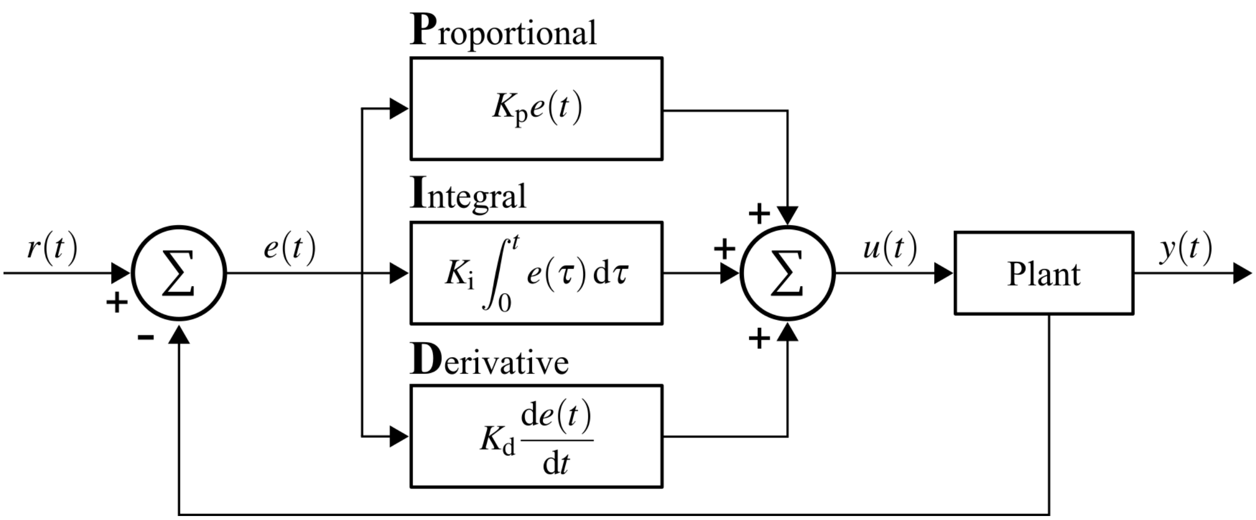
Engineering Control Systems

A Practical Introduction for Mechanical, Civil, and Aerospace Engineers



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Preface

This open-source text is designed to offer the reader a complete text on the basics of control theory within the contest of Mechanical Engineering.

Cover Art

A proportional-integral-derivative (PID) controller is a feed-back controller used in a variety of applications that require continuously modulated control. The Russian American engineer Nicolas Minorsky was arguably the first to develop the theoretical analysis of the PID controller in 1922 while he was researching and designing automatic ship steering for the US Navy. He based his work on watching how a ship's helmsman responds to wave loading on a ship, with a delayed input to the helm that not only considered the current ship course, but also past error and the desired rate of change for the ship. For a helmsman, the goal is stability, not absolute control, which simplifies how one thinks about the challenge of control.

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Source Code

The source code for this text is available at https://github.com/austindowney/Engineering-Control-Systems.

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1.1 Basics of Control System

A control system consisting of interconnected components is designed to achieve a desired purpose. To understand the purpose of a control system, it is useful to examine examples of control systems through the course of history. These early systems incorporated many of the same ideas of feedback that are in use today.

Modern control engineering practice includes the use of control design strategies for improving manufacturing processes, the efficiency of energy use, advanced automobile control, including rapid transit, among others.

System - An interconnection of elements and devices for a desired purpose.

Control System - An interconnection of components forming a system configuration that will provide a desired response.

Process - The device, plant, or system under control. The input and output relationship represents the cause-and effect relationship of the process.

Figure 1.1: Process to be controlled.

Open-Loop Control Systems - utilize a controller or control actuator to obtain the desired response.

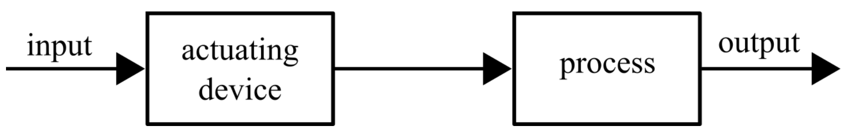


Figure 1.2: Open-loop control system (without feedback).

Closed-Loop Control Systems - utilizes feedback to compare the actual output to the desired output response.

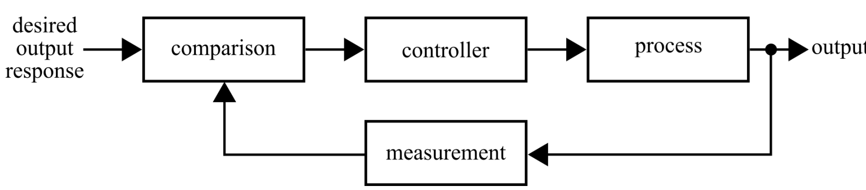


Figure 1.3: Closed-loop control system (with feedback).

Multivariable Control System - a control system with multiple desired response controlled by multiple output variables.

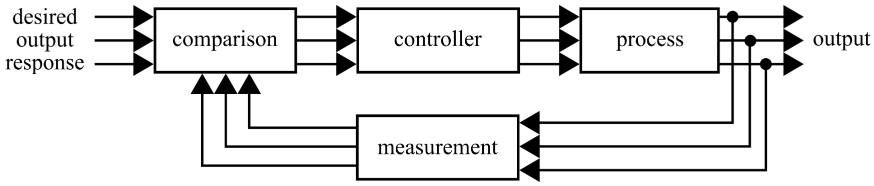


Figure 1.4: Multivariable control system with many inputs and many outputs to the system.

Positive and negative feedback - A positive feedback (also called Regenerative feedback) adds control outputs to the system in the same direction as the original perpetuation while a negative feedback (Degenerative feedback) provides a counteracting force to the original permutations.

1.2 Videos of control systems

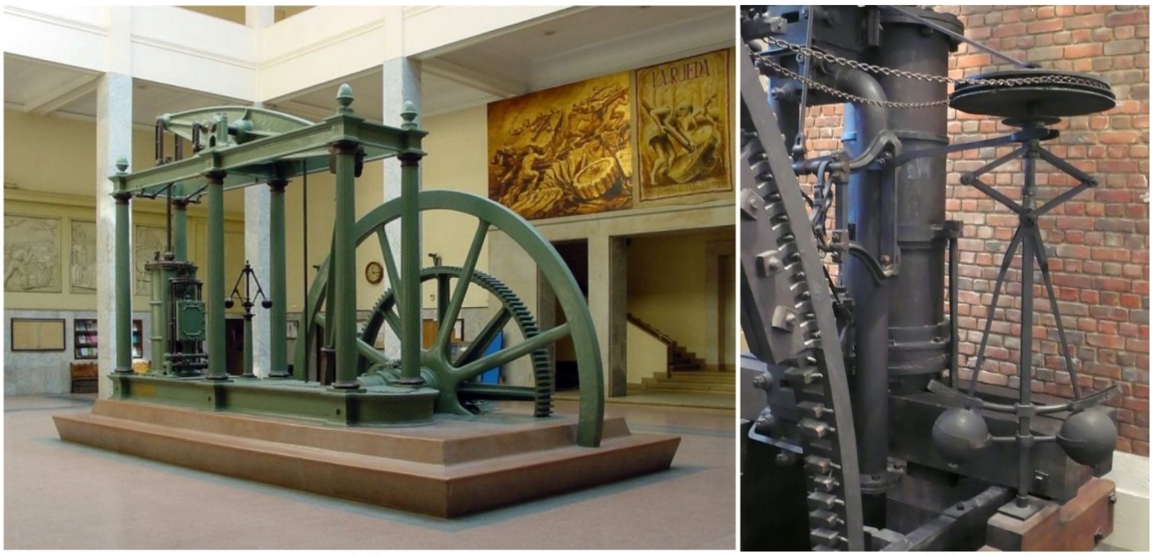
Ball and Plate PID control with 6 DOF Stewart platform
https://www.youtube.com/watch?v=j4OmVLc_oDw&t=95s&ab_channel=FullMotionDynamics
Ball and Plate PID control with 6 DOF Stewart platform
https://www.youtube.com/watch?v=meMWfva-Jio&ab_channel=StepanOzana
SpaceX first landing
https://www.youtube.com/watch?v=1sJlFzUQVmY&ab_channel=BloombergQuicktake

1.3 History

- Greece (BC) - Float regulator mechanism.
- Holland (16th Century) - Temperature regulator.
- 18th Century James Watt's centrifugal governor for the speed control of a steam engine.
- 1920s Minorsky worked on automatic controllers for steering ships.
- 1930s Nyquist developed a method for analyzing the stability of controlled systems.
- 1940s Frequency response methods made it possible to design linear closed-loop control systems.
- 1950s Root-locus method due to Evans was fully developed.
- 1960s State space methods, optimal control, adaptive control.
- 1980s Learning controls are begun to investigated and developed.
- Present and on-going research fields. Recent application of modern control theory includes such non-engineering systems such as biological, biomedical, economic and socio-economic systems

1.4 Examples of control systems

Controllers in the form of centrifugal governors for steam engines set off the industrial revolution by allowing steam engines to produce reliable and controllable power. Figure 1.5 shows centrifugal governor on steam engines.

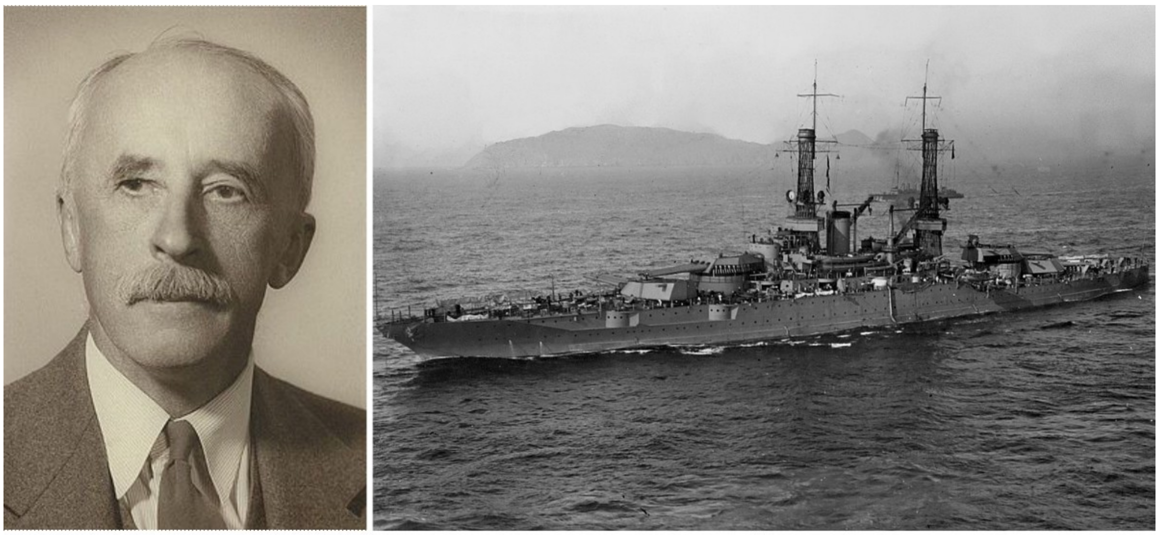


(a) (b)

Figure 1.5: Centrifugal governors on a: (a) Boulton & Watt steam engine from 1788 [1], and; (b) D. Napier & Son from 1832 double-acting steam engine [2].

[1] Dr. Mirko Junge, CC BY 3.0 <https://creativecommons.org/licenses/by/3.0>, via Wikimedia Commons.
[2] Nicolás Pérez, CC BY-SA 3.0 <http://creativecommons.org/licenses/by-sa/3.0/>, via Wikimedia Commons.

Arguably, the Russian American engineer Nicolas Minorsky was the first to develop the theoretical analysis for the three-term control we now call PID. This was done in 1922 while he was researching and designing automatic ship steering for the US Navy. He based his work on watching how a ship's helmsman responds to wave loading on a ship, with a delayed input to the helm that not only considered the current ship course, but also past error and the desired rate of change for the ship. For a helmsman, the goal is stability, not absolute control, which simplifies how one thinks about the challenge of control.



(a) (b)

Figure 1.6: Historical prospective of PID control showing: (a) Portrait of Nicolas Minorsky [1] and (b) the battleship USS New Mexico (BB-40) of the United States Navy which was the first to implement PID control in its steering [2].

[1] Peter Minorsky, grandson of Nicolas Minorsky, CC BY-SA 1.0 <https://creativecommons.org/licenses/by-sa/1.0>, via Wikimedia Commons

[2] U.S. Navy, Public domain, via Wikimedia Commons

2 Systems

2.1 Basic Concepts in Vibrations

The study of vibrations, within the broader field of classical mechanics, is the investigation of oscillations that occur about an equilibrium point. Vibrations, both desired and undesired, are present in all mechanical systems and can be helpful (e.g. a soil sieve, rotary sander) or destructive (e.g. an aircraft frame in resonance). The oscillations that form a vibrating system may be periodic (e.g., pendulum) or random (e.g. turbulence in an airplane), or a combination of the two.

Vibrations impact our daily lives in a variety of ways, from the sound made by banjo strings that vibrates between 140 and 400 Hz to the 4-6 Hz vibration felt by a passenger in a car seat. The consideration of the vibrations and their associated mathematical modeling are an important factor in the design of mechanical systems. In the material that follows, the fundamental theories of vibration are presented and modeled using fundamental physical principles such and Newton's three laws of motion. These models and analyzed using the mathematical tools of calculus and differential equations.

Review 2.1 Newton's three laws of motion:

1. In an inertial frame of reference, an object either remains at rest or continues to move at a constant velocity, unless acted upon by a force.

2. In an inertial reference frame, the vector sum of the forces F on an object is equal to the mass m of that object multiplied by the acceleration of the object: $F = ma$. (It is assumed here that the mass m is constant)

3. When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.

2.2 Single Degree-of-Freedom Systems

In its simplest form, the phenomenon of vibration is the exchange of energy between potential and kinetic energy. Therefore, a vibrating system must have a component that stores potential energy. This component must also be capable of releasing the energy as kinetic energy. This kinetic energy is stored in the movement of a mass where the measure of this movement is the velocity of the system and the continuous interchange between potential and kinetic energy is the vibration of the system. The simplest vibrating systems can be modeled as a single-degree-of-freedom (1-DOF) system. In a 1-DOF system, one variable can describe the motion of a system. Potential examples of 1-DOF systems include:

1. yo-yo

2. pogo stick

3. door swinging on axis

4. throttle (gas pedal)

Variables often used for describing 1-DOF systems are $x(t)$, $y(t)$, $z(t)$, and $\theta(t)$. Examples of 1-DOF systems are presented in figure 2.1 where the assumption of small displacements is made. Note: we will often drop the "(t)" for simplicity in this material.

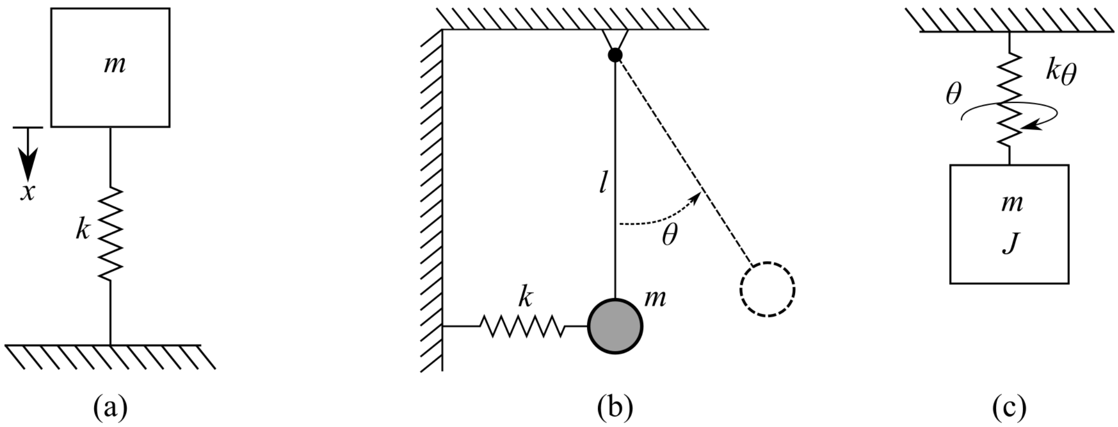


Figure 2.1: Examples of single degree of freedom (DOF) systems showing: (a) a vertical spring-mass system; (b) a simple pendulum; and (c) a rotational spring-mass system.

2.2.1 Spring-Mass Model

> "All models are wrong, but some are useful"
>
> George E.P. Box (1919 - 2013)

Newtonian physics describes the motion of particles in terms of displacement x, velocity $\dot{x}$, and acceleration $\ddot{x}$ vectors. Moreover, from Newton's second law of motion says that the change in the velocity of mass in motion is a product of the force acting on the mass. A simple way to express this phenomenon is though a spring-mass model as presented in figure 2.2. These spring-mass models neglect the mass of the spring and concentrate all the mass of the system into a single point. Note that in this case the force vector and mass-acceleration vectors lie on the same axis and as such are collinear. Therefore, these vectors can be easily treated as scalers simplifying the math used in the modeling of the system.

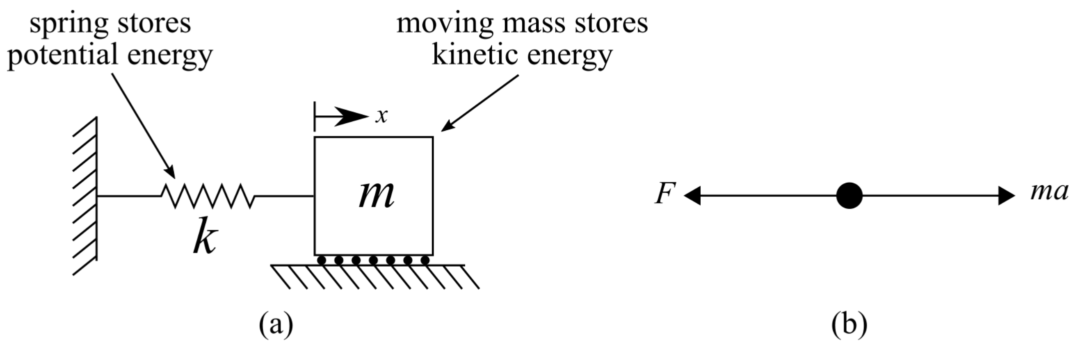


Figure 2.2: A single-degree-of-freedom (1-DOF) spring mass model showing: (a) annotated schematic of a mass-spring system; and (b) the equivalent free-body diagram represented as a point-mass system.

2.2.2 Linear Springs

Springs are mechanical devices that store energy, moreover, ideal spring is a theoretical representation of this mechanical device that is massless and responds with a linear increase in force for a unit increase in displacement (i.e. $F = kx$). For simplicity, the spring in the spring-mass model considered here is assumed aways ideal linear springs. A graphical representation of the idealized linear spring is presented in figure 2.3 where a unit force F applied to the free end of the spring results in a unite displacement x of the spring. The resulting mathematical relationships, $F = kx$, is known as Hooke's Law. Nonlinear springs add considerable complexity to the modeling of spring-mass systems, therefore, these are not considered in this introductory work.

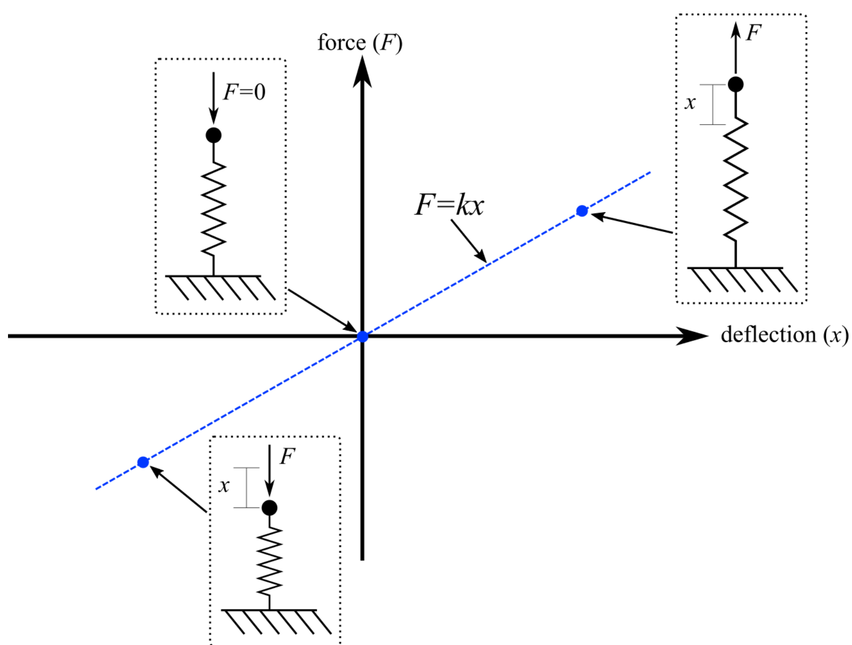


Figure 2.3: Force-displacement plot for a linear spring.

2.2.3 Equation of Motion for an Oscillating System

An Equation of Motion (EOM) is an equation that provides a basis for modeling a vibrating system about its equilibrium point and relates the transfer of the potential energy from the spring to the kinetic energy mass. In developing the EOM we assume that any surfaces are frictionless and as such, no energy is extracted from the vibrating system. Referencing the 1-DOF system in figure 2.4(a), and assuming the mass only moves in the x direction, the only force acting on the mass in the x direction is the force that results from the elongation of the spring as annotated in figure 2.4(b). Therefore, the sum of forces in along the x axis must equal the mass (m) times the acceleration of the mass ($a\dot{x}$).

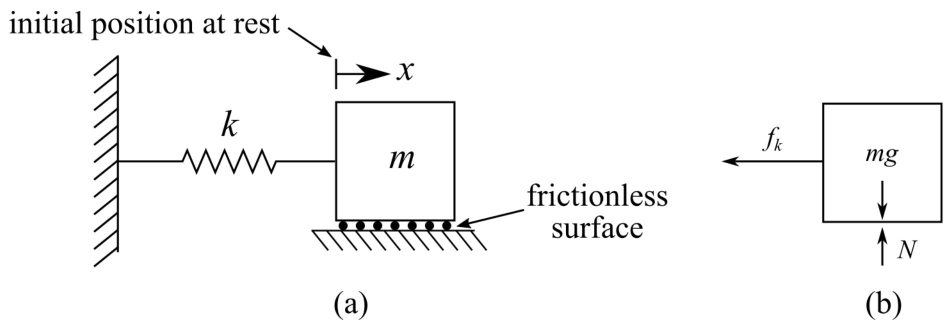


Figure 2.4: A spring mass model of a 1-DOF system showing: (a) a schematic of the system; (b) free-body diagram of the system at its initial position.

Considering that positive displacements are to the right, the standard form of the equation of motion for an undamped system without any excitation is expressed as:

$$s_1\ddot{x} + s_2 x = 0 \tag{1}$$

where s_1 and s_2 are constants to be determined for the specific system. A systematic approach to obtaining free-body diagram (FBD) of a system under vibration can be expressed in three steps:

1. Draw a free-body diagram (FBD) at the system's equilibrium and displaced position (without a displacing force).
2. Apply Newton's second law to both FBDs (equilibrium and displaced).
3. Combine the equations to write the EOM in standard form with the forcing component on the right-hand side. For free vibration, the forcing component is 0.

Solving these three steps for 1-DOF system presented in figure 2.4 results in the EOM:

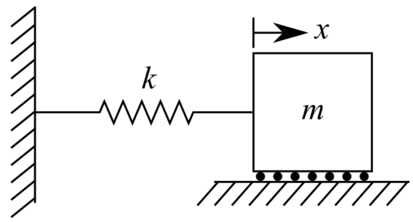
$$m\ddot{x} + kx = 0 \tag{2}$$

Review 2.2 A second-order linear homogeneous differential equation has the form:

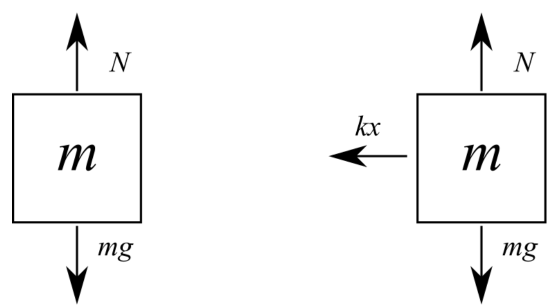
$$a\ddot{x} + b\dot{x} + cx = 0 \tag{3}$$

The EOM for a 1-DOF system under a free vibration is a a second-order differential equation due to acceleration ($\ddot{x}$) being the second derivative of displacement (x) and homogeneous as the forcing function (right-hand side of the equations) is zero. In EOM's current form, $m = k$, $b = 0$, and $c = k$. In future work, b will account for damping in the vibrating system.

Example 2.1 Considering the system:



Step-1 Define the direction of displacement, and draw the FBD for the equilibrium and displaced state.



equilibrium state displaced state

The equation for the equilibrium state is:

$$\overset{+}{\rightarrow}\sum F_x = 0 \tag{4}$$

and in the displaced state:

$$\overset{+}{\rightarrow}\sum F_x = -kx \tag{5}$$

This equation does not equal zero as the FBD does not account for the restoring force.
Step-2 Apply Newton's second law (we want to store energy in the kinetic state) of motion to the sum of forces for the displaced position we get:

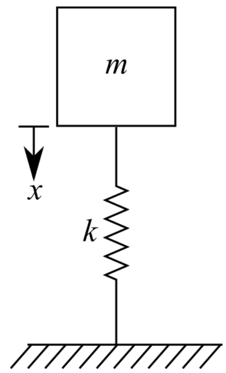
$$ma = m\ddot{x} = \overset{+}{\rightarrow}\sum F_x = -kx \tag{6}$$

$$m\ddot{x} = -kx \tag{7}$$

Step-3 Rearrange in the Equation to construct an EOM:

$$m\ddot{x} + kx = 0 \tag{8}$$

Example 2.2 Some systems will have an initial displacement, as the system will oscillate around this position we need to define the EOM about this position. Considering the system:



Step-1 Define the direction of displacement, and draw the FBD for the equilibrium and displaced state.

equilibrium state displaced state

The equation for the equilibrium state is:

$$+\downarrow \sum F_x = mg - k\delta = 0 \tag{9}$$

and in the displaced state:

$$+\downarrow \sum F_x = mg - k(\delta + x) \tag{10}$$

This equation does not equal zero as the FBD does not account for the restoring force.
Step-2 Apply Newton's second law (we want to store energy in the kinetic state) of motion to the sum of forces for the displaced position we get:

$$m\ddot{x} = +\downarrow \sum F_x = mg - k\delta - kx \tag{11}$$

We can than use the information from the equilibrium state to cancel out some terms, this becomes:

$$m\ddot{x} = -kx \tag{12}$$

Step-3 Rearrange in the Equation to construct an EOM:

$$m\ddot{x} + kx = 0 \tag{13}$$

2.3 Forcing Function

2.3.1 Step Function

A step function is a common loading situation and can represent the dropping of a load into a truck, a car going over a curve, or a motor starting up. The step function (Φ) is also know as the Heaviside function

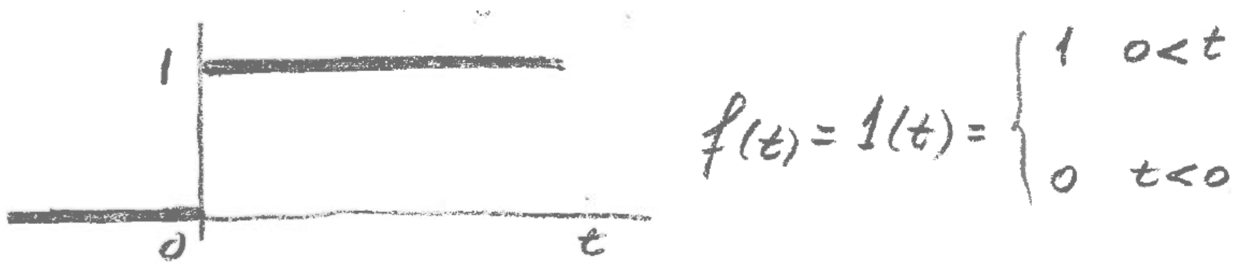


Figure 2.5: Step function.

2.3.2 Pulse Function

Pulse function ($\rho(t;\tau)$) consists of a step up at $t = 0$ followed by a step down at $t = \tau$. The amplitude is $^1/_\tau$ so that the area under the pulse is constant and equal to unity ($A = 1$).

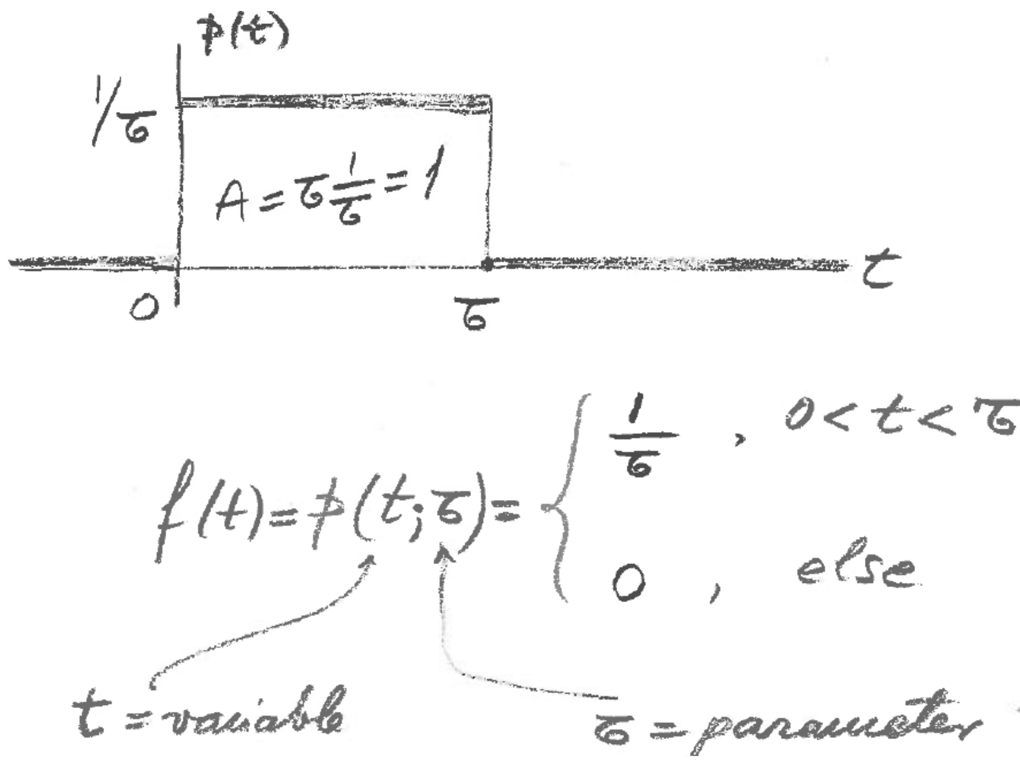
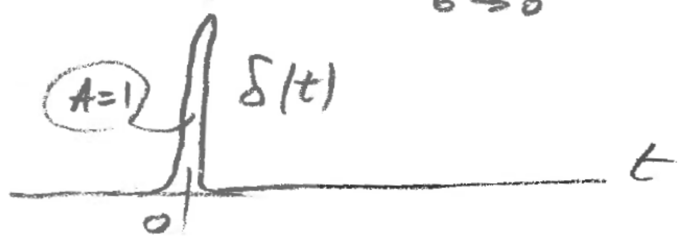


Figure 2.6: Pulse function.

2.3.3 Impulse Function

Shock loads on mechanical systems represent a very common source of vibration. These short-duration forces are also called called an impulse. An impulse excitation is defined as a force that is applied for a very short, or infinitesimal, length of time. An impulse is a nonperiodic force that is represented by the symbol δ. Impulse function ($\delta(t)$) is also known as the "Dirac function", "Dirac delta function", "Dirac impulse function", or "delta function". The impulse function $\delta(t)$ is obtained from the pulse function $\rho(t;\tau)$ by letting τ become infinitesimally small, i.e.,

$$\delta(t) = \lim_{\tau \to 0} p(t;\tau)$$



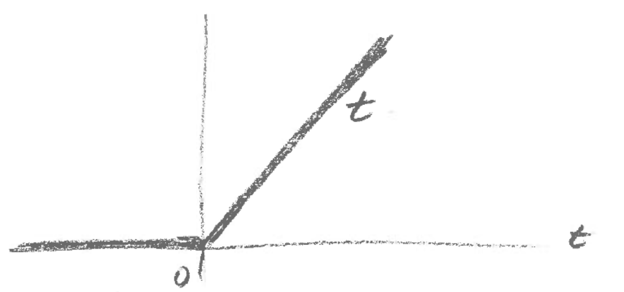
$$A = \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

Figure 2.7: Impulse function.

The area under the *delta* function is equal to unity ($A = 1$) just like the pulse function $\rho(t;\tau)$.

2.3.4 Ramp Function

The ramp function is zero for $t < 0$ and equal to t for $t > 0$.



$$f(t) = \begin{cases} t, & t > 0 \\ 0, & t < 0 \end{cases}$$

Figure 2.8: Ramp function.

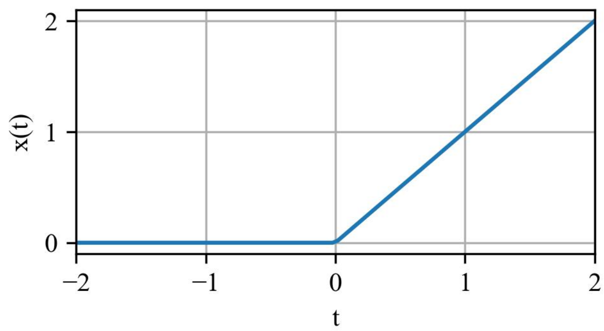


Figure 2.9: Ramp function.

$$f(t) = \begin{cases} t & t > 0 \\ 0 & t < 0 \end{cases} \tag{14}$$

2.4 Introduction to Stability

The stability of a system is explained though "poles" and "zeros". The poles and the zeros of a system determine whether the system is stable, and how well the system performs.

1. A system at a pole has an output that is infinite even though the input to the system was finite (i.e. unstable).

2. A system at a zero has an output that is finite even though the input to the system was infinite (i.e. stable).

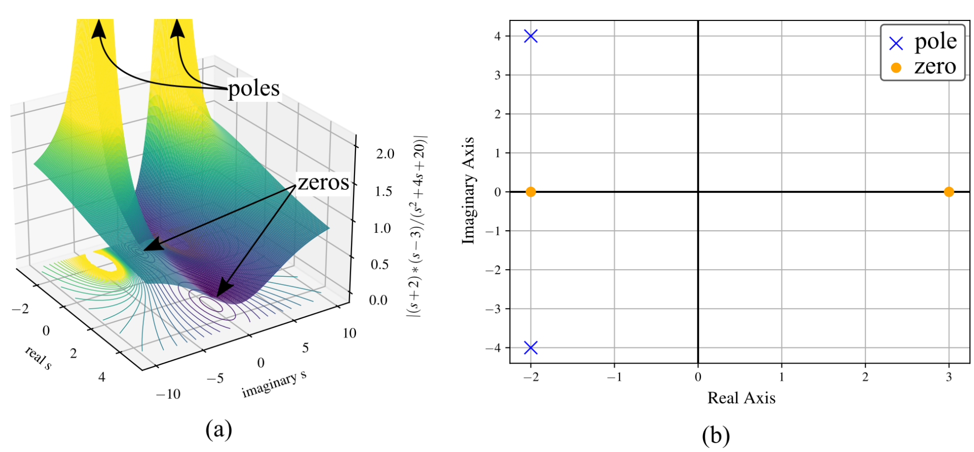


Figure 2.10: Poles and zeros.

Control systems can be designed simply by assigning specific values to the poles and zeros of the system. Physically realizable control systems must have a number of poles greater than the number of zeros. Systems that satisfy this relationship are called Proper. For a rational polynomial transfer function: The poles of a transfer function are defined as the roots of the denominator polynomial of the transfer function. The zeros of a transfer function are simply the roots of the numerator polynomial. Visualizing of poles and zeros in the complex space can be helpful in understanding the stability of the system. Figure 2.10 shows the complex space for an arbitrary transfer function ${(s+2)(s-3)}/{s^2+4s+20}$ where figure 2.10(a) shows the 3D space while figure 2.10(b) just reports the poles and zeros of the transfer function. Figure 2.11 reports the transfer functions for 3 additional transfer functions.

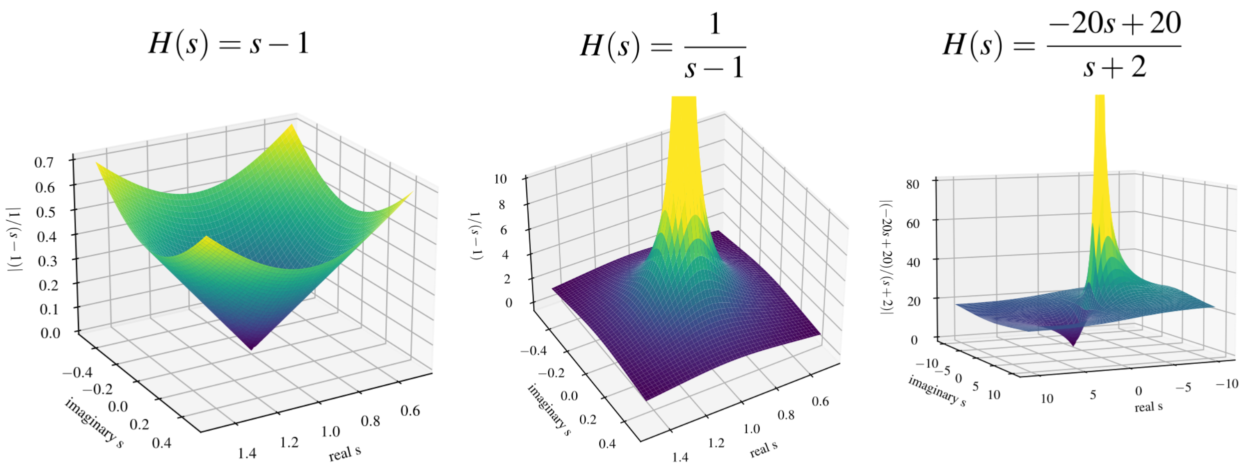


Figure 2.11: Poles and zeros in the complex space, plotted for 3 transfer functions.

2.5 1st Order Systems

2.5.1 Spring-damper mechanism

Consider the system:

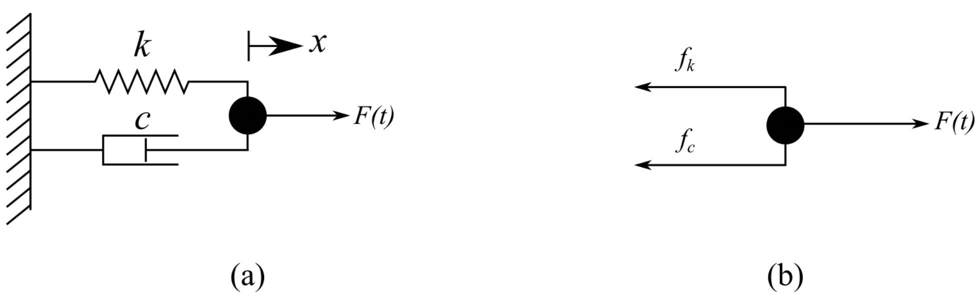


Figure 2.12: Damped 1-DOF system with an external force ($F(t)$) applied, showing: (a) the system configuration; and (b) the free body diagram

Making the assumption that this system does not have any mass, the force balance equation is written as:

$$-kx - c\dot{x} + f^* = 0 \tag{15}$$

this becomes the equation of motion

$$c\dot{x} + kx = f^* \tag{16}$$

normalizing the equation of motion by the stiffness k,

$$\frac{c}{k}\dot{x} + x = \frac{1}{k}f^* \tag{17}$$

define the *time constant* T as

$$T = \frac{c}{k} \tag{18}$$

and the normalized forcing function $f(t) = \frac{1}{k}f^*(t)$. This results in the 1st-order ODE in standard form:

$$T\dot{x}(t) + x(t) = f(t) \tag{19}$$

A general solution for this ODE is written as:

$$x(t) = x_c(t) + x_p(t) \tag{20}$$

where: x_c is called the "complementary solution" and satisfies the homogeneous equation and satisfies the homogeneous equation

$$T\dot{x}_c + x_c = 0 \tag{21}$$

The homogeneous equation is obtained from equation 19 by making zero the right had side. The complementary solution x_c is the free response.

$x_p(t)$ is called the "particular solution" and is used to satisfy the right-hand side of equation 19. Substituting equation 20 into equation 19 yields:

$$T(\dot{x}_c + \dot{x}_p) + x_c + x_p = f(t) \tag{22}$$

or

$$T\dot{x}_c + x_c + T\dot{x}_p + x_p = f(t) \tag{23}$$

As $T\dot{x}_c + x_c = 0$:

$$T\dot{x}_p + x_p = f(t) \tag{24}$$

This shows that:

- $T\dot{x}_c + x_c$ is the solution to the free response.
- $T\dot{x}_p + x_p$ is the solution to the forced response.

2.5.2 Homogeneous Equation (Free Response)

The Homogeneous equation is obtained from equation 19 and setting the right had side to zero:

$$T\dot{x}_c + x_c = 0 \tag{25}$$

To solve, assume:

I really think c should be a or it confuses with damping.

$$\begin{aligned} x_c(t) &= ce^{pt} \\ \dot{x}_c(t) &= pce^{pt} \end{aligned} \tag{26}$$

Inserting equation 26 into equation 25 gives:

$$T(pce^{pt}) + ce^{pt} = 0 \tag{27}$$

canceling out like terms results in the "characteristic equation":

$$Tp + 1 = 0 \tag{28}$$

therefore:

$$p = \frac{-1}{T} \tag{29}$$

where this is a "pole" of the system. Inserting equation 29 into equation 26 results in the "free response" of the system:

$$x_c(t) = ce^{-t/T} \tag{30}$$

as shown in figure 2.13. Note that the lack of mass in the system prevents it from oscillate as there is no way for it to hold kinetic energy, and therefore no way for it to exchange energy between potential and kinetic forms.

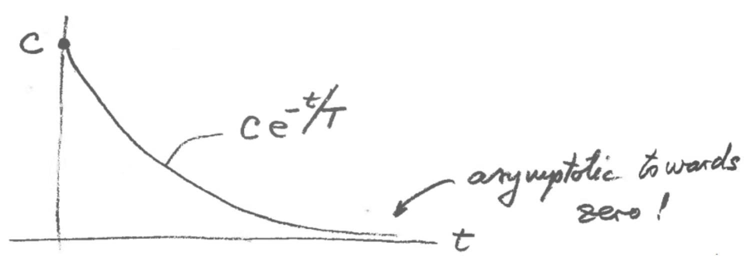


Figure 2.13: Free decay of a 1-DOF system

2.5.3 Stability of 1^{st}-order systems

Using the general free response form eq 30 where p is a root of the charismatic equation (equation 28) and is called the "pole". The stability of the system is dictate by the sign of of p, i.e., its location in the complex p plane.

NOTE

A **Stable System** is a system where if a disturbance is applied, then the system returns to its initial state.

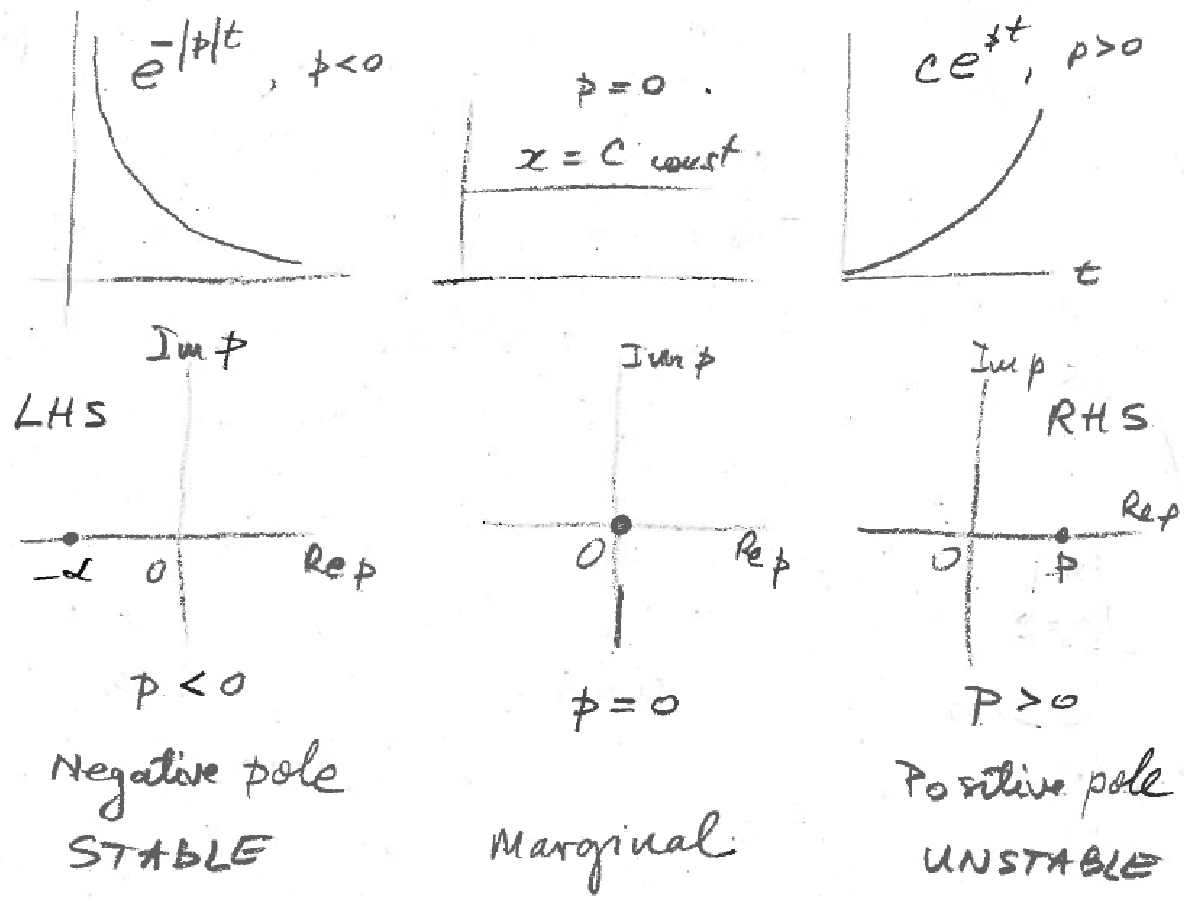


Figure 2.14: Stability of a first order system for different p values.

2.5.4 Stability of 1st-order Systems Under Forced Response

The total solution of equation 19 is given as:

$$x(t) = x_c(t) + x_p(t) \tag{31}$$

or as:

$$x(t) = ce^{-t/T} + x_p(t) \tag{32}$$

The constant c and the function $x_p(t)$ have to be determined depending on initial conditions and the forms of $f(t)$. In control theory, the initial conditions are usually assumed go be zero.

This is incomplete and needs more context.

2.6 2nd Order Systems

Consider the system:

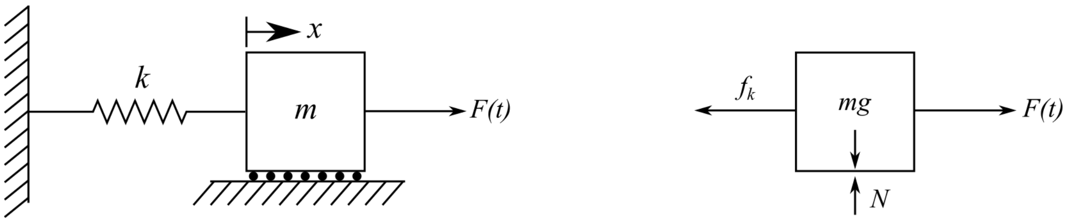


Figure 2.15: 1-DOF system with an external force ($F(t)$) applied, showing: (a) the system configuration; and (b) the free body diagram

Newton's second law of motion states:

$$m\ddot{x} = -k + f^* \tag{33}$$

this can be rearranged to the standard equation of motion for a system with forced vibrations:

$$m\ddot{x}(t) + kx(t) = f^* \tag{34}$$

It is an 2^{nd}-order inhomogeneous ordinary differential equation in time t. Similar to the solutions to the EOM for a 1^{st}-order system, the solution for a 2^{nd}-order system ae made of two parts, the "complementary solution" and the "particular solution". Again, this is written as:

$$x(t) = x_{\text{c}}(t) + x_{\text{p}}(t) \tag{35}$$

where $x_{\text{c}}(t)$ satisfies the homogeneous equation ($m\ddot{x}(t) + kx(t) = 0$) and represent the "free" response of the system. $x_{\text{p}}(t)$ satisfies the complete equation $m\ddot{x}(t) + kx(t) = f^*$ and represents the "forced" response of the system.

2.6.1 Homogeneous Equation (Free Vibration Response)

The homogeneous equation $x_{\text{c}}(t)$ is obtained by setting the forcing function of the EOM to zero:

$$m\ddot{x}(t) + kx(t) = 0 \tag{36}$$

normalizing by the mass returns:

$$\ddot{x}(t) + \frac{k}{m}x(t) = 0 \tag{37}$$

denoting the natural frequency of the system:

$$\omega_{\text{n}}^2 = \frac{k}{m} \tag{38}$$

or more simply (and traditionally)

$$\omega_{\text{n}} = \sqrt{\frac{k}{m}} \tag{39}$$

Inserting the natural frequency of the system back into the EOM yields the homogeneous equation in standard form:

$$\ddot{x} + \omega_{\text{n}}^2 x = 0 \tag{40}$$

To solve the homogeneous equation, assume the form:

$$x(t) = ce^{pt} \tag{41}$$

hence

$$\dot{x}(t) = pce^{pt} = px \tag{42}$$
$$\ddot{x}(t) = p^2ce^{pt} = p^2x$$

inserting these back into the EOM for the 2nd-order system results in:

$$p^2x + \omega_n^2 x = 0 \tag{43}$$

this simplifies to the characteristic equation:

$$p^2 + \omega_n^2 = 0 \tag{44}$$

Rearranging this equations lead to

$$p^2 = -\omega_n^2 \tag{45}$$

and considering that $i = \sqrt{-1}$ to the solution

$$p_{1,2} = \pm i\omega_n \tag{46}$$

inserting these back into the EOM for the 2nd-order system results in:

$$x_c(t) = c_1e^{i\omega_n t} + c_2e^{-i\omega_n t} \tag{47}$$

which is the the complementary solution where c_1 and c_2 are constants to be determine. The complementary solution can also be written as:

$$x_c(t) = c\sin(\omega_n t + \phi) \tag{48}$$

where c is the amplitude of oscillation and ϕ is the phase angle.

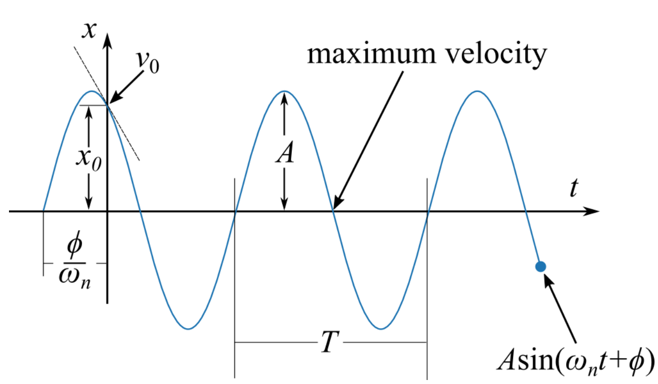


Figure 2.16: Summary of the temporal response for a 1-DOF system.

where x_0 and v_0 are the is the displacement and velocity at t=0 (i.e. the initial displacement).

Review 2.3 Euler's (pronounced oy-ler) formula, named after Swiss engineer and mathematician Leonhard Euler (1707-1783), is a mathematical formula in complex analysis that establishes the fundamental relationship between the trigonometric functions and the complex exponential function. Euler's formula states that for any real number x,

$$e^{j\psi} = \cos(\psi) + j\sin(\psi) \tag{49}$$

where $j = \sqrt{-1}$. This equation can also be expressed as:

$$e^{-j\psi} = \cos(\psi) - j\sin(\psi) \tag{50}$$

This can be expressed in terms of polar coordinates as:

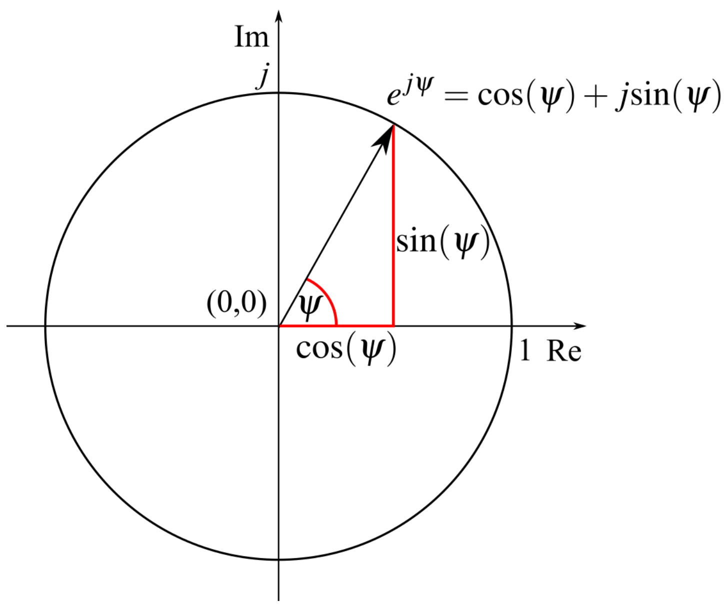


Figure 2.17: Euler's formula illustrated on the unit circle in the complex plane.

Proof

Showing that $x_c(t) = c_1 e^{i\omega_n t} + c_2 e^{-i\omega_n t}$ is the same as $x_c(t) = C\sin(\omega_n t + \phi)$, consider Euler's identity,

$$e^{i\alpha} = \cos(\alpha) + i\sin(\alpha) \tag{51}$$

therefore:

$$\begin{aligned} c_1 e^{i\omega_n t} &= c_1\cos(\omega_n t) + c_1 i\sin(\omega_n t) \\ c_2 e^{-i\omega_n t} &= c_2\cos(\omega_n t) + c_2 i\sin(\omega_n t) \end{aligned} \tag{52}$$

rearranging terms yields:

$$c_1 e^{i\omega_n t} + c_2 e^{-i\omega_n t} = (c_1 + c_2)\cos(\omega_n t) + (c_1 + c_2) i\sin(\omega_n t) \tag{53}$$

which is equivalent to:

$$c_1 e^{i\omega_n t} + c_2 e^{-i\omega_n t} = A\cos(\omega_n t) + B\sin(\omega_n t) \tag{54}$$

where:

$$A = c_1 + c_2; \;\; B = i(c_1 - c_2) \tag{55}$$

Recall that $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$. Therefore, $x_c(t) = c\sin(\omega_n t + \phi)$ can be written as (with a little rearranging):

$$x_c(t) = c\cos(\phi)\sin(\omega_n t) + c\sin(\phi)\cos(\omega_n t) \tag{56}$$

Therefore we can show that:

$$A\cos(\omega_n t) + B\sin(\omega_n t) = c\sin(\omega_n t + \phi) \tag{57}$$

as

$$B = c\cos(\phi); \;\; A = c\sin(\phi) \tag{58}$$

Moreover,

$$A^2 + B^2 = C^2\sin^2(\phi) + C^2\cos^2(\phi) = C^2\left(\sin^2(\phi) + \cos^2(\phi)\right) = C^2 \tag{59}$$

and

$$C = \sqrt{A^2+B^2}, \quad \frac{A}{B} = \frac{c\sin(\phi)}{c\cos(\phi)} = c\tan(\phi), \quad \phi = \tan^{-1}\frac{A}{B} \tag{60}$$

The Standard form of Inhomogeneous Equation can be found, starting at the EOM:

$$m\ddot{x} + kx = f^* \tag{61}$$

divide by m, therefore:

$$\ddot{x} + \frac{k}{m}x = \frac{1}{m}f^* \tag{62}$$

Recall that $\frac{k}{m} = \omega_n^2$ and define the normalized forcing function as $f(t) = \frac{1}{k}f^*$, this results in:

$$\frac{1}{m}f^*(t) = \frac{1}{m}kf(t) = \omega_n^2 f(t) \tag{63}$$

inserting this back into the the EOM yields a 2$^{\text{nd}}$-order inhomogeneous ODE in standard form:

$$\ddot{x} + \frac{k}{m}x = \omega_n^2 f(t) \tag{64}$$

2.7 Forced Vibration Response

The total vibration $x(t)$ of the system can be explained as:

$$x(t) = x_c(t) + x_p(t) = C\sin(\omega_n t + \phi) + x_p(t) \tag{65}$$

inserting the first part of this expression for $x(t)$ into equation 64 yields:

$$\ddot{x}_\mathrm{c} + \ddot{x}_\mathrm{p} + \omega_\mathrm{n}^2(x_\mathrm{c} + x_\mathrm{p}) = \omega_\mathrm{n}^2 f(t) \tag{66}$$

assuming that the complementary solution equals zero, this expression simplifies to:

$$\ddot{x}_\mathrm{p} + \omega_\mathrm{n}^2 x_\mathrm{p} = \omega_\mathrm{n}^2 f(t) \tag{67}$$

This expressions requires that we find c, ϕ, and $x_\mathrm{p}(t)$; this is not easy.

2.8 Spring-mass-damper oscillator

Consider the system:

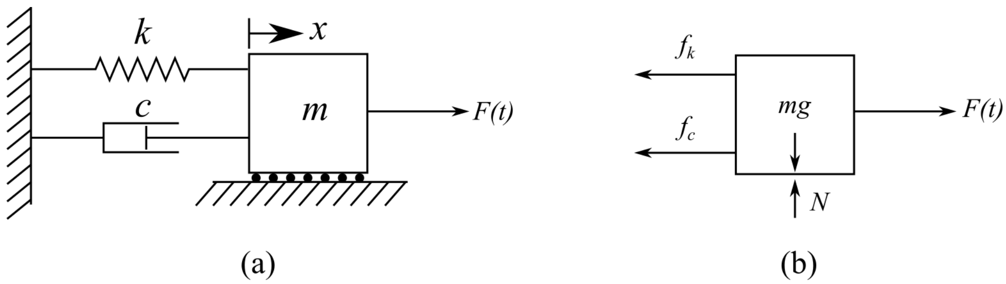


Figure 2.18: Damped 1-DOF system with an external force ($F(t)$) applied, showing: (a) the system configuration; and (b) the free body diagram

Newtons second law of motion tells us

$$m\ddot{x} = -kx - c\dot{x} + f^* \tag{68}$$

therefore, we can write the EOM as:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f^*(t) \tag{69}$$

for the considered damped forced vibration problem. This is a 2^{nd} order inhomogeneous ODE. As before, the solution of $x(t)$ for equation 69 is the sum of the complementary and particular solution where the complementary solution satisfies the homogeneous equation while the particular solution satisfies the inhomogeneous equation.

To solve the homogeneous equation, set the right hand side of equation 69 to zero to get the equation for damped free vibration:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0 \tag{70}$$

Again, assume $x(t) = Ce^{pt}$, therefore,

$$\begin{aligned} \dot{x}(t) &= pce^{pt} = px \\ \ddot{x}(t) &= p^2ce^{pt} = p^2x \end{aligned} \tag{71}$$

Putting these expressions back into the EOM yields:

$$mp^2x + cpx + kx = 0 \tag{72}$$

or the characteristic equation

$$mp^2 + cp + k = 0 \tag{73}$$

solving this expression leads to:

$$p_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\frac{c}{2m} - \frac{k}{m}} \tag{74}$$

or:

$$p_{1,2} = -\frac{c}{2m} \pm i\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} \tag{75}$$

The critical damping value c_{cr} for the system is defined as the value of c that results in a 0 for the radicand (number under the square root). Therefore, we need:

$$\frac{k}{m} - \left(\frac{c_{\mathrm{cr}}}{2m}\right)^2 = 0 \tag{76}$$

$$c_{\mathrm{cr}} = \frac{km^2k}{m} = 4mk \tag{77}$$

this results in:

$$c_{\mathrm{cr}} = 2\sqrt{mk} \tag{78}$$

The damping ratio ζ is the ratio between the damping c and the critical damping ratio c_{cr}, i.e.,

$$\zeta = \frac{c}{c_{\mathrm{cr}}} = \frac{c}{2\sqrt{mk}} = \frac{c}{2m\omega_{\mathrm{n}}} \tag{79}$$

Recall that the natural frequency of a system is $\omega_{\mathrm{n}} = {}^{k}/_{m}$. Damping slightly reduces the natural frequency of the system, resulting in the damped natural frequency ω_{d}

$$\omega_d = \omega_{\mathrm{n}}\sqrt{1 - \zeta^2} \tag{80}$$

Using this expression for the damped natural frequency, the poles of the system can be identified as:

$$\begin{aligned} p_{1,2} &= -\zeta\omega_{\mathrm{n}} \pm i\sqrt{\omega_{\mathrm{n}}^2 - \zeta^2\omega_{\mathrm{n}}^2} \\ &= -\zeta\omega_{\mathrm{n}} \pm i\sqrt{1 - \zeta^2} \\ &= -\zeta\omega_{\mathrm{n}} \pm i\omega_{\mathrm{d}} \end{aligned} \tag{81}$$

The damped free vibration response is:

$$x(t) = c_1 e^{(-\zeta\omega_{\mathrm{n}} + i\omega_{\mathrm{d}})t} + c_2 e^{(-\zeta\omega_{\mathrm{n}} - i\omega_d)t} \tag{82}$$

This simplifies to:

$$x(t) = e^{-\zeta\omega_n t}(c_1 e^{i\omega_d t} + c_2 e^{-i\omega_d t}) \tag{83}$$

or using Euler's equation,

$$x_c(t) = Ce^{-\zeta\omega_n t}\sin(\omega_d t + \phi) \tag{84}$$

This equation breaks down into two parts, one for the decay of the system and one for the oscillations as shown below:

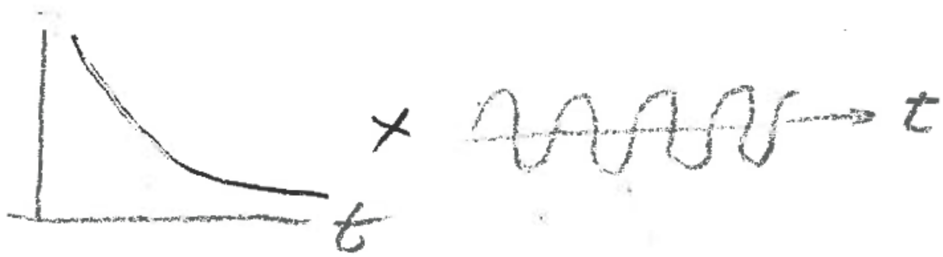


Figure 2.19: Decay and oscillations of the system.

Together, these make the free damped vibration response, as computed by the complementary solution $x_c(t)$

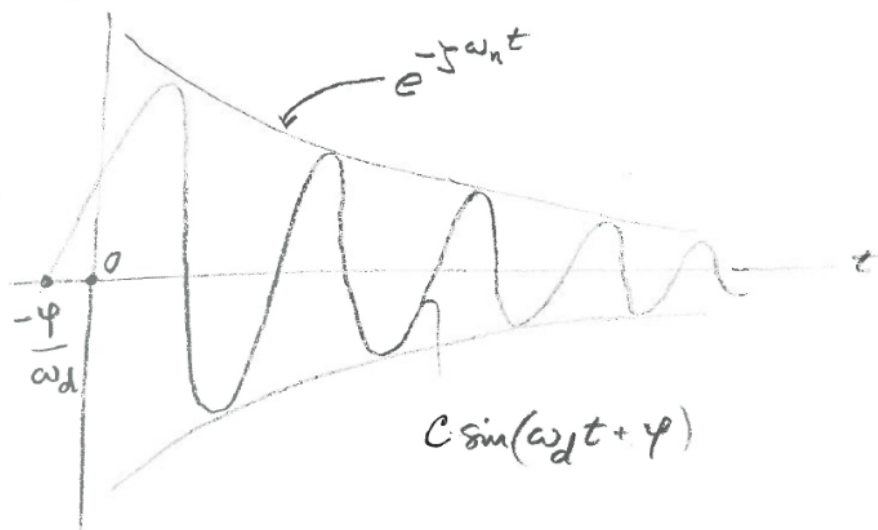


Figure 2.20: Free decay with oscillations.

Generating the standard form of the damped forced vibration problem is done by dividing the EOM by what leads the first term, mainly:

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{1}{m}f^*(t) \tag{85}$$

The expression c/m can be rewritten as:

$$\frac{c}{m} = \frac{\zeta c_{cr}}{m} = \frac{\zeta 2\sqrt{mk}}{m} = 2\zeta\omega_n \tag{86}$$

while the previously defined normalized forcing function is $f(t) = \frac{1}{k}f^*(t)$, this results in a right had side of EOM written as:

$$\frac{1}{m}f^*(t) = \frac{1}{m}kf(t) = \omega_\mathrm{n}^2 f(t) \tag{87}$$

therefore, the standard form of the EOM for the 1-DOF damped forced vibration problem is written as a 2_{nd} order ODE in standard form:

$$\ddot{x} + 2\zeta\omega_\mathrm{n}\dot{x} + \omega_\mathrm{n}^2 x = \omega_\mathrm{n}^2 f(t) \tag{88}$$

The timed resolved solution for the forced vibration problem is:

$$\begin{aligned} x(t) &= x_\mathrm{c}(t) + x_\mathrm{p}(t) \qquad (89) \\ &= e^{-\zeta\omega_\mathrm{n}t} C\sin(\omega_d t + \phi) + x_\mathrm{p}(t) \end{aligned}$$

Putting equation 89 back into equation 88 results in the expression:

I don't like the rest of this, why not just use the method of undetermined coefficients?

$$(\ddot{x}_\mathrm{c} + 2\zeta\omega_\mathrm{n}\dot{x}_\mathrm{c} + \omega_\mathrm{n}^2 x_\mathrm{c}) + \ddot{x}_\mathrm{p} + 2\zeta\omega_\mathrm{n}\dot{x}_\mathrm{p} + \omega_\mathrm{n}^2 x_\mathrm{p} = \omega_\mathrm{n}^2 f(t) \tag{90}$$

where we need to find C, ϕ, $x_\mathrm{p}(t)$ based on the initial conditions of the system. This is not an easy task.

2.9 Stability of 2nd order systems

Recall that the damped free vibration response:

$$x_\mathrm{c}(t) = Ce^{-\zeta\omega_\mathrm{n}t}\sin(\omega_\mathrm{d}t + \phi) \tag{91}$$

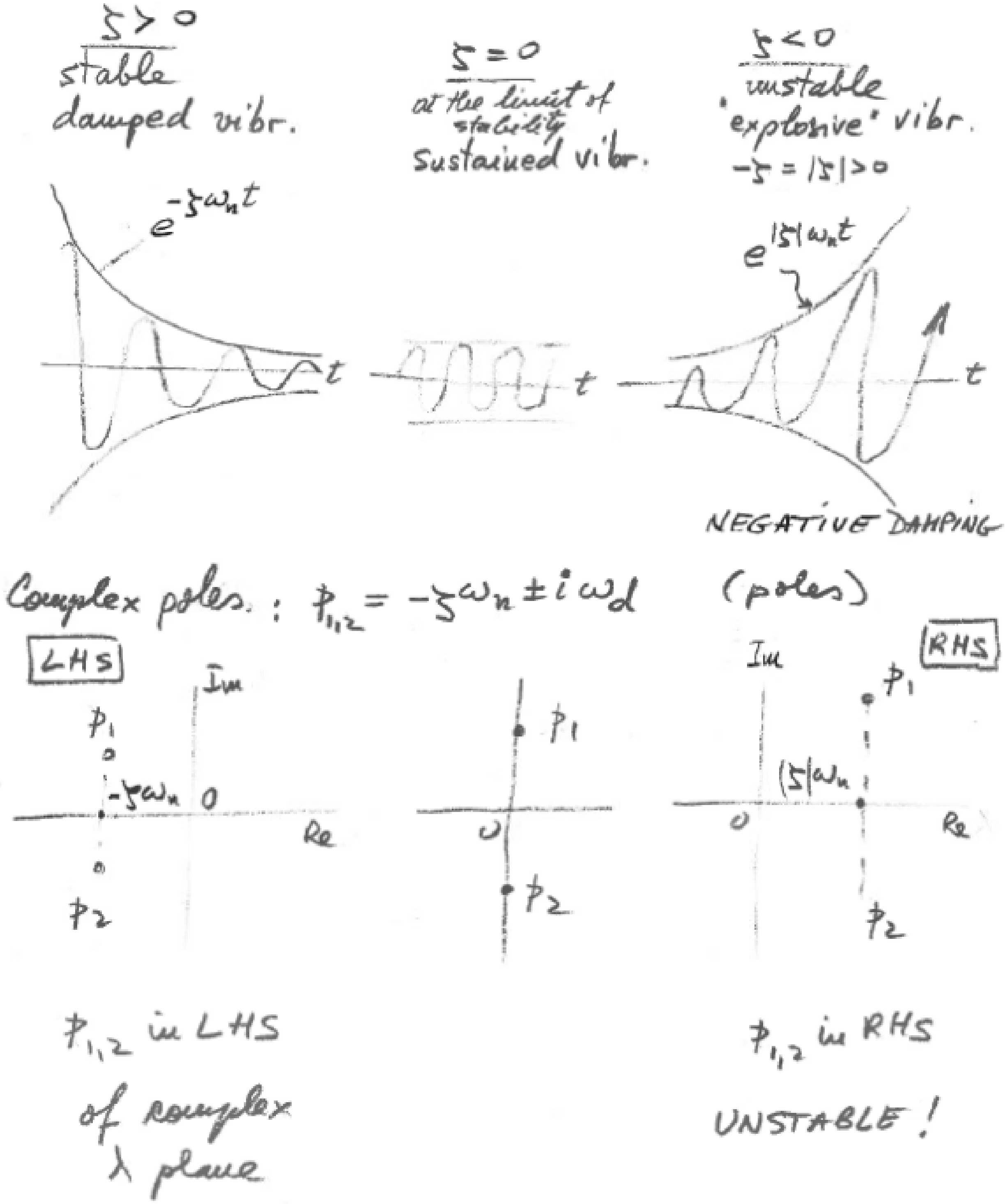


Figure 2.21: Stability of 2$^{\text{nd}}$ order systems.

Underdamped, critical-damped, overdamped 2$^{\text{nd}}$ order systems and vibration response.

$$p_{1,2} = -\zeta\omega_n \pm i\sqrt{1-\zeta^2} \tag{92}$$

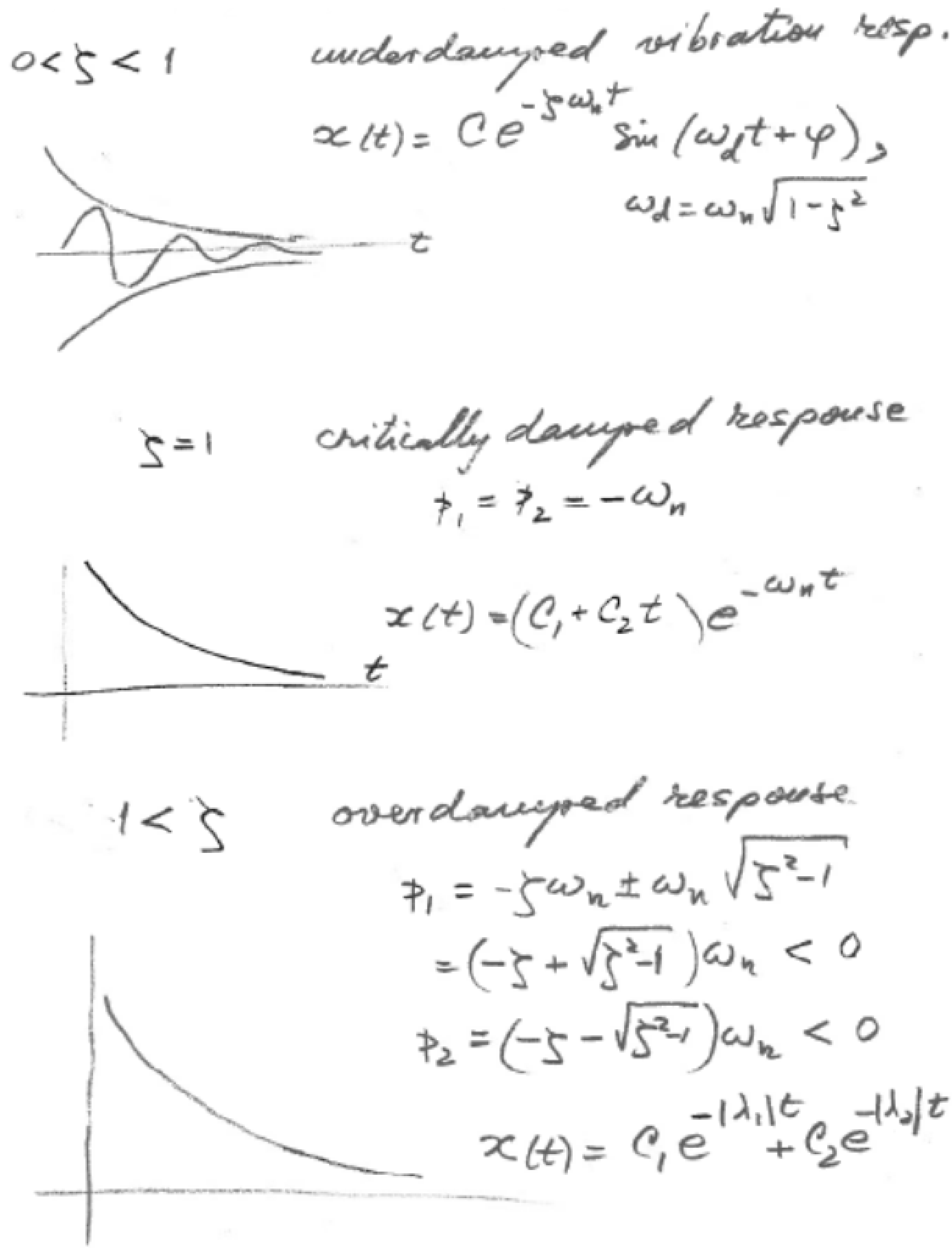


Figure 2.22: vibration responses.

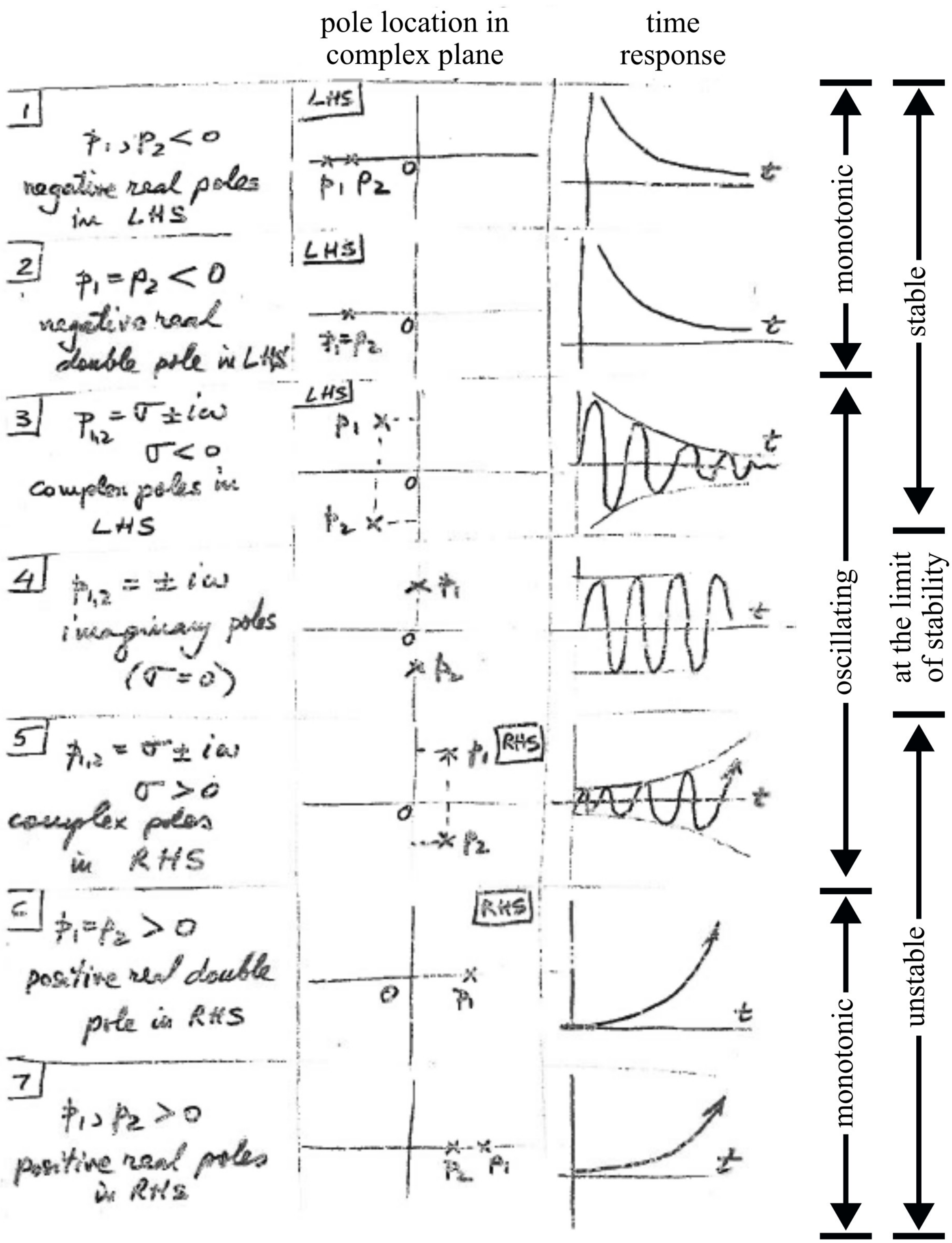


Figure 2.23: 2nd order poles.

Example 2.3 Do example on stability of 2^{nd} order systems

3 Transfer Functions

Thus-far, this text has only considered forced vibrations for 1-DOF systems excited with forcing functions that can be easily expressed using either sin or cos examples. Therefore, the previously developed solutions are only acceptable for systems with known and simple excitations. This chapter will introduce the concept of transfer functions for solving vibration related problems. The transfer function, in particular the Laplace transfer function, is an important tool in the study of vibrations as it allows the practitioner to solve for the temporal response of a system for a variety of inputs using a single approach. Examples of force excitation that can be calculated include using this method include:

- sinusoidal
- base excitation
- impulse
- arbitrary input
- arbitrary periodic input

Consider the following system

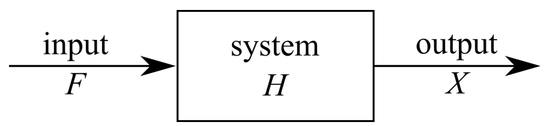


Figure 3.1: Generic system H subjected to an input F and its corresponding output X.

where F is the input, H is the system, and X is the output from the system. This formulation is called the transfer-function approach and is commonly used for the formulation and solution of dynamic problems in the control literature. It can also be used for solving the various forced-vibration problems including those from complex or stochastic inputs.

3.1 Laplace Transforms

Review 3.1 Laplace transforms, or more boradly integral transform, are a procedure for integrating the time (t) dependence of a function into a function of of position or space (s). By transforming the whole whole differential equation from the time domain into a lower order function of space the problem becomes easier to solve as the function can often be manipulated algebraically.

Figure 3.2: Portrait of Pierre-Simon Laplace by Johann Ernst Heinsius (1775).[1]

The Laplace transform is named after mathematician and astronomer Pierre-Simon Laplace (23 March 1749 - 5 March 1827). Pierre-Simon Laplace was one of the greatest scientists of all time and is often considered the French Newton. He taught Napoleon at the École Militaire in 1784, became a count of the empire in 1806, and a marquis in 1817 after the restoration of the monarchy. He is credited with advancements in engineering, mathematics, statistics, physics, astronomy, and philosophy; however, maybe his greatest achievement is not only surviving but benefiting from the change from the Ancien Régime → Bonaparte → Bourbon Restoration.

Of interest to this class is the Laplace transform ($\mathscr{L}[\ ]$) of the function $f(t)$, expressed as $\mathscr{L}[f(t)]$. Here, a Laplace transform is used as a method of solving the differential equations of motion by reducing the computation needed to that of integration and algebraic manipulation.

The definition of the Laplace transform of the function $f(t)$ is:

$$\mathscr{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt \tag{93}$$

where s represents a variable in the complex plane (also called the s-plane) and $f(t) = 0$ for all values of $t < 0$. Here, the s is a complex value. Lastly, the term $F(s)$ is a generic term that represents the input to a system. As this class needs the derivatives of the base function, we will calculate these next:

$$\mathscr{L}\left[\dot{f}(t)\right] = \int_0^\infty \dot{f}(t)e^{-st}dt = \int_0^\infty e^{-st}\frac{d[f(t)]}{dt}dt \tag{94}$$

integration by parts yields:

$$\mathscr{L}\left[\dot{f}(t)\right] = e^{-st}f(t)\Big|_0^\infty + s\int_0^\infty e^{-st}f(t)dt \tag{95}$$

Astutely, it can be noticed that the second term $s\int_0^\infty e^{-st}f(t)dt$ is the input to the system $F(s)$. Therefor, with a little rearranging this becomes:

$$\mathscr{L}\left[\dot{f}(t)\right] = sF(s) - f(0) \tag{96}$$

Taking the derivative of again yields:

$$\mathscr{L}\left[\ddot{f}(t)\right] = s^2F(s) - sf(0) - \dot{f}(0) \tag{97}$$

A few key points of the Laplace transforms are:

- The domain of the problem changes from the real number line (t) to the complex plane (s-plane).
- The integration of the Laplace transform changes differentiation into multiplication.
- The transform procedure is linear. Therefore, the transform of the linear combination of two transforms is the same as the linear transformation of these functions.
- To move from the time domain to the complex number plane we typically use tables of pre-solved integral.
- The function $x(t)$ can be obtained by taking the inverse Laplace transform defined as $x(t) = \mathscr{L}[X(s)]^{-1}$

The Laplace transform can be calculated in symbolic form. In particular interest to this text is the Laplace form of the system input $F(s)$ and output $X(s)$. To expand the symbolic form of the Laplace transform for the system inputs are and for system outputs:

$$\mathscr{L}[f(t)] = F(s) \tag{98}$$

$$\mathscr{L}\left[\dot{f}(t)\right] = sF(s) - f(0) \tag{99}$$

$$\mathscr{L}\left[\ddot{f}(t)\right] = s^2F(s) - sf(0) - \dot{f}(0) \tag{100}$$

here, $f(0)$ and $\dot{f}(0)$ are the initial values of the function $f(t)$. Furthermore, the for system outputs are:

$$\mathscr{L}[x(t)] = X(s) \tag{101}$$

$$\mathscr{L}[\dot{x}(t)] = sX(s) - x(0) \tag{102}$$

$$\mathscr{L}[\ddot{x}(t)] = s^2X(s) - sx(0) - \dot{x}(0) \tag{103}$$

here, $x(0)$ and $\dot{x}(0)$ are the initial values of the function $x(t)$.

[a]Johann Ernst Heinsius, CC BY-SA 4.0 <https://creativecommons.org/licenses/by-sa/4.0>, via Wikimedia Commons

3.2 Transfer Function Method for Solving Vibrating Systems

As mentioned in the introduction to this chapter, a variety of systems can be solved for using the transfer function method. The procedure for using the Laplace transform to solve equations of motion expressed as an inhomogeneous ordinary differential equation is:

1. Take the Laplace transform of both sides of the EOM while treating the time derivatives symbolically.
2. Solve for $X(s)$ in the obtained equation.
3. Apply the inverse transform $x(t) = \mathscr{L}[X(s)]^{-1}$

3.2.1 Free Vibration for Undamped Systems

Consider the undamped single-DOF system:

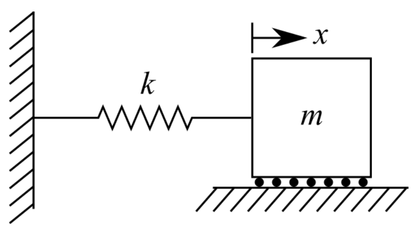


Figure 3.3: A spring mass model of a 1-DOF system.

The EOM for this system is a homogeneous differential equation becasue the right-hand side is equal to zero:

$$m\ddot{x}(t) + kx(t) = 0 \tag{104}$$

Here we will leave the "(t)" for clarity to differentiate the time domain solution from Laplace solution "(s)" in the s-plane, as discussed in review 6.1. The EOM can be rewritten in standard form as:

$$\ddot{x}(t) + \omega_n^2 x(t) = 0 \tag{105}$$

where the initial conditions at $t = 0$ are $x(0) = x_0$ and $\dot{x}(0) = v_0$. Taking the Laplace transforms, in symbolic form using equations 101 - 103, of both sides of the EOM yields:

$$\left[s^2X(s) - sx_0 - v_0\right] + \left[\omega_n^2 X(s)\right] = 0 \tag{106}$$

using equations 101 and 103 from section 6.1. Solving for the output of the system $X(s)$ yields:

$$X(s) = \frac{sx_0 + v_0}{s^2 + \omega_n^2} \tag{107}$$

We can expand this form of $X(s)$ to obtain equations listed in our Laplace Transform table:

$$X(s) = \frac{sx_0}{s^2 + \omega_n^2} + \frac{v_0}{s^2 + \omega_n^2} \cdot \frac{\omega_n}{\omega_n} \tag{108}$$

This becomes:

$$X(s) = x_0 \frac{s}{s^2 + \omega_n^2} + \left(\frac{v_0}{\omega_n}\right) \cdot \frac{\omega_n}{s^2 + \omega_n^2} \tag{109}$$

Next, using the inverse Laplace transform $x(t) = \mathscr{L}[X(s)]^{-1}]$ and the two following Laplace transforms (#5 and #6):

$$f(t) \text{ is } \cos(\omega t) \text{ when } F(s) \text{ is } \frac{s}{s^2 + \omega^2} \tag{110}$$

$$f(t) \text{ is } \sin(\omega t) \text{ when } F(s) \text{ is } \frac{\omega}{s^2 + \omega^2} \tag{111}$$

Therefore, we can obtain the solution for the system output $X(s)$ as:

$$x(t) = x_0 \cos(\omega_n t) + \frac{v_0}{\omega_n} \sin(\omega_n t) \tag{112}$$

The same procedure can be used to calculate the under damped and forced responses. However, when calculating these responses the algebraic solution for $X(s)$, s often contains quotients of polynomials. These Polynomial ratios may not be found in simple Laplace tables and must be solved using the method of partial fractions. An example of this procedure can be found in Appendix B of Inman.

3.2.2 Forced Vibration (Impulse) for damped Systems

Shock loads on mechanical systems represent a very common source of vibration. These short-duration forces are also called called an impulse. An impulse excitation is defined as a force that is applied for a very short, or infinitesimal, length of time. An impulse is a nonperiodic force that is represented by the symbol δ. The response of a system to an impulse load is the same as the system's free response provided that the correct initial conditions are applied. This is illustrated in the following where the applied force $F(t)$ is impulsive in nature (i.e., large magnitude over a very short time).

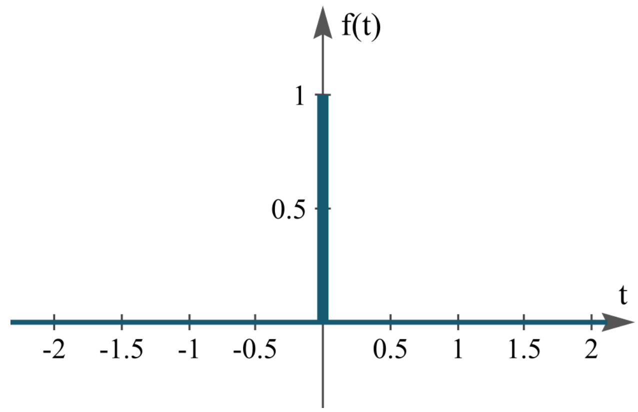


Figure 3.4: An impulse function with the impulse at $t = 0$.

The impulse response function can be solved for analytically, however, we will solve it using the transfer function approach. Here we will consider the under-damped spring-mass system. First, assume that the system is at rest (no initial conditions). Next, we write the EOM as:

$$m\ddot{x} + c\dot{x} + kx = \delta(t) \tag{113}$$

Taking the Laplace transform of both sides of the equation yields

$$m\big(s^2X(s) - sx(0) - \dot{x}(0)\big) + c\big(sX(s) - x(0)\big) + kX(s) = 1 \tag{114}$$

note that the $\mathscr{L}[\delta] = 1$ per #1 in the transform table. However, if we assume zero initial conditions (a system at rest when the impulse happens), the equation simplifies to.

$$ms^2X(s) + csX(s) + kX(s) = 1 \tag{115}$$

or

$$(ms^2 + cs + k)X(s) = 1 \tag{116}$$

Solving this equation for $X(s)$:

$$X(s) = \frac{1}{m} \cdot \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{117}$$

Again, the mass is extracted to develop a formulation that can be found in the Laplace tables. Setting the constraint that $\zeta < 1$ and consulting #10 in the table for Laplace transforms results in:

$$x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t) \tag{118}$$

where this is the general solution for a damped system subjected to an impulse loading function. For the undamped case a solution can be obtained by setting $\zeta = 0$. This Results in the following form for the undamped case:

$$x(t) = \frac{1}{m\omega_n} \sin(\omega_n t) \tag{119}$$

Below is a typical response for both a undamped and underdamped 1-DOF system subject to an impulse response at $t = 0$ seconds.

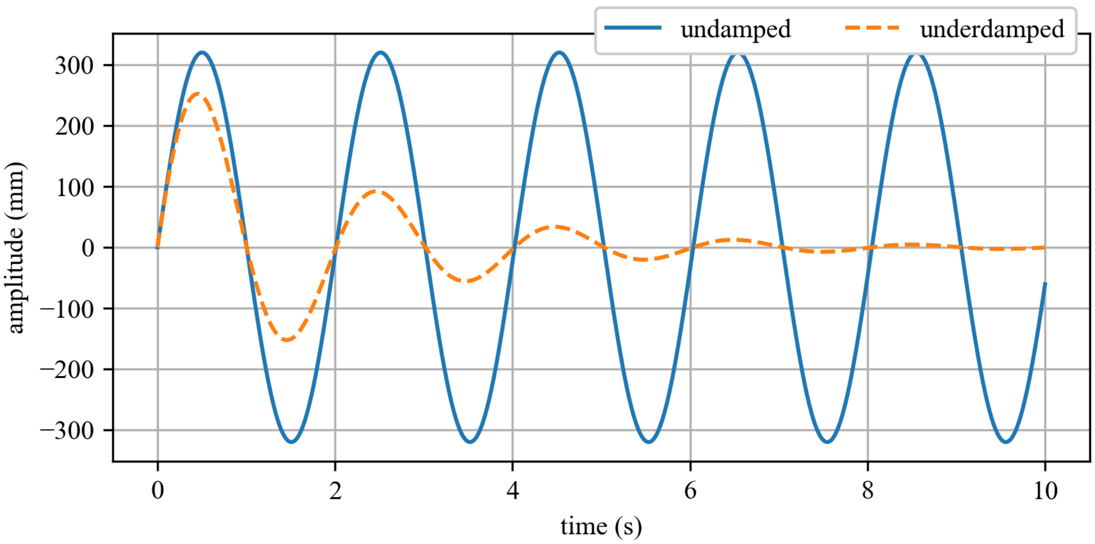


Figure 3.5: Temporal responses from a underdamped and undamped 1-DOF systems to a impulse response function.

3.2.3 Arbitrary Inputs to a System

The time domain response of a system to an arbitrary input force in time can be calculated using a series of impulses as shown in figure 3.6. This method allows the practitioner to easily calculate the response of an arbitrary input to a system using a single expression executed in a "for loop". This type of analysis is often more efficient in terms of programming than more direct methods such as the transfer functions sown in this text.

