varphi(0, \infty)$ has measure 0, the interval is integrable. By change of variables,

$$\int_{\varphi(0, \infty)} e^u du = \int_{(-\infty, 0)} e^u du = 1 = \int_0^{\infty} e^{-1/2x_k^2} x_k dx_k.$$

Hence,

$$= \frac{2}{n} \frac{1}{\sqrt{2\pi}} \sum_{k=1}^n = \sqrt{\frac{2}{\pi}}.$$

(b) Skipping since part (a) was so long with all the theorems.

(5.15)

(a) follows from definition

(b) TODO I got a different answer

(5.16)

These are straight forward, though I forgot the t distribution.

(5.17) TODO

(5.19) TODO

(5.21) TODO

(5.22) TODO

(5.23) TODO

(5.24) TODO

(5.25) TODO

(5.30) TODO

(5.32) TODO

(5.34) Straight forward.

(5.42) TODO

(5.44)

(a) Follows directly from CLT.

(5.47) TODO

(5.49) TODO

(5.50) TODO

(5.52) TODO

(5.53) TODO

NOTES

8 / 4 _____

TODO convolution proof.

8 / 5 _____

The real insight to the normal distribution is in the gamma function. The normal distribution is a special case of the gamma function by change of variables.

The gamma distribution is more fundamental in continuous distributions.

Actually, the gamma distribution is the most fundamental continuous distribution.

Change of variables is a super important theorem in statistics. TODO go through the proof.

THE GAMMA FUNCTION IS THE MOST FUNDAMENTAL THING IN CONT
DISTS GAMMA = Normal with change of variables GAMMA = Chi Square

ALL OF THE CONTINUOUS STUFF FOLLOWS FROM THE GAMMA FUNCTION AND CHANGE OF VARIABLES!

Convolution is used over and over.

TODO Convolution proof

DONT FORGET Probability is just integration over indicators using change of variables.

$$P(X \in A) = \int I_A dP$$

TODO PROVE CHANGE OF VARIABLES and CHANGE OF VARIABLES IN MEASURES (See Folland)

TODO study Change of Variables extremely well. It is probably the most important concept.

Since $x + y \mapsto e^x e^y$ is a homomorphism between the additive group and the multiplicative group, combined with independence an exponential function makes use of this property such as the Gamma function.

8 / 6

Chebyshev is a really useful inequality.

basically all the discrete distributions can be approximated by the CLT with some algebra manipulations.

TODO study properties of characteristic functions because of this:

$$P(X \leq a) = \int_{\Omega} 1_{x: X \leq a} dP$$

8 / 7

So many applications of change of variables!!

Think of the joint distribution as the identity function for the vector (X_k) .

8 / 8
