

# Combining $l^1$ Penalization with Higher Moment Constraints in Regression Models

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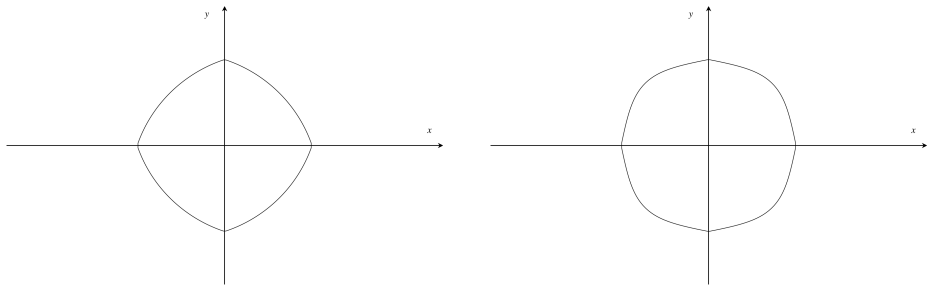
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# Motivation

- In statistics and probability theory it is common to impose moment assumptions on a random variable  $X : \Omega \rightarrow \mathbb{R}^n$  such as  $E(\|X\|^k) < \infty$  for  $k \in \mathbb{R}$ .
- These constraints correspond to the  $L^p$  spaces which allow control over the width and the height of such random variables. Thus, these norms give us some "statistics" about the random variable.
- If statisticians so freely impose such constraints then we should build a tool to allow scientists and researchers to impose similar constraints on their real problems.
- In this project, I built an R package to expand upon the idea of ElasticNet [8] that can hopefully be useful to scientists and researchers.

## Geometric Motivation

Consider for example an Elasticnet [8] penalty  $Q(x) = \frac{1}{2}|x| + \frac{1}{2}|y| + \frac{1}{2}|x|^2 + \frac{1}{2}|y|^2 \leq 1$  shown on the left. Extending this idea, a new penalty  $P(x) = \frac{1}{2}|x| + \frac{1}{2}|y| + \frac{1}{2}|x|^4 + \frac{1}{2}|y|^4 \leq 1$  shown on the right.



It is possible that a scientist or researcher may want the option to "bow" out the feasible set even more.

# The Setup

Define a new penalty to try to impose more "stability" for scientists and researchers (try to extend the Elasticnet [8] idea). Let

$$L_{\lambda}(\beta) = \frac{1}{2} \|y - X\beta\|_2^2 + \lambda P(\beta)$$

with penalty

$$\lambda P(\beta) = \lambda \alpha_0 \|\beta\|_1 + \lambda \sum_{k=1}^5 \alpha_k \|\beta\|_{2k}^{2k}$$

where  $y \in \mathbb{R}^n$ ,  $X \in M_{n \times p}(\mathbb{R})$ , and  $\beta \in \mathbb{R}^p$ ,  $\lambda \in \mathbb{R}_+$  and  $\alpha$ 's are convex combinations or use separate tuning parameters instead.

An important property is that this is a convex and completely separable penalty.

# Algorithm Implementation 1

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**Algorithm 1:** Subgradient Coordinate Method

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Choose  $\beta^0 \in \mathbb{R}^p$  and tolerance  $\delta > 0$ ;

Set  $k \leftarrow 0$

**repeat**

    Randomly choose  $i_k \in \{1, \dots, p\}$

$h_i^k \leftarrow \frac{R}{\sqrt{1+k}}$  for some  $R > 0$  or use  $\frac{R}{\sqrt{1+N}}$  where  $N$  is the length of the algorithm.

$\beta_{i_{k+1}} \leftarrow \beta_{i_k} - h^k g^{i_k}$  where  $g^{i_k} \in (\partial L)_{i_k}$

$k \leftarrow k + 1$

**until** *Until the loss difference  $\Delta L$  is less than  $\delta$ ;*

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# Drawbacks

- Not a descent method, so you cannot do line search.
- No good stopping criterion.
- There are many choices for the subgradient.
- Tends to produce really small values instead of truly sparse solutions.