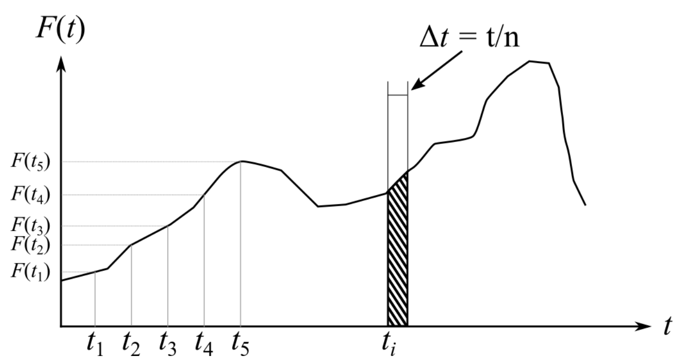


Figure 3.6: Generalized response showing that any signal can be represented as a series of impulse signals.

Example 3.1 In testing, an hammer is used to excite a 1-DOF system with an impact (i.e. impulse), however, the hammer ascendantly impacts the system twice. The first impact has a force of 0.2 N, while the second has a force of 0.1 N and happens 0.1 seconds after the first impact. Plot the response for the double impact. The system has the parameters m = 1 kg, c = 0.5 kg/s, k = 4 N/m.
Solution: First, we can define the forcing function as:

$$F(t) = 0.2\delta(t) + 0.1\delta(t-\tau) \tag{120}$$

where τ is the offset between the first and second impacts. Next, considering that the unit impulse has a magnitude of 1 we can obtain solutions for the first impact by first writing it's EOM:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0.2\delta(t) \tag{121}$$

Taking the Laplace transform of both sides of the equation yields

$$m\big(s^2X(s) - sx(0) - \dot{x}(0)\big) + c\big(sX(s) - x(0)\big) + kX(s) = 0.2 \tag{122}$$

However, assuming zero initial conditions, the equation simplifies to.

$$(ms^2 + cs + k)X(s) = 0.2 \tag{123}$$

Solving this equation for $X(s)$:

$$X(s) = \frac{0.2}{m} \cdot \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{124}$$

Again, consulting #10 in the table for Laplace transforms results in:

$$x_1(t) = \frac{0.2}{m\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t) \tag{125}$$

where this is the general solution for a damped system subjected to an impulse loading function. The second impact can now be solved for using the same method. However, now the time (t) must be offset by (τ) to allow the impact to still be located at $t = 0$ in terms of the second impact. This results in:

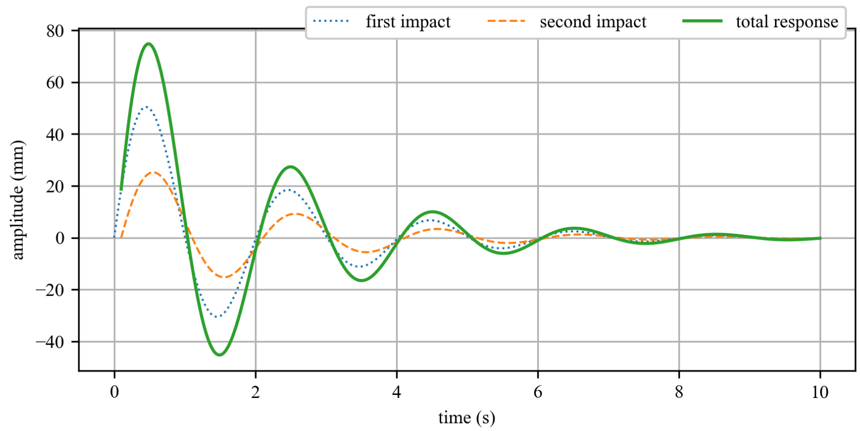
$$x_1(t) = \frac{0.2}{m\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t) \tag{126}$$

$$x_2(t) = \frac{0.1}{m\omega_d} e^{-\zeta\omega_n (t-\tau)} \sin\big(\omega_d (t-\tau)\big) \tag{127}$$

Next, using the knowledge that the systems are linear and that the Laplace transform of a linear combination of two transforms is the same as the linear transformation of these functions we can build the piecewise function:

$$x(t) = \begin{cases} \frac{0.2}{m\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t) & \text{if } t < \tau \\ \frac{0.2}{m\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t) + \frac{0.1}{m\omega_d} e^{-\zeta\omega_n (t-\tau)} \sin\big(\omega_d (t-\tau)\big) & \text{if } \tau \le t \end{cases}$$

For the mass, damping, and stiffness values given above this can be plotted as:



3.3 Laplace method in Controls

Again, consider the Laplace system:

$$\mathscr{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt \tag{128}$$

where:

$$\int_0^\infty = \lim_{p\to\infty} \int_0^p \tag{129}$$

exists. For now, $s \in \mathbb{R}$ (real). Later, we will let $s \in \mathbb{C}$ (complex). For any given $F(s)$, we can find the original $F(t)$ using the pairing method.

$$f(t) \underset{ILT}{\overset{LT}{\rightleftarrows}} F(s) \tag{130}$$

this can also be written as:

$$\text{LT pair} = \begin{cases} F(s) & \mathscr{L}[f(t)] \\ f(t) & \mathscr{L}[F(s)]^{-1} \end{cases} \tag{131}$$

3.3.1 Laplace transform of exponential function

Consider the exponential function,

$$\mathscr{L}\left[e^{p_0 t}\right] = \frac{1}{s-p_0} \tag{132}$$

where $f(t) = e^{p_0 t}$ and $F(s) = \frac{1}{s-p_0}$. As $F(s) \to \infty$, $s \to p_0$, therefore p_0 is a pole of the system. This is shown in Fig. 3.7.

$$\text{LT pair} = \begin{cases} f(t) & e^{p_0 t} \\ F(s) & \frac{1}{s-p_0} \end{cases} \tag{133}$$

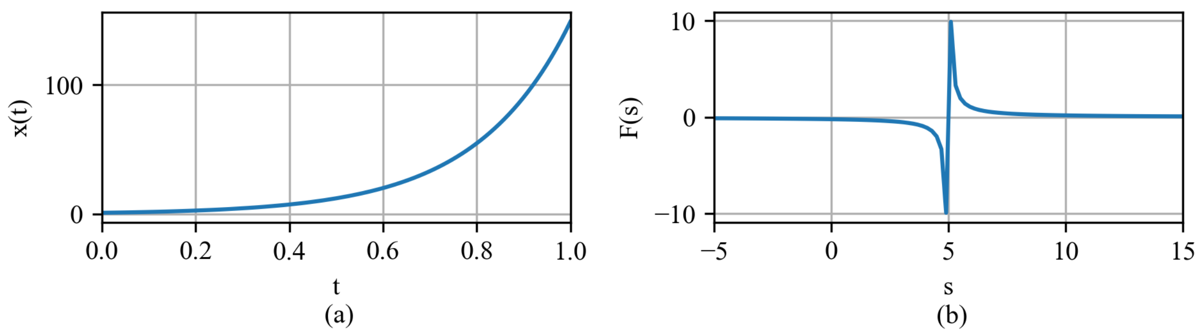


Figure 3.7: Function $e^{p_0 t}$ with $p_0 = 5$; showing the (a) time domain and; (c) the s-space.

Proof

Show that

$$\mathscr{L}\left[e^{p_0 t}\right] = \frac{1}{s-p_0} \tag{134}$$

Again, consider the Laplace system:

$$\mathscr{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt \tag{135}$$

therefore,

$$\int_0^\infty f(t)e^{-st}dt = \int_0^\infty e^{p_0 t}e^{-st}dt = \int_0^\infty e^{-(s-p_0)t}dt \tag{136}$$

Change of variables can be used such that:

$$\begin{aligned}(s-p_0)t &= t^* \\ (s-p_0)dt &= dt^* \\ dt &= \frac{1}{s-p_0}dt^*\end{aligned} \tag{137}$$

therefore

$$\begin{aligned}\int_0^\infty e^{-(s-p_0)}dt &= \int_0^\infty e^{-t^*}\frac{1}{s-p_0}dt^* \\ &= \frac{1}{s-p_0}\int_0^\infty e^{-t^*}dt^* \\ &= \frac{1}{s-p_0}\left(-e^{-t^*}\right)\Big|_0^\infty \\ &= \frac{-1}{s-p_0}\left(e^{-t^*}\right)\Big|_0^\infty \\ &= \frac{-1}{s-p_0}\left[e^{-\infty-e^0}\right] \\ &= \frac{-1}{s-p_0}(-1) \\ &= \frac{1}{s-p_0}\end{aligned} \tag{138}$$

Therefore, it can be shown that:

$$\mathscr{L}\left[e^{p_0 t}\right] = \frac{1}{s-p_0} \tag{139}$$

3.3.2 Laplace transform of step function

$$\text{LT pair} = \begin{cases} f(t) & 1(t), \quad t > 0 \\ F(s) & \frac{1}{s} \end{cases} \tag{140}$$

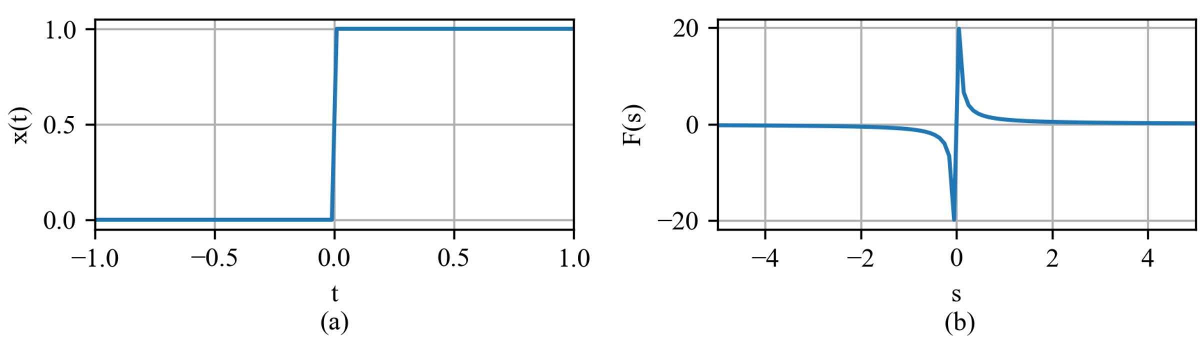


Figure 3.8: Step function; showing the (a) time domain and; (c) the s-space.

Proof

Show that

$$\mathscr{L}[1(t)] = \frac{1}{s} \tag{141}$$

This can be done as:

$$\begin{aligned}\mathscr{L}[1(t)] &= \int_0^\infty 1\cdot e^{-st}dt \\ &= \int_0^\infty e^{-st}dt \\ &= \frac{-1}{s}e^{-st}\Big|_0^\infty \\ &= \frac{1}{s}\end{aligned} \tag{142}$$

3.3.3 Laplace transform of ramp function

$$\text{LT pair} = \begin{cases} f(t) & t, \quad t>0 \\ F(s) & \frac{1}{s^2}\end{cases} \tag{143}$$

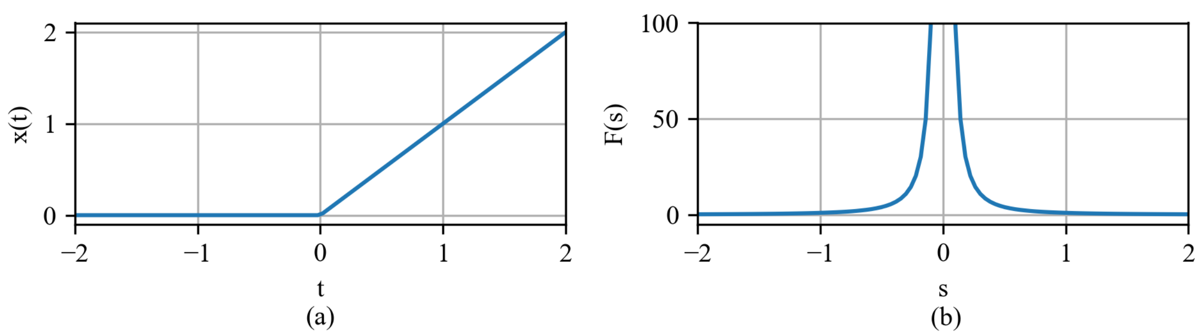


Figure 3.9: Ramp function; showing the (a) time domain and; (c) the s-space.

Proof

Show that

$$\mathscr{L}[t] = \frac{1}{s^2} \tag{144}$$

This can be done as:

$$\mathscr{L}[t] = \int_0^\infty t\cdot e^{-st}dt \tag{145}$$

integration by parts leads to:

$$d[uv] = udv + vdu \rightarrow vdu = d[uv] - udv \tag{146}$$

where

$$\int_a^b vdu = uc\Big|_a^b - \int_a^b udv \tag{147}$$

which leads to:

$$du = e^{-st}; \quad u = \frac{1}{-s}e^{-st} \tag{148}$$

or more simply,

$$du = 1; \quad u = t \tag{149}$$

Using these expressions, we get

$$\begin{aligned}
\int_0^\infty t \cdot e^{-st} dt &= \frac{t}{-s}e^{-st}\Big|_0^\infty - \int_0^\infty \frac{-1}{-s}e^{-st} dt \\
&= \frac{1}{s}\int_0^\infty e^{-st} dt \\
&= \left[\frac{1}{s}\right]\left[\frac{1}{s}\right], \text{ as } \mathscr{L}[1(t)] = \int_0^\infty e^{-st} dt = \frac{1}{s} \\
&= \frac{1}{s^2}
\end{aligned} \tag{150}$$

3.3.4 Laplace transform shifted step function

$$\text{LT pair} = \begin{cases} f(t) & 1(t-\tau) \\ F(s) & e^{-\tau s}\frac{1}{s} \end{cases} \tag{151}$$

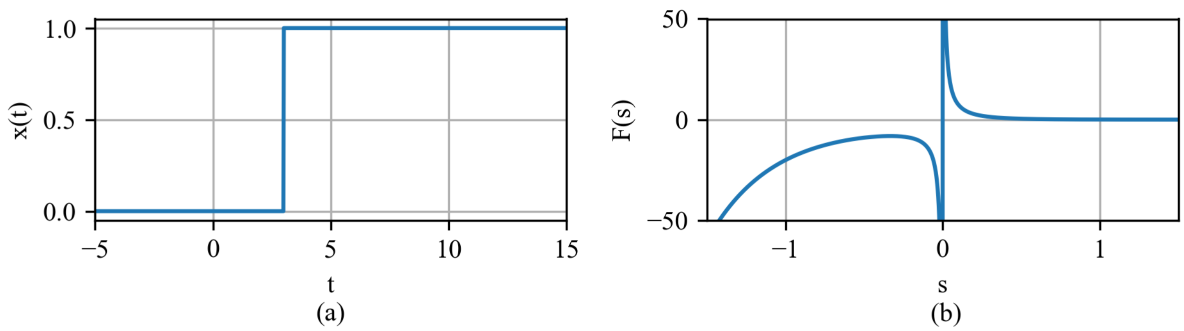


Figure 3.10: Shifted step function; showing the (a) time domain and; (c) the s-space.

Proof

Show that

$$\mathscr{L}[1 \cdot (t-\tau)] = e^{-\tau s}\frac{1}{s} \tag{152}$$

This can be done as:

$$\tag{153}$$

$$\begin{aligned}\mathscr{L}[1\cdot(t-\tau)] &= \int_0^\infty 1(t-\tau)e^{-st}dt \\ &= \int_0^\tau 0\cdot e^{-st}dt + \int_\tau^\infty e^{-st}dt \\ &= \int_\tau^\infty e^{-st}dt\end{aligned} \tag{154}$$

using a change of variable substitution,

$$t^* = t-\tau;\quad t = t^*+\tau;\quad dt^* = dt \tag{155}$$

Therefore

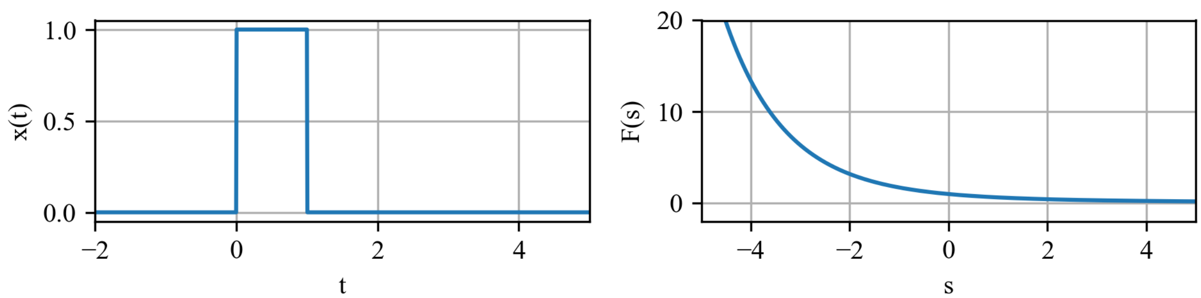
$$\begin{aligned}\int_\tau^\infty e^{-st}dt &= \int_0^\infty e^{-s(t^*+\tau)}dt^* \\ &= e^{-s\tau}\int_0^\infty e^{-s(t+\tau^*)}dt^* \\ &= e^{-s\tau}\frac{1}{s}\end{aligned} \tag{156}$$

connecting the start of proof to the end shows:

$$\mathscr{L}[1\cdot(t-\tau)] = e^{-\tau s}\frac{1}{s} \tag{157}$$

3.3.5 Laplace transform pulse function

$$\text{LT pair} = \begin{cases} f(t) & p(t;\ \tau) \\ F(s) & \frac{1-e^{-s\tau}}{s\tau}\end{cases} \tag{158}$$



(a) (b)

Figure 3.11: Pulse function; showing the (a) time domain and; (c) the s-space.

Proof

Show that

$$\mathscr{L}[p(t;\tau)] = \frac{1-e^{-st\tau}}{s\tau} \tag{159}$$

This can be done by writing the pulse as a step up followed by a step down at $t=\tau$ and scaled to be $\frac{1}{\tau}$, i.e.,:

$$p(t;\tau) = \frac{1}{\tau}1(t) - \frac{1}{\tau}1(t-\tau) \tag{160}$$

given that

$$\mathscr{L}[1(t)] = \frac{1}{s}, \quad \mathscr{L}[1(t-\tau)] = e^{-s\tau}\frac{1}{s} \tag{161}$$

the s domain of $p(t;\tau)$ is expressed as

$$F(s) = \frac{1}{\tau}\frac{1}{s} - \frac{1}{\tau}e^{-s\tau}\frac{1}{s} \tag{162}$$

$$\begin{aligned} F(s) &= \frac{1}{\tau}\frac{1}{s} - \frac{1}{\tau}e^{-s\tau}\frac{1}{s} \\ &= \frac{1-e^{-s\tau}}{s\tau} \end{aligned} \tag{163}$$

connecting the start of proof to the end shows:

$$\mathscr{L}[p(t;\tau)] = \frac{1-e^{-s\tau}}{s\tau} \tag{164}$$

3.3.6 Laplace transform impulse function

$$\text{LT pair} = \begin{cases} f(t) & \delta(t) \\ F(s) & 1 \end{cases} \tag{165}$$

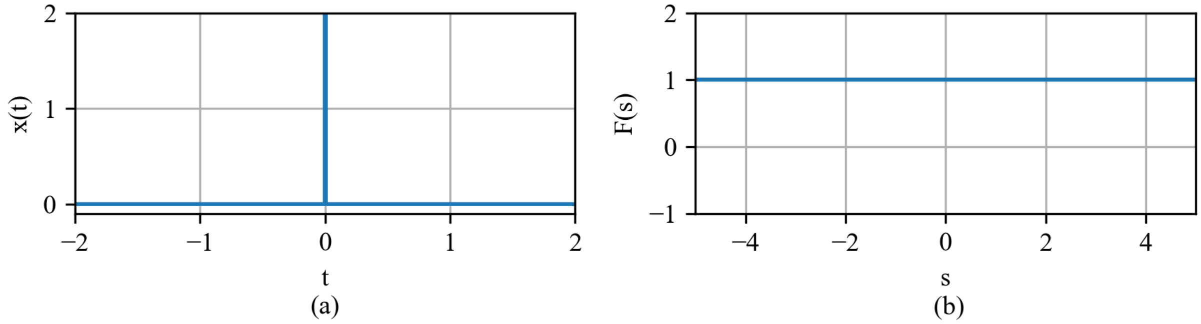


Figure 3.12: Impulse function; showing the (a) time domain and; (c) the s-space.

Proof

Show that

$$\mathscr{L}[\delta(t)] = 1 \tag{166}$$

Consider $\delta(t)$ as the limit of $p(t;\tau)$ as $\tau \to 0$ and take the Laplace Transform.

$$\delta(t) = \lim_{\tau\to 0} p(t;\tau) \tag{167}$$

taking the Laplace transform

$$\begin{aligned}\mathscr{L}[\delta(t)] &= \lim_{\tau\to 0} \mathscr{L}[p(t;\tau)] \\ &= \lim_{\tau\to 0} \frac{1-e^{-st}}{s\tau}\end{aligned} \tag{168}$$

taking the limit of this in $1-1/0 = 0/0$. Applying L'Hospital's (loh-pee-TAHL) rule,

$$\begin{aligned}\lim_{\tau\to 0} \frac{\frac{\partial}{\partial\tau}\left(1-e^{-st}\right)}{\frac{\partial}{\partial\tau}(s\tau)} &= \lim_{\tau\to 0} \frac{-\left(-se^{-s\tau}\right)}{s} \\ &= \lim_{\tau\to 0} e^{-s\tau} \\ &= 1\end{aligned} \tag{169}$$

connecting the start of proof to the end shows:

$$\mathscr{L}[\delta(t)] = 1 \tag{170}$$

Moreover, this can also be done useint the localization property of the delta function. Recall that

$$\int_0^\infty \delta(t)g(t)dt = g(0) \tag{171}$$

then

$$\begin{aligned}\mathscr{L}[\delta(t)] &= \int_0^\infty \delta(t)e^{-st}dt;\ \text{where } g(t) = e^{-st} \\ &= e^{-st}\Big|_{t=0} \\ &= e^0 \\ &= 1\end{aligned} \tag{172}$$

Again, connecting the start of proof to the end shows:

$$\mathscr{L}[\delta(t)] = 1 \tag{173}$$

3.4 Properties of Laplace Transforms

There are a variety of Laplace transforms that can assist in moving between $f(t)$ and $F(s)$. Importantly, the differentiation, integration, time shift, and final value theorems are useful as they allow you to expand a Laplace table by taking the Laplace transform of functions without using the Laplace transform definition. Moreover, these properties are used extensively during control system design so being fluent in them makes you a better designer. A subset of these are shown in Table 1 while some of the more important ones are derived in the following sub sections.

The goal of studying the Laplace transform properties is to be able to Laplace Transform a differential equation with initial conditions and to relate differentiation and integration in the time domain to their counterparts in the s-domain domain.

Table 1: A list of important Laplace Transform theorems used in control.

number	theorem	name
1	$\mathscr{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt$	definition
2	$\mathscr{L}[kf(t)] = kF(s)$	linearity theorem
3	$\mathscr{L}[f_1(t)+f_2(t)] = F_1(s)+F_2(s)$	linearity theorem
4	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$	frequency shift theorem
5	$\mathscr{L}[f(t-t_0)] = e^{-t_0 s}F(s)$	time shift theorem
6	$\mathscr{L}[e^{s_0 t}f(t)] = F(s-s_0)$	time shift theorem
7	$\mathscr{L}[f(at)] = \frac{1}{a}F\frac{s}{a}$	scaling theorem
8	$\mathscr{L}[f'(t)] = sF(s)$, if $f(0)=0$	differentiation theorem
9	$\mathscr{L}[f''(t)] = s^2F(s)$, if $f'(0)=f(0)=0$	differentiation theorem
10	$\mathscr{L}[f^n(t)] = s^nF(s)$, if $f^{n-1}(0)=\cdots=f'(0)=f(0)=0$	differentiation theorem
11	$\mathscr{L}\left[\int_0^t f(t^*)dt^*\right] = \frac{1}{s}F(s)$	integration theorem
12	$f(\infty) = \lim\limits_{s\to 0}[sF(s)]$	final value theorem[1]
13	$f(0+) = \lim\limits_{s\to\infty}[sF(s)]$	initial value theorem[2]

[1]This requires all roots of the denominator of $F(s)$ to have negative real parts, and no more than one can be on the origin of the real axis.

[2]This requires that $f(t)$ be continuous or have a step discontinuity at $t=0$. This does not allow for impulses at $t=0$.

3.4.1 Differentiation Property

The Laplace transforms can be simplified if we assume $f(0)=0$, where these are the terms associated with the initial conditions. This is shown as:

$$\begin{aligned}\mathscr{L}\left[f'(t)\right] &= sF(s), \text{ if } f(0)=0 \\ \mathscr{L}\left[f''(t)\right] &= s^2F(s), \text{ if } f'(0)=f(0)=0 \\ &\vdots \\ \mathscr{L}\left[f^{(n)}(t)\right] &= s^{(n)}F(s), \text{ if } f^{(n-1)}(0)=\cdots=f'(0)=f(0)=0\end{aligned} \tag{174}$$

Recall that differentiation in the time domain becomes multiplication by s in the s-domain.

Proof

Method 1: Show that $\mathscr{L}[f'(t)]=sF(s)$ if $f(0)=0$;

$$\mathscr{L}\left[f'(t)\right]=\int_0^\infty f'(t)e^{-st}dt \tag{175}$$

integration by parts $udv=d(uv)-vdu$ where

$$\begin{aligned}e^{-st} &= u \\ f'(t)dt &= dv \\ f(t) &= v \\ -se^{-st} &= du\end{aligned} \tag{176}$$

leads to

$$\mathscr{L}\left[f'(t)\right]=f(t)e^{-st}\big|_0^\infty-\int_0^\infty f(t)(-se^{-st})dt \tag{177}$$

further simplification results in:

$$\begin{aligned}\mathscr{L}\left[f'(t)\right] &= f(\infty)e^{-\infty}-f(0)+s\int_0^\infty fe^{-st}dt \\ &= sF(s)-f(0) \\ &= sF(s)\end{aligned} \tag{178}$$

where $F(s)=\int_0^\infty fe^{-st}dt$, $e^{-\infty}=0$, and $f(0)=0$. Therefore,

$$\mathscr{L}\left[f'(t)\right]=sF(s) \tag{179}$$

Method 2: Show that $\mathscr{L}[f'(t)] = sF(s)$ if $f(0) = 0$. Denote $f(t) = f'(t)$, therefore:

$$G(s) = \mathscr{L}[g(t)] = \mathscr{L}\left[f'(t)\right] = sF(s) \tag{180}$$
$$\mathscr{L}\left[f''(t)\right] = \mathscr{L}\left[g'(t)\right] = s\mathscr{L}[G(s)] = s^2F(s)$$

by induction it can be shown that

$$\mathscr{L}\left[f'(t)\right] = sF(s) \tag{181}$$

Bottom line: "to differentiate, multiply by s", provided $f(0) = 0$, $f'(0) = 0$, etc. Else, need to subtract them $\mathscr{L}[f'(t)] = sF(s) - f(0)$ etc.

3.4.2 Integration Property

The integral of $f(t)$ is

$$\mathscr{L}\left[\int_0^t f(t^*)dt^*\right] = \frac{1}{s}F(s) \tag{182}$$

where t^* is the dummy variable to integrate over as t is used for the limit. The main take away here is that integration in the time domain becomes division by s in the s-domain.

Proof

Show that $\mathscr{L}\left[\int_0^t f(t^*)dt^*\right] = \frac{1}{s}F(s)$. Denote $g(t) = \int_0^t f(t^*)dt^*$, then:

$$g'(t) = f(t);\ g(0) = 0 \tag{183}$$

taking the Laplace of the left hand side of this expression yields:

$$\mathscr{L}\left[g'(t)\right] = s\mathscr{L}[g(t)] = s\mathscr{L}\left[\int_0^t f(t^*)dt^*\right] \tag{184}$$

the Laplace of the right hand side of this expression yields:

$$\mathscr{L}[f(t)] = F(s) \tag{185}$$

therefore, by equation 183

$$s\mathscr{L}\left[\int_0^t f(t^*)dt^*\right] = F(s) \tag{186}$$

divide by s to get

$$\mathscr{L}\left[\int_0^t f(t^*)dt^*\right] = \frac{1}{s}F(s) \tag{187}$$

Bottom line: "to integrate, divide by s".

3.4.3 Time Shift Property

To shift in t

$$\mathscr{L}[f(t-t_0)] = e^{-t_0 s}F(s) \tag{188}$$

To shift in s

$$\mathscr{L}\left[e^{s_0 t} f(t)\right] = F(s - s_0) \tag{189}$$

A shift in the t domain

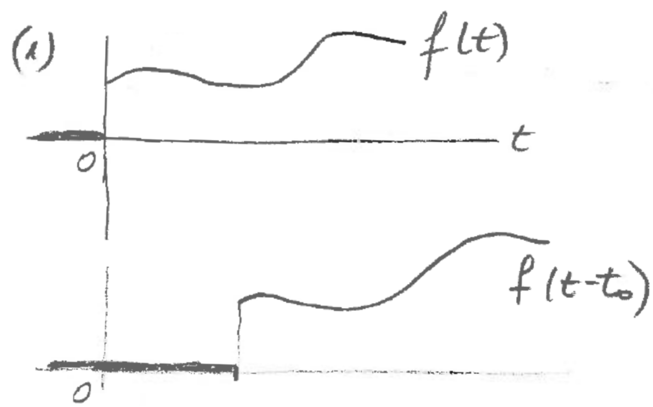


Figure 3.13: Shift of a signal in the t domain.

Note that:
- Function $f(t)$ is zero for $t < 0$ arrangement
- Function $f(t - t_0)$ is zero for $t < t_0$ arrangement

Proof

t-domain

For a shift in t, show that $\mathscr{L}[f(t - t_0)] = e^{-t_0 s} F(s)$

$$\begin{aligned}\mathscr{L}[f(t - t_0)] &= \int_0^\infty f(t - t_0) e^{-st} dt \\ &= \int_0^{t_0} 0 \cdot e^{-st} dt + \int_{t_0}^\infty f(t - t_0) e^{-st} dt \\ &= \int_{t_0}^\infty f(t - t_0) e^{-st} dt\end{aligned} \tag{190}$$

using a change of variables:

$$\begin{aligned} t* &= t - t_0 \rightarrow t = t * + t_0 \\ dt* &= dt \end{aligned} \tag{191}$$

therefore,

$$\begin{aligned}\mathscr{L}[f(t - t_0)] &= f(t^*) e^{-s(t^* + t_0)} dt^* \\ &= e^{-st_0} \int_0^\infty f(t^*) e^{-st^*} dt^* \\ &= e^{-st_0} F(s)\end{aligned} \tag{192}$$

lastly, we can show that:

$$\mathscr{L}[f(t-t_0)] = e^{-t_0 s}F(s) \tag{193}$$

s-domain

For a shift in s, show that $\mathscr{L}[e^{s_0 t}f(t)] = F(s-s_0)$

$$\begin{aligned}\mathscr{L}\left[e^{s_0 t}f(t)\right] &= \int_0^\infty e^{-s_0 t}f(t)e^{-st}dt \\ &= \int_0^\infty f(t)e^{-(s-s_0)t}dt \\ &= F(s-s_0)\end{aligned} \tag{194}$$

3.4.4 Final Value Theorem

This applies to the steady state response $x(\infty)$ which can also be denoted with a subscript ss as x_{ss} where

$$x_{ss} = x(\infty) = \lim_{t\to\infty}[x(t)] = \lim_{s\to 0}[sX(s)] \tag{195}$$

therefore,

$$x_{ss} = \lim_{s\to 0}[sX(s)] \tag{196}$$

Proof

Prove that $x_{ss} = \lim_{s\to 0} sX(s)$. Starting with,

$$\mathscr{L}[\dot{x}(t)] = sX(s), x(0) = 0 \tag{197}$$

by definition

$$\mathscr{L}[\dot{x}(t)] = \int_0^\infty \dot{x}(t)e^{-st}dt \tag{198}$$

combing the last two equations yields

$$\int_0^\infty \dot{x}(t)e^{-st}dt = sX(s) \tag{199}$$

taking the limit as $s \to 0$ results in

$$\lim_{s\to 0}\left[\int_0^\infty \dot{x}(t)e^{-st}dt\right] = \lim_{s\to 0}[sX(s)] \tag{200}$$

but

$$\lim_{s\to 0}[e^{-st}] = e^0 = 1 \tag{201}$$

therefore, the left hand side of equation 200 simplifies to

$$\int_0^\infty \dot{x}(t)dt = x(t)\big|_0^\infty = x(\infty) = x_{ss} \tag{202}$$

inserting this term back into equation 200 yields the desired proof

$$x_{ss} = \lim_{s\to 0}[sX(s)] \tag{203}$$

3.5 Convolutional property of Laplace Transforms

$$\begin{aligned}\mathscr{L}[G(s)\cdot F(s)]^{-1} &= \mathscr{L}[F(s)\cdot G(s)]^{-1} \\ &= \int_0^t f(\tau)g(t-\tau)d\tau \\ &= (f*g)(t) \\ &= x(t)\end{aligned} \tag{204}$$

where $f(\tau)$ is the excitation function and $g(t-\tau)$ is the impulse response of the function. Convolution expresses the system response $x(t)$ to a complicated excitation $f(\tau)$ as an integral using the impulse response $g(t)$ shifted by τ to $t-\tau$. The integral in equation 204 is not easily computed in the time domain, however, Laplace transforms make it easy using a three step process:

1. Calculate $F(s) = \mathscr{L}[f(t)]$, $G(s) = \mathscr{L}[g(t)]$
2. Multiply $F(s)G(s)$
3. Take the inverse Laplace transform to get $x(t)$

3.6 Transfer Function for Response to Random Inputs

Consider the following system

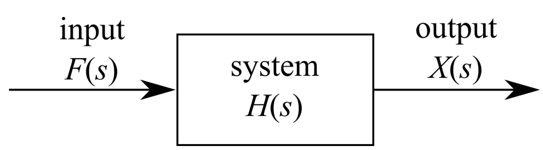


Figure 3.14: Generic block diagram of a system $H(s)$ subjected to an input $F(s)$ and its corresponding output $X(s)$ where the (s) denotes that the considered system is in the s-plane.

where $F(s)$ is the input, $H(s)$ is the system, and $X(s)$ is the output from the system. This formulation is called the transfer-function approach and is commonly used for the formulation and solution of dynamic problems in the control literature. It can also be used for solving the various forced-vibration problems including those from complex or stochastic inputs.

3.6.1 Defining the transfer function H(s)

Again, consider the generic system represented in figure 3.14. For this system representation, $F(s)$ is the Laplace of the transform of the driving force and $H(s)$ is the Laplace transform of the response of the system $h(t)$.

We need to define transfer function $H(s)$ for a generic system. To do this let us show the reasoning behind the transfer function. Here we will show that the output of any system ($x(t)$) can be related to the input of the system ($f(t)$) through a series of polynomial coefficients (a and b). Consider the general n^{th}-order linear, time-invariant differential equation that governs the behavior of the dynamic system.

$$a_n \frac{d^n x(t)}{dt^n} + a_{n-1} \frac{d^{n-1} x(t)}{dt^{n-1}} + \ldots + a_0 x(t) = b_m \frac{d^m f(t)}{dt^m} + b_{m-1} \frac{d^{m-1} f(t)}{dt^{m-1}} + \ldots + b_0 f(t) \tag{205}$$

where $x(t)$ is the output and $f(t)$ is the input. Note that this is similar to the formulation we have had before for the EOM. Taking the Laplace transformation of both side of the above equation yields

$$\begin{aligned} a_n s^n X(s) + a_{n-1} s^{n-1} X(s) + \ldots + a_0 X(s) + \text{initial condition for } x(t) = \\ b_m s^m F(s) + b_{m-1} s^{m-1} F(s) + \ldots + b_0 F(s) + \text{initial condition for } f(t) \end{aligned} \tag{206}$$

It can be seen that this equation is a purely algebraic expression. If we assume the initial conditions to be zero, the equation reduces to the following:

$$(a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0) X(s) = (b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0) F(s) \tag{207}$$

if we rearrange equation 207 to solve for the relationship between the Laplace variables $\big(X(s)$ and $F(s)\big)$ and the algebraic expressions we get:

$$\frac{X(s)}{F(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0} \tag{208}$$

this shows that the ratio of the input algebraic expressions over the output algebraic expressions is equal to the ratio of the output Laplace variable over the input Laplace variable. This shows that we can relate the Laplace variables to the algebraic expressions. Therefore, we can define the transfer function $H(s)$ as:

$$G(s) = H(s) = \frac{X(s)}{F(s)} \tag{209}$$

In a more formal term, the transfer function that is defined as: “The ratio of the Laplace transforms of the output or response function to the Laplace transform of the input or forcing function assuming zero initial conditions”. Note that many texts define $H(s)$ as $G(s)$ and this is simply a matter of syntax.

Equation 209 can be rearranged to show that the output of the system $X(s)$, can be obtained if we know the input $F(s)$ and the transfer function $H(s)$:

$$X(s) = H(s) F(s) \tag{210}$$

3.6.2 Transfer Function method (Steady-State solution)

Considering the forced system:

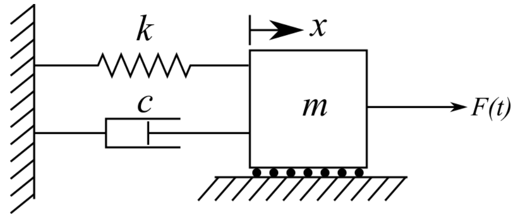


Figure 3.15: A spring-dashpot-mass model of a 1-DOF system with external excitation.

that can be expressed as the equation of motion

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=F_0\cos(\omega t) \tag{211}$$

Here $F_0\cos(\omega t)$, is used at the input but any input will develope the same transfer function as the transfer function is bounded to the system and not the input. From the #6 in the table for Laplace Transforms, we know that

$$\mathscr{L}[\cos(\omega t)]=\frac{s}{s^2+\omega^2} \tag{212}$$

Therefore,

$$F(s)=\frac{F_0 s}{s^2+\omega^2} \tag{213}$$

Ignoring the initial conditions, and therefore considering only the particular solution, and taking the Laplace transform of the EOM equation yields:

$$(ms^2+cs+k)X(s)=\frac{F_0 s}{s^2+\omega^2} \tag{214}$$

where $X(s)$ denotes the Laplace transform of the unknown function $x(t)$ and s is the complex transform variable. Rearranging the above equation for $X(s)$ yields:

$$X(s)=\frac{F_0 s}{(ms^2+cs+k)(s^2+\omega^2)} \tag{215}$$

Now that we have $F(s)$ and $X(s)$ we can obtain $H(s)$ as

$$H(s)=\frac{X(s)}{F(s)}=\frac{F_0 s}{(ms^2+cs+k)(s^2+\omega^2)}\cdot\frac{s^2+\omega^2}{F_0 s}=\frac{1}{ms^2+cs+k} \tag{216}$$

or

$$H(s)=\frac{1}{ms^2+cs+k} \tag{217}$$

This ratio is termed the transfer function of a system and is an important tool in vibration analysis.

Sometimes, how the system responds to a inputs with certain frequency components is important in understanding they system in general, therefore, we want to solve for the frequency response function of the system. The frequency response function is denoted as $H(j\omega)$ where the complex number s is replaced by the frequency component of the system while considering the imaginary portion in the complex plane (i.e., $s = j\omega$). Therefor, the frequency response function of the system becomes:

$$H(j\omega) = \frac{1}{m(j\omega)^2 + cj\omega + k} = \frac{1}{-m\omega^2 + cj\omega + k} \tag{218}$$

rearranging into a standard form yields:

$$H(j\omega) = \frac{1}{k - m\omega^2 + c\omega j} \tag{219}$$

recall that $j^2 = -1$. This is the frequency response function of the system. Therefore, it can be seen that the frequency response function of the system is the transfer function of the system evaluated along the imaginary axis $s = j\omega$. However, this expression contains imaginary values (that help to account for the phase in the system) and therefore can be challenging to work with. As the amplitude $|H(j\omega)|$ of the response (the real portion of the equation) is useful to the practitioner, it is prudent to consider the special case of amplitude response while neglecting the phase response. Consider that:

$$H(j\omega) = \mathrm{R} + \mathrm{I}j \tag{220}$$

so

$$|H(j\omega)| = \sqrt{\mathrm{R}^2 + \mathrm{I}^2} \tag{221}$$

multiplying $H(j\omega)$ by 1 that is represented by the its unit complex conjugate yields:

$$\begin{aligned} H(j\omega) &= \left(\frac{1}{k - m\omega^2 + c\omega j}\right)\left(\frac{k - m\omega^2 - c\omega j}{k - m\omega^2 - c\omega j}\right) \\ &= \left(\frac{k - m\omega^2}{(k - m\omega^2)^2 (c\omega)^2}\right)\left(\frac{-c\omega}{(k - m\omega^2)^2 (c\omega)^2} j\right) \end{aligned} \tag{222}$$

therefore, $\mathrm{R} = \frac{k-m\omega^2}{(k-m\omega^2)^2(c\omega)^2}$ and $\mathrm{I} = \frac{-c\omega}{(k-m\omega^2)^2(c\omega)^2}$. Now, calculating the amplitude of $H(j\omega)$ we get:

$$\begin{aligned} H(\omega) &= |H(j\omega)| \\ &= \sqrt{\mathrm{R}^2 + \mathrm{I}^2} \\ &= \sqrt{\frac{(k - m\omega)^2 + (-c\omega)^2}{\left((k - m\omega^2)^2 + (c\omega)^2\right)^2}} \\ &= \sqrt{\frac{1}{(k - m\omega^2)^2 + c^2\omega^2}} \\ &= \frac{1}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \end{aligned} \tag{223}$$

where $H(\omega)$ represents only the amplitude of the frequency response function and therefore drops the j term from the expression.

To recap, for a single DOF damped spring-mass system the transfer function is:

$$H(s) = \frac{1}{ms^2 + cs + k} \tag{224}$$

And the frequency response function is:

$$H(j\omega) = \frac{1}{k - m\omega^2 + c\omega j} \tag{225}$$

While the amplitude of the frequency response is:

$$H(\omega) = |H(j\omega)| = \frac{1}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \tag{226}$$

Example 3.2 Considering the forced system:

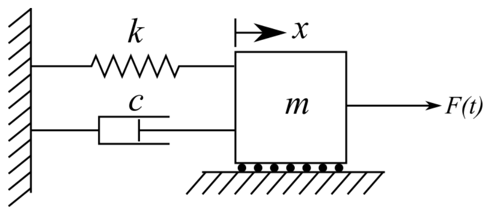


Figure 3.16: A spring-dashpot-mass model of a 1-DOF system with external excitation.

Set the forcing function to be $F_0 \sin(\omega t)$ and calculate the transfer function.
Solution: The equation of motion for the system is:

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin(\omega t) \tag{227}$$

From the #6 in the table for Laplace Transforms, we know that:

$$\mathscr{L}[\sin(\omega t)] = \frac{\omega}{s^2 + \omega^2} \tag{228}$$

Therefore,

$$F(s) = \frac{F_0\omega}{s^2 + \omega^2} \tag{229}$$

Ignoring the initial conditions and taking the Laplace transform of the EOM equation yields:

$$(ms^2 + cs + k)X(s) = \frac{F_0\omega}{s^2 + \omega^2} \tag{230}$$

Solving algebraically for the $X(s)$ yields:

$$X(s) = \frac{F_0\omega}{(ms^2 + cs + k)(s^2 + \omega^2)} \tag{231}$$

Now that we have $F(s)$ and $X(s)$ we can obtain $H(s)$ as

$$H(s) = \frac{X(s)}{F(s)} = \frac{F_0\omega}{(ms^2+cs+k)(s^2+\omega^2)} \cdot \frac{s^2+\omega^2}{F_0\omega} = \frac{1}{ms^2+cs+k} \tag{232}$$

or

$$H(s) = \frac{1}{ms^2+cs+k} \tag{233}$$

This is identical to the solution obtained using $F_0\cos(\omega t)$ as would be expected because the transfer function is related to the system and not to the input.

Review 3.2 Frequency and Time Domains
The frequency domain is a mathematical representation of a signal or data in terms of its frequency components, as opposed to its temporal or time-based representation. The frequency domain provides a different perspective on the signal by decomposing it into its constituent sinusoidal signals at discrete frequencies and their respective magnitudes. A 3D rerensation of this process is shown in figure 3.17. The transformation between the time domain and the frequency domain is typically achieved using mathematical techniques such as the Fourier Transform or the Fast Fourier Transform (FFT).

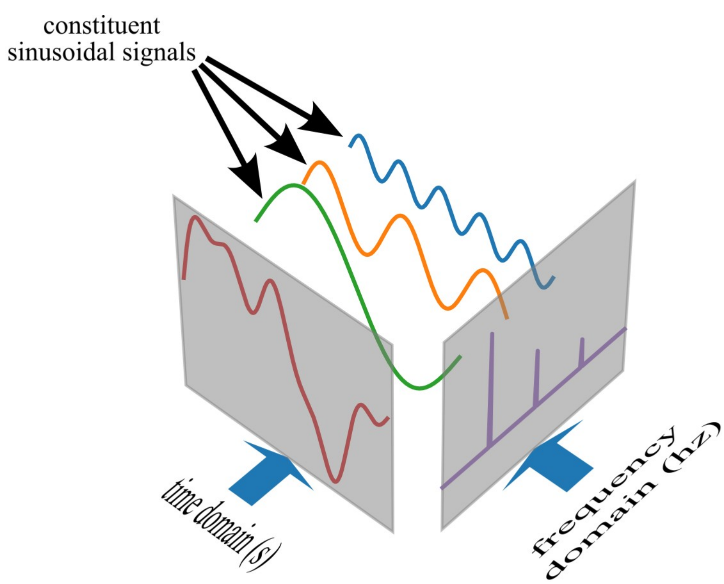


Figure 3.17: 3D visualization of time and frequency domains where a temporal signal is decomposed into constituent sinusoidal signals.

3.6.3 Response to Random Inputs

The transfer and frequency response functions can be very useful for determining the system's response to random inputs. Up to this point we have solved for deterministic input.

- **Deterministic**-For a known time t, the value of the input force $F(t)$ is precisely known.

- **Random** For a known time t, the value of the input force $F(t)$ is known only statistically.

To expand, a random signal is a signal with no obvious pattern. For these types of it is not possible to focus on the details of the input signal, as is done with a deterministic signal, rather the signal is classified and manipulated in terms of its statistical properties.

Randomness in vibration analysis can be thought of as the result of a series of results obtained from testing a system repeatability for various inputs under varying conditions. In these cases, one record or time history is not enough to describe the system. Rather, an ensemble of various tests are used to describe how the system will respond to the various inputs.

First, let us consider two inputs, a deterministic input (typical sin wave), and a random input (white noise). These inputs are shown in figure 3.18.

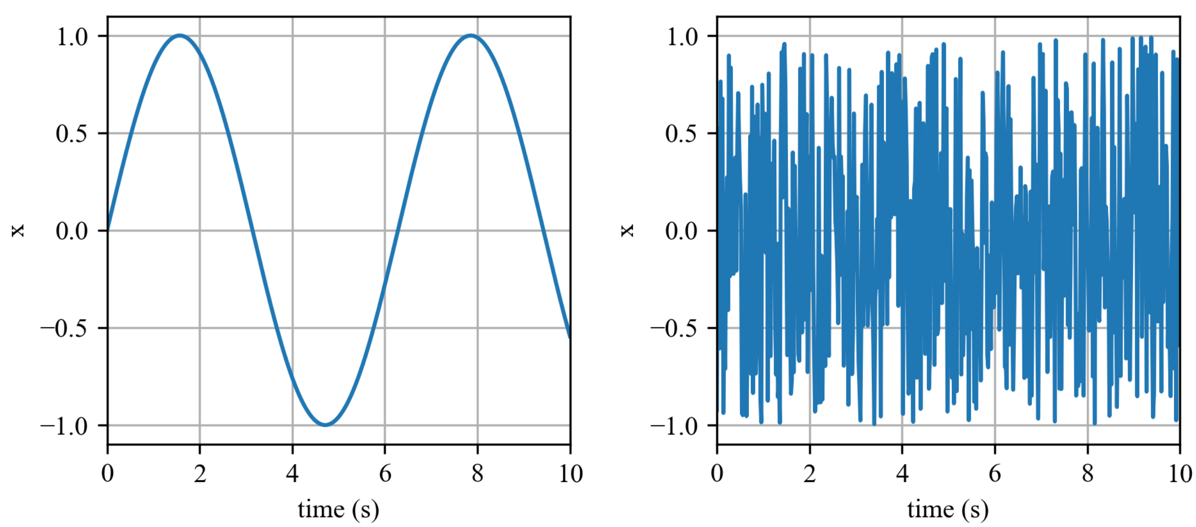


Figure 3.18: Two arbitrary inputs: (a) sinusoidal; and (b) uniform random noise.

One of the first factors to consider is the mean of the random signal $x(t)$, defined as:

$$E[x] = \bar{x} = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t)dt \tag{234}$$

where T is the length in time of the data collected. However, for random signals we often want to consider signals with an average mean of zero (i.e. $\bar{x}(t) = 0$). Therefore, for signals not centered around zero we can obtain a zero centered signal if the signal is stationary and we subtract the mean value from $\bar{x}$ from the signal $x(t)$. This can be written as:

$$x'(t) = x(t) - \bar{x} \tag{235}$$

where the $x'(t)$ is now centered around zero. As mentioned before, it is important to consider whether or not the input signals are stationary. A signal is stationary if its statistical properties (usually expressed by its mean) do not change with time. Here, it can be seen that for our inputs considered the signals are stationary if a long enough time period is considered.

Another important variable is variance (or mean-square value) of the random variable $x(t)$ defined as:

$$E[(x-\bar{x})^2] = \lim_{T\to\infty} \frac{1}{T} \int_0^T (x(t)-\bar{x})^2 dt \tag{236}$$

and provides a measure of the magnitude of the fluctuations in the signal $x(t)$. If the signal has an expected value of zero, or $E[x] = 0$, this simplifies to.

$$E[x^2] = \overline{x^2} = \lim_{T\to\infty} \frac{1}{T} \int_0^T x^2(t) dt \tag{237}$$

This expression leads to the calculation of the root-mean-square (RMS) of the signal:

$$x_{\text{rms}} = \sqrt{\overline{x^2}} \tag{238}$$

Considering a nonstationary signal, a important measure of interests is how fast the value of the variables change. This is important to understand as it provides context for how long a signal must be sampled to before a meaningful representation of the signal can be calculated in a statistical sense. One way to quantify how fast the values of signal change is the autocorrelation function:

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t)x(t+\tau) dt \tag{239}$$

The subscript xx denotes that this is a measure of the response for the variable xx, τ is the time difference between the values at which the signal $x(t)$ is sampled. The auto collation for the two inputs considered above are expressed in figure 3.19.

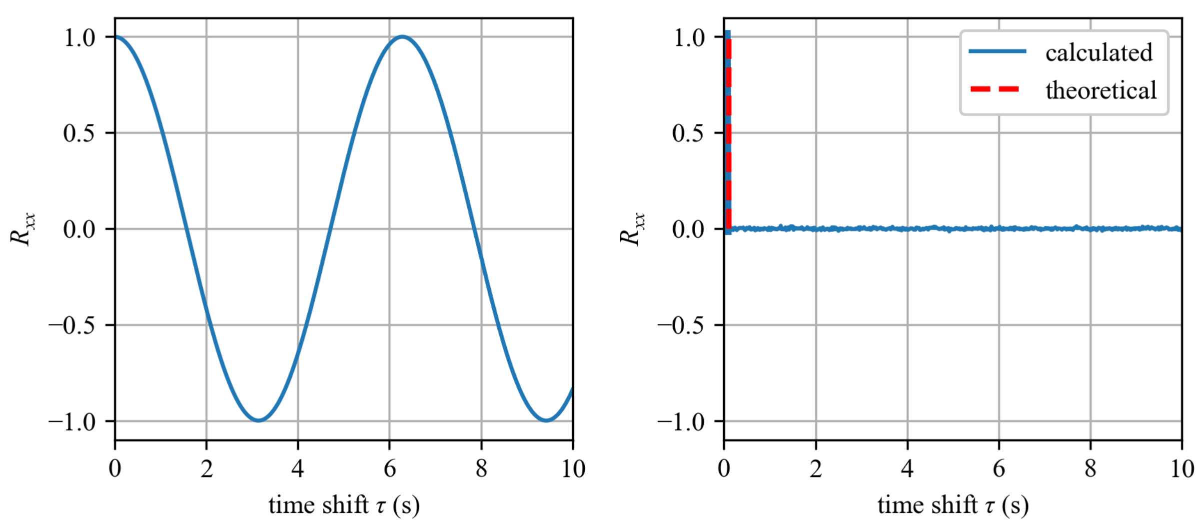


Figure 3.19: Responses from the autocorrelation function for the inputs shown in figure 3.18 showing: (a) a sinusoidal; and (b) uniform random noise.

Note that the value of τ selected in the auto correlation function greatly affects its response for the sinusoidal input. This is because the values for the sinusoidal are highly correlated. To expand, the value at any time t is greatly effected by the values immediately before and after it. This is not the case for the random input where the signal is not correlated and therefore there is little difference in changing the value of τ on the response of the autocorrelation function.

Next, if we take the Fourier transform of the autocorrelation function we obtain the power spectral density (PSD) defined as:

$$S_{xx}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{xx}(\tau)e^{-j\omega\tau}d\tau \tag{240}$$

where the integral of $R_{xx}(\tau)$ changes the real number τ into the frequency-domain value ω. The frequency spectrum is denoted with S and the subscript of the considered variable (e.g., $S_{xx}(\omega)$). The frequency spectrum for the two input cases considered are plotted in figure 3.20.

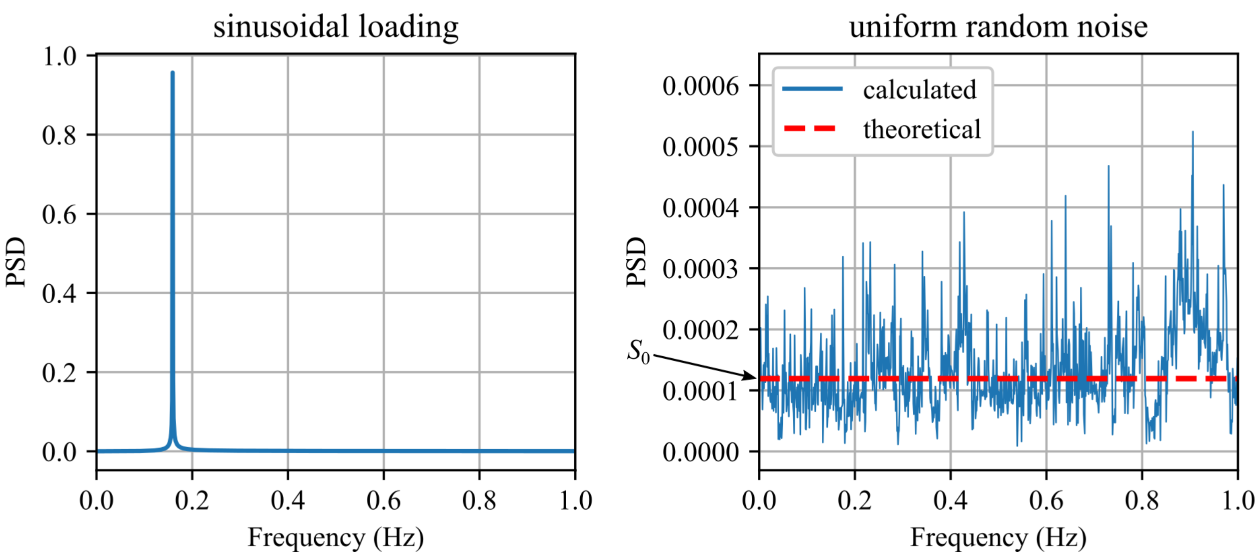


Figure 3.20: Power spectral density plots for the inputs shown in figure 3.18 showing: (a) a sinusoidal; and (b) uniform random noise.

where the flat frequency response for the random input denotes that the random input is white noise input. This flat frequency response in the frequency domain can be denoted S_0, such that $S_{ff}(\omega) = S_0$ or $S_{xx}(\omega) = S_0$, depending on whether the frequency spectrum of the input (ff) or output (xx) is being considered. While a true white noise input would be perfectly flat, white noise is really just a theoretical concept as all real-world data will have some variation in the frequency domain as diagrammed in figure 3.20(b).

Recall that S_{xx} is the spectrum of the response of the system. For the one-DOF system considered here, we can express the arbitrary input as a series of impulse inputs as shown in section 3.2.3. This knowledge, along with the frequency response function can be used to relate the spectrum of the input $S_{ff}(\omega)$ to the output through the transfer function as:

$$S_{xx}(\omega) = |H(j\omega)|^2\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} R_{ff}(\tau)e^{-j\omega\tau}d\tau\right] \tag{241}$$

This can also be expressed in symbolic form as:

$$S_{xx}(\omega) = |H(j\omega)|^2 S_{ff}(\omega) \tag{242}$$

where R_{ff} denotes the autocorrelation function of $F(t)$ and S_{ff} denotes the PSD of the forcing function $F(t)$. The notation $|H(j\omega)|^2$ is the square of the magnitude of the complex frequency response function. A more detailed derivation can be found in [Engineering Vibrations, Inman (2001)],[Random Vibrations, Spectral & Wavelet Analysis, Newland (1993)], but here it is more important to study the results rather than the derivations.

Example 3.3 Consider the following system

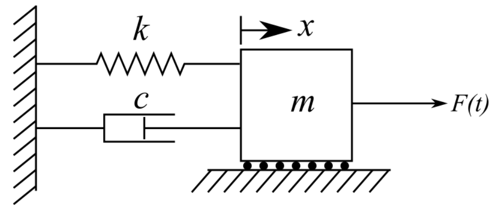


Figure 3.21: A spring-dashpot-mass model of a 1-DOF system with external excitation.

Calculate the PSD of the response $x(t)$ given that the PSD of the applied force $S_{ff}(\omega)$ is white noise.
Solution: From the system we know that the EOM is

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=F(t) \tag{243}$$

The frequency response function for this system is

$$H(j\omega)=\frac{1}{k-m\omega^2+c\omega j} \tag{244}$$

while the amplitude of the response is:

$$H(\omega)=|H(j\omega)|=\frac{1}{\sqrt{(k-m\omega^2)^2+c^2\omega^2}} \tag{245}$$

Applying the equation that relates $S_{ff}(\omega)$ to $S_{xx}(\omega)$ we get:

$$S_{xx}(\omega)=|H(j\omega)|^2S_{ff}(\omega)=\left|\frac{1}{k-m\omega^2+c\omega j}\right|^2S_{ff}(\omega) \tag{246}$$

White noise means the forcing function $S_{ff}(\omega)$ is constant across the frequency spectrum, therefore, $S_{ff}(\omega)=S_0$. Additionally as:

$$|H(j\omega)|^2=\left|\frac{1}{k-m\omega^2+c\omega j}\right|^2=\frac{1}{(k-m\omega^2)^2+c^2\omega^2} \tag{247}$$

where the absolute value is the amplitude of the system. Therefore, we obtain:

$$S_{xx}(\omega) = |H(j\omega)|^2 S_0 = \frac{1}{(k - m\omega^2)^2 + c^2\omega^2} S_0 = \frac{S_0}{(k - m\omega^2)^2 + c^2\omega^2} \tag{248}$$

Using various values for the elements in the system, the PSD for the system considered looks like:

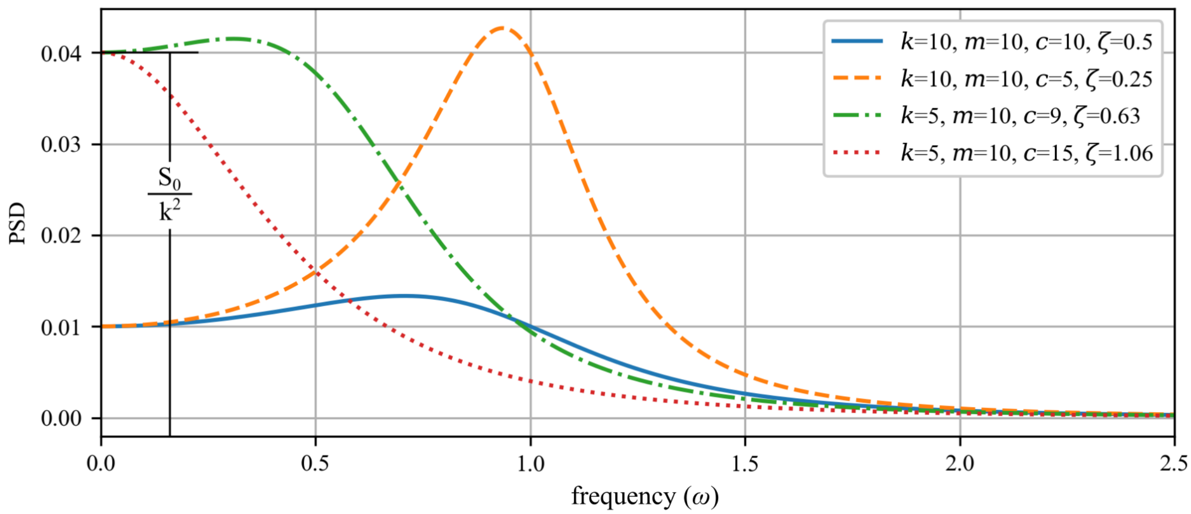


Figure 3.22: Response for considered 1-DOF systems subjected to a white noise input.

Another useful quantity to consider is the expected output, in terms of its mean and variance, for a given input. Working within the constraint that the system will oscillate about zero, $E[x] = 0$, the mean-square value can be directly related to the PSD function as:

$$E[x^2] = \overline{x^2} = \int_{-\infty}^{\infty} |H(j\omega)|^2 S_{ff}(\omega) d\omega \tag{249}$$

For a constant input S_0, as diagrammed in figure 3.20(b), the mean-square value can be expressed as:

$$E[x^2] = \overline{x^2} = S_0 \int_{-\infty}^{\infty} |H(j\omega)|^2 d\omega \tag{250}$$

After inspecting the above equation, it becomes clear that to obtain the square of the expected value, a solution for $\int_{-\infty}^{\infty} |H(j\omega)|^2 d\omega$ must be obtained. For cases where $S_{ff}(\omega) = S_0$ and as such $S_{ff}(\omega)$ can be pulled out of the integral, these integrals have been solved [Random Vibrations, Spectral & Wavelet Analysis, Newland (1993)]. For example, given $\int_{-\infty}^{\infty} |H(j\omega)|^2 d\omega$:

$$\int_{-\infty}^{\infty} \left| \frac{B_0}{A_0 + j\omega A_1} \right|^2 d\omega = \frac{\pi B_0^2}{A_0 A_1} \tag{251}$$

and

$$\int_{-\infty}^{\infty} \left| \frac{B_0 + j\omega B_1}{A_0 + j\omega A_1 - \omega^2 A_2} \right|^2 d\omega = \frac{\pi (A_0 B_1^2 + A_2 B_0^2)}{A_0 A_1 A_2} \tag{252}$$

When combined with equation 250, these integrals allow for the easy calculation of the expected values.

Example 3.4 For system below, calculate the mean-square response of the system given that the the spectrum of the input force $F(t)$ is a perfect theoretical white noise.

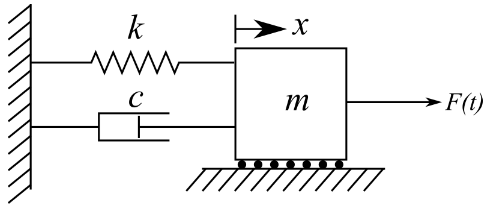


Figure 3.23: A spring-dashpot-mass model of a 1-DOF system with external excitation.

Solution: Again, as the forcing function $S_{ff}(\omega)$ is constant across the frequency spectrum $S_{ff}(\omega) = S_0$ the mean-square response can be calculated as:

$$E[x^2] = \overline{x^2} = S_0 \int_{-\infty}^{\infty} |H(j\omega)|^2 d\omega \tag{253}$$

Using the already tabulated response:

$$\int_{-\infty}^{\infty} \left| \frac{B_0 + j\omega B_1}{A_0 + j\omega A_1 - \omega^2 A_2} \right|^2 d\omega = \frac{\pi(A_0 B_1^2 + A_2 B_0^2)}{A_0 A_1 A_2} \tag{254}$$

and the frequency response function for the system as derived in equation 219:

$$H(j\omega) = \frac{1}{k - m\omega^2 + c\omega j} \tag{255}$$

when $B_0 = 1$, $B_1 = 0$, $A_0 = k$, $A_1 = c$, and $A_2 = m$. Therefore, using the tabulated expression we can show that:

$$E[x^2] = S_0 \frac{\pi m}{kcm} = \frac{S_0 \pi}{kc} \tag{256}$$

3.7 Inverse Laplace Transform

Consider that X(s) can be expanded in partial fractions as

$$X(s) = \frac{B(s)}{A(s)} = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + \cdots + \frac{r_n}{s - p_n} = \sum_{i=1}^{n} \frac{r_i}{s - p_i} \tag{257}$$

recall that $\mathscr{L}[e^{p_0 t}] = \frac{1}{s - p_0}$ for p_0. The inverse Laplace Transform of $X(s)$ is a number:

$$x(t) = r_1 e^{p_1 t} + r_2 e^{p_2 t} + \cdots + r_n e^{p_n t} = \sum_{i=1}^{n} r_i e^{p_i t} \tag{258}$$

where the values $p_1,\ p_2,\ \cdots,\ p_n$ are the roots of the demonstrator equation and are the poles of the system and may be complex. The values $r_1,\ r_2,\ \cdots,\ r_n$ are called “residuals” and are defined as,

$$r_i = \lim_{s \to p_i} \left[(s - p_i)X(s)\right] \tag{259}$$

the MATLAB function `residue` will perform partial fraction expansion (PFD) and can be used to find the values.

When complex poles appear in the system, they are conjugate pairs, i.e.,

$$p_{1,2} = \sigma \pm i\omega_d \tag{260}$$

where σ is the position of the poles on the real axis in the S-domain. This can also be written as $\sigma = \zeta\omega_n$. Partial fraction decomposition gives us:

$$X(s) = X_1(s) + X_2(s) = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} \tag{261}$$

where $A(s)$ and $B(s)$ are polynomial expansions, similar to equation 207. The linearity of the system allows us to define $X(s) = X_1(s) + X_2(s)$ and therefore, $x(t) = x_1(t) + X_2(t)$ in the time domain.

From equation 261, r_1 and r_1 represent the residuals of the solution as a whole, but we want to extract the steady state (harmonic component) and transient state (exponential component) of the signals. For that, we need a joint residual value that we will call r_0. With this in mind, it can be shown that $r_1 = ir_0$, and that $r_2 = ir_0$

$$\begin{aligned} x_1(t) &= \frac{-ir_0}{s - p_1},\ p_1 = \sigma + i\omega_d \\ &= \mathscr{L}[X_1(s)]^{-1} \\ &= \mathscr{L}\left[\frac{-ir_0}{s - p_1}\right]^{-1} \\ &= -ir_0 e^{p_1 t} \\ &= -ir_0 e^{(\sigma + i\omega_d)t} \\ &= -ir_0 e^{\sigma t} e^{i\omega_d t} \\ &= -ir_0 e^{\sigma t}(\cos(\omega_d t) + i\sin(\omega_d t)) \\ &= r_0 e^{\sigma t}(-i\cos(\omega_d t) + \sin(\omega_d t)) \end{aligned} \tag{262}$$

similarly,

$$\begin{aligned} x_2(t) &= \frac{ir_0}{i - p_1},\ p_2 = \sigma + i\omega_d \\ &= r_0 e^{\sigma t}(i\cos(\omega_d t) + \sin(\omega_d t)) \end{aligned} \tag{263}$$

the time series response can be rebuilt as

$$\begin{aligned} x_1(t) + x_2(t) &= r_0 e^{\sigma t}\big(-i\cos(\omega_d t) + \sin(\omega_d t)\big) + r_0 e^{\sigma t}\big(i\cos(\omega_d t) + \sin(\omega_d t)\big) \\ &= 2r_0 e^{\sigma t}\sin(\omega_d t) \end{aligned} \tag{264}$$

For stable systems, we we expect $\sigma > 0$ so, $\sigma = -|\sigma|$ as the poles are on the left hand side for a stable response

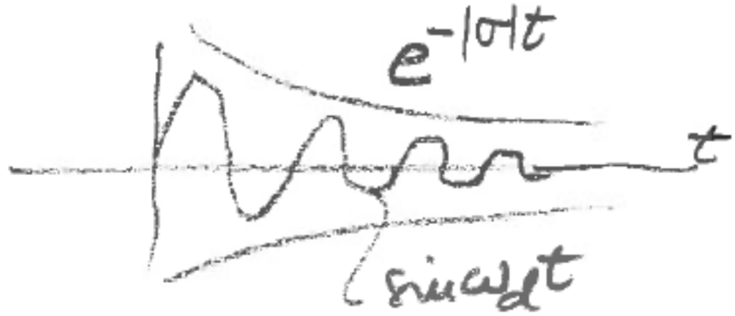


Figure 3.24: Time series response for the signal.

Example 3.5 For the system in the S-domain expressed as,

$$X(s) = \frac{1}{s+2}\frac{2}{s+1} \tag{265}$$

use MATLAB to compute the roots and residuals, and use the roots and residuals to compute $X(s)$.

$$\begin{aligned} X(s) &= \frac{1}{s+2}\frac{2}{s+1} \\ &= \frac{(s+1)+2(s+2)}{(s+1)(s+2)} \\ &= \frac{3s+5}{s^2+3s+2} \\ &= \frac{B(s)}{A(s)} \end{aligned} \tag{266}$$

The `residue` command can be used such as:

Listing 1: MATLAB code to find poles and resuduals.

```
% Define B and A
B = [3, 5];
A = [1, 3, 2]

% find the poles and the residuales
[r, p, k] = residue(B,A)

% use poles and residuals to calculate the A and B
[B_new, A_new] = residue(r, p, k)
```

this returns $r_1 = 1,\ r_2 = 2,\ p_1 = -2,\ p_2 = -1,\ k = void$. Inserting this back into the expression yields

$$\frac{r_1}{s-p_1}+\frac{r_2}{s-p_2}=\frac{1}{s+2}+\frac{2}{s+1} \tag{267}$$

which checks out! Now, converting the partial fraction expansion back to the ratio of two polynomials results in $A(s) = 1,\ 3,\ 2$, and $B(s) = 3,\ 5$. So this also check outs!

3.8 Dominant Poles

Given

$$X(s)=\frac{B(s)}{A(s)}=\frac{r_1}{s-p_1}+\frac{r_2}{s-p_2}+\cdots+\frac{r_k}{s-p_k}+\cdots \tag{268}$$

where p_k is either a complex number ($p_k \in \mathbb{C}$) and a conjugate pair

$$p_k = \sigma_k \pm i\omega_k \tag{269}$$

or a real number ($p_k \in \mathbb{R}$)

$$p_k = \sigma_k \tag{270}$$

Again, knowing that the inverse Laplace of $X(s)$ results in

$$x(t) = r_1 e^{p_1 t} + r_2 e^{p_2 t} + \cdots + r_k e^{p_k t} + \cdots \tag{271}$$

we can solve for the time series response. For controls, we are interested in the long term behavior of $x(t)$, i.e., to find $x_{ss} = \lim_{t\to\infty}[x(t)]$ which is the steady state response of the system. Note, that every term in the expansion will have the following form

$$r_k e^{p_k t} = r_k e^{\sigma_k t} \cdot e^{i\omega_k t} \tag{272}$$

where:
- $r_k e^{\sigma_k t}$ is the exponential function
- $e^{i\omega_k t}$ is the harmonic oscillation

we distinguish the following possible cases

3.8.1 Case A (Unstable):

If at least one σ_k is positive, then $e^{\sigma_k t} \to \infty$, unstable system.

3.8.2 Case B (Stable):

If system is stable, $\sigma_k < 0$ (all poles on the left hand side) then:
Case B1 - Dominant Poles on the imaginary axis: If terms with $\sigma_k = 0$ exist, than these will be maintain oscillations as the rest of the poles die out. This means the poles placed on the imaginary axis, if they exist, are dominant poles.

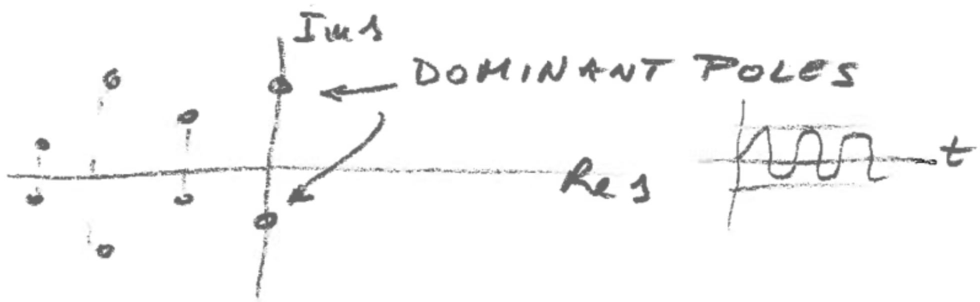


Figure 3.25: System with dominant poles on the imaginary axis.

Case B2 - Dominant Poles to the left of the imaginary axis: If terms with $\sigma_k = 0$ do not exist, than the dominant poles are the poles closest to the imaginary axis because they take the longest to die out, having small σ_k values (small damping).

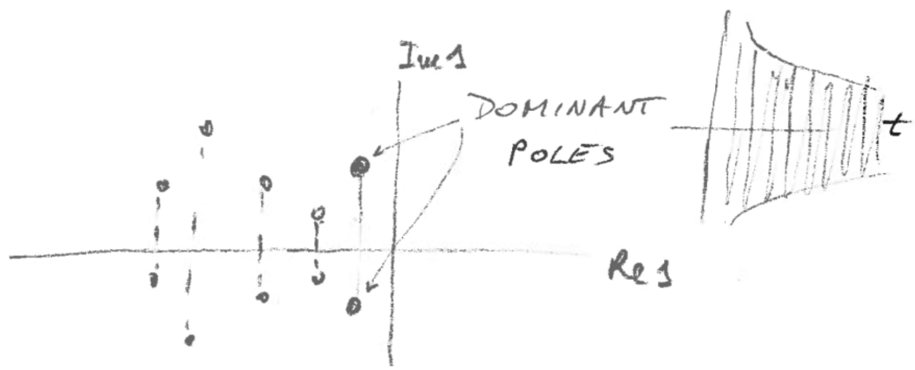


Figure 3.26: System with dominant poles to the left of the imaginary axis.

Table of Laplace Transforms for Vibrations

This is a partial lists of important Laplace transforms for vibrations that assumes zero initial conditions, $0 < t$, and $\zeta < 1$.

$f(t)$	$\mathscr{L}[f(t)] = F(s)$	
$\delta(t)$	1	(1)
$\delta(t-t_0)$	e^{-st_0}	(2)
1	$\frac{1}{s}$	(3)
e^{at}	$\frac{1}{s-a}$	(4)
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	(5)
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	(6)
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$	(7)
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$	(8)
$\frac{1}{\omega^2}\big(1-\cos(\omega t)\big)$	$\frac{1}{s(s^2+\omega^2)}$	(9)
$\frac{1}{\omega_d}e^{-\zeta\omega t}\sin(\omega_d t)$	$\frac{1}{s^2+2\zeta\omega s+\omega^2}$	(10)
$1-\frac{\omega}{\omega_d}e^{-\zeta\omega t}\sin(\omega_d t+\phi),\ \phi=\cos^{-1}(\zeta)\ldots$	$\frac{\omega^2}{s(s^2+2\zeta\omega s+\omega^2)}$	(11)
$\frac{t^{n-1}}{(n-1)!},\ n=1,2,\ldots$	$\frac{1}{s^n}$	(12)
$t^n,\ n=1,2,\ldots$	$\frac{n!}{s^{n+1}}$	(13)
$t^n e^{\omega t},\ n=1,2,\ldots$	$\frac{n!}{(s-\omega)^{n+1}}$	(14)
$\frac{1}{\omega}(1-e^{-\omega t})$	$\frac{1}{s(s+\omega)}$	(15)
$\frac{1}{\omega^2}(e^{-\omega t}+\omega t-1)$	$\frac{1}{s^2(s+\omega)}$	(16)

$f(t)$	$\mathscr{L}[f(t)] = F(s)$	
$\frac{1}{\omega^3}\big(\omega t-\sin(\omega t)\big)$	$\frac{1}{s^2(s^2+\omega^2)}$	(17)
$\frac{1}{2\omega^3}\big(\sin(\omega t)-\omega t\cos(\omega t)\big)\ldots$	$\frac{1}{(s^2+\omega^2)^2}$	(18)
$\frac{t}{2\omega}\sin(\omega t)$	$\frac{s}{(s^2+\omega^2)^2}$	(19)
$t\sin(\omega t)$	$\frac{2\omega s}{(s^2+\omega^2)^2}$	(20)
$t\cos(\omega t)$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$	(21)
$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$	(22)
$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$	(23)
$e^{at}\sinh(\omega t)$	$\frac{\omega}{(s-a)^2-\omega^2}$	(24)
$e^{at}\cosh(\omega t)$	$\frac{s-a}{(s-a)^2-\omega^2}$	(25)
$\frac{1}{\omega_2}\sin(\omega_2 t)-\frac{1}{\omega_1}\sin(\omega_1 t)\ldots$	$\frac{\omega_1^2-\omega_2^2}{(s^2+\omega_1^2)(s^2+\omega_2^2)}$	(26)
$\cos(\omega_2 t)-\cos(\omega_1 t)$	$\frac{s(\omega_1^2-\omega_2^2)}{(s^2+\omega_1^2)(s^2+\omega_2^2)}$	(27)
$e^{at}f(t)$	$F(s-a)$	(28)
$f(t-a)\Phi(t-a)$	$e^{-as}F(s)$	(29)
$\Phi(t-a)$	$\frac{e^{-as}}{s}$	(30)
$f'(t)$	$sF(s)-f(0)$	(31)

4 Time Series Response

4.1 Transfer Functions Block Diagrams

Transfer functions can be created using:

- Polynomial model (numerator / denominator)

$$G(s) = \frac{B(s)}{A(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + ... + b_0}{a_n s^n + a_{n-1}s^{n-1} + ... + a_0} \tag{273}$$

  - n = order of transfer function models where $m < n$
- Zero-pole-gain model (numerator / denominator)

$$G(s) = k\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \tag{274}$$

  - zeros: $z_1, z_2, \cdots, z_m$ roots of $B(s) = 0$
  - poles: $p_1, p_2, \cdots, p_n$ roots of $A(s) = 0$
  - gain: k
- Time constant model

$$G(s) = \frac{k}{s^N} \cdot \frac{(T_a+1)(T_b+1)\cdots}{(T_1+1)(T_2+1)\cdots} \tag{275}$$

  - N = type of transfer function model

4.1.1 Create transfer function using numerator and denominator coefficients

MATLAB can be used to create continuous-time single-input, single-output (SISO) transfer functions from their numerator and denominator coefficients using `tf`. To use the tf function, you must have the Control System Toolbox licensed and installed. To find out if you do, type: `ver control` in your Command Window or a script.

Method I

Create the transfer function

$$G(s) = \frac{s}{s^2+3s+2} \tag{276}$$

Listing 2: MATLAB code for Method I.

```
num = [1 0];
dem = [1 3 2];
G = tf(num, dem)
% To use the tf function, you must have the Control System Toolbox
% licensed and installed. To find out if you do, type:
% ver control in your Command Window or a script.
```

where `nem` and `dem` are the numerator and denominator polynomial coefficients in descending powers of s. For example, `den = [1 3 2]` represents the denominator polynomial $\frac{s}{s^2+3s+2}$

`G` is a `tf` model object, which is a data container for representing the transfer function in polynomial form.

Method II

Alternatively, you can specify the transfer function $G(s)$ as an expression in s-domain.

1. Create a transfer Function model for the variable s
2. Specify $G(s)$ as a ratio of polynomials in s

Listing 3: MATLAB code for Method II.

```
s = tf('s');
G = s/(s^2+3*s+2)
```

Therefore, the full expression of $G(s)$ can be written as

$$G(s) = \frac{B(s)}{A(s)} = \frac{s}{s^2+3s+2} = \frac{b_1 s + b_0}{a_1 s^2 + a_2 s + a_0} \tag{277}$$

where

$$B(s) = b_1 s + b_0 \tag{278}$$

resulting in $b_1 = 1$, and $b_0 = 0$; or, `B = [1 0]`. Moreover,

$$A(s) = a_1 s^2 + a_2 s + a_0 \tag{279}$$

where $a_1 = 1$, $a_1 = 3$, and $a_0 = 1$; or, `A = [1 3 1]`.

4.1.2 Create transfer function using Zeros, Poles, and Gain

MATLAB can be used to create continuous-time single-input, single-output (SISO) transfer functions in factored form using `zpk`. Create the factored transfer function

$$G(s) = 5\frac{s}{(s-1-i)(s-1-i)(s-2)} \tag{280}$$

Listing 4: MATLAB code to create a transfer function using Zeros, Poles, and Gain.

```
Z = 0;
P = [-1-1i -1+1i -2];
K = 5;
G = zpk(Z,P,K)
```

where `Z` and `P` are zeros and poles (the roots of the numerator and denominator respectively). K is the gain of the factored from. Solving fore the poles p_1, p_2, and p_3 of $G(s)$;

$$G(s) = 5\frac{s}{(s-1-i)(s-1-i)(s-2)} \tag{281}$$

$$= 5\frac{s-0}{\left[s-(-1-i)\right]\left[(s-(-1-i)\right]\left[(s-(-2)\right]} \tag{282}$$

where $K = 5$, $s - 0 = s - z_1 \rightarrow z_1 = 0$, and $Z = [0]$. Therefore,

$$\left[s-(-1-i)\right]\left[(s-(-1-i)\right]\left[(s-(-2)\right] = (s-p_1)(s-p_2)(s-p_3) \tag{283}$$

this leads to

$$G(s) = k\frac{s-z_1}{(s-p_1)(s-p_2)(s-p_3)} \tag{284}$$

therefore, $G(s)$ has a real pole at $s = -2$ and a pair of complex poles as $s = -1 \pm i$. The vector `P = [-1-1i -1+1i -2]` specifies these pole locations.

4.2 Order verse Type

need to find some real-world examples of orders and types.

A system has a "Type" and an "Order", which have different meanings.

- Order = n, highest exponent of s in the denominator. n is the number of poles.
- Type = N, exponent of the factored out s in the denominator. N is the number of poles in origin ($p = 0$).

Consider the 1st-order mass-damper system (no stiffness) as shown in figure X with the transfer function

make a figure

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s} \tag{285}$$

The transfer function can easily be written in the basic form

$$G(s) = \frac{b_0}{a_2 s^2 + a_1 s} \tag{286}$$

where $a_2 = 1$, $a_1 = 2\zeta\omega_n$, $b_0 = \omega_n^2$. There the presence of a_2 means its a 2^{nd} order system. To solve for the type of the system, s must be factored of of the denominator, leading to:

$$\begin{aligned} G(s) &= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s} \\ &= \frac{\omega_n^2}{s(s + 2\zeta\omega_n)} \\ &= \frac{\frac{\omega_n}{2\zeta}}{s} \cdot \frac{1}{\frac{1}{2\zeta\omega_n}s + 1} \\ &= \frac{K}{s^N} \cdot \frac{1}{T_1 s + 1} \end{aligned} \tag{287}$$

where $K = \frac{\omega_n}{2\zeta}$, $N = 1$, and $T_1 = \frac{1}{2\zeta\omega_n}$. $N = 1$ means that is a Type 1 system. Therefore, this is a 2^{nd} order system of Type 1, "Type" and "Order" have different meanings. Table ?? reports the types and orders for different transfer functions.

Table 2: Examples of types and orders for different transfer functions.

transfer function	Type	Order
$G(s) = \frac{1}{Ts+1}$	0	1
$G(s) = \frac{1}{cs} = \frac{1/c}{s}$	1	1
$G(s) = \frac{1}{Js^2} = \frac{1/J}{s^2}$	2	2
$G(s) = \frac{1}{Js^2+cs} = \frac{K/c}{s} \cdot \frac{1}{J/c s+1}$	1	2
$G(s) = \frac{K}{J^4} \frac{T_a s+1}{(T_1 s+1)(T_2 s+1)(T_3 s+1)} = \frac{b_1 s+b_0}{s^4(a_3 s^3+a_2 s^2+a_1 s+a_0)} = \frac{b_1 s+b_0}{(a_3 s^7+a_2 s^6+a_1 s^5+a_0 s^4)}$	4	7

4.3 Time Response

Time response calculations are obtained using the Laplace transforms where the Laplace transform is

$$X(s) = G(s)F(s) \tag{288}$$

and the time response is

$$x(t) = \mathscr{L}[X(s)]^{-1} \tag{289}$$

4.3.1 Time Series Response for a Step Function

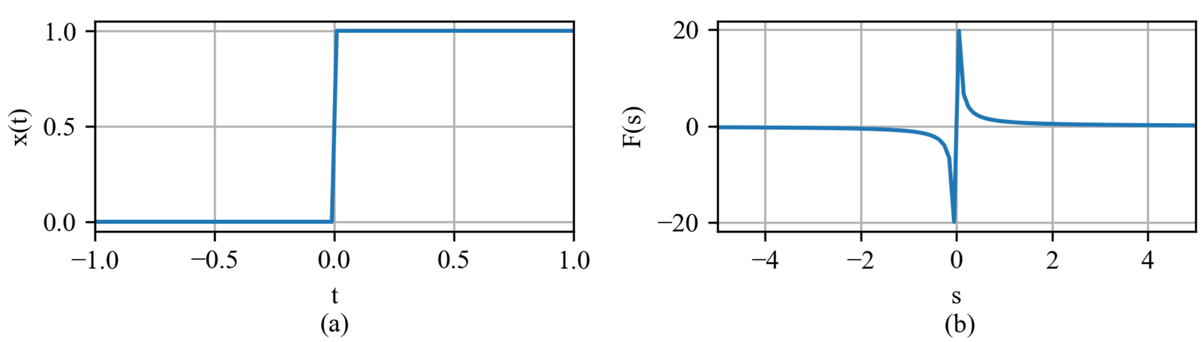


Figure 4.1: Step function; showing the (a) time domain and; (c) the s-space.

A step function is expressed as the following Laplace pair:

$$\text{LT pair} = \begin{cases} f(t) & 1(t), \quad t > 0 \\ F(s) & \frac{1}{s} \end{cases} \tag{290}$$

therefore, the time response of the systems is expressed as

$$x(t) = \mathscr{L}\left[G(s)\frac{1}{s}\right]^{-1} \tag{291}$$

In MATLAB, this is expressed as:

Listing 5: MATLAB code for the time-series resposne of a step function.

```
x_t = step(G)
```

4.3.2 Time Series Response for a Impulse Function

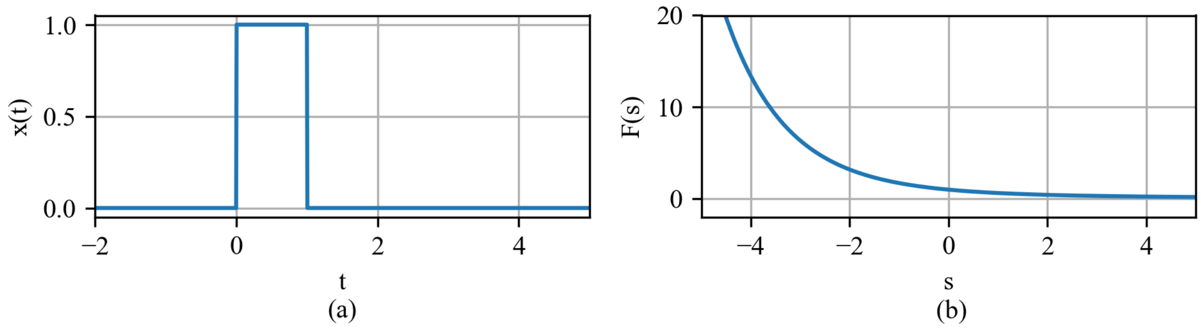


Figure 4.2: Pulse function; showing the (a) time domain and; (c) the s-space.

An impulse function is expressed as the following Laplace pair:

$$\text{LT pair} = \begin{cases} f(t) & p(t;\ \tau) \\ F(s) & \frac{1-e^{-st\tau}}{s\tau} \end{cases} \tag{292}$$

therefore, the time response of the systems is expressed as

$$x(t) = \mathscr{L}\left[G(s)\frac{1-e^{-st\tau}}{s\tau}\right]^{-1} \tag{293}$$

In MATLAB, this is expressed as:

Listing 6: MATLAB code for the time-series resposne of a step function.

```
x_t = impulse(G)
```

4.3.3 Time Series Response for a Ramp Function

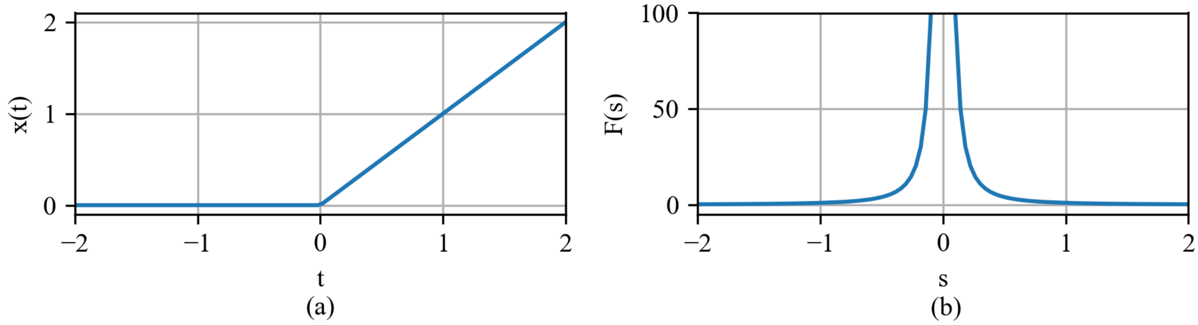


Figure 4.3: Ramp function; showing the (a) time domain and; (c) the s-space.

An ramp function is expressed as the following Laplace pair:

$$\text{LT pair} = \begin{cases} f(t) & t, \quad t > 0 \\ F(s) & \frac{1}{s^2} \end{cases} \tag{294}$$

therefore, the time response of the systems is expressed as

$$x(t) = \mathscr{L}\left[G(s)\frac{1}{s^2}\right]^{-1} \tag{295}$$

In MATLAB, this is expressed as:

Listing 7: MATLAB code for the time-series resposne of a step function.

```
x_t = impulse(G/(s^2))
```

Note that for the MATLAB code, we used the property:

$$X(s) = G(s)\frac{1}{s^2} = \left(\frac{G(s)}{s^2}\right) \cdot 1 \tag{296}$$

where 1 is the Laplace transform of an impulse. Note that `ramp` is not an option in MATLAB as this command is already used to generate a time-series ramp signal.

