CHANGE OF VARIABLES IN INTEGRATION

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1. Change of Variables in Integration

The main idea behind change of variables is changing the integration from the domain to the codomain of some function.

COV 1

See Linear Algebra Done Right for some insight.

Let $f : \mathbb{R}^n \to [0, +\infty]$ be measurable. Let $T \in GL_n(\mathbb{R})$. Since T is linear, $vol(T(\mathbb{R}^n)) = |det(T)|vol(\mathbb{R}^n)$. Hence,

$$\int_{T(\mathbb{R}^n)} f d\mu = |det(T)| \int_{R^n} f \circ T d\mu.$$

COV 1 for differentiation

Let $f : \mathbb{R}^n \to [0, +\infty]$ be measurable.

Let $\varphi : U \to \mathbb{R}^n$ be a diffeomorphism.

Since the differential $\varphi(x) + D\varphi(x) : U \to \mathbb{R}^n$ is a linear approximation to φ near x, shifting by $\varphi(x)$ does not change volume, and since $D\varphi(x)$ is linear,

$$\int_{\varphi(U)} f(y)dy = \int_U f \circ \varphi(x)|det(D\varphi(x))|dx.$$

COV 2

Lebesgue integral version:

This is the pushforward change of variable. The idea is that the way that this pushforward measure is defined, there is no change in volume.

Let $f : X \to [0, +\infty]$ be a measurable map. Let $g : X \to Y$ be a measureable map. If μ is a measure on X, then μg^{-1} is a pushforward measure or image measure on the sigma algebra of Y. Then

$$\int_{g(X)} f(y)d\mu(g^{-1}(y)) = \int_X f(g(x))d\mu(gg^{-1}(x)). = \int_X f(g(x))d\mu(x).$$

Example

The main change of variables for random variables is this. If $f = id$, and $X : \Omega \to \mathbb{R}$ is a RV, then

$$E(X) = \int_{\Omega} X(\omega) dP(\omega) = \int_{X(\Omega)} x dPX^{-1}(x).$$

By definition,

$$\int_{X(\Omega)} x dPX^{-1}(x) = \int_{X(\Omega)} x dP_X(x) = \int_{X(\Omega)} x dF_X(x).$$

COV 3

Riemann-Stieltjes integral version:

Let f :[a, b]$\to \mathbb{R}$ be Riemann integrable. Let $g : [a, b] \to \mathbb{R}$ be continuous and g' is Riemann integrable. Then

$$\int_{[a,b]} f(x) d(g(x)) = \int_{[a,b]} f(x) g'(x) dx$$

Lebesgue integral version:

Let g be absolutely contiuous and it will hold.

Examples:

This one is useful when $g = F_X$ is the distribution function of a random variable.