# A Better Algorithm

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**Algorithm 2:** Proximal Gradient Coordinate Descent

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Choose  $\beta^0 \in \mathbb{R}^p$  and tolerance  $\delta > 0$ ;

Set  $k \leftarrow 0$

**repeat**

    Set the step size  $h^k > 0$  constant, diminishing or by line search.

    Randomly choose  $i_k \in \{1, \dots, p\}$

$\beta_i^{k+1} \leftarrow (\mathbf{prox}_{h^k L})_i(\beta_i^k - h^k \langle X_{i_k}, y - X\beta \rangle)$

$k \leftarrow k + 1$

**until** *Until the Moreau-Yoshida mapping  $M_{h_k, f} < \delta$ ;*

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# Benefits

- A descent method, so line search can be implemented.
- Can be accelerated.
- Good stopping criterion at approximately  $O(p)$  flops.
- Convergence theory is unchanged from differentiable functions.
- Coordinate descent theory is developed by Nesterov (2012).



# Cross-Validation Implementation

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**Algorithm 3:** Warm Start Cross-Validation

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Choose a sequence of Lagrangian dual variables  $\lambda_1, \dots, \lambda_N$ , and initial value  $\beta^0$ .

Order  $\lambda_{(1)}, \dots, \lambda_{(N)}$  descending.

$\beta^{Warm} \leftarrow \beta^0$

**for**  $k \in 1, \dots, N$  **do**

$\beta^k \leftarrow$  Using Cross-Validation with  $\lambda_{(k)}$  warm started with  $\beta^{Warm}$ .

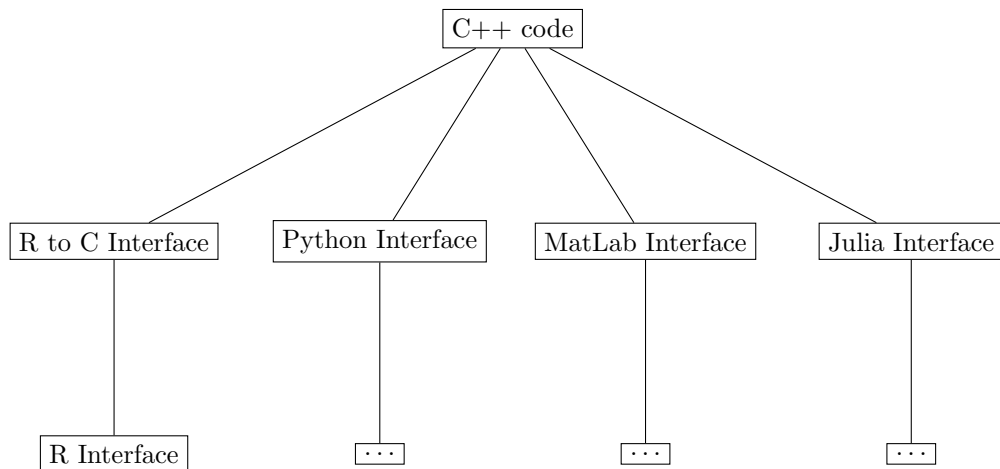
$\beta^{Warm} \leftarrow \beta^k$

**end**

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# Package Architecture

Written entirely in C++ for speed and uses Eigen [6] for linear algebra. It can be interfaced to many other popular languages.



# Penalized Regression on Steroids

Installation:

```
> devtools::install_github("austindavidbrown/pros/R-package")
```

A single fit function with prediction

```
> fit <- pros(X, y, alpha, lambda)
> predict(fit, new_X)
```

A cross-validation fit function with prediction

```
> cv <- cv.pros(X, y, alpha)
> predict(cv, new_X)
```

There is even a [reference manual](#).

# The Boston Housing Dataset Analysis

The Boston Housing dataset is popular from Harrison et al. [1]. There are 13 predictors and the response is the median value of owner-occupied homes in \$1000s.