4.4 1^{st} Order System Time Response

The first order equation of motion is

$$T\dot{x}(t) + x(t) = f(t) \tag{297}$$

where $x(0) = 0$ is the initial condition and T is a time constant for the first order system. The Laplace transform gives us

$$\begin{aligned} x &\to X(s) \\ \dot{x} &\to sX(s) \\ f(t) &\to F(s) \end{aligned} \tag{298}$$

therefore, the s-domain equation is:

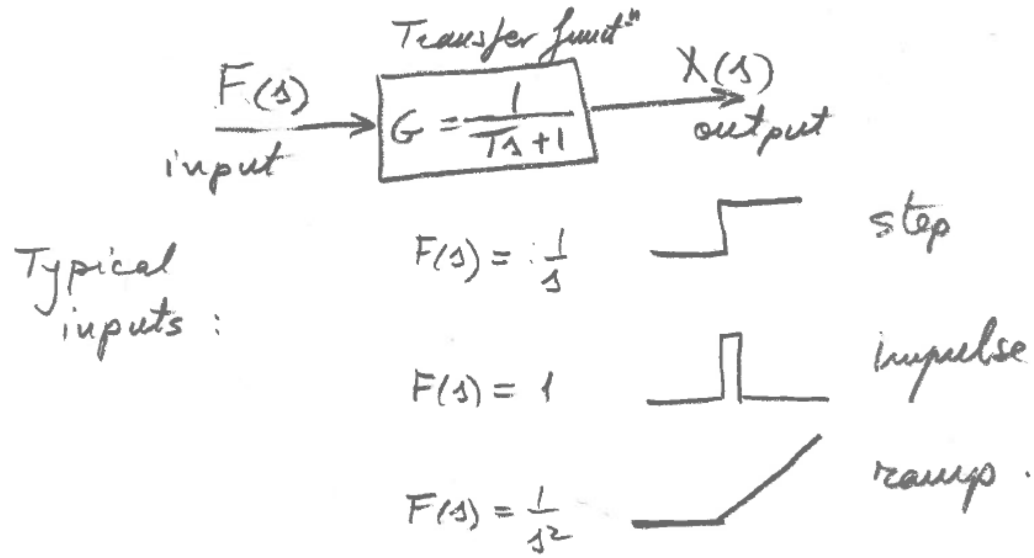
$$TsX(s) + X(s) = F(s) \tag{299}$$

where:

$$X(s) = \frac{F(s)}{Ts+1} = \frac{1}{Ts+1}F(s) = G(s)F(s) \tag{300}$$

therefore, the transfer function is:

$$G(s) = \frac{1}{Ts+1} \tag{301}$$



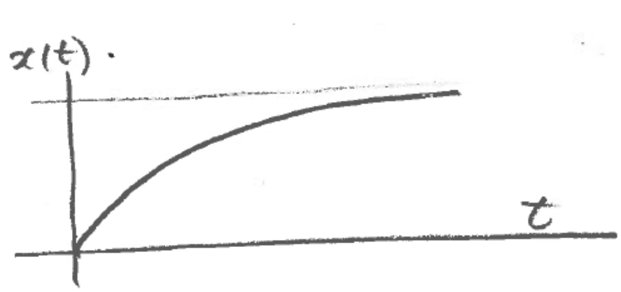
4.4.1 Step response of a 1st Order System

$$\begin{aligned} X(s) &= G(s)F(s) \qquad (302) \\ &= \frac{1}{Ts+1}\cdot\frac{1}{s} \\ &= \frac{1}{s(Ts+1)} \qquad (303) \end{aligned}$$

Therefore, solving for $\mathscr{L}[X(s)]^{-1}$ yields

$$x(t) = 1 - e^{-t/T} \tag{304}$$

or



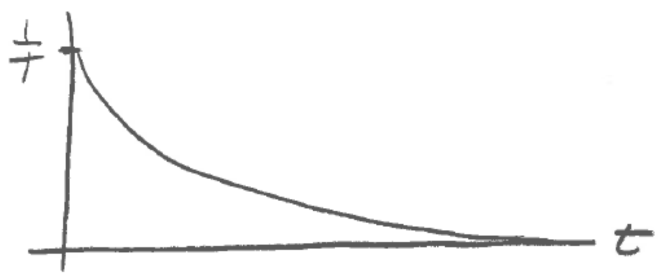
4.4.2 Impulse response of a 1st Order System

$$\begin{aligned} X(s) &= G(s)F(s) \qquad (305)\\ &= \frac{1}{Ts+1}\cdot 1 \\ &= \frac{1}{Ts+1} \qquad (306)\end{aligned}$$

Therefore, solving for $\mathscr{L}[X(s)]^{-1}$ yields

$$x(t) = \frac{1}{T}e^{-t/T} \qquad (307)$$

or



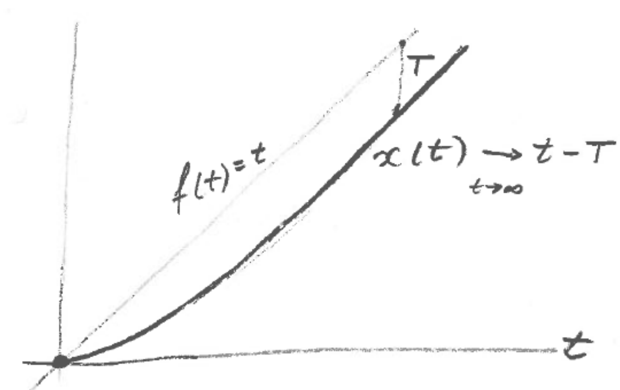
4.4.3 Ramp response of a 1st Order System

$$\begin{aligned} X(s) &= G(s)F(s) \qquad (308)\\ &= \frac{1}{Ts+1}\cdot \frac{1}{s^2} \\ &= \frac{1}{s^2(Ts+1)} \qquad (309)\end{aligned}$$

Therefore, solving for $\mathscr{L}[X(s)]^{-1}$ yields

$$\begin{aligned} x(t) &= t - T + Te^{-t/T} \\ &= t - T\left(1 - e^{-t/T}\right) \qquad (310)\end{aligned}$$

or



Moreover,

$$x(t) = t - T + Te^{-t/T} \xrightarrow[t\to\infty]{} t - T \tag{311}$$

4.4.4 Summary of the First Order System Responses

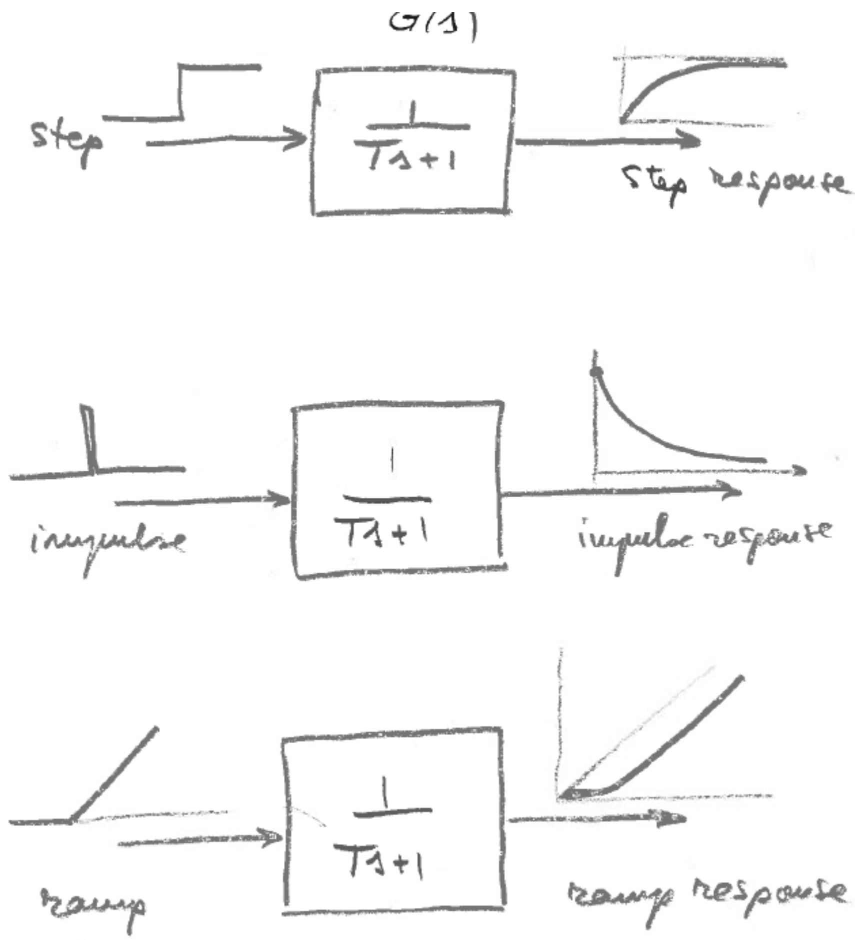


Figure 4.4: A summary of the first order system responses.

Listing 8: MATLAB code for time series responses of 1st order system.

```
%{
This program studies time response of 1st order systems
%}
clc
clear
close all

format compact

%% Given data
T=2.5; % time response for the 1D system

%% time range setup
T_max = 10;          % run the test to 10 seconds
dt = T_max*1e-4;     % find the delta-t value
t = 0:dt:T_max;      % build the time vector

%% Define system
B = [1];
A = [T 1];
G = tf(B,A);

%% create the figure enviorment
figure(1)

%% step response
subplot(3,1,1)
hold on
step(G,t)
ylim([0 1.2])

%% Impulse response
subplot(3,1,2)
impulse(G,t)

%% Ramp response
F_ramp = tf([1],[1 0 0])
subplot(3,1,3)
impulse(G*F_ramp,t)
title('Ramp Response') % need to set manually.
```

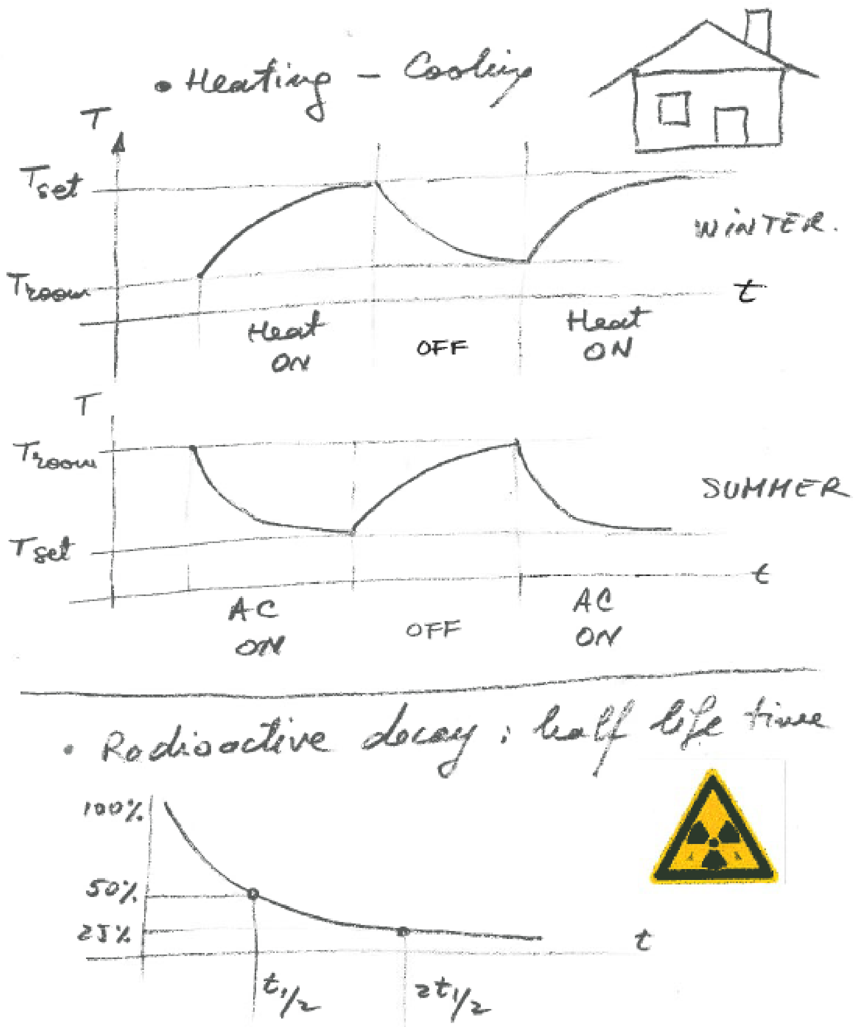


Figure 4.5: Examples of first order systems in the real world.

4.5 2nd Order System Time Response

The ordinary differential equation for the equation of motion of a 2nd order system can be expressed as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 f(t) \tag{312}$$

with the initial conditions $\ddot{x}(0) = 0$, and $\dot{x}(0) = 0$. The Laplace transform gives us

$$\begin{aligned} x &\to X(s) \\ \dot{x} &\to sX(s) \\ \ddot{x} &\to s^2X(s) \\ f(t) &\to F(s) \end{aligned} \tag{313}$$

Taking the Laplace transform of the equation of motion yields

$$s^2X(s) + 2\zeta\omega_n sX(s) + \omega_n^2 X(s) = \omega_n^2 F(s) \tag{314}$$

Pulling $X(s)$ out of the first equation results in

$$(s^2 2\zeta\omega_n s+\omega_n^2)X(s)=\omega_n^2 F(s) \tag{315}$$

next, we can solve for $X(s)$

$$X(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}F(s) \tag{316}$$

As $X(s)=G(s)F(s)$, it we can show that

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \tag{317}$$

4.5.1 Step Response for a 2nd Order System

$$X(s)=G(s)F(s) \tag{318}$$

$$=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\cdot\frac{1}{s}$$

$$=\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)} \tag{319}$$

Therefore, solving for $\mathscr{L}[X(s)]^{-1}$ yields

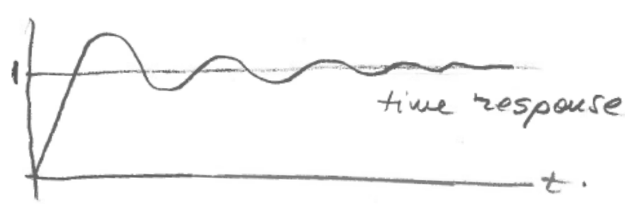
$$x(t)=1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_d t+\phi) \tag{320}$$

where

$$\phi=\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} \tag{321}$$

$$=\sin^{-1}\sqrt{1-\zeta^2}$$

or



Proof

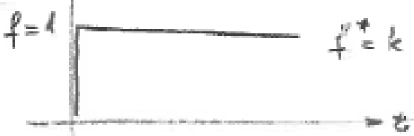
ODE derivation

7] $x_c(t) = e^{-\zeta\omega_n t}(A\cos\omega_d t + B\sin\omega_d t)$; A, B

$= e^{-\zeta\omega_n t} C \sin(\omega_d t + \varphi)$; C, φ

$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 f(t)$; $\omega_n^2 f = \frac{f^*}{m} \rightarrow f = \frac{f^*}{k}$

Step input, $f(t) = 1(t)$



$x_p(t) = D$, $\dot{x}_p - \ddot{x}_p = 0$

$\omega_n^2 D = \omega_n^2 \rightarrow D = 1$

$x_p(t) = 1$

$x(t) = e^{-\overbrace{\zeta\omega_n}^{\alpha} t} C \sin(\overbrace{\omega_d t + \varphi}^{\psi}) + 1$

$\dot{x}(t) = (-\alpha\sin\psi + \omega_d\cos\psi)\, C e^{-\alpha t}$

$x(0) = C\sin\varphi + 1 = 0$

$\dot{x}(0) = (-\alpha\sin\varphi + \omega_d\cos\varphi) = 0 \rightarrow \tan\varphi = \frac{\omega_d}{\alpha}$

$\tan\varphi = \frac{\omega_n\sqrt{1-\zeta^2}}{\zeta\omega_n} = \frac{\sqrt{1-\zeta^2}}{\zeta} \rightarrow \varphi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$

$\sin^2\varphi = \frac{\tan^2\varphi}{1+\tan^2\varphi} = \frac{1-\zeta^2}{\zeta^2+1-\zeta^2} = 1-\zeta^2$

$\sin\varphi = \sqrt{1-\zeta^2}$ $\qquad \varphi = \sin^{-1}\sqrt{1-\zeta^2}$

$C = -\frac{1}{\sin\varphi} = -\frac{1}{\sqrt{1-\zeta^2}}$

$x(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_d t + \varphi)$ unit response

for $f^* = k$

For $f^* = F_o$

$x^*(t) = \frac{F_o}{k}\left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_d t + \varphi)\right] = x_{st}\cdot x(t)$

$x_{st} = F_o/k$ static displacement

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14) Step response $f(t) = 1(t)$, $F(s) = \frac{1}{s}$

$$X(s) = \frac{\omega_n^2}{(s+\alpha)^2 + \omega_d^2} \cdot \frac{1}{s} = \frac{A}{s} + \frac{Ds+E}{(s+\alpha)^2 + \omega_d^2}$$

ILT by partial fraction expansion

$A(s+\alpha)^2 + A\omega_d^2 + Ds^2 + Es$ = win^2

$A(s^2 + 2s\alpha + \alpha^2 + \omega_d^2) + Ds^2 + Es = 1$

$s^2: \; A + D = 0$

$s^1: \; 2\alpha A + E = 0$

$s^0: A(\alpha^2 + \omega_d^2) = \omega_n^2 \qquad A(\cancel{\zeta^2}\omega_n^2 + (1-\cancel{\zeta^2})\omega_n^2) = \omega_n^2$

$A = 1$

$D = -A = -1$

$E = -2\alpha A = -2\alpha$

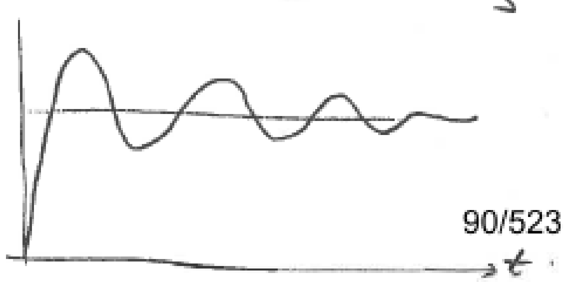
$$X(s) = \left[\frac{1}{s} - \frac{s+2\alpha}{(s+\alpha)^2 + \omega_d^2}\right] = \left[\frac{1}{s} - \frac{\alpha + (s+\alpha)}{(s+\alpha)^2 + \omega_d^2}\right]$$

$$x(t) = 1 - e^{-\alpha t}\left[\frac{\alpha}{\omega_d}\sin\omega_d t + \cos\omega_d t\right]$$

$$x(t) = 1 - e^{-\alpha t}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t + \cos\omega_d t\right)$$

$$x(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_d t + \varphi) \qquad \varphi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$x_{ss} = 1$



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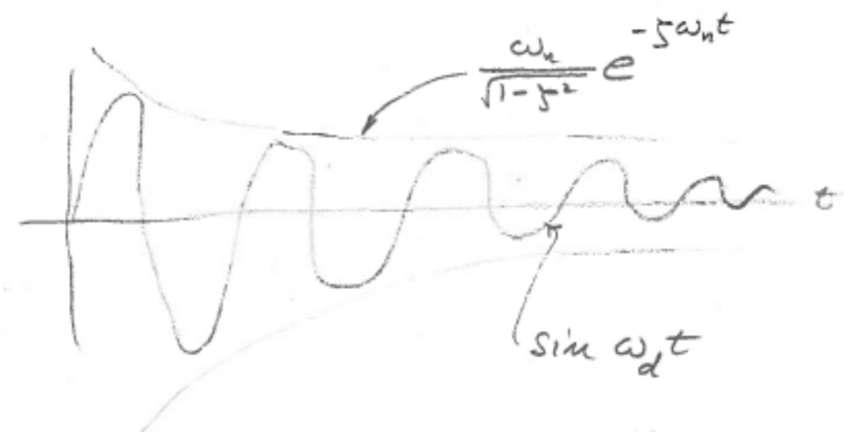
4.5.2 Impulse response of a 2nd Order System

$$X(s) = G(s)F(s) \tag{322}$$
$$= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot 1$$
$$= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{323}$$

Therefore, solving for $\mathscr{L}[X(s)]^{-1}$ yields

$$x(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n^2} \sin(\omega_d t) \tag{324}$$

or



Proof

13a) Long hand solution

ILT by partial fraction expansion

$$\frac{X}{\omega_n^2} = \frac{1}{(s+p_1)(s+p_2)} = \frac{A}{s+p_1} + \frac{B}{s+p_2}$$

$$As + Ap_2 + Bs + Bp_1 = 1$$

$$s^1: \quad A + B = 0 \longrightarrow B = -A$$

$$s^0: \quad Ap_2 + Bp_1 = 1 \longrightarrow A = \frac{1}{p_2 - p_1}, \quad B = \frac{1}{p_1 - p_2}$$

$$\frac{1}{(s+p_1)(s+p_2)} = \frac{1}{p_2 - p_1}\left(\frac{1}{s+p_1} - \frac{1}{s+p_2}\right)$$

$$x(t) = \frac{1}{p_2 - p_1}\left(e^{-p_1 t} - e^{-p_2 t}\right)$$

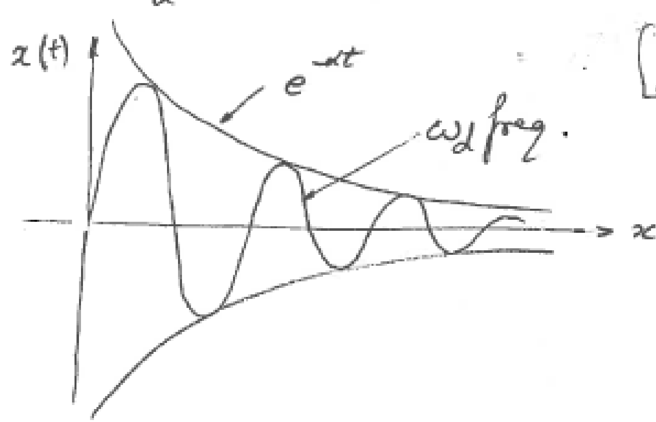
$$p_{1,2} = \zeta\omega_n \mp i\omega_n\sqrt{1-\zeta^2} = \alpha \mp i\omega_d.$$

$$p_2 - p_1 = i2\omega_d$$

$$\frac{x(t)}{\omega_n^2} = \frac{1}{2i\omega_d}\left[e^{-(\alpha - i\omega_d)t} - e^{-(\alpha + i\omega_d)t}\right]$$

$$= \frac{1}{2i\omega_d} e^{-\alpha t}\left[e^{i\omega_d t} - e^{-i\omega_d t}\right]$$

$$= \frac{1}{\omega_d} e^{-\alpha t} \frac{e^{i\omega_d t} - e^{-i\omega_d t}}{2i} = \frac{e^{-\alpha t}}{\omega_d} \sin\omega_d t$$



$$\begin{cases} \alpha = \zeta\omega_n \\ \omega_d = \omega_n\sqrt{1-\zeta^2} \end{cases}$$

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Alternative way

$$\frac{1}{(s+p_1)(s+p_2)} \xrightarrow[\mathcal{L}^{-1}]{} \frac{1}{p_2-p_1}\left(e^{-p_1 t}-e^{-p_2 t}\right)$$

Another way Residue Theorem

$$\frac{1}{(s+p_1)(s+p_2)} = \frac{a_1}{s+p_1}+\frac{a_2}{s+p_2}$$

$$a_1 = \left[\cancel{(s+p_1)}\frac{1}{\cancel{(s+p_1)}(s+p_2)}\right]_{s=-p_1} = \frac{1}{-p_1+p_2}$$

$$a_2 = \left[\cancel{(s+p_2)}\frac{1}{(s+p_1)\cancel{(s+p_2)}}\right]_{s=-p_2} = \frac{1}{p_1-p_2} = -a_1$$

4.5.3 Ramp response of a 2nd Order System

$$\begin{aligned} X(s) &= G(s)F(s) && (325)\\ &= \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\cdot\frac{1}{s^2} \\ &= \frac{\omega_n^2}{s^2(s^2+2\zeta\omega_n s+\omega_n^2)} && (326)\end{aligned}$$

Therefore, solving for $\mathscr{L}[X(s)]^{-1}$ yields

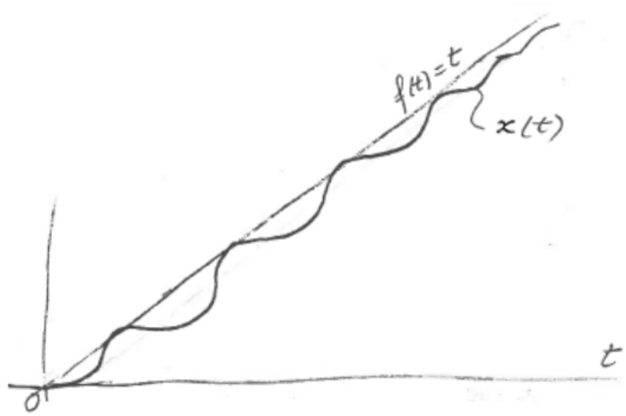
$$x(t) = t-\frac{2\zeta}{\omega_n}\left(1+\frac{1}{\sin\gamma_1}e^{-\zeta\omega_n t}\sin(\omega_d t-\gamma_1)\right) \quad (327)$$

where

$$\gamma_1 = \tan^{-1}\frac{2\zeta\sqrt{1-\zeta^2}}{1-2\zeta^2} \tag{328}$$

$$= \sin^{-1} 2\zeta\sqrt{1-\zeta^2} \tag{329}$$

or



Moreover,

$$\sin^2\gamma_1 = \frac{\tan^2\gamma_1}{1+\tan^2\gamma_1} \tag{330}$$
$$= \frac{4s^2(1-\zeta^2)}{(1-2\zeta^2)+4\zeta^2(1-\zeta^2)}$$
$$= \frac{4s^2(1-\zeta^2)}{1-4s^2+4s^4+4s^2-4s^4}$$
$$= 4s^2(1-\zeta^2)$$

and

$$\sin\gamma_1 = 2\zeta\sqrt{1-\zeta^2} \tag{331}$$

Proof

4 2016 10 21

PROOF of 2nd order system Ramp response

ODE solution

$$x_p(t) = Dt + E$$

$$\dot{x}_p = D \; ; \; \ddot{x}_p = 0$$

$$\ddot{x}_p + 2\zeta\omega_n \dot{x}_p + \omega_n^2 x_p = \omega_n^2 f(t)$$

$$2\zeta\omega_n D + \omega_n^2 (Dt + E) = \omega_n^2 t$$

$$t^0: \; 2\zeta\omega_n D + \omega_n^2 E = 0 \longrightarrow E = -\frac{2\zeta}{\omega_n} D$$

$$t^1: \; \omega_n^2 D = \omega_n^2 \longrightarrow D = 1 \longrightarrow E = \frac{2\zeta}{\omega_n}$$

$$x_p(t) = t - 2\zeta/\omega_n$$

$$x(t) = e^{-\underset{\alpha}{\zeta\omega_n} t} C \sin(\overbrace{\omega_d t + \varphi_1}^{\psi}) + t - \frac{2\zeta}{\omega_n}$$

$$\dot{x}(t) = (-\alpha \sin\psi + \omega_d \cos\psi) C e^{-\alpha t} + 1$$

$$x(0) = 0: \; C \sin\varphi_1 - \frac{2\zeta}{\omega_n} = 0 \longrightarrow C\omega_n \sin\varphi_1 = 2\zeta \quad (a)$$

$$\dot{x}(0) = 0: \; (-\alpha \sin\varphi_1 + \omega_d \cos\varphi_1) C + 1 = 0$$

$$C(\alpha \sin\varphi_1 - \omega_d \cos\varphi_1) = 1$$

$$C\zeta\omega_n \sin\varphi_1 - C\omega_d \cos\varphi_1 = 1$$

$$(a): \quad 2\zeta^2 - C\omega_d \cos\varphi_1 = 1$$

$$C\omega_n \sqrt{1-\zeta^2} \cos\varphi_1 = 2\zeta^2 - 1$$

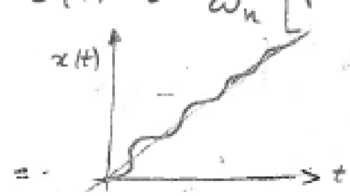
$$C\omega_n \cos\varphi_1 = -\frac{1-2\zeta^2}{\sqrt{1-\zeta^2}} \quad (b)$$

$$\frac{(a)}{(b)} = \tan\varphi_1 = -\frac{2\zeta\sqrt{1-\zeta^2}}{1-2\zeta^2}$$

Note: φ_1 is different from $\varphi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$ of step response

$$\varphi_1 = -\tan^{-1}\frac{2\zeta\sqrt{1-\zeta^2}}{1-2\zeta^2}, \quad C = \frac{2\zeta}{\omega_n} \cdot \frac{1}{\sin\varphi_1}$$

$$x(t) = t - \frac{2\zeta}{\omega_n}\left[1 - \frac{1}{\sin\varphi_1} e^{-\zeta\omega_n t} \sin(\omega_d t + \varphi_1)\right]$$



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4.5.4 Summary of the Second Order System Responses

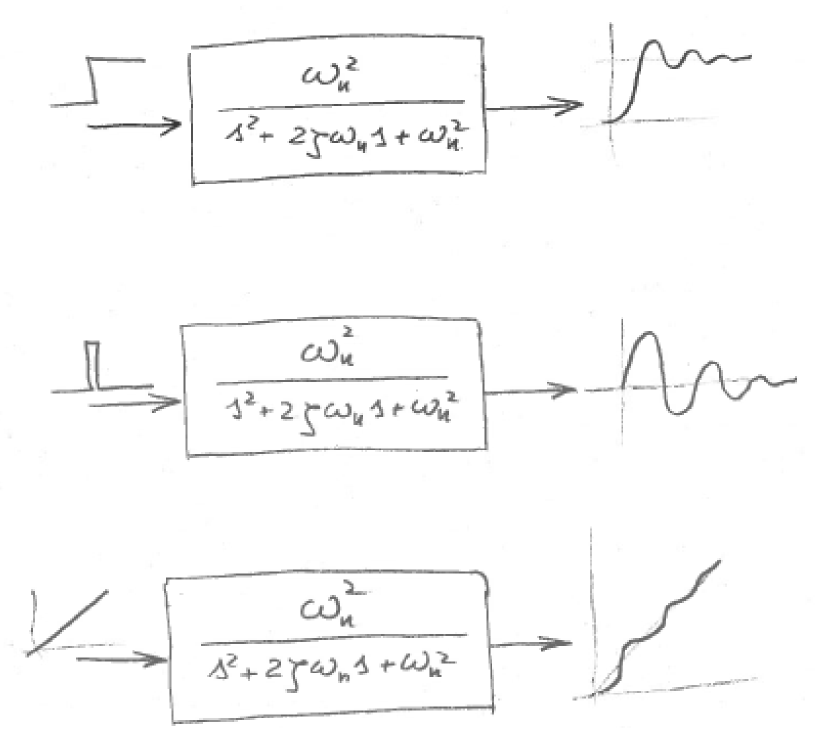


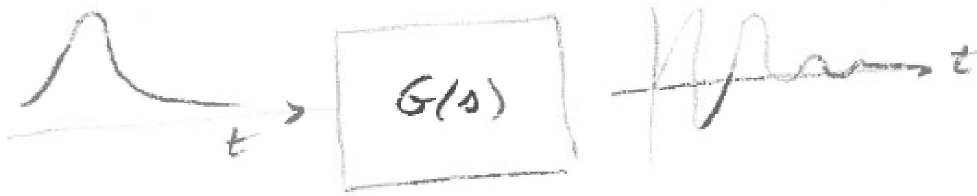
Figure 4.6: A summary of the second order system responses.

Listing 9: MATLAB code for time series responses of 2nd order system.

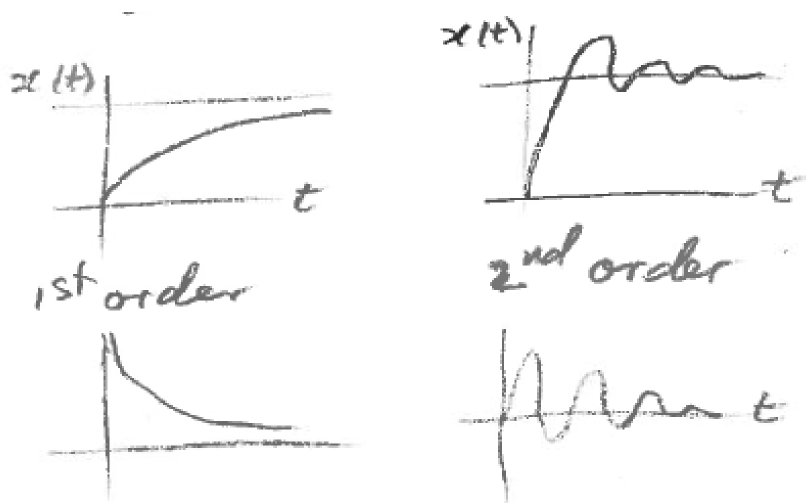
```
%{
This program studies time response of 2nd order systems
%}

clc
clear
close all

format compact

%% Given data
fn = 5; % time response for the 1D system
wn = 2*pi()*fn
z = 0.035

%% time range setup
T_max = 10;          % run the test to 10 seconds
dt = T_max*1e-4;     % find the delta-t value
t = 0:dt:T_max;      % build the time vector

%% Define system
B = [wn^2];
A = [1 2*z*wn wn^2];
G = tf(B,A);

%% create the figure enviorment
figure(1)
xlim([0 1])

%% step response
subplot(3,1,1)
hold on
step(G,t)
ylim([0 2])

%% Impulse response
subplot(3,1,2)
impulse(G,t)
ylim([-35 35])

%% Ramp response
F_ramp = tf([1],[1 0 0])
subplot(3,1,3)
impulse(G*F_ramp,t)
xlim([0 1])
ylim([0 1])
title('Ramp Response') % need to set manually.
```

4.6 Stability of response

A system is stable if any stable input excitation produces a stable output response.



A response is stable if it remains bounded at $t \to \infty$



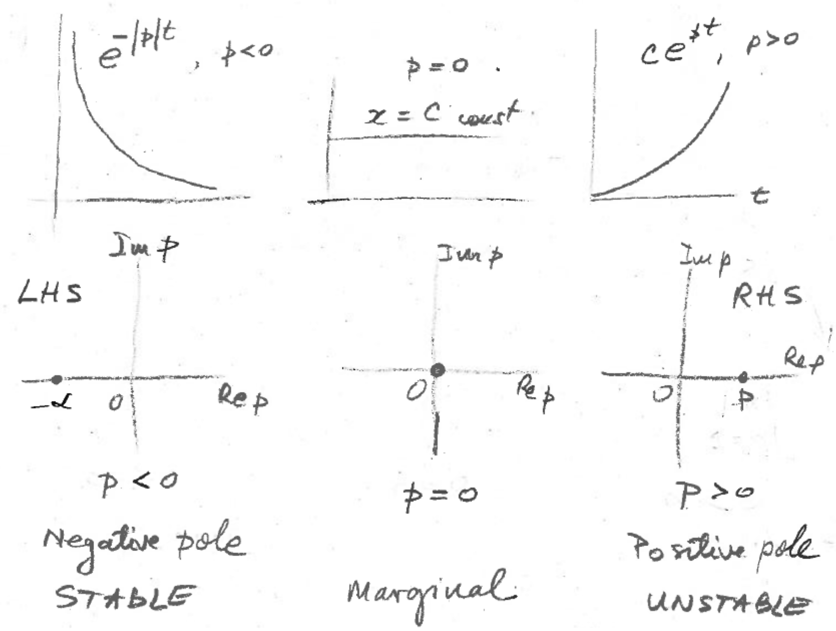
4.6.1 Stability of 1st-Order Responses

$$X(s) = \frac{K}{s-p} \tag{332}$$

$$x(t) = Ke^{pt} \tag{333}$$

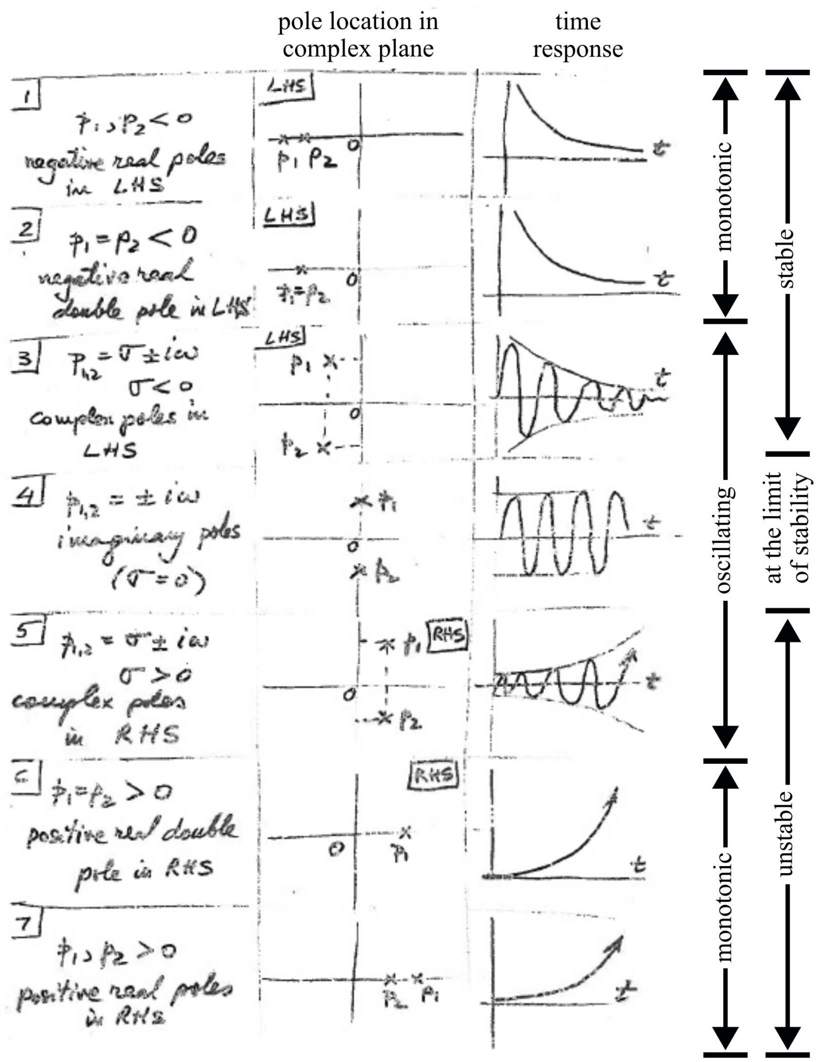
where p is the pole of $X(s)$.

The stability is dictated by the sign of p, i.e. its location in the complex p-plane. If $P < 0$ (or p is in the left-hand-side), the system is stable. Therefore, if a disturbing force is applied, the system will return to its initial state.



4.6.2 Stability of 2$^{\text{nd}}$-Order Responses

$$X(s) = \frac{k(s-z_1)}{(s-p_1)(s-p_2)} \tag{334}$$



4.6.3 Stability of Higher-order Responses

Starting with a general expression for the output of a system

$$X(s) = \frac{B(s)}{A(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0} \tag{335}$$

partial fraction expansion results in

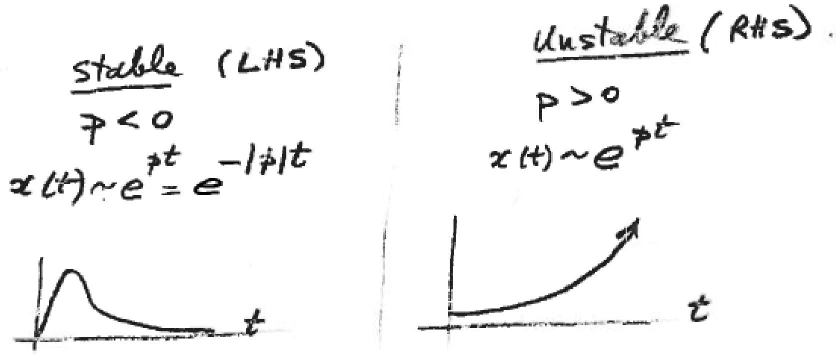
$$X(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \cdots + \frac{r_n}{s-p_n} \tag{336}$$

where $p_1,\ p_2,\ \cdots\ p_n$ are the poles of the system (i.e. roots of $A(s) = 0$) and $r_1,\ r_2,\ \cdots\ r_n$ are the residues of the system. Note that the poles can either be real or complex. Again, MATLAB can be used to solve for the roots, poles, and gains of the system using `[r,p,k]=residue(B,A)`. The real poles can be

- single pole: $\frac{r}{s-p} \xrightarrow{\mathscr{L}[\ ]^{-1}} re^{pt}$

- double poles: $\frac{r}{(s-p)^2} \xrightarrow{\mathscr{L}[\ ]^{-1}} rte^{pt}$
- multiple poles: $\frac{r}{(s-p)j} \xrightarrow{\mathscr{L}[\ ]^{-1}} \frac{1}{(j-1)!}t^{j-1}e^{pt}$

where stable and unstable responses are



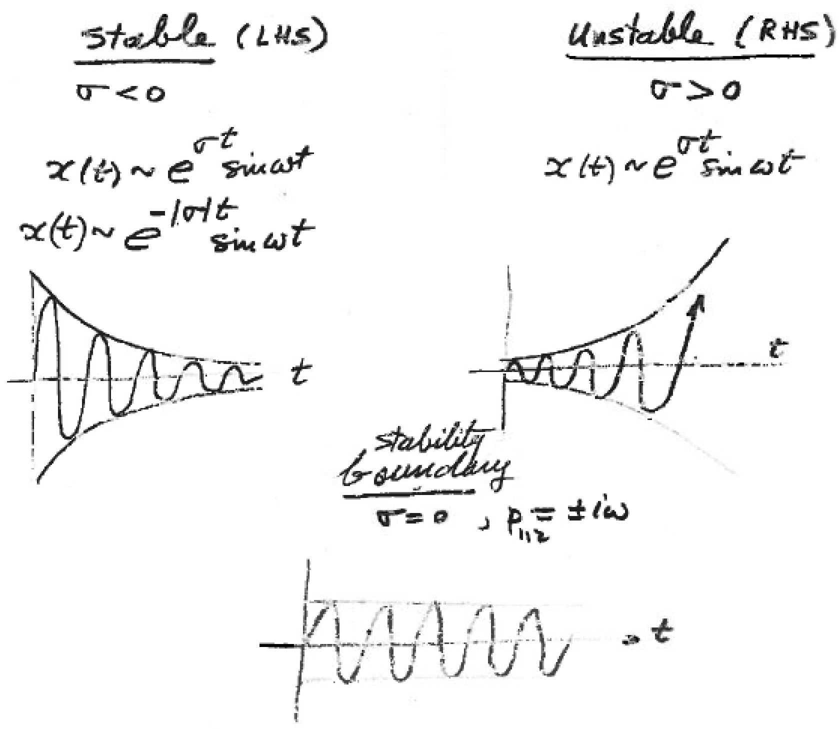
The complex poles are always in conjugate pairs, and are governed by $p_{1,2} = \sigma \pm i\omega$ where

- first pole: $\frac{1}{s-p_1} = \frac{1}{s-(\sigma+i\omega)} \xrightarrow{\mathscr{L}[\ ]^{-1}} e^{(\sigma+i\omega)t} = e^{\sigma t}e^{i\omega t}$
- second pole: $\frac{1}{s-p_2} = \frac{1}{s-(\sigma-i\omega)} \xrightarrow{\mathscr{L}[\ ]^{-1}} e^{(\sigma-i\omega)t} = e^{\sigma t}e^{-i\omega t}$

Using Euler's formula, this results in

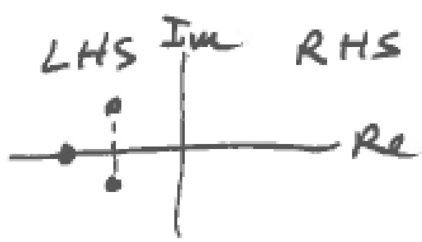
$$x(t) = e^{\sigma t}(e^{i\omega t} - e^{-i\omega t}) = e^{\sigma t}\sin(\omega t) \tag{337}$$

where stable and unstable responses are



4.6.4 Absolute Stability

A necessary and sufficient condition for a system to be stable is that its poles are placed in the Left hand side of the complex plane.



4.6.5 Marginal Stability

If the poles are purely imaginary (i.e. placed on the imaginary axis) then the system as marginal stability.

- bounded impulse response, the system is stable
- unbounded impulse response, or other inputs, the system is not stable.

4.6.6 Relative Stability

Would a stable system still be stable if its parameters are slightly changed? What margin of safety is there?

5 Performance Indicators

Performance indicators are used to judge the quality of a control system.

5.1 1st-order System Generic Performance Indicators

Generic performance indicators are:

- steady-state value $x_{ss} = \lim_{t \to \infty} x(t)$
- steady-state error $e_{ss} = \lim_{t \to \infty} e(t)$ where $e(t) = f(t) - x(t)$

To expand on the error definition, $e(t) = f(t) - x(t)$, consider a step response where $x(\infty) = 1$ and $f(\infty) = 1$, therefore $e(\infty) = 0$.

For a 1st-order system, x_{ss} and e_{ss} can be found using the final value theorem and exist in both the time domain and the s-domain. Recall the final value theorem for x_{ss} and e_{ss} leads to

$$x_{ss} = \lim_{s \to 0} sX(s) \tag{338}$$

$$e_{ss} = \lim_{s \to 0} sE(s) \tag{339}$$

Therefore, starting at the transfer function of a 1st-order system

$$G(s) = \frac{1}{Ts+1} \tag{340}$$

we can expand on this to show

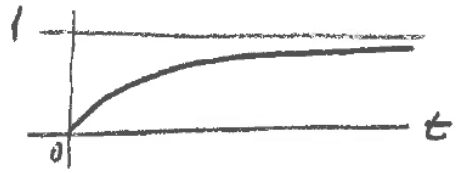
$$\begin{aligned} X(s) &= G(s)F(s) && (341) \\ &= \frac{1}{Ts+1}F(s) \end{aligned}$$

next, the error is s-domain is shown to be

$$\begin{aligned} E(s) &= F(s) - X(s) && (342) \\ &= F(s) - G(s)F(s) && (343) \\ &= \big(1 - G(s)\big)F(s) \\ &= \frac{Ts}{Ts+1}F(s) \end{aligned}$$

Again, these are general terms for a 1st-order system.

5.1.1 Step response 1st-order system performance indicators



For a 1st-order system subjected to a step response, we want to find x_{ss} and e_{ss} and we know the transfer function is defined as

$$G(s) = \frac{1}{Ts+1} \tag{344}$$

while the equation of motion is

$$T\dot{x} + x = F(s) \tag{345}$$

Where the step function is defined as

$$f(t) = 1,\ t > 0 \tag{346}$$

therefore, the system response is

$$x(t) = 1 - e^{-t/T} \tag{347}$$

while the steady-state response is

$$\begin{aligned} x_{ss} &= \lim_{t\to\infty} \\ &= 1 \end{aligned} \tag{348}$$

Next, the error as a function of time is

$$\begin{aligned} e(t) &= 1 - (1 - e^{-t/T}) \\ &= e^{-t/T} \end{aligned} \tag{349}$$

while the steady-state error is

$$\begin{aligned} e_{ss} &= \lim_{t\to\infty} e(t) \\ &= \lim_{t\to\infty} e^{t/T} \\ &= 0 \end{aligned} \tag{350}$$

These same solutions can be found in the s-domain where the step function is $F(s) = \frac{1}{s}$. Consider the s-domain expression solved for $X(s)$,

$$X(s) = \frac{1}{Tx+1} \cdot \frac{1}{s} \tag{351}$$

the steady-state error is shown to be

$$\begin{aligned} x_{ss} &= \lim_{s\to 0} s\frac{1}{Ts+1}\cdot\frac{1}{s} \\ &= \lim_{s\to 0}\frac{1}{Ts+1} \\ &= 1 \end{aligned} \tag{352}$$

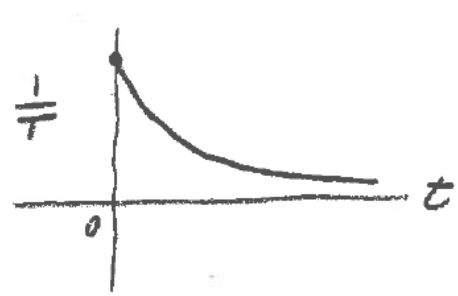
Next, we can build the s-domain representation of the error as

$$\begin{aligned} E(s) &= \frac{Ts}{Ts+1}\cdot\frac{1}{s} \\ &= \frac{T}{Ts+1} \end{aligned} \tag{353}$$

solving for the steady-state error results in

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} s\frac{T}{Ts+1} \\ &= 0 \end{aligned} \tag{354}$$

5.1.2 Impulse response 1^{st}-order system performance indicators



For a 1^{st}-order system subjected to a impulse response, we want to find x_{ss} and e_{ss}. The impulse function is defined as

$$f(t) = \delta(t) \tag{355}$$

therefore, the response is

$$x(t) = \frac{1}{T}e^{-t/T} \tag{356}$$

where the steady-state response is

$$x_{ss} = 0 \tag{357}$$

The error is

$$e(t) = \delta(t) - \frac{1}{T}e^{-t/T} \tag{358}$$

Lastly, the steady-state error is

$$\begin{aligned} e_{ss} &= \lim_{t\to\infty} e(t) \\ &= 0 \end{aligned} \tag{359}$$

These same solutions can be found in the s-domain where the impulse function is $F(s) = 1$. Consider the s-domain expression solved for $X(s)$,

$$X(s) = \frac{1}{Ts+1} \tag{360}$$

the steady-state error is shown to be

$$\begin{aligned} x_{ss} &= \lim_{s\to 0} s\frac{1}{Ts+1} \\ &= 0 \end{aligned} \tag{361}$$

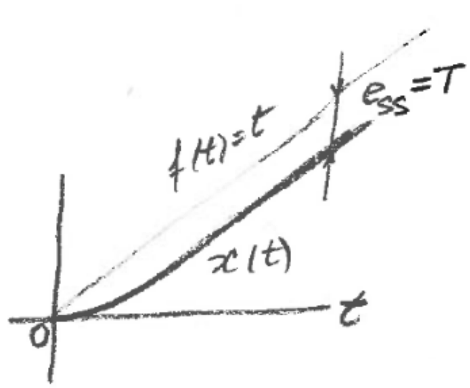
The s-domain error is expressed as

$$E(s) = \frac{T}{Ts+1} \tag{362}$$

which leads to the steady-state error value

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} s\frac{T}{Ts+1} \\ &= 0 \end{aligned} \tag{363}$$

5.1.3 Ramp response 1st-order system performance indicators



For a 1st-order system subjected to a ramp response, we want to find x_{ss} and e_{ss}. The impulse function is defined as

$$f(t) = t \tag{364}$$

therefore, the response is

$$x(t) = t - T(1 - e^{-t/T}) \tag{365}$$

which leads to the steady-state response

$$x_{ss} = \infty \tag{366}$$

which means that there is no steady-state value. The error is

$$e(t) = T(1 - e^{-t/T}) \tag{367}$$

Lastly, the steady-state error is

$$\begin{aligned} e_{ss} &= \lim_{t\to\infty} e(t) \\ &= T - \lim_{t\to\infty} e^{-t/T} \\ &= T \end{aligned} \tag{368}$$

as $\lim_{t\to\infty} e^{-t/T} = 0$.

These same solutions can be found in the s-domain where the ramp function is $F(s) = \frac{1}{s^2}$. Consider the s-domain expression solved for $X(s)$,

$$X(s) = \frac{1}{Ts+1} \cdot \frac{1}{s^2} \tag{369}$$

the steady-state error is shown to be

$$\begin{aligned} x_{ss} &= \lim_{s\to 0} s \frac{1}{Ts+1} \cdot \frac{1}{s^2} \\ &= \lim_{s\to 0} \frac{1}{Ts+1} \cdot \frac{1}{s} \\ &= \infty \end{aligned} \tag{370}$$

therefore, there is no steady-state value. The s-domain error is expressed as

$$\begin{aligned} E(s) &= \frac{Ts}{Ts+1} \cdot \frac{1}{s^2} \\ &= \frac{T}{Ts+1} \cdot \frac{1}{s} \end{aligned} \tag{371}$$

which leads to the steady-state error value

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} s \frac{T}{Ts+1} \cdot \frac{1}{s} \\ &= \lim_{s\to 0} \frac{T}{Ts+1} \\ &= T \end{aligned} \tag{372}$$

5.2 1st-order System Specific Performance Indicators

Specific performance indicators exist. They depend on system order and excitation type. Examples are:

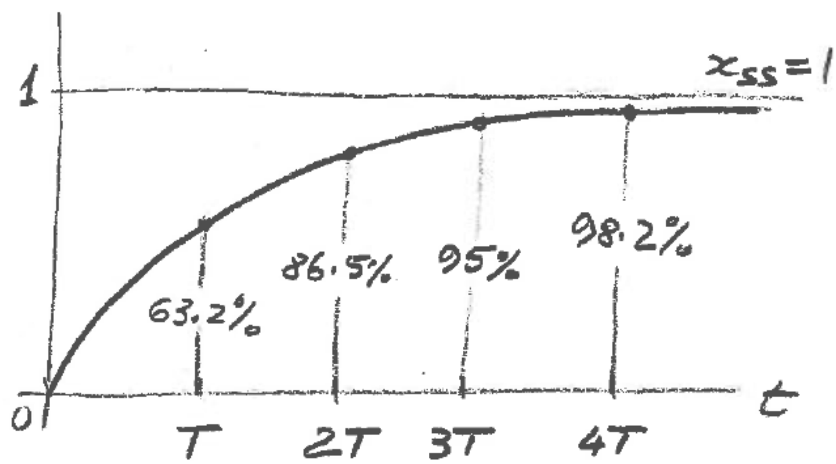
- rise-time $\rightarrow t_r$
- delay time $\rightarrow t_d$
- settling time $\rightarrow t_s$

- decay time / half-time $\rightarrow t_{1/2}$

For a step response, consider the system displacement

$$x(t) = 1 - e^{-t/T} \tag{373}$$

that is plotted as



The rise-time (t_r) of a system

- to rise 63.2% of $x_s s$ is $t_{r\_63.2\%} = T$
- to rise 86.5% of $x_s s$ is $t_{r\_86.5\%} = 2T$
- to rise 95% of $x_s s$ is $t_{r\_95\%} = 3T$
- to rise 98.2% of $x_s s$ is $t_{r\_98.2\%} = 4T$

in general, $1 - e^{-t/T} = x \rightarrow t = -T\ln(1-x)$

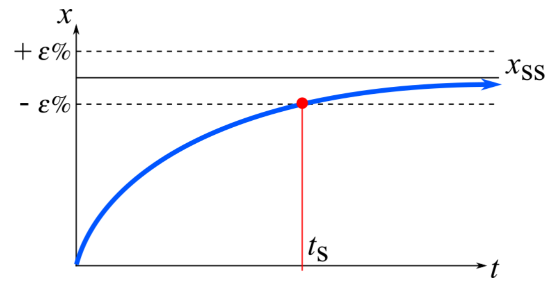
The delay time (t_d) is the time it takes to rise to 50% of x_{ss},

$$\begin{aligned} x(t) &= 1 - e^{-t/T} \\ &= 0.5 \end{aligned} \tag{374}$$

therefore, solving for $t = t_d$,

$$\begin{aligned} e^{-t_d/T} &= 0.5 \\ \frac{-t_d}{T} &= \ln(0.5) \\ t_d &= -T\ln(0.5) \\ t_d &= 0.693T \\ t_d &\approx 0.7T \end{aligned} \tag{375}$$

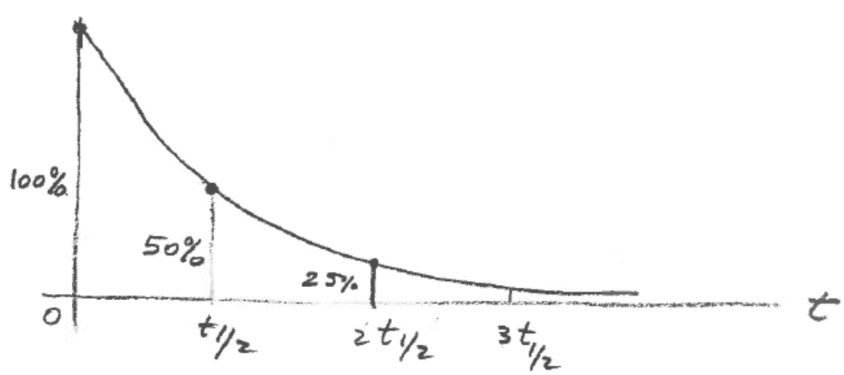
The settling time (t_s) is the time its takes x to get within $\varepsilon\%$ of x_{ss}.



A more precise estimator is the rise time to any selected x which can be defined as

$$
\begin{aligned}
x_{ss}(1-e^{-t/T}) &= x \qquad (376)\\
e^{(-t/T)} &= 1-\frac{x}{x_{ss}}\\
-t/T &= \log\left[1-\frac{x}{x_{ss}}\right]\\
t &= -T\log\left[1-\frac{x}{x_{ss}}\right]
\end{aligned}
$$

5.2.1 Impulse Response Specific Performance Indicators



The half-cycle decay time ($t_{1/2}$) is an impulse response specific performance indicator. Given that the system response to an impulse is

$$x(t) = \frac{1}{T}e^{-t/T} \qquad (377)$$

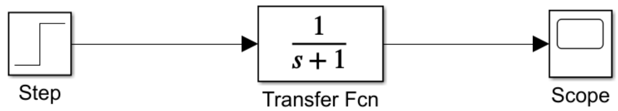
the half-life decay time is where $e^{-t/T} = \frac{1}{2}$, therefore

$$
\begin{aligned}
t_{1/2} &= -\ln\left(\frac{1}{2}\right)T \qquad (378)\\
&\approxeq 0.693T
\end{aligned}
$$

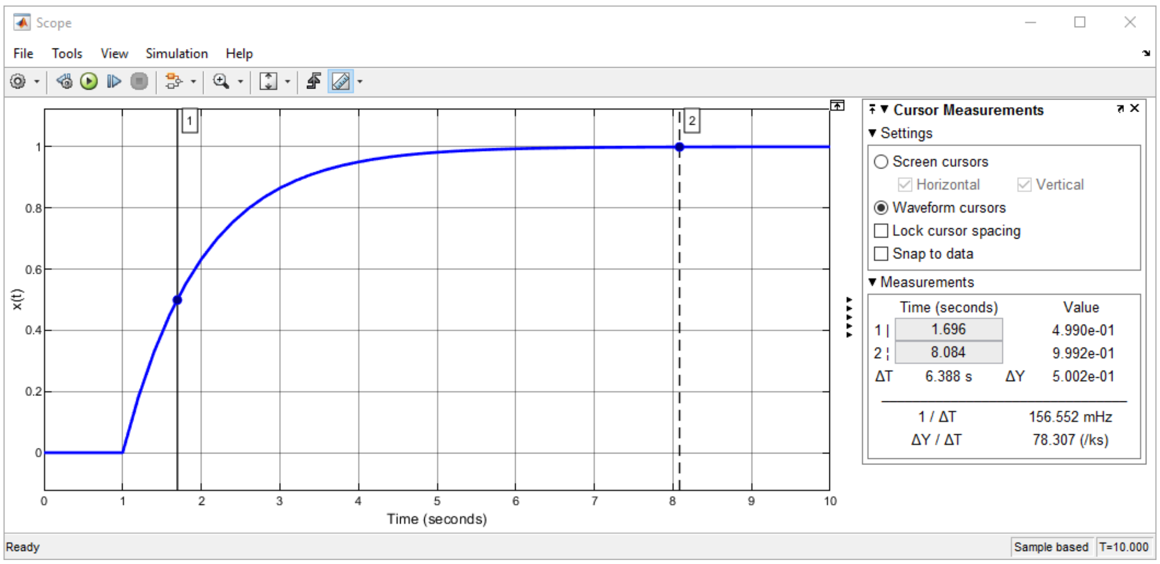
Note that the signal continues to decay by half of its value after each additional $t_{1/2}$.

Example 5.1 SIMULINK Tutorial on Measuring Performance Indicators

Build the simple 1^{st}-order system as shown below



Open the scope and press the 'Cursor Measurements' button to activate the cursors.



To measure the delay time t_d, place the first cursor around the point where 'Value' measurement is closest to 0.5. $x(t) = 0.5$. Read the 'Time' value. This is estimate for t_d . It gives the value $t_d = 0.70$ sec, as the step function happens at 1 sec.

The second cursor can be used to get the settling time t_s . We are going to use the 1% definition of t_d . This means that the response should be around 0.99, or 9.9e-1. Reading the corresponding time value, we get $t_s \approx 7.0$ sec.

5.3 2^{nd}-order System Generic Performance Indicators

Again, starting at the equation of motion

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 f(t) \tag{379}$$

and transfer function of a 2^{nd}-order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{380}$$

we can expand on this to show

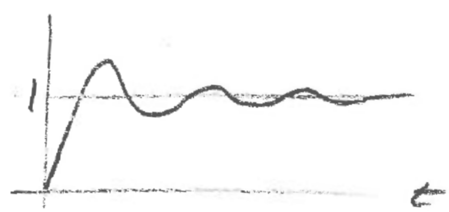
$$\begin{aligned} X(s) &= G(s)F(s) \\ &= \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}F(s) \end{aligned} \tag{381}$$

next, the error in the s-domain is shown to be

$$\begin{aligned} E(s) &= F(s)-X(s) \\ &= F(s)-G(s)F(s) \\ &= \big(1-G(s)\big)F(s) \\ &= \frac{s^2+2\zeta\omega_n s}{s^2+2\zeta\omega_n s+\omega_n^2}F(s) \end{aligned} \tag{382}$$

Again, these are general terms for a 2$^{\text{nd}}$-order system. Note that the definitions for the steady-state value x_{ss} and steady-state error e_{ss} remain largely unchanged from the 1$^{\text{st}}$-order system.

5.3.1 Step response 2$^{\text{nd}}$-order system performance indicators



For a 2$^{\text{nd}}$-order system subjected to a step response, we want to find x_{ss} and e_{ss}. Using the transfer function method to solve for the response, we know the system response is

$$x(t) = 1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_d t+\phi) \tag{383}$$

where

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \tag{384}$$

and

$$\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \sin^{-1}\sqrt{1-\zeta^2} \tag{385}$$

next, the steady-state displacement can be solved for as

$$\begin{aligned} x_{ss} &= \lim_{t\to\infty} x(t) \\ &= 1-\frac{\sqrt{1-\zeta^2}}{\zeta}\lim_{t\to\infty} e^{-\zeta\omega_n t}\sin(\omega_d t+\phi) \\ &= 1 \end{aligned} \tag{386}$$

as $e^{-\zeta\omega_n t}$ goes to 0. Next, the error as a function of time is

$$\begin{aligned} e(t) &= f(t) - x(t) \\ &= \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_d t + \phi) \end{aligned} \tag{387}$$

while the steady-state error is

$$\begin{aligned} e_{ss} &= \lim_{t\to\infty} e(t) \\ &= \frac{\sqrt{1-\zeta^2}}{\zeta}\lim_{t\to\infty} e^{-\zeta\omega_n t}\sin(\omega_d t + \phi) \\ &= 0 \end{aligned} \tag{388}$$

as $e^{-\zeta\omega_n t}$ goes to 0.

These same solutions can be found in the s-domain where the step function is $F(s) = \frac{1}{s}$. Consider the s-domain expression solved for $X(s)$,

$$X(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\cdot\frac{1}{s} \tag{389}$$

the steady-state displacement is shown to be

$$\begin{aligned} x_{ss} &= \lim_{s\to 0} s\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\cdot\frac{1}{s} \\ &= \frac{\omega_n^2}{\omega_n^2} \\ &= 1 \end{aligned} \tag{390}$$

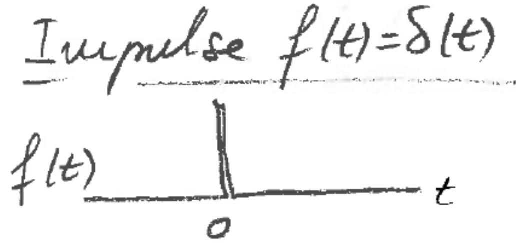
Next, we can build the s-domain representation of the error as

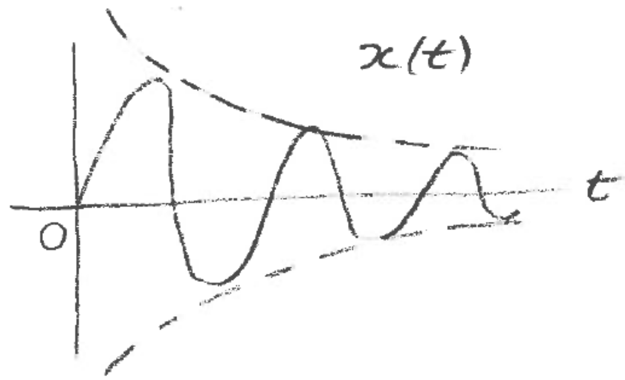
$$E(s) = \frac{s^2 + 2\zeta\omega_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2}\cdot\frac{1}{s} \tag{391}$$

solving for the steady-state error results in

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} s\frac{s^2 + 2\zeta\omega_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2}\cdot\frac{1}{s} \\ &= \frac{0}{\omega_n} \\ &= 0 \end{aligned} \tag{392}$$

5.3.2 Impulse response 2^{nd}-order system performance indicators





For a 2^{nd}-order system subjected to a impulse response, we want to find x_{ss} and e_{ss}. The impulse function is defined as

$$f(t) = \delta(t) \tag{393}$$

therefore, the response is

$$x(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t) \tag{394}$$

where the steady-state response is

$$\begin{aligned} x_{ss} &= \lim_{t\to\infty} x(t) \\ &= \frac{\omega_n}{\sqrt{1-\zeta^2}} \lim_{t\to\infty} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) \\ &= 0 \end{aligned} \tag{395}$$

as $e^{-\zeta\omega_n t}$ goes to 0. Next, the error as a function of time is

$$\begin{aligned} e(t) &= f(t) - x(t) \\ &= \delta(t) - \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) \end{aligned} \tag{396}$$

Lastly, the steady-state error is

$$\begin{aligned} e_{ss} &= \lim_{t\to\infty} e(t) \\ &= \lim_{t\to\infty} \delta(t) - \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) \\ &= 0 \end{aligned} \tag{397}$$

as $\delta(t)$ and $e^{-\zeta\omega_n t}$ both go to 0.

These same solutions can be found in the s-domain where the impulse function is $F(s) = 1$. Consider the s-domain expression solved for $X(s)$,

$$X(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{398}$$

the steady-state error is shown to be

$$x_{ss} = \lim_{s\to 0} s\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \\ = 0 \tag{399}$$

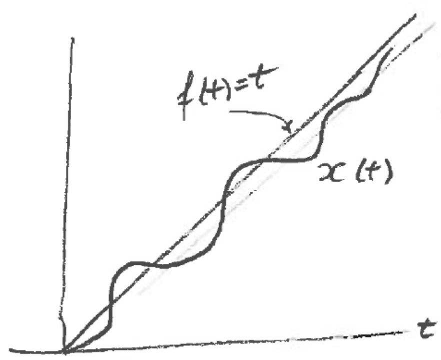
The s-domain error is expressed as

$$E(s) = \frac{s^2+2\zeta\omega_n s}{s^2+2\zeta\omega_n s+\omega_n^2} \tag{400}$$

which leads to the steady-state error value

$$e_{ss} = \lim_{s\to 0} s\frac{s^2+2\zeta\omega_n s}{s^2+2\zeta\omega_n s+\omega_n^2} \\ = 0 \tag{401}$$

5.3.3 Ramp response 2nd-order system performance indicators



For a 2nd-order system subjected to a ramp response, we want to find x_{ss} and e_{ss}. The impulse function is defined as

$$f(t) = t \tag{402}$$

therefore, the response can be found through the transfer function approach as

$$x(t) = t - \frac{2\zeta}{\omega_n}\left[1+\frac{1}{\sin\phi_1}e^{-\zeta\omega_n t}\sin(\omega_d t+\phi_1)\right] \tag{403}$$

where

$$\phi_1 = \tan^{-1}\frac{2\zeta\sqrt{1-\zeta^2}}{1-2\zeta^2} \tag{404}$$

which leads to the response

$$x_{ss} = \lim_{t\to\infty} x(t) \\ = t - \frac{2\zeta}{\omega_n} \\ = \infty \tag{405}$$

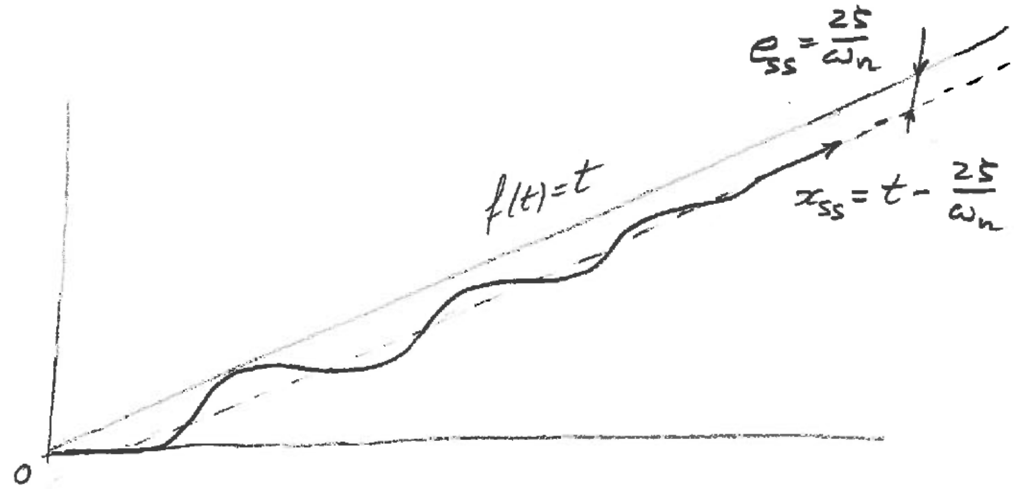
which means that there is no steady-state value. Next, the error as a function of time is

$$\begin{aligned} e(t) &= f(t) - x(t) \\ &= t - \left[t - \frac{2\zeta}{\omega_n} \left[1 + \frac{1}{\sin\phi_1} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi_1) \right] \right] \\ &= \frac{2\zeta}{\omega_n} \left[1 + \frac{1}{\sin\phi_1} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi_1) \right] \end{aligned} \tag{406}$$

Lastly, the steady-state error is

$$\begin{aligned} e_{ss} &= \lim_{t\to\infty} e(t) \\ &= \frac{2\zeta}{\omega_n} \lim_{t\to\infty} \left[1 + \frac{1}{\sin\phi_1} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi_1) \right] \\ &= \frac{2\zeta}{\omega_n} \end{aligned} \tag{407}$$

as the term in the brackets goes to 1.



These same solutions can be found in the s-domain where the ramp function is $F(s) = \frac{1}{s^2}$. Consider the s-domain expression solved for $X(s)$,

$$X(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s^2} \tag{408}$$

the steady-state error is shown to be

$$
\begin{aligned}
x_{ss} &= \lim_{s\to 0} s \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\cdot\frac{1}{s^2} \\
&= \lim_{s\to 0} \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\cdot\frac{1}{s} \\
&= \lim_{s\to 0} \frac{1}{s} \\
&= \infty
\end{aligned}
\tag{409}
$$

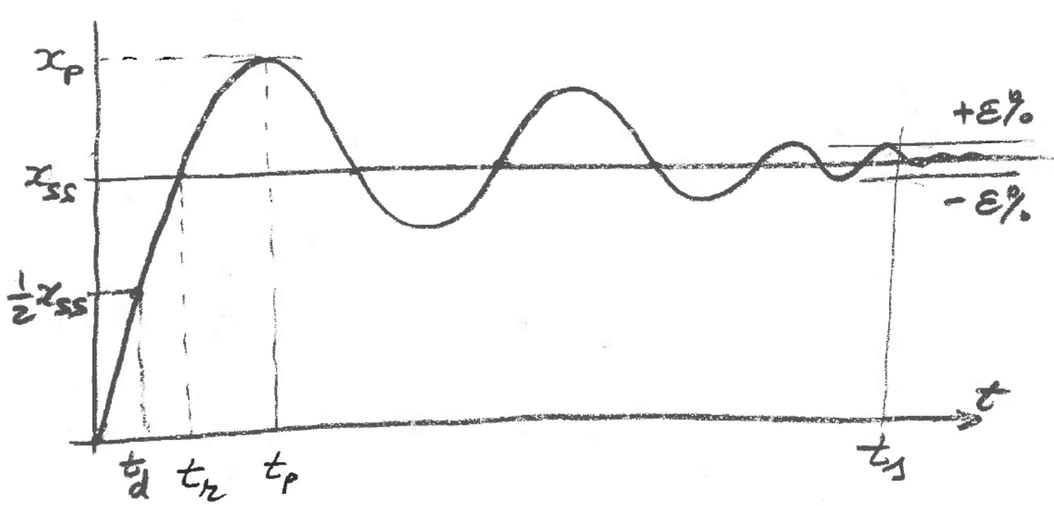
therefore, there is no steady-state value. The s-domain error is expressed as

$$
E(s) = \frac{s^2+2\zeta\omega_n s}{s^2+2\zeta\omega_n s+\omega_n^2}\cdot\frac{1}{s^2} \tag{410}
$$

which leads to the steady-state error value

$$
\begin{aligned}
e_{ss} &= \lim_{s\to 0} s \frac{s^2+2\zeta\omega_n s}{s^2+2\zeta\omega_n s+\omega_n^2}\cdot\frac{1}{s^2} \\
&= \lim_{s\to 0} \frac{s^2(s+2\zeta\omega_n)}{s^2+2\zeta\omega_n s+\omega_n^2}\cdot\frac{1}{s^2} \\
&= \frac{2\zeta\omega_n}{\omega_n^2} \\
&= \frac{2\zeta}{\omega_n}
\end{aligned}
\tag{411}
$$

5.4 2nd-order System Specific Performance Indicators



Specific performance indicators exist. They depend on system order and excitation type. Examples are:

- rise time $\rightarrow t_r$
- peak time $\rightarrow t_p$
- peak value $\rightarrow x_p$
- settling time $\rightarrow t_s$
- delay time $\rightarrow t_d$
- max percentage overshoot $\rightarrow M_p$

Many of these are the same as a 1st-order system or taken directly from the system response. The max percentage overshoot (M_p) is defined as

$$M_p = \left(\frac{x_p}{x_{ss}} - 1\right) \cdot 100 \tag{412}$$

5.4.1 Procedure for a Step Response

Given the 2nd-order system, with the transfer function

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{413}$$

where ω_n is the natural frequency and ζ is the critical damping ratio. Considering that the system is subjected to a step response, we know the system response is

$$x(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) \tag{414}$$

where we can calculate:

$$\omega_d = \omega_n \sqrt{1-\zeta^2} \tag{415}$$

and

$$\phi = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta} = \sin^{-1} \sqrt{1-\zeta^2} \tag{416}$$

Next, compute the values for the performance indicators.

First the rise time is calculated by the fact that $x(0) = 0$ and $x(t_r) = 1$. Form equation 414, the mean height of the system is shown to be when $\sin(\omega_d t + \phi) = 0$, therefore, as $\sin(\pi) = 0$, we can show that $\omega_d t + \phi = \pi$, rearranging this yields

$$t_r = \frac{\pi - \phi}{\omega_d} \tag{417}$$

To find the peak time (t_p), we need to find the peak, which is defined as $\frac{dx}{dt}\big|_{t=t_p} = 0$, or drawn as

where we consider only the decaying part of the signal caused by the step function, or $x = e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$. Therefore

$$\begin{aligned}
0 &= \frac{dx}{dt} \\
&= \frac{d}{dt}\left[e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)\right] \\
&= -\zeta\omega_n e^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + \omega_d e^{-\zeta\omega_n t}\cos(\omega_d t + \phi) \\
\zeta\omega_n \sin(\omega_d t + \phi) &= \omega_d \cos(\omega_d t + \phi) \\
\frac{\sin(\omega_d t + \phi)}{\cos(\omega_d t + \phi)} &= \frac{\omega_d}{\zeta\omega_n}
\end{aligned} \tag{418}$$

By converting the left has side of this equation to $\tan(\omega_d t_p + \phi)$ yields

$$\begin{aligned}
\tan(\omega_d t_p + \phi) &= \frac{\omega_d}{\zeta\omega_n} \\
&= \frac{\sqrt{1-\zeta^2}}{\zeta} \\
&= \tan(\phi)
\end{aligned} \tag{419}$$

when considering the definition of tan provided by equation 416. We must find t_p such that $\tan(\omega_d t_p + \phi) = \tan(\phi)$. Given that the function $\tan(\alpha)$ repeats itself after π, 2π, 3π, $\cdots$ as shown in figure 5.1, we know $\tan(\alpha) = \tan(\alpha + \pi) = \tan(\alpha + 2\pi), \cdots$.

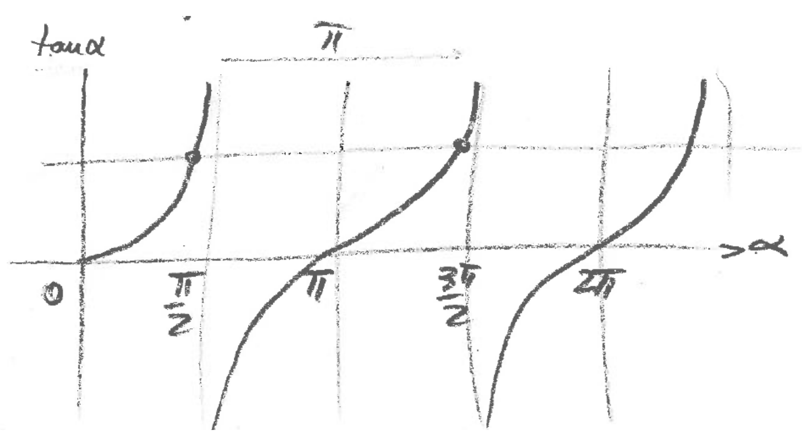


Figure 5.1: Plotting the $\tan(\alpha)$

Hence, picking back up from equation 419 lead to

$$\begin{aligned}
\omega_d t_p + \phi &= \phi + \pi \\
\omega_d t_p &= \pi
\end{aligned} \tag{420}$$

which results in

$$t_p = \frac{\pi}{\omega_d} \tag{421}$$

The peak value (x_p) can be found as

$$\begin{aligned} x_p &= x(t_p) \\ &= 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n \frac{\pi}{\omega_d}} \sin\left(\omega_d(\pi/\omega_d) + \phi\right) \\ &= 1 + \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n \frac{\pi}{\omega_d}} \sin(\phi) \\ &= 1 + \frac{1}{\sqrt{1-\zeta^2}} \left(e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}\right) \sqrt{1-\zeta^2} \\ &= 1 + e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi} \end{aligned} \tag{422}$$

when considering that $\sin(\phi + \pi) = -\sin(\phi)$ and $\sin(\phi) = \sqrt{(1-\zeta^2)}$.

The max overshoot (M_p) can be found as

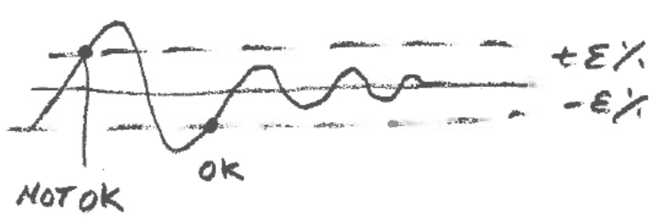
$$\begin{aligned} M_p &= \frac{x_p - x_{ss}}{x_{ss}} \\ &= \frac{1 + e^{\frac{\zeta}{\sqrt{1-\zeta^2}}\pi} - 1}{1} \\ &= e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi} \end{aligned} \tag{423}$$

for a few select damping ratios, the overshoot percentages are shown in Table 3. However, the typical range of damping is $0.4 < \zeta < 0.8$ and therefore the typical max overshoot is $0.25\% < M_p < 1.5\%$.

Table 3: Overshoot percentages for select damping ratios.

ζ	0	0.2	0.4	0.6	0.8	1
M_p	100.00%	52.68%	25.40%	9.49%	1.52%	0.00%

The definition of settling time (t_s) is to "get withing $\pm\varepsilon\%$ of x_{ss} and stay so".



The settling time is defined as the time when both

$$|x_{ss} - x(t_s)| < \Delta \tag{424}$$

and

$$|x_{ss} - x(t > t_s)| < \Delta \tag{425}$$

are true where $\Delta = \varepsilon \cdot x_{ss}$. This is shown in figure 5.2.

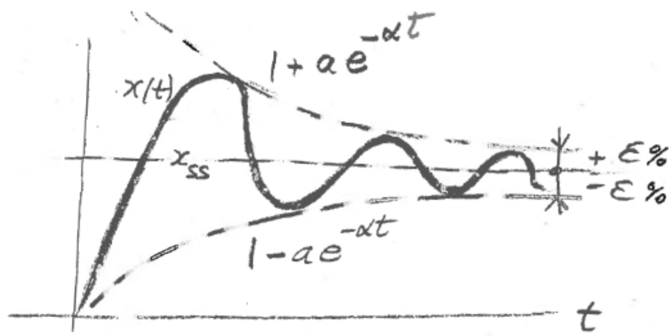


Figure 5.2: Settling time for a 2^{nd}-order system.

We can find a generalized expression for t_s if we consider the system response for $x_{ss} = 1$ as

$$x(t) = 1 - \left[\frac{1}{\sqrt{1-\zeta^2}}\right] e^{-[\zeta\omega_n]t} \sin(\omega_d t + \phi) \tag{426}$$

redefining the items in the brackets as a and α, respectively, the system response can be simplified to

$$x(t) = 1 - ae^{-\alpha t} \sin(\omega_d t + \phi) \tag{427}$$

Therefore, the envelop of the system response is $1 \pm ae^{-\alpha t}$ while the settling condition is $ae^{-\alpha t} = \Delta$. Considering that the final peak aboove the error range will happen when $a \approxeq 1$, we make the approximate calculation

$$a = \left.\frac{1}{\sqrt{1-\zeta^2}}\right|_{\zeta << 1} \approxeq 1 \tag{428}$$

and considering that t_s is when the system decays under the value ε, we need to find the t value when

$$e^{-\alpha t} = \varepsilon \tag{429}$$

therefore, setting $\varepsilon = 2\%$, we can find

$$e^{-\alpha t} = 0.02 \tag{430}$$

$$\begin{aligned} -\alpha t &= \log(0.02) \qquad (431) \\ &= -3.9 \\ &\approxeq -4 \end{aligned}$$

Therefore, knowing that $\alpha = \zeta\omega_n$, we can deduce

$$\begin{aligned} -\alpha t_s &\approxeq -4 \qquad (432) \\ -\zeta\omega_n t_s &\approxeq -4 \\ t_s &\approxeq \frac{4}{\zeta\omega_n} \end{aligned}$$

when $\zeta << 1$ this simplified expression is within $\pm 2\%$.

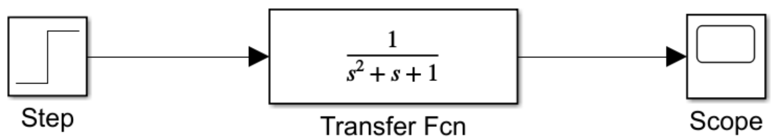
Importantly, one should always consider the effects of ζ and ω_n on performance.

- An increase in ω_n shortens rise time (t_r), peak time (t_p), and settling time (t_s).
- An increase in ζ reduces max overshoot (M_p) and shortens settling time (t_s).

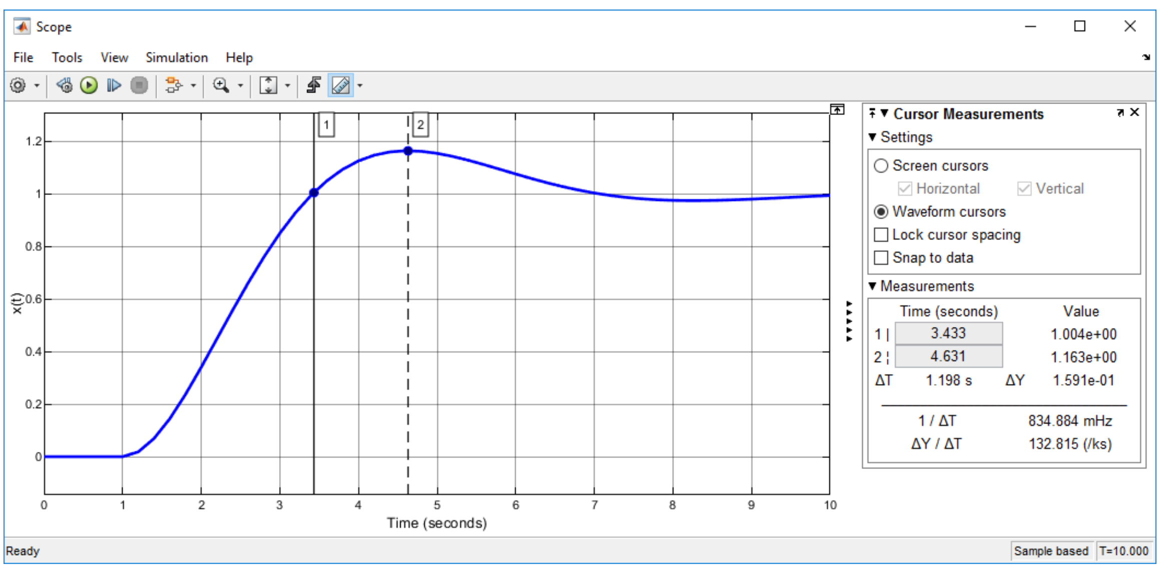
As before, the definition of time delay (t_d) for a 2nd-order system is the time it takes to rise to 50% of x_{ss} the first time. Similarly, the max percentage overshoot (M_p) for a 2nd-order system is calculated in the same way as a 1st-order system.

Example 5.2 SIMULINK Tutorial on Measuring Performance Indicators

Build the simple 2nd-order system as shown below



Open the scope and press the 'Cursor Measurements' button to activate the cursors.



Use the first cursor to find the first crossing of $x_{ss} = 1$. Read the time as $t_r = 2.433$ sec as the step function starts at 1 sec. Place the second cursor at peak value. Read the peak time $t_r = 3.631$ sec and peak amplitude $x_p = 1.163$. Calculate $M_p = 16.3\%$.

6 System Identification

Given an experimental signal, the task is to find system parameters.

Review 6.1 Harry Nyquist (February 7, 1889 ? April 4, 1976) was a Swedish physicist and electronic engineer. His parents emigrated to the U.S. in 1907. He attended the University of North Dakota starting in 1912 where he obtained a B.S. in 1914 and a M.S. in 1915, both in electrical engineering (entry to M.S. was 3 years!). Thereafter, he went to to Yale University where he received a Ph.D. in physics in 1917.

Figure 6.1: Picture of Harry Nyquist from the American Institute of Physics. Fair use, via Wikimedia Commons

6.1 Experimental Signal Processing

to be updated based on Open Vibrations

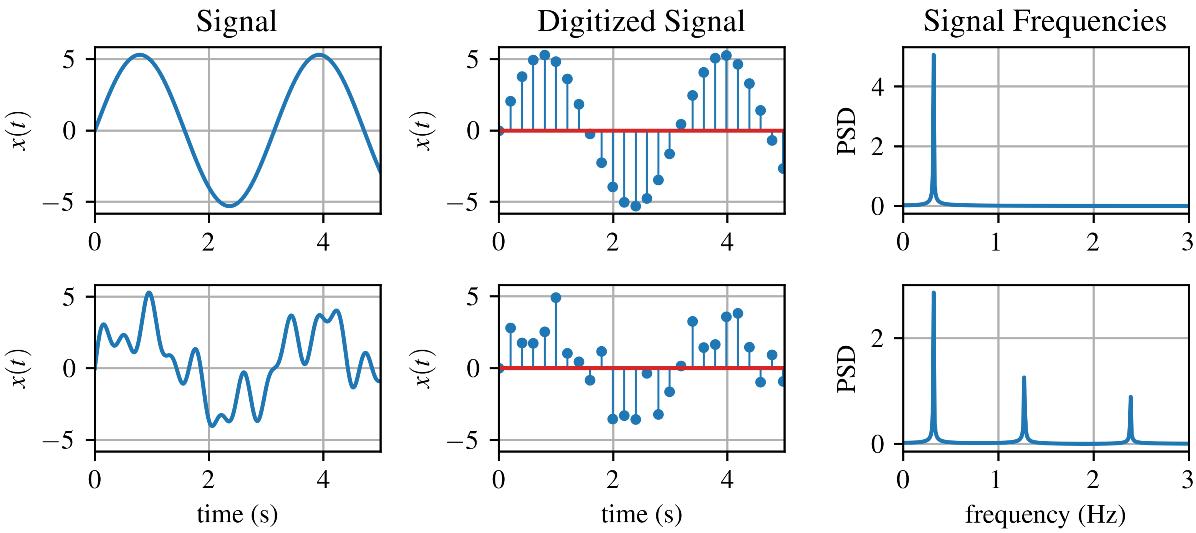


Figure 6.2: Digitization of two continuous time-series signals sampled at 5 S/s.

The Nyquist-Shannon sampling theorem is a theorem in the field of signal processing that defines the sample rate that permits a discrete sequence of samples (i.e. discrete-time) to sample a continuous-time signal of a finite bandwidth.

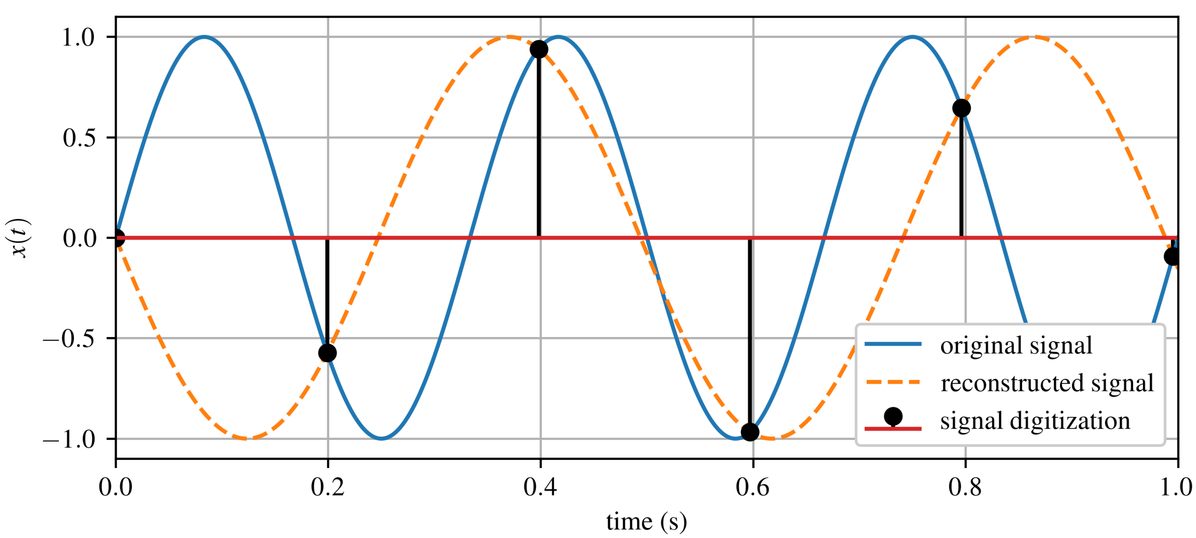
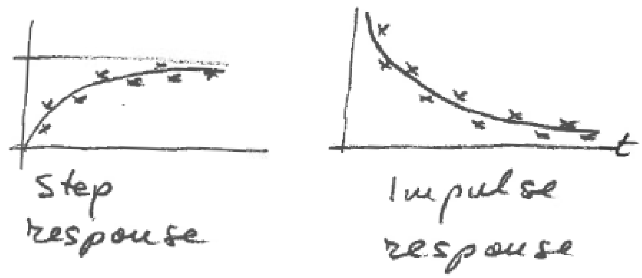


Figure 6.3: Aliasing of a 3 Hz signal that is sampled at 5 S/s.

In signal processing, aliasing is an effect that causes different signals to become indistinguishable from each other. In this way, the signals become an aliases of one another when sampled. Aliasing also accounts for the development of distortion or artifact in a reconstructed signal when compared to the original continuous signal.

6.2 System Identification for 1^{st}-order systems



Given a 1^{st}-order system subjected to either a step or impulse input, there is a need to find the time constant (T) for the first-order transfer function

$$G(s) = \frac{1}{Ts+1} \tag{433}$$

6.2.1 Option 1 Optimization

Here, the task is to fit a curve to the experimental data. To do this, minimize the error in the function

$$x(t;T) = 1 - e^{-t/T} \tag{434}$$

for a series of experiments

$$\begin{aligned} x_{\exp} &= x_1,\ x_2,\ x_3, \cdots \\ t_{\exp} &= t_1,\ t_2,\ t_3, \cdots \end{aligned} \tag{435}$$

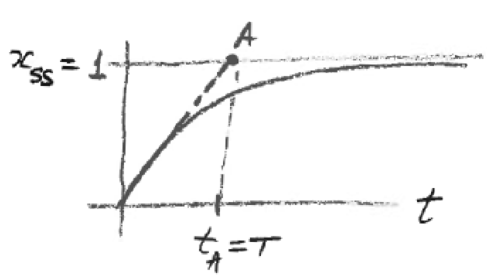
using any number of available curve fitting software to find T.

add example with curve fitting in Matlab

6.2.2 Option 2 Graphical Methods

In this method graphical methods can be used to provide quick estimates of the system parameter.

