MEASURE AND PROBABILITY MEASURE

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Abstract. Some personal notes on measure and probability measure.

1. Axioms of Measure

Probability is a special type of volume ranging from $[0, 1]$. In order to handle volume in mathematics, there is Measure Theory. Probability is a special type of measure.

There is an issue with uncountable sets (Banach-Tarski Paradox), and hence we need to restrict the type of sets for which we can define volume for.

Definition. A σ-algebra $\mathcal{M}$ is a collection of subsets of a space Ω satisfying the following axioms:

- (Nonempty) $\mathcal{M} \neq \varnothing$.
- (Closure under compliments) If $A \in \mathcal{M}$ then $A^c \in \mathcal{M}$.
- (Closure under countable unions) If $\{A_i\}_i \in \mathcal{M}$ then $\bigcup_i A_i \in \mathcal{M}$.

The sigma algebra is a formalization of the properties of measurable sets since the compliment of a measurable set is measurable and the countable union of measurable sets is measurable.

Definition. Let X be a topological space. The σ-algebra $\mathcal{B}_X$ generated by the open sets is the Borel σ-algebra.

The idea is volume on a set should be non-negative, nothing should have volume 0, and if we add more stuff to the set, the volume should grow additively.

Definition. A Measure is a map $\mu : \mathcal{M} \to [0, \infty]$ where $\mathcal{M}$ is a σ-algebra and satisfies the following axioms:

- $\mu(\varnothing) = 0$
- Countable additivity: $\mu(\bigcup A_i) = \sum_i \mu(A_i)$

Definition. A Probability Measure is a measure $P : \mathcal{M} \to [0,1]$ where $\mathcal{M}$ is a σ-algebra, and $P(\mathcal{M}) = 1$.

Example. Let X be a nonempty set. By the axioms, the power set $\mathcal{P}(X)$ is a σ-algebra.

Example. Let $\Omega = \{0, 1\}$. By the axioms, the power set $\mathcal{P}(X)$ of Ω is a σ-algebra and $p(\omega) = 1/2$ for $\omega \in \Omega$ is a probability measure on the power set. This example extends to the n case.

Example. Use the example above to measure probability of the top side of dice rolls.

2. Properties of Measure

Let μ be a measure on $(X, \mathcal{M})$ where $\mathcal{M}$ is a σ-algebra for the set X.

Proposition. (Monotonicity) If $A, B \in \mathcal{M}$ with $A \subset B$, then $\mu(A) \leq \mu(B)$.

Proof. $\mu(A) \leq \mu(A) + \mu(B \setminus A) = \mu(A \coprod B \setminus A) = \mu(B)$. ∎

Proposition. (Subadditivity) If $\{E_k\}_k \subset \mathcal{M}$, then $\mu(\bigcup_{k\in\mathbb{N}} E_k) \leq \sum_{k\in\mathbb{N}} \mu(E_k)$.

Proof. The idea is to put the problem in terms of a disjoint union and use the additive property of measure. Indeed,

$$\bigcup_{k=1}^{\infty} E_k = \coprod_{k=1}^{\infty} \left(E_k \setminus \bigcup_{j=1}^{k-1} E_j \right)$$

and the right-hand side is a disjoint union. By the additive property of measure,

$$\mu\left(E_k \setminus \bigcup_{j=1}^{k-1} E_j \right) = \mu(E_k) - \mu\left(\bigcup_{j=1}^{k-1} E_j \right).$$

Applying the additive property of measure again and using the above results, we have

$$\begin{aligned}
\mu\left(\bigcup_{k=1}^{\infty} E_k\right) &= \mu\left(\coprod_{k=1}^{\infty}\left(E_k \setminus \bigcup_{j=1}^{k-1} E_j\right)\right) \\
&= \sum_{k=1}^{\infty} \mu\left(E_k \setminus \bigcup_{j=1}^{k-1} E_j\right) \\
&= \sum_{k=1}^{\infty} \mu(E_k) - \mu\left(\bigcup_{j=1}^{k-1} E_j\right) \\
&\leq \sum_{k=1}^{\infty} \mu(E_k).
\end{aligned}$$

□

Proposition. (Continuity from Below) If $\{E_k\}_k \subset \mathcal{M}$ with $E_1 \subset \cdots$, then $\mu(\bigcup_{k\in\mathbb{N}} E_k) = \lim_{n\to\infty} \mu(E_n)$.

Proof. The idea is to construct a sequence in $\mathcal{M}$ whose union telescopes and is disjoint. Define $(G_k)_k = (E_k \setminus E_{k-1})_{k\in\mathbb{N},k\geq 2}$ and $G_1 = \varnothing$. By construction, $\bigcup_{k=1}^{n} G_k = E_n \setminus E_0 = E_n$ since it telescopes, and is disjoint $\coprod_{k\in\mathbb{N}} G_k$. By properties of measures,

$$\mu(\lim_{n\to\infty} \bigcup_{k=1}^{n} G_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(G_k) = \lim_{n\to\infty} \mu(\bigcup_{k=1}^{n} G_k) = \lim_{n\to\infty} \mu(E_n).$$

□

Proposition. (Continuity from Above) If $\{E_k\}_k \subset \mathcal{M}$, $\mu(E_k) < \infty$, and $E_1 \supset \cdots$, then $\mu(\bigcap_{k\in\mathbb{N}} E_k) = \lim_{n\to\infty} \mu(E_n)$.

Proof. The idea is to put this intersection in terms of a disjoint union and use additive property of measure. Hence, by the additive property of measure,

$$\mu(E_1) = \mu\left(E_1 \setminus \bigcap_{k\in\mathbb{N}} E_k \coprod \bigcap_{k\in\mathbb{N}} E_k\right) = \mu(E_1 \setminus \bigcap_{k\in\mathbb{N}} E_k) + \mu(\bigcap_{k\in\mathbb{N}} E_k).$$

Now, $\{E_1 \setminus E_k\}_k \subset \mathcal{M}$, and $E_1 \setminus E_2 \subset \cdots$, so we apply the continuity from above property, and the additivity property of measure. Hence, we have

$$\begin{aligned}
\mu(E_1 \setminus \bigcap_{k \in \mathbb{N}} E_k) &= \mu(\bigcup_{k \in \mathbb{N}} E_1 \setminus E_k) \\
&= \lim_{k \to \infty} \mu(E_1 \setminus E_k) \\
&= \lim_{k \to \infty} \mu(E_1) - \mu(E_k) \\
&= \mu(E_1) - \lim_{k \to \infty} \mu(E_k).
\end{aligned}$$

Since $\mu(E_1) < \infty$ and combining the above results,

$$\mu(\bigcap_{k \in \mathbb{N}} E_k) = \lim_{k \to \infty} \mu(E_k).$$

$\square$

Proposition. If $A, B \in \mathcal{M}$, then $\mu(A \bigcup B) = \mu(A) + \mu(B) - \mu(A \bigcap B)$.

Proof. The idea is to put this union in terms of a disjoint union and use additive property of measure. Since $A \bigcup B = A \bigcup B \setminus A$, by properties of measure,

$$\mu(A \bigcup B) = \mu(A \bigcup B \setminus A) = \mu(A) + \mu(B \setminus A).$$

Since $B \setminus A = B \setminus A \bigcap B$,

$$\mu(B \setminus A) = \mu(B \setminus A \bigcap B) = \mu(B) - \mu(A \bigcap B).$$

Combine both results.

$\square$

References

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[2] R. Durett. *Probability: Theory and Examples*. Cambridge, New York. 2010.