- The data is randomly split into a training set with 404 observations and a test set with 102 observations.
- The predictor data is standardized.
- Basic 10-fold cross-validation and no special tuning.
- The **glmnet** [7] library was used for the Lasso and the ElasticNet.
- **pros** [3] was used to fit the new penalty.
- The random seed was set to 8989.
- The code is available to you at the **pros** repository [3].

# The Boston Housing Dataset Results

Penalty	Tuning	Test MSE
Lasso Penalty (glmnet)	$\alpha = (1, 0)$	28.52329
ElasticNet Penalty (glmnet)	$\alpha = (1/2, 1/2)$	27.46224
New Penalty (pros)	$\alpha = (1/2, 0, 1/2, 0, 0, 0)$	26.31973
New Penalty (pros)	$\alpha = (1/2, 0, 0, 0, 0, 1/2)$	26.3852
Tuned ElasticNet Penalty (glmnet)	$\alpha = (.25, .75)$	26.80933
New Penalty (pros)	$\alpha = (.25, 0, .75, 0, 0, 0)$	26.25065
Tuned New Penalty (pros)	$\alpha = (.1, 0, .9, 0, 0, 0)$	26.21775

# The Boston Housing Dataset Notes

- It is interesting that the un-tuned 4th moment Penalty and un-tuned Elasticnet both removed age, but have different solutions otherwise.
- The 10th moment penalty produces no sparsity.

# The Prostate Cancer Dataset Analysis

This is a popular dataset from Stamey et al. [2] and analyzed in the ElasticNet paper by Zou and Hastie [8]. There are 8 predictors and the response is the log of the prostate specific antigen.

- The data is split into a training set with 67 observations and a test set with 30 observations.
- The predictor data is standardized.
- The **glmnet** [7] library was used to fit and tune the Lasso and a naive ElasticNet.
- **pros** [3] was used to fit the new penalty.
- 10-fold cross-validation and crude manual tuning were used.
- The code is available to you at the **pros** repository [3].

# The Prostate Cancer Dataset Results

Varying results for **glmnet**: sometimes lower sometimes higher.

Penalty	Tuning	Test MSE
Lasso Penalty (glmnet)	$\alpha = (1, 0)$	0.4558642
ElasticNet Penalty (glmnet)	$\alpha = (1/2, 1/2)$	0.4554527
New Penalty (pros)	$\alpha = (1/5, 0, 0, 0, 0, 4/5), \lambda = 54.02$	0.4442196
New Penalty (pros)	$\alpha = (1/5, 0, 1/5, 1/5, 1/5, 1/5), \lambda = 54$	0.4474082
New Penalty (pros)	$\alpha = (1/2, 0, 0, 0, 0, 1/2), \lambda = 21.05$	0.4441578



# Conclusion

- I think that there are many "good" solutions to real problems.
- We can get "statistics" on these solutions by imposing more norms on the feasible sets building upon the Elasticnet [8] idea.
- I would like to explore even more sparsity inducing penalizations such as

$$\lambda P'(\beta) = \lambda \alpha_1 \|\beta\|_1 + \lambda \alpha_2 \|\beta\|_\infty$$

$$\lambda P''(\beta) = \lambda \alpha_1 \|\beta\|_1 + \lambda \sum_{k=2}^{10} \alpha_k \|\beta\|_k$$

## Things to Address

- Currently, you have to tune a step size. I need to implement a fast line search. This is not too difficult actually just ran out of time.
- Look into convergence analysis.
- C++ offers many random number generators. I use the 64-bit Mersenne Twister by Matsumoto and Nishimura (2000).
- minor bugs like divergence and overflow.
- passing data between R and C in the fastest way.

# References

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- [8] Zou, H. and Hastie, T. (2005). Regularization and variable selection via the elastic net. *Journal of the Royal Statistical Society: Series B*, 67, 301–320.
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