- **Tangent at Origin** This method shows us that $T = t_A$, as shown in the figure



$$x(t) = 1 - e^{-t/T} \tag{436}$$

and

$$\dot{x} = \frac{dx}{dt} = \left(-\frac{1}{T}\right)\left(-e^{-t/T}\right) = \frac{1}{T}e^{-t/T} \tag{437}$$

therefore

$$\dot{x}_0 = \frac{dx}{dt}\Big|_{t=0} = \frac{1}{T} \tag{438}$$

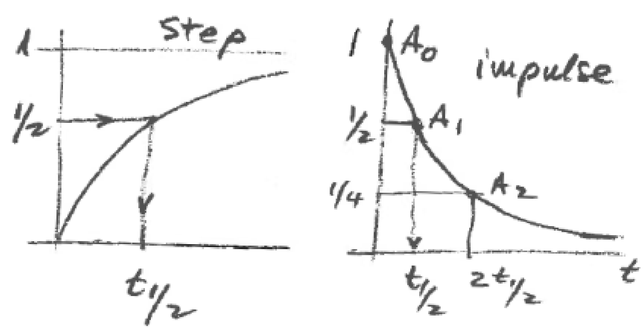
therefore, the tangent at the origin is

$$y(t) = \dot{x}_0 t = \frac{1}{T}t \tag{439}$$

where $y(t)$ intersects $x_{ss} = 1$ at $\frac{1}{T}t_{\mathrm{A}} = 1$; therefore

$$T = t_{\mathrm{A}} \tag{440}$$

- **Half Time** Thought measuring the half time of a 1st-order systems subjected to either a step or impulse input, the time constant can be obtained.



  - For a step function

$$x(t) = 1 - e^{-t/T} \tag{441}$$

solving for $t = t_{1/2}$ leads to

$$\begin{aligned} x(t_{1/2}) &= 1/2 \\ 1/2 &= 1 - e^{-t_{1/2}/T} \\ &= e^{-t_{1/2}/T} \end{aligned} \tag{442}$$

solving for T results in

$$\begin{aligned} \frac{-t_{1/2}}{T} &= \ln(1/2) \\ T &= \frac{-t_{1/2}}{\ln(1/2)} \\ &= \frac{t_{1/2}}{0.693} \\ &\approxeq 1.4t_{1/2} \end{aligned} \tag{443}$$

– For a impulse function

$$x(t) = \frac{1}{T}e^{-t/T} \tag{444}$$

where

$$\begin{aligned} x(t_{1/2}) &= 1/2 \\ 1/2 &= \frac{1}{T}e^{-t_{1/2}/T} \end{aligned} \tag{445}$$

therefore

$$\begin{aligned} T &= \frac{-t_{1/2}}{\ln(1/2)} \\ T &\approxeq 1.4t_{1/2} \end{aligned}$$

Note that you may use the consecutive points A_1, A_2 if A_0 is not easy to determine, important is that signal halves between A_1 and A_2.

This equation is correct, but I don't have a proof for it.

Proof

For the expression

$$\begin{aligned} x(t_{1/2}) &= 1/2 \\ 1/2 &= 1 - e^{-t_{1/2}/T} \\ &= e^{-t_{1/2}/T} \end{aligned} \tag{446}$$

we can show the rest of this is only true for T=1, need a more robust proof.

$$T = -\frac{t_{1/2}}{W_0\left(-\frac{t_{1/2}}{2}\right)} \tag{447}$$

where W_0 is the upper branch of the Lambert Function and $W_0\left(-\frac{t_{1/2}}{2}\right) \neq 0$, $t_{1/2} \neq 0$ and $n \in \mathbb{Z}$

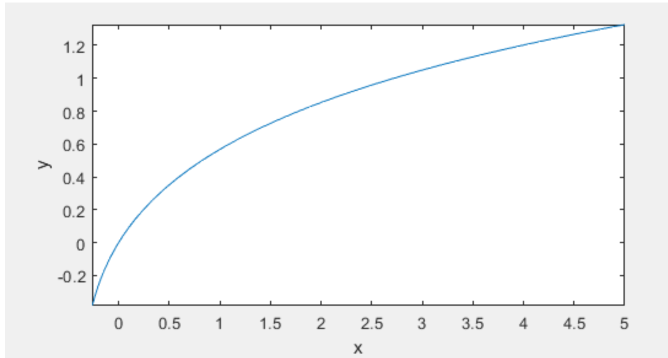


Figure 6.4: The upper branch of the Lambert Function graphed for $y = W_0(x)$ for real components.

6.2.3 Option 3 Performance Indicators

Two methods can be used to estimate the time period:

- **Delay time** (t_d) can be used to estimate the time period, as

$$t_d = -T \ln(0.5) \tag{448}$$

  and therefore

$$T = -\frac{t_d}{\ln(0.5)} \tag{449}$$

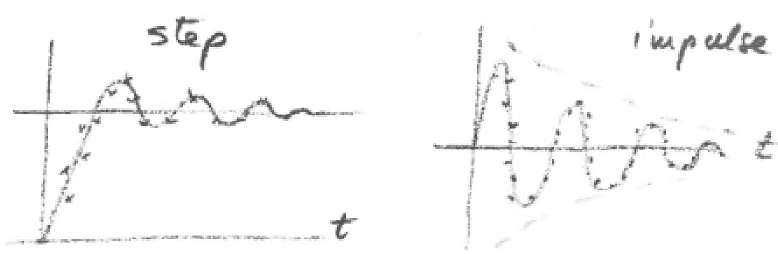
- **Settling time** t_s can be uses to estimate the time period, here consider the 2% settling time value

$$t_s^{2\%} = -T \ln(0.02) \tag{450}$$

  and therefore

$$T = -\frac{t_s^{2\%}}{\ln(0.5)} \tag{451}$$

6.3 System Identification for 2$^{\text{nd}}$-order systems



Given a 2nd-order system subjected to either a step or impulse input, there is a need to find the natural frequency (ω_n) and damping ratio (ζ) for the 2nd-order transfer function

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{452}$$

6.3.1 Optimization

Here, the task is to fit a curve to the experimental data. To do this, minimize the error in the time series response of the system for a series of experiments

$$\begin{aligned} x_{\text{exp}} &= x_1,\ x_2,\ x_3, \cdots \\ t_{\text{exp}} &= t_1,\ t_2,\ t_3, \cdots \end{aligned} \tag{453}$$

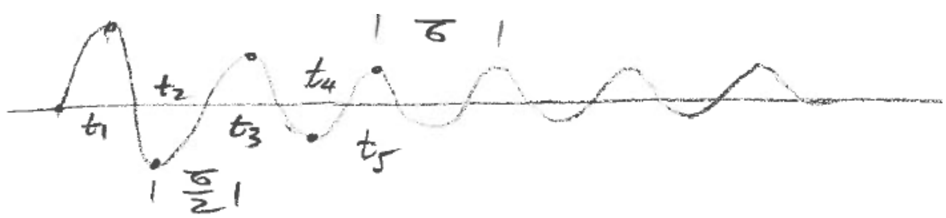
using any number of available curve fitting software to find the natural frequency (ω_n) and damping ratio (ζ).

add Matlab example, maybe with beam.

6.3.2 Frequency Estimation

Can be performed by finding the frequency at which the damped system crosses a given point, termed at f_d. This is either

- **Peak Detection**



Is done by taking the average of the half period

$$f_d = \frac{\tau}{2} = \text{avg}\left[(t_2 - t_1), (t_2 - t_1), \cdots\right] \tag{454}$$

- **Zero Crossing**

Is done by taking the average of the half period

$$f_d = \frac{\tau}{2} = \text{avg}\left[(t_2 - t_1), (t_2 - t_1), \cdots\right] \tag{455}$$

In either case,

$$\omega_d = 2\pi f_d \tag{456}$$

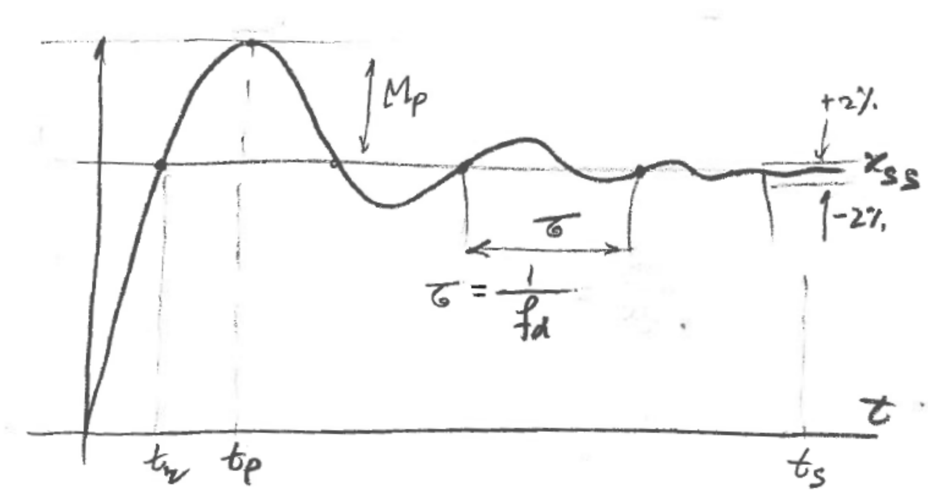
where

$$f_n = \frac{f_d}{\sqrt{1-\zeta^2}} \tag{457}$$

and

$$\omega_n = 2\pi f_n \tag{458}$$

6.3.3 Step Response Analysis ($\zeta << 1$)



Recall that the rise time is

$$t_r = \frac{\pi - \phi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi - \phi}{\omega_d} \tag{459}$$

with

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \tag{460}$$

and

$$\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \sin^{-1}\sqrt{1-\zeta^2} \tag{461}$$

while peak time is

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d} \tag{462}$$

The max percentage overshoot is

$$M_p = e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi} = \frac{x_p - x_{ss}}{x_{ss}} 100\% \tag{463}$$

and lastly, settling time is

$$t_s \approxeq \frac{4}{\zeta\omega_n} \tag{464}$$

From this, there are only two unknowns, ω_n and ζ. There is more information than minimally required to obtain these from t_r, t_p, M_p,and t_s.

- Obtain ω_n and ζ from t_r and t_p.

$$t_p = \frac{\pi}{\omega_d} \rightarrow \omega_d = \frac{\pi}{t_p} \tag{465}$$

and

$$\begin{aligned} t_r = \frac{\pi - \phi}{\omega_d} \rightarrow \phi &= \pi - t_r \omega_d \\ &= \pi - \frac{t_r}{t_p}\pi \\ &= \pi(1 - \frac{t_r}{t_p}) \end{aligned} \tag{466}$$

recall that

$$\phi = \sin^{-1}\sqrt{1-\zeta^2} \tag{467}$$

$$1 - \zeta^2 = \sin^2\phi$$

$$\zeta = \sqrt{1 - \left[\sin(\phi)\right]^2} \tag{468}$$

and

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} \tag{469}$$

- Obtain ζ from M_p.

$$M_p = e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi} \tag{470}$$

$$\begin{aligned} -\frac{\zeta}{\sqrt{1-\zeta^2}}\pi &= \ln(M_p) \\ \zeta^2\pi^2 &= (1-\zeta^2)\big(\ln(M_p)\big)^2 \\ \zeta^2\pi^2 &= \big(\ln(M_p)\big)^2 - \zeta^2\big(\ln(M_p)\big)^2 \\ \zeta^2\left[\pi^2 + \big(\ln(M_p)\big)^2\right] &= \big(\ln(M_p)\big)^2 \\ \zeta^2 &= \frac{(\ln(M_p))^2}{\pi^2 + \big(\ln(M_p)\big)^2} \\ \zeta &= \frac{|\ln(M_p)|}{\sqrt{\pi^2 + \big(\ln(M_p)\big)^2}} \end{aligned}$$

- Obtain ω_n and ζ from t_p and t_s.

$$t_p = \frac{\pi}{\omega_d} \rightarrow \omega_d = \frac{\pi}{t_p} \tag{471}$$

next

$$\begin{aligned}\omega_n &= \frac{\omega_d}{\sqrt{1-\zeta^2}} \\ &= \frac{\frac{\pi}{t_p}}{\sqrt{1-\zeta^2}} \\ &= \frac{\pi}{t_p\sqrt{1-\zeta^2}}\end{aligned} \tag{472}$$

looking at the settling time

$$\begin{aligned}t_s &= \frac{4}{\zeta\omega_n} \\ &= \frac{4t_p\sqrt{1-\zeta^2}}{\zeta\pi} \\ t_s\zeta\pi &= 4t_p\sqrt{1-\zeta^2} \\ t_s^2\zeta^2\pi^2 &= 16t_p^2(1-\zeta^2) \\ (16t_p^2-\pi^2t_s^2)\zeta^2 &= 16t_p \\ \zeta^2 &= \frac{16t_p}{16t_p^2-\pi^2t_s^2} \\ &= \frac{1}{1-\left(\frac{\pi}{4}\frac{t_s}{t_p}\right)^2} \\ \zeta &= \frac{1}{\sqrt{1-\left(\frac{\pi}{4}\frac{t_s}{t_p}\right)^2}}\end{aligned} \tag{473}$$

- Obtain ω_n and ζ from M_p and t_r.

$$\zeta = \frac{|\ln M_p|}{\sqrt{\pi^2+(\ln M_p)^2}} \tag{474}$$

Recall that

$$t_r = \frac{\pi-\phi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi-\phi}{\omega_d} \tag{475}$$

Therefore

$$\omega_n = \frac{\pi-\phi}{t_r\sqrt{1-\zeta^2}} \tag{476}$$

where

$$\phi = \sin^{-1}\sqrt{1-\zeta^2} \tag{477}$$

System identification can be done through measuring performance indicators using Simulink.

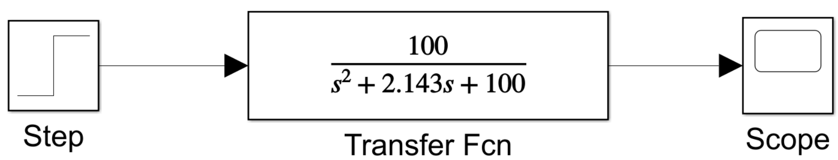
Example 6.1 SIMULINK Tutorial on Measuring Performance Indicators

System identification can be done through measuring performance indicators using Simulink. Consider a 2^{nd}-order system with $k = 700$ N/m, $c = 15$ kg/s, and $m = 7$ kg. To implement this in Simulink, one could use MATLAB to generate the transfer function:

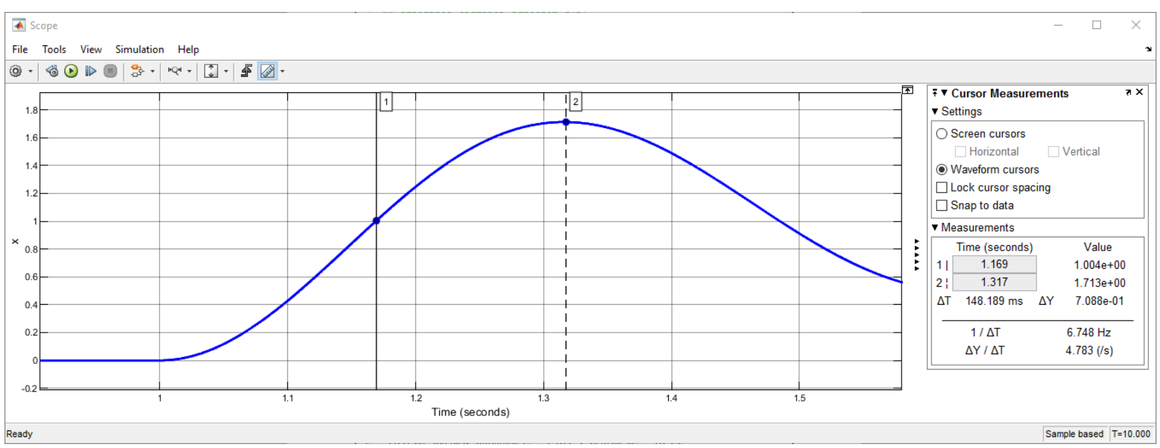
Listing 10: MATLAB code for Method II.

```
%% calculate system values
c = 15;      % N/(m/sec)
k = 700;     % N/m,
m = 7;       % kg
w_n = sqrt(k/m)      % natural frequency in rad/sec
z = c/(2*sqrt(k*m)) % damping ratio
w_d = w_n*sqrt(1-z^2);    % damped natural frequency in rad/sec

%% calculate transfer function G(s)
B = [w_n^2];
A = [1 2*z*w_n w_n^2];
G = tf(B,A) % build the transfer function
```

and than deploy the transfer function to SIMULINK as shown below



Using the scope and waveform cursors, several performance indicators can be read



these include

- $t_r = 0.169$ sec
- $t_p = 0.317$ sec

- $x_p = 1.173$ m
- $M_p = 71.3\%$

From these, measurements the damping ratio of the system can be obtained using one of two methods,

- **Method 1** Estimate damping ratio using t_r ,t_p. Knowing that

$$\phi = \pi(1 - t_r/t_p) \tag{478}$$
$$= 1.46674 \tag{479}$$

and

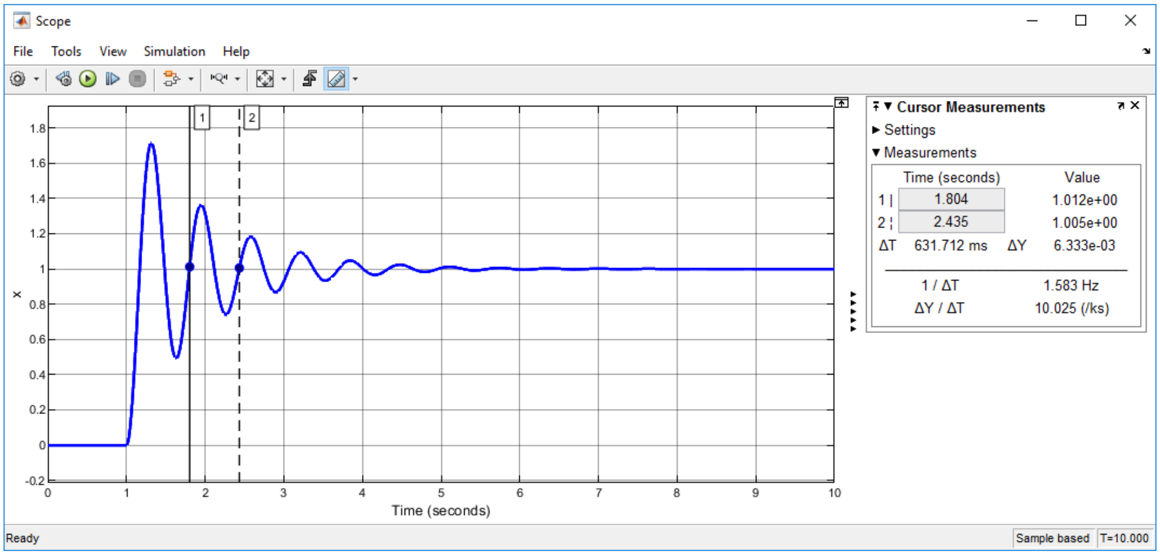
$$\zeta = \sqrt{1 - \left[\sin(\phi)\right]^2} \tag{480}$$
$$= 0.1039 \tag{481}$$

- **Method 2** Estimate damping ratio using M_p. Knowing that

$$\zeta = \frac{|\ln M_p|}{\sqrt{\pi^2 + (\ln M_p)^2}} \tag{482}$$
$$= \frac{|\ln(0.72)|}{\sqrt{\pi^2 + (\ln M 0.72)^2}}$$
$$= 0.1071 \tag{483}$$

These values give an error of 3.4% and 0% when compared to the correct value of $\zeta = 0.1071$, respectively. This is considered very good, but these are ideal systems with no noise and therefore excellent predictions are to be expected.

The SIMULINK model can also be used to estimate the natural frequency of the system using the zero crossing method by placing the coursers as close as possible to the "zero" value, 1 cycle apart. Here, "zero" is one due to the system being subjected to a step response.



The ΔT of the cursors is 0.631712 sec, or 1.583 Hz. This is 0.53$ off from the true value of 1.5915 Hz.

6.3.4 Logarithmic decrement

Logarithmic decrement ($\zeta << 1$) For a vibrating system, the mass (m) and stiffness (k) can be measured using scales and static deflection tests. However, the damping coefficient (c) is a more difficult quantity to determine. From k and m we can compute the natural frequency (ω_n) and the critical damping coefficient (c_cr). Therefore, knowing that the critical damping ratio (ζ) is defined as:

$$\zeta = \frac{c}{c_\text{cr}} = \frac{c}{2\sqrt{km}} = \frac{c}{2m\omega_\text{n}} \tag{484}$$

if we calculate ζ, we can obtain c for the system of interest. This is made possible because c_cr can be calculated from k and m. Observing the temporal response for the underdamped system,

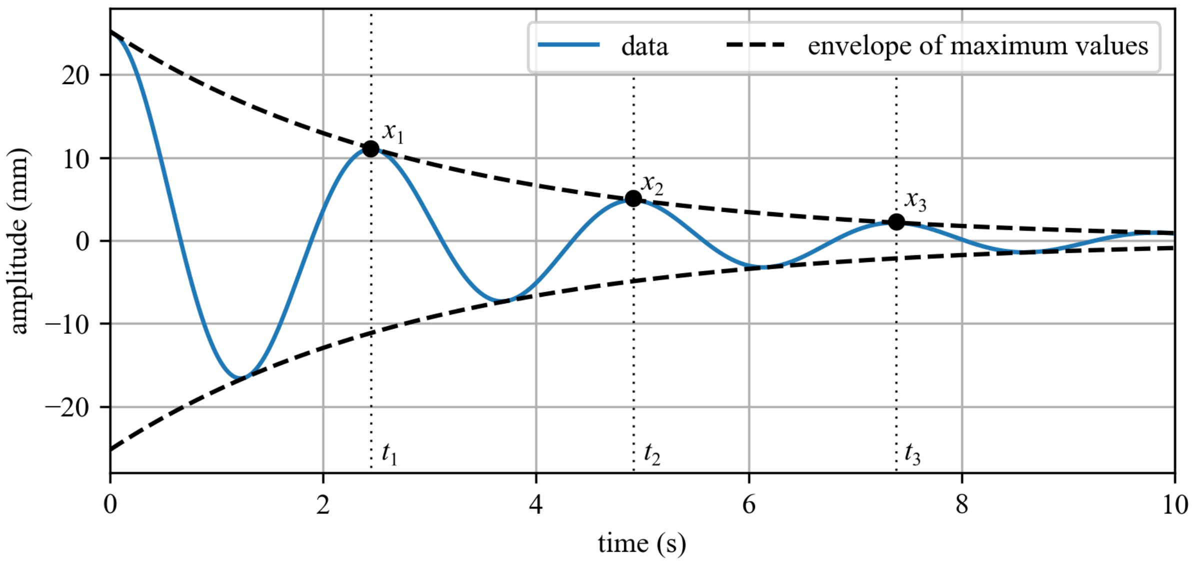


Figure 6.5: Measuring the peak displacements points in an experimental system with decay caused by damping.

we mark three points of maximum amplitude, x_1, x_2, and x_3 that happen at t_1, t_2, and t_3, respectively. Considering displacement values for the first two points x_1 and x_2, separated by a complete period (T). Knowing that one cycle is 2π, the time period for this complete cycle is given by:

$$t_2 - t_1 = \frac{2\pi}{\omega_\mathrm{d}} = \frac{2\pi}{\omega_\mathrm{n}\sqrt{1-\zeta^2}} \tag{485}$$

where ω_d is the damped natural frequency. This is the time period (T) of damped oscillations. If we derive an equation for the values of the peaks, also called the envelope of maximum values, we get:

$$x_\mathrm{peaks} = Ae^{-\zeta\omega_\mathrm{n}t} \tag{486}$$

Knowing that the system is underdamped, A can be solved for using the initial conditions x_0 and v_0, therefore:

$$A = \frac{\sqrt{(v_0 + \zeta\omega_\mathrm{n}x_0)^2 + (x_0\omega_\mathrm{d})^2}}{\omega_\mathrm{d}} \tag{487}$$

In terms of t_1 and t_2, we can express the displacement at these times as:

$$x_1 = Ae^{-\zeta\omega_\mathrm{n}t_1} \tag{488}$$

and

$$x_2 = Ae^{-\zeta\omega_\mathrm{n}t_2} \tag{489}$$

therefore:

$$\frac{x_1}{x_2} = \frac{e^{-\zeta\omega_\mathrm{n}t_1}}{e^{-\zeta\omega_\mathrm{n}t_2}} = e^{\zeta\omega_\mathrm{n}(t_2-t_1)} \tag{490}$$

However, from before we know that $t_2 - t_1 = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$. Therefore, we can express this last equation as:

$$\frac{x_1}{x_2} = e^{\left(\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\right)} \tag{491}$$

Next, we take the natural log of both sides to get the logarithmic decrement, denoted by δ:

$$\delta = \ln\left(\frac{x_1}{x_2}\right) = \ln\left(\frac{x(t_1)}{x(t_1+T)}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{492}$$

This shows us that the ratio of any two successive amplitudes for an underdamped system, vibrating freely, is constant and is a function of the damping only. Sometimes, in experiments, it is more convenient/accurate to measure the amplitudes after say "n" peaks rather than two successive peaks (because if the damping is very small, the difference between the successive peaks may not be significant). The logarithmic decrement can then be given by the equation

$$\delta = \frac{1}{n}\ln\left(\frac{x_1}{x_{n+1}}\right) = \frac{1}{n}\ln\left(\frac{x(t_1)}{x(t_1+nT)}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{493}$$

Once we use the experimental data to obtain δ, and knowing that:

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{494}$$

we can calculate the value of ζ:

$$\zeta = \frac{\delta}{\sqrt{4\pi^2+\delta^2}} \tag{495}$$

Therefore, having ζ we can solve for the coefficient of damping, c, as:

$$c = \zeta 2\sqrt{km} \tag{496}$$

Example 6.2 Calculate the damping coefficient for the system with the measured amplitude as expressed below given that m = 3 kg and k = 43 N/m. Use t_1 = 1 sec, and $t_{n+1} = t_4 = 6$ sec. Use the peaks as marked in figure 6.6.

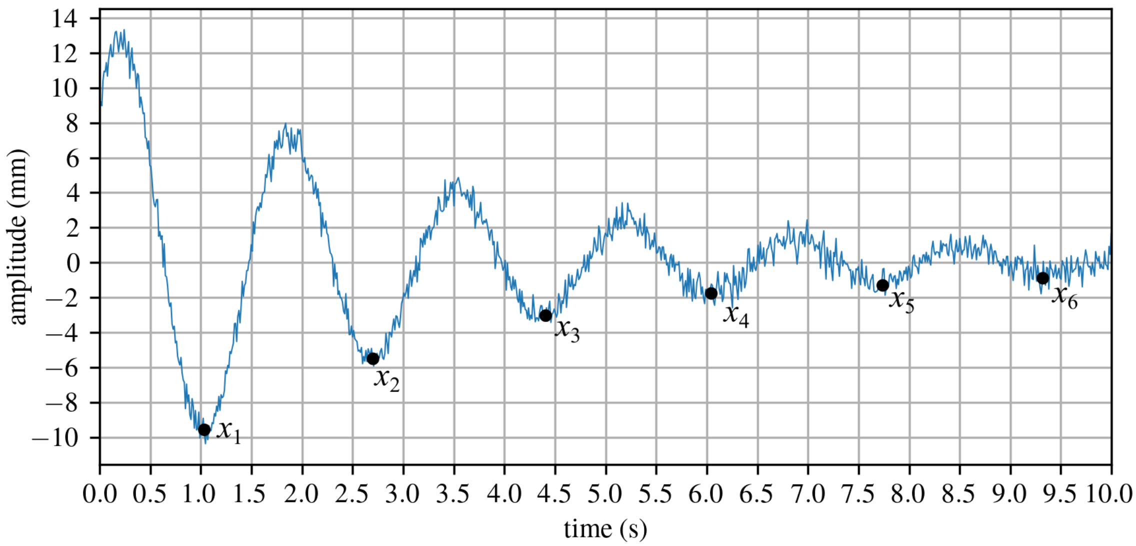


Figure 6.6: Response from an experimental system with noise.

Solution:

First, from the plot we can determine that $x_1 = -9.5$ mm and $x_4 = -1.8$ mm where $n = 3$. Thereafter, we can solve for δ:

$$\delta = \frac{1}{3}\ln\left(\frac{x_1}{x_4}\right) = \frac{1}{3}\ln\left(\frac{-9.5}{-1.8}\right) = 0.554 \tag{497}$$

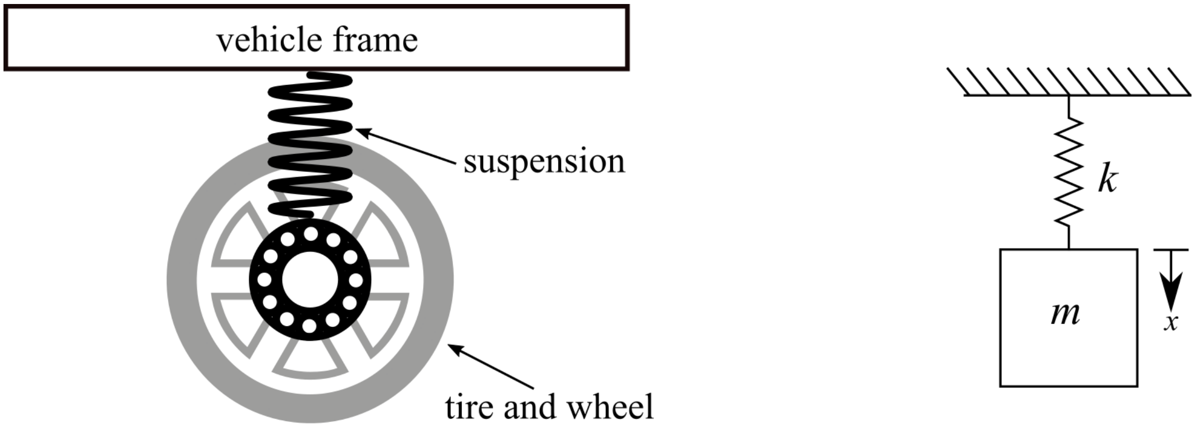
Next, we can calculate ζ, as:

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \frac{0.554}{\sqrt{4\pi^2 + 0.554^2}} = 0.0879 \tag{498}$$

And lastly:

$$c = \zeta 2\sqrt{km} = 0.0879 \cdot 2\sqrt{43 \cdot 3} = 2.0 \text{ kg/s} \tag{499}$$

Example 6.3 A vehicle wheel, tire, and suspension can be modeled as a SDOF spring and mass as depicted below:



(a) (b)

Figure 6.7: Modeling of a vehicle wheel, tire, and suspension showing: (a) Graphical representation; and (b) a spring-mass model.

The free response of a 1000-kg automobile with a stiffness of k = 400,000 N/m is observed to be underdamped. Modeling the automobile as a single-degree-of-freedom oscillation in the vertical direction, as annotated in figure 6.7, determine the damping coefficient if the displacement at t_1 is measured to be 2 cm and 0.22 cm at t_2.

Solution:

Knowing x_1 = 2 cm and x_2 = 0.22 cm and $t_2 = T + t_1$, therefore:

$$\delta = \ln\frac{x_1}{x_2} = \ln\frac{2}{0.22} = 2.207 \tag{500}$$

and:

$$\zeta = \left(\frac{\delta}{\sqrt{4\pi^2+\delta^2}}\right) = \left(\frac{2.207}{\sqrt{4\pi^2+2.207^2}}\right) = 0.331 \tag{501}$$

therefore, we can obtain the damping coefficient as

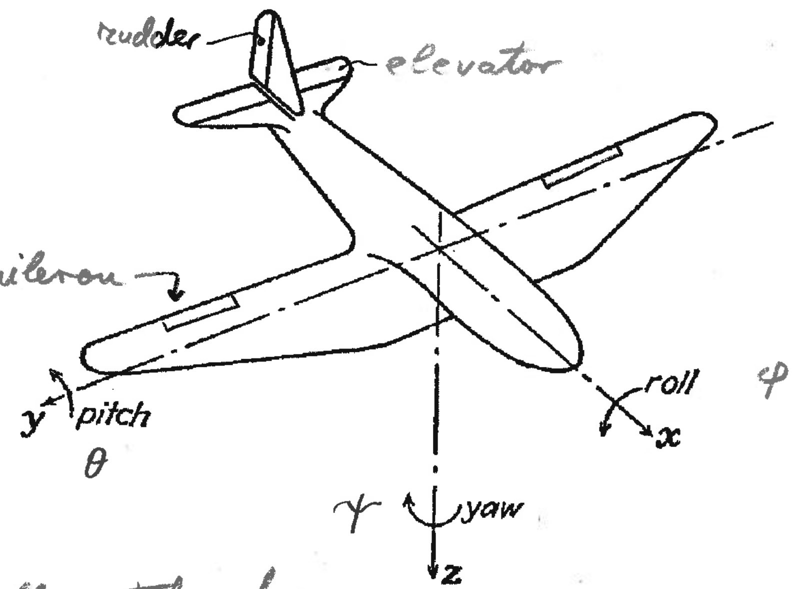
$$c = 2\zeta\sqrt{km} = 2(0.331)\sqrt{400,000 \cdot 1,000} = 13,256 \text{ kg/s} \tag{502}$$

7 Control Systems

7.1 Control Systems

2016 11 03

AIRCRAFT CONTROLS



Aircraft control surfaces

AILERONS control the rolling motion
φ = roll angle (~ bank angle)
pilot stick moves left-right to deflect ailerons

ELEVATOR controls the pitch motion
nose up / nose down
θ = pitch angle
pilot stick moves forward-backward to deflect elevator

RUDDER controls yaw motion
ψ = yaw angle
pilot uses rudder pedals to deflect rudder

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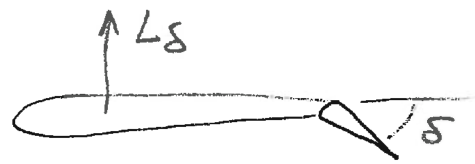
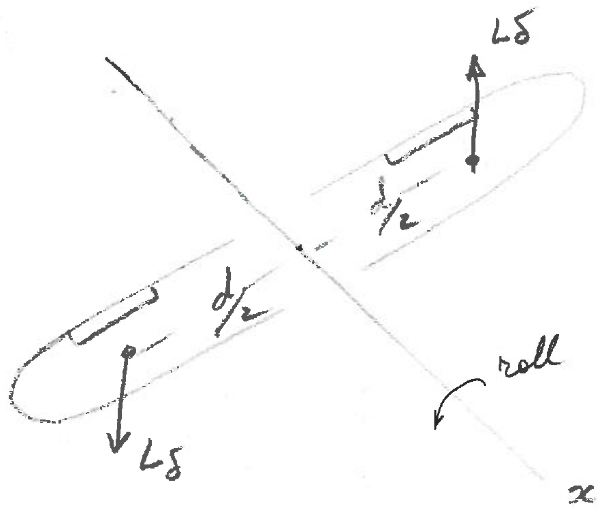
AILERON DEFLECTION
AIRCRAFT ROLL MOTION

$$L_\delta = \frac{\partial L}{\partial \delta}\delta$$





- Aileron deflection δ produces additional lift L_δ
- L_δ is up/down because ailerons left/right move up/down
- Net effect is a rolling moment,

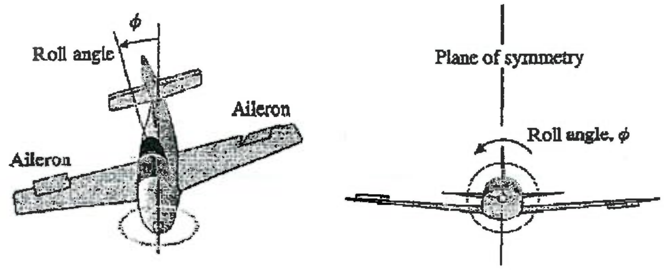
$$M = L_\delta d = d \cdot \frac{\partial L}{\partial \delta} \cdot \delta \qquad (1)$$

- Essentially, the rolling moment M is proportional to aileron deflection δ, i.e.

$$M = K\,\delta \qquad (K = \text{gain}) \qquad (2)$$

RTF

ROLL TRANSFER FUNCTION



EOM: $J\ddot{\varphi} + c\dot{\varphi} = M$ (1)

where

J = inertia : mass moment of inertia about roll axis

c = damping : air resistance to roll motion

M = rolling moment produced by aileron deflection δ

Recall:

$M = K\delta$ (2).

(2) → (1) : $J\ddot{\varphi} + c\dot{\varphi} = K\delta$ (3)

Take LT of Eq.(3) to get

$\mathcal{L}(3)$: $(Js^2 + cs)\tilde{\varphi}(s) = K\tilde{\delta}(s)$ (4)

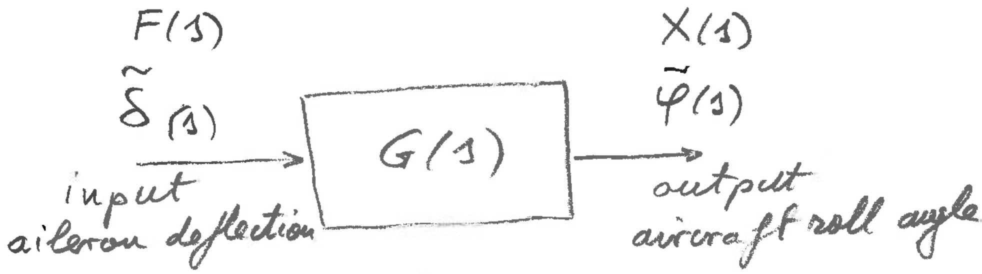
where $\tilde{\varphi}(s) = \mathcal{L}\,\varphi(t)$

$\tilde{\delta}(s) = \mathcal{L}\,\delta(t)$ (5)

Solution of Eq.(4) yields:

RTP

$$\tilde{\varphi}(s) = \frac{K}{Js^2 + cs}\,\tilde{\delta}(s) \qquad (6)$$



$$G(s) = \frac{K}{Js^2 + cs} \qquad (7)$$

$$X(s) = G(s)\,F(s). \qquad (8)$$

The system described by Eq.(7) is an uncontrolled 2nd order dynamic system.

- This system is "uncontrolled" because a constant input will produce a continuously growing response (see roll response on next page)
- Another example of uncontrolled 2nd order dynamic system is the DC Motor where an applied constant voltage produces continuous rotation

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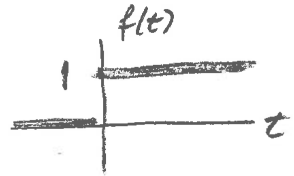
ROLL RESPONSE TO CONSTANT AILERON DEFLECTION

Assume the pilot moves the stick laterally such as to create a constant aileron deflection $\delta = \text{const}$. This corresponds to a step input, i.e.,



$$\begin{cases} f(t) = 1 \\ F(s) = 1/s \end{cases} \quad (9)$$

(9) → (8): $$X(s) = \frac{K}{Js^2 + cs} \cdot \frac{1}{s} = \frac{K}{s^2(Js+c)} \quad (10)$$

Table 2-1, # 19 has the pair

$$t - T\left(1 - e^{-t/T}\right) \underset{ILT}{\overset{LT}{\rightleftarrows}} \frac{1}{s^2(Ts+1)} \quad (11)$$

Write (10) such as to look like (11), i.e.,

$$X(s) = \frac{K}{c} \cdot \frac{1}{s^2(Ts+1)}, \quad T = \frac{J}{c} \quad (12)$$

ILT of (12) gives:

$$x(t) = \frac{K}{c}\left[t - T\left(1 - e^{-t/T}\right)\right] \quad (13)$$

$$x_{ss} = \lim_{t\to\infty} x(t) = \frac{K}{c}\left(t - T\right) \quad (14)$$

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RR
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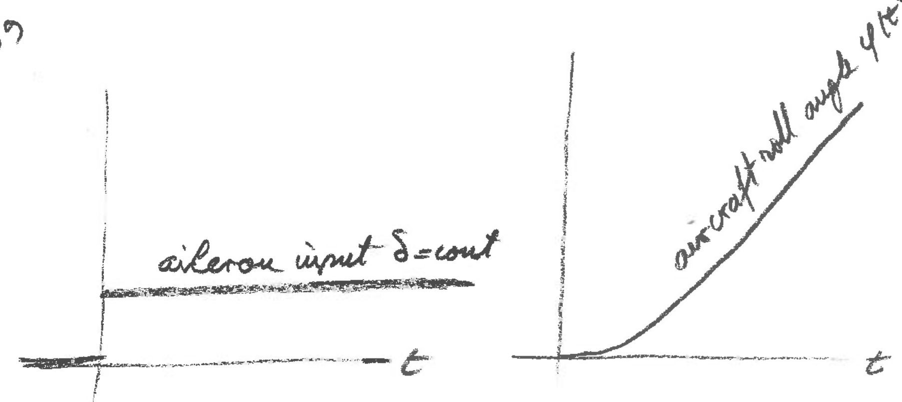

Plot of eq. (13) indicates that the aircraft will roll continously with a constant roll rate.

Thus, a constant aileron input δ = const produces continuously increasing roll angle of aircraft.

This is a general property of Type 1 system: they cannot maintain position. The step response of a Type 1 system is unconstrained

- DC motor spins continuously under constant voltage input
- Aircraft rolls continuously under constant aileron input

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3 RR 2016 11 09

Step response of Type 1 systems

$$G(s) = \frac{K}{s} \cdot \frac{(T_a s+1)(T_b s+1)\dots}{(T_1 s+1)(T_2 s+1)\dots} \qquad \text{Type 1}$$

$$F(s) = \frac{1}{s} \qquad \text{step excitation}$$

$$X(s) = G(s)\, F(s)$$

$$= \frac{K}{s} \frac{(T_a s+1)(\quad)\dots}{(T_1 s+1)(\quad)\dots} \cdot \frac{1}{s}$$

$$X(s) = \frac{K}{s^2} \frac{(\quad)(\quad)\dots}{(\quad)(\quad)\dots} \qquad (1)$$

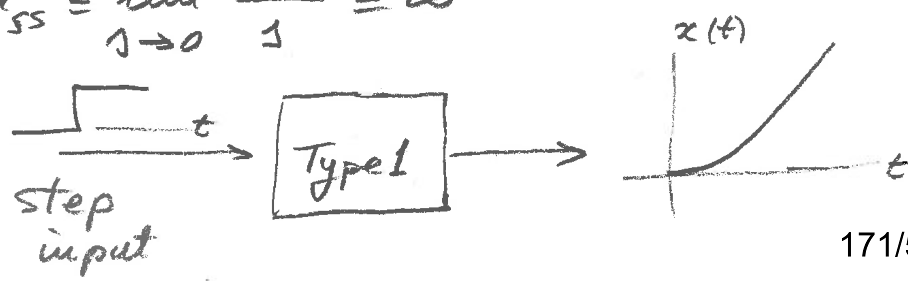
Steady state response is calculated with Final Value Theorem, i.e.,

$$x_{ss} = \lim_{t\to\infty} x(t) = \lim_{s\to 0} s\, X(s) \qquad (2)$$

(1) → (2) :

$$x_{ss} = \lim_{s\to 0} \cancel{s} \frac{K}{s^{\cancel{2}}} \frac{\overbrace{(T_a s+1)(T_b s+1)\dots}^{s\to 0 \;\to 1}}{\underbrace{(T_1 s+1)(T_2 s+1)\dots}_{\to 1}}$$

$$x_{ss} = \lim_{s\to 0} \frac{K}{s} = \infty$$



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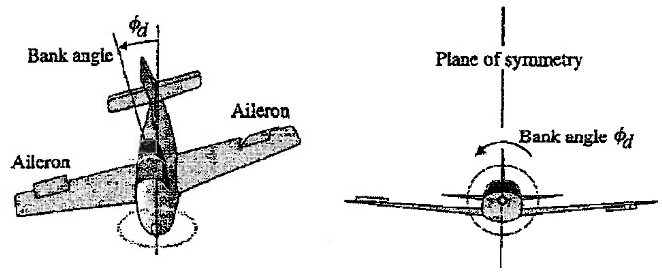
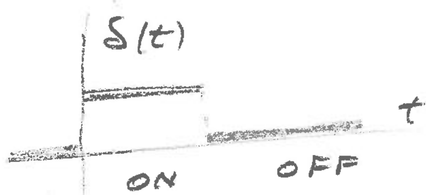
BAC

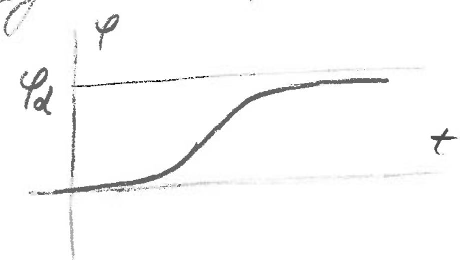
BANK ANGLE CONTROL



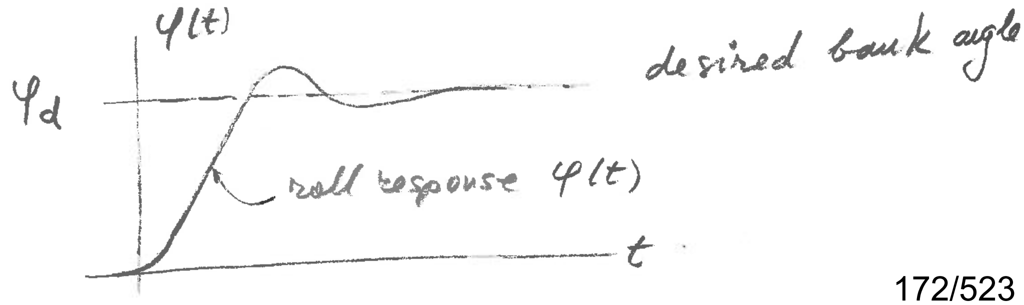
We want the aircraft to roll from flying straight & level to flying inclined with a bank angle φ_d = const.

- Manually, the pilot creates a bank angle by an on/off deflection of aileron





- We desire to build a FB control system to achieve this transition in a smooth way <u>automatically</u>.



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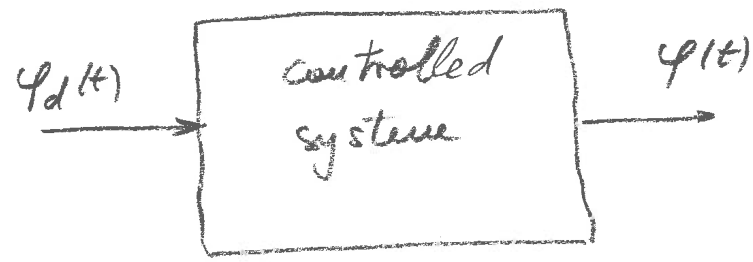
2 BAC



The control system design process needs specifications. We choose two performance indicators, M_p and t_p and define their values as design specs.

<u>Control design specifications</u>

DS1 : Fast response time as measured by rise time, $t_r \leq 1.5\,sec$.

DS2 : Maximum percentage overshoot for step input less than 20%

$M_p \leq 20\%$

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BAC Details of:

MANUAL BANK ANGLE CONTROL

- Aircraft flies straight & level

- Pilot wants to bank 15°

- Pilot moves stick sideways.

- Aircraft starts to respond

- Pilot's eyes see the aircraft rolling and estimates actual value φ
- Pilot's mind processes the measured value φ, compares it with desired value $\varphi_d = 15°$, and sends action order to hand muscles:
  - if $\varphi < \varphi_d$, then continue to push stick
  - if $\varphi \sim \varphi_d$, move stick back to neutral position to reduce rolling moment from ailerons
  - if $\varphi > \varphi_d$, move stick the other way because the aircraft overshot the target angle and needs to roll backwards

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1
FB
2016.1109

FEEDBACK CONTROL

FB concept:

- adjust the system input to obtain a desired output

FB implementation in Laplace s-domain:

- measure output
- calculate "error", i.e. difference between "desired" and "measured"
- feed "error" to the system to adjust itself until the "error" is reduced to zero.

(i.e., "measured" = "desired")

In the time domain, the FB process is a transient process of repeated adjustments until output matches input (error → 0).

The process is based on Convolution Theorem of Laplace Transform

$$\mathcal{L}^{-1} G(s) F(s) = \int_0^t f(\tau) g(\tau - t) d\tau$$

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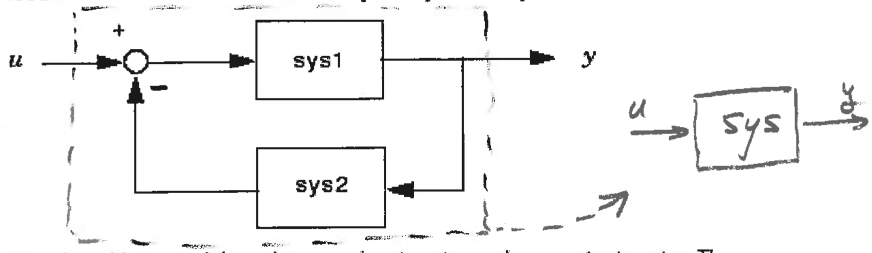
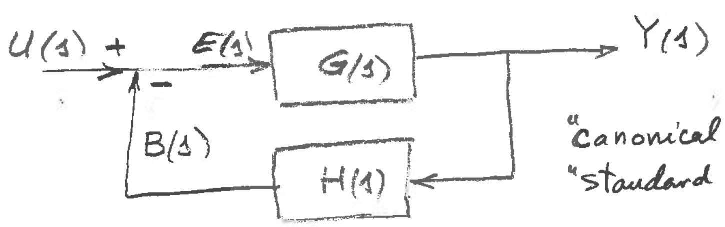
2 FB 2016/11/10

MATLAB:

`sys = feedback(sys1,sys2)` returns a model object `sys` for the negative feedback interconnection of model objects `sys1` and `sys2`.

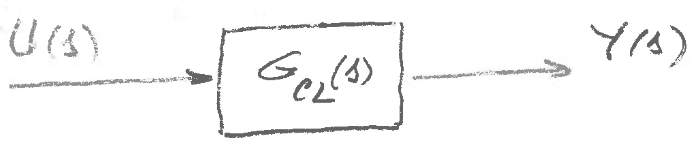


The closed-loop model `sys` has u as input vector and y as output vector. The models `sys1` and `sys2` must be both continuous or both discrete with identical sample times. Precedence rules are used to determine the resulting model type



FB sys.

"canonical form"
"standard form"



Equivalent FB sys.

$$G_{CL}(s) = \frac{G(s)}{1 + G(s)H(s)}$$

closed loop transfer funct.

Proof

$$B(s) = H(s)Y(s)$$

$$E(s) = U(s) - B(s) = U(s) - H(s)Y(s)$$

$$Y(s) = G(s)E(s) = G(s)U(s) - G(s)H(s)Y(s)$$

$$Y(s) + G(s)H(s)Y(s) = G(s)U(s)$$

$$\left[1 + G(s)H(s)\right]Y(s) = G(s)U(s)$$

$$Y(s) = \frac{G(s)}{1 + G(s)H(s)}U(s)$$

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3 FB 2016 11 10

FB Nomenclature

$U(s)$: input reference signal (desired result)

$Y(s)$: output signal

$B(s)$: feedback signal

$E(s)$: error signal

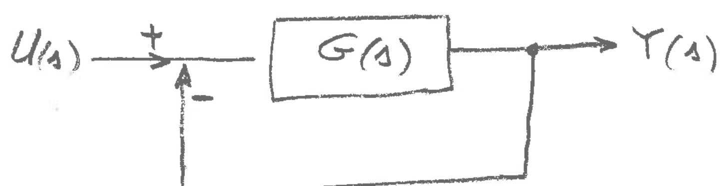
Transfer functions

feedforward T.F: $G(s) = \frac{Y(s)}{E(s)}$

open loop TF: $G(s)H(s) = \frac{B(s)}{E(s)}$

closed loop TF: $G_{CL}(s) = \frac{Y(s)}{U(s)}$.

UNIT FEEDBACK



"unit feedback" is obtained for $H(s) = 1$

$$G_{CL} = \frac{G}{1+G}$$

unit feedback closed loop TF.

feedback

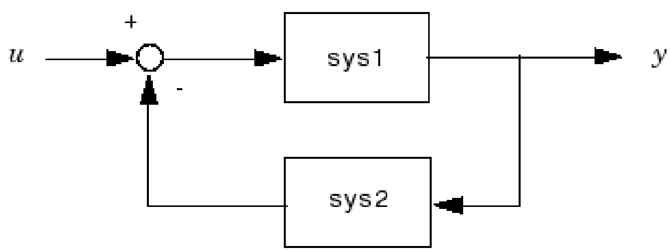
Feedback connection of two models

Syntax

```
sys = feedback(sys1,sys2)
```

Description

`sys = feedback(sys1,sys2)` returns a model object `sys` for the negative feedback interconnection of model objects `sys1` and `sys2`.



The closed-loop model `sys` has *u* as input vector and *y* as output vector. The models `sys1` and `sys2` must be both continuous or both discrete with identical sample times. Precedence rules are used to determine the resulting model type (see "Rules That Determine Model Type").

To apply positive feedback, use the syntax

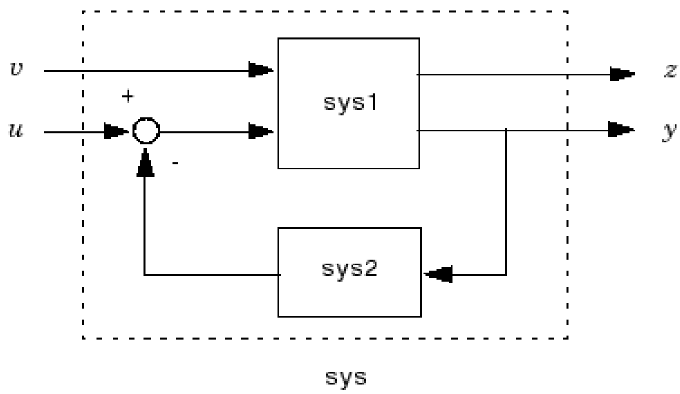
```
sys = feedback(sys1,sys2,+1)
```

By default, `feedback(sys1,sys2)` assumes negative feedback and is equivalent to `feedback(sys1,sys2,-1)`.

Finally,

```
sys = feedback(sys1,sys2,feedin,feedout)
```

computes a closed-loop model sys for the more general feedback loop.



The vector feedin contains indices into the input vector of sys1 and specifies which inputs u are involved in the feedback loop. Similarly, feedout specifies which outputs y of sys1 are used for feedback. The resulting model sys has the same inputs and outputs as sys1 (with their order preserved). As before, negative feedback is applied by default and you must use

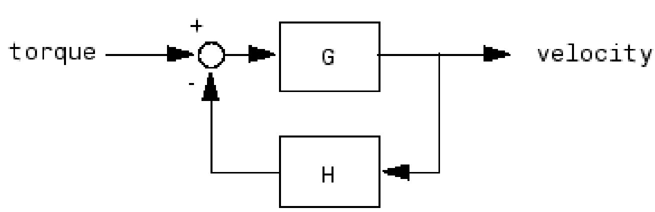
```
sys = feedback(sys1,sys2,feedin,feedout,+1)
```

to apply positive feedback.

For more complicated feedback structures, use append and connect.

Examples

Example 1



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To connect the plant

$$G(s) = \frac{2s^2 + 5s + 1}{s^2 + 2s + 3}$$

with the controller

$$H(s) = \frac{5(s+2)}{s+10}$$

using negative feedback, type

```
G = tf([2 5 1],[1 2 3],'inputname','torque',...
                         'outputname','velocity');
H = zpk(-2,-10,5)
Cloop = feedback(G,H)
```

These commands produce the following result.

```
Zero/pole/gain from input "torque" to output "velocity":
0.18182 (s+10) (s+2.281) (s+0.2192)
-----------------------------------
 (s+3.419) (s^2 + 1.763s + 1.064)
```

The result is a zero-pole-gain model as expected from the precedence rules. Note that `Cloop` inherited the input and output names from `G`.

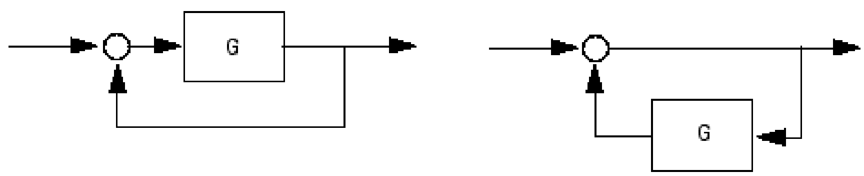
Example 2

Consider a state-space plant `P` with five inputs and four outputs and a state-space feedback controller `K` with three inputs and two outputs. To connect outputs 1, 3, and 4 of the plant to the controller inputs, and the controller outputs to inputs 4 and 2 of the plant, use

```
feedin = [4 2];
feedout = [1 3 4];
Cloop = feedback(P,K,feedin,feedout)
```

Example 3

You can form the following negative-feedback loops



by

```
Cloop = feedback(G,1)       % left diagram
Cloop = feedback(1,G)      % right diagram
```

Limitations

The feedback connection should be free of algebraic loop. If D_1 and D_2 are the feedthrough matrices of `sys1` and `sys2`, this condition is equivalent to:

- $I + D_1D_2$ nonsingular when using negative feedback
- $I - D_1D_2$ nonsingular when using positive feedback.

See Also

`series` | `parallel` | `connect`

Introduced before R2006a

1 FB

Feedback control of Type 1 systems.

- Type 1 system response is unconstrained
- Feedback can be used to control the response.

Examples

- DC motor cannot hold position; it rotates continuously under constant voltage
  With FB, a DC motor becomes a servomotor and holds position
- Aircraft rolls continuously; with FB, aircraft can maintain a constant bank angle.

Type 1 system transfer function

$$G(s) = \frac{K}{Js^2 + cs} \quad \left\{ \begin{array}{l} K = \text{gain} \\ J = \text{inertia} \\ c = \text{damping} \end{array} \right.$$

2nd order system : s^2 is highest power in denominator

Type 1 system : $G(s) = \frac{K}{s} \cdot \frac{1}{Js + c}$

"s to power 1"

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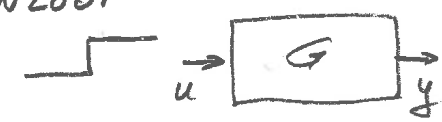
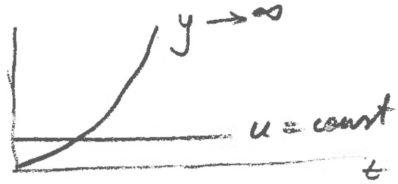
8 FB

FB Step response of Type 1 system

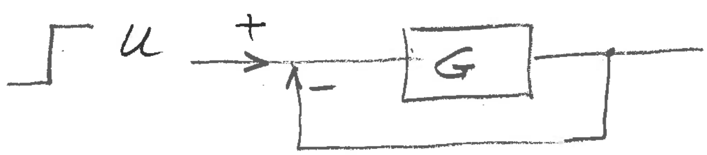
$$G(s) = \frac{K}{Js^2 + cs}$$

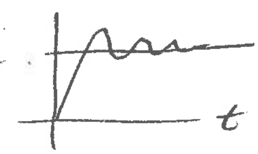
OPEN LOOP





CLOSED LOOP w FB





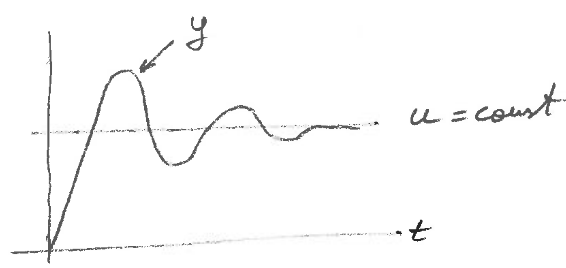
$$G_{CL} = \frac{G}{1+G}$$

$$= \frac{\frac{K}{Js^2+cs}}{1 + \frac{K}{Js^2+cs}} = \frac{K}{Js^2+cs+K} = \frac{\frac{K}{J}}{s^2 + \frac{c}{J}s + \frac{K}{J}}$$

$$G_{CL} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \left\{ \begin{array}{l} 2^{nd} \text{ order system} \\ \text{Type 0 system} \end{array} \right.$$

$$\omega_n^2 = \frac{K}{J}$$

$$\zeta = \frac{c}{2\sqrt{JK}}$$



- FB has controlled the Type 1 system
- Type 1 system w FB can hold position

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9
FB

```
unit FB example
input data
    K  |  J  |   c =
   114    10      4
======================
calculated results

G =

      114
  ------------
  10 s^2 + 4 s

Continuous-time transfer function.


G_CL =

         114
  ----------------
  10 s^2 + 4 s + 114

Continuous-time transfer function.

poles =
  -0.2000 + 3.3705i
  -0.2000 - 3.3705i

    fn,Hz    |  f,Hz   |  zeta% =
    0.5374     0.5364     5.9235
    0.5374     0.5364     5.9235
```

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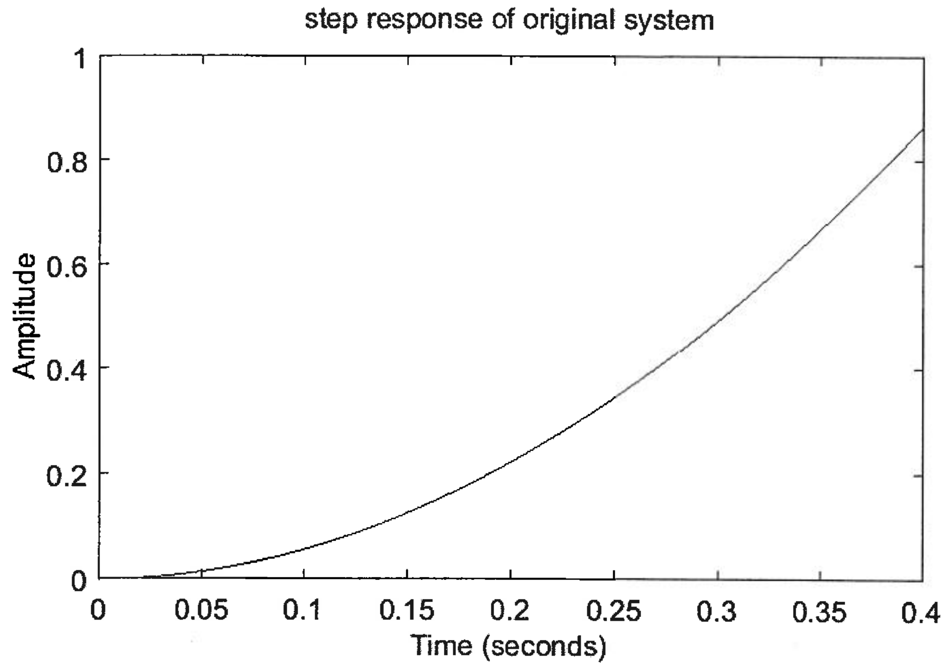
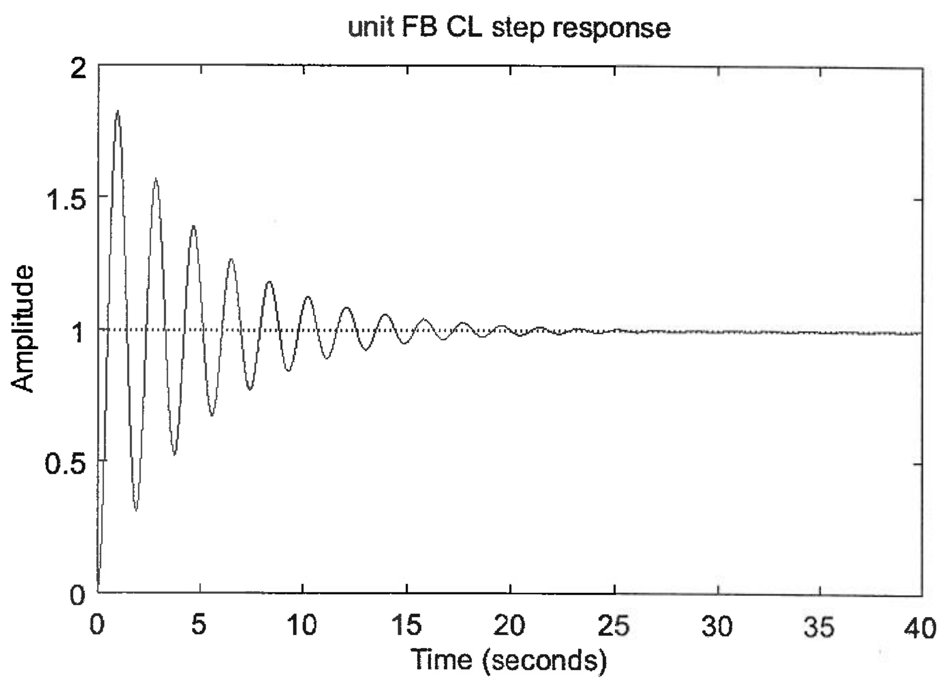
10
FB


step response of original system
Amplitude
Time (seconds)
0
0.2
0.4
0.6
0.8
1
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4


unit FB CL step response
Amplitude
Time (seconds)
0
0.5
1
1.5
2
5
10
15
20
25
30
35
40

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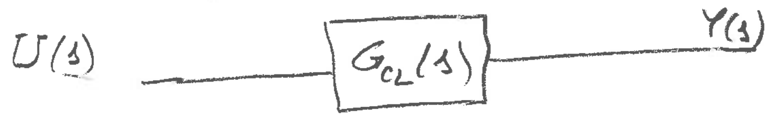
FBE 20161202

Steady state error of feedback systems



Objective: calculate steady state error e_{ss} of CL system without calculating $G_{CL}(s)$

Method: Use FVT or $E(s)$

Details: $E(s) = U(s) - Y(s)$ (1)

$Y(s) = G(s)E(s)$ (2)

(2) → (1) $E = U - GE \longrightarrow E(s) = \frac{1}{1+G(s)} U(s)$ (3)

FVT: $\lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$

$e_{ss} = \lim_{s \to 0} \frac{s}{1+G(s)} U(s)$ steady state error of FB system (4)

SS error depends on:

- input function $\begin{cases} \text{step} & U(s) = \frac{1}{s} \\ \text{ramp} & U(s) = \frac{1}{s^2} \end{cases}$

- system type

$G(s) = \frac{K}{s^N} \frac{(T_a s+1)(\dots)\dots}{(T_1 s+1)(\dots)\dots}$ $\begin{cases} \text{Type 0}, & N = 0 \\ \text{Type 1}, & N = 1 \\ \text{Type 2}, & N = 2 \end{cases}$

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FBE

Step error of FB systems

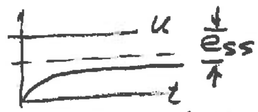
Figure of merit: "static position error constant" $= \lim_{s\to 0} G(s)$

misnomer: in fact, the larger, the better!

• **Derivation**: $u^{step}(s) = \frac{1}{s}$

$$e_{ss}^{step} = \lim_{s\to 0} \frac{\cancel{s}}{1+G(s)}\frac{1}{\cancel{s}} = \lim_{s\to 0}\frac{1}{1+G(s)} = \frac{1}{1+\lim_{s\to 0} G(s)}$$

$\boxed{\text{Type 0 system}}$ $(N=0)$: $G(s) = K\frac{(T_a s+1)(\ldots}{(T_1 s+1)(\ldots} \xrightarrow[s\to 0]{} K = \text{const}$

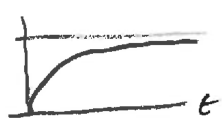


$$e_{ss}^{step} = \lim_{s\to 0}\frac{1}{1+G(s)} \xrightarrow[s\to 0]{} \frac{1}{1+K} \neq 0$$

Feedback creates non zero ss error for Type 0 sys!

$\boxed{\text{Type 1 system}}$ $(N=1)$: $G(s) = \frac{K}{s}\frac{(T_a s+1)\ldots}{(T_1 s+1)\ldots} \xrightarrow[s\to 0]{} \frac{1}{0}$

$$e_{ss}^{step} = \lim_{s\to 0}\frac{1}{1+G(s)} = \frac{1}{1+\frac{1}{0}} = \frac{0}{0+1} = 0$$



$e_{ss} = 0$

$\boxed{\text{Type 2 system}}$ $(N=2)$: $G(s) = \frac{K}{s^2}\frac{(T_a s+1)\ldots}{(T_1 s+1)\ldots} \xrightarrow[s\to 0]{} \frac{1}{0}$

$$e_{ss}^{step} = \lim_{s\to \infty}\frac{1}{1+G(s)} = \cdots = 0$$

$$e_{ss}^{step} = 0 \text{ for } N \geq 1$$

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Ramp error of FB systems

Figure of merit: "static velocity error constant" $= \lim_{s\to 0} s\,G(s)$

misnomer: in fact, the larger, the better!

Derivation: $U^{ramp}(s) = \frac{1}{s^2}$

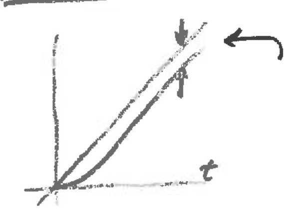
$$e_{ss}^{ramp} = \lim_{s\to 0} \frac{\cancel{s}}{1+G(s)}\frac{1}{s^{\cancel{2}}} = \lim_{s\to 0}\frac{1}{\cancel{s}+s\,G(s)} = \frac{1}{\lim_{s\to 0} s\,G(s)}$$

$\boxed{\text{Type 0 sys}}$ $\quad s\,G(s) = s\,K\frac{(T_a s+1)\cdots}{(T_1 s+1)\cdots} \xrightarrow[s\to 0]{} 0$

$$e_{ss}^{ramp} = \frac{1}{\lim_{s\to 0} s\,G(s)} = \frac{1}{0} = \infty$$

infinite error! cannot follow!

$\boxed{\text{Type 1 sys}}$ $\quad s\,G(s) = \cancel{s}\frac{K}{\cancel{s}}\frac{(T_a s+1)\cdots}{(T_1 s+1)\cdots} \xrightarrow[s\to 0]{} K = \text{const}$



$$e_{ss}^{ramp} = \frac{1}{\lim_{s\to 0} s\,G(s)} = \frac{1}{K}$$

ramp offset.

$\boxed{\text{Type 2 sys}}$ $\quad s\,G(s) = \cancel{s}\frac{K}{s^{\cancel{2}}}\frac{(T_a s+1)\cdots}{(T_1 s+1)\cdots} \xrightarrow[s\to 0]{} \frac{1}{0}$

$$e_{ss}^{ramp} = \frac{1}{\lim_{s\to 0} s\,G(s)} = 0$$

$e_{ss}^{ramp} = 0$ for $N \geq 2$

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FBE
20161202

System		Steady State errors	
Type	Expression	Step error e_{ss}^{step}	Ramp error e_{ss}^{ramp}
0	$K\frac{(T_a s+1)(\dots)\dots}{(T_1 s+1)(\dots)\dots}$	$\frac{1}{1+K}$	∞
1	$\frac{K}{s}\frac{(T_a s+1)(\dots)\dots}{(T_1 s+1)(\dots)\dots}$	0	$\frac{1}{K}$
2	$\frac{K}{s^2}\frac{(T_a s+1)(\dots)\dots}{(T_1 s+1)(\dots)\dots}$	0	0
⋮ $N>2$	$\frac{K}{s^N}\frac{(T_a s+1)(\dots)\dots}{(T_1 s+1)(\dots)\dots}$	0	0

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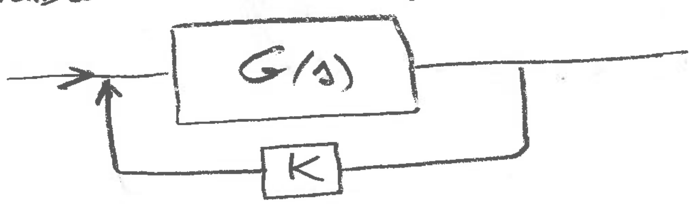
7.2 Stability of feedback control systems and root locus

1
SFB

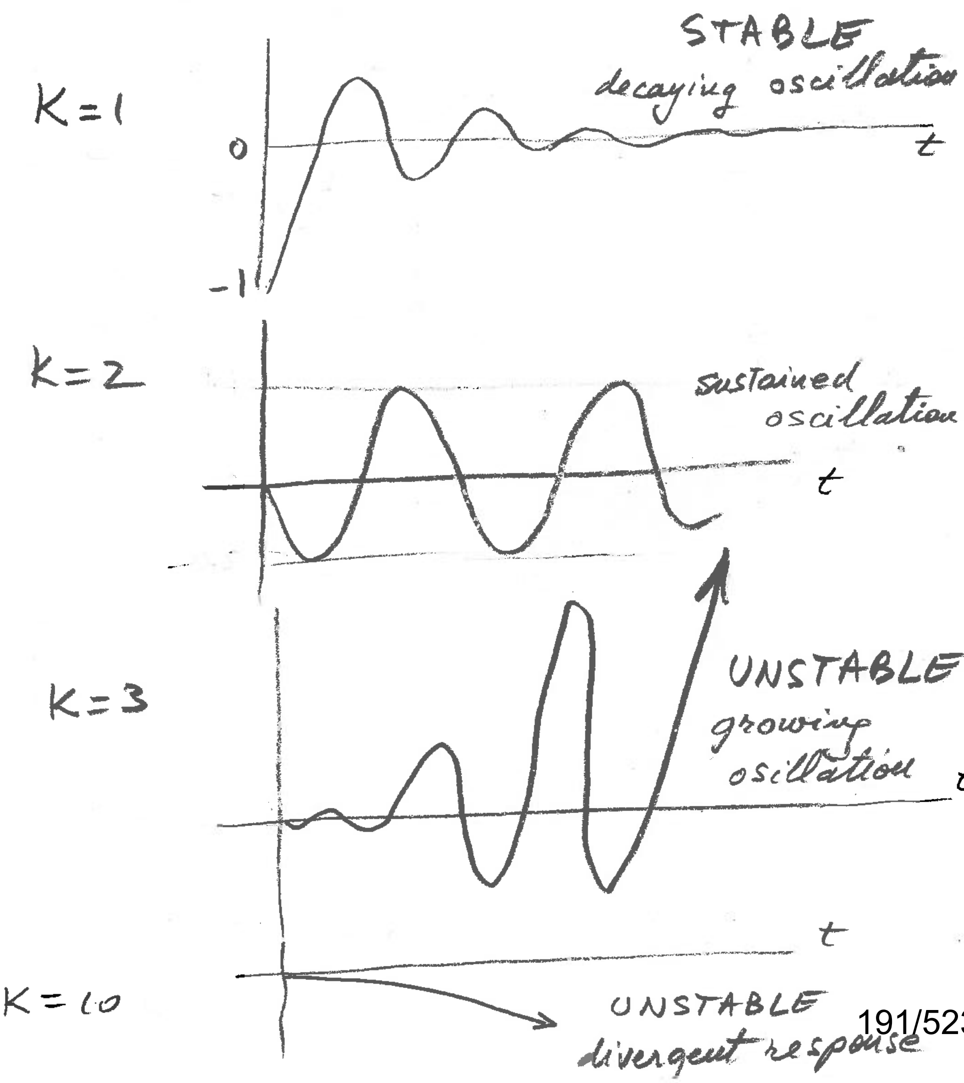
STABILITY OF FEEDBACK SYSTEMS

EXAMPLE

Run MATLAB code FB_stability_ex1



$$G(s) = \frac{1-s}{s^2+2s+4}$$



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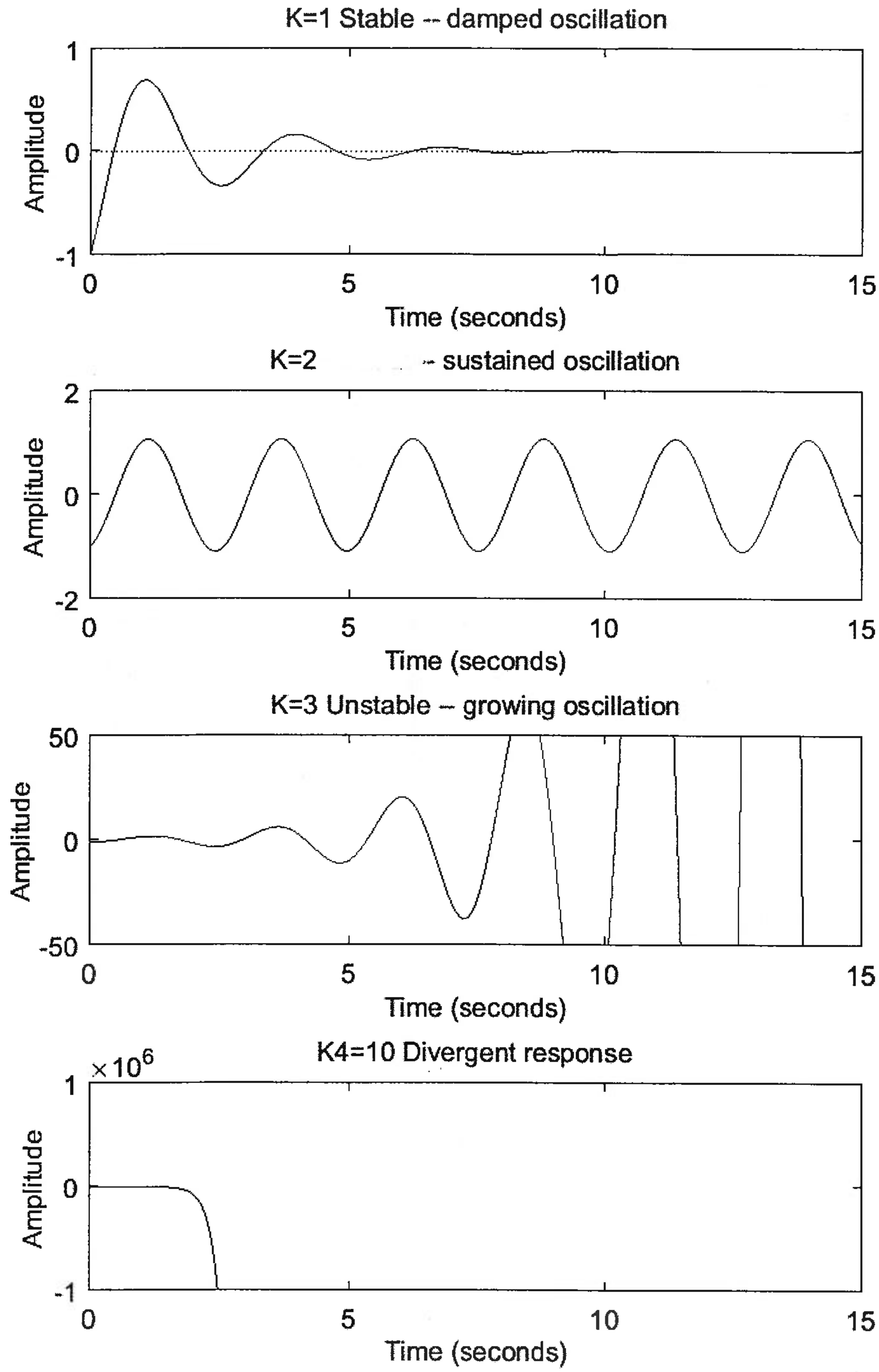
SFB


K=1 Stable -- damped oscillation
Amplitude
Time (seconds)
K=2 -- sustained oscillation
Amplitude
Time (seconds)
K=3 Unstable -- growing oscillation
Amplitude
Time (seconds)
K4=10 Divergent response
×10^6
Amplitude
Time (seconds)

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SFB

Explanation

$$G_{CL} = \frac{G}{1+GK} = \frac{\dfrac{1-s}{s^2+2s+4}}{1+K\dfrac{1-s}{s^2+2s+4}}$$

$$= \frac{\dfrac{1-s}{s^2+2s+4}}{s^2+2s+4+K(1-s)} = \frac{1-s}{s^2+(2-K)s+(K+4)}$$

Impulse response: $X(s) = G_{CL}(s) = \dfrac{1-s}{s^2+(2-K)s+(K+4)}$

Response type depends on characteristic eq.

$$s^2+(2-K)s+(K+4) = 0$$ characteristic eqn.

$K=1$ $\quad s^2+s+5=0$

$$p_{1,2} = \frac{-1 \pm \sqrt{1-4\times 5}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{19}}{2}$$

LHS
decaying osc.
STABLE

$K=2$ $\quad s^2+6=0$

$$p_{1,2} = \pm i\sqrt{6}$$

imag. axis
sustained oscillation

$K=3$ $\quad s^2-s+7=0$

$$p_{1,2} = \frac{1 \pm \sqrt{1-28}}{2} = \frac{1}{2} \pm i\frac{\sqrt{27}}{2}$$

RHS
growing osc.
UNSTABLE

$K=10$ $\quad s^2-8s+14=0$

$$p_{1,2} = 4 \pm \sqrt{4^2-14} = 4 \pm \sqrt{2} \begin{cases} 5.41 \\ 2.59 \end{cases}$$

RHS
UNSTABLE
non-oscillatory

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4

```
K1 =
     1

G1CL =

    -s + 1
  -----------
  s^2 + s + 5

Continuous-time transfer function.

p1 =
  -0.5000 + 2.1794i
  -0.5000 - 2.1794i
wn1, rad/sec | zeta1 =
    2.2361    0.2236
    2.2361    0.2236
=========================
K2 =
     2

G2CL =

  -s + 1
  -------
  s^2 + 6

Continuous-time transfer function.

p2 =
   0.0000 + 2.4495i
   0.0000 - 2.4495i
wn2, rad/sec | zeta2 =
    2.4495         0
    2.4495         0
```

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```
K3 =
     3

G3CL =

    -s + 1
  -----------
  s^2 - s + 7

Continuous-time transfer function.

p3 =
   0.5000 + 2.5981i
   0.5000 - 2.5981i
wn3, rad/sec | zeta3 =
     2.6458   -0.1890
     2.6458   -0.1890
=========================
K4 =
    10

G4CL =

      -s + 1
  --------------
  s^2 - 8 s + 14

Continuous-time transfer function.

p4 =
    5.4142
    2.5858
wn4, rad/sec | zeta4 =
    5.4142   -1.0000
    2.5858   -1.0000
```

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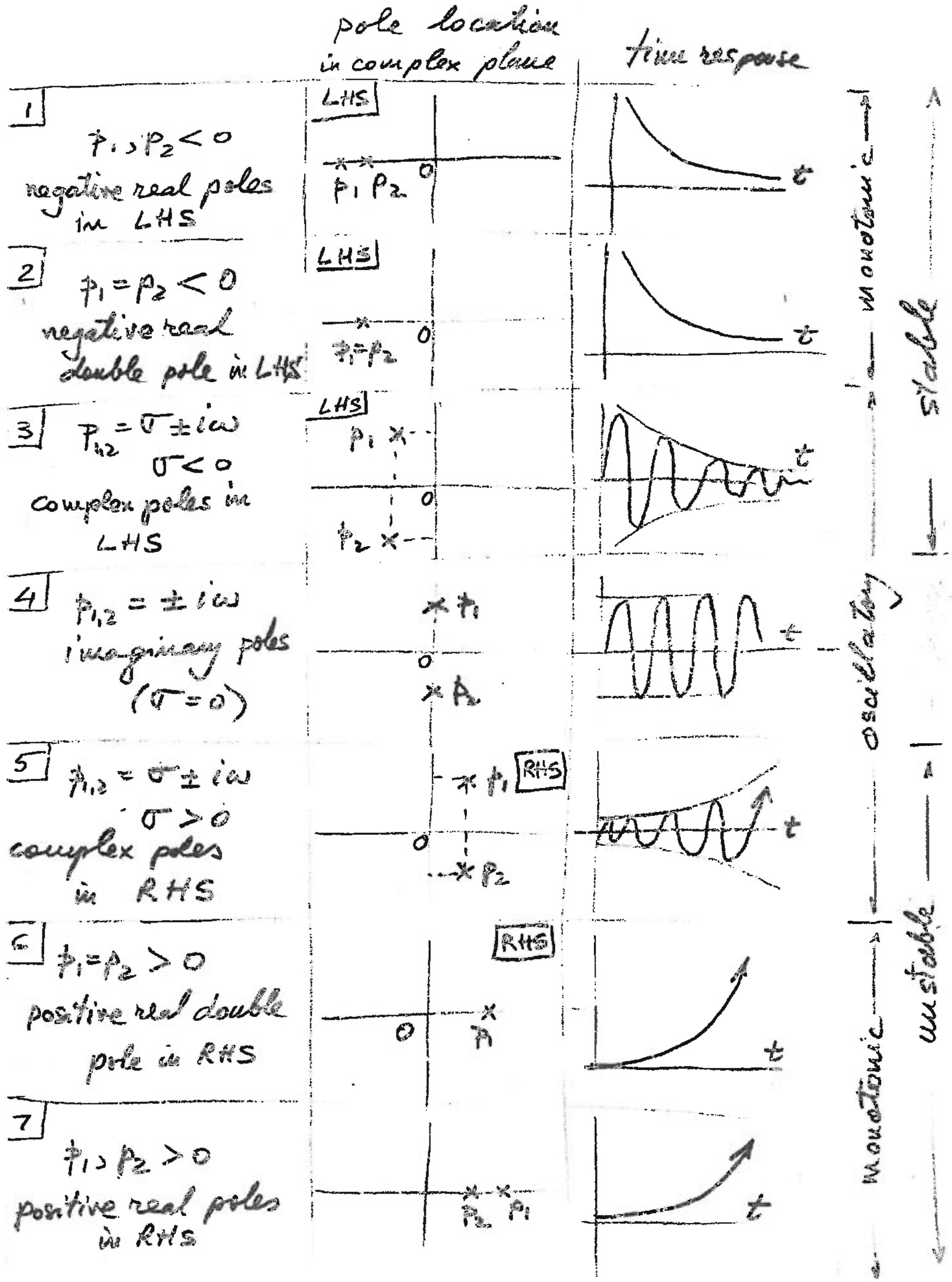
SFb

Recall stability analysis as function of pole location:



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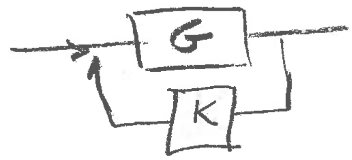
20140305

Root Locus

Given $G(s)$



Trace the poles of $G_{CL}(s)$ as K increases

$$G(s) = \frac{B(s)}{A(s)} = \frac{num(s)}{den(s)}$$

$$G_{CL} = \frac{G}{1+KG} = \frac{B(s)}{A(s)+KB(s)}$$

The root locus method looks at the root of the denominator of the CL system G_{CL}

$$A(s)+KB(s) = 0$$

The roots are traced in the s-plane for $K = [0, \infty)$

For $K=0$, the poles of G_{CL} are the same as the poles of G.

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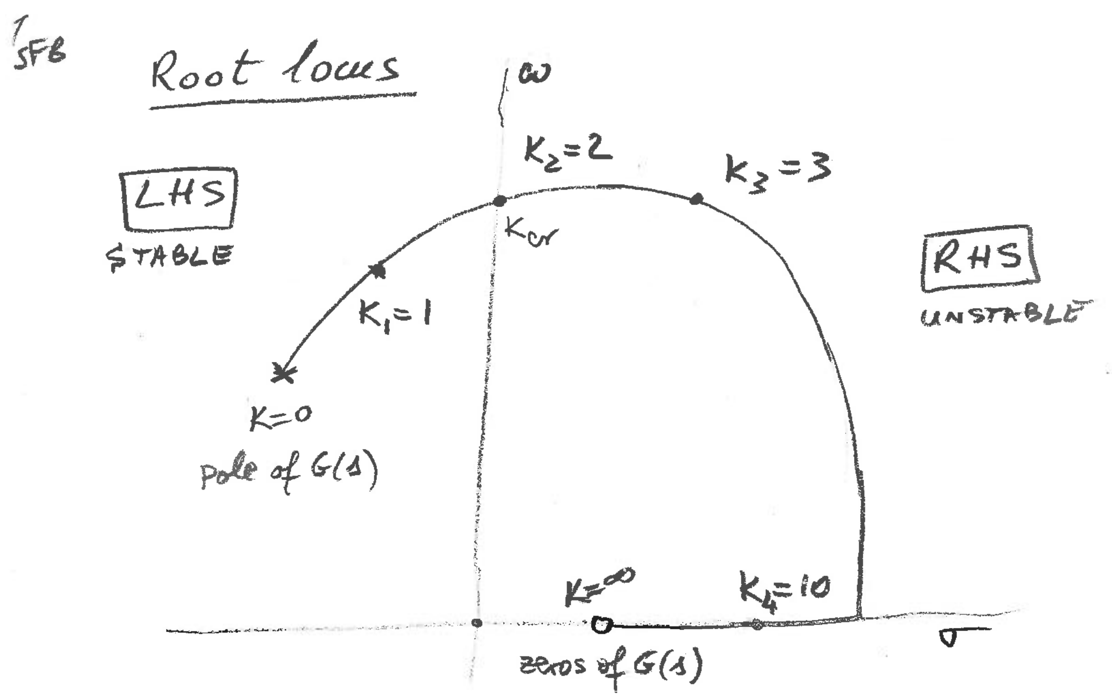


$$p_{1,2} = \sigma \pm i\omega_d = -\zeta\omega_n \pm i\,\omega_d$$

$$x(t) = C\,e^{-\zeta\omega_n t}\sin(\omega_d t + \varphi)$$

K_1 : $\sigma < 0 \longrightarrow \zeta > 0$

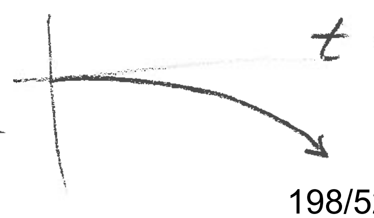
K_2 : $\sigma = 0 \longrightarrow \zeta = 0$
K_{cr}

K_3 : $\sigma > 0 \longrightarrow \zeta < 0$

K_4 : $\omega = 0$

$p_1 = \sigma_1$, $p_2 = \sigma_2 > 0$

$$x(t) = C_1 e^{p_1 t} + C_2 e^{p_2 t}$$



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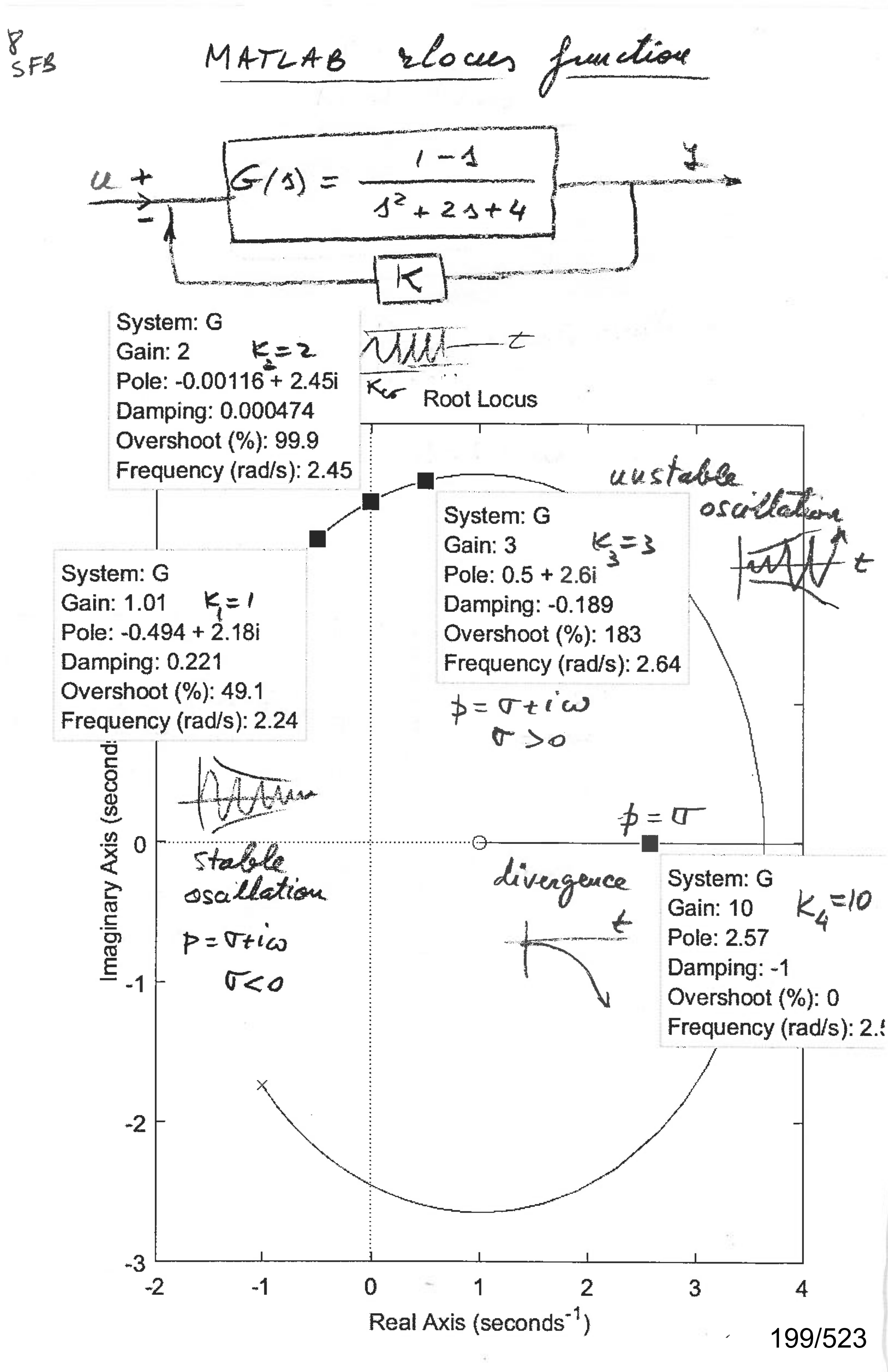
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MATLAB rlocus function
G(s) = (1 − s)/(s² + 2s + 4)
u
y
K
System: G
Gain: 2
K₂ = 2
Pole: -0.00116 + 2.45i
Damping: 0.000474
Overshoot (%): 99.9
Frequency (rad/s): 2.45
t
Kcr
Root Locus
unstable oscillation
System: G
Gain: 3
K₃ = 3
Pole: 0.5 + 2.6i
Damping: -0.189
Overshoot (%): 183
Frequency (rad/s): 2.64
System: G
Gain: 1.01
K₁ = 1
Pole: -0.494 + 2.18i
Damping: 0.221
Overshoot (%): 49.1
Frequency (rad/s): 2.24
p = σ + iω
σ > 0
p = σ
stable oscillation
p = σ + iω
σ < 0
divergence
System: G
Gain: 10
K₄ = 10
Pole: 2.57
Damping: -1
Overshoot (%): 0
Frequency (rad/s): 2.
Imaginary Axis (second
Real Axis (seconds⁻¹)
-3
-2
-1
0
1
2
3
4
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sFB

rlocus

Evans root locus

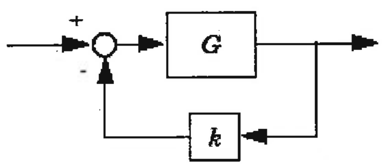
Syntax

```
rlocus
rlocus(sys)
rlocus(sys1,sys2,...)
```

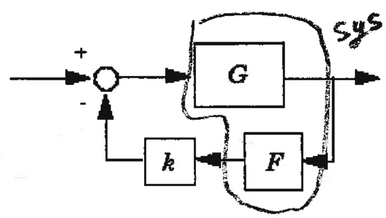
Description

`rlocus` computes the Evans root locus of a SISO open-loop model. The root locus gives the closed-loop pole trajectories as a function of the feedback gain k (assuming negative feedback). Root loci are used to study the effects of varying feedback gains on closed-loop pole locations. In turn, these locations provide indirect information on the time and frequency responses.

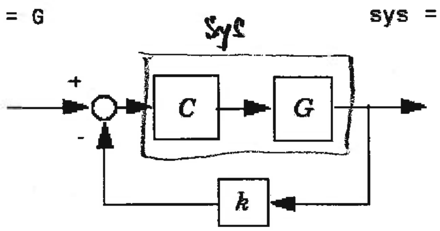
`rlocus(sys)` calculates and plots the root locus of the open-loop SISO model `sys`. This function can be applied to any of the following *negative* feedback loops by setting `sys` appropriately.



`sys = G`



`sys = F * G`



`sys = G * C`

If `sys` has transfer function

$$h(s) = \frac{n(s)}{d(s)}$$

the closed-loop poles are the roots of

$$d(s) + k\ n(s) = 0$$

`rlocus` adaptively selects a set of positive gains k to produce a smooth plot. Alternatively,

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. FB

```
rlocus(sys,k)
```

uses the user-specified vector k of gains to plot the root locus.

rlocus(sys1,sys2,...) draws the root loci of multiple LTI models sys1, sys2,... on a single plot. You can specify a color, line style, and marker for each model, as in

```
rlocus(sys1,'r',sys2,'y:',sys3,'gx').
```

When invoked with output arguments,

```
[r,k] = rlocus(sys)
r = rlocus(sys,k)
```

return the vector k of selected gains and the complex root locations r for these gains. The matrix r has length(k) columns and its jth column lists the closed-loop roots for the gain k(j).

Remarks

You can change the properties of your plot, for example the units. For information on the ways to change properties of your plots, see Ways to Customize Plots .

Example

Find and plot the root-locus of the following system.

$$h(s) = \frac{2s^2 + 5s + 1}{s^2 + 2s + 3}$$

```
h = tf([2 5 1],[1 2 3]);
rlocus(h)
```

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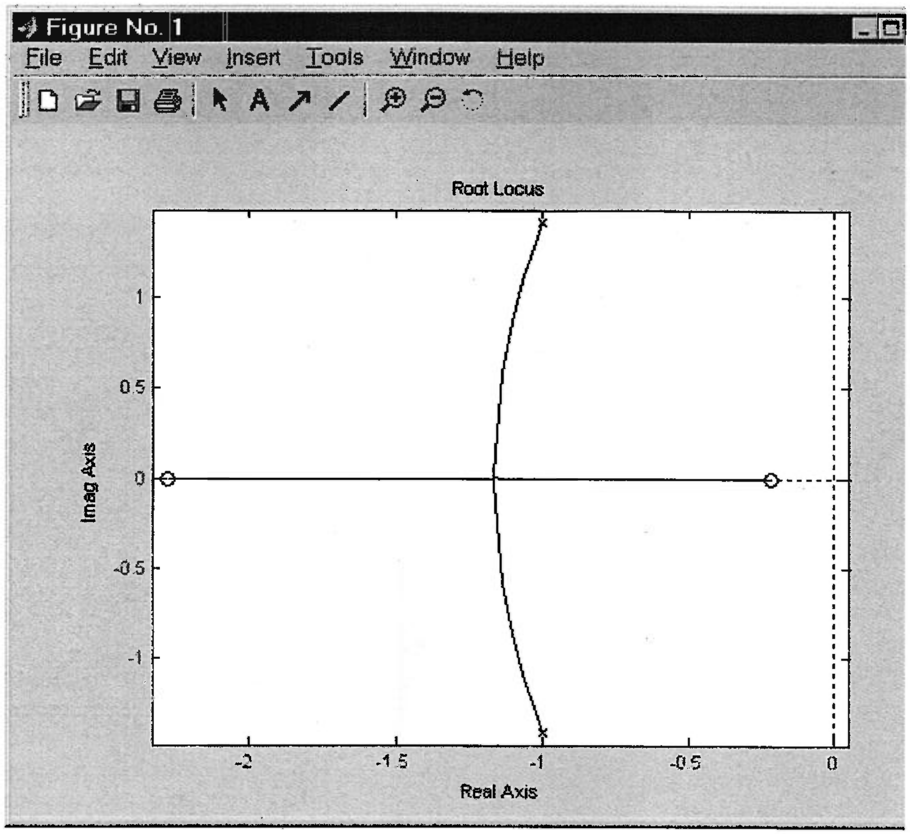
5FB



You can use the right-click menu for rlocus to add grid lines, zoom in or out, and invoke the Property Editor to customize the plot. Also, click anywhere on the curve to activate a data marker that displays the gain value, pole, damping, overshoot, and frequency at the selected point.

See Also

pole, pzmap

Provide feedback about this page

reshape rlocusplot

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ROOT LOCUS METHOD

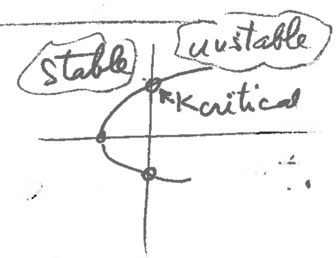
$G(s) = \frac{B(s)}{A(s)} = \frac{num}{den}$

$$G_{CL} = \frac{KG}{1+KG}$$

$$G(s) = \frac{(s+z_1)(s+z_2) + \cdots (s+z_m)}{(s+p_1)(s+p_2) + \cdots (s+p_n)} = \frac{num}{den}$$

$-z_1, \ldots, -z_m$ zeros of open loop transfer function

$-p_1, \ldots, -p_n$ poles ——— " ———



charact. eq^n: $1 + KG = 0$.

$A(s) + KB(s) = 0$

$(s+p_1)(s+p_2) \cdots (s+p_n) + K(s+z_1) \cdots (s+z_m) = 0$.

Solution of this equation gives the CL poles $s = \sigma + i\omega$

Root locus: trajectory of these poles as $K = 0 \longrightarrow \infty$

rlocus (num, den) ⟶ automatic K generation $K \in [0, \infty)$

rlocus (num, den, K) ⟶ must give K values

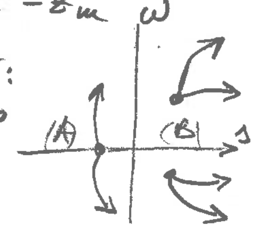
$K=0$: roots are the OL poles: $-p_1, -p_2, \ldots -p_n$

$K=\infty$: roots are the OL zeros: $-z_1, -z_2 \ldots -z_m$

Breakaway: multiple roots branch out:

(A) from real poles on real axis

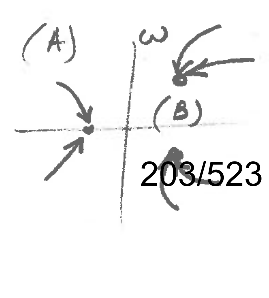
(B) from conjugate poles



Break in: branches coalesce into multiple roots

(A) into real poles

(B) into conjugate poles



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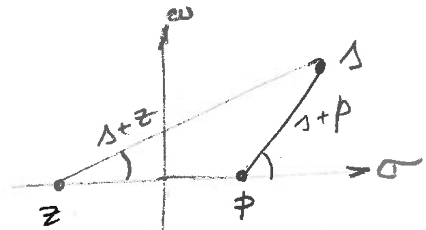
SFB

Angle condition



$$1+KG=0$$

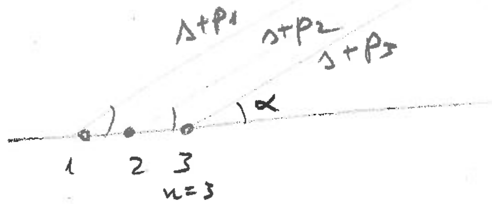
$$KG=-1$$

$$\angle KG = \angle -1 = \pm 180^\circ(2k+1),\ k=0,1,\dots$$

$$\angle KG = \angle K\frac{(s+z_1)(s+z_2)\cdots}{(s+p_1)(s+p_2)\cdots}$$

$$= \angle s+z_1 + \angle s+z_2 + \dots - \angle s+p_1 - \angle s+p_2 - \angle s+p_3 \dots = \pm 180^\circ(2k+1).$$

Asymptotes $s \to \infty$



- angles of asymptotes

$$\angle asymp = \frac{\pm 180^\circ(2k+1)}{3}$$

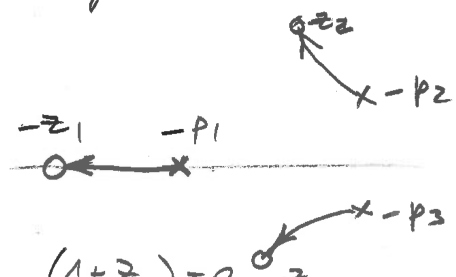
$$= 60^\circ;\ 180^\circ,\ -60^\circ$$

- axis crossing of asymptotes

14 SFB $(s+p_1)(s+p_2)\cdots + K(s+z_1)(s+z_2)\cdots = 0$

$K=0 \longrightarrow (s+p_1)(s+p_2)\cdots(s+p_n)=0$

roots are the OL poles



$K \to \infty$: $K(s+z_1)(s+z_2)\cdots(s+z_m)=0$

roots are the OL zeros

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7.3 Stability Criteria

RH 20140306

ROUTH CRITERION FOR HIGHER ORDER POLYNOMIALS

Given: $A(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0$

Find if any roots of $A(s)$ are in the RHS

Solution by Routh criterion

(1) If any of the coefficients $a_0, a_1, \ldots, a_n$ is negative, then at least one root is in RHS and the system is UNSTABLE. $\boxed{\text{STOP}}$

(2) If all coefficients are positive, do the Routh table:

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2
RH

R Table

$$A(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \cdots a_1 s + a_0 = 0$$

$$\begin{array}{l|lll}
s^n & a_n & a_{n-2} & a_{n-4} \; \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} \; \cdots \\
s^{n-2} & b_1 & b_2 & \\
s^{n-3} & c_1 & c_2 & \\
\vdots & & &
\end{array}$$

$$b_1 = \frac{a_{n-1} a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$b_2 = \frac{a_{n-1} a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$$c_1 = \frac{b_1 a_{n-3} - a_{n-1} b_2}{b_1}$$

$$\vdots$$

Count the number of <u>sign changes</u> and <u>zeros</u> in the first column to get the number of roots in RHS

Use MATLAB file:
Routh_Hurwitz_Stability_criterion_modified_vg1.m

3 RH

R. criterion Example VG1

$$G(s) = \frac{4s+2}{s^3+3s^2+4s+2} = \frac{B(s)}{A(s)}$$

Examine $A(s) = s^3 + 3s^2 + 4s + 2$

$a_3 = 1 \quad a_1 = 4$

$a_2 = 3 \quad a_0 = 2$

$$\begin{array}{c|cc} s^3 & 1 & 4 \\ s^2 & 3 & 2 \\ s^1 & 10/3 & \\ s^0 & 2 & \end{array}$$

$$b_1 = \frac{3\times 4 - 1\times 2}{3} = 10/3$$

$$b_2 = \frac{0-0}{3} = 0$$

$$c_1 = \frac{\frac{10}{3}\times 2 - 3\times 0}{10/3} = 2$$

Routh criterion: number of sign changes = zero (0)

i.e. NO root in RHS $\longrightarrow$ STABLE!

Verification by MATLAB

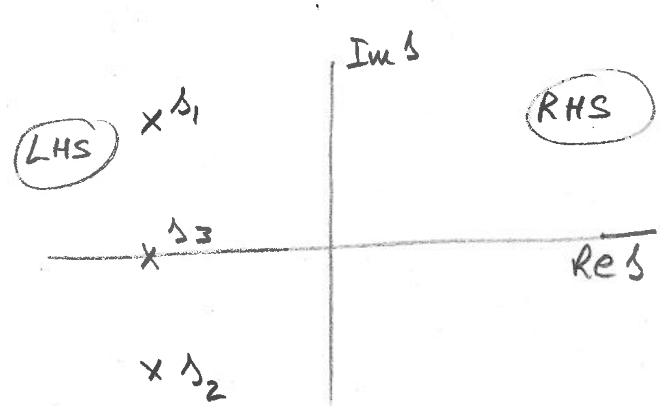
A = [1 3 4 2]

roots(A):

$s_1 = -1 + i$

$s_2 = -1 - i$

$s_3 = -1$



All roots are in LHS $\longrightarrow$ System is STABLE!

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4 RH 20140303

R Criterion

Example VG2

$$G(s) = \frac{31}{s^3 + 5s^2 + 6s + 31}$$

$a_3 = 1$ $a_2 = 5$ $a_1 = 6$ $a_0 = 31$.

$$\begin{array}{c|cc} s^3 & 1 & 6 \\ s^2 & 5 & 31 \\ s^1 & -0.2 & \\ s^0 & 31 & \end{array}$$

$$b_1 = \frac{5 \times 6 - 1 \times 31}{5} = -0.2$$

$$c_1 = \frac{-0.2 \times 31}{-0.2} = 31$$

$\rightarrow$ 2 sign changes .

2 roots in RHS $\rightarrow$ Unstable !

See Matlab print out on next page .

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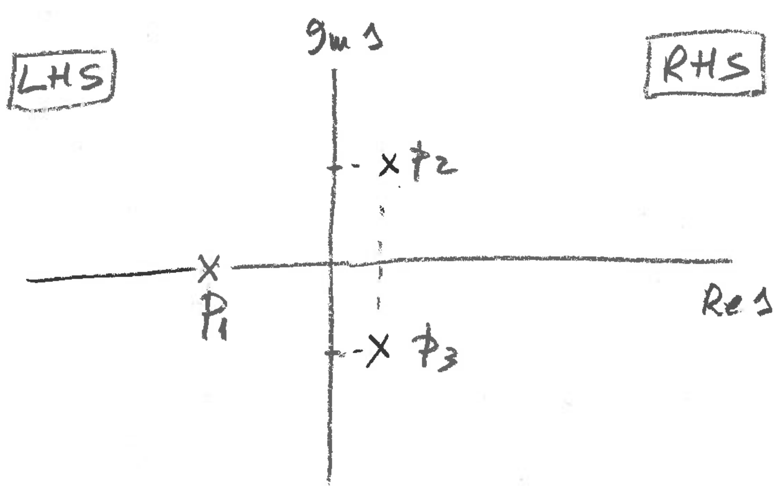
RH
20140303

MATLAB Command Window

```
Input coefficients of characteristic equation
[an an-1 an-2 ... a0]= [1 5 6 31]

----------------------------------------
Roots of characteristic equation is:

ans =

p1   -5.0319
p2    0.0160 + 2.4820i
p3    0.0160 - 2.4820i

--------The R                array is:--------

m =

    1.0000    6.0000
    5.0000   31.0000
   -0.2000         0
   31.0000         0

        ----> System is Unstable <----
>>
```



5a
RH R criterion Example VG3

$$G(s) = \frac{5s^2+8s+3}{s^6+3s^5-s^4-7s^3+10s^2+14s-20}$$

A(s) has some negative coefficients
At least one root of A(s) is in RHS
System is unstable

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RH

R criterion Example VG4

$$G(s) = \frac{5s^2+8s+3}{s^6+3s^5+s^4+7s^3+10s^2+14s+20}$$

$a_6=1$ $a_4=1$ $a_2=10$ $a_0=20$
$a_5=3$ $a_3=7$ $a_1=14$

s^6	1	1	10	20
s^5	3	7	14	
s^4	-1.33	5.33	20	
s^3	19	59		
s^2	9.4737	20		
s^1	18.8889			
s^0	20			

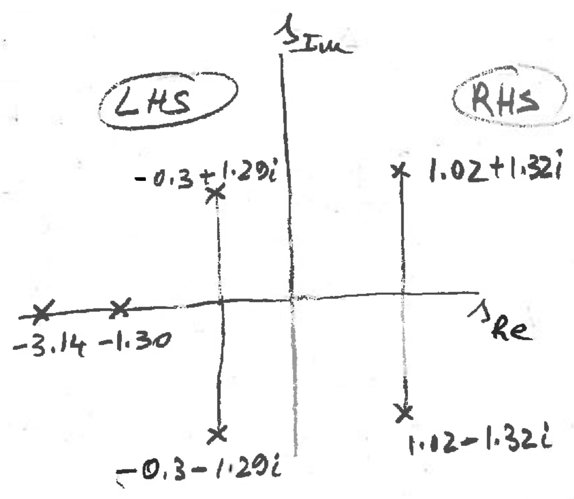
1
2

Two (2) sign changes
2 roots in RHS
UNSTABLE!

MATLAB
A=[1 3 1 7 10 14 20]
roots(A):

-3.14 + 0i
-1.30 + 0i
-0.30 + 1.29i
-0.30 - 1.29i
LHS

1.02 + 1.32i
1.02 - 1.32i
RHS



See book 5. for more about Routh criterion

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RH

Input coefficients of characteristic equation
[an an-1 an-2 ... a0]= [1 3 1 7 10 14 20]

Roots of characteristic equation are:

ans =

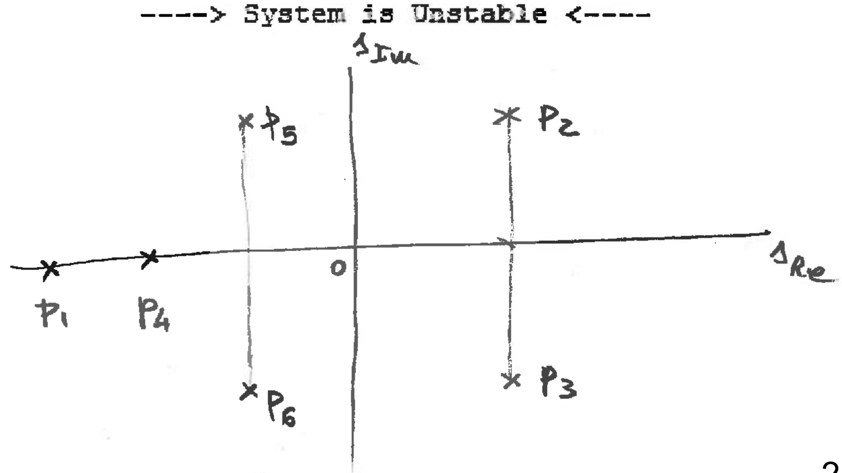
p1	-3.1462 + 0.0000i	LHS
p2	1.0202 + 1.3244i	RHS
p3	1.0202 - 1.3244i	RHS
p4	-1.2962 + 0.0000i	LHS
p5	-0.2990 + 1.2905i	LHS
p6	-0.2990 - 1.2905i	LHS

--------The Routh array is:--------

m =

1.0000	1.0000	10.0000	20.0000
3.0000	7.0000	14.0000	0
-1.3333	5.3333	20.0000	0
19.0000	59.0000	0	0
9.4737	20.0000	0	0
18.8889	0	0	0
20.0000	0	0	0

----> System is Unstable <----



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RH

(page 116)

5.4 HURWITZ STABILITY CRITERION

The Hurwitz criterion is another method for determining whether all the roots of the characteristic equation of a continuous system have negative real parts. This criterion is applied using determinants formed from the coefficients of the characteristic equation. It is assumed that the first coefficient, a_n, is positive. The determinants Δ_i, $i = 1, 2, \ldots, n-1$, are formed as the principal minor determinants of the determinant

$$\Delta_n = \begin{vmatrix} a_{n-1} & a_{n-3} & \cdots & \begin{bmatrix} a_0 & \text{if } n \text{ odd} \\ a_1 & \text{if } n \text{ even} \end{bmatrix} & 0 & \cdots & 0 \\ a_n & a_{n-2} & \cdots & \begin{bmatrix} a_1 & \text{if } n \text{ odd} \\ a_0 & \text{if } n \text{ even} \end{bmatrix} & 0 & \cdots & 0 \\ 0 & a_{n-1} & a_{n-3} & \cdots & \cdots & \cdots & 0 \\ 0 & a_n & a_{n-2} & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & a_0 \end{vmatrix}$$

The determinants are thus formed as follows:

$$\Delta_1 = a_{n-1}$$

$$\Delta_2 = \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix} = a_{n-1}a_{n-2} - a_n a_{n-3}$$

$$\Delta_3 = \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} \\ a_n & a_{n-2} & a_{n-4} \\ 0 & a_{n-1} & a_{n-3} \end{vmatrix} = a_{n-1}a_{n-2}a_{n-3} + a_n a_{n-1} a_{n-5} - a_n a_{n-3}^2 - a_{n-4}a_{n-1}^2$$

and so on up to Δ_{n-1}.

Hurwitz Criterion: *All the roots of the characteristic equation have negative real parts if and only if* $\Delta_i > 0$, $i = 1, 2, \ldots, n$.

EXAMPLE 5.5. For $n = 3$,

$$\Delta_3 = \begin{vmatrix} a_2 & a_0 & 0 \\ a_3 & a_1 & 0 \\ 0 & a_2 & a_0 \end{vmatrix} = a_2a_1a_0 - a_0^2 a_3, \qquad \Delta_2 = \begin{vmatrix} a_2 & a_0 \\ a_3 & a_1 \end{vmatrix} = a_2a_1 - a_0a_3, \qquad \Delta_1 = a_2$$

Thus all the roots of the characteristic equation have negative real parts if

$$a_2 > 0 \qquad a_2a_1 - a_0a_3 > 0 \qquad a_2a_1a_0 - a_0^2a_3 > 0$$

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RH
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Q : What is R criterion ?

A : R criterion is a tabular method to determine if a polynomial

$A(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$

has roots in the RHS

Q : What is R criterion good for ?

A : R criterion is used to evaluate the stability of a system $G(s) = B(s)/A(s)$

If $A(s)$ has roots in RHS, then the system is UNSTABLE

Q : How do I use R criterion ?

A : - If $A(s)$ has at least one -ve coefficient, then the system is UNSTABLE. $\boxed{\text{STOP}}$

- If all coefficients of $A(s)$ are +ve, then do R Table to find if any roots are in RHS

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RH

Q: What is H criterion?

A: H criterion is another method of using the coefficients of a polynomial equation to determine whether all the roots have negative real parts

---

Q: How is H criterion different from R criterion?

A: H criterion uses determinants, whereas R criterion uses a table

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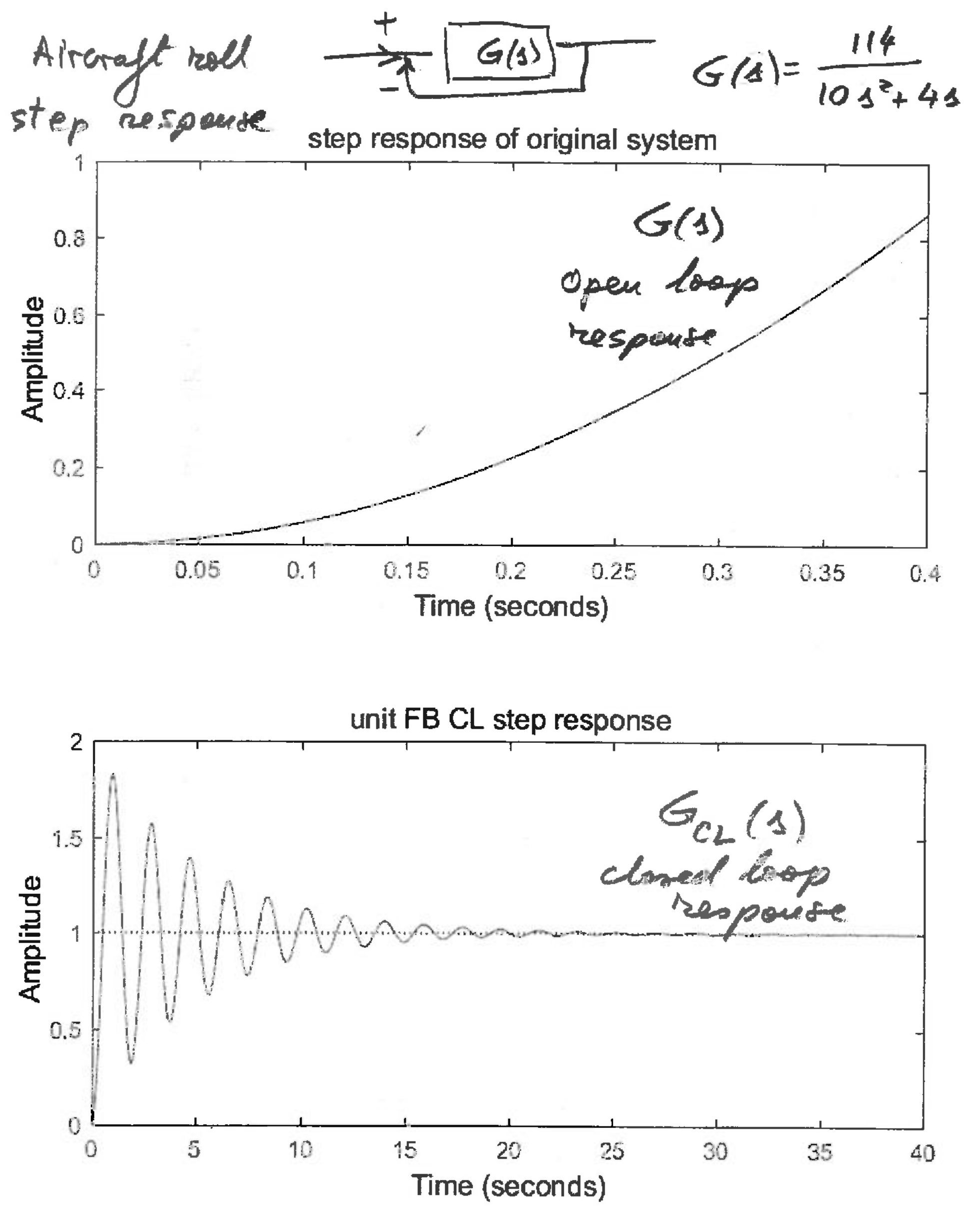
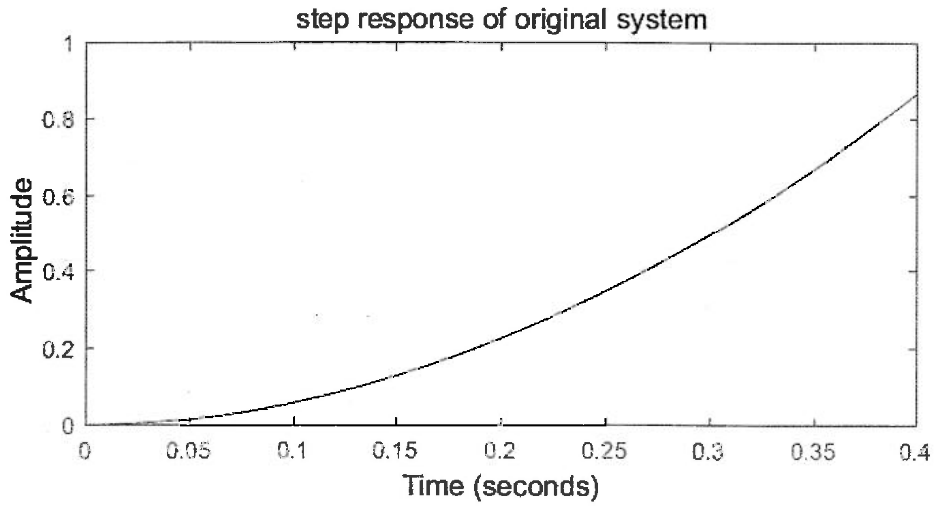
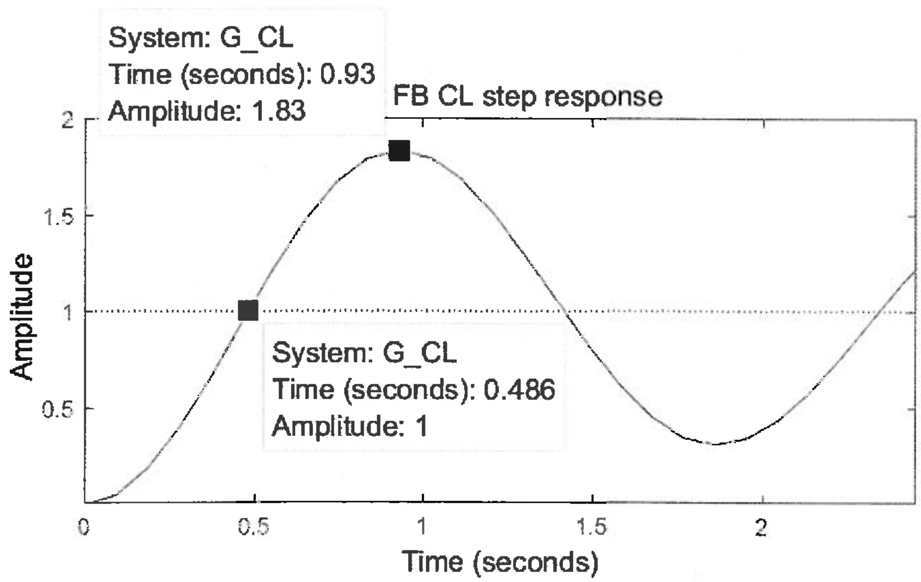
7.4 Feedback Controllers


Aircraft roll step response
+
−
G(s)
$G(s) = \frac{114}{10s^2+4s}$
step response of original system
G(s) Open loop response
Amplitude
Time (seconds)
0
0.2
0.4
0.6
0.8
1
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
unit FB CL step response
$G_{CL}(s)$ closed loop response
0.5
1.5
2
5
10
15
20
25
30
35
40

Ob c





Measured: $t_r = 0.486$ sec

$M_p = 83\%$

Design Specs

DS1: $t_r \leq 1.5$ sec

DS2: $M_p \leq 20\%$

DS1: satisfied

DS2: NOT satisfied

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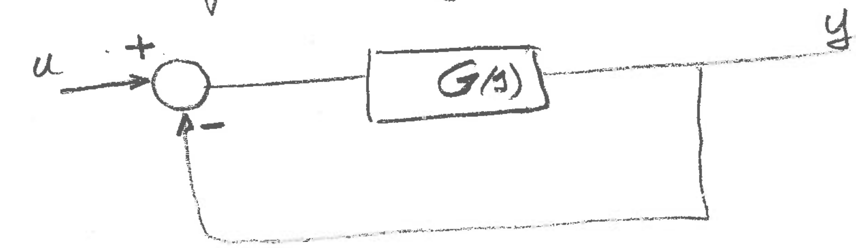
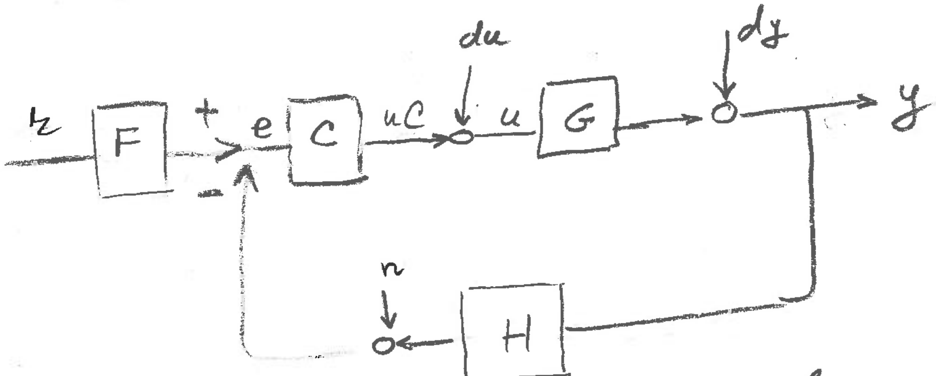
C 20140317

CONTROLLERS

Basic feedback system:



Enhanced FB control systems



Addition "boxes" with specific transfer function may be added to improve performance

Examples:

- PID controllers
- Filters
- Compensators
- Pre-filters
- Post-filters

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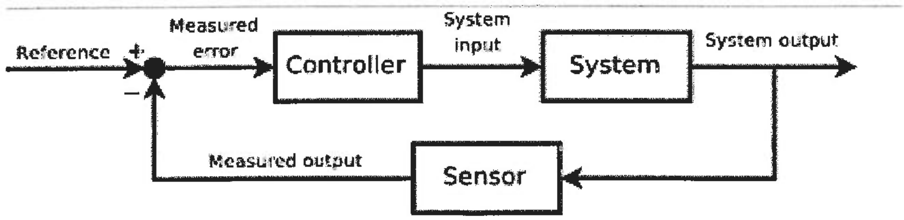
2
C

CONTROLLERS



Controllers modulate the feedback error to improve the performance of the feedback control system.

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4c

Filters

$$G_c(s) = \frac{1}{Ts+1} \quad \text{(low pass filter)}$$

$$G_c(s) = \frac{Ts}{Ts+1} \quad \text{(high pass filter)}$$

Compensators

$$G_c(s) = \frac{T_a s + 1}{T_1 s + 1}$$

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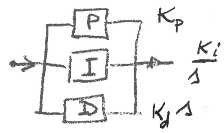
PID controllers



$$\boxed{G_c = K_p + \frac{K_i}{s} + K_d s}$$

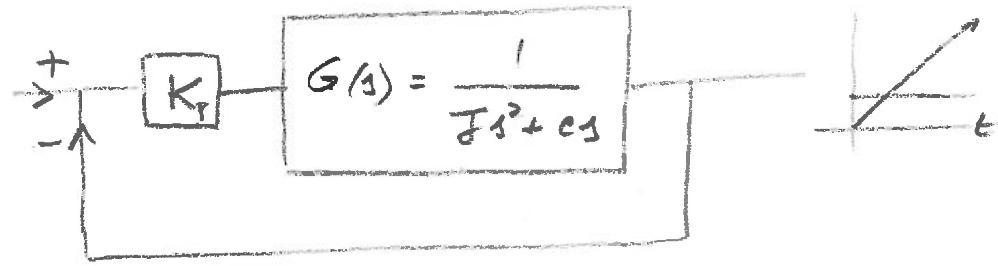
$$\boxed{G_c = K\left(1 + \frac{1}{T_i s} + T_d s\right)}$$

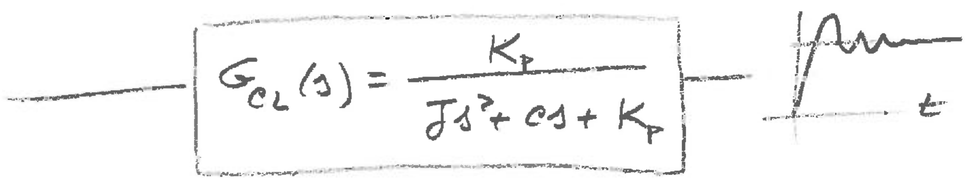
	t-domain	s-domain		comments	
P (proportional)	K	K		• simplest • adds stiffness, increases frequency	
I (integral)	$\int dt$	$\frac{K_i}{s}$	$\frac{1}{T_i s}$	• eliminates offsets • increases system order • may be unstable	"under compensator"
D (derivative)	d/dt	$K_d s$	$T_d s$	• adds damping • decreases system order • increases sensitivity • used as PD or PID	"anticipates & corrects the error"

Note : D control is not physically realisable. It is done as $\frac{Ns}{s+N} \xrightarrow[N \gg 1]{} s$ 224/523

P control of Type 1 system.

Recall:





$$G_{CL}(s) = \frac{K_p}{Js^2 + cs + K_p} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = \frac{K_p}{J} \quad , \quad \omega_n = \sqrt{\frac{K_p}{J}}$$

$$\zeta = \frac{c}{2\sqrt{JK_p}}$$

K = factor of proportionality ; "P-control"

Modification of K_p can modify frequency ω_n and damping ζ

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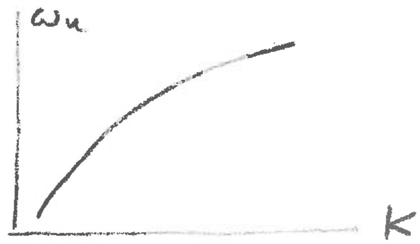
v P

P-control of frequency and damping

K_p = factor of proportionality $\rightarrow$ P-control

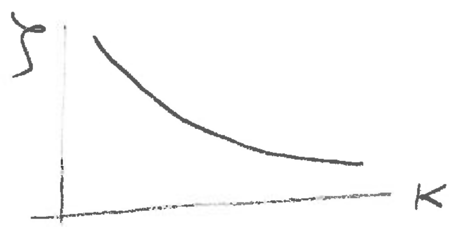
Modification of K_p can modify the frequency ω_n and the damping ζ

$$\omega_n^2 = \frac{K_p}{J}$$



Frequency increases with K

$$\zeta = \frac{c}{2\sqrt{J K_p}}$$

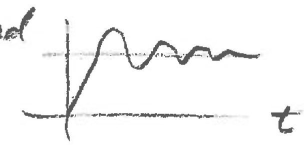


Damping decreases with K_p

Critical damping: $\zeta = 1 \rightarrow \frac{c}{2\sqrt{J K_{cr}}} = 1$: $K_{cr} = \frac{c^2}{4J}$

$0 < K_p \leq K_{cr}$. overdamped converging . exponential response .

$K_{cr} < K_p$ underdamped oscillatory response



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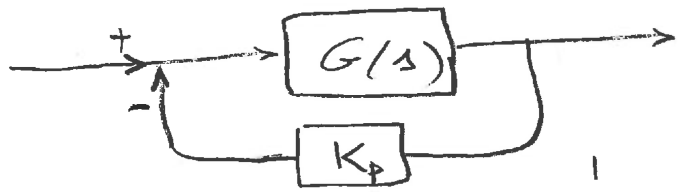
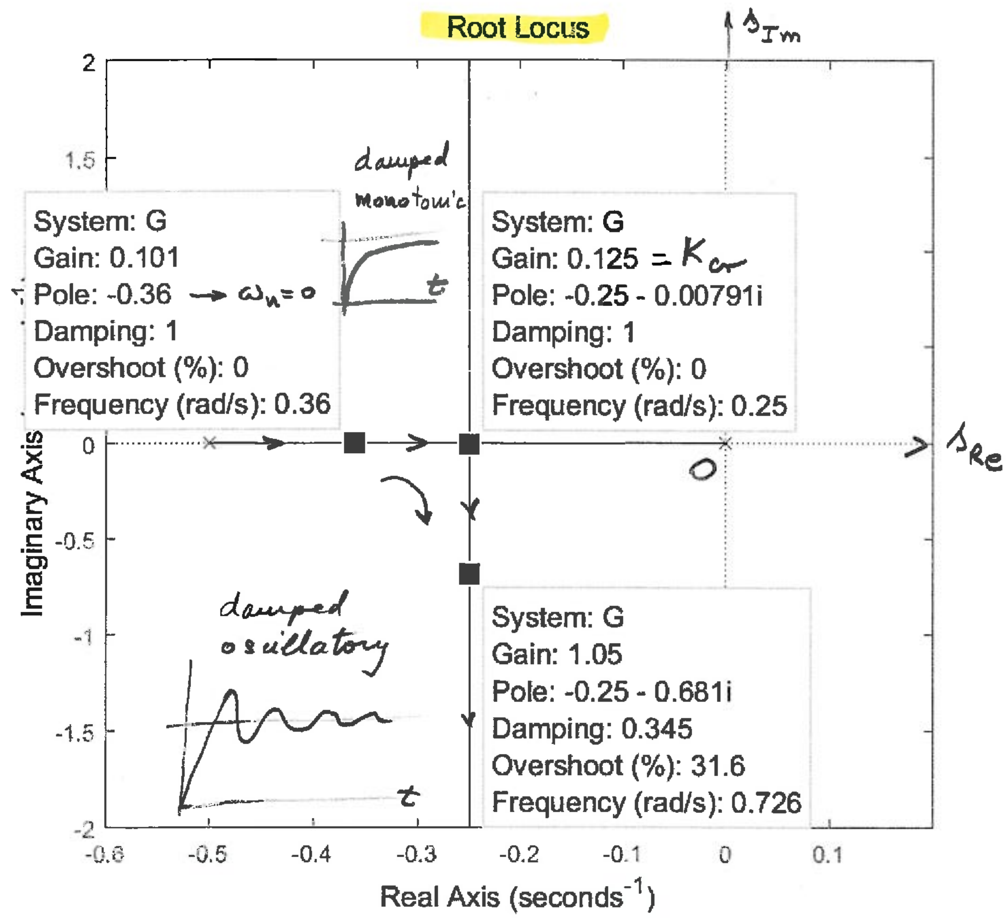
3 P

$$G(s) = \frac{1}{2s^2 + s}$$

$$K_{cr} = \frac{c^2}{4J} = \frac{1}{8} = 0.125$$





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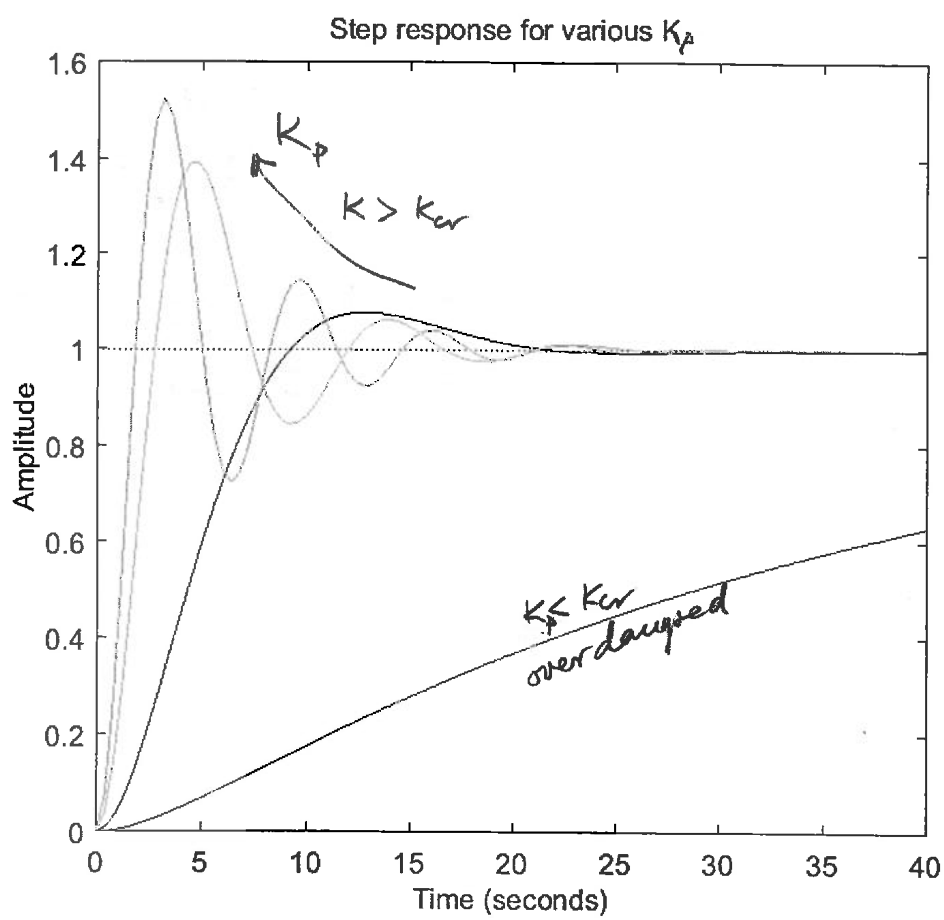
k
p

P-CONTROL

Type 1 sys



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5p

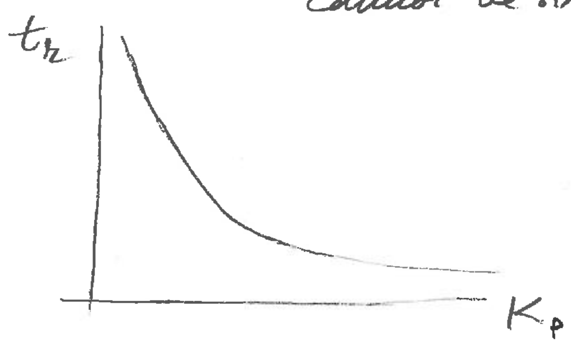
P-control of t_r and M_p

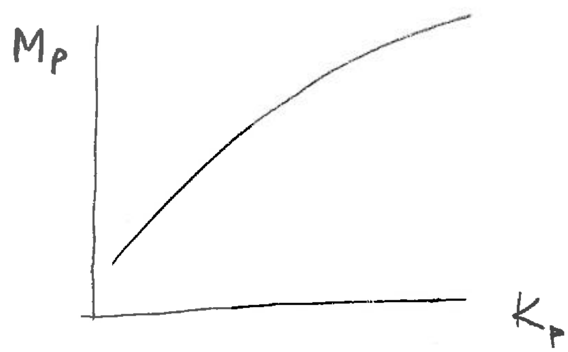
Recall:

rise time $$t_r = \frac{\pi - \varphi}{\omega_d} = \frac{\pi - \varphi}{\omega_n \sqrt{1-\zeta^2}} \quad , \quad \varphi = \sin^{-1}\sqrt{1-\zeta^2}$$

overshoot $$M_p = e^{-\pi \frac{\zeta}{\sqrt{1-\zeta^2}}}$$

Small overshoot M_p and small rise time t_r cannot be simultaneously met!





Run MATLAB program

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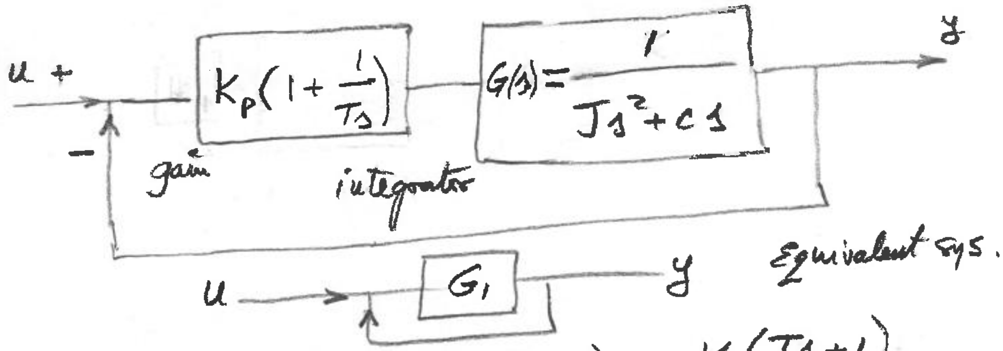
PI

PI Control, Principle



$$G_1(s) = K_p\left(1+\frac{1}{Ts}\right)\left(\frac{1}{Js^2+cs}\right) = \frac{K_p(Ts+1)}{Ts(Js^2+cs)}$$

$$G_{CL} = \frac{G_1}{1+G_1} = \frac{K_p(Ts+1)}{(TJ)s^3+(Tc)s^2+(K_pT)s+K_p}$$

- 3rd order system
- May become unstable
- One could use R–H criterion to predict critical T for instability, i.e.,

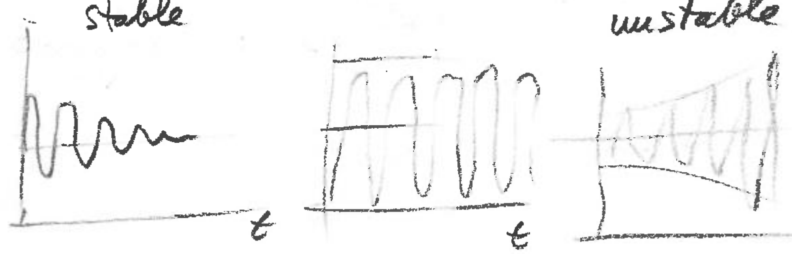
$$T > T_{cr} = J/c$$

(see next page.)

Example: $J=2$, $c=1$, $T_{cr}=\frac{2}{1}=2$

(see results next page)

$T=4$ stable $\qquad$ $T=T_{cr}=2$ $\qquad$ $T=1$ unstable



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10
P5 RH (Routh-Hurwitz) Criterion Table.

$$\overset{a_0}{(TJ)}s^3 + \overset{a_1}{(Tc)}s^2 + \overset{a_2}{(K_pT)}s + \overset{a_3}{K} = 0$$

s^3	TJ	K_pT
s^2	Tc	K_p
s^1	b_1	
s^0	K_p	

$$b_1 = \frac{(Tc)(K_pT) - (TJ)K_p}{Tc} = \frac{(cT-J)K_p}{c}$$

$$c_1 = \frac{\cancel{b_1}K_p}{\cancel{b_1}} = K_p$$

Discussion

s^3	TJ	+ve
s^2	Tc	+ve
s^1	b_1	may be +ve or -ve.
s^0	K_p	+ve

If b_1 is -ve, then sign change, i.e., INSTABILITY

For stability, $b_1 > 0$, i.e. $cT - J > 0$.

Need $cT > J$

$$T > \frac{J}{c}$$

Denote $T_{cr} = \frac{J}{c}$

Need $T > T_{cr}$ for stability.

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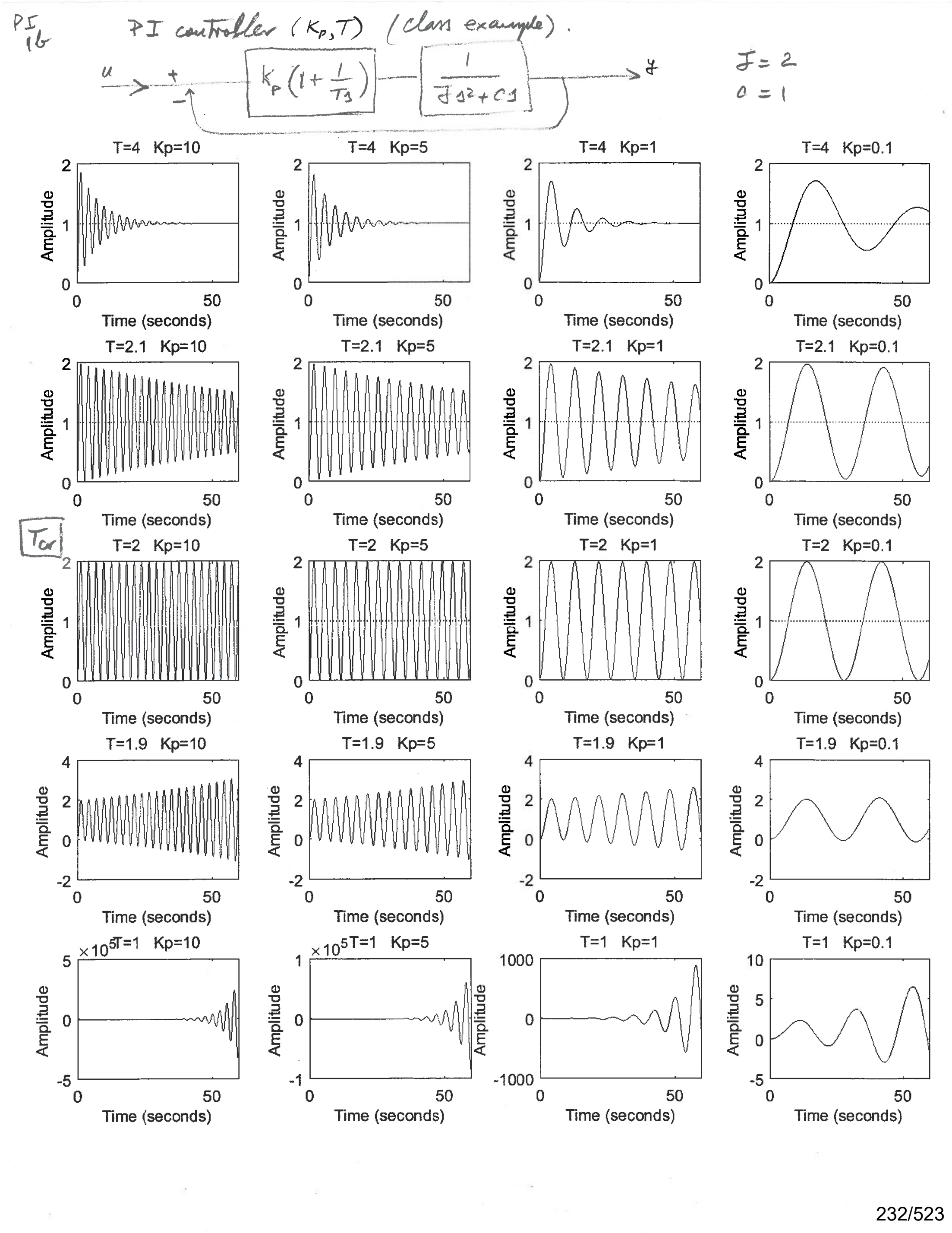
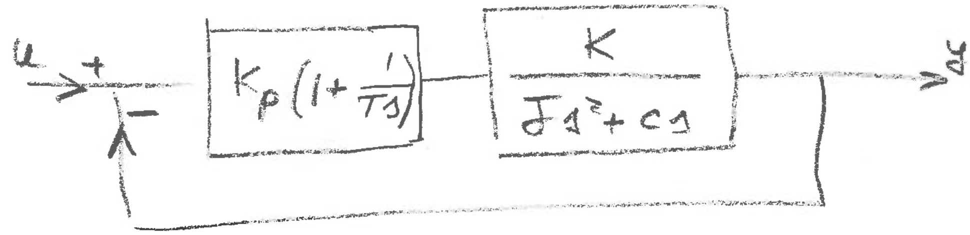

PI 1b
PI controller (K_p, T) (class example).
u
+
−
$K_p\left(1+\frac{1}{Ts}\right)$
$\frac{1}{Js^2+cs}$
y
J = 2
c = 1
T_{cr}
T=4 Kp=10
T=4 Kp=5
T=4 Kp=1
T=4 Kp=0.1
T=2.1 Kp=10
T=2.1 Kp=5
T=2.1 Kp=1
T=2.1 Kp=0.1
T=2 Kp=10
T=2 Kp=5
T=2 Kp=1
T=2 Kp=0.1
T=1.9 Kp=10
T=1.9 Kp=5
T=1.9 Kp=1
T=1.9 Kp=0.1
T=1 Kp=10
T=1 Kp=5
T=1 Kp=1
T=1 Kp=0.1
×10^5
Amplitude
Time (seconds)
232/523

2 PI

PI Controller (K_p, T)



$$G(s) = K_p\left(1+\frac{1}{Ts}\right)\left(\frac{K}{Js^2+cs}\right) = \frac{K_pK(Ts+1)}{Ts(Js^2+cs)}$$

$$G_{CL} = \frac{G}{1+G} = \frac{K_pK(Ts+1)}{TJs^3+Tcs^2+K_pKTs+K_pK}$$

- 3rd order system
- May become unstable
- R-H stability criterion requires $T > T_{cr}$,
  $T_{cr} = \frac{J}{c}$, for stability (next page)

<u>Example</u> : $K=114$, $J=10$, $c=4$, $T_{cr}=\frac{10}{4}=2.5$
(aircraft)

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20
P1

R-H Table

$$(TJ)s^3 + (Tc)s^2 + (K_p K T)s + (K_p K) = 0$$

s^3	TJ	$K_p K T$
s^2	Tc	$K_p K$
s^1	b_1	
s^0	$K_p K$	

$$b_1 = \frac{\cancel{T}c K_p K T - \cancel{T} J K_p K}{\cancel{T}c} = \frac{Tc - J}{c} K_p K$$

Discussion

s^3	TJ	+ve
s^2	Tc	+ve
s^1	b_1	may be +ve or −ve
s^0	$K_p K$	+ve

INSTABILITY happens if $b_1 < 0$, i.e., $Tc < J$.

For stability, need $Tc > J$ or

$$T > T_{cr}, \quad T_{cr} = \frac{J}{c}$$

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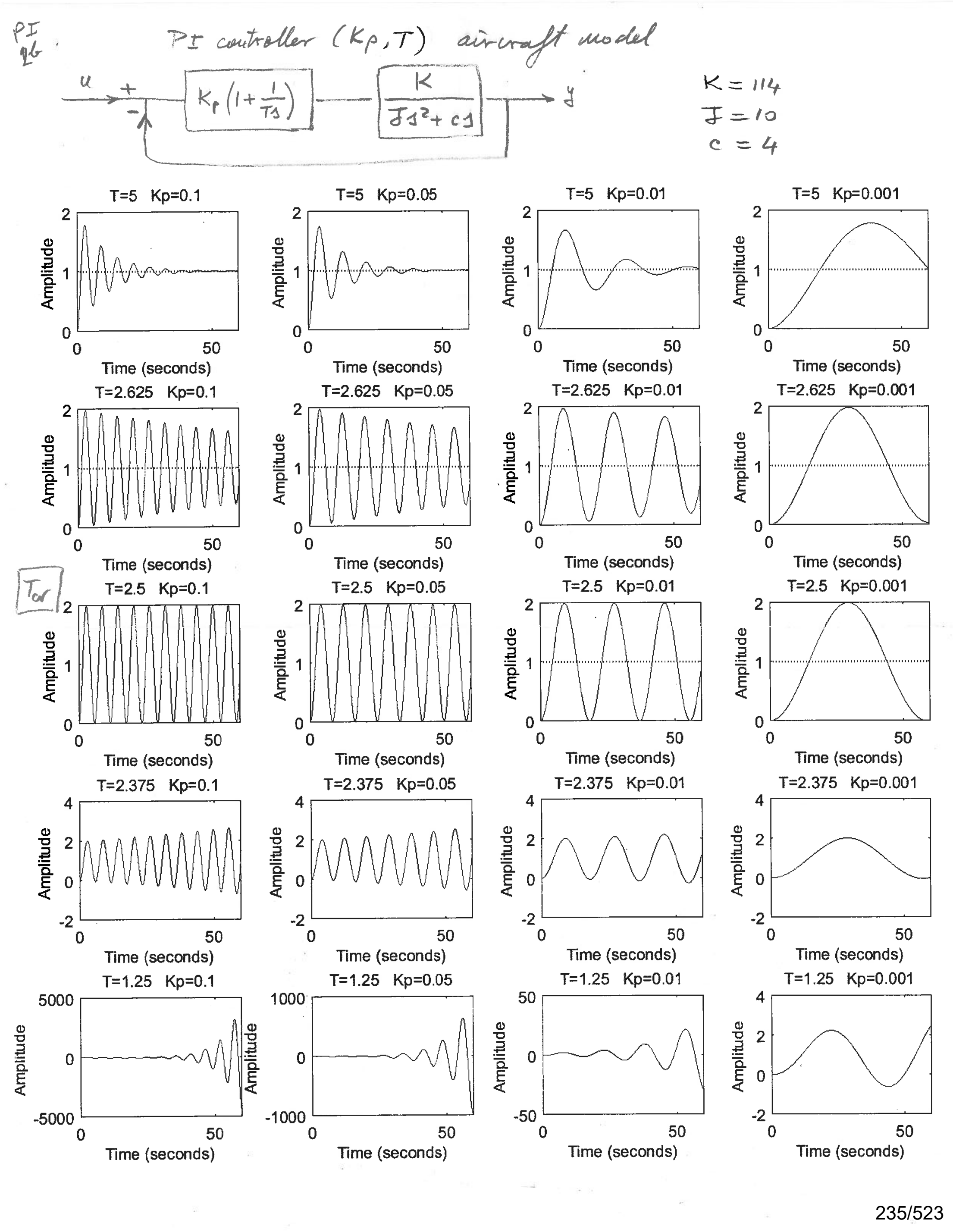

PI controller (Kp, T) aircraft model
$K_p\left(1+\frac{1}{Ts}\right)$
$\frac{K}{Js^2+cs}$
K = 114
J = 10
c = 4
T=5 Kp=0.1
T=5 Kp=0.05
T=5 Kp=0.01
T=5 Kp=0.001
T=2.625 Kp=0.1
T=2.625 Kp=0.05
T=2.625 Kp=0.01
T=2.625 Kp=0.001
T=2.5 Kp=0.1
T=2.5 Kp=0.05
T=2.5 Kp=0.01
T=2.5 Kp=0.001
T=2.375 Kp=0.1
T=2.375 Kp=0.05
T=2.375 Kp=0.01
T=2.375 Kp=0.001
T=1.25 Kp=0.1
T=1.25 Kp=0.05
T=1.25 Kp=0.01
T=1.25 Kp=0.001
Amplitude
Time (seconds)
235/523

2c
PI

Aircraft roll response

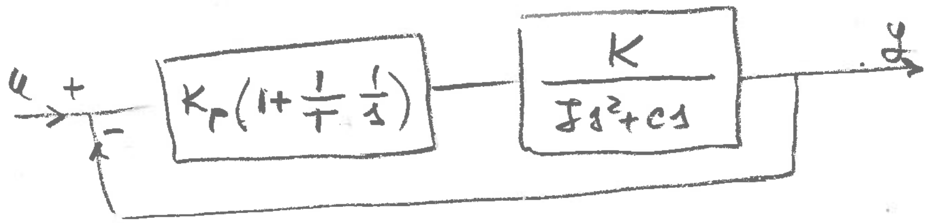
with (K_p, T) PI controller

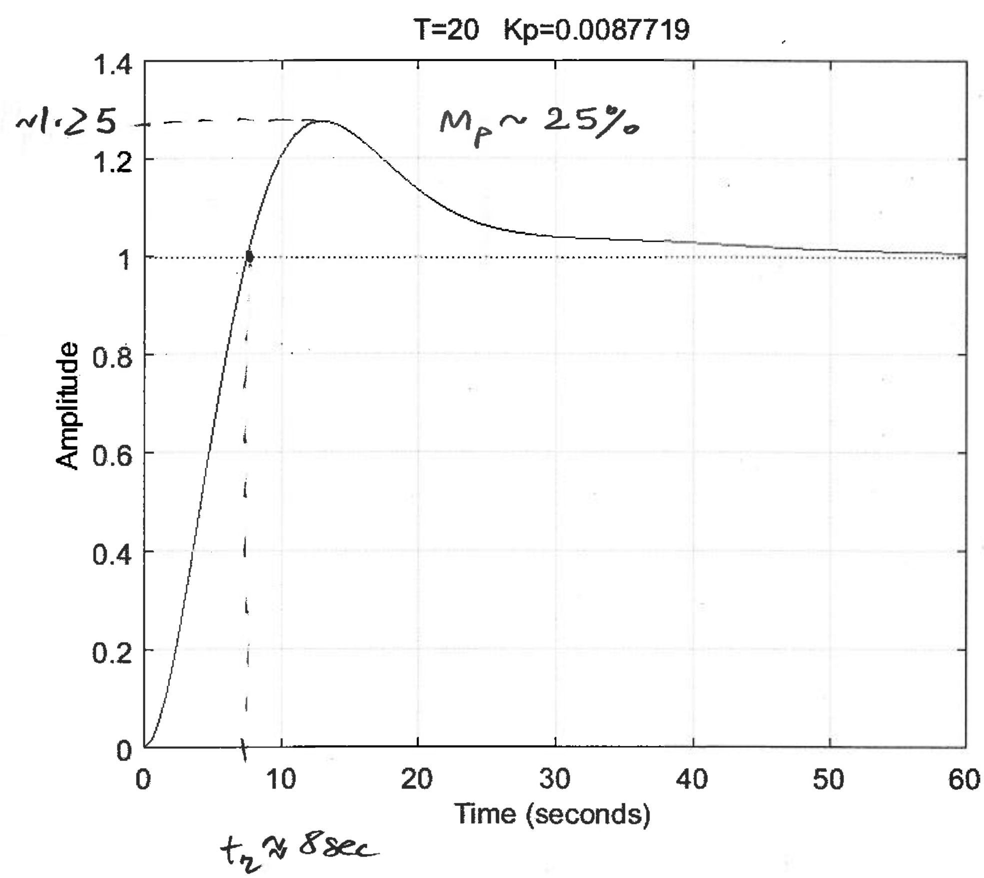
$K = 114$
$J = 10$
$c = 4$

$K_p = 1/K = 1/114$

$T_{cr} = \frac{J}{c} = 2.5$

$T = 20 > T_{cr}$





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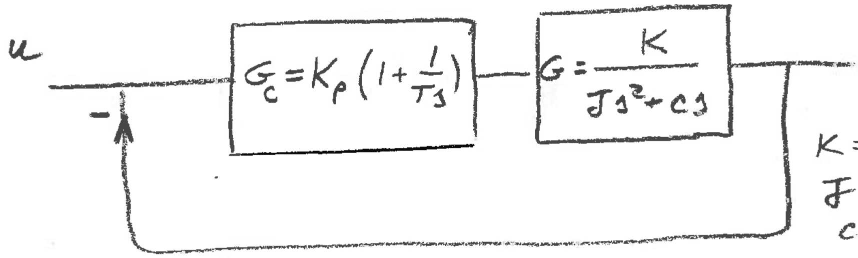
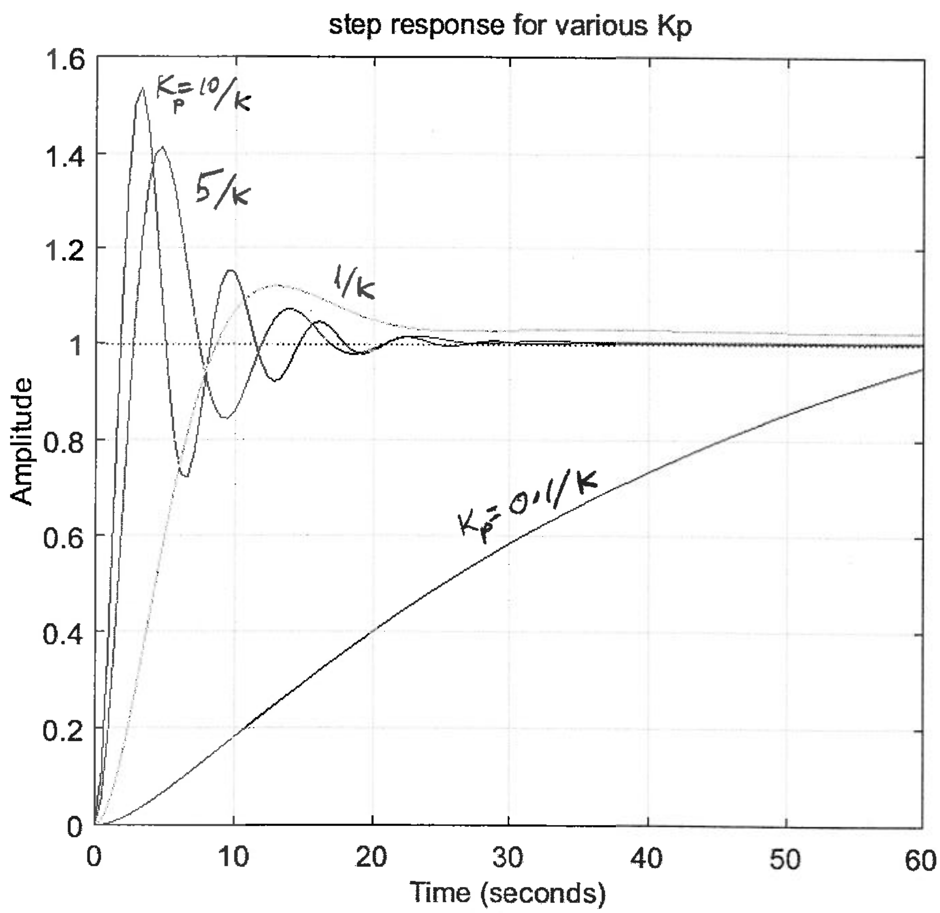
3

PI – control



$K = 114$
$J = 10$
$c = 4$
$T_{cr} = \frac{10}{4} = 2.5$

$T = 100 > T_{cr}$



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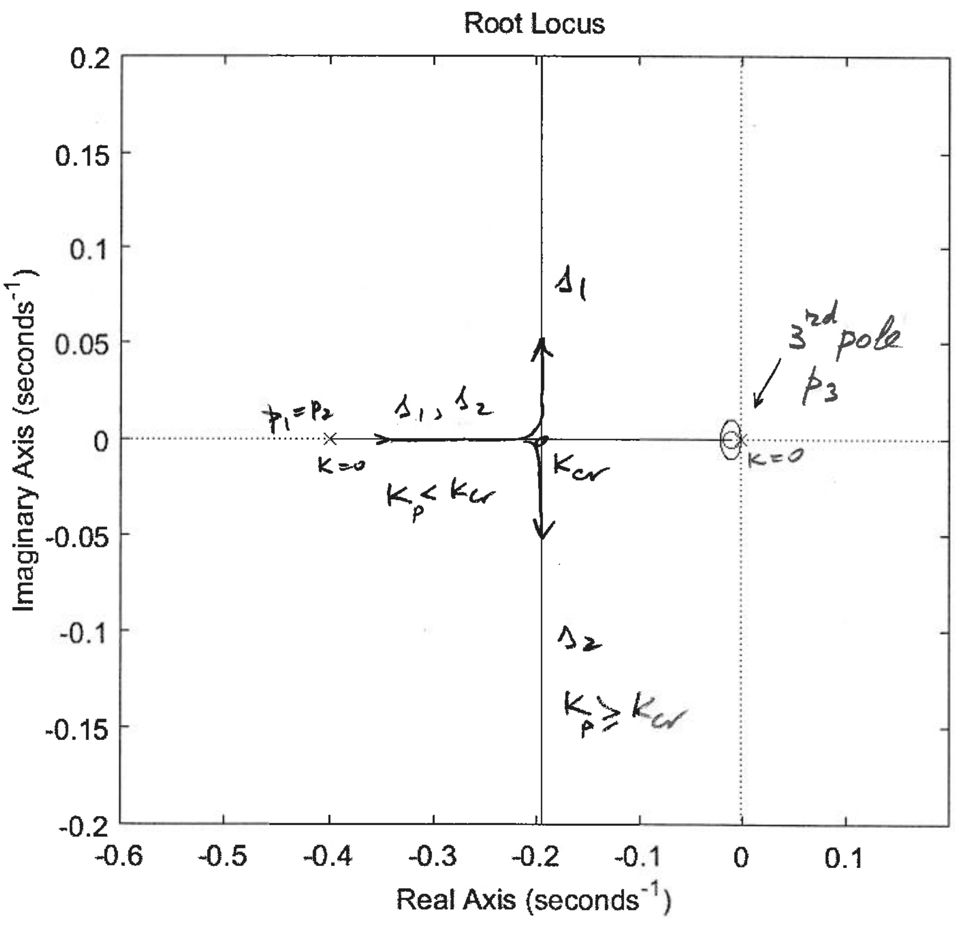
4

PI control

Root locus

aircraft $\left\{ \begin{array}{l} K=114 \\ J=10 \\ c=4 \\ T_{cr}=2.5 \end{array} \right.$

$T=100 > T_{cr}$



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5

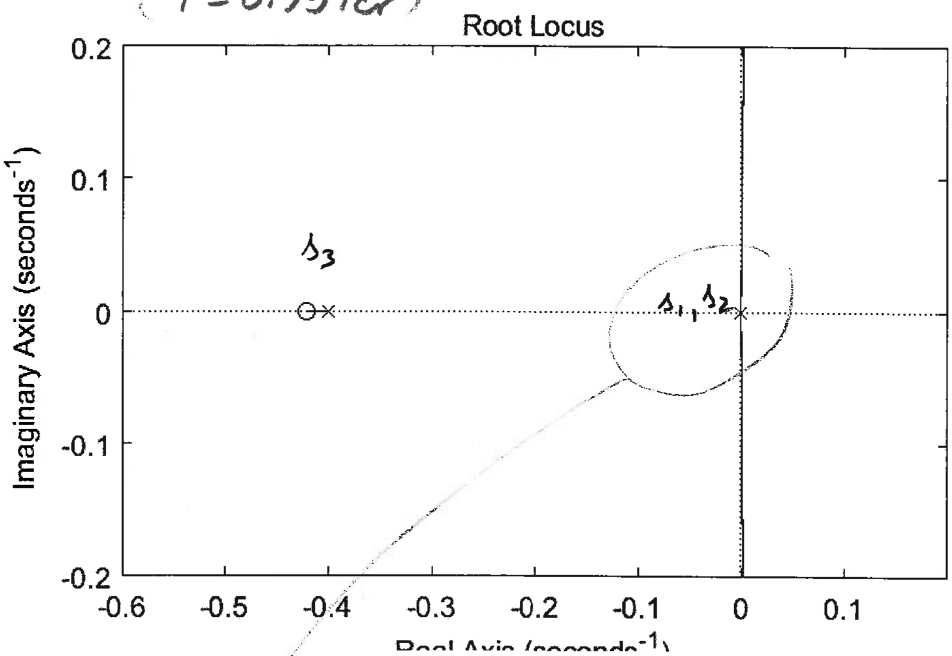
UNSTABLE PI Control
Root locus

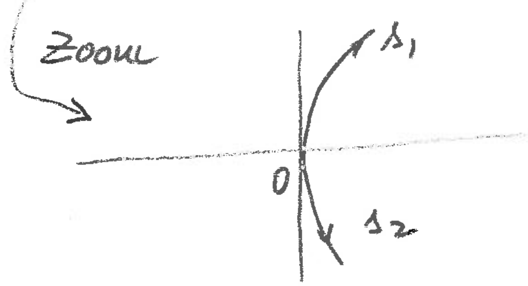
aircraft $\begin{cases} K = 114 \\ J = 10 \\ c = 4 \end{cases}$
$T_{cr} = 2.5$

UNSTABLE

$T < T_{cr}$

$(T = 0.95 T_{cr})$





unstable

s_1, s_2 in RHS

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6

UNSTABLE PI Control

Step response

aircraft

K = 114

J = 10

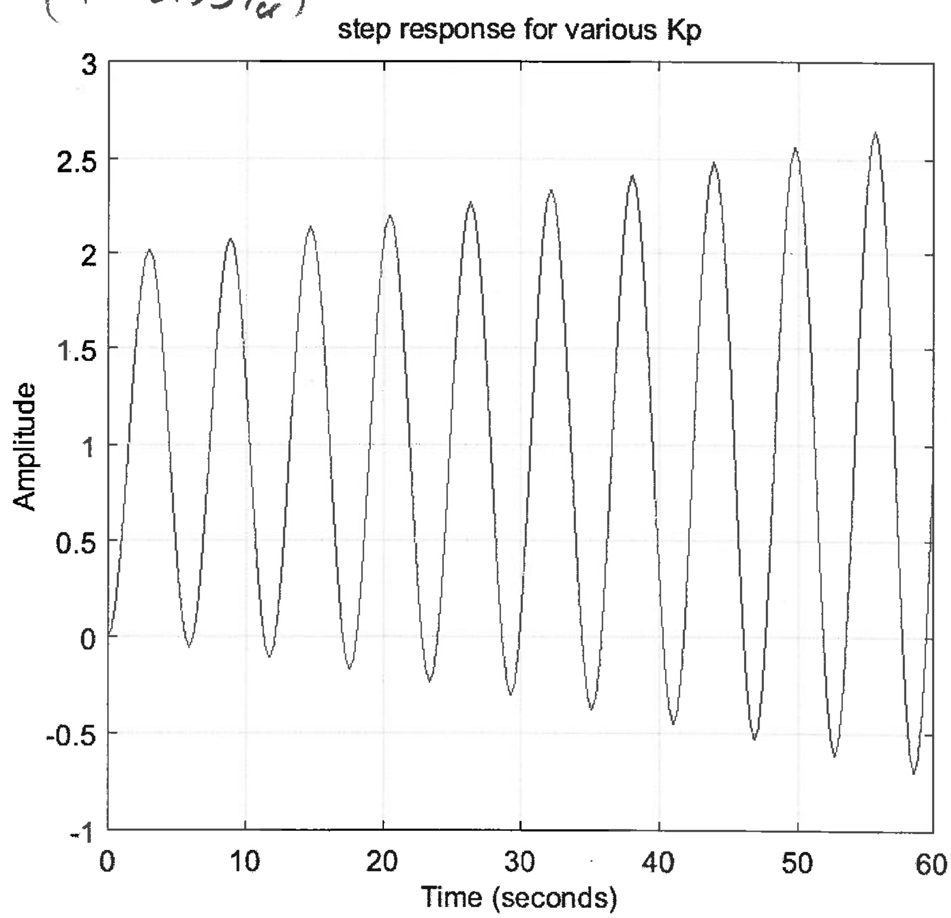
c = 4

$T_{cr} = 2.5$

UNSTABLE

$T < T_{cr}$

$(T = 0.95 T_{cr})$



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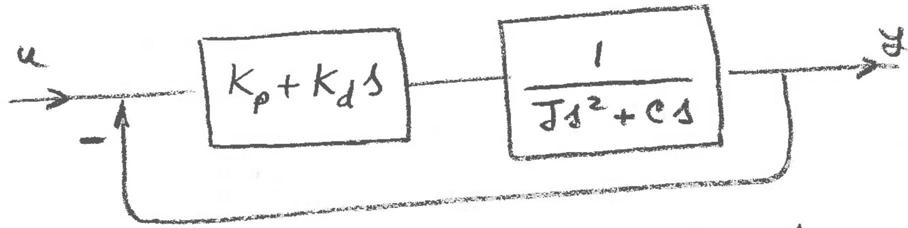
PD

PD - Control



$$G(s) = \frac{1}{Js^2 + cs}$$ (assume $K=1$ for ease)

$$G_c(s) = K_p + K_d s$$

$$G_1(s) = \frac{K_p + K_d s}{Js^2 + cs}$$

$$G_{CL}(s) = \frac{K_p + K_d s}{Js^2 + cs + K_p + K_d s}$$

$$= \frac{K_p + K_d s}{Js^2 + \underline{(c + K_d)} s + K_p}$$

damping is augmented by K_d

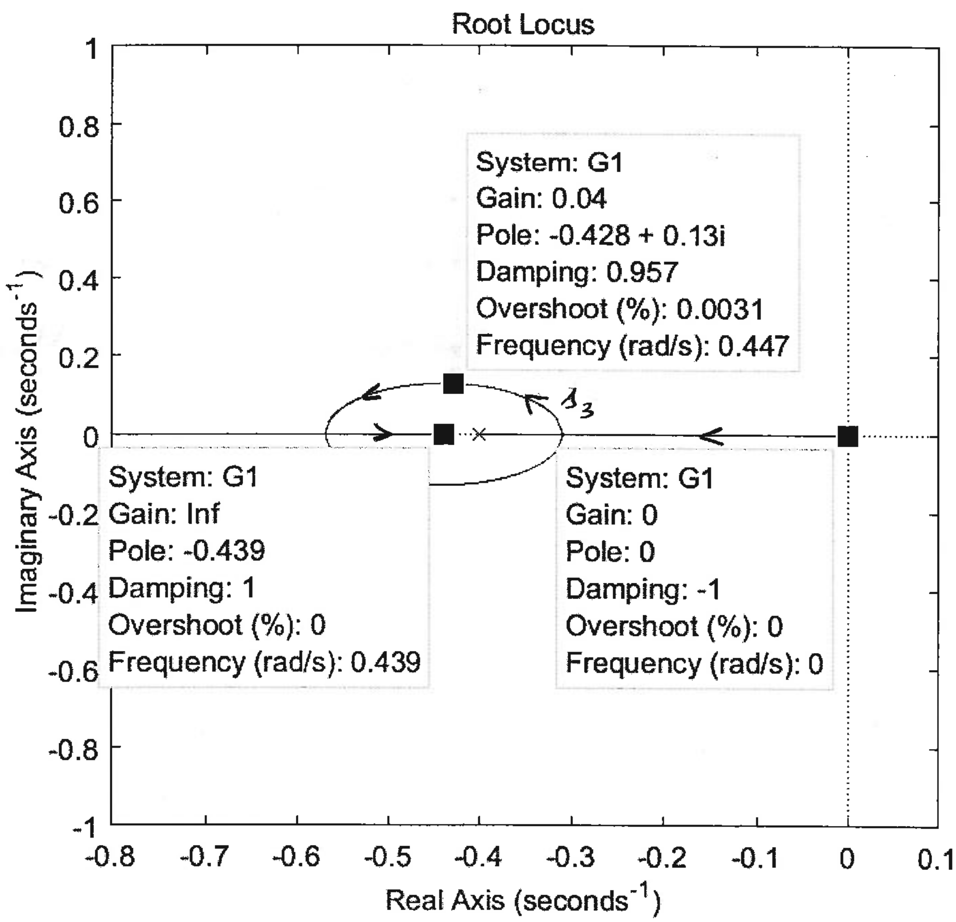
Strategy

- Adjust frequency with K_p to get small t_r
- Reduce overshoot by increasing damping (with K_d)

2
PD

PD controller

Interesting behavior of the poles s_1, s_2
in the root locus



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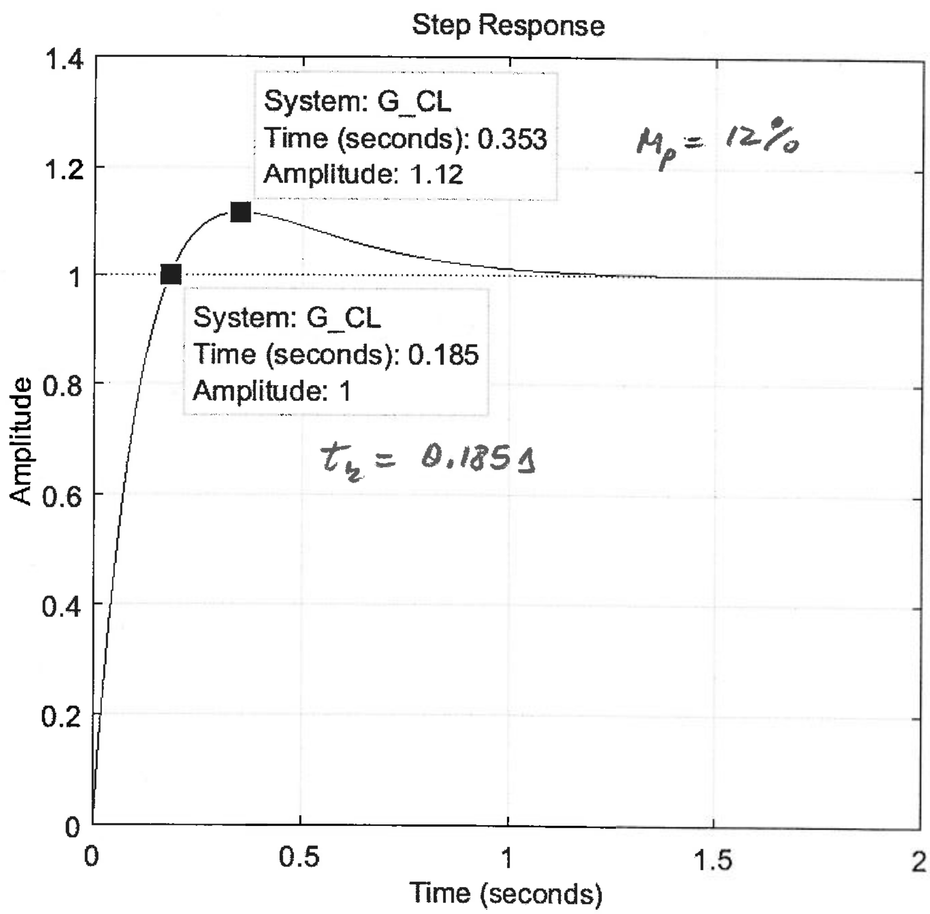
3
PD

PD control

$K = 114$
$J = 10$
$c = 4$

$K_p = 3$
$K_d = 1$



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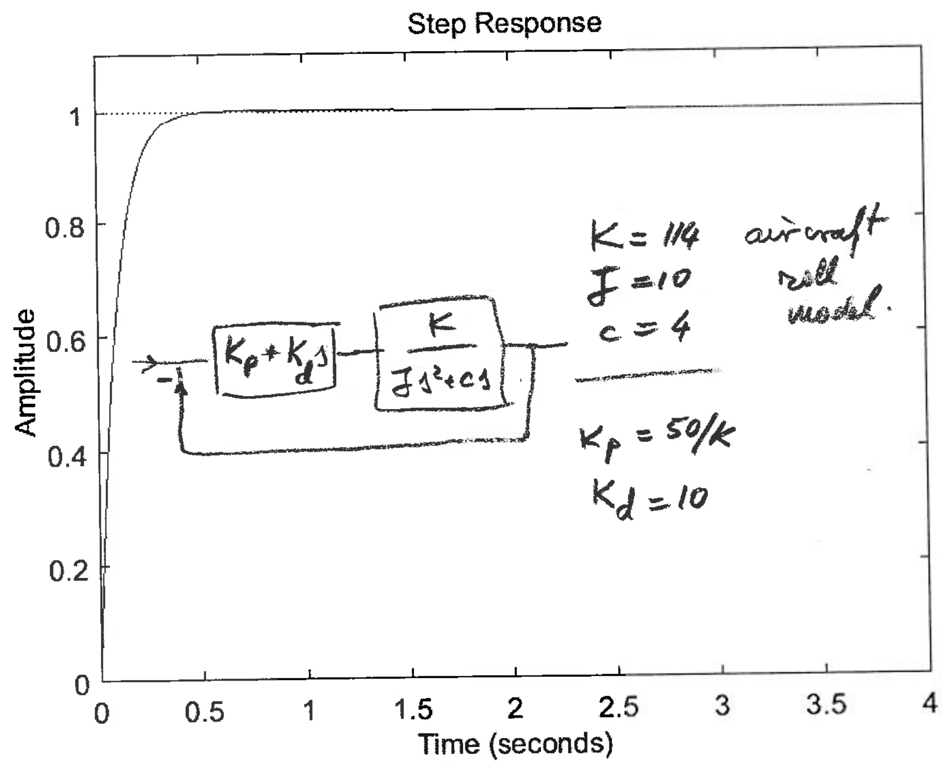
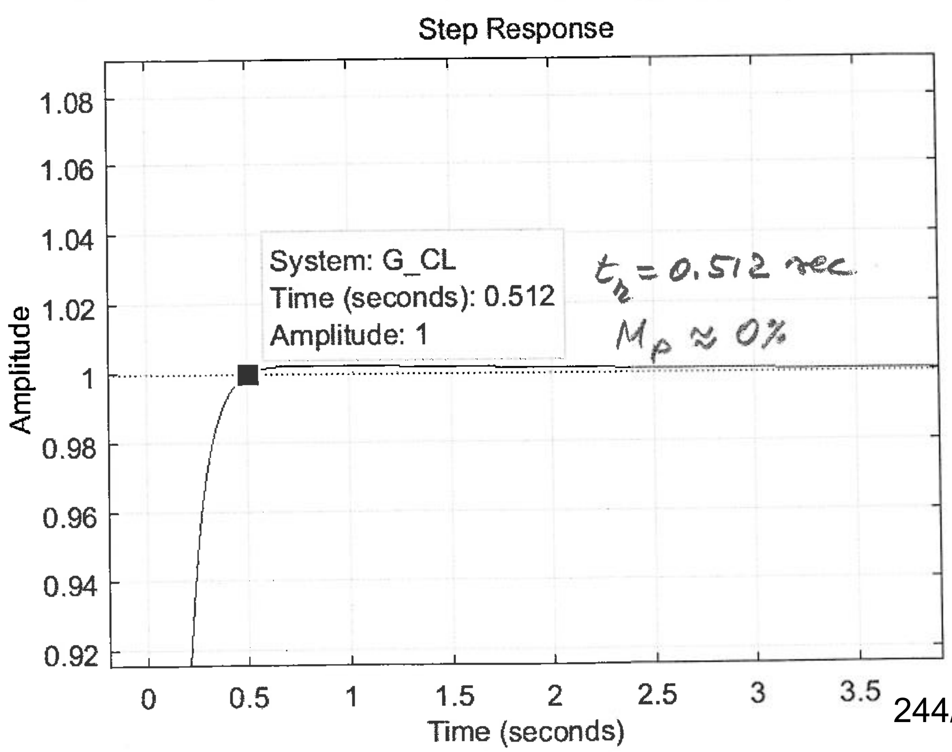
u
PD

PD control





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7.5 SIMULINK Aircraft Roll Motion

SIMULINK

Aircraft Roll Motion Autopilot Development

Table of Contents:

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1 CONTROL SYSTEM DESIGN OBJECTIVES

Consider the aircraft roll transfer function $G(s) = \dfrac{K}{Js^2 + cs} = \dfrac{114}{10s^2 + 4s}$

Design a feedback control system to control the aircraft roll motion with the following control design objectives:

Design Objective 1: Control the unconstraint aircraft motion resulting from an aileron input. Have a autopilot system that can maintain the aircraft at a constant bank angle

We wish to achieve this objective through feedback (FB)

Design Objective 2: Achieve a reasonable aircraft roll response.

We define 'reasonable response' using two control design specifications:

- DS1: Fast response time as measured by rise time
  $t_r \le 1.5$ sec
- DS2: maximum percentage overshoot for step input less than 20%
  $M_p \le 20\%$

2 UNCONSTRAINED AIRCRAFT ROLL MOTION

2.1 MODEL

Open a new model canvas and save it as 'SIMULINK_airplane_roll_unitFB'

From Simulink Library Browser drag onto the new canvas the following blocks:

- select 'Step' from 'Sources'
- select 'Transfer Fcn' from 'Continuous'
- select 'Scope' from 'Sinks', enter 'Number of inputs' = 2

Create annotation text boxes above each box as follows:

- 'reference signal' above the 'Step' box
- 'aircraft roll dynamics' above 'Transfer Fcn' box
- 'display' above 'Scope'

Change properties of 'Transfer Fcn' block to represent the aircraft roll transfer function $G(s) = \dfrac{K}{Js^2 + cs} = \dfrac{114}{10s^2 + 4s}$

Connect the blocks:

- 'reference signal' → 'aircraft dynamics' → 'display' (1st port)
- 'reference signal' → 'display' (2nd port)

Use the pull down menu: 'Edit → Copy Current View to Clipboard → Metafile' to capture only the model as shown in Figure 1.

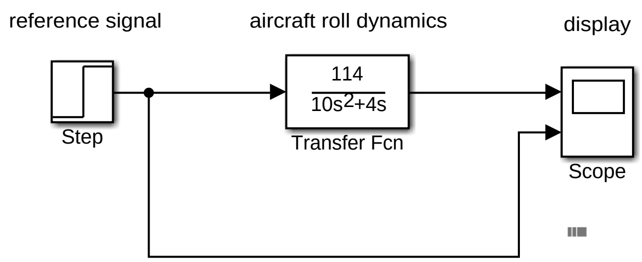


Figure 1

2.2 SIMULATION PARAMETERS

Use the pull down menu to open Configuration Parameters window, i.e.,
Simulation → Model Configuration Parameters
Make:

- Max step size: 1e-4
- Stop time: 2

The rest should remain unchanged (verify that they are the same as inFigure 2)

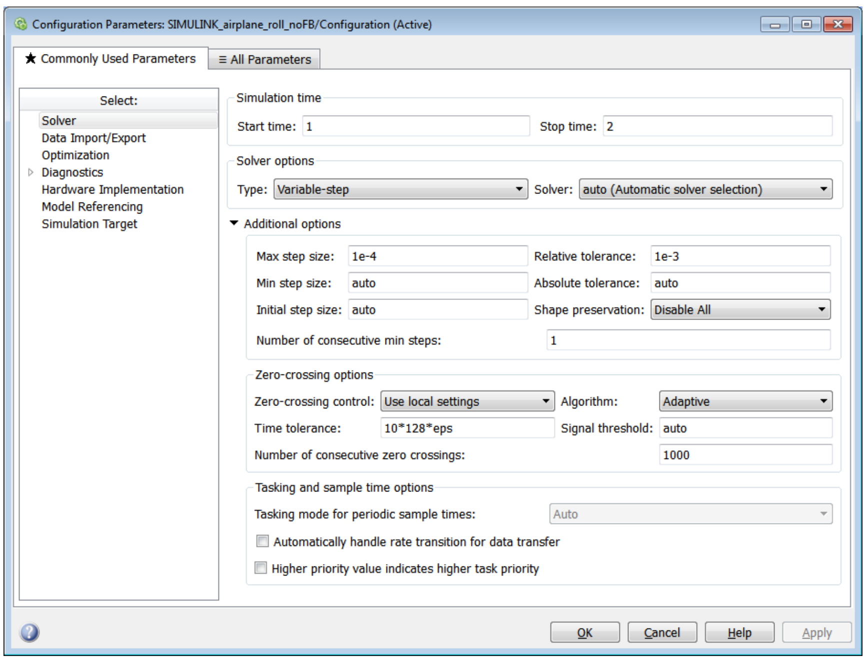


Figure 2

2.3 Display Setup

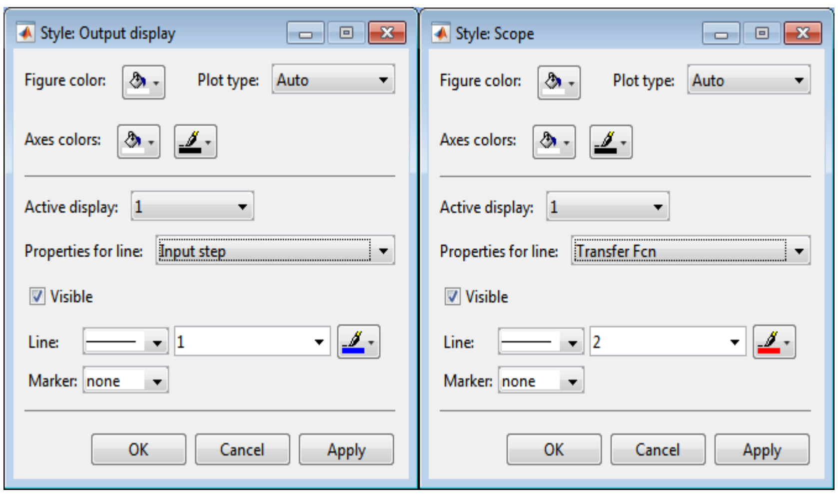
Double click on the 'Scope' block in the SIMULINK model. The 'Scope' display should open in a new window.

Select 'View →Style' to open 'Style Scope' dialog box (Figure 3a).
Choose color white for:

- Figure color: 'bucket'
- Axes colors: 'bucket'

Choose color black for Axes colors: 'brush'

Choose color blue for 'Properties for line: Input step' (Figure 3a)
Choose color red for 'Properties for line: Transfer Fcn' (Figure 3b)



(a) (b)

Figure 3

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The 'Scope' display should look as shown in Figure 4:

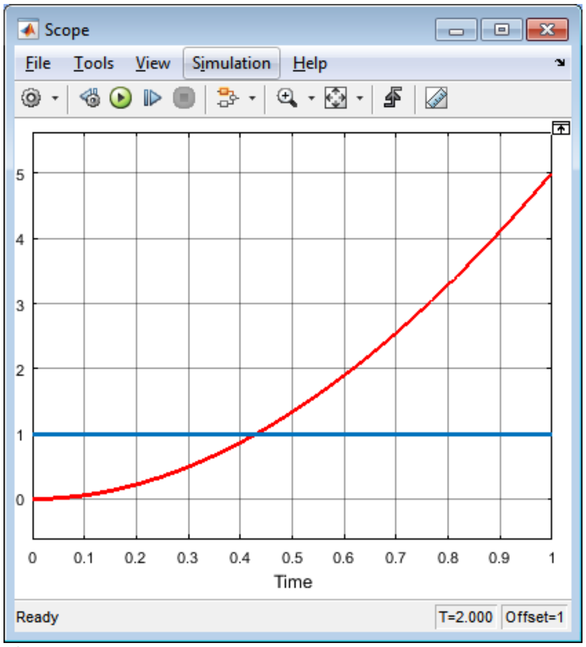

Figure 4

Note that the blue line represents the step input whereas the red line represent the aircraft roll response, which growth continuously. This situation is unacceptable. To counteract it, use feedback control as shown in next section.

3 UNIT FEEDBACK CONTROL OF AIRCRAFT ROLL MOTION

Open a new model canvas and save it as 'SIMULINK_airplane_roll_unitFB'
From Simulink Library Browser drag onto the new canvas the following blocks:

- 'Step' from 'Sources'
- 'Sum' from 'Commonly Used Blocks'
- 'Scope' from 'Sinks'
- 'Transfer Fcn' from 'Continuous'

Create annotation text boxes above each box as follows:

- 'reference signal' above the 'Step' box
- 'aircraft roll dynamics' above 'Transfer Fcn' box
- 'output signal' above 'Scope'

Change properties of 'Transfer Fcn' block to represent the aircraft roll transfer function $G(s) = \frac{114}{10s^2 + 4s}$

Change the second port of the 'Sum' block to negative (-). Do this by using 'right-click' → 'Block Parameters: Sum' and then putting '+-' in 'List of signs'

Connect the blocks:

- 'reference signal' → 'Sum' → 'aircraft dynamics' → 'display'
- output from 'aircraft dynamics' to negative (-) port of the 'Sum block' (this is the unit feedback closed loop)

Use the pull down menu: 'Edit → Copy Current View to Clipboard → Metafile' to capture only the model (Figure 5).

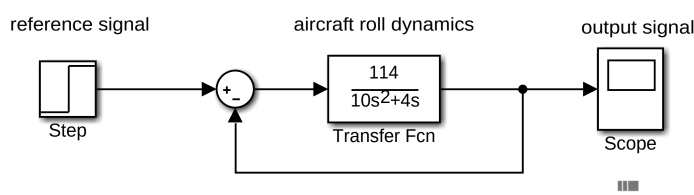

Figure 5

3.1 SIMULATION PARAMETERS

Use the pull down menu to open Configuration Parameters window, i.e.,
Simulation → Model Configuration Parameters
Make:

- Max step size: 1e-4
- Stop time: 40

The rest should remain unchanged (verify that they are the same as in the figure)

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3.2 DISPLAY SETUP

Double click on the 'Scope' block in the SIMULINK model. The 'Scope' display should open in a new window.

Select 'View →Style' to open 'Style Scope' dialog box.
Choose color white for:

- Figure color
- Axes colors: bucket

Choose color black for Axes colors: brush
'Properties for line: Transfer Fcn': 1.5 weight, red (Figure 6).

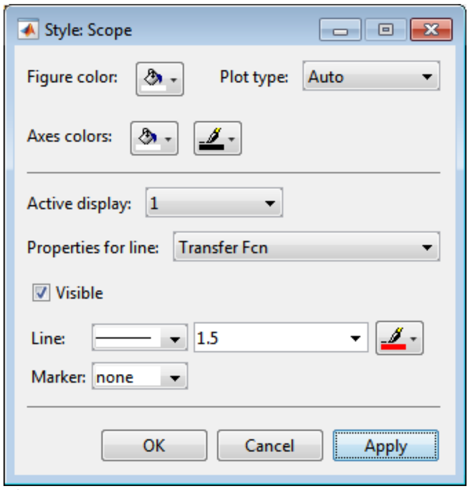

Figure 6

3.3 Aircraft Roll Response with Unit FB

Press the ‘Run’ button. The aircraft roll response with unit FB is displayed (Figure 7).

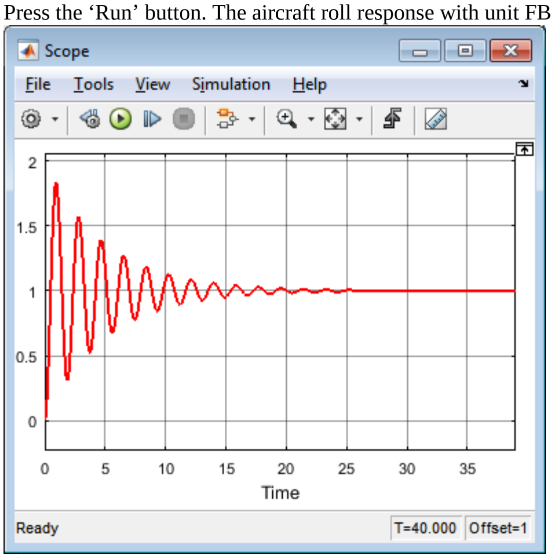

Figure 7

It is apparent that the response is unsatisfactory because:

- many oscillations until it settles down to $x_{ss} = 1$
- large overshoot ($x_p \approx 1.8$, $M_p \approx 80\%$)

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4 VARIABLE GAIN FB CONTROL OF AIRCRAFT ROLL MOTION (P CONTROL)

To improve the aircraft FB response, we can try to use variable gain control. This is known as proportional control or 'P control'.

4.1 P-CONTROLLED AIRCRAFT ROLL MODEL

Add a gain box to the model to obtain the P-controller (Figure 8).

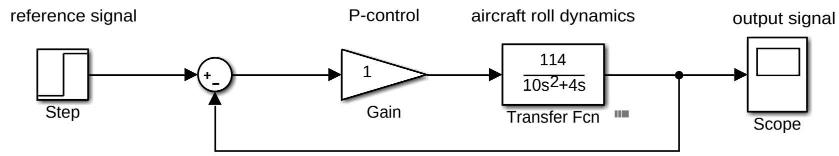


Figure 8

Save this model as 'SIMULINK_aircraft_roll_P_control'

4.2 DISPLAY SETUP

Have the display setup as before.
In addition, stop automatic axes scaling by doing the following:

- Tools → Axes Scaling → Axis Scaling Properties

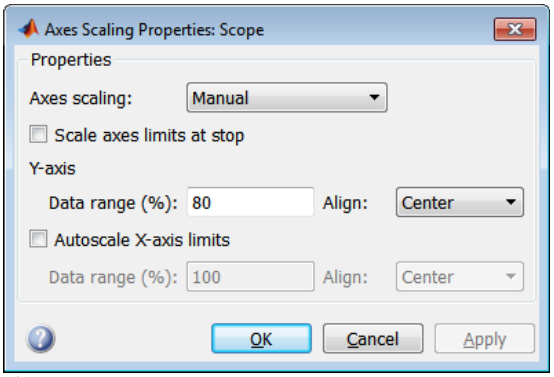


Figure 9

4.3 P-CONTROL GAIN CHANGES

The P-control gain Kp can be modified as follows:

Right click ‘Gain’ box, select ‘Block Parameters’. The ‘Block Parameters: Gain’ dialog box open (Figure 10).

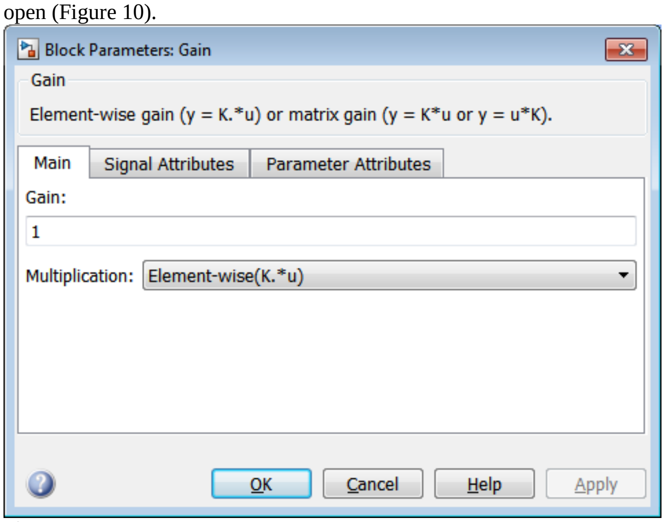


Figure 10

The gain default value is ‘Gain: 1’. , i.e., Kp=1. After entering a new gain value, press ‘Apply’. Keep the dialog box open for the next gain value. (simulation runs OK with the dialog box open.)

Try a number of different gains to see their effect on the response. Gains to be tried are:

- Kp=1
- Kp=10
- Kp=0.1
- Kp=0.01

4.3.1 Kp=1 Response

Put ‘Gain = 1’ in ‘Block Parameters: Gain’ dialog box and press ‘Apply’. The response is shown in Figure 11.

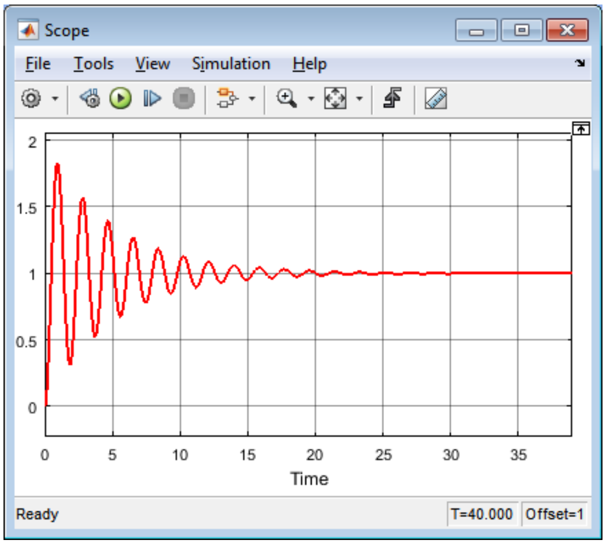


Figure 11

This Kp=1 case corresponds to Unit FB which we have already studied. It is unsatisfactory because

- many oscillations until it settles down to $x_{ss} = 1$
- large overshoot ($x_p \approx 1.8$, $M_p \approx 80\%$)

4.3.2 Kp=10 Response

Put ‘Gain = 10’ in ‘Block Parameters: Gain’ dialog box and press ‘Apply’. The response is shown in Figure 12.

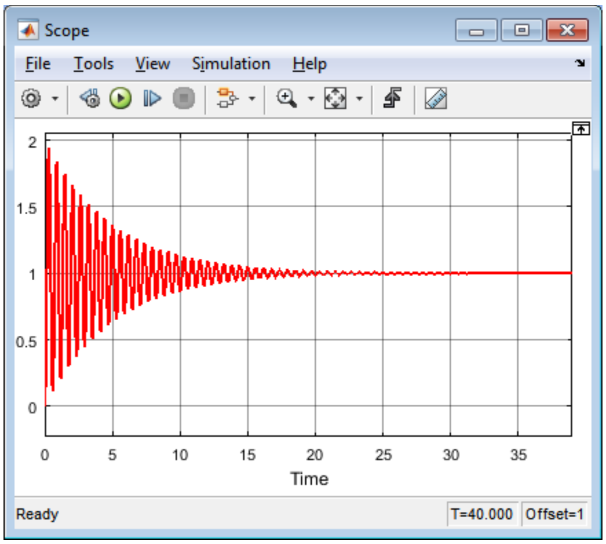


Figure 12

This Kp=10 response is unsatisfactory because

- very many oscillations until it settles down to $x_{ss} = 1$
- large overshoot ($x_p \approx 2$, $M_p \approx 100\%$)

4.3.3 Kp=0.1 Response

Put 'Gain = 0.1' in 'Block Parameters: Gain' dialog box and press 'Apply'. The response is shown in Figure 13.

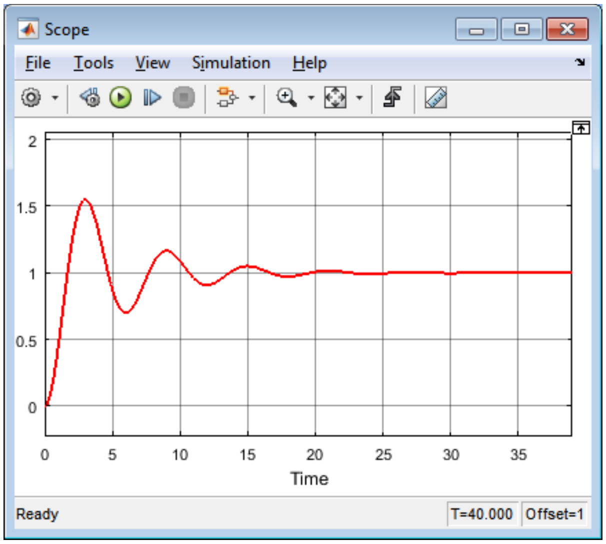


Figure 13

The response for Kp=0.1 seems better because:

- fewer oscillations until it settles down to $x_{ss} = 1$
- smaller overshoot ($x_p \approx 1.55$, $M_p \approx 55\%$)

4.3.4 Kp=0.01 Response

Put 'Gain = 0.01' in 'Block Parameters: Gain' dialog box and press 'Apply'. The response is shown in Figure 14.

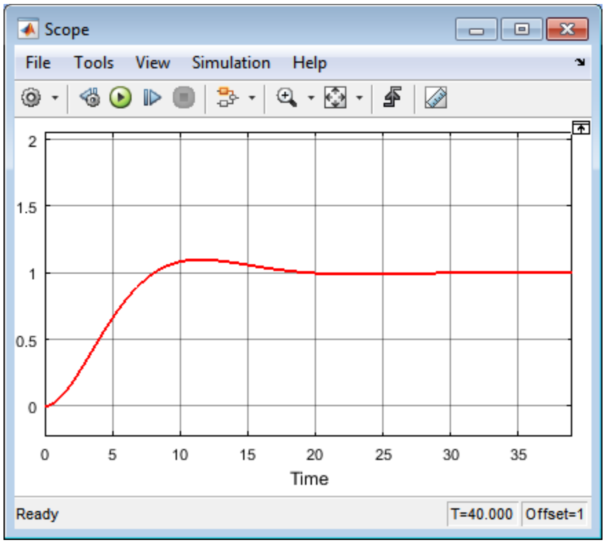


Figure 14

The response for Kp=0.01 seems even better because:

- almost no oscillation until it settles down to $x_{ss} = 1$
- much smaller overshoot ($x_p \approx 1.1$, $M_p \approx 10\%$)

However, the rise time is much longer than before ($t_r \approx 8$ sec). This is unacceptable because it makes the aircraft very sluggish.

We need to examine the mathematics of P-control. Recall:

$$\omega_n = \sqrt{\frac{K}{J}},\ \zeta = \frac{c}{2\sqrt{JK}}$$

where K is the overall forward gain (K=Kp*K1, with K1=114 for our aircraft roll model). Hence we can calculate:

- rise time: $t_r = \dfrac{\pi - \varphi}{\omega_n \sqrt{1-\zeta^2}}$ where $\varphi = \sin^{-1}\sqrt{1-\zeta^2}$
- overshoot: $M_p = e^{-\pi \frac{\zeta}{\sqrt{1-\zeta^2}}}$

A plot of these values is given in Figure 15.

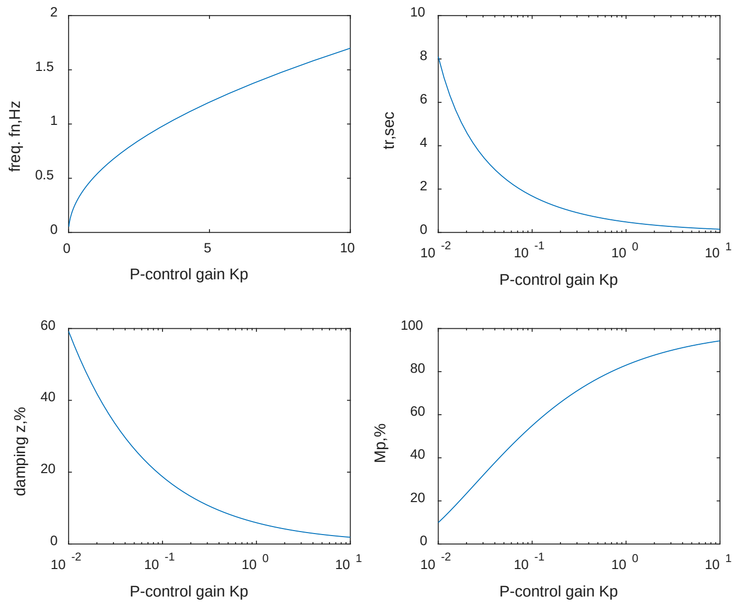


Figure 15

The above plot shows that small overshoot M_p happens at low gain values. However, at these low gain values, the rise time t_r becomes large. Hence, we conclude that **P-control gain has an opposite effect on the two performance indicators, M_p and t_r considered in this controller design.** We need to explore other control options.

5 PI CONTROL

PI control stands for 'proportional + integrative control'.

5.1 PI CONTROL SETUP

The aircraft model with PI control has two branches, one for the P block (which is just a simple gain) and the other for the I block which consists of a gain followed by an integrator. The PI controlled aircraft model looks as shown in Figure 16.

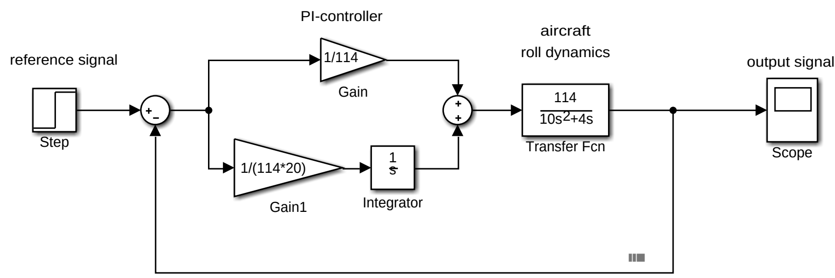


Figure 16

Construct this model and save it as 'SIMULINK_aircraft_roll_PI_control_Kp_Ki'

Observe that the model has two gains:
- Proportional gain, Kp=1/114
- Integrative gain Ki=1/(114*20)

The resulting PI controller has the expression:

$$G_{PI}(s) = K_p + K_i \frac{1}{s}$$

5.2 Aircraft Roll Response with PI Control

Run the PI control model. The resulting response will show up in the 'Scope' window (Figure 17):

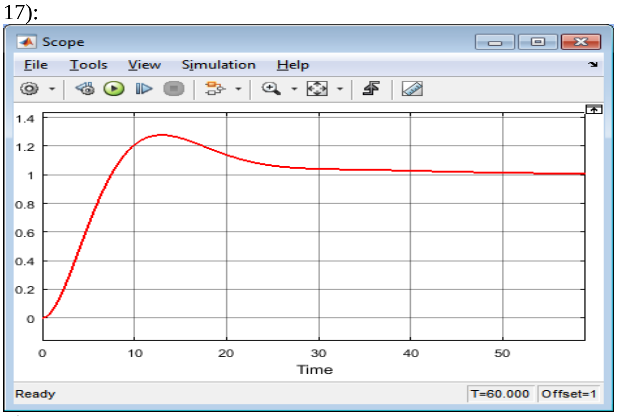


Figure 17

Press the 'Cursor Measurements' button and set the cursors to measure rise time and overshoot (Figure 18).

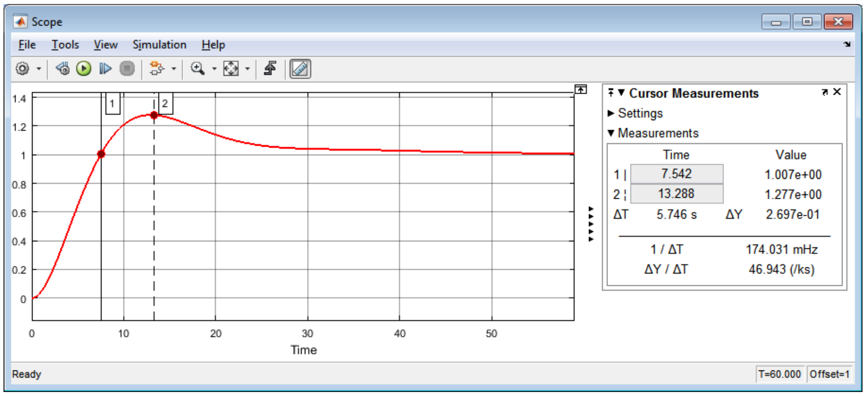


Figure 18

The results are:

- rise time $t_r = 7.542$ sec
- overshoot $M_p = 27.7\%$

These values are better than the values obtained with P-control. However, they are still below the target values. Need to try something else.

6 PD CONTROL

PD control stands for ‘proportional + derivative control’.

6.1 PD Control Setup

The aircraft model with PI control has two branches, one for the proportional P-block (which is just a simple gain) and the other for the derivative D-block. The PD controlled aircraft model looks as shown in Figure 19.

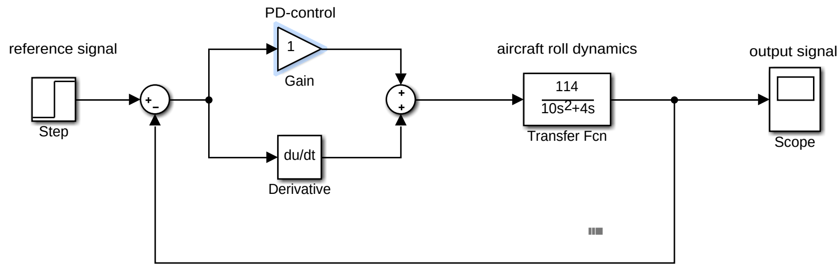


Figure 19

Construct this model and save it as ‘SIMULINK_aircraft_roll_PD_control’

6.2 Aircraft Roll Response with PD Control

Set the model run time to 20 sec and run it. The response is as shown in Figure 20.

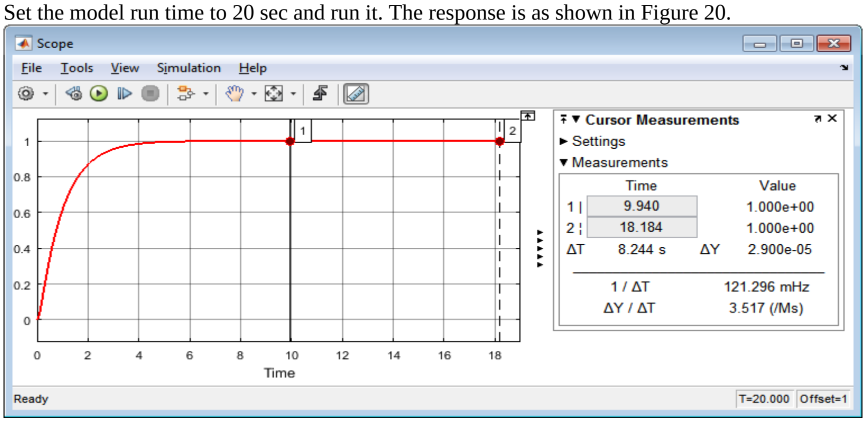


Figure 20

The response convergence continuously towards the target value 1, but the rise time is around 10 sec.

This is unacceptable. But we can now try different value of the P gain Kp.

Increase P control gain to Kp=10. The model looks as shown in Figure 21.

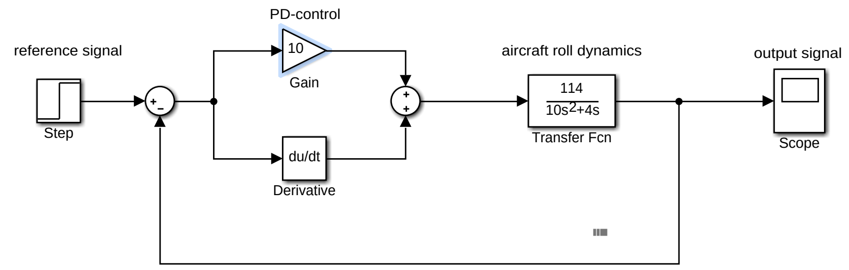


Figure 21

The response of this model with Kp=10 is as shown in Figure 22.

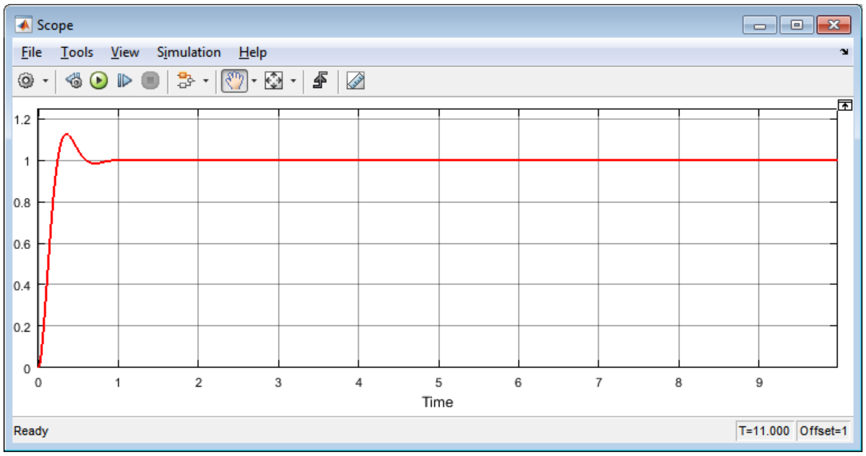


Figure 22

Note that the response has become much crispier but overshoot is now present.

Press the 'Cursor Measurements' button to get measurements (Figure 23).

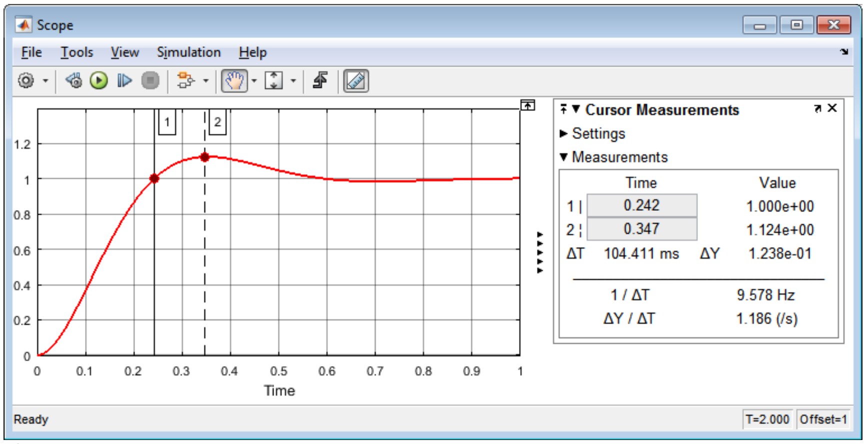


Figure 23

The cursor measurements allow us to determine $t_r = 0.242$ sec and $M_p = 12.4\%$.

<u>Conclusion</u>:
PD control with Kp=10 seems to give a reasonable response:

- short rise time $t_r = 0.242$ sec
- small overshoot $M_p = 12.4\%$

Note of caution: I cannot reproduce these results exactly in MATLAB control systems toolbox. It is perhaps because PD control is not a stable control. In practice, the D part of a PD controller is difficult to implement because differentiation is not easy to obtain with electrical system; therefore, the D part of the controller is implemented as a compensator $s/(T_1 s+1)$.

7 PID CONTROL

PID control stands for 'proportional + integrative + derivative control'.

7.1 PID CONTROL SETUP

The aircraft model with PID control has two branches, uses the PID block which can be found in 'Continuous' part of the library. The PI controlled aircraft model looks as shown in Figure 24.

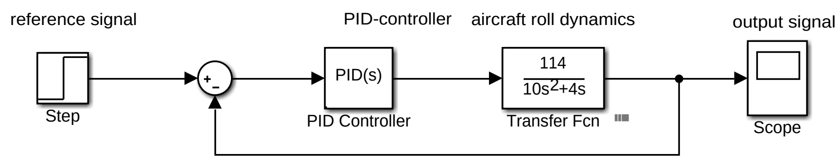


Figure 24

Construct this model and save it as 'SIMULINK_aircraft_roll_PID_control'.
The PID block represents the following controller transfer function:

$$G_c(s) = K_P + \frac{K_I}{s} + K_D s$$

The default values at startup are $K_p = 1,\ K_I = 1,\ K_D = 0$. The response with these default values is as shown in Figure 25.

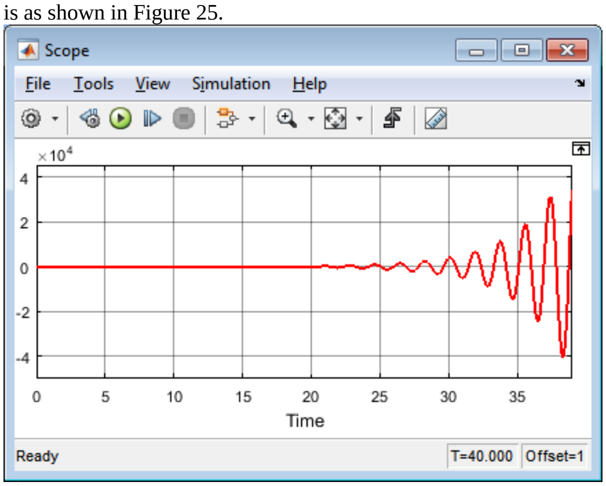


Figure 25

This is an unstable response and needs to be corrected! We will adjust the PID controller parameters to obtain a stable response which meets the design specifications. This process of adjustment is called 'tuning'.

7.2 AUTOMATIC TUNING OF THE PID CONTROLLER

Double click the PID box and open it (Figure 26).

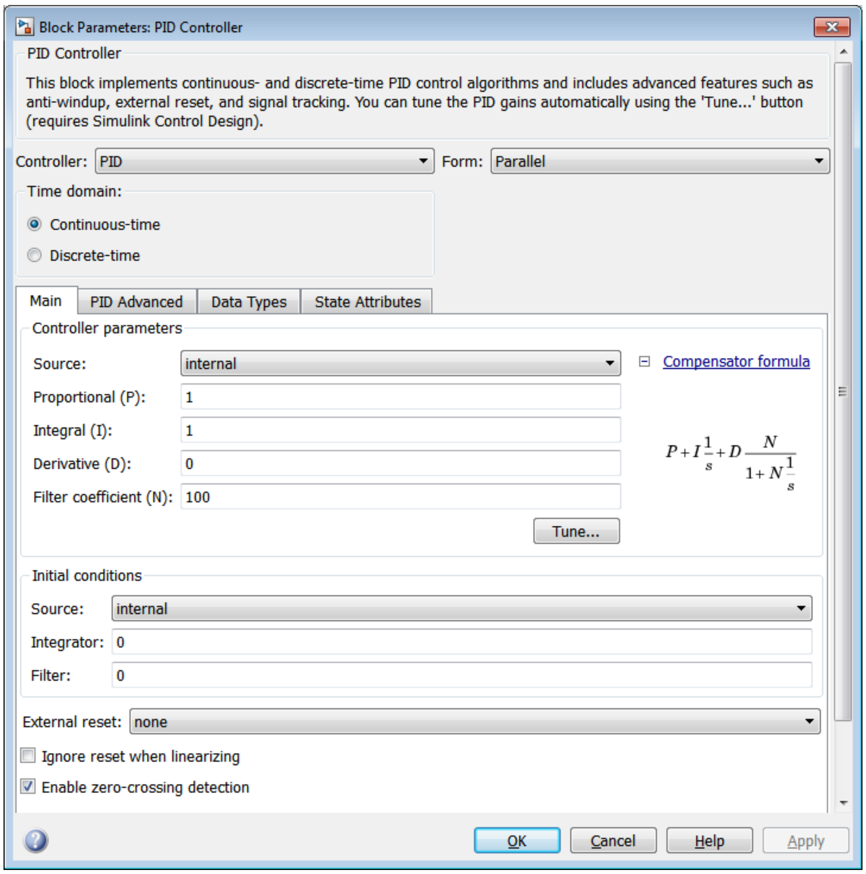


Figure 26

Next, press the ‘Tune…’ button. This will open a separate window for tuning, i.e., the ‘PID Tuner’ window shown in Figure 27.

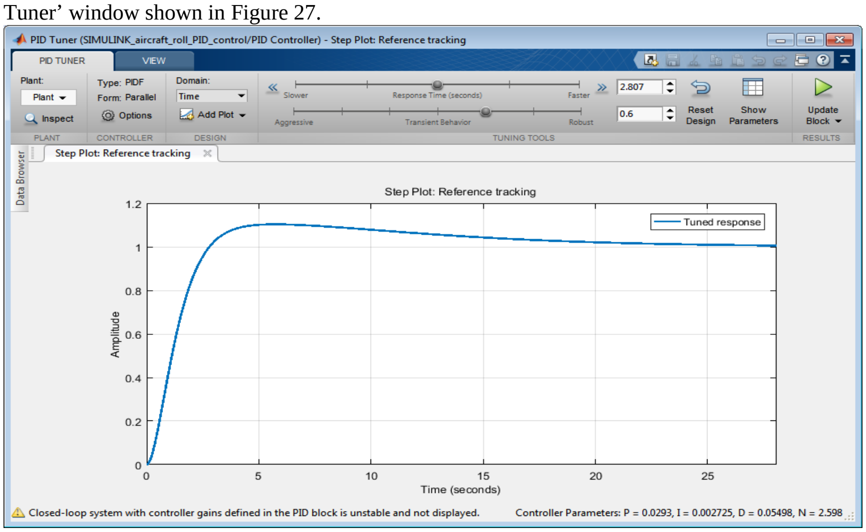
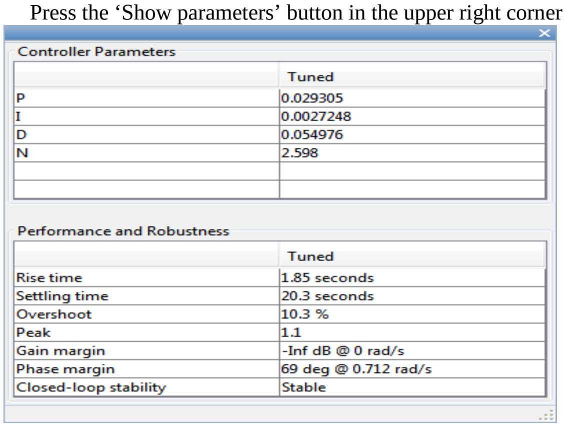


Figure 27

Note the message at the bottom of the window in the left corner: ‘Closed-loop system with controller gains defined in the PID block is unstable and not displayed’.

In the right bottom corner, we read: ‘Controller Parameters: P=0.0293, I=0.002725, D=0.05498, N=2.598’. These controller parameters have been obtained by the tuner through an internal algorithm.

Press the ‘Show parameters’ button in the upper right corner to open Figure 28.


Controller Parameters

	Tuned
P	0.029305
I	0.0027248
D	0.054976
N	2.598

Performance and Robustness

	Tuned
Rise time	1.85 seconds
Settling time	20.3 seconds
Overshoot	10.3 %
Peak	1.1
Gain margin	-Inf dB @ 0 rad/s
Phase margin	69 deg @ 0.712 rad/s
Closed-loop stability	Stable

Figure 28

In this window, we read Rise time = 1.85 sec and Overshoot = 10.3%. Note that MATLAB defines the rise time in the 10%--90% range To evaluate the 0--100% rise time t_r,. read the upper-right side window next the ‘Response Time (seconds) Faster’. Here, we read $t_r = 2.807$ sec.

It is apparent that the overshoot is within specifications but the rise time is to long. Need to do some tuning adjustments manually.

7.3 MANUAL ADJUSTMENT OF THE PID TUNER

To tune the system manually, use the ‘Response time (seconds)’ slider. When the slider is moved to the right, the response time becomes faster.

Under ‘Options’, check the box ‘Show Block Response’ to display the original response (‘Block response’) besides the ‘Tuned response’.

Move the slider until the time t_r shown in the small window on the right is less than 1.5 sec (Figure 29).

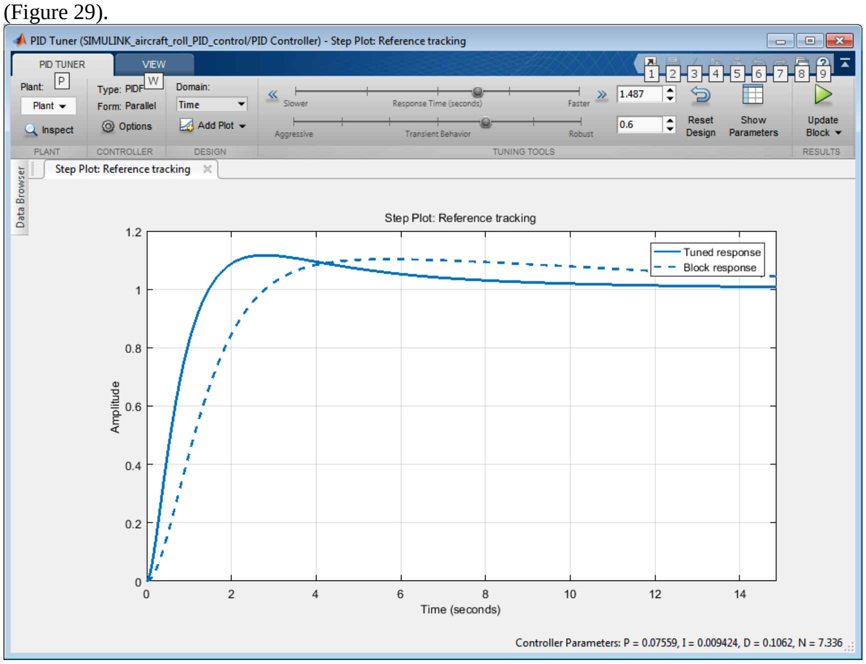
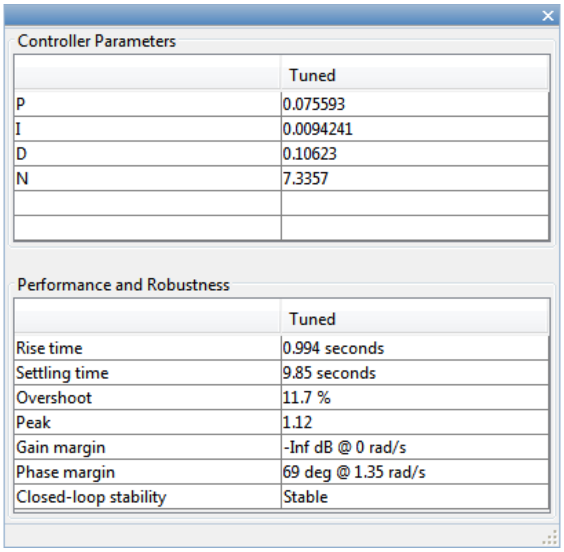


Figure 29

We read $t_r = 1.487$ sec

At this moment, the ‘Block Parameters PID’ window looks as shown in Figure 30.



Controller Parameters

	Tuned
P	0.075593
I	0.0094241
D	0.10623
N	7.3357

Performance and Robustness

	Tuned
Rise time	0.994 seconds
Settling time	9.85 seconds
Overshoot	11.7 %
Peak	1.12
Gain margin	-Inf dB @ 0 rad/s
Phase margin	69 deg @ 1.35 rad/s
Closed-loop stability	Stable

Figure 30

In this window, we read ‘Overshoot = 11.7%’ which means $M_p = 11.7\%$ which is less than the required 20%.
It is now apparent that the design specification have been met. The final PID settings are read from the ‘Block Parameters PID’ window as:

- ‘P= 0.075593’, i.e., Kp = 0.075593
- ‘I = 0.0094241’, i.e., Ki = 0.0094241
- ‘D = 0.10623’, Kd = 0.10623
- N = 7.3357

The aircraft roll response now meets the design specifications, i.e.,

- DS1: Fast response time as measured by rise time
  $t_r = 1.487 < 1.5$ sec
- DS2: maximum percentage overshoot for step input less than 20%
  $M_p = 11.7 < 20\%$

7.4 Verification of the PID Tuning Process

In the ‘PID Tuner’ window, press the ‘Update Block’ green arrow. Look in the ‘Block Parameters: PID Controller’ window to verify that the PID parameters have been updated. The window should look like Figure 31.

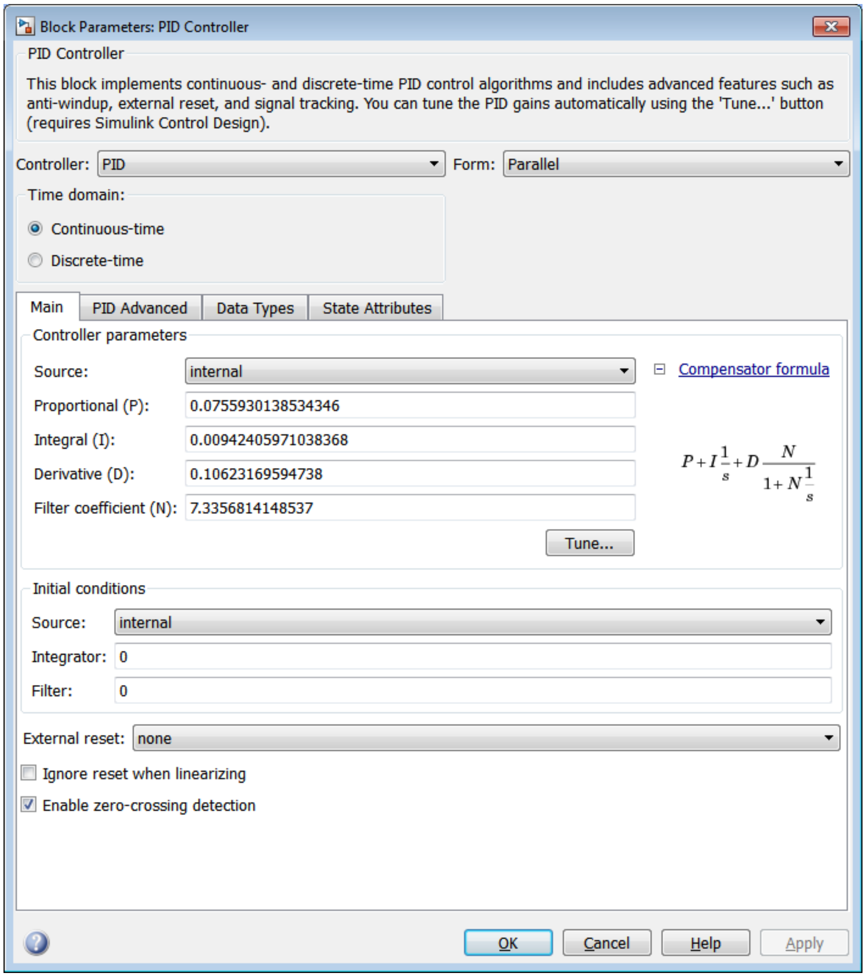


Figure 31

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Now, reduce the SIMULINK run time to 20 sec and run the model (Figure 32).

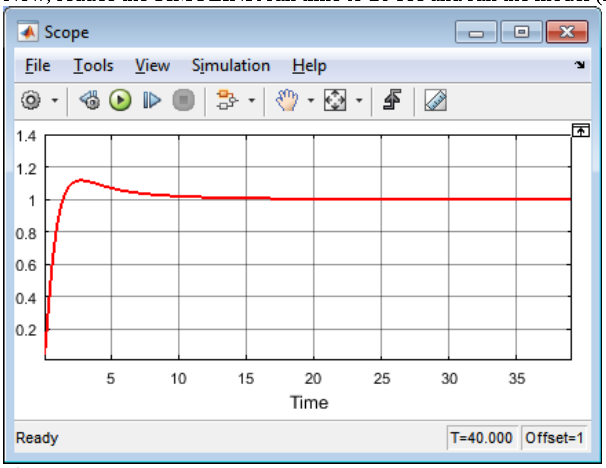

Figure 32

Press the ‘Cursor Measurements’ button. The window should look like Figure 33.

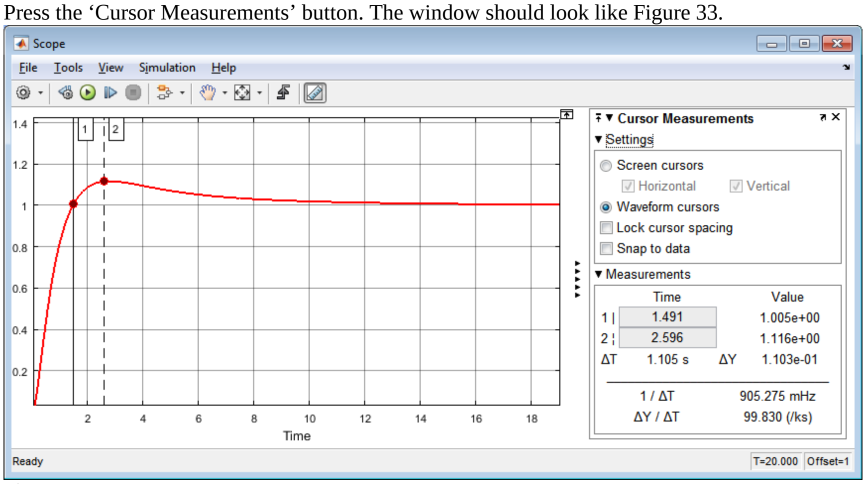

Figure 33

Measure the rise time and overshoot.

- $t_r = 1.483 \le 1.5$ sec
- $M_p = 11.6 \le 20\%$

It apparent that the two design specifications are satisfied.

To verify the MATLAB 10%--90% rise time of 0.994 seconds indicated in the previously shown ‘Controller Parameters’ window, try to measure this rise time in the ‘Scope’ window.

To do this measurement, we need higher resolution; hence, reduce the run time to 5 sec and run again. The Scope window should look like Figure 34.

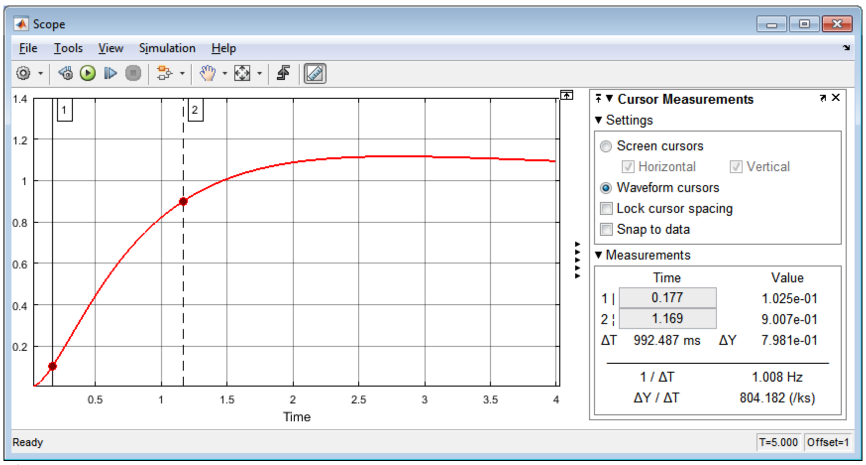

Figure 34

The cursors 1 and 2 are used to identify the 10% and 90% points on the curve. To determine the rise time from this image, read the time difference between the two cursors, i.e., $\Delta T = 992.487$ ms. This is the rise time between 10% and 90%. It agrees with the 0.994 seconds indicated in the previously shown ‘Controller Parameters’ window.

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7.6 Feedback Controllers (continued)

PID

(See SIMULINK first)

PID Control

$K_p = 0.094$

$K_i = 0.0094241$

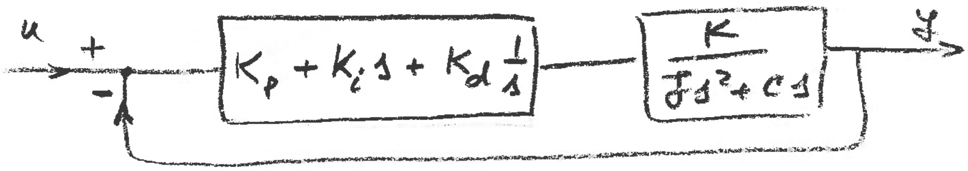
$K_d = 0.10623$

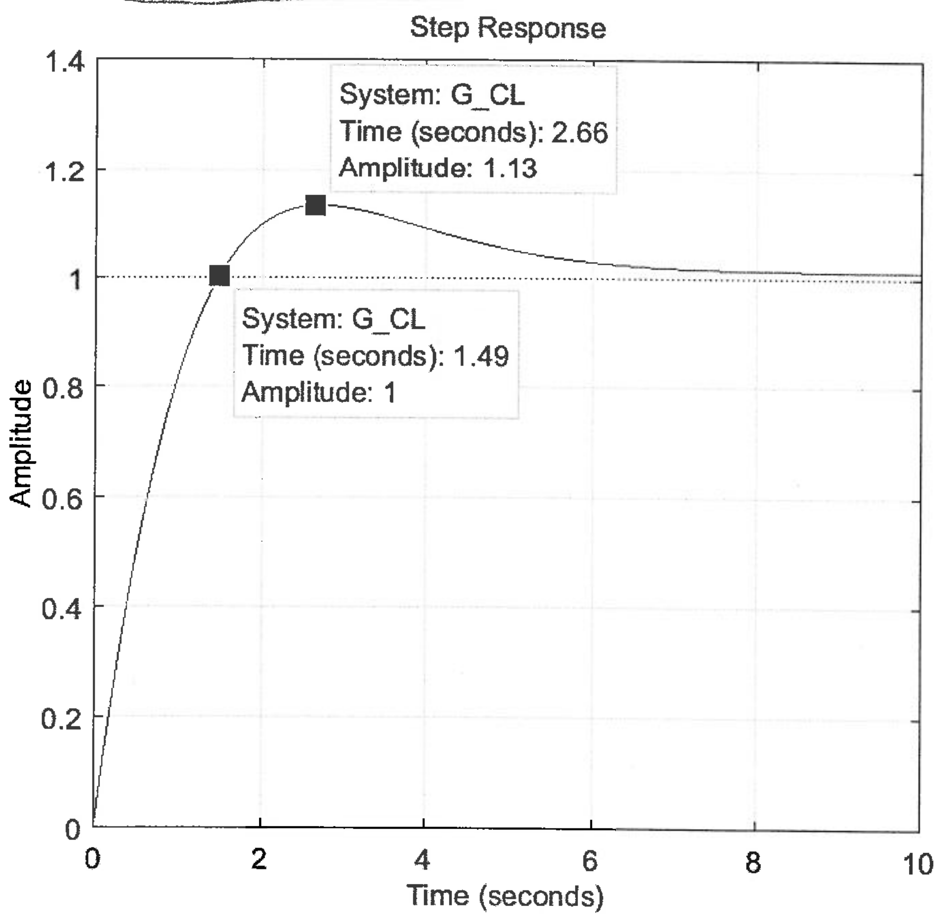
Aircraft model

$K = 114$

$J = 10$

$c = 4$

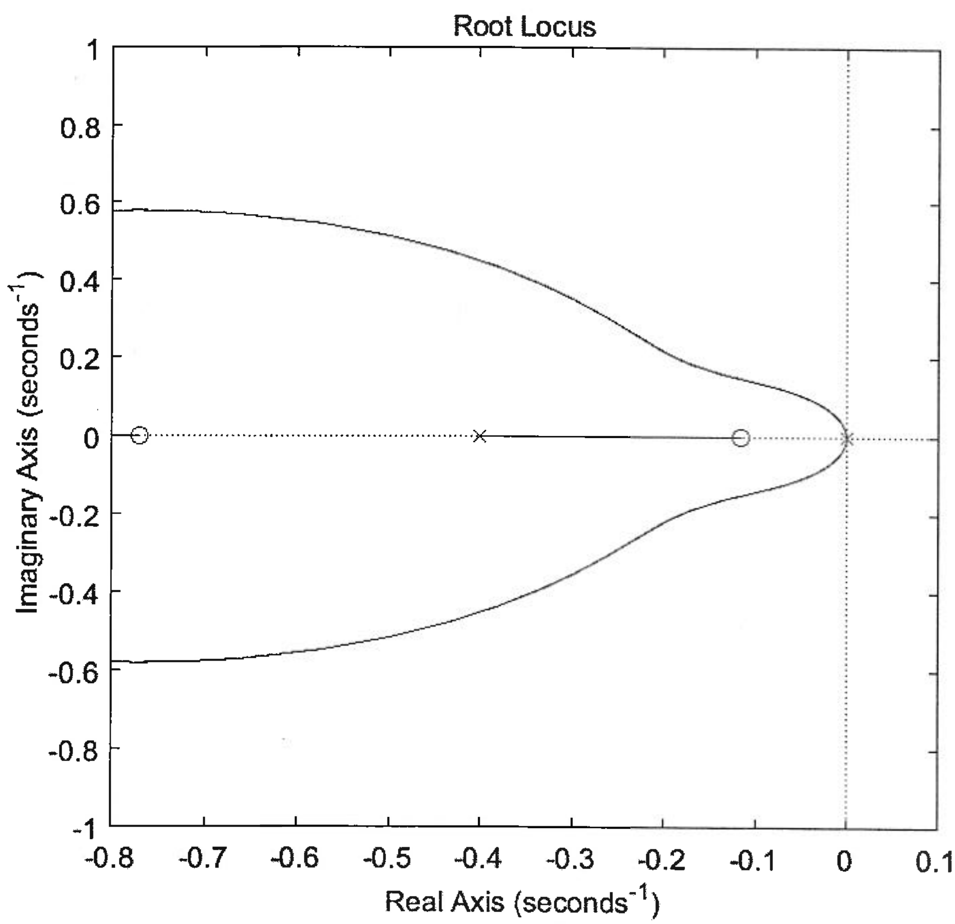




2
PID

PID control

Root locus



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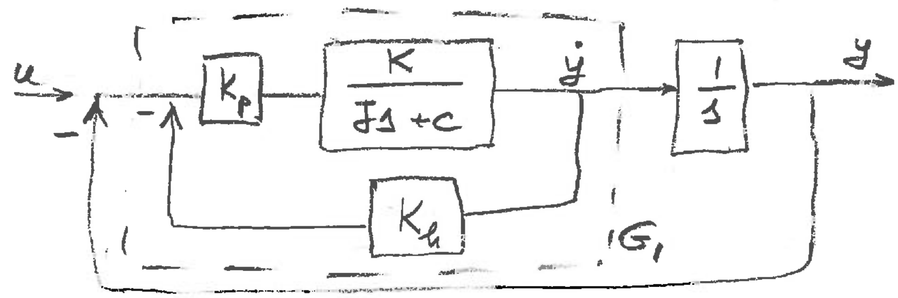
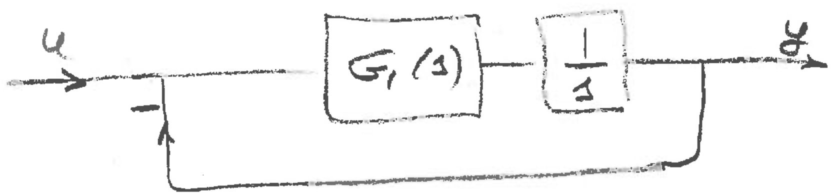
Velocity Feedback P-Controller

$$G(s) = \frac{K}{Js^2 + cs} = \frac{K}{Js + c} \cdot \frac{1}{s}$$



$$G_1(s) = \frac{\frac{K_p K}{Js + c}}{1 + \frac{K_p K}{Js + c} K_h} = \frac{K_p K}{Js + c + K_p K K_h} \qquad (1)$$



$$G_2 = G_1(s)\frac{1}{s} = \frac{K_p K}{Js^2 + (c + K_p K K_h)s} \qquad (2)$$

$$G_{CL} = \frac{G_2}{1 + G_2} = \frac{K_p K}{Js^2 + \underbrace{(c + K_p K K_h)}_{c^*}s + K_p K}$$

$$G_{CL} = \frac{K_p K}{Js^2 + c^* s + K_p K} \qquad (3)$$

$$c^* = c + K_p K K_h \tag{4}$$

$$G_{CL} = \frac{\frac{K_p K}{J}}{s^2 + \frac{c^*}{J}s + \frac{K_p K}{J}} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = \frac{K_p K}{J} \longrightarrow \omega_n = \sqrt{\frac{K_p K}{J}} \tag{5}$$

$$2\zeta\omega_n = \frac{c^*}{J} \rightarrow \zeta = \frac{c^*}{2\omega_n J} = \frac{c^*}{2\sqrt{K_p K J}}$$

$$\zeta = \frac{c + K_p K K_h}{2\sqrt{K_p K J}}$$

- Velocity feedback gain K_h increases ζ (damping)
- P-control gain K_p modifies both freq. ω_n and damping ζ

Strategy
- modify frequency with K_p
- add more damping with K_h

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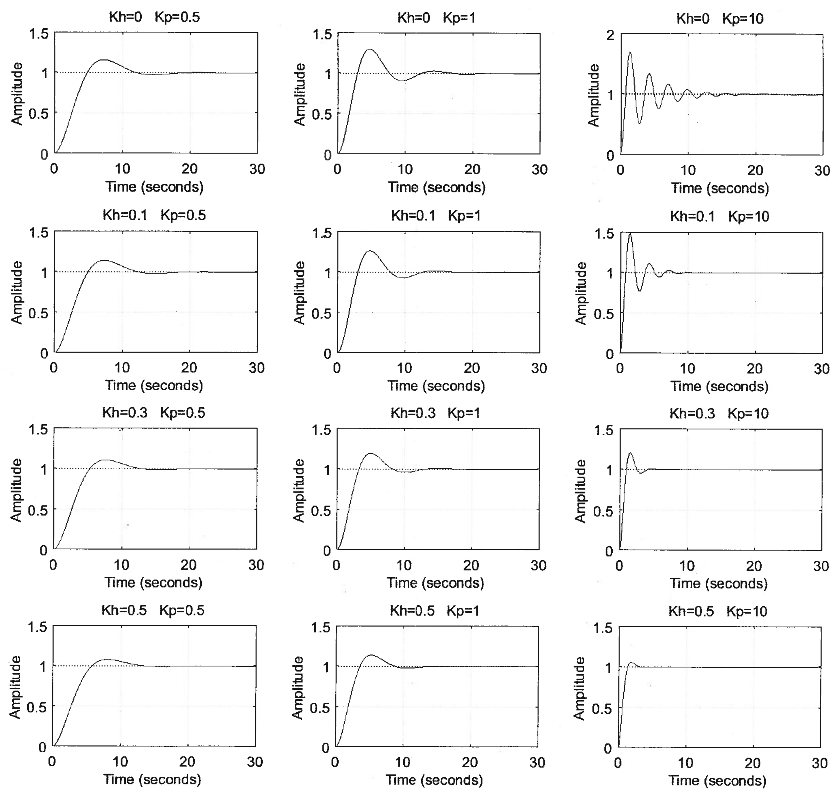

class example.
$K=1$
$J=2$
$c=1$
$\frac{K}{Js+c}\cdot\frac{1}{s}$

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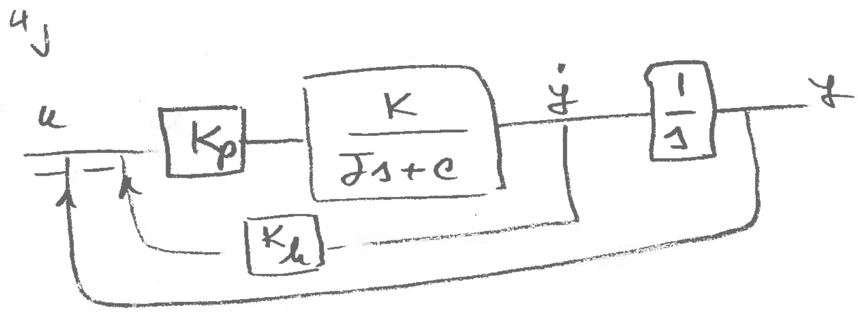
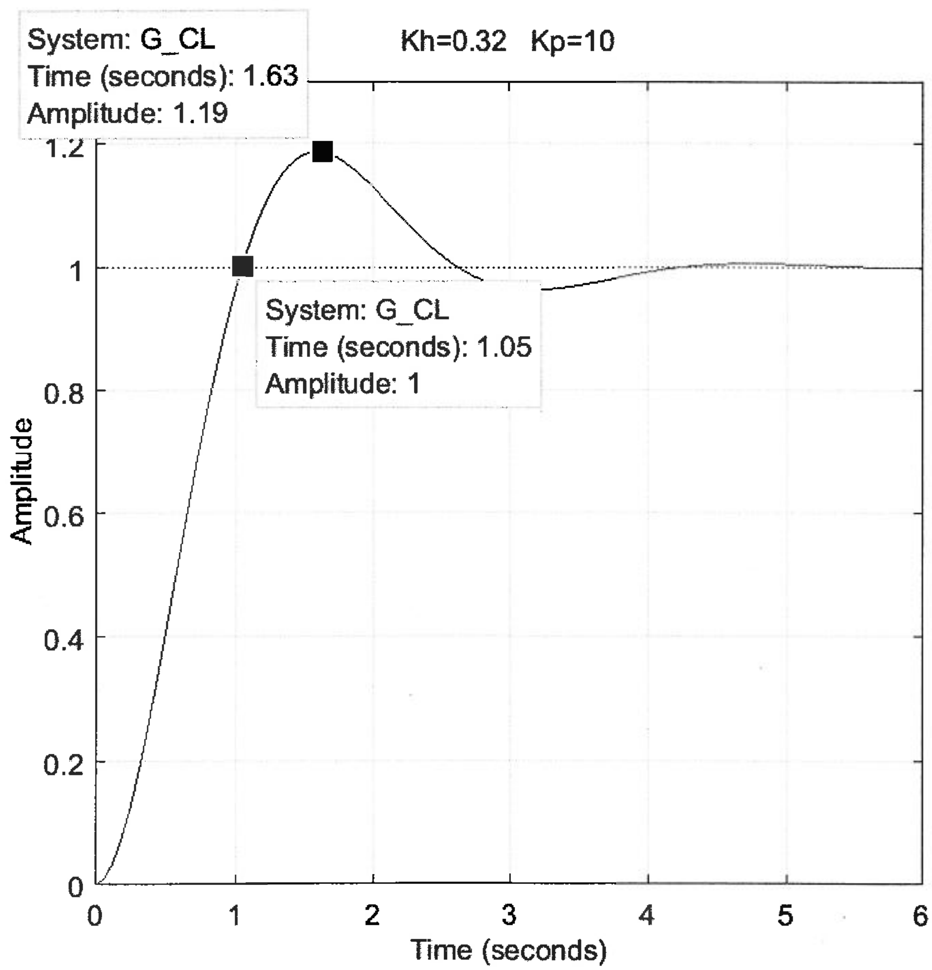


K = 1
J = 2
c = 1

class example



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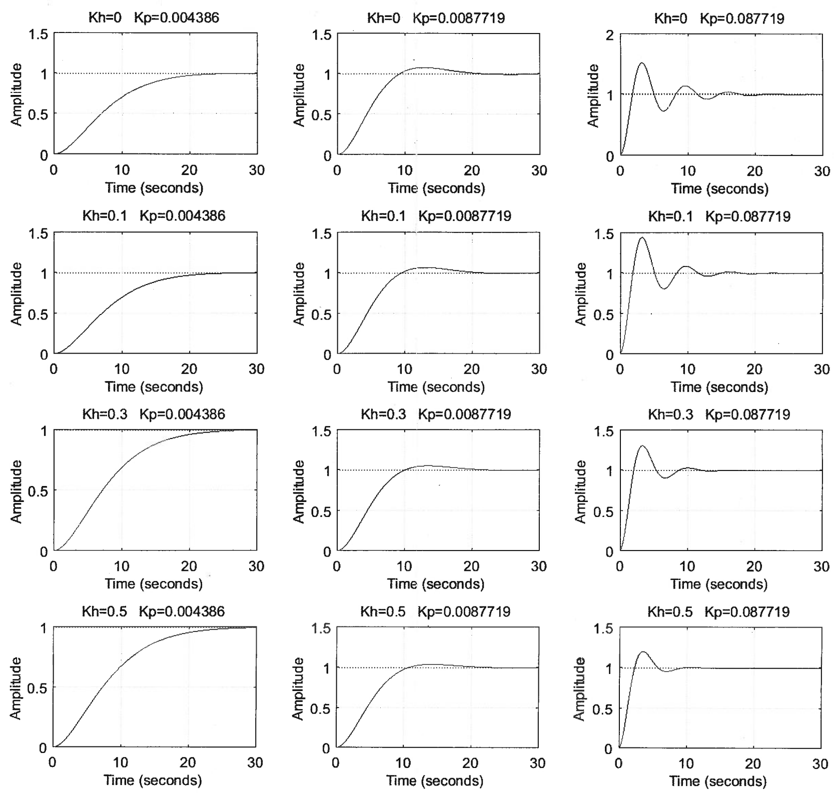

aircraft

$\frac{K}{Js+c} \cdot \frac{1}{s}$

$K = 114$

$J = 10$

$c = 4$

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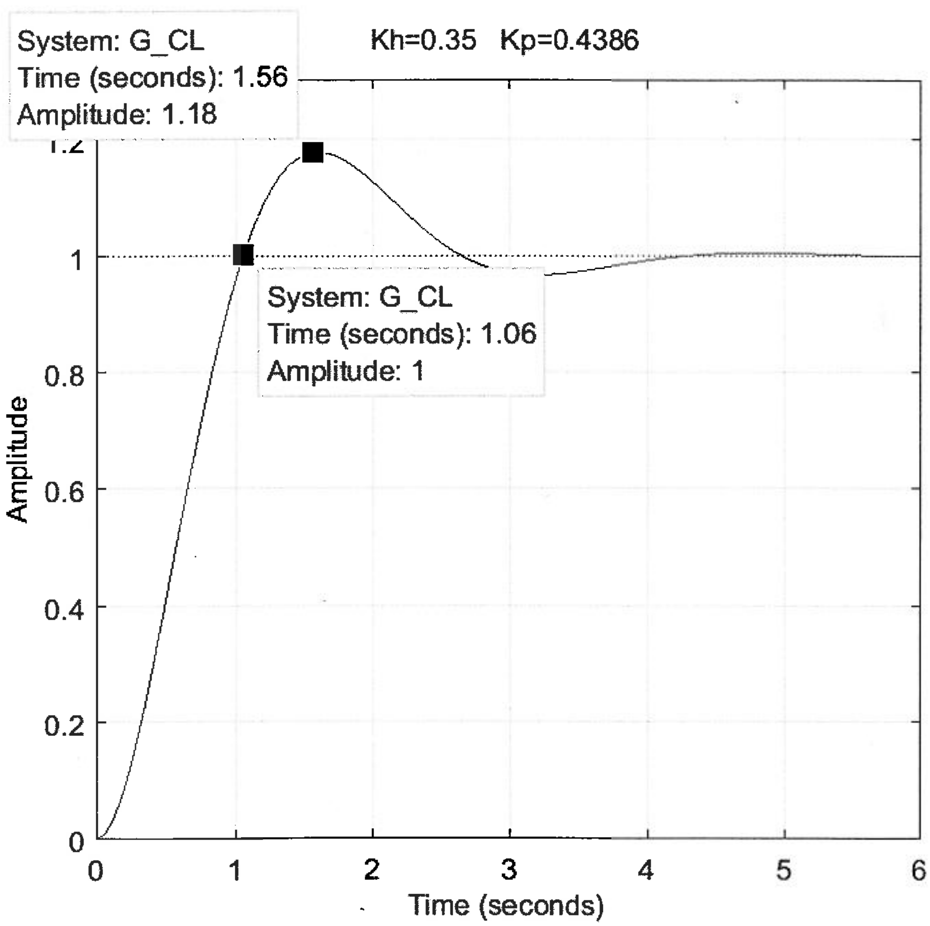
6) velocity feedback

$$\frac{K}{Js+C} \cdot \frac{1}{s}$$

$K = 114$
$J = 10$
$C = 4$

aircraft roll model



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7.7 SIMULINK Aircraft Roll Motion (continued)

8 VELOCITY FEEDBACK CONTROL

Velocity feedback control concept consists of two controls:

- P-control with gain Kp
- velocity control with gain Kh

The advantage of this control combination is that one can use the P-control gain Kp to improve the response time by increasing the natural frequency and then use the velocity feedback gain Kh to reduce the overshoot by increasing damping.

The initial values for the model are Kp=1/114, Kh=0. With these values, the SIMULINK model for velocity feedback looks like Figure 35.

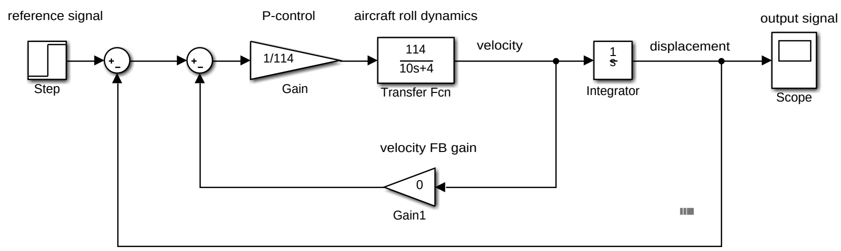

Figure 35

Note: when inserting ‘Gain1’ for velocity FB, use right click to flip the box to point backward.

The aircraft response is sluggish with $t_p \approx 9$ sec, as shown in Figure 36.

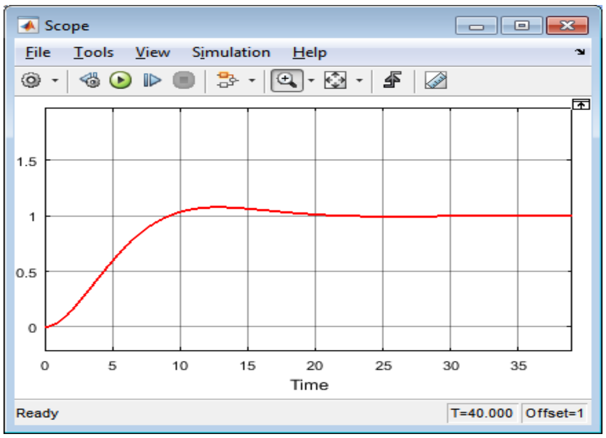

Figure 36

To accelerate the aircraft response, increase the P-control gain to Kp=50/114 (Figure 37).

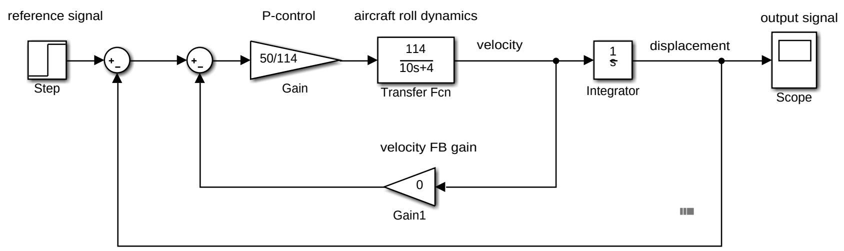

Figure 37

The aircraft response now looks like Figure 38.

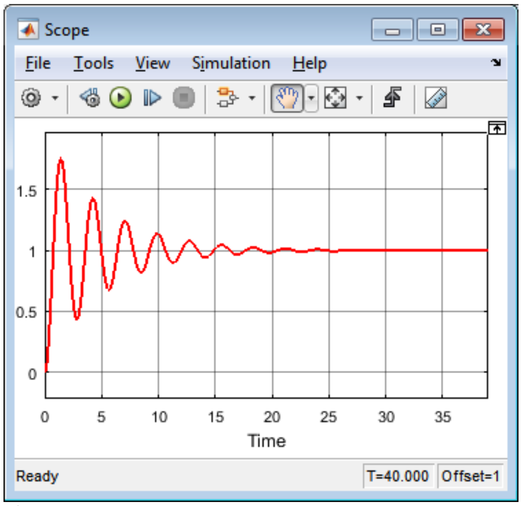

Figure 38

Notice that the aircraft response is much more rapid now. However, it has a large overshoot which must be reduced.

To reduce the overshoot, increase the velocity feedback gain to Kh=0.2 (Figure 39).

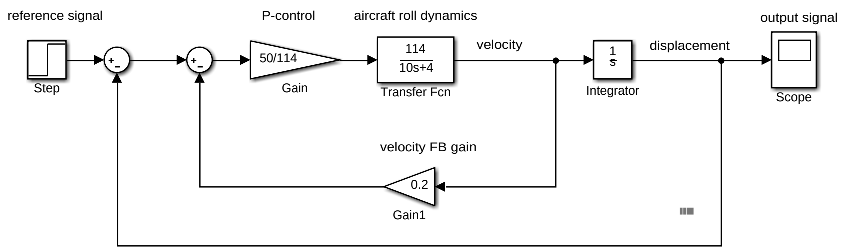

Figure 39

The response has become less oscillatory while remaining relatively fast (Figure 40).

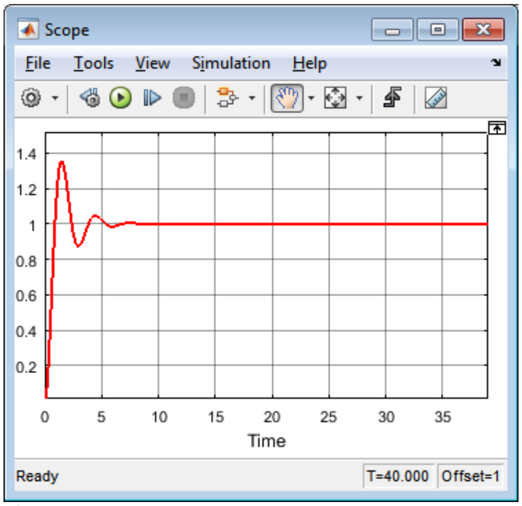

Figure 40

Further adjustments of Kh can ensure that the design specifications are met.

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Reduce the computation time to 6 sec in order to expand the initial response zone and get a better reading of rise time and overshoot.

Increase Kh to various values until a satisfactory reduction of overshoot is obtained. For Kh=0.35, we get Figure 41:

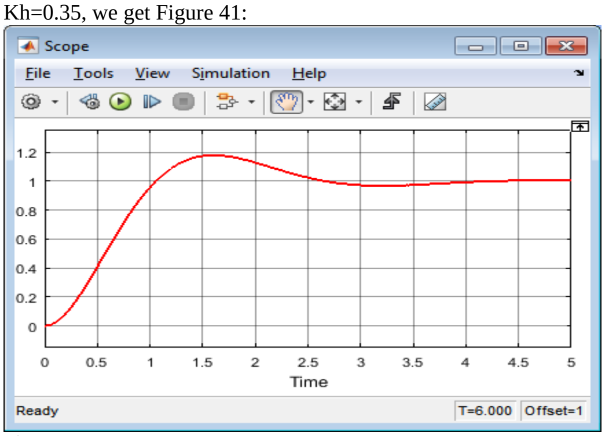


Figure 41

Open the cursors to read the rise time and overshoot(Figure 42).

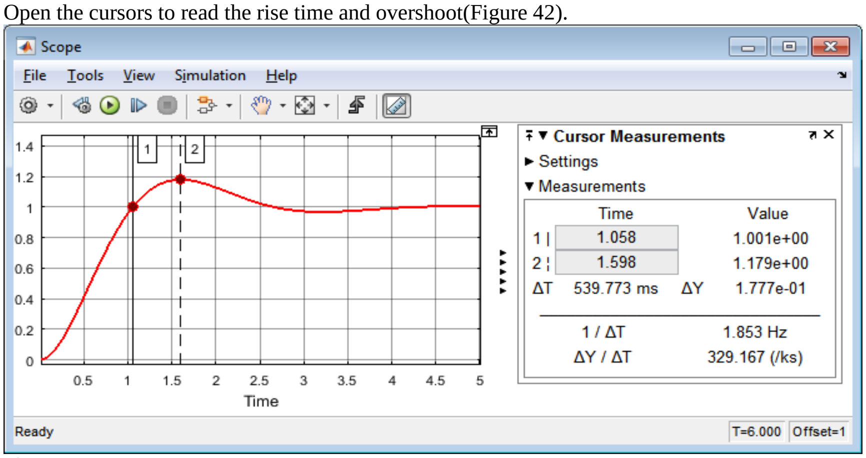


Figure 42

The readings indicate satisfactory results, i.e.,

- $t_r = 1.058 < 1.5$ sec
- $M_p = 17.9 < 20\%$

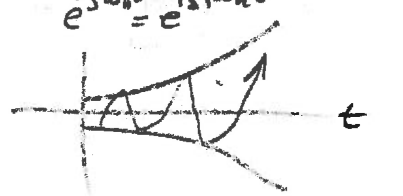
7.8 Instability Suppression with velocity feedback

VFB

20160613

INSTABILITY SUPPRESSION WITH VELOCITY FEEDBACK

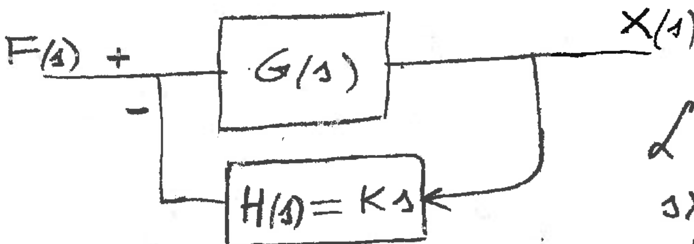
Assume a 2^{nd} order system with negative damping, $\zeta < 0$. Hence, the response will be unstable: $e^{-\zeta\omega_n t} = e^{|\zeta|\omega_n t}$



$\zeta < 0$

$-\zeta\omega_n t = |\zeta|\omega_n$

We can use FB to suppress this instability.



$\mathcal{L}\,\dot{x} = sX$

sX is the LT of velocity!

$$G_{CL} = \frac{G}{1+GH} = \frac{\dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}}{1+\dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}Ks}$$

$$= \frac{\omega_n^2}{s^2+(2\zeta\omega_n+K\omega_n^2)s+\omega_n^2} \qquad (1)$$

$$G_{CL} = \frac{\omega_n^2}{s^2+2\zeta_{CL}\omega_n s+\omega_n^2} \qquad (2)$$

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FB4
20181022 (1)&(2) :

$$2\zeta_{CL} = 2\zeta + K\omega_n$$

$$\zeta_{CL} = \zeta + \frac{K\omega_n}{2} \qquad (3)$$

1) Velocity feedback increases damping.

2) If the initial system is unstable because it has –ve damping ($\zeta < 0$), then velocity feedback can give the CL system a positive damping, $\zeta_{CL} > 0$.

Solve Eq. (3) to get

$$K = \frac{2}{\omega_n}(\zeta_{CL} - \zeta) \qquad (4)$$

Critical gain K_{cr} is the gain that makes $\zeta_{CL} = 0$.
Recall Eq (3) and set it to zero to get

$$\zeta + \frac{K\omega_n}{2} = 0 \longrightarrow K_{cr} = -\frac{2\zeta}{\omega_n} \qquad (5)$$

Example:

$\zeta = -5\% = -0.05$ $\qquad f_n = 4\ Hz$

$\omega_n = 2\pi \times 4 = 8\pi$ rad/s.

$$K_{cr} = -\frac{2 \times (-0.05)}{8\pi} \cong 0.004$$

For $K > K_{cr}$ the system is stable

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FB4a
20181022

Calculation of FB gain K

Recall (4) $K = \frac{2}{\omega_n}(\zeta_{CL} - \zeta)$

Factor out $-\zeta$ to get $K = -\frac{2\zeta}{\omega_n}\left(1 - \frac{\zeta_{CL}}{\zeta}\right)$

Recall (5) and write

$$K = K_{cr}\left(1 - \frac{\zeta_{CL}}{\zeta}\right) \qquad (6)$$

Define ζ_{ratio} as

$$\zeta_{ratio} = \frac{\zeta_{CL}}{\zeta} \qquad (7)$$

(7) → (6): $$K = K_{cr}(1 - \zeta_{ratio}) \qquad (8)$$

Eq(8) gives the FB gain K as a function of ζ_{ratio}

Gain ratio K_{ratio}

Divide Eq(8) by K_{cr} to get

$$K_{ratio} = 1 - \zeta_{ratio} \qquad (9)$$

Discussion of Eq.(8)

(a) if $\zeta < 0$, then $\zeta_{ratio} < 0$, $K_{cr} > 0$ → $\boxed{K > 0}$

(b) if $0 < \zeta < \zeta_{CL}$, then $K_{cr} < 0$, $\zeta_{ratio} > 1$, $(1 - \zeta_{ratio}) < 0$, $\boxed{K > 0}$

(c) $\zeta_{CL} < \zeta$, then $K_{cr} < 0$, $\zeta_{ratio} < 1$, $1 - \zeta_{ratio} < 0$, $\boxed{K < 0}$ reduce ζ_{CL}

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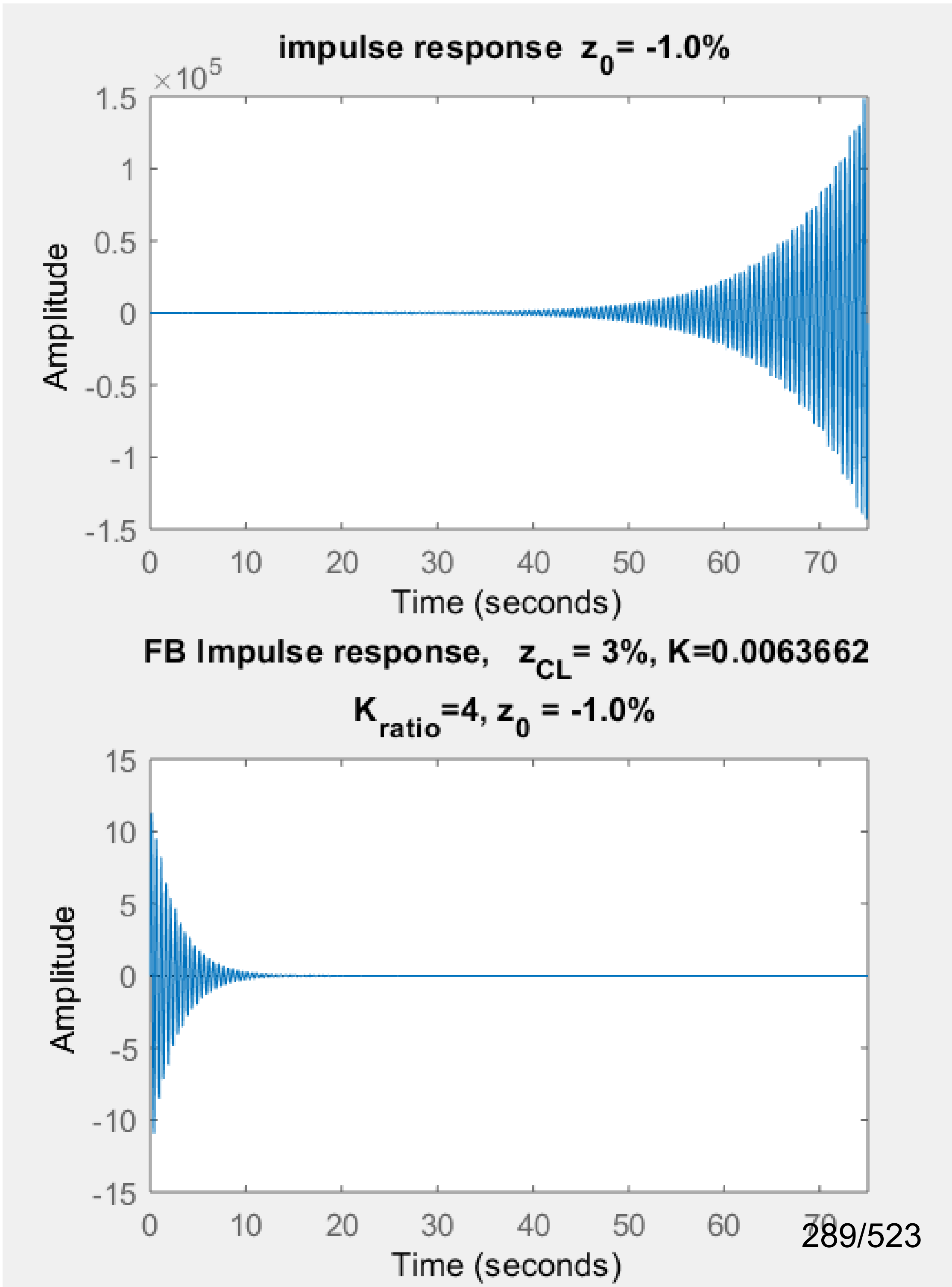
impulse response z_0= -1.0%
×10^5
1.5
1
0.5
0
-0.5
-1
-1.5
Amplitude
0 10 20 30 40 50 60 70
Time (seconds)
FB Impulse response, z_{CL}= 3%, K=0.0063662
K_{ratio}=4, z_0 = -1.0%
15
10
5
0
-5
-10
-15
Amplitude
0 10 20 30 40 50 60 70
Time (seconds)
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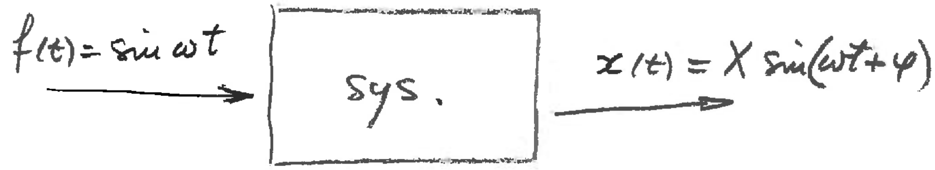
8 Frequency Analysis

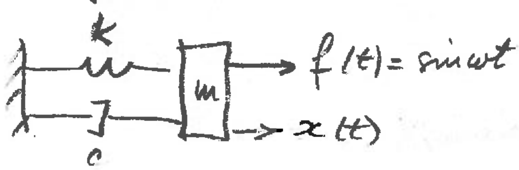
8.1 Time Response Under Harmonic Excitations

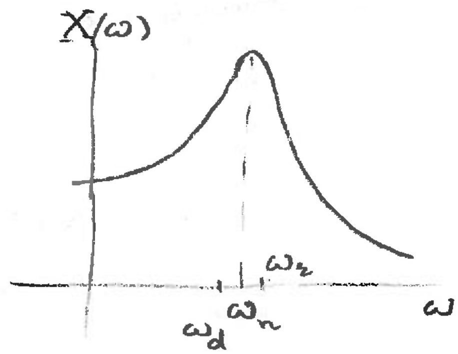
Time Response to harmonic excitation



One is interested in finding how the system responds to harmonic excitation of different frequency values, $\omega = 2\pi f$

Ex: For a 2^{nd} order system, one is concerned with avoiding resonance where the response may become very large





$$x(t) = X(\omega)\sin(\omega t + \varphi(\omega))$$

$$\zeta = \frac{c}{2\sqrt{km}}, \quad \omega_n = \sqrt{\frac{k}{m}}.$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

We notice that the amplitude and phase of the time response varies with excitation freq. ω

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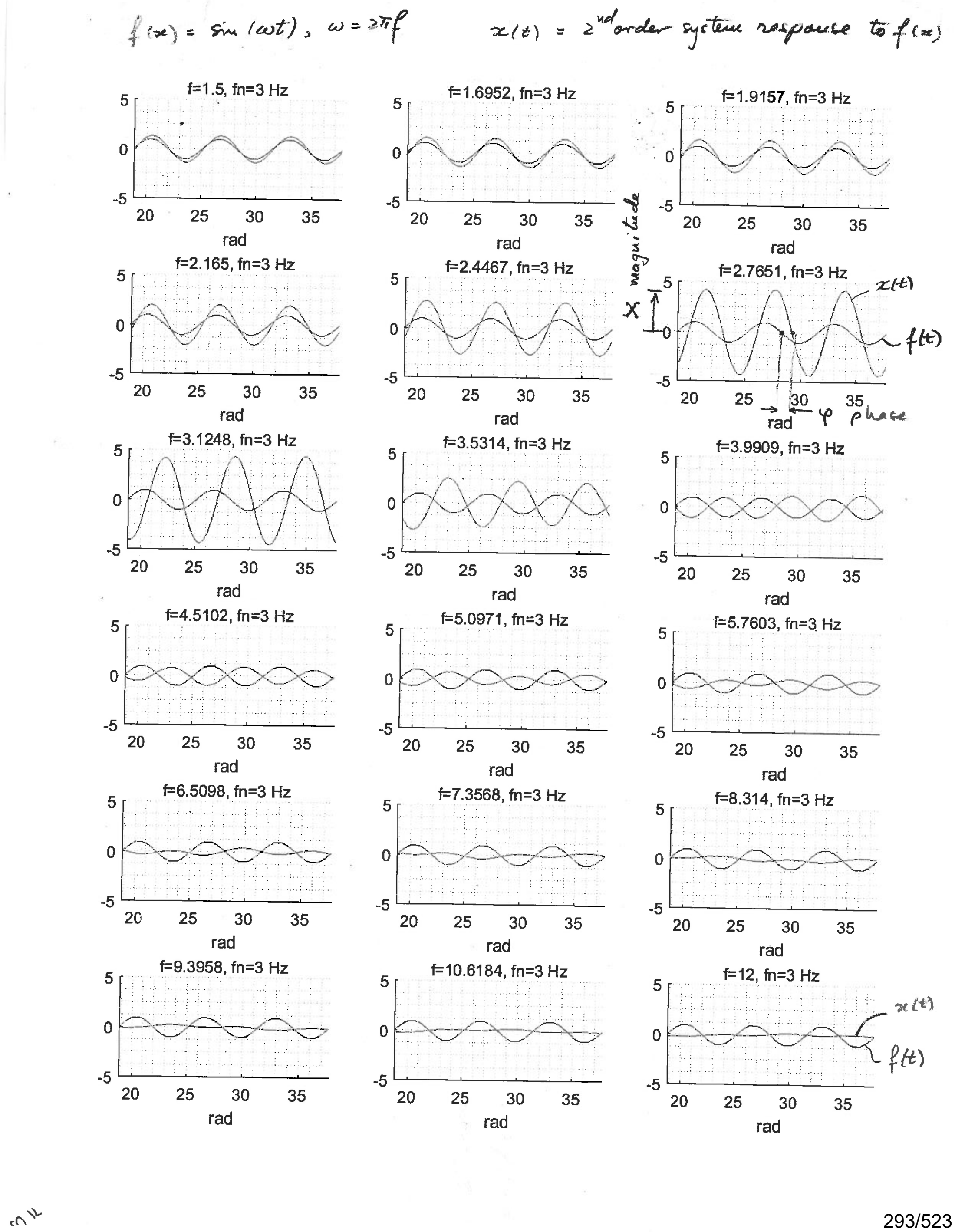

f(x) = sin(ωt), ω = 2πf
x(t) = 2nd order system response to f(x)
f=1.5, fn=3 Hz
f=1.6952, fn=3 Hz
f=1.9157, fn=3 Hz
f=2.165, fn=3 Hz
f=2.4467, fn=3 Hz
f=2.7651, fn=3 Hz
magnitude
X
x(t)
f(t)
φ phase
f=3.1248, fn=3 Hz
f=3.5314, fn=3 Hz
f=3.9909, fn=3 Hz
f=4.5102, fn=3 Hz
f=5.0971, fn=3 Hz
f=5.7603, fn=3 Hz
f=6.5098, fn=3 Hz
f=7.3568, fn=3 Hz
f=8.314, fn=3 Hz
f=9.3958, fn=3 Hz
f=10.6184, fn=3 Hz
f=12, fn=3 Hz
x(t)
f(t)
rad
293/523

a F

$$f(x) = \sin \omega t$$

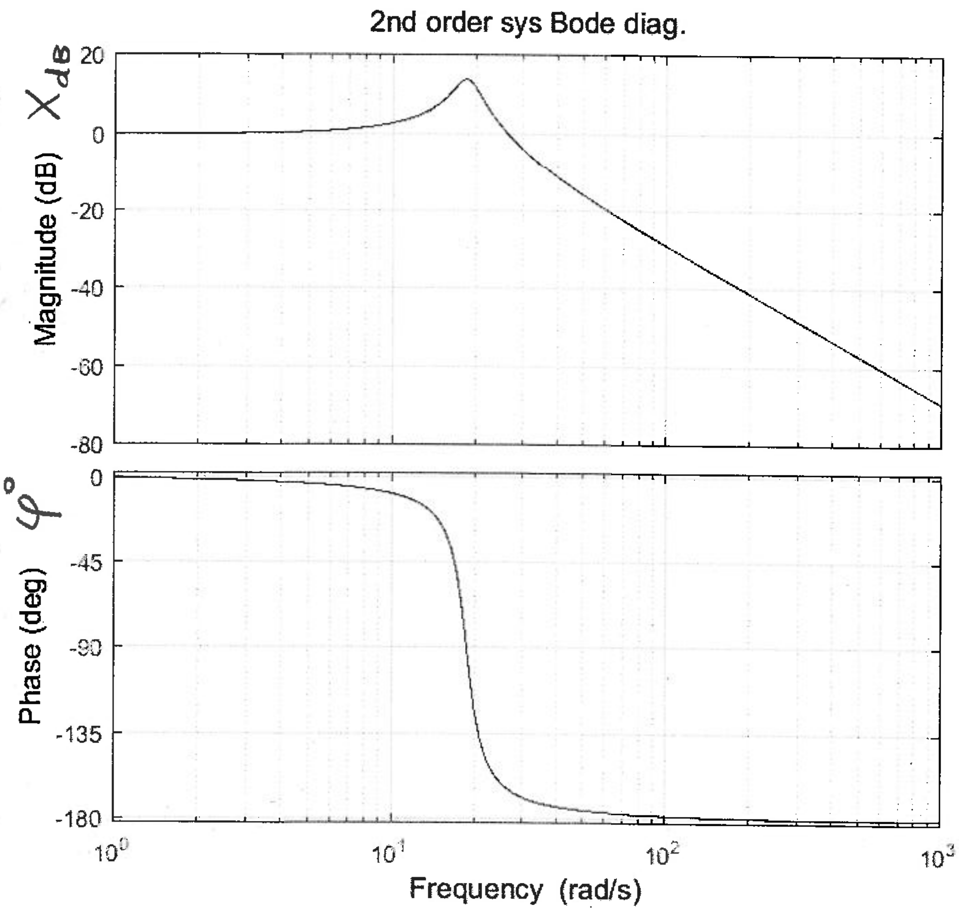
$$x(t) = X(\omega) \sin[\omega t + \varphi(\omega)]$$

$X(\omega)$ = magnitude of response

$\varphi(\omega)$ = phase of response

Magnitude X and phase φ vary with excitation freq. ω.

Bode diagram plots: variation of $X(\omega)$ & $\varphi(\omega)$



Note that φ is $-ve$, i.e. the response lags behind the excitation.

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C:\Mydata\1_USC\c...\FRF_2ndOrderSys_20161214.m Page 1

```
% Amplitude_and_phase_2ndOrderSys.m
% AMPLITUDE AND PHASE IN FREQUENCY RESPONSE
clc
clear
close all
s=tf('s');
% f=input('f=');
figure (1);
%% 2nd order system
fn=3; z=10e-2; wn=2*pi*fn; G2=wn^2/(s^2+2*z*wn*s+wn^2);
M=6; N=3; Nplots=M*N;
fmin=fn/2; fmax=fn*4;
a=log10(fmin); b=log10(fmax); f=logspace(a,b,Nplots);
%% plotting setup
Na=1e3; amax=10*2*pi; da=amax/Na; angle=0:da:amax;
xmin=0.3*Na*da; xmax=0.6*Na*da;
%% plot respose at various frequencies
for i=1:Nplots
w=2*pi*f(i);    % excitation frequency
t=angle/w;      % time steps at this excitation freq.
A=1;  % forcing function amplitude
F=A*w/(s^2+w^2);    % Laplace transform of sine forcing function
fe=impulse(F,t);    % time response of forcing function
X2=G2*F;            % Laplace transform of 2nd order system response
subplot(M,N,i);
x2=impulse(X2,t);   % time response of 2nd order system
plot(angle,fe,angle,x2); hold  on
title(['f=' num2str(f(i)) ', fn=' num2str(fn) ' Hz' ],...
    'FontSize', 10,'FontWeight','normal')
xlabel('rad'); xlim([xmin xmax]); ylim([-5*A 5*A]);
% grid on
grid minor; box off
```

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C:\Mydata\1_USC\c...\FRF_2ndOrderSys_20161214.m Page 2

```
end
%% FRF Bode plots
figure(2)
bode(G2); grid on; title('2nd order sys Bode diag.' );
aw=log10(2*pi*fn/2); bw=log10(2*pi*fn*2); N=1e4; wBode=logspace(aw,bw,N);
figure(3)
bode(G2,wBode); grid on; title('zoom 2nd order sys Bode diag.' );
```

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8.2 Frequency Response Function (FRF)

FRF

dB scale deci bell ($\frac{1}{10}$)

dB scale is a $\log_{10}$ scale magnified by 20

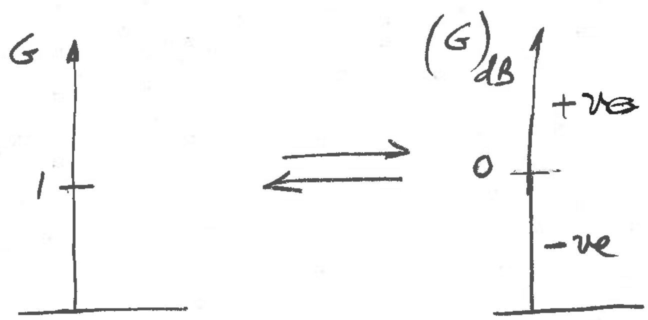
$$(G)_{dB} = 20 \log_{10} G$$

Properties

(1) Only +ve numbers have dB value
(cannot take log of −ve numbers!)

(2) $\left(\frac{1}{G}\right)_{dB} = -(G)_{dB}$ (reciprocal numbers)

(3)



$G > 1$	$(G)_{dB}$ is +ve
$G = 1$	$(G)_{dB} = 0$
$G < 1$	$(G)_{dB}$ is −ve

2 FRF

(4) Scale up or scale down in physical values means shift up or shift down in dB

(a) double (octave up) = +6 dB

(b) "half" (octave down) = − 6 dB

(c) "ten times" (decade) = +20 dB

octave = (A B C D E F G) A_1 8th musical note has double the frequency $f_{A_1} = 2 f_A$

Proof

(a) $\log_{10} 2 = 0.303$; $20 \log 2 \cong 6$

$G_2 = 2G_1$

$(G_2)_{dB} = 20 \log_{10}(2G_1) = 20 \log_{10} 2 + 20 \log_{10} G_1$

$= \underbrace{20 \times 0.303}_{\sim 6} + (G_1)_{dB}$

$(G_2)_{dB} = (G_1)_{dB} + 6\,dB$

```
>> mag2db(2)
ans
     6.0206
```

(b) $G_2 = \frac{1}{2} G_1$

$(G_2)_{dB} = (G_1)_{dB} + 20 \log_{10}\left(\frac{1}{2}\right)$

$= (G_1)_{dB} + 20\left(- \log_{10} 2\right)$

$= (G_1)_{dB} - 6\,dB$

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3 FRF

(c) $G_2 = 10\,G_1$

$$(G_2)_{dB} = (G_1)_{dB} + 20 \underbrace{\log_{10}(10)}_{=1}$$

$$= (G_1)_{dB} + 20\,dB$$

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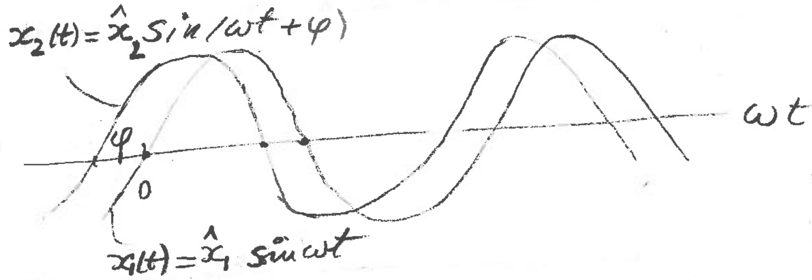
FRF

Phase

$x_1(t) = \hat{x}_1 \sin \omega t$ reference signal

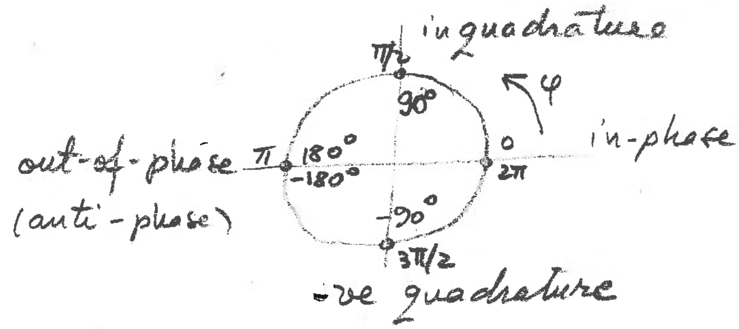
$x_2(t) = \hat{x}_2 \sin(\omega t + \varphi)$



$\psi_1 = \omega t$ phase of x_1 (reference)

$\psi_2 = \omega t + \varphi$ phase of x_2

$\varphi = \psi_2 - \psi_1$
- phase difference
- x_2 leads by φ.



Periodic signals

$\left.\begin{matrix} 2\pi \\ 360° \end{matrix}\right\}$ periodicity:
- $x(\omega t + 2\pi) = x(\omega t)$
- $x(\omega t + \pi) = x(\omega t - \pi)$

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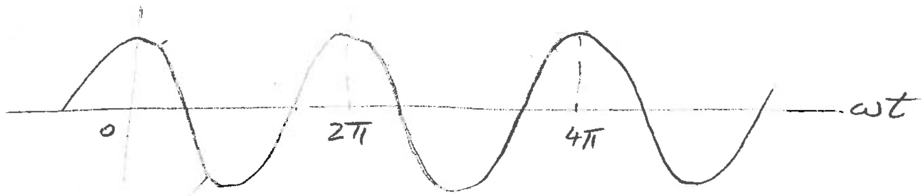
5 FRF

COMPLEX NUMBER REPRESENTATION OF HARMONIC SIGNALS



Cos ωt

$$e^{i\omega t} = \cos\omega t + i\sin\omega t \quad \text{(Euler)}$$

$$x(t) = \hat{x}\cos(\omega t + \varphi)$$

$$x(t) = \mathrm{Re}\left[\hat{x}e^{i(\omega t+\varphi)}\right]$$

$$x(t) = \hat{x}\sin(\omega t + \varphi)$$

$$x(t) = \mathrm{Im}\left[\hat{x}e^{i(\omega t+\varphi)}\right]$$

In general $\quad x(t) = \hat{x}e^{i(\omega t+\varphi)}$

$$\hat{x}e^{i(\omega t+\varphi)} = \hat{x}e^{i\varphi}e^{i\omega t} = X e^{i\omega t}$$

$$X = \hat{x}e^{i\varphi} \quad \text{(phasor)}$$

$|X| = \hat{x}$

magnitude of $x(t)$

$|X| = \mathrm{abs}(X)$

$\angle X = \varphi$

phase of $x(t)$

$\angle X = \mathrm{angle}(X)$, rad

rad2deg$(\pi) = 180°$.

6 FRF

Frequency response function (FRF)

- $G(s)$ TF: Transfer function in Laplace domain s

- $G(i\omega)$ FRF: Transfer function in frequency domain ω

$$f(t) = e^{i\omega t} \longrightarrow \boxed{G(i\omega)} \longrightarrow x(t) = G(i\omega)\, e^{i\omega t}$$

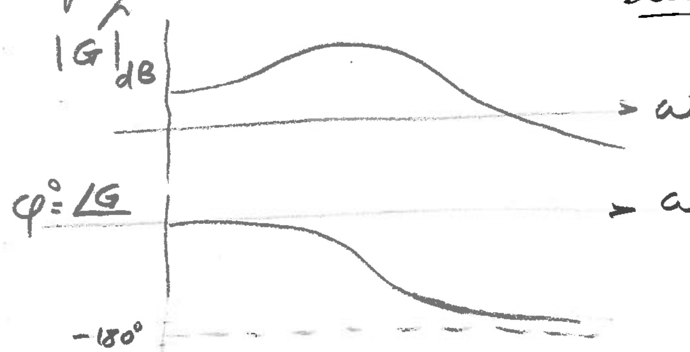
$G(i\omega)$ is a complex number

$|G(i\omega)|$ = magnitude, dB

$\angle G(i\omega)$ = phase, deg

Frequency response function (FRF) is $G(i\omega)$ measured over a range of frequencies ω

Bode diagram



Magnitude plot

Phase plot

usually $\varphi < 0$ "phase lag"

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20181205

Harmonic Response via Laplace Transform

$$f(t) = e^{i\omega t} \xrightarrow{\mathcal{L}} F(s) = \frac{1}{s - i\omega}$$

$$X(s) = G(s)\,F(s) = G(s)\,\frac{1}{s - i\omega} \qquad (1)$$

$$\text{Assume: } G(s) = K\,\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \qquad (2)$$

Poles in LHS
$-p_n \cdots -p_1 < 0$

STABLE

(1) & (2) yields the PFE:

$$X(s) = \frac{a}{s - i\omega} + \frac{b_1}{s+p_1} + \frac{b_2}{s+p_2} + \cdots + \frac{b_n}{s+p_n} \qquad (3)$$

$$x(t) = a e^{i\omega t} + \underbrace{b_1 e^{-p_1 t} + b_2 e^{-p_2 t} + \cdots b_n e^{-p_n t}}_{\text{Transient response} \xrightarrow[t\to\infty]{} 0} \qquad (4)$$

steady state response.

$$x_{ss}(t) = a e^{i\omega t} \qquad (5)$$

To find a, multiply (3) by $s - i\omega$ and make $s = i\omega$ to get:

$$\Big[(s-i\omega)X(s)\Big]_{s=i\omega} = \left[a + \cancel{\left[\frac{b_1}{s+p_1} + \frac{b_2}{s+p_2} + \cdots \frac{b_n}{s+p_n}\right](s-i\omega)}\right]_{s=i\omega}$$

$$a = \Big[(s-i\omega)X(s)\Big]_{s=i\omega} \qquad (6)$$

$$= \left[\cancel{(s-i\omega)}\,G(s)\,\frac{1}{\cancel{(s-i\omega)}}\right]_{s=i\omega} = G(i\omega).$$

$$(6)\to(5): \qquad x_{ss}(t) = \underbrace{\boxed{G(i\omega)}}_{\text{complex amplitude}} e^{i\omega t}.$$

$$\text{FRF} = G(i\omega) = |G(i\omega)|\,e^{i\varphi}, \qquad \varphi = \angle G(i\omega)$$

FR14

Harmonic Response via Laplace Transform

$$G(s)=\frac{1}{Ts+1}\quad,\quad f(t)=e^{i\omega t}\longrightarrow F(s)=\frac{1}{s-i\omega}$$

$$X(s)=G(s)F(s)=\frac{1}{Ts+1}\cdot\frac{1}{s-i\omega}=\frac{1}{T}\cdot\frac{1}{s+\frac{1}{T}}\cdot\frac{1}{s-i\omega}$$

PFE, p 30, Eqn (2.6), modified: $\quad s_1=-\frac{1}{T}\quad s_2=i\omega$

$$X(s)=\frac{a_1}{s-s_1}+\frac{a_2}{s-s_2}+\cdots+\frac{a_k}{s-s_k}+\cdots$$

$$a_k=\Big[(s-s_k)X(s)\Big]_{s=s_k}$$

$$a_1=\left[\cancel{\left(s+\frac{1}{T}\right)}\frac{1}{T}\frac{1}{\cancel{s+\frac{1}{T}}}\frac{1}{s-i\omega}\right]_{s=-\frac{1}{T}}=\frac{1}{T}\,\frac{1}{-\frac{1}{T}-i\omega}=-\frac{1}{i\omega T+1}$$

$$a_2=\left[\cancel{(s-i\omega)}\frac{1}{T}\frac{1}{s+\frac{1}{T}}\frac{1}{\cancel{s-i\omega}}\right]_{s=i\omega}=\frac{1}{T}\,\frac{1}{i\omega+\frac{1}{T}}=\frac{1}{i\omega T+1}$$

$$X(s)=-\frac{1}{i\omega T+1}\cdot\frac{1}{s+\frac{1}{T}}+\frac{1}{i\omega T+1}\cdot\frac{1}{s-i\omega}$$

$$x(t)=-\frac{1}{i\omega T+1}e^{-t/T}+\frac{1}{i\omega T+1}e^{i\omega t}$$

| transient | steady state |

$$x_{ss}(t)=\frac{1}{i\omega T+1}e^{i\omega t}=G(i\omega)\,e^{i\omega t}.$$

$$FRF=G(i\omega)=|G(i\omega)|e^{i\varphi}\ ,\ \varphi=\angle G(i\omega)$$

True for stable systems (i.e., transients vanish).

FR15 Harmonic Response via Laplace Transform for sine excitation

$$G(s) = \frac{1}{Ts+1}, \quad f(t) = \sin \omega t \rightarrow F(s) = \frac{\omega}{s^2+\omega^2}$$

$$X(s) = G(s)F(s) = \frac{1}{Ts+1} \cdot \frac{\omega}{s^2+\omega^2} = \frac{1}{T}\frac{1}{s+\frac{1}{T}} \cdot \frac{\omega}{(s-i\omega)(s+i\omega)}$$

Poles: $s_1 = -1/T \quad s_{2,3} = \pm i\omega$

PFE, p.30, Eq (2.6), modified:

$$X(s) = \frac{a_1}{s-s_1} + \frac{a_2}{s-s_2} + \dots + \frac{a_k}{s-s_k} + \dots$$

$$a_k = \Big[(s-s_k)\,X(s)\Big]_{s=s_k}$$

$$X(s) = \frac{a_1}{s+\frac{1}{T}} + \frac{a_2}{s-i\omega} + \frac{a_3}{s+i\omega}$$

$$a_1 = (s+\tfrac{1}{T})G(s)\Big|_{s=s_1} = \cancel{(s+\tfrac{1}{T})}\frac{1/T}{\cancel{s+\frac{1}{T}}}\frac{\omega}{s^2+\omega^2}\Bigg|_{s=-\frac{1}{T}}$$

$$a_1 = \frac{1}{T}\frac{\omega}{\frac{1}{T^2}+\omega^2} = \frac{\omega T}{\omega^2T^2+1}$$

$$a_2 = (s-i\omega)\,G(s)\Big|_{s=i\omega} = \cancel{(s-i\omega)}\frac{1}{Ts+1}\frac{\omega}{\cancel{(s-i\omega)}(s+i\omega)}\Bigg|_{s=i\omega}$$

$$a_2 = \frac{1}{i\omega T+1} \cdot \frac{\omega}{i\omega+i\omega} = \frac{1}{i\omega T+1}\frac{1}{2i} = G(i\omega)\frac{1}{2i}$$

FR16

$$a_3 = (s+i\omega)G(s)\Big|_{s=-i\omega} = \cancel{(s+i\omega)}\frac{1}{Ts+1}\,\frac{\omega}{(s-i\omega)\cancel{(s+i\omega)}}\Bigg|_{s=-i\omega}$$

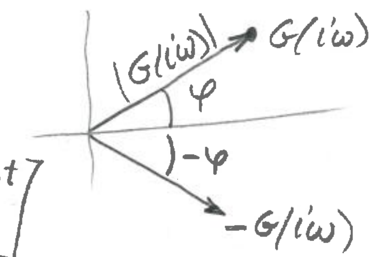
$$= \frac{1}{-i\omega T+1}\,\frac{1}{-2i} = -G(-i\omega)\,\frac{1}{2i}$$

$$X(s) = \underbrace{\frac{\omega T}{\omega^2T^2+1}\,\frac{1}{s+\frac{1}{T}}}_{\text{transient}} + \underbrace{\frac{1}{2i}\left[\frac{G(i\omega)}{s-i\omega} - \frac{G(-i\omega)}{s+i\omega}\right]}_{\text{steady state}}$$

$$x(t) = \frac{\omega T}{\omega^2T^2+1}\,e^{-t/T} + \frac{1}{2i}\left[G(i\omega)\,e^{i\omega t} - G(-i\omega)\,e^{-i\omega t}\right]$$

But $G(i\omega) = |G(i\omega)|\,e^{i\varphi}$, $\varphi = \angle G(i\omega)$

$G(-i\omega) = |G(i\omega)|\,e^{-i\varphi}$



Hence

$$x_{ss}(t) = \frac{1}{2i}\left[G(i\omega)\,e^{i\omega t} - G(-i\omega)\,e^{-i\omega t}\right]$$

$$= \frac{1}{2i}\,|G(i\omega)|\left(e^{i\varphi}e^{i\omega t} - e^{-i\varphi}e^{-i\omega t}\right), \quad \varphi = \angle G(i\omega)$$

$$= |G(i\omega)|\,\frac{e^{i\psi} - e^{-i\psi}}{2i} \qquad \psi = \omega t + \varphi$$

$$= |G(i\omega)|\,\sin\psi$$

$$x_{ss}(t) = |G(i\omega)|\,\sin(\omega t + \varphi), \quad \varphi = \angle G(i\omega)$$

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FR17

Lemma:

If $G(s)$ is polynomial (or fraction of)

and $G(i\omega) = |G(i\omega)| e^{i\varphi}$

Then $G(-i\omega) = |G(i\omega)| e^{-i\varphi}$.

Proof

(a) $G(s) = s + a$

$G(i\omega) = i\omega + a = \sqrt{a^2+\omega^2}\, e^{i\varphi}, \quad \varphi = \tan^{-1}\frac{\omega}{a}$.

$G(-i\omega) = -i\omega + a = \sqrt{a^2+\omega^2}\, e^{i\varphi^*}, \quad \varphi^* = \tan^{-1}\frac{-\omega}{a} = -\varphi$

(b) $G(s) = (s+a_1)(s+a_2)$

$G(i\omega) = (i\omega + a_1)(i\omega + a_2)$

$= \sqrt{a_1^2+\omega^2}\sqrt{a_2^2+\omega^2}\, e^{i\varphi_1} e^{i\varphi_2}, \quad \varphi_1 = \tan^{-1}\frac{\omega}{a_1}$

$\varphi_2 = \tan^{-1}\frac{\omega}{a_2}$

$G(-i\omega) = (-i\omega + a_1)(-i\omega + a_2)$

$= \sqrt{a_1^2+\omega^2}\sqrt{a_2^2+\omega^2}\, e^{i\varphi_1^*} e^{i\varphi_2^*}$

$\varphi_1^* = \tan^{-1}\frac{-\omega}{a_1} = -\varphi_1$

$\varphi_2^* = \tan^{-1}\frac{-\omega}{a_2} = -\varphi_2$

etc.

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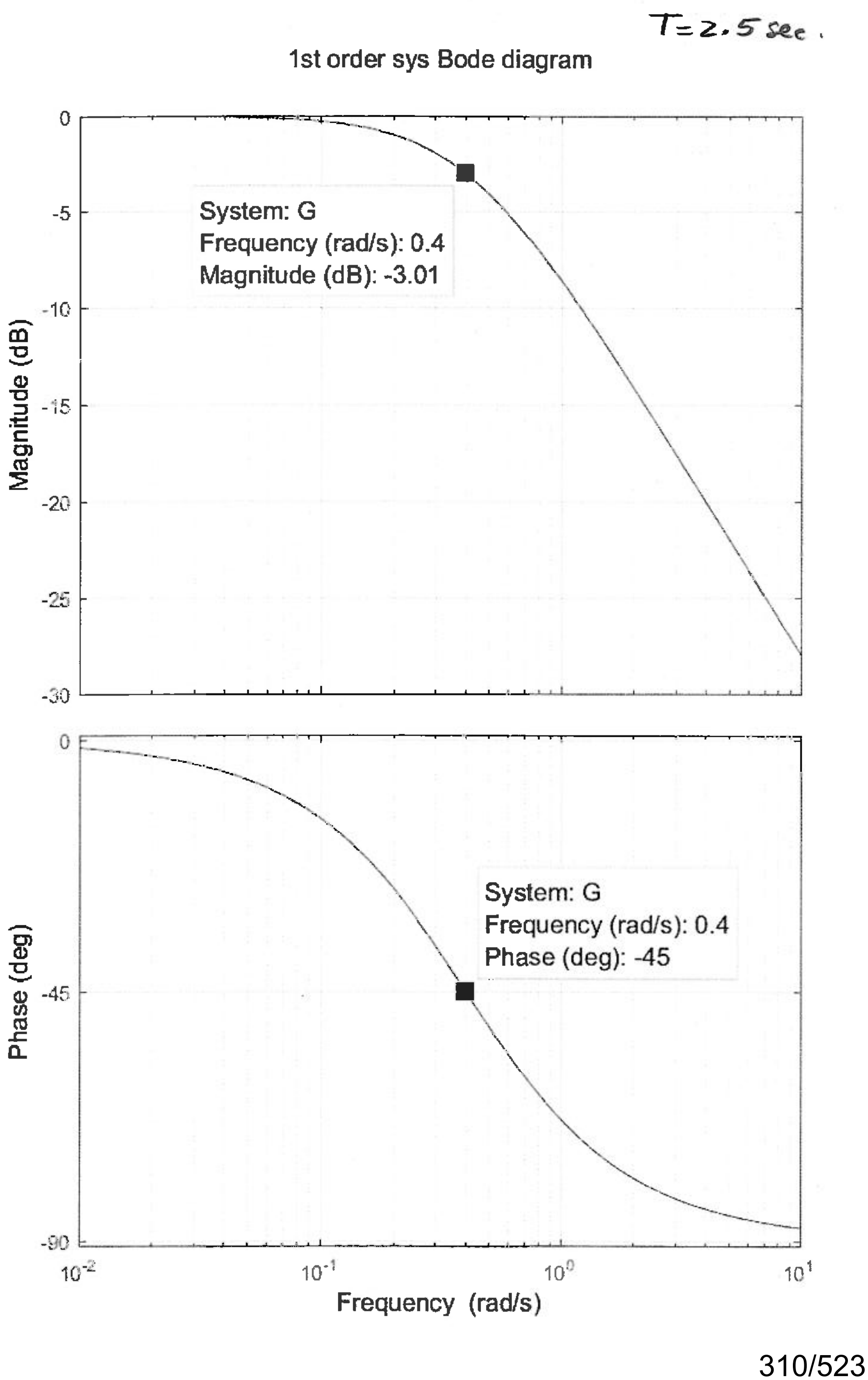
1st order system FRF


T=2.5 sec.
1st order sys Bode diagram
System: G
Frequency (rad/s): 0.4
Magnitude (dB): -3.01
System: G
Frequency (rad/s): 0.4
Phase (deg): -45
Magnitude (dB)
Phase (deg)
Frequency (rad/s)
0
-5
-10
-15
-20
-25
-30
0
-45
-90
10^{-2}
10^{-1}
10^{0}
10^{1}
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1st Order System FRF

$$G(s) = \frac{1}{Ts+1} \quad \text{transfer function (TF)}$$

$$G(i\omega) = \frac{1}{i\omega T+1} \quad \text{frequency response function (FRF)}$$

Asymptotes

Define $\omega_c = 1/T$

Low freq. asymptote $\omega \ll \omega_c$, $\omega T \ll 1$ $\quad G(i\omega) \to G_{LF}$

$$G(i\omega) = \frac{1}{i\omega T+1} \xrightarrow[\omega T \ll 1]{} \frac{1}{1} = 1$$

$$\therefore \; G_{LF}(i\omega) = 1, \quad |G_{LF}|_{dB} = 0 \quad \angle G_{LF} = 0^\circ$$

High freq. asymptote $\omega \gg \omega_c$, $\omega T \gg 1$ $\quad G(i\omega) \to G_{HF}$

$$G(i\omega) = \frac{1}{i\omega T+1} \xrightarrow[\omega T \gg 1]{} \frac{1}{i\omega T}$$

$$\therefore \; G_{HF}(i\omega) = \frac{1}{i\omega T} = \frac{1}{\omega T} e^{-i\frac{\pi}{2}}$$

$$|G_{HF}| = \frac{1}{\omega T} \quad \angle G_{HF} = -\frac{\pi}{2} = -90^\circ$$

Intersection of G_{LF} & G_{HF}

$$1 = \frac{1}{\omega_c T} \quad \omega_c = \frac{1}{T}$$ cutoff frequency, decay starts at ω_c

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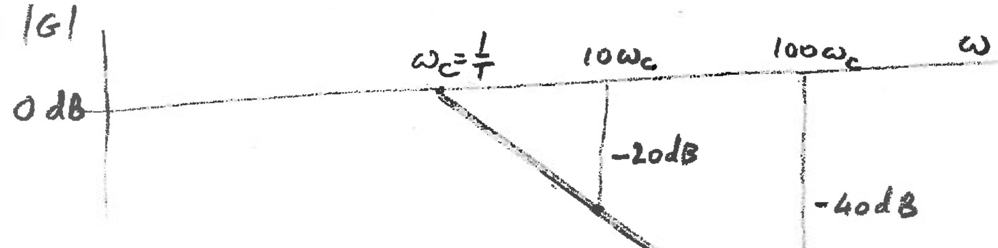
Slope of G_{HF}

$$|G_{HF}|_{dB} = -20 \log_{10}(\omega T)$$



$$|G_{HF}| = \frac{1}{\omega T}$$

$\omega_1 = \omega_c = \frac{1}{T}$, $\omega_1 T = 1$, $|G_{HF}(\omega_1 T)| = 0\,dB$

$\omega_2 = 10\omega_c$, $\omega_2 T = 10$, $|G_{HF}(\omega_2 T)| = \frac{1}{10} = -20\,dB$

$\omega_3 = 100\,\omega_c$, $\omega_3 T = 100$, $|G_{HP}(\omega_3 T)| = \frac{1}{100} = -40\,dB$

$|G_{HF}|$ drops $-20\,dB/\text{decade}$

For octave, take ω_c, $2\omega_c$, $4\omega_c$

ω_1	ω_2	ω_3
1	$\frac{1}{2}$	$\frac{1}{4}$
$0\,dB$	$-6\,dB$	$-12\,dB$.

$|G_{HF}|$ drops $-6\,dB/\text{octave}$

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3 I

Exact amplitude and phase at ω_c

$$\omega_c = \frac{1}{T}$$

$$G(i\omega_c) = \frac{1}{i\underbrace{\omega_c T}_{1}+1} = \frac{1}{1+i} = \frac{1}{\sqrt{2}} e^{-i\frac{\pi}{4}}$$

$$|G(i\omega_c)| = \frac{1}{\sqrt{2}} = -3dB$$

mag2db(1/sqrt(2))
3.0103

$$\angle G(i\omega_c) = -\frac{\pi}{4} = -45^\circ$$

Error of using asymptotes

Maximum error occurs at ω_c

Appox. value $= 0dB$

Exact value $= -3dB$

Error $= 3dB$

Bandwidth ω_B

Bandwidth is defined as the frequency below which signal does not decrease more than 3dB

For 1st order sys, $\omega_B = \omega_c = \frac{1}{T}$

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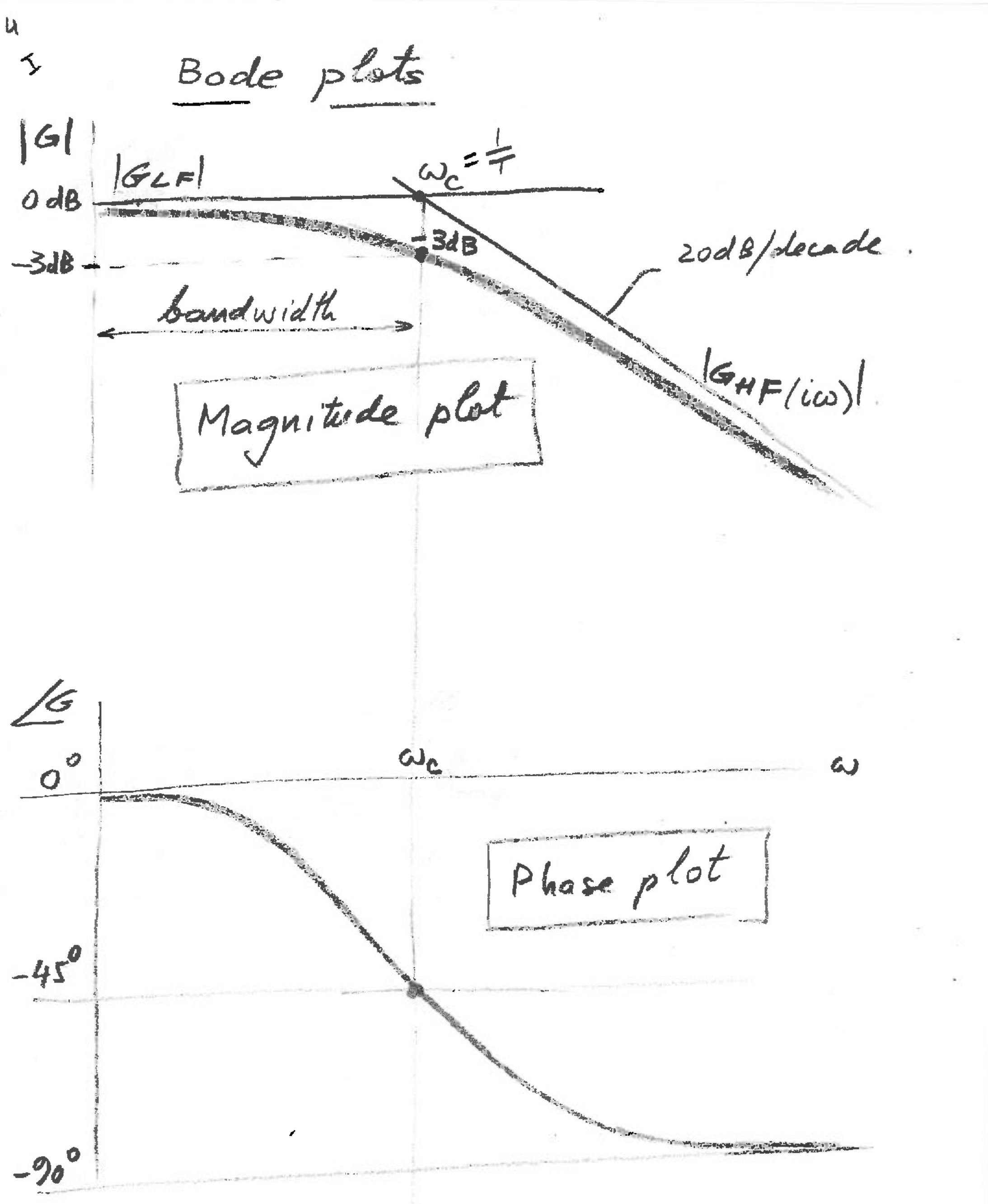
Bode plots
|G|
0 dB
-3dB
|GLF|
ωc = 1/T
-3dB
20dB/decade
bandwidth
|GHF(iω)|
Magnitude plot
∠G
0°
ωc
ω
Phase plot
-45°
-90°

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2nd order system FRF

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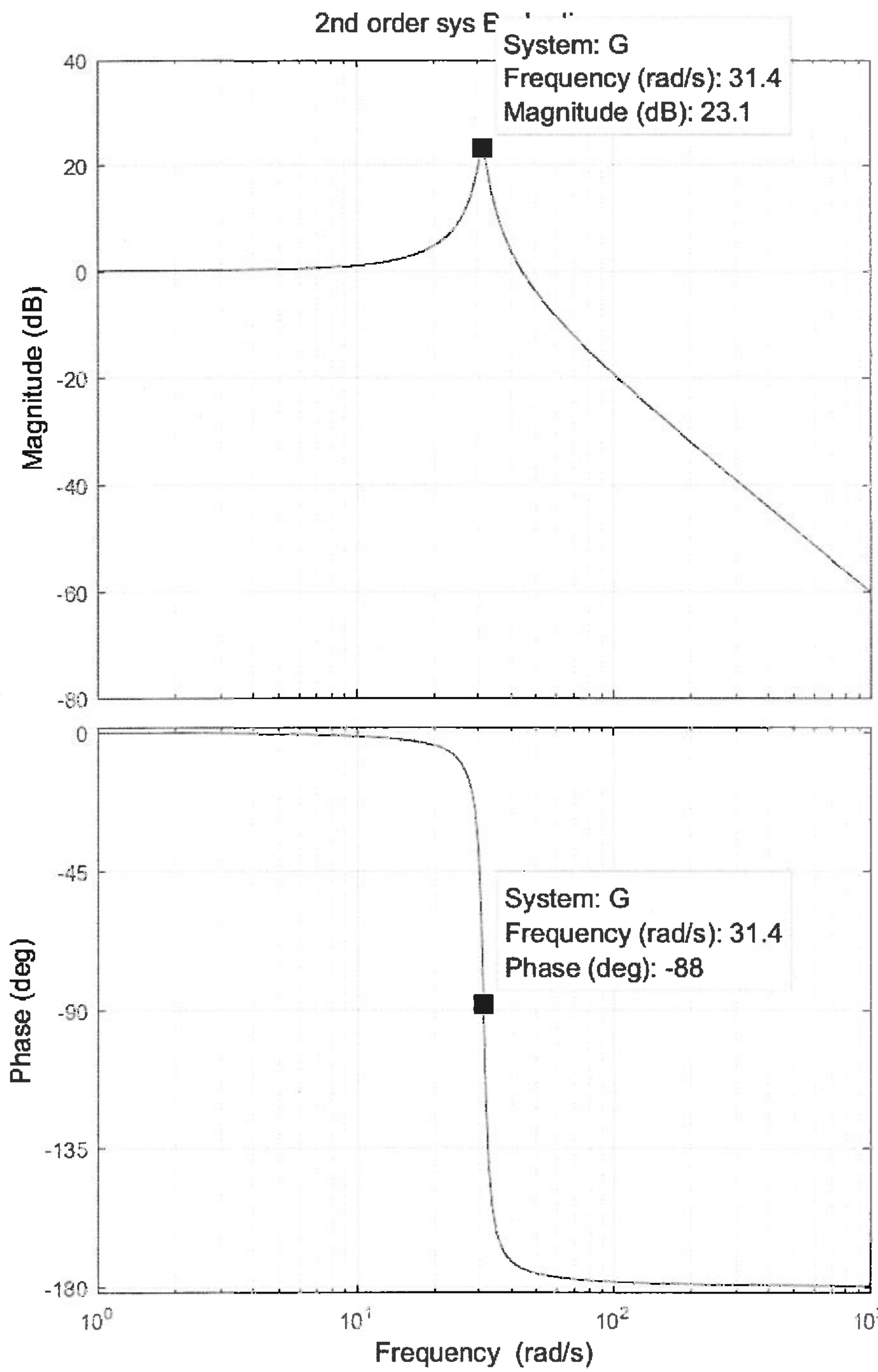
$f_n = 5\,Hz$
$\zeta = 3.5\%$



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2nd order Syst. FRF

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$G(i\omega) = \frac{\omega_n^2}{-\omega^2 + 2i\zeta\omega_n\omega + \omega_n^2}$$

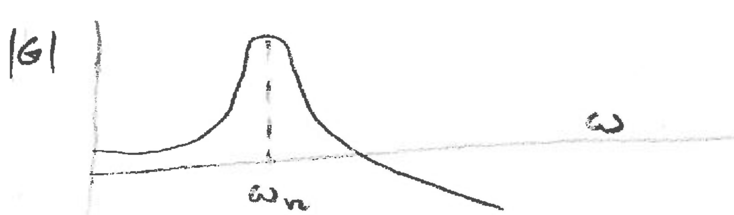
$$= \frac{\omega_n^2}{(-\omega^2+\omega_n^2) + i\,2\zeta\omega_n\omega}$$

Phase resonance: $\omega = \omega_n$ (natural freq.)

$$G(i\omega_n) = \frac{\omega_n^2}{(\cancel{-\omega_n^2+\omega_n^2}) + i\,2\zeta\omega_n\omega_n} = \frac{1}{i2\zeta}$$

$$|G(i\omega_n)| = \frac{1}{2\zeta}$$

$$\angle G(i\omega_n) = \angle \frac{1}{i2\zeta} = -90^\circ$$



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2/II

LF asymptote

$$G(i\omega) \xrightarrow[\omega \ll \omega_n]{} G_{LF}(i\omega)$$

$$G(i\omega) = \frac{\omega_n^2}{(\cancel{-\omega^2}+\omega_n^2)+\cancel{i2\zeta\omega\omega_n}} \xrightarrow[\omega \ll \omega_n]{} \qquad \frac{\omega_n^2}{\omega_n^2} = 1$$

$$G_{LF}(i\omega) = 1 \qquad |G_{LF}|_{dB} = 0\,dB$$

$$\angle G_{LF} = 0^\circ$$

HF asymptote

$$G(i\omega) \xrightarrow[\omega \gg \omega_n]{} G_{HF}(i\omega)$$

$$G(i\omega) = \frac{\omega_n^2}{(-\omega^2+\cancel{\omega_n^2})+\cancel{i2\zeta\omega\omega_n}} \xrightarrow[\omega \gg \omega_n]{} -\frac{\omega_n^2}{\omega^2}$$

$$G_{HF}(i\omega) = -\frac{\omega_n^2}{\omega^2} = \frac{\omega_n^2}{\omega^2}e^{-i\pi}$$

$$|G_{HF}(i\omega)| = \frac{\omega_n^2}{\omega^2} = \left(\frac{\omega_n}{\omega}\right)^2 = 1\Big/\left(\frac{\omega}{\omega_n}\right)^2$$

$$|G_{HF}(i\omega)|_{dB} = -40\log_{10}\left(\frac{\omega}{\omega_n}\right)$$

$\omega_2 = 10\omega_1$
$\omega_2/\omega_1 = 10$; $\log_{10}\omega_2/\omega_1 = 1$ "-40 dB/decade"

$$\angle G_{HF}(i\omega) = -\pi = -180^\circ$$

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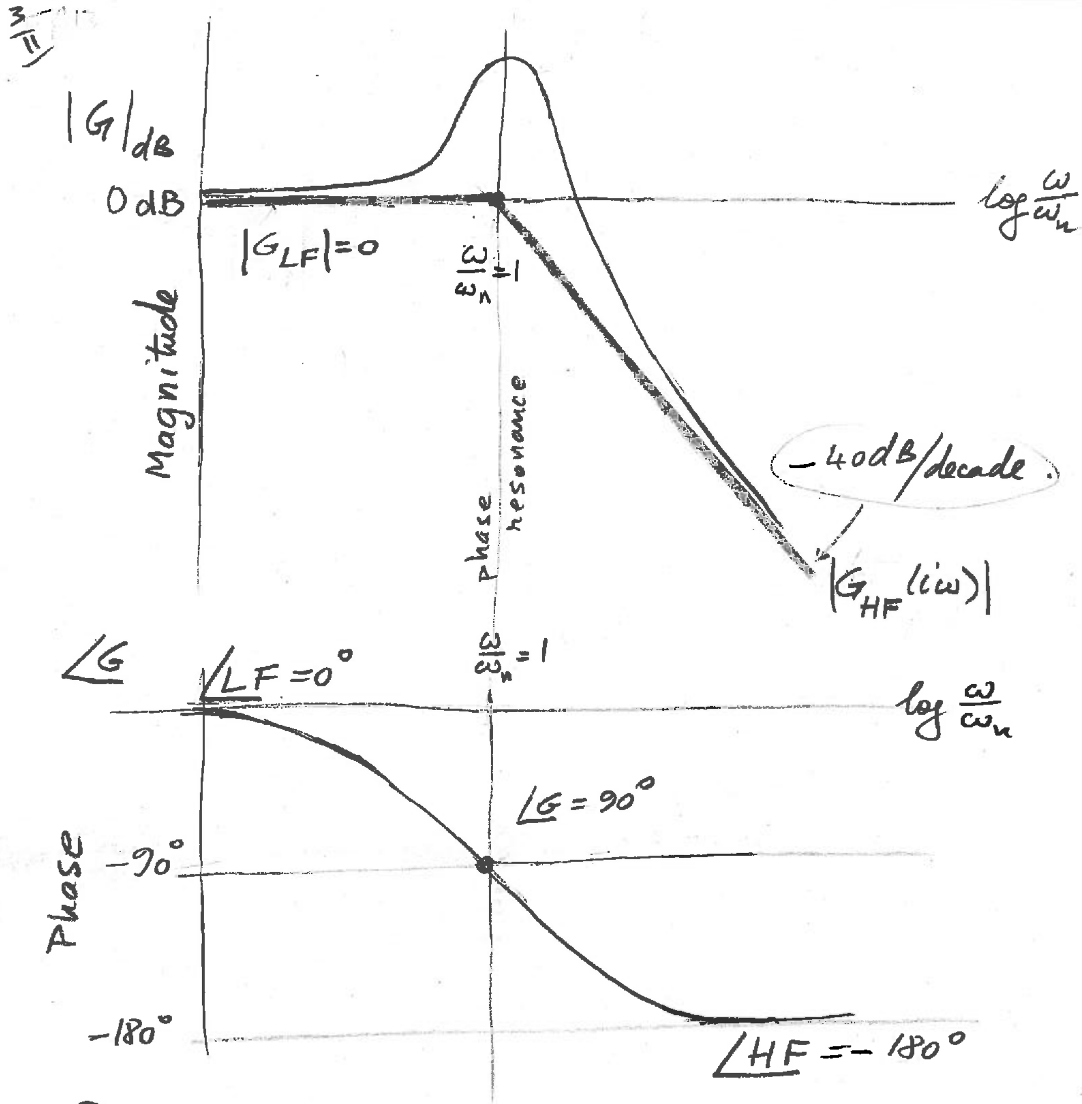

Phase

$$\angle G = -\tan^{-1}\left(\frac{2\zeta\omega\omega_n}{-\omega^2+\omega_n^2}\right) = -\tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega^2}}$$

$\frac{\omega}{\omega_n} \ll 1$, $\angle G = 0^\circ$ (LF)

$\frac{\omega}{\omega_n} = 1$, $\angle G = -90^\circ$ 'phase resonance'. ($\tan^{-1}\infty = 90^\circ$)

$\frac{\omega}{\omega_n} \gg 1$, $\angle G = -180^\circ$ (HF)

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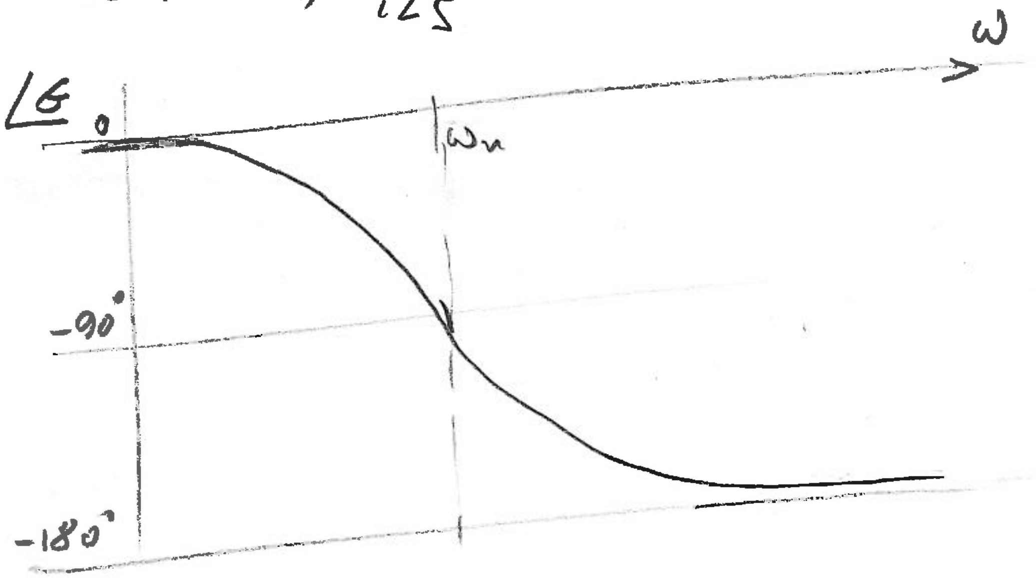
Phase diagram.

$$G(i\omega) = \frac{\omega_n^2}{(-\omega^2+\omega_n^2) + i2\zeta\omega\omega_n}$$

$$G_{LF}(\omega) = 1 \qquad \angle 1 = 0^\circ$$

$$G_{HF}(\omega) = -\frac{\omega_n^2}{\omega^2} \qquad \angle -1 = -180^\circ$$

$$G(\omega=\omega_n) = \frac{1}{i2\zeta} \qquad \angle \frac{1}{i} = -90^\circ$$



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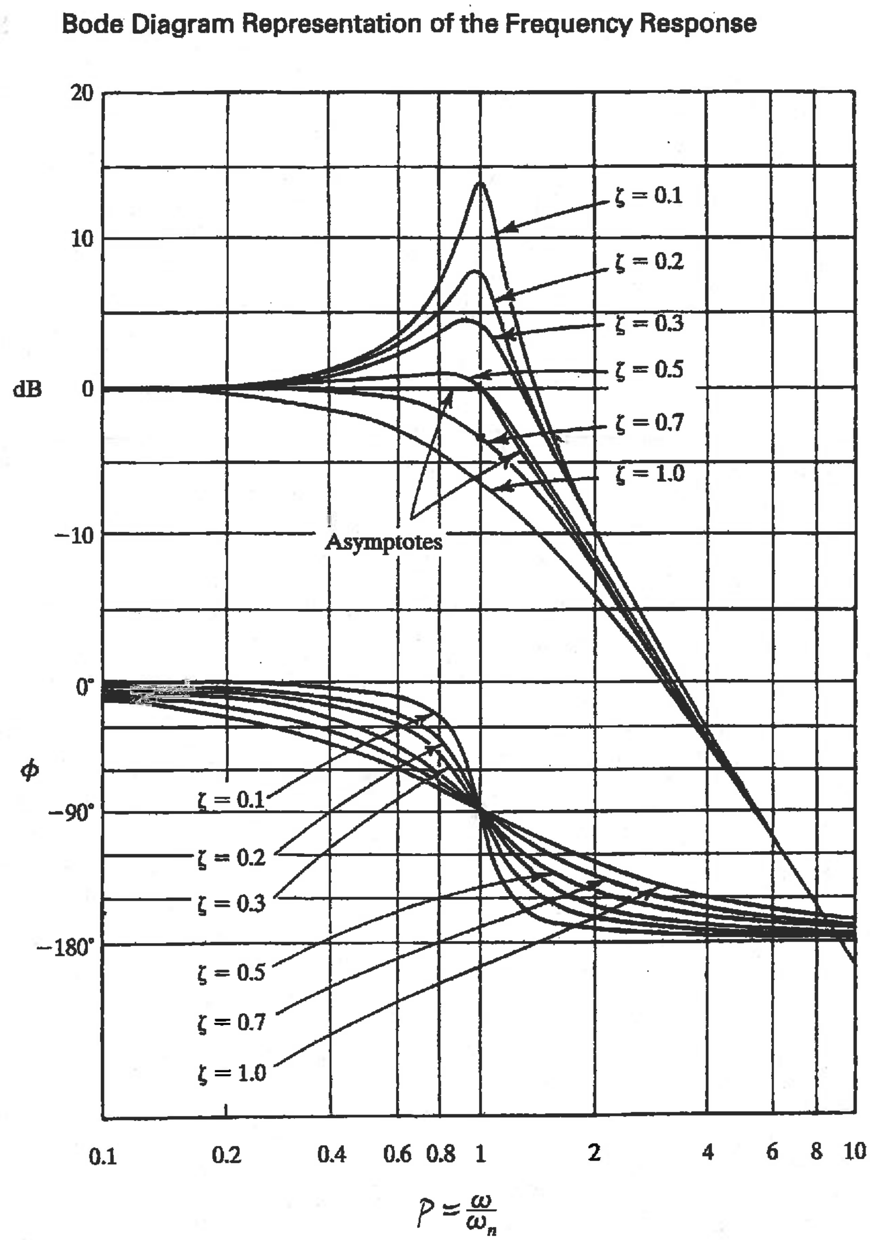
4 a
II



Log-magnitude curves together with the asymptotes and phase-angle curves of the quadratic sinusoidal transfer function

MATLAB FD2

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8.3 Performance Indicators in Frequency Domains

PIF

Generic Performance
Indicators in Freq. Domain

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p18

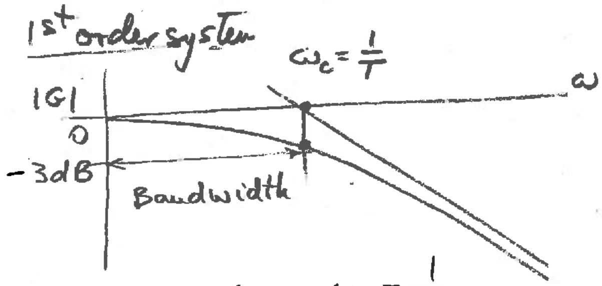
Generic Performance Indicators in Freq. Domain

Bandwidth and cutoff frequency, ω_B

$$|G(\omega_B)| = G(0) - 3dB$$

has declined 3dB below LF value

1st order system

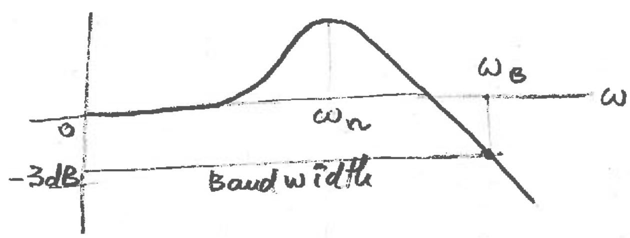


$$G(s) = \frac{1}{Ts+1}$$

$$\omega_B = \omega_c = \frac{1}{T}$$

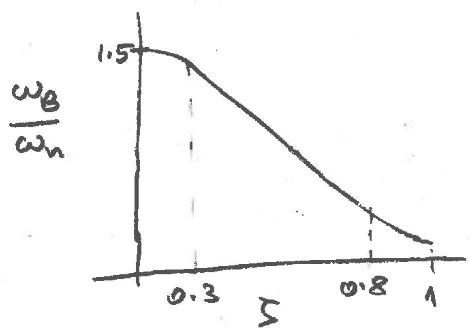
2nd order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



$$|G(i\omega)|_{dB} = -20\log_{10}\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2} = -3dB$$

ω_B is determined graphically or solved numerically



$$\frac{\omega_B}{\omega_n} \approx -1.19\zeta + 1.85$$

$$0.3 < \zeta < 0.8$$

$\omega_B \uparrow$ as $\zeta \downarrow$

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3 PIF Ex. 11.2 : 1st order system performance

Given: two systems:

$$G_1 = \frac{1}{s+1} \quad ; \quad G_2 = \frac{1}{3s+1}$$

Find: (a) bandwidth

(b) frequency response

(c) step response

(d) ramp response

(e) discuss results

Solution:

$$G_1(i\omega) = \frac{1}{i\omega+1} = \frac{1}{i\omega T_1+1} \rightarrow T_1 = 1$$

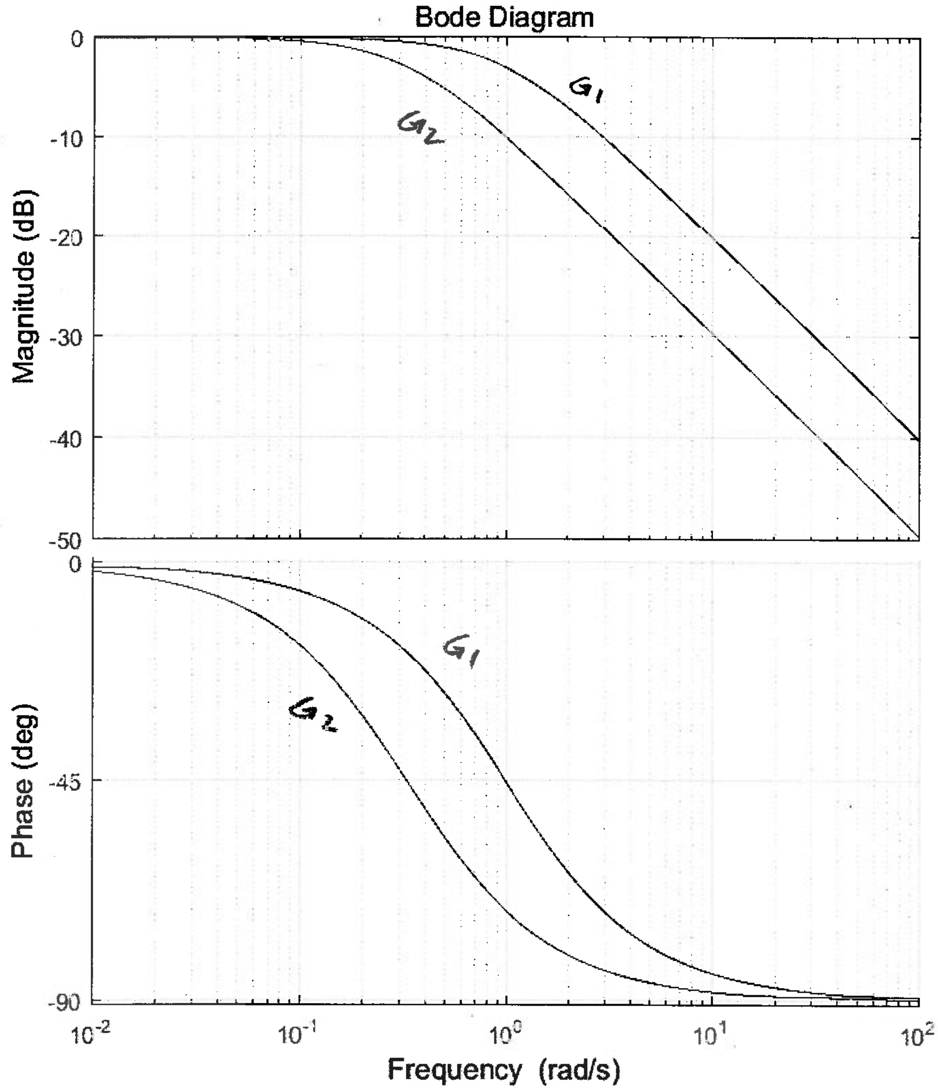
$$G_2(i\omega) = \frac{1}{3i\omega+1} = \frac{1}{i\omega T_2+1} \longrightarrow T_2 = 3$$

(a) $\omega_B = \frac{1}{T}$ $\quad \omega_1^B = \frac{1}{1} = 1\ \text{rad/sec} \quad \omega_2^B = \frac{1}{3}\ \text{rad/sec}$

(b),(c),(d): see MATLAB Ex. 11.2

(e) system 1 has bandwidth three times larger than system 2 ($\omega_1 = 1$ vs. $\omega_2 = 1/3$ rad/sec).
Sys. 1 has faster step response and follows the ramp input much better / smaller ramp error)

4
PIF


Bode Diagram
Magnitude (dB)
0
-10
-20
-30
-40
-50
G1
G2
Phase (deg)
0
-45
-90
10-2
10-1
100
101
102
Frequency (rad/s)

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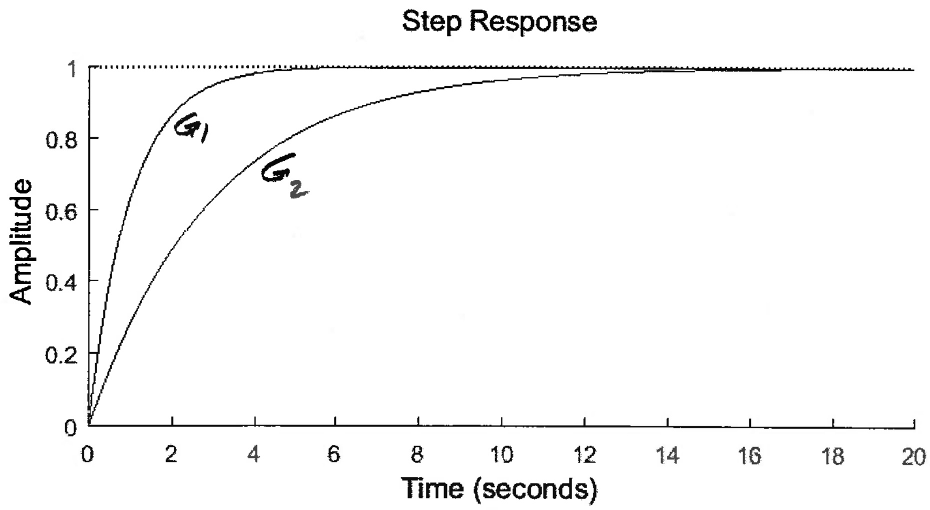
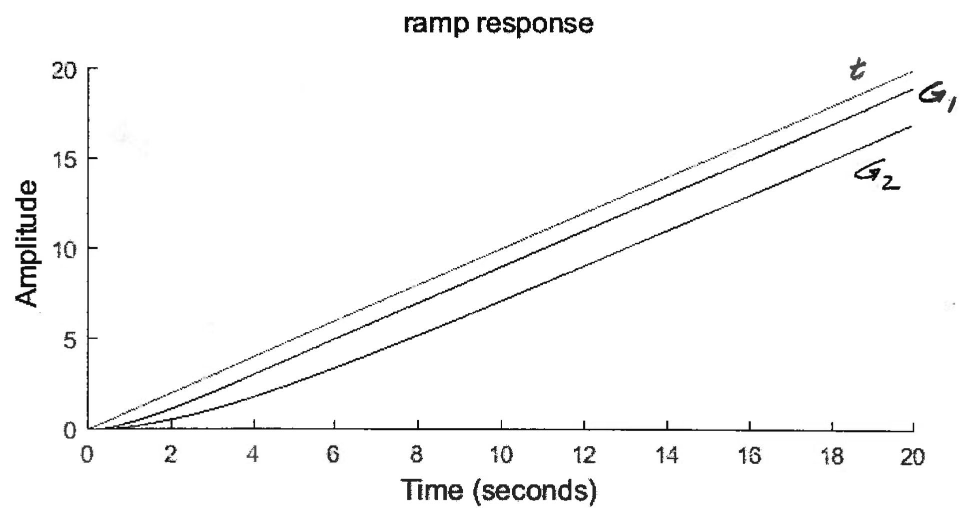
3
11.2


Step Response
Amplitude
Time (seconds)
G1
G2


ramp response
Amplitude
Time (seconds)
t
G1
G2

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C:\Mydata\1_USC...\Example11_2_p628_20161216.m Page 1

```
%{
 EXAMPLE 11.2
 1st Order System analysis
%}
%% initialization
clc
clear all
% close all
s=tf('s');
%% system definition
G1=1/(s+1); G2=1/(3*s+1);
%% Bode plots
figure(1)
bode(G1,G2)
box off
grid on
%% time response
figure(2)
Tfinal=20;
dt=0.1; t=0:dt:Tfinal; u=1.*t;
subplot(2,1,1)
step(G1,G2,Tfinal)
box off
subplot(2,1,2)
lsim(G1,G2,u,t)
axis([0 Tfinal 0 Tfinal]); title( 'ramp response' )
box off
```

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6PIF

Ex: 2nd order Sys. Performance

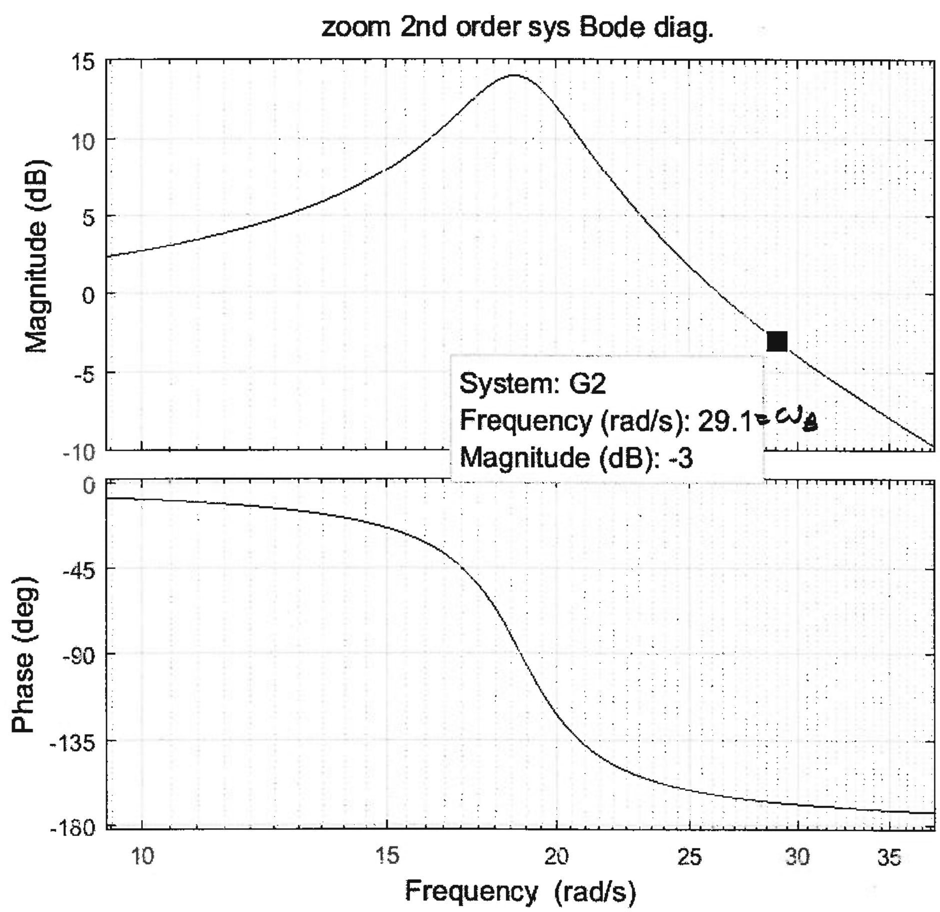
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$\omega_n = 2\pi f_n$

$f_n = 3\ Hz$

$\zeta = 10\%$

$\omega_B = 29.1\ Hz.$



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7
PIF

Specific Performance Indicators in Freq. Domain for 2nd Order Systems.

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8 PIF

Resonance: peak response
max. response

$$|G(i\omega)|^2 = \frac{1}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}.$$

$$\frac{d}{d\omega}|G(i\omega)|^2 = 0 \quad \text{for peak}$$

$$\frac{d}{d\omega}\left[\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2\right] = 0.$$

use auxiliary variable $p = \frac{\omega}{\omega_n}$ and write

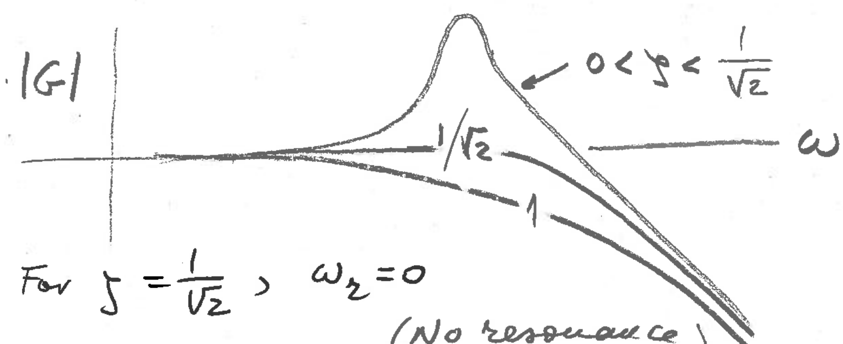
$$\frac{d}{dp}\left[(1-p^2)^2 + (2\zeta p)^2\right] = 0 .$$

$$2(-2p)(1-p^2) + 2(2\zeta)(2\zeta p) = 0 .$$

$$(1-p^2)\cancel{p} = 2\zeta^2\cancel{p}$$

$$p^2 = 1-2\zeta^2 \longrightarrow p_r = \sqrt{1-2\zeta^2}$$

$$\omega_r = \omega_n\sqrt{1-2\zeta^2}$$



For $\zeta = \frac{1}{\sqrt{2}}$, $\omega_r = 0$
(No resonance).

Peak exist only for $0<\zeta<\frac{1}{\sqrt{2}}$

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9 PIF

Amplitude at resonance

$$M_r^2 = |G(i\omega_r)|^2 = \frac{1}{[1-(1-2\zeta^2)]^2 + (2\zeta)^2(1-2\zeta^2)}$$

$$= \frac{1}{(\cancel{1}-\cancel{1}+2\zeta^2)^2 + 4\zeta^2(1-2\zeta^2)}$$

$$= \frac{1}{4\zeta^4 + 4\zeta^2 - 8\zeta^4} = \frac{1}{4\zeta^2(1-\zeta^2)}$$

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

$$\zeta = \frac{1}{\sqrt{2}}$$

$$M_r = \frac{1}{2\frac{1}{\sqrt{2}}\sqrt{1-\frac{1}{2}}} = 1$$

Phase at resonance, φ_r

$$G(i\omega) = \frac{\omega_n^2}{(i\omega)^2 + 2i\zeta\omega\omega_n + \omega_n^2} = \frac{1}{(1-p^2)+2i\zeta p}$$

$$p_r = \sqrt{1-2\zeta^2}, \quad 1-p_r^2 = 1-(1-2\zeta^2) = 2\zeta^2$$

$$G(i\omega_r) = \frac{1}{2\zeta^2 + 2i\zeta\sqrt{1-2\zeta^2}}$$

$$\varphi_r = \angle G(i\omega_r) = -\tan^{-1}\frac{\sqrt{1-2\zeta^2}}{\zeta}$$

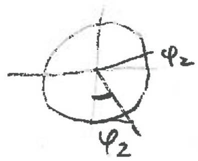
$$\varphi_r = -90^\circ + \underbrace{\sin^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}}}_{=\varphi_2} = -90^\circ + \varphi_2$$

Proof

$$\tan(-90^\circ + \varphi_2) = -\cot\varphi_2$$

$$\sin\varphi_2 = \frac{\zeta}{\sqrt{1-\zeta^2}}$$

$$\cos^2\varphi_2 = 1 - \sin^2\varphi_2 = \frac{1-2\zeta^2}{1-\zeta^2}$$



$$\cot\varphi_2 = \frac{\cos\varphi_2}{\sin\varphi_2} = \frac{\sqrt{1-2\zeta^2}/\sqrt{1-\zeta^2}}{\zeta/\sqrt{1-\zeta^2}} = -\tan\varphi_r$$

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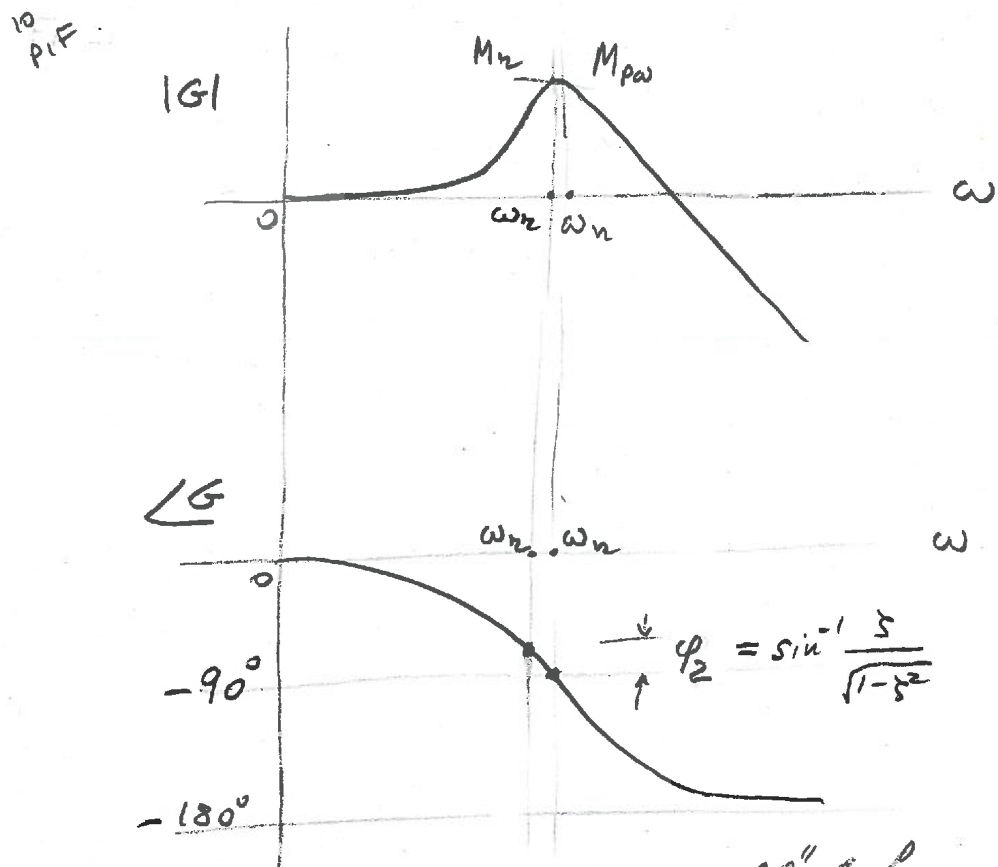

Resonance happens "slightly" before ω_n

Two definitions of "resonance":

(1) 90° phase @ ω_n, $M_{p\omega} = \frac{1}{2\zeta}$
(phase resonance)

(2) peak value @ $\omega_r = \omega_n \sqrt{1-2\zeta^2}$

$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$ "RESONANCE"

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PERFORMANCE INDICATORS VS. ζ

ζ	0.01	0.1	0.4	$1/\sqrt{2}$	0.8	0.99	1
$M_{p\omega} = \frac{1}{2\zeta}$	50	5	1.25	0.7	0.625	0.505	0.5
$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$	50	5.025	1.36	1	—	—	—
$1+M_p = 1+e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$	1.97	1.73	1.25	1.04	1.02	1.00	1
$\frac{\omega_r}{\omega_n} = \sqrt{1-2\zeta^2}$	1	0.99	0.825	0	—	—	—
$\frac{\omega_d}{\omega_n} = \sqrt{1-\zeta^2}$	1	0.995	0.917	$1/\sqrt{2}$	0.6	0.141	0
$\frac{t_s}{T_n} = \frac{2}{\pi\zeta}$	63	6.37	1.59	0.9	0.8	0.64	0.64
$\varphi_r = -\tan^{-1}\frac{\sqrt{1-2\zeta^2}}{\zeta}$	-89°	-84°	-64°	0°	—	—	—

$$t_s = \frac{4}{\zeta\omega_n}$$

$$T_n = \frac{1}{f_n} = \frac{2\pi}{\omega_n} \text{ (osc. period)}$$

$$\frac{t_s}{T_n} = \frac{4}{\zeta}\frac{1}{2\pi} = \frac{2}{\pi\zeta}$$

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12
PIF

Specific PIs vs ζ 2nd order system

```
performance indicators comparison, 2nd order sys
z =
    0.0100    0.1000    0.4000    0.7071    0.8000    0.9900    1.0000
Mpw =
   50.0000    5.0000    1.2500    0.7071    0.6250    0.5051    0.5000
Mr =
   50.0025    5.0252    1.3639    1.0000         0         0       NaN
1+Mp =
    1.9691    1.7292    1.2538    1.0432    1.0152    1.0000    1.0000
wr/wn =
    0.9999    0.9899    0.8246    0.0000         0         0         0
wd_wn =
    0.9999    0.9950    0.9165    0.7071    0.6000    0.1411         0
ts_tn =
   63.6620    6.3662    1.5915    0.9003    0.7958    0.6431    0.6366
phi_r =
  -89.4270  -84.2318  -64.1233    0.0000         0         0         0
```

$M_{p\omega} = \frac{1}{2\zeta}$ magnitude at phase res.

$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$ resonance peak

$1+M_p = 1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}}$ step response peak

$\frac{\omega_r}{\omega_n} = \sqrt{1-2\zeta^2}$ resonance freq. ratio

$\frac{\omega_d}{\omega_n} = \sqrt{1-\zeta^2}$ damped freq / natural freq

$t_s/\tau_n = \frac{2}{\pi\zeta}$ settling time / osc. period

$\varphi_r = -\tan^{-1}\frac{\sqrt{1-2\zeta^2}}{\zeta}$ phase at resonance

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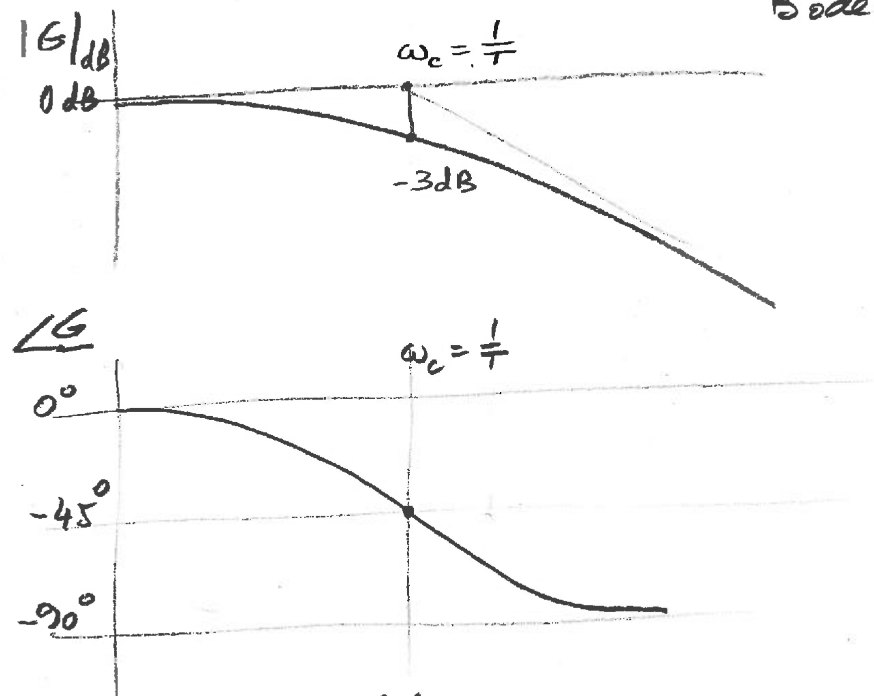
8.4 System Identification in Frequency Domains

IDF

1st order sys ID in Freq Domain

$$G(i\omega) = \frac{1}{i\omega T + 1}$$

Bode plots



Given : Bode plots

Find : T

Sol^{n} : Read: -3dB point on $|G|_{dB}$ plot

45° point on $\angle G$ plot

Estimate ω_c

Calculate $T = \frac{1}{\omega_c}$

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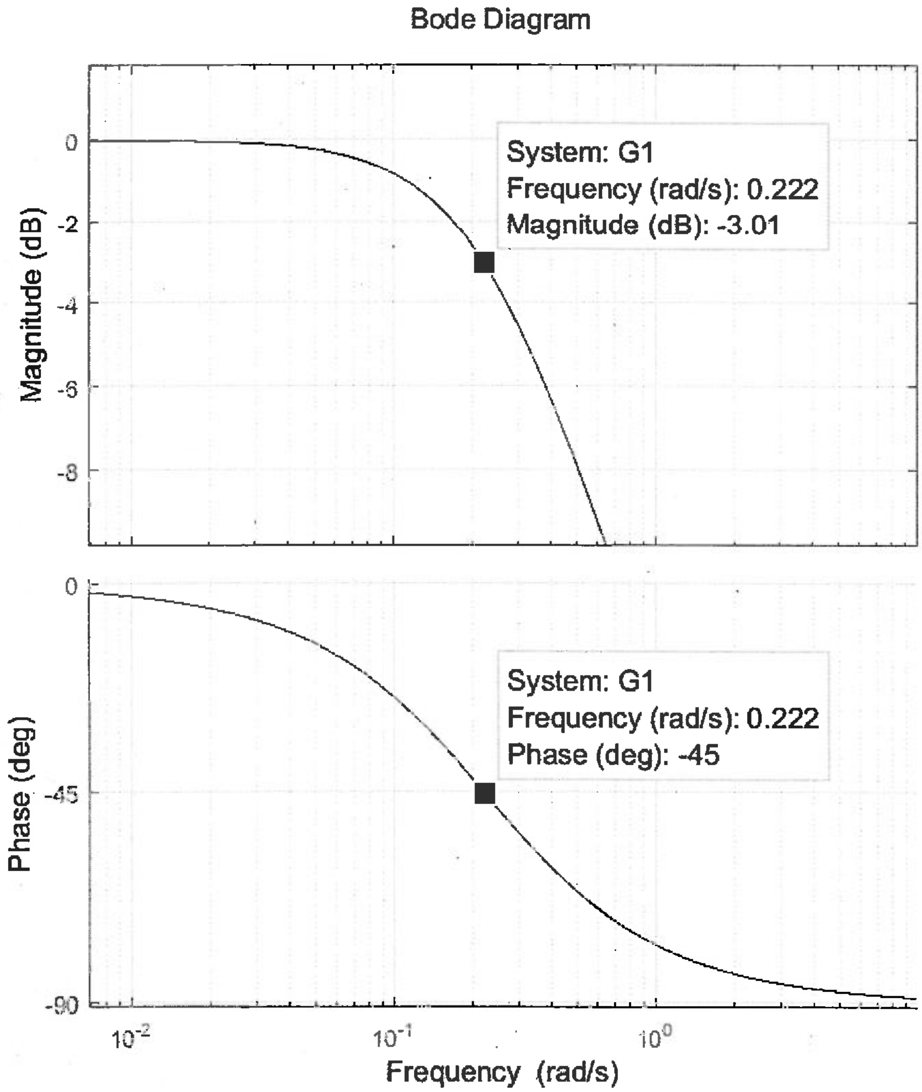
Ex: ID 1st order sys. $G(s) = \frac{1}{4.5s+1}$



$\omega_c = 0.222$

$T = \frac{1}{\omega_c} = 4.5045$

error : -0.1%

In practice, error may be larger due to noise.

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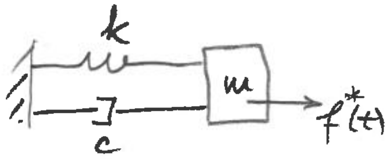
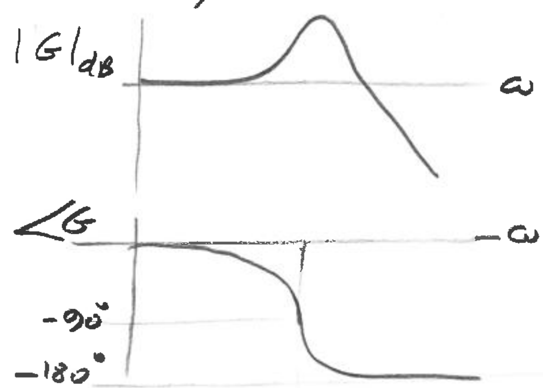
3 IDF Ex: ID 2nd order sys





Given: Bode diagram FRF

Find: m, c, k.

Solution: recall FBD, EOM

$$m\ddot{x} + c\dot{x} + kx = f^*(t)$$

$$\mathcal{L}T \quad (ms^2 + cs + k)X(s) = F^*(s)$$

$$G(s) = \frac{X(s)}{F^*(s)} = \frac{1}{ms^2 + cs + k}$$

$$G(i\omega) = \frac{1}{-m\omega^2 + i\omega c + k} \quad ; \quad G(i0) = \frac{1}{k}$$

$$\omega_n^2 = k/m \qquad G(i\omega_n) = \frac{1}{i\omega_n c} \; ; \; |G(i\omega_n)| = \frac{1}{\omega_n c}$$

Numerical example: $m = 2\,kg$, $c = 4\,\frac{N}{m/s}$, $k = 20\,\frac{N}{m}$

Run MATLAB SYS_ID_2ndOrder_Ex11_1

Read on Bode diagram:

$$|G(i0)|_{dB} = -26\,dB$$

$$\omega_n = 3.16\ rad/sec$$

$$|G(i\omega_n)|_{dB} = -22\,dB$$

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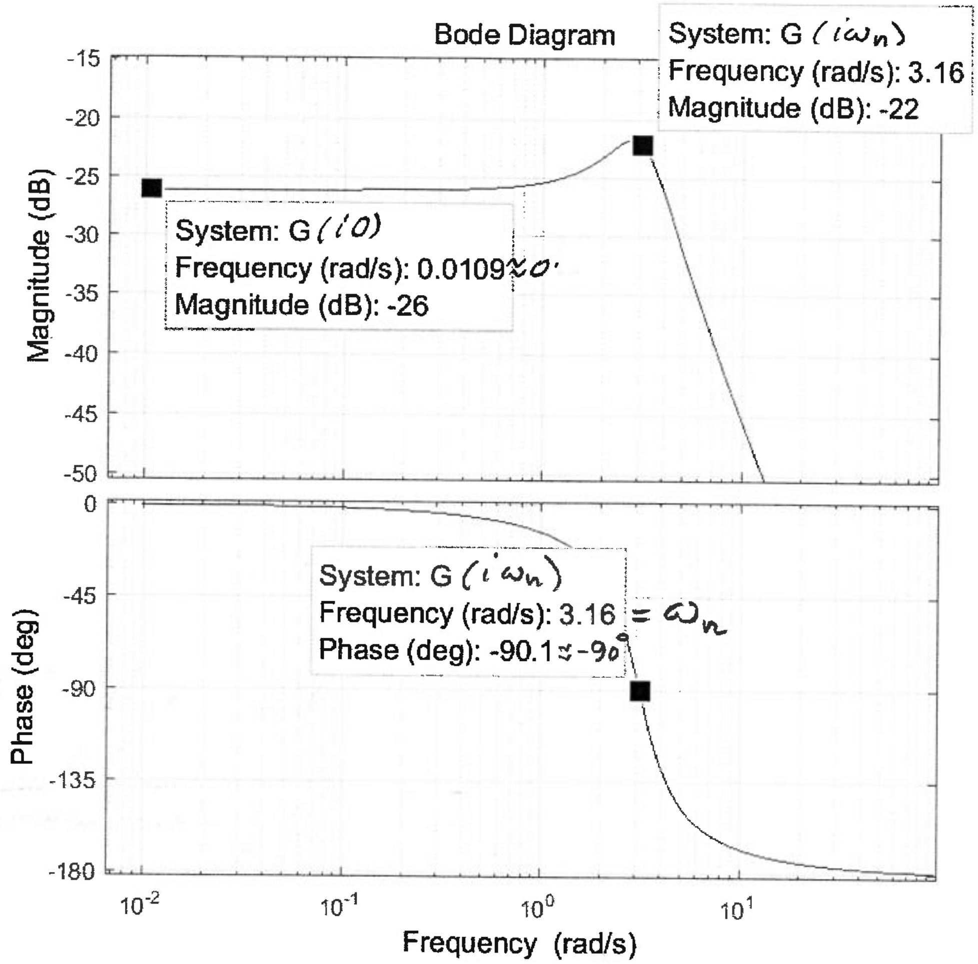
4
IDF



$$(M_{p\omega})_{dB} = |G(i\omega_n)|_{dB} - |G(i0)|_{dB}$$

$$= -22dB - (-26dB) = +4dB$$

5
1DF calculate · k_1, ω_1, c_1

$$k_1 = \frac{1}{G(i0)} = 1/db2mag(\,G(i0)_{dB})$$

$$= 19.9526\ N/m \approx 20\ N/m \quad \checkmark$$

$\omega_n^2 = k/m$

$$m_1 = \frac{k_1}{\omega_n^2} = 1.9981\ kg \approx 2\ kg \quad \checkmark$$

$$c_1 = \frac{1}{\omega_n\,|G(i\omega_n)|} = 1/\omega_n/db2mag\left(G(i\omega_n)_{dB}\right) \doteq 3.9839$$

or

$$\approx 4\ \frac{N}{m/s}$$

$$(M_{p\omega})_{dB} = |G(i\omega_n)|_{dB} - |G(i0)|_{dB}$$

$$M_{p\omega} = db2mag\left(|G(i\omega_n)|_{dB} - |G(i0)|_{dB}\right)$$

$$M_{p\omega} = \frac{1}{2\zeta} \longrightarrow \zeta = \frac{1}{2M_{p\omega}}$$

$$c_2 = 2\zeta\,\omega_n\,m_1 = 3.9920\ \frac{N}{m/s} \approx 4\ \frac{N}{m/s} \quad \checkmark$$

All three parameters of the system, k, m, c have been recovered with quite acceptable error.

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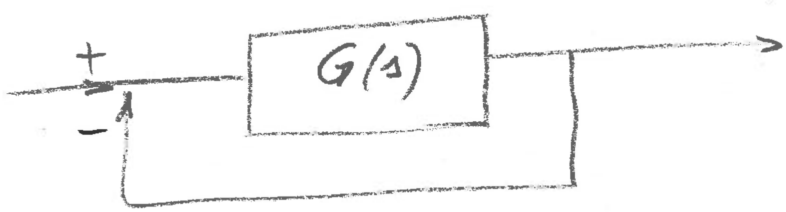
8.5 Frequency Domain Analysis of Feedback System Stability

STABILITY ANALYSIS
IN FREQ. DOMAIN

Evaluate stability of CL system by analysing OL system in freq. domain



Two methods:

- Nyquist criterion
- Gain & phase margins

NYQUIST ANALYSIS OF FEEDBACK STABILITY

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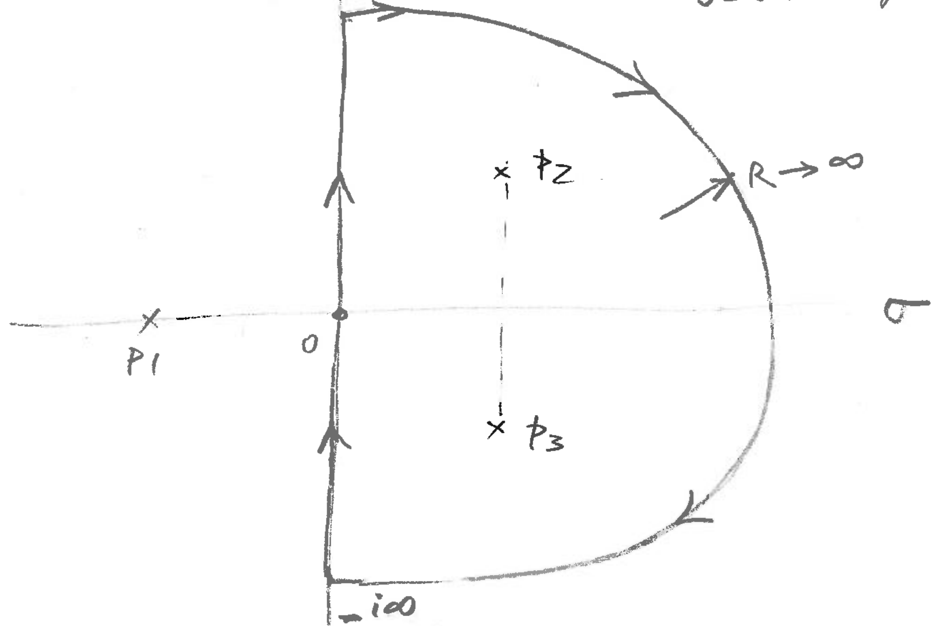
Nyquist circuit

$$G(s) = \frac{1}{(s-p_1)(s-p_2)(s-p_3)}$$

Assume
$p_1 \in \mathbb{R}$, $p_1 < 0$
$p_2, p_3 \in \mathbb{C}$ in RHS
$p_3 = \bar{p}_2$

$s = \sigma + i\omega$ plane



Nyquist circuit is:

- semi-circle path with $R \to \infty$
- in the RHS of s-plane
- along vertical axis from $-i\infty$ to $+i\infty$
- traveled clockwise (CW)

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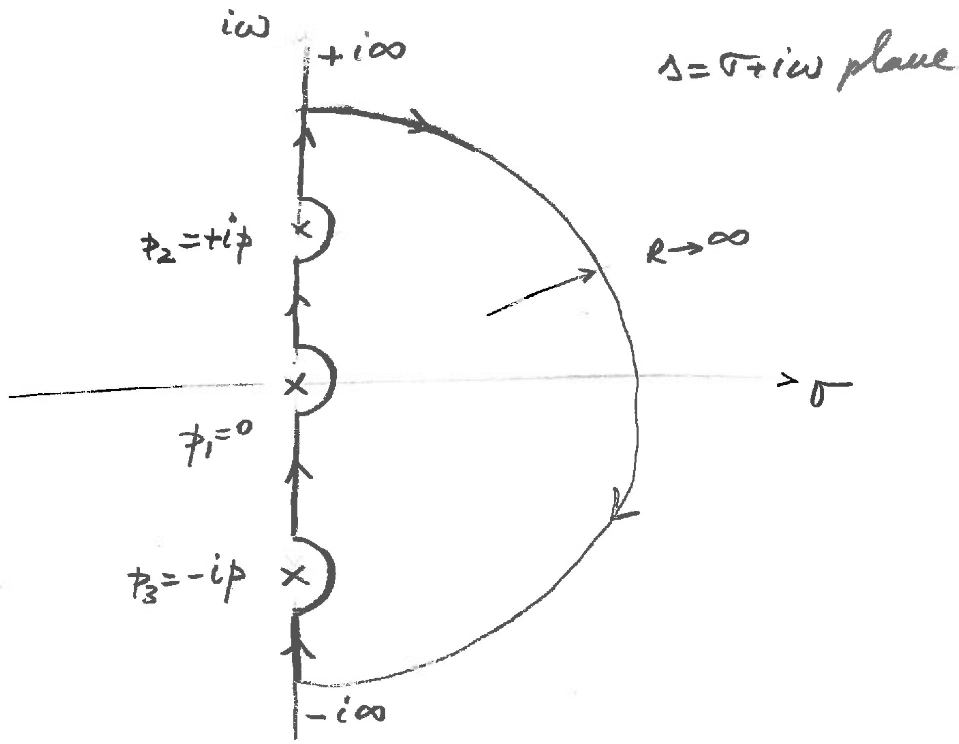
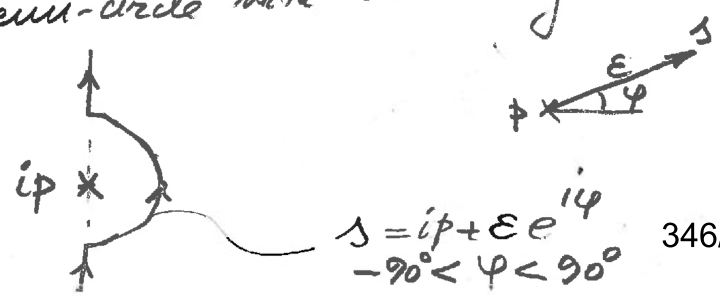
Nyquist circuit with poles on the vertical axis

$$G(s) = \frac{1}{s(s-ip)(s+ip)}$$



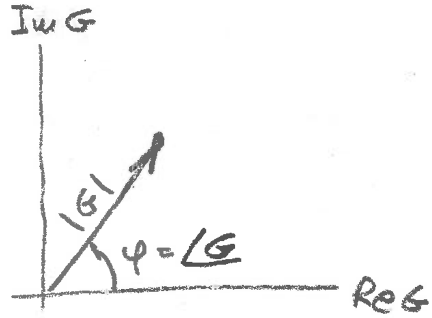
The poles on the vertical axis must be excluded. We travel around them in the RHS. To do so, we follow a small semi-circle with vanishing radius ε

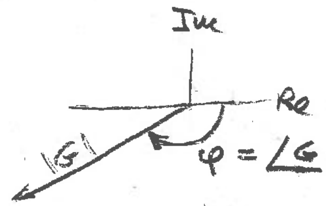


$$s = ip + \varepsilon e^{i\varphi}$$
$$-90^\circ < \varphi < 90^\circ$$

POLAR PLOTS

$$G(i\omega) = \mathrm{Re}\, G(i\omega) + i\, \mathrm{Im}\, G(i\omega)$$



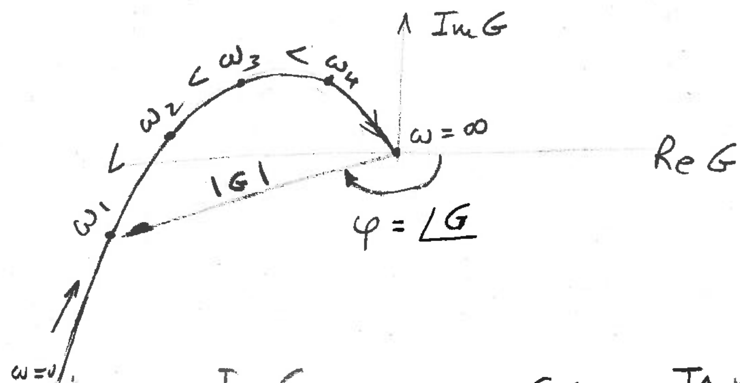


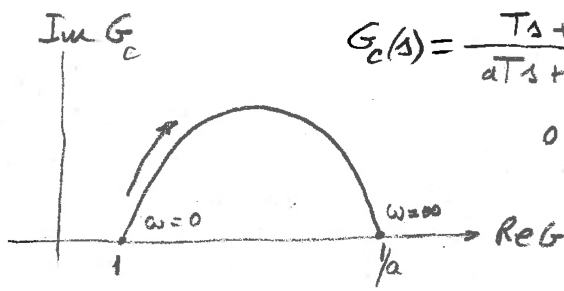
-ve value

$-\pi < \text{angle}(G) < \pi$

-180° 180°

(see MATLAB Help)





$$G_c(s) = \frac{Ts+1}{\alpha Ts+1}$$

$$0 < \alpha < 1$$

Run MATLAB examples.

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NPP

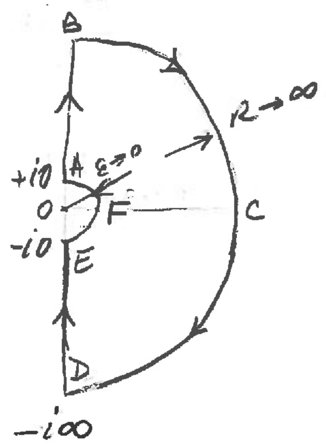
NYQUIST POLAR PLOTS

"s follows N-circuit ; G(s) follows N-polar plot"

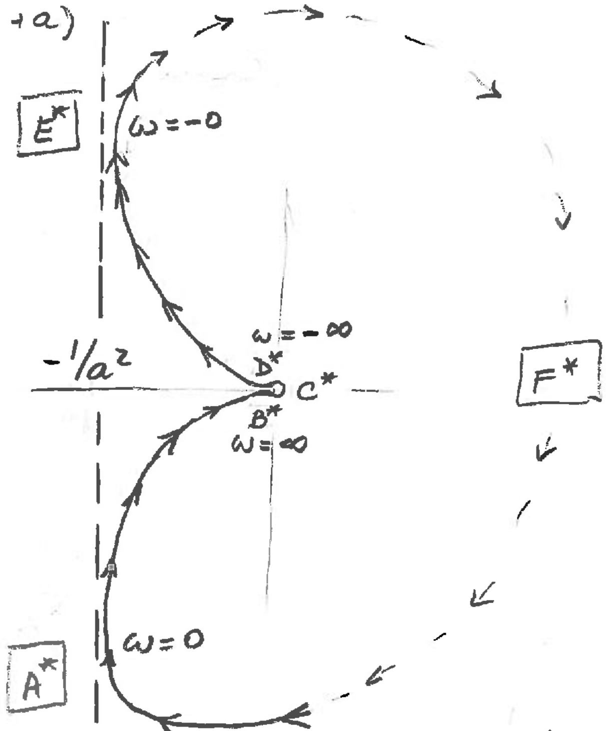
Example: $G(s) = \dfrac{1}{s(s+a)}$

2 poles: $p_1 = 0$, $p_2 = -a$

Nyquist circuit



Nyquist polar plot



- The variable s follows the Nyquist circuit ABCDEFA clockwise in the s-plane
- The function G(s) follows the resulting circuit in the G-plane

In this example, we distinguish 4 segments to be analyzed individually and then assembled in one continuous circuit

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2 NPP

Segment $(\overset{A}{+i\varepsilon}, \overset{B}{+iR})$ $\quad \varepsilon \to 0 \quad R \to \infty$

$$G(s) = \frac{1}{s(s+a)} = \frac{1}{as} - \frac{1}{a(s+a)}$$

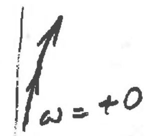
$$\boxed{A} \quad G(i\varepsilon) = \frac{1}{ai\varepsilon} - \frac{1}{a(\cancel{i\varepsilon}+a)} \quad \varepsilon \ll a$$

$$G(i\varepsilon) \simeq -\frac{1}{a^2} - i\frac{1}{a\varepsilon}$$

$$G(+i0) = \lim_{\varepsilon \to 0} G(i\varepsilon) = -\frac{1}{a^2} - i\infty$$

$$|G(+i0)| = \infty$$

$$\angle G(+i0) = \angle -i = -90^\circ$$

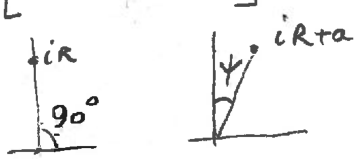


$$\boxed{B} \quad G(iR) = \frac{1}{iR(iR+a)}$$

$$|G(iR)| \underset{R \gg a}{\approx} \frac{1}{R^2} \xrightarrow[R \to \infty]{} 0$$

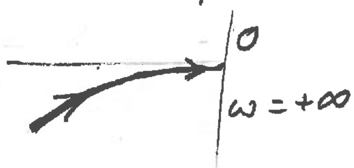
$$\angle G(iR) = -\left[\angle iR + \angle iR+a\right]$$

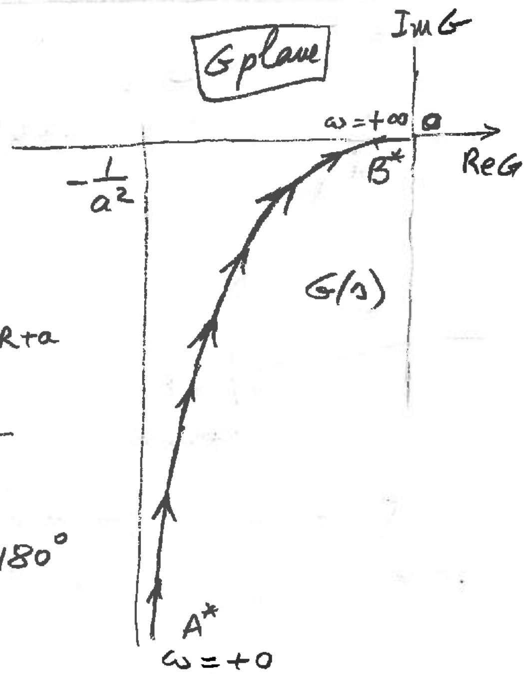
$\varphi = \tan^{-1} \frac{a}{R}$



$$\angle G(iR) = -(90^\circ + 90^\circ - \psi)$$

$$= -180^\circ + \psi \xrightarrow[\substack{R \to \infty \\ \psi \to 0}]{} -180^\circ$$





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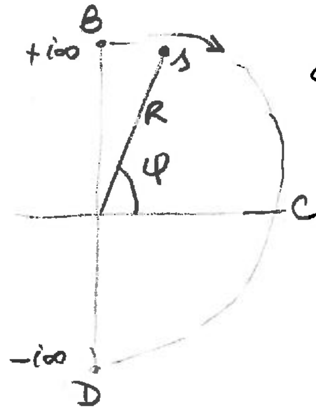
3 NPP Segment $(\overset{B}{iR}, \overset{C}{R}, \overset{D}{-iR})$ Big circle
$R \to \infty$

$$s = Re^{i\varphi}, \quad \varphi \in (90^\circ \;\; 0 \;\; -90^\circ)$$

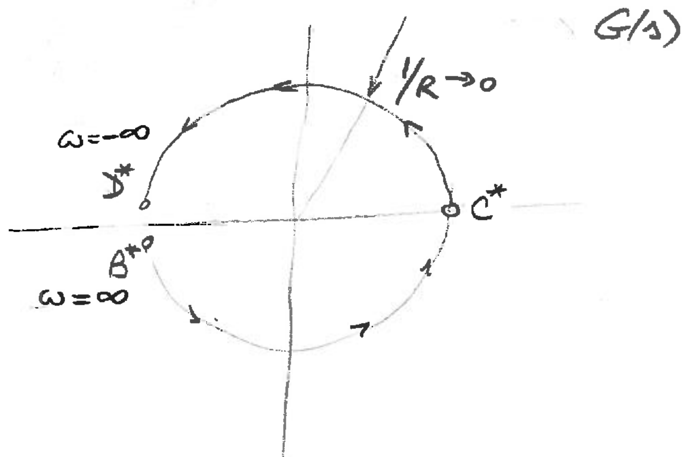


$$G(s) = \frac{1}{Re^{i\varphi}(Re^{i\varphi}+a)}$$

$$\underset{R\to\infty}{\approx} \frac{1}{R^2 e^{2i\varphi}} = \frac{1}{R^2} e^{-i2\varphi}$$

$$\angle G(s) = -2\varphi$$

$$\angle G(s) \in (\underset{B^*}{-180^\circ} \;\; \underset{C^*}{0^\circ} \;\; \underset{D^*}{180^\circ})$$



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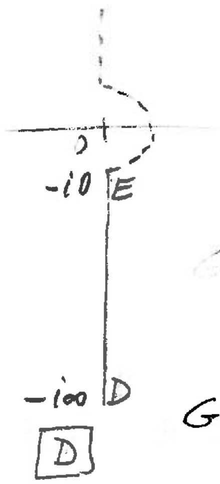
NPP k

Segment $(\overset{D}{\underset{R\to\infty}{-iR}},\ \overset{E}{\underset{\varepsilon\to 0}{-i\varepsilon}})$

s-plane



$$G(s)=\frac{1}{s(s+a)}=\frac{1}{as}-\frac{1}{a(s+a)}$$

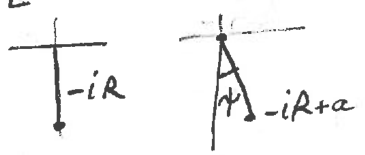
$$G(s)\Big|_{s=-iR}=\frac{1}{-iR(-iR+a)}$$

$\boxed{D}$

$$|G(s)|=\frac{1}{R^2}\xrightarrow[R\to\infty]{}0$$

$$\angle G(s)=-\left(\angle -iR+\angle -iR+a\right)$$

$$=-\left[-90^\circ+(-90^\circ+\psi)\right]=180^\circ-\psi\xrightarrow[\substack{R\to\infty\\ \psi\to 0}]{}180^\circ$$



$$\psi=\tan^{-1}\frac{a}{R}$$

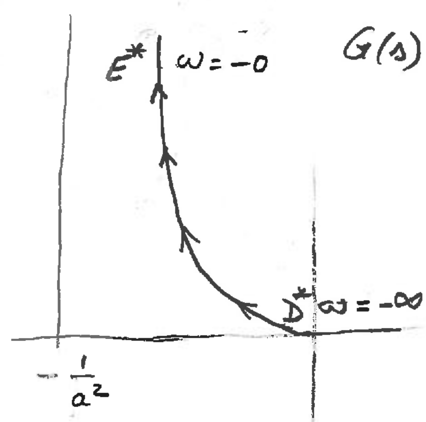
$\boxed{E}$ $$G(-i\varepsilon)=-\frac{1}{a(-i\varepsilon)}-\frac{1}{a(-i\varepsilon+a)}=\frac{i}{a\varepsilon}-\frac{1}{a^2}$$

$$G(-i0)=\lim_{\varepsilon\to 0}G(-i\varepsilon)$$

$$G(-i0)=-\frac{1}{a^2}+\frac{i}{a\varepsilon}\Big|_{\varepsilon\to 0}=-\frac{1}{a^2}+i\infty$$

$$|G(-i0)|=\infty$$

$$\angle G(-i0)=90^\circ$$



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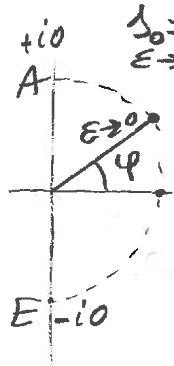
5 NPP

Segment $\overset{E\ \ F\ \ A}{(-i\varepsilon, \varepsilon, i\varepsilon)}$ small circle

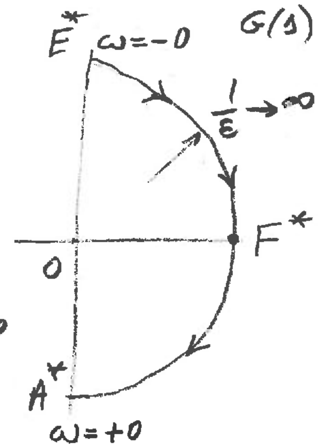
$\varepsilon \to 0$



$$s_0 = \varepsilon e^{i\varphi}, \quad \varepsilon \to 0, \quad \varphi \in (-90^\circ, 0, 90^\circ)$$

D E F

$$G(s) = \frac{1}{\varepsilon e^{i\varphi}(\varepsilon e^{i\varphi} + a)} \underset{\varepsilon \to 0}{\approx} \frac{1}{a\varepsilon e^{i\varphi}} = \frac{1}{a\varepsilon} e^{-i\varphi}$$



$$|G(s_0)| = \frac{1}{a\varepsilon} \xrightarrow[\varepsilon \to 0]{} \infty$$

$$\angle G(s_0) = -\varphi \in (90^\circ \quad 0^\circ \quad -90^\circ)$$

E^* F^* A^*

$\varphi = -90^\circ$ $\varphi = 0^\circ$ $\varphi = +90^\circ$

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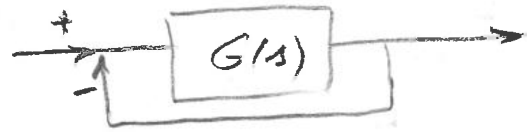
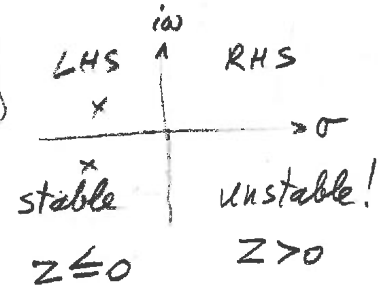
NYQUIST STABILITY CRITERION



Find stability of $G_{CL} = \frac{G}{1+G}$ by analyzing the polar plot of $G(s)$ as s follow the Nyquist circuit.

$$G_{CL}(s) = \frac{G(s)}{1+G(s)}$$

Poles $G_{CL}(s)$



Poles $G_{CL}(s) \Rightarrow$ zeros of $1+G(s)$

Nyquist Criterion:

$$\boxed{Z = P + N \leq 0}$$

P = number of $G(s)$ poles in RHS

$$G(s) = \frac{B(s)}{A(s)} \rightarrow p_1, p_2, \ldots$$

N = number of clockwise encirclements of the $(-1,0)$ point as s follows the Nyquist path (circuit)

Z = number of zeros of $1+G(s)$ in RHS

STABLE if $Z \leq 0$

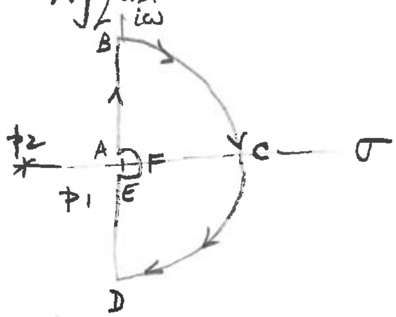
Example : aircraft roll model

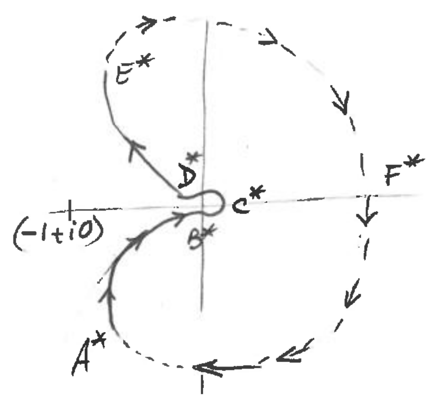
$$G(s) = \frac{114}{10s^2 + 4s} = \frac{114}{s(10s+4)}$$

$$p_1 = 0 \quad , \; P_2 = -2/5 \; : \; P = 0$$

Recall $G(s) = \frac{1}{s(s+a)} = \frac{1}{s^2 + sa}$ no ~~poles~~ in RHS

Nyquist circuit





$P = 0$ no poles in RHS

$N = 0$ no encirclements of $(-1+i0)$ point

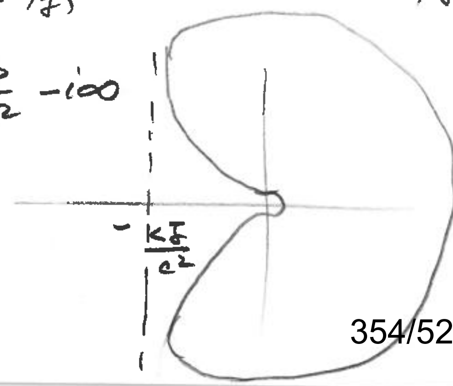
$Z = N + P = 0$ NO CL poles in RHS

Conclusion : system is unconditionally stable

Note : $G = \frac{K}{Js^2 + cs} = \frac{K}{J}\frac{1}{s(s + c/J)} = B\frac{1}{s(s+a)}$ $\quad B = K/J \quad a = c/J$

$$G(i\varepsilon)_{\varepsilon \to 0} = B\left(-\frac{1}{a^2} - i\infty\right) = -\frac{B}{a^2} - i\infty$$

$$\frac{B}{a^2} = \frac{K}{J}\frac{J^2}{c^2} = \frac{KJ}{c^2}$$



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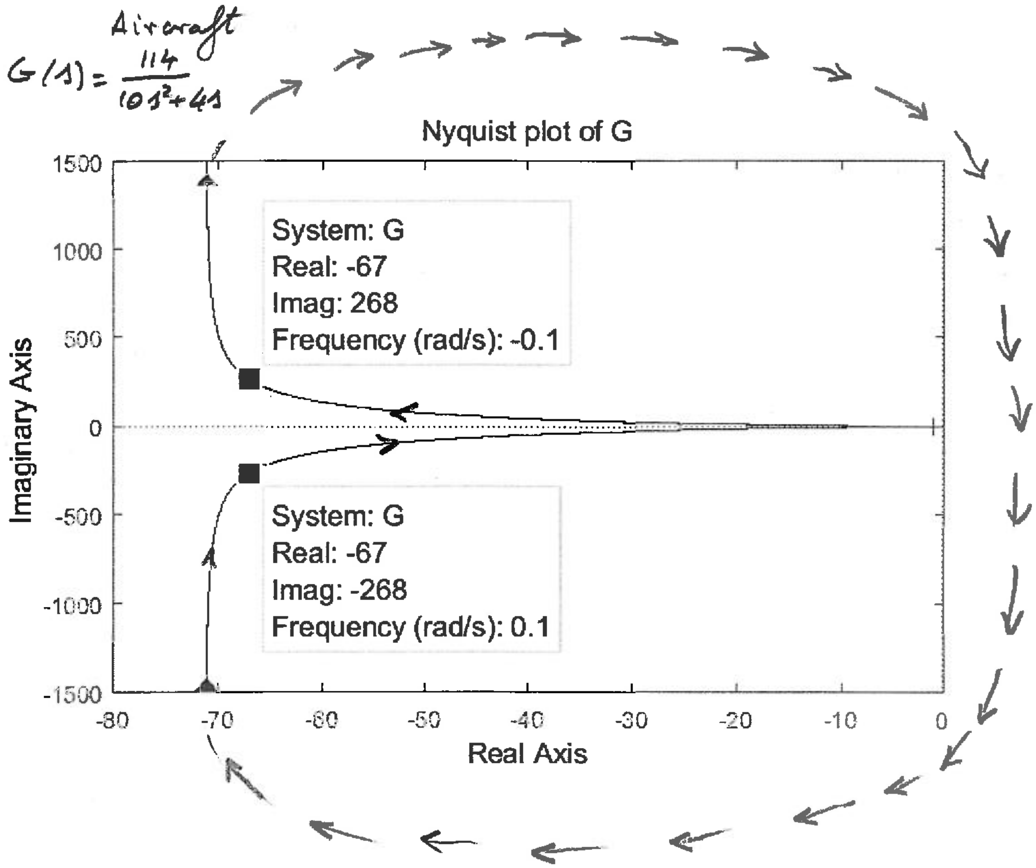
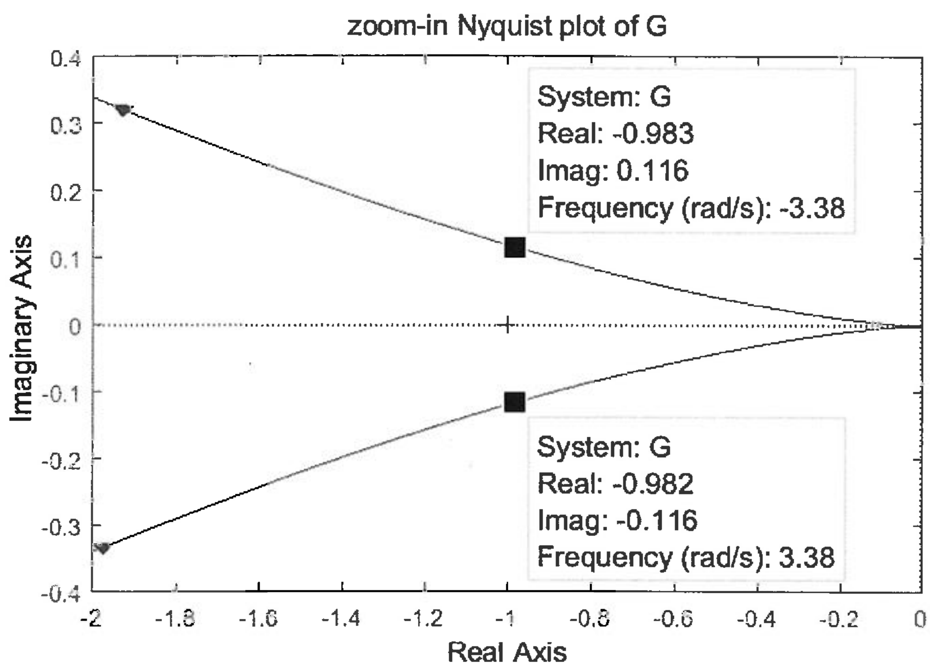


Aircraft
G(s) = 114 / (10s²+4s)
Nyquist plot of G
System: G
Real: -67
Imag: 268
Frequency (rad/s): -0.1
System: G
Real: -67
Imag: -268
Frequency (rad/s): 0.1
Imaginary Axis
Real Axis
1500
1000
500
0
-500
-1000
-1500
-80
-70
-60
-50
-40
-30
-20
-10
0



zoom-in Nyquist plot of G
System: G
Real: -0.983
Imag: 0.116
Frequency (rad/s): -3.38
System: G
Real: -0.982
Imag: -0.116
Frequency (rad/s): 3.38
Imaginary Axis
Real Axis
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-2
-1.8
-1.6
-1.4
-1.2
-1
-0.8
-0.6
-0.4
-0.2
0

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Example AM.8

Given: $G = K\dfrac{1}{s-1}$

Note: pole in RHS, $s=1$

Find: critical value of K for stability using Nyquist criterion

Solution: do Nyquist plot of $G(s)$

$G(0) = -\frac{K}{1} = -K$

$G(i\infty) = \lim_{\omega\to\infty} \dfrac{K}{i\omega - 1} \simeq \dfrac{K}{i\omega}$, $|G(i\omega)| = 0$

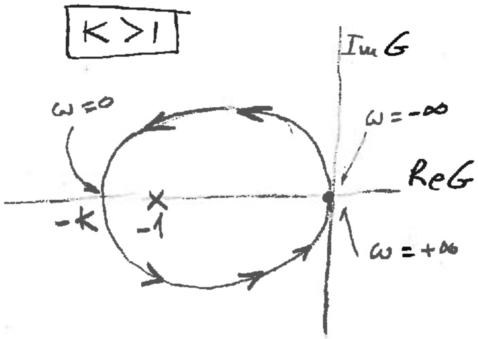
$\angle G(i\omega) = -90^\circ$

$G(-i\infty) \rightarrow |G(-i\omega)| = 0$

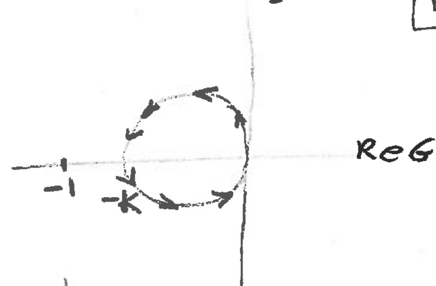
$\angle G(-i\omega) = 90^\circ$

Plot in MATLAB

$\boxed{K>1}$



Im G $\boxed{K<1}$



$N=-1$ (counterclockwise)

$P=1$ ($s=1$ pole of G in RHS)

$Z = -1+1 = 0$

STABLE for $K>1$

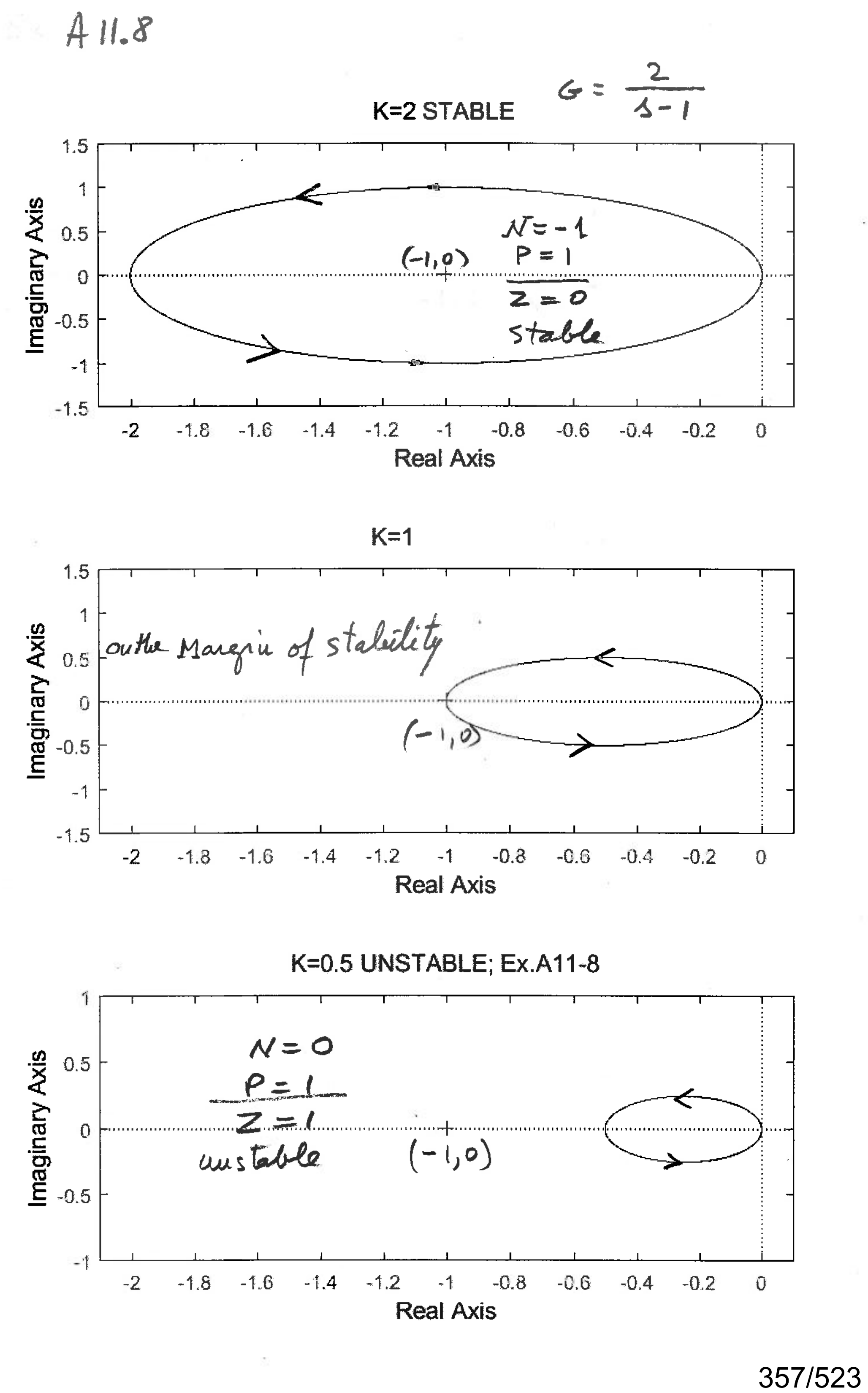
$N=0$

$P=1$

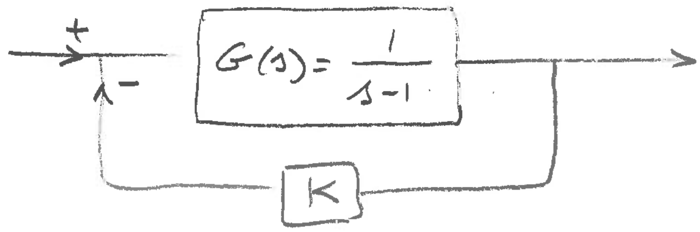
$Z=1$

UNSTABLE for $K<1$

$\boxed{\text{CRITICAL VALUE: } K=1}$ 356/523


A11.8
G = 2/(s-1)
K=2 STABLE
N = -1
P = 1
Z = 0
stable
(-1,0)
Imaginary Axis
Real Axis
K=1
on the Margin of stability
(-1,0)
Imaginary Axis
Real Axis
K=0.5 UNSTABLE; Ex.A11-8
N = 0
P = 1
Z = 1
unstable
(-1,0)
Imaginary Axis
Real Axis
357/523

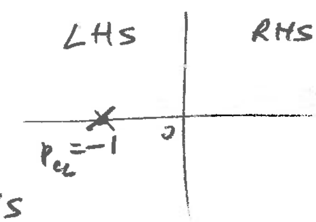
A11.8



$$G_{CL} = \frac{G}{1+GH} = \frac{1}{s-1+K} = \frac{1}{s-(1-K)}$$

$p_{CL} = 1-K$

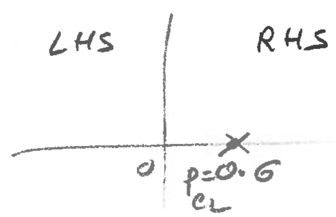
$K = 2,\ p_{CL} = -1$



STABLE

$p_{CL} \in LHS$

$K = 0.4,\ p_{CL} = 0.6$

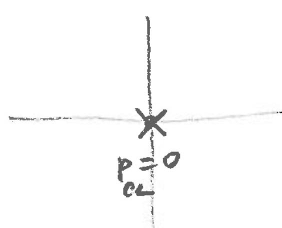


UNSTABLE

$p_{CL} \in RHS$

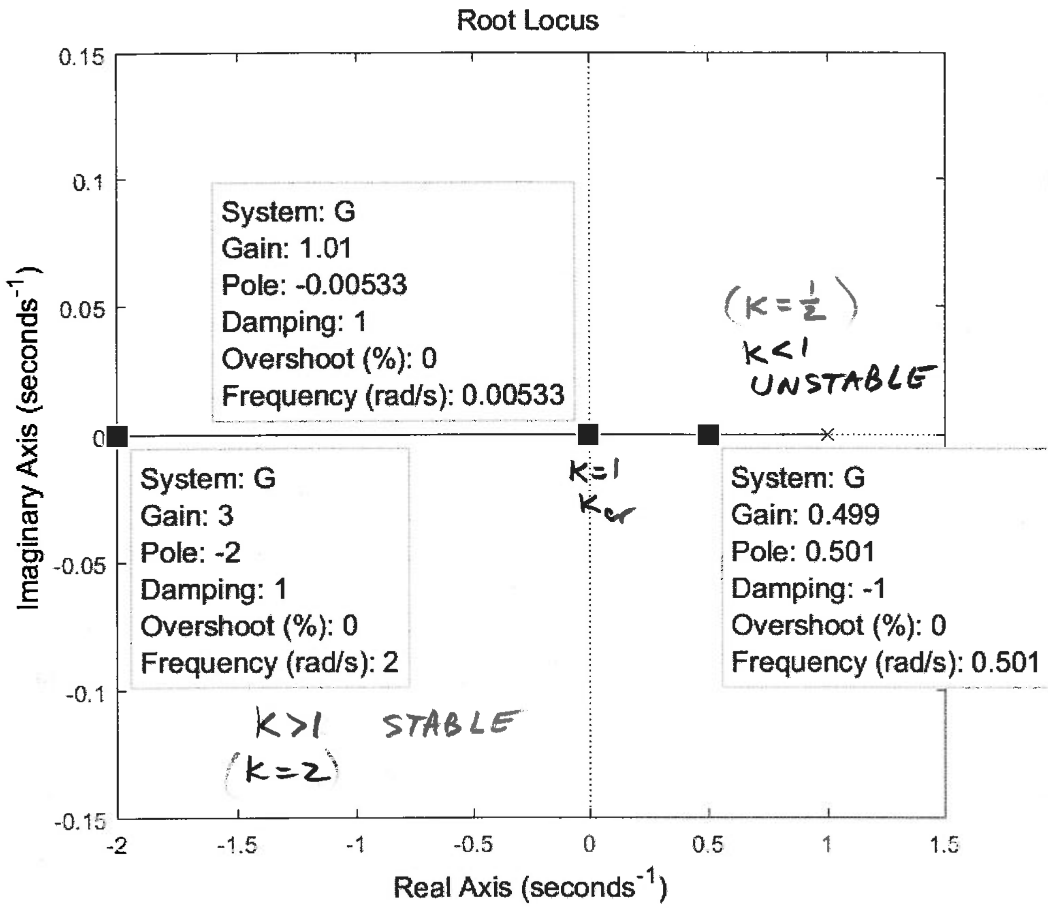
$p_{CL} = 1 - K_{cr} = 0$

$K_{cr} = 1$



critical K value

A11.8


Root Locus
System: G
Gain: 1.01
Pole: -0.00533
Damping: 1
Overshoot (%): 0
Frequency (rad/s): 0.00533
System: G
Gain: 3
Pole: -2
Damping: 1
Overshoot (%): 0
Frequency (rad/s): 2
System: G
Gain: 0.499
Pole: 0.501
Damping: -1
Overshoot (%): 0
Frequency (rad/s): 0.501
(K = 1/2)
K<1
UNSTABLE
K=1
Kcr
K>1 STABLE
(K=2)
Imaginary Axis (seconds^-1)
Real Axis (seconds^-1)
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-2
-1.5
-1
-0.5
0
0.5
1
1.5

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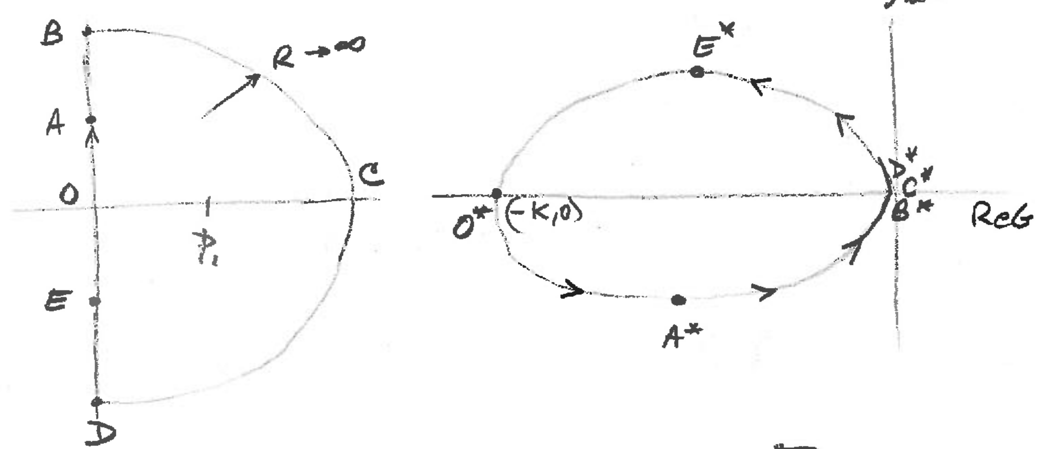
All 8a MANUAL PLOT.

$$G = k\frac{1}{s-1} \qquad p_1 = 1$$



$$O: \qquad G(0) = K\frac{1}{0-1} = -K = Ke^{i\pi}$$

$$A = i \qquad G(A) = K\frac{1}{i-1} = -K\frac{1+i}{2} = K\frac{\sqrt{2}}{2}e^{i(\pi+\frac{\pi}{4})}$$

$$\angle G(A) = \frac{5\pi}{4}$$

$$B = iR \qquad G(B) = K\frac{1}{iR-1} = -K\frac{1+iR}{1+R^2} = \frac{K}{\sqrt{1+R^2}}e^{i(\pi+\varphi)}$$

$$\varphi = \tan^{-1}R \xrightarrow[R\to\infty]{} \frac{\pi}{2}$$

$$|G(B)| \xrightarrow[R\to\infty]{} 0$$

$$\angle G(B) \xrightarrow[R\to\infty]{} \frac{3\pi}{2}$$

$$C = R \qquad G(C) = K\frac{1}{R-1} \xrightarrow[R\to\infty]{} 0$$

$$D = -iR \qquad G(D) = K\frac{1}{-iR-1} = -K\frac{1-iR}{1+R^2} = \frac{K}{\sqrt{1+R^2}}e^{i(\pi-\varphi)}$$

$$\varphi = \tan^{-1}R \xrightarrow[R\to\infty]{} \frac{\pi}{2}$$

$$|G(D)| \xrightarrow[R\to\infty]{} 0$$

$$\angle G(D) \xrightarrow[R\to\infty]{} \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

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A11.8b

$$E = -i \qquad G(E) = K\frac{1}{-i-1} = -K\frac{1-i}{2} = K\frac{\sqrt{2}}{2}e^{i(\pi-\frac{\pi}{4})}$$

$$\angle G(E) = \frac{3\pi}{4}$$

Nyquist criterion gives $K < 1$ UNSTABLE

$K > 1$ STABLE

To verify, calculate the poles of G_{CL}

$$G_{CL} = \frac{G}{1+G} = \frac{K}{s-1+K} = \frac{K}{s-(1-K)}$$

$$p_{CL} = 1-K$$

for $K<1$, $p_{CL} > 0$, in RHS, UNSTABLE

$K>1$, $p_{CL} < 0$, in LHS, STABLE

QED

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p634

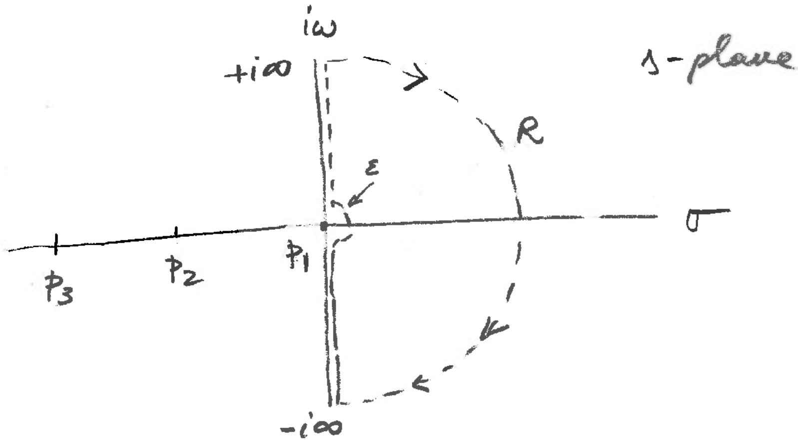
Example p634

$$G(s) = K \frac{1}{s(T_1 s+1)(T_2 s+1)} \qquad T_2 > T_1$$

Poles: $p_1 = 0$

$p_2 = -1/T_1$

$p_3 = -1/T_2$



Nyquist circuit will have to go around the origin on a small circle of radius $\varepsilon \to 0$. We study four segments:

- positive iω axis $s \in (+i0, +i\infty)$
- big circle $R \to \infty$
- negative iω axis $s \in (-i\infty, -i0)$
- small circle $\varepsilon \to 0$

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2
p634

$+i\omega$ axis.

$s = (+i\varepsilon\ ,\ iR)$

$\varepsilon \to 0$
$R \to \infty$

$$G(+i\varepsilon) = -K(T_1+T_2) - i\frac{K}{\varepsilon}$$

$$G(+i\varepsilon) \xrightarrow[\varepsilon\to 0]{} -K(T_1+T_2) - i\infty$$

$$|G(i\varepsilon)| \xrightarrow[\varepsilon\to 0]{} \infty$$

$$\angle G(i\varepsilon) \xrightarrow[\varepsilon\to 0]{} -90^\circ$$

Proof

$$\frac{K}{i\varepsilon(i\varepsilon T_1+1)(i\varepsilon T_2+1)} = \frac{K}{i\varepsilon\left[i^2\varepsilon^2 T_1T_2 + i\varepsilon(T_1+T_2)+1\right]} =$$

$$= \frac{K}{i\varepsilon}\,\frac{1}{i\varepsilon(T_1+T_2)+1}$$

$$= \frac{K}{i\varepsilon}\,\frac{-i\varepsilon(T_1+T_2)+1}{\varepsilon^2\cancel{(T_1+T_2)}+1}$$

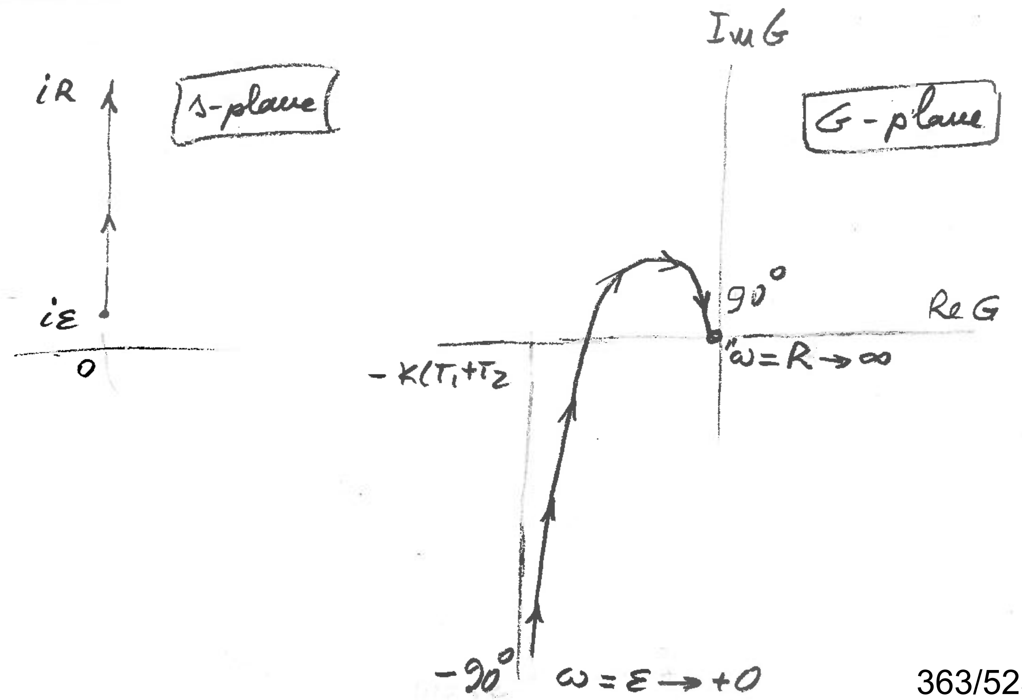
$$= -K(T_1+T_2) - i\frac{K}{\varepsilon}$$

$$G(iR) = \frac{K}{iR(T_1 iR+1)(T_2 iR+1)} \approx \frac{K}{T_1T_2R^3i^3} = \frac{K}{T_1T_2R^3}e^{i\frac{\pi}{2}}$$

$$\frac{1}{i^3} = \frac{i^4}{i^3} = i = e^{i\frac{\pi}{2}}$$

$$|G(iR)| = \frac{K}{T_1T_2R^3} \xrightarrow[R\to\infty]{} 0$$

$$\angle G(iR) = \frac{\pi}{2} = 90^\circ$$



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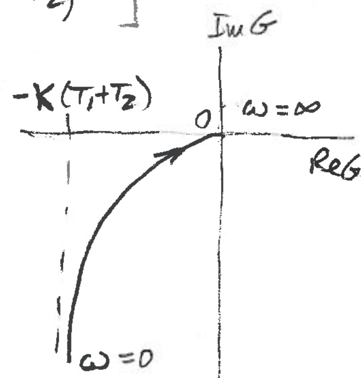
$$G(i\varepsilon) = \frac{K}{i\varepsilon(i\varepsilon T_1+1)(i\omega T_2+1)}$$

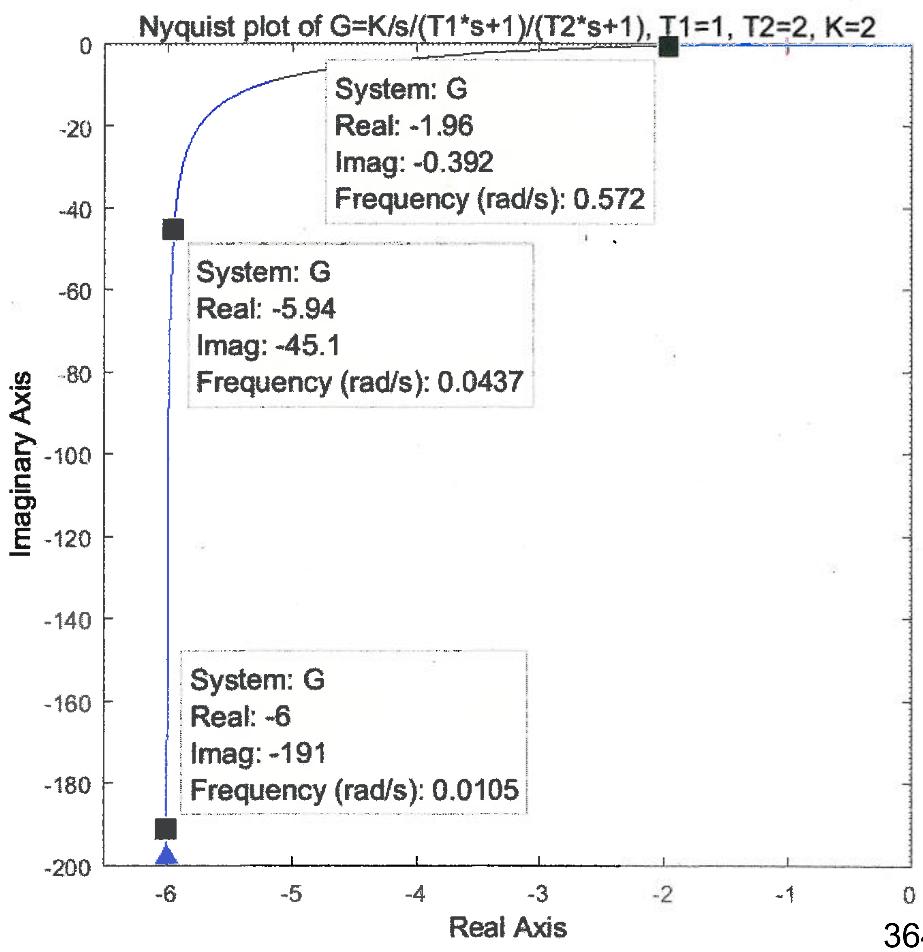
$$= \frac{K}{i\varepsilon\left(\cancel{i^2\varepsilon^2 T_1T_2} + i\varepsilon(T_1+T_2)+1\right)} = \frac{K}{\varepsilon}\,\frac{1}{-\varepsilon(T_1+T_2)+i}$$

$$= \frac{K}{\varepsilon}\,\frac{-\varepsilon(T_1+T_2)-i}{\cancel{\varepsilon^2(T_1+T_2)^2}+1} = -\frac{K}{\varepsilon}\left[\varepsilon(T_1+T_2)+i\right]$$

$$G(i\varepsilon) = -K(T_1+T_2) - i\frac{K}{\varepsilon}$$

$$\xrightarrow[\varepsilon\to 0]{} \underbrace{-K(T_1+T_2)}_{ReG = Const} - i\infty$$





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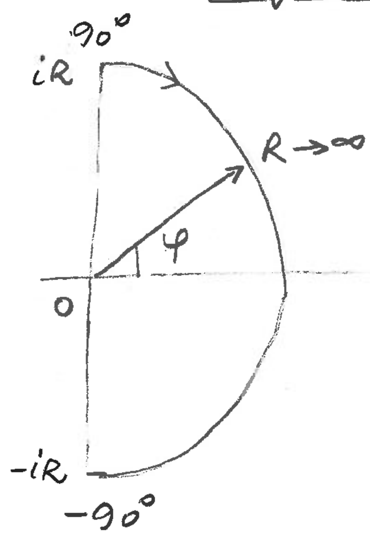
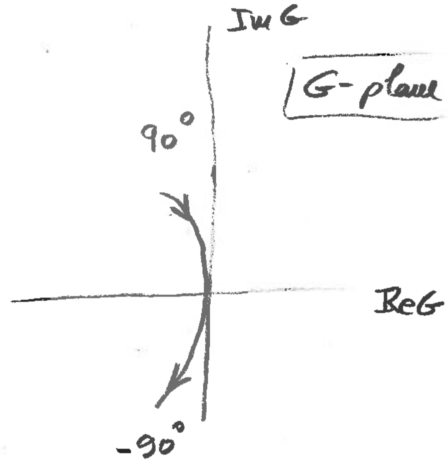
Big circle





$$G(s) = \frac{K}{Re^{i\varphi}(T_1 Re^{i\varphi}+1)(T_2 Re^{i\varphi}+1)}$$

$$G(s) \underset{R>>1}{\approx} \frac{K}{T_1 T_2 R^3 e^{3i\varphi}} = \frac{K}{T_1 T_2 R^3} e^{-3i\varphi}$$

$$|G(s)| = \frac{K}{T_1 T_2 R^3} \xrightarrow[R\to\infty]{} 0$$

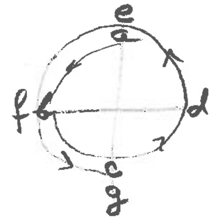
$$\angle G(s) = -3\varphi = (-270^\circ \quad +270^\circ)$$

$$\varphi = 90 \qquad -90$$

$(-270+360=90^\circ;\ 270-360=-90)$

$$\angle G(s) = (90^\circ, -90^\circ)$$

$\varphi=$	90°	60°	30°	0°	−30°	−60°	−90°
$\angle G$	−270	−180	−90	0	90	180	270
	90	180	−90	0	90	180	−90
	a	b	c	d	e	f	g



G(s) goes round origin 1.5 times

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$-i\omega$ axis

$$s = (-iR \quad -i\varepsilon)$$

$\varepsilon \to 0$
$R \to \infty$

$$G(-i\varepsilon) \cong -K(T_1+T_2) + i\frac{K}{\varepsilon}$$

$$G(-i\varepsilon) \xrightarrow[\varepsilon \to 0]{} -K(T_1+T_2) + i\infty$$

$$|G(-i\varepsilon)| \xrightarrow[\varepsilon \to 0]{} \infty$$

$$\angle G(-i\varepsilon) \xrightarrow[\varepsilon \to 0]{} 90^\circ$$

Proof

$$\frac{K}{-i\varepsilon(-i\varepsilon T_1+1)(-i\varepsilon T_2+1)} = \frac{K}{-i\varepsilon[\cancel{i^2\varepsilon^2 T_1 T_2} - i\varepsilon(T_1+T_2)+1]}$$

$$= i\frac{K}{\varepsilon}\,\frac{i\varepsilon(T_1+T_2)+1}{\cancel{\varepsilon^2(T_1+T_2)^2}+1}$$

$$= i\frac{K}{\varepsilon}\left[i\varepsilon(T_1+T_2)+1\right]$$

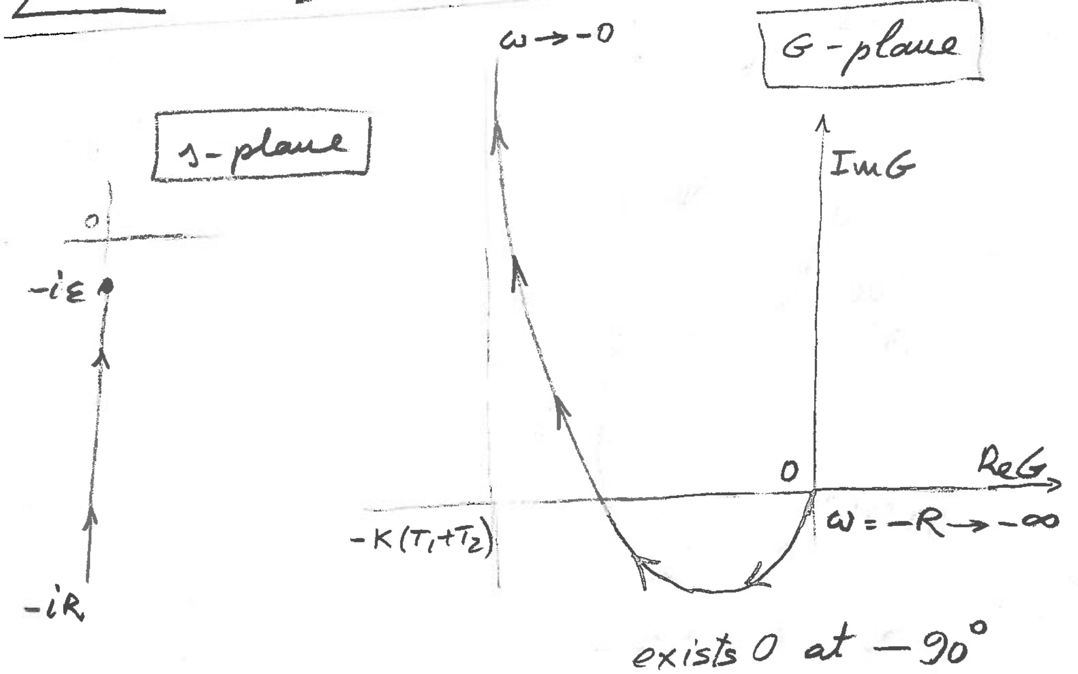
$$= -K(T_1+T_2) + i\,K/\varepsilon$$

$$G(-iR) = \frac{K}{-iR(-T_1 iR+1)(-T_2 iR+1)} \approx \frac{K}{-T_1T_2R^3 i^3} = \frac{K}{T_1T_2R^3}e^{-i\frac{\pi}{2}}$$

$$\frac{-1}{i^3} = \frac{i^2}{i^3} = \frac{1}{i} = -i$$

$$|G(-iR)| = \frac{K}{T_1T_2R^3} \xrightarrow[R \to \infty]{} 0$$

$$\angle G(iR) = -\frac{\pi}{2} = -90^\circ$$



exists 0 at -90°

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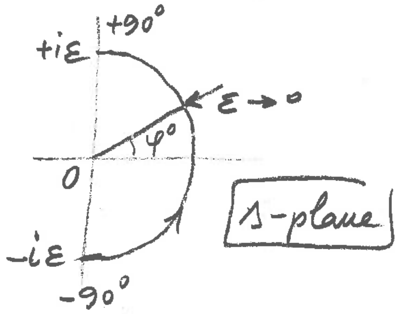
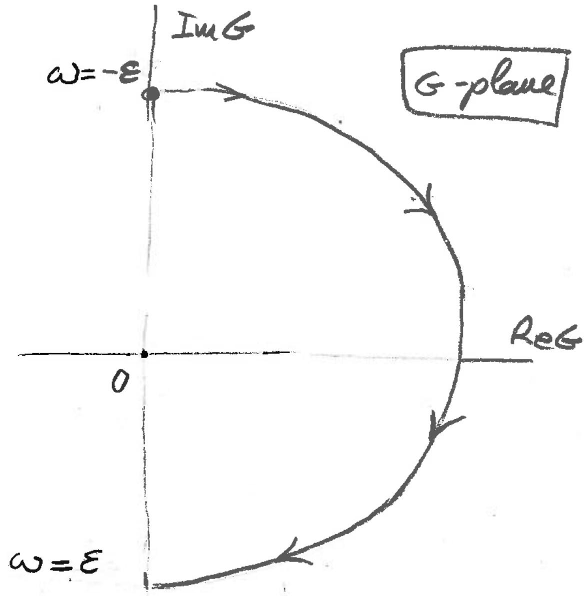
Small circle

$s_0 = \varepsilon e^{i\varphi_0}$

$\varphi_0 = (-90°, 90°)$





$\varphi_0 = (-90°, 90°)$

$$G(s_0) = \frac{K}{\varepsilon e^{i\varphi_0}(T_1 \varepsilon e^{i\varphi_0}+1)(T_2 \varepsilon e^{i\varphi_0}+1)}$$

$$G(s_0) \approx \frac{K}{\varepsilon e^{i\varphi_0}} = \frac{K}{\varepsilon} e^{-i\varphi_0}$$

$$|G(s_0)| = \frac{K}{\varepsilon} \xrightarrow[\varepsilon \to 0]{} \infty$$

$$\angle G(s_0) = -\varphi_0 = (90°, -90°)$$

$$\varphi_0 = (\underset{-i\varepsilon}{-90°}, \underset{i\varepsilon}{90°})$$

$G(s_0)$ travels a big semicircle from 90° to −90°.

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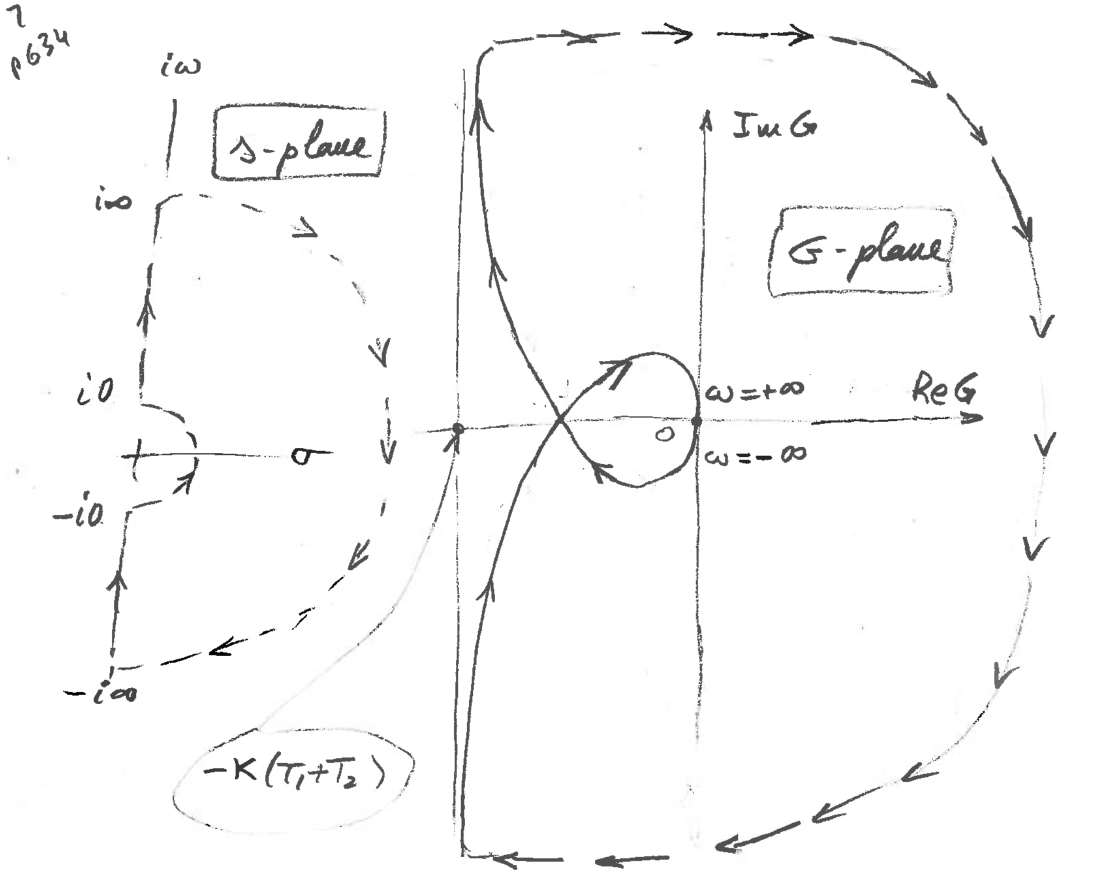
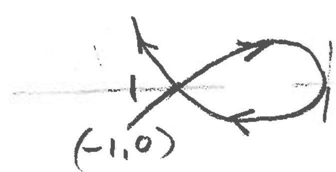
7 p634

Finally, assemble the four segments:

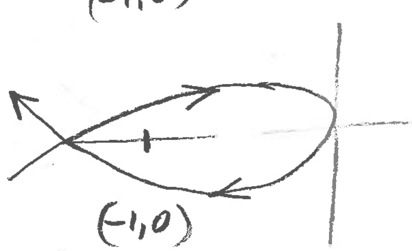


Note that $G(s)$ goes around two times!

Depending on K, it may or may not enclose the point (−1,0).

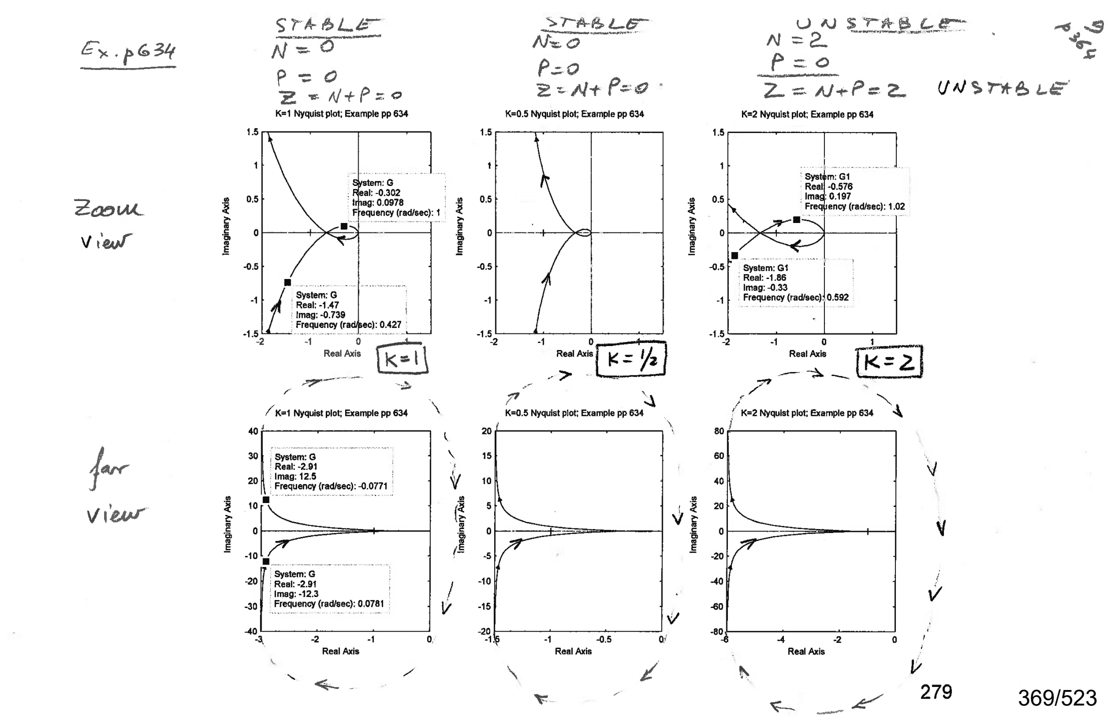


STABLE $\begin{array}{c} N=0 \\ P=0 \\ \hline Z=0 \end{array}$



UNSTABLE $\begin{array}{c} N=2 \\ P=0 \\ \hline Z=2 \end{array}$

(see MATLAB plot next page) 368/523


Ex. p634
STABLE
N = 0
P = 0
Z = N+P = 0
STABLE
N=0
P=0
Z=N+P=0
UNSTABLE
N = 2
P = 0
Z = N+P=2 UNSTABLE
K=1 Nyquist plot; Example pp 634
K=0.5 Nyquist plot; Example pp 634
K=2 Nyquist plot; Example pp 634
Zoom
view
far
view
Imaginary Axis
Real Axis
System: G
Real: -0.302
Imag: 0.0978
Frequency (rad/sec): 1
System: G
Real: -1.47
Imag: -0.739
Frequency (rad/sec): 0.427
System: G1
Real: -0.576
Imag: 0.197
Frequency (rad/sec): 1.02
System: G1
Real: -1.86
Imag: -0.33
Frequency (rad/sec): 0.592
K=1
K=1/2
K=2
System: G
Real: -2.91
Imag: 12.5
Frequency (rad/sec): -0.0771
System: G
Real: -2.91
Imag: -12.3
Frequency (rad/sec): 0.0781
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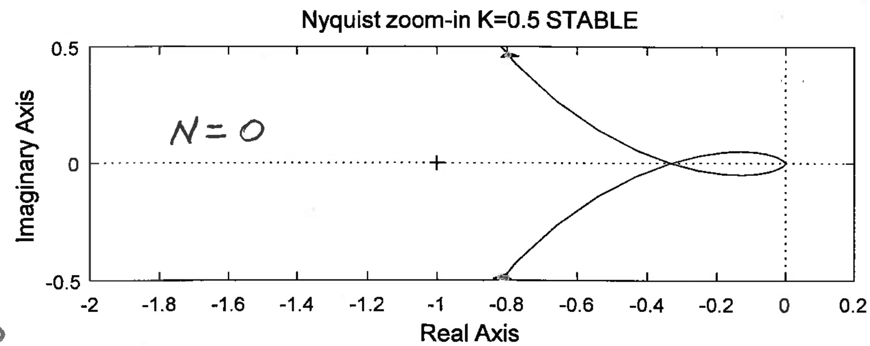
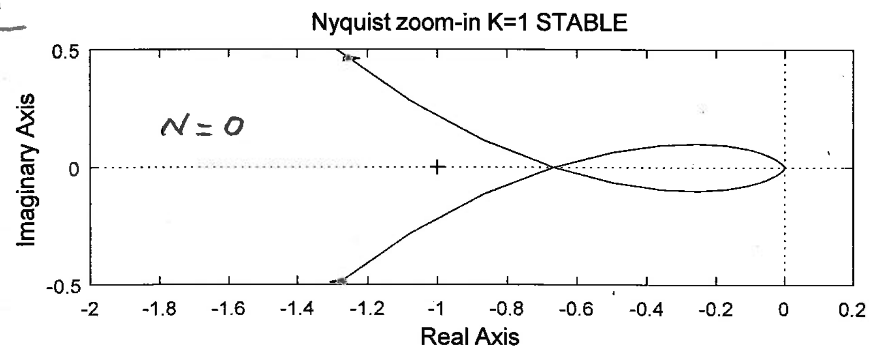
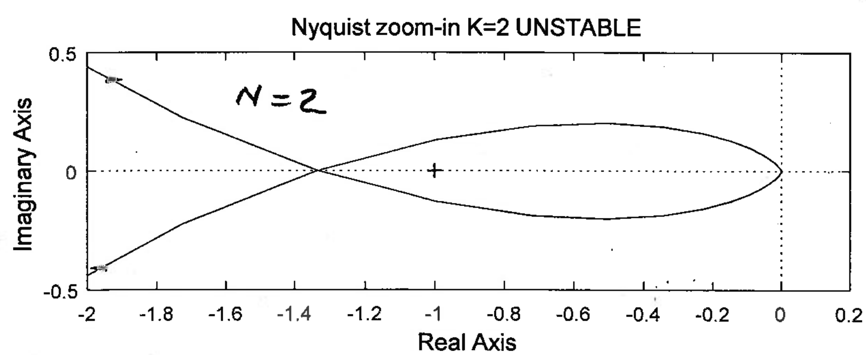
pG34



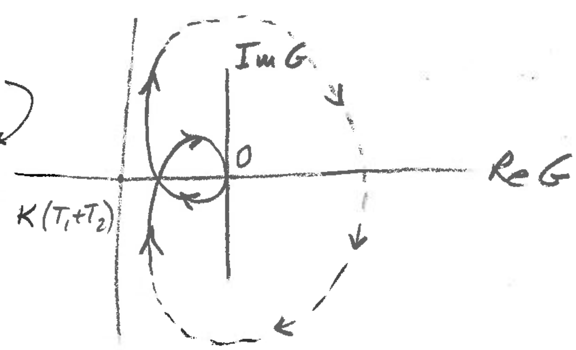
$N=0$
$P=0$
$Z=0$
stable



$N=2$
$P=0$
$Z=2$
unstable



two times around



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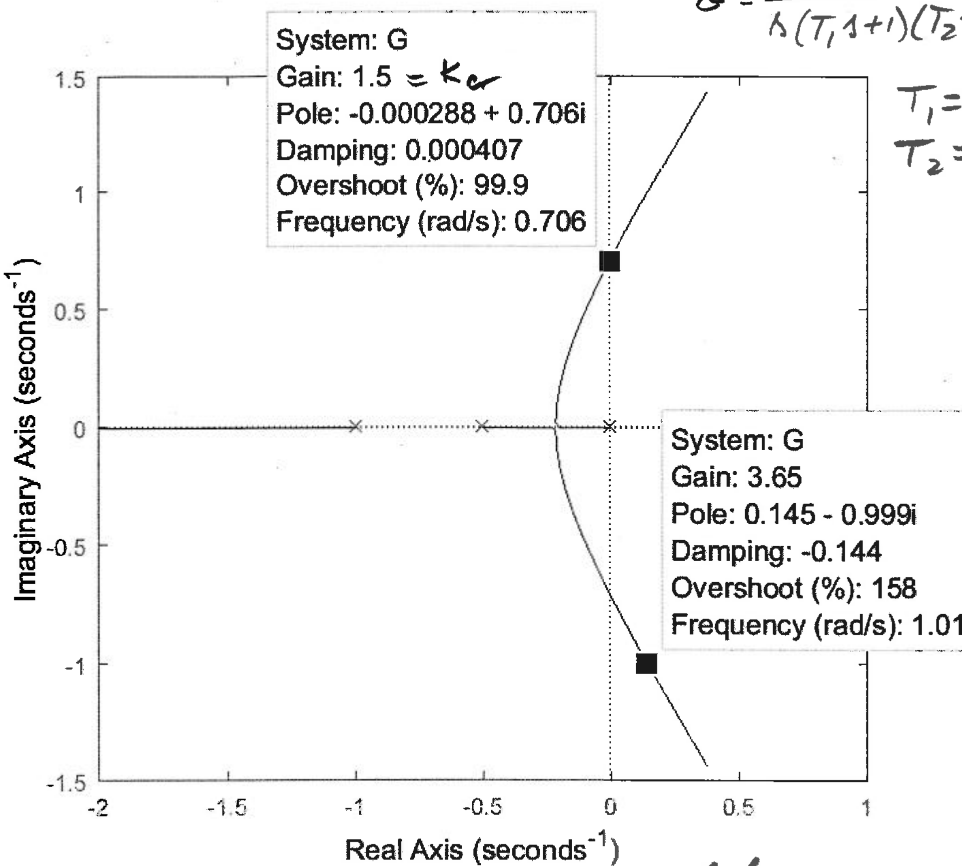
ROOT LOCUS

$$G = \frac{K}{s(T_1 s+1)(T_2 s+1)}$$

$T_1 = 1$

$T_2 = 2$



$K_{cr} = 1.5$ according to rlocus plot

verify using $G_{CL} = \dfrac{G}{1+G} = \dfrac{K}{s(T_1 s+1)(T_2 s+1)+K}$

$$s(T_1 s+1)(T_2 s+1)+K\Big|_{\substack{T_1=1\\T_2=2}} = s(s+1)(2s+1)+K$$

$$s(2s^2+3s+1)+K = 2s^3+3s^2+s+K$$

Routh criterion

s^3	2	1
s^2	3	K
s^1	$\frac{3-2K}{3}$	
s^0	K	

For sign change:

$3-2K=0$

$K_{cr} = \frac{3}{2} = 1.5$

QED.

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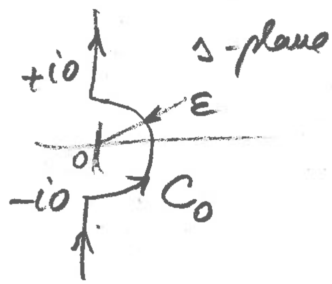
N3d

Tips on Nyquist Plot



$$s_0 = \varepsilon e^{i\varphi}$$

$$-90^\circ < \varphi < 90^\circ.$$

$G = \frac{1}{s}$, $G(s_0) = \frac{1}{\varepsilon e^{i\varphi}} = \frac{1}{\varepsilon} e^{-i\varphi}$

(Type 1 system)

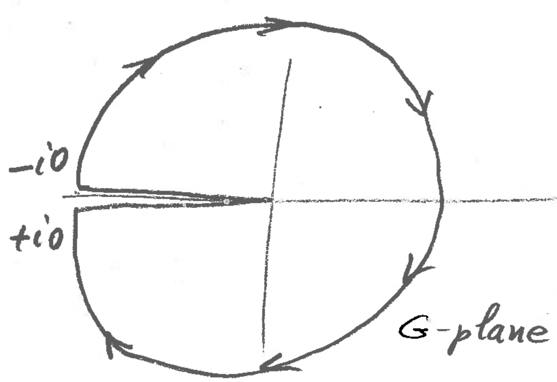
$$+90^\circ > \angle G(s_0) > -90^\circ.$$

Half-circle, clockwise.

$G = \frac{1}{s^2}$, $G(s_0) = \frac{1}{(\varepsilon e^{i\varphi})^2} = \frac{1}{\varepsilon^2} e^{-i2\varphi}$.

(Type 2 system)

$$+180^\circ > \angle G > -180^\circ.$$



Full circle, clockwise.

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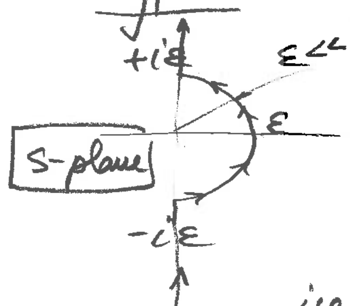
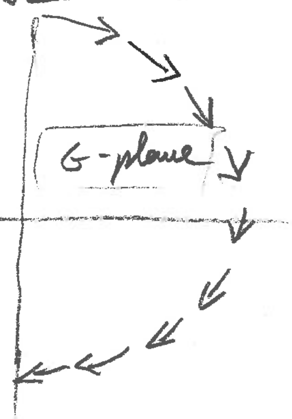
Details on behavior of common functions on the Nyquist circuit

Type 1

$$G(s) = \frac{1}{s}$$





$$s_0 = \varepsilon e^{i\varphi}$$

$$-90^\circ < \varphi < 90^\circ$$

$$G(s_0) = \frac{1}{\varepsilon e^{i\varphi}} = \frac{1}{\varepsilon} e^{-i\varphi}$$

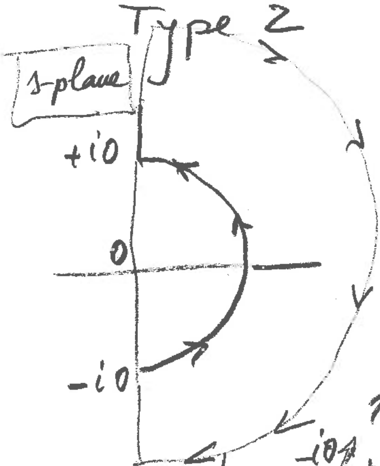
$$90^\circ > \angle G(s_0) > -90^\circ$$

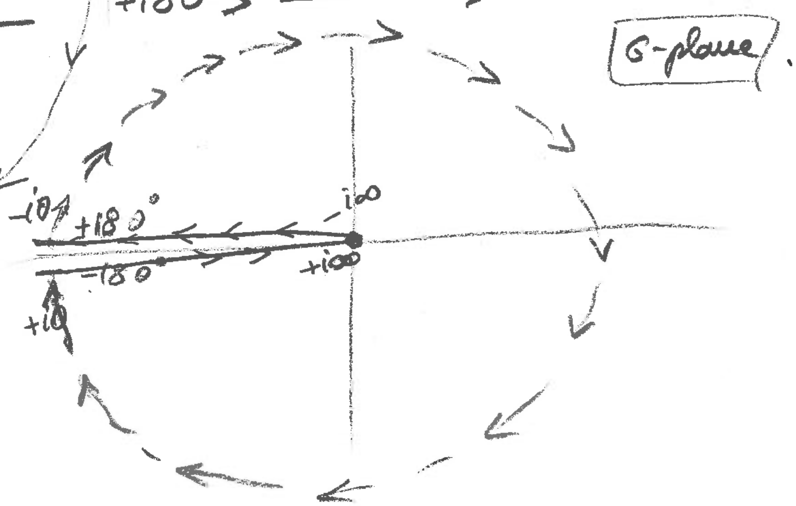
Type 2

$$G(s) = \frac{1}{s^2}$$

$$G(s_0) = \frac{1}{\varepsilon^2} e^{-i2\varphi}, \quad -90^\circ < \varphi < +90^\circ$$

$$+180^\circ > \angle G(s_0) > -180^\circ$$





$$G(-i0) = \frac{1}{(-i0)^2}$$

$$G(+i0) = \frac{1}{(i0)^2}$$

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N1a

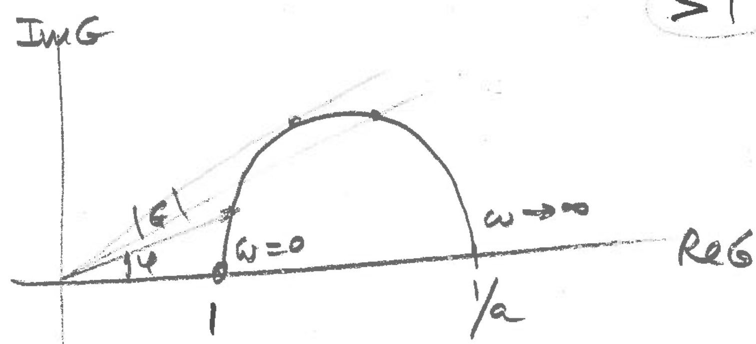
$$G(s) = \frac{Ts+1}{aTs+1} \qquad 0 < a \leq 1$$

$$G(i\omega) = \frac{T\cdot(i\omega)+1}{a\cdot T\cdot(i\omega)+1}$$

$$\omega \to 0, \quad G(i0) = 1$$

$$\omega \to \infty, \quad G(i\infty) = \lim_{\omega\to\infty} \frac{T(i\omega)+1}{a\cdot T\cdot(i\omega)+1} = \frac{T}{aT} = \frac{1}{a}$$

$$> 1$$



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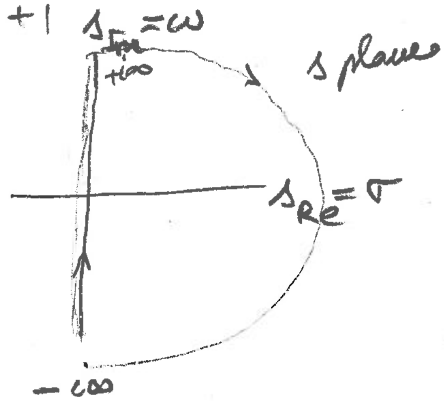
NIG. Ex 11.4, p 641

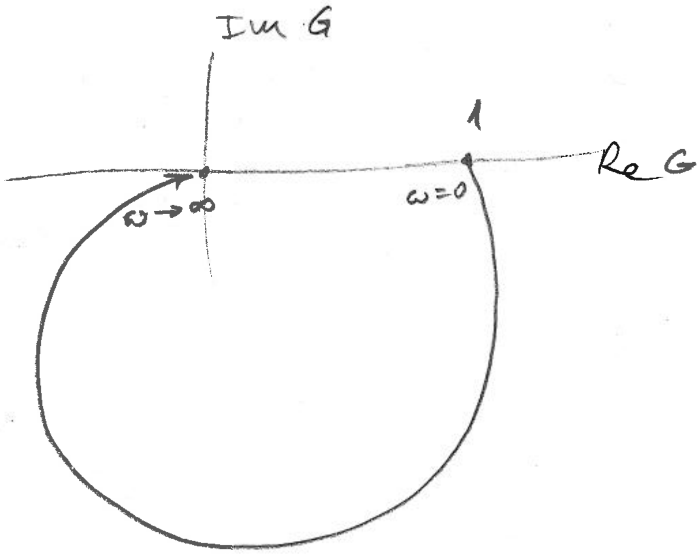
$$G(s) = \frac{1}{s^2+0.8s+1}$$

$G = 1/$— — —

nyquist(G)

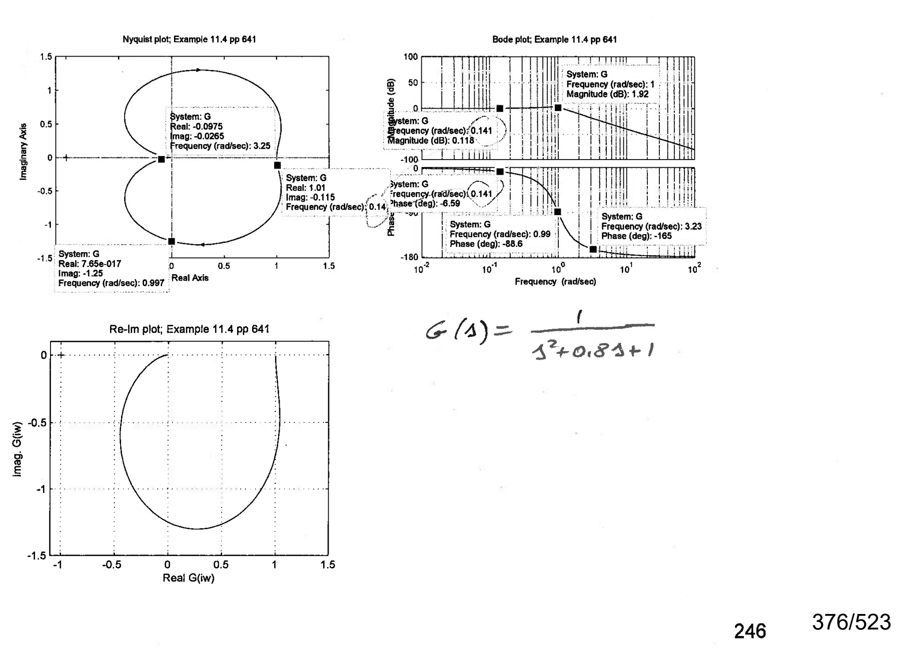
Nyquist circuit for ω





Ex. 11.5 p 642

$$G(s) = \frac{1}{s(s+1)}.$$


Nyquist plot; Example 11.4 pp 641
Imaginary Axis
Real Axis
System: G
Real: -0.0975
Imag: -0.0265
Frequency (rad/sec): 3.25
System: G
Real: 1.01
Imag: -0.115
Frequency (rad/sec): 0.14
System: G
Real: 7.65e-017
Imag: -1.25
Frequency (rad/sec): 0.997
Bode plot; Example 11.4 pp 641
System: G
Frequency (rad/sec): 1
Magnitude (dB): 1.92
System: G
Frequency (rad/sec): 0.141
Magnitude (dB): 0.118
System: G
Frequency (rad/sec): 0.141
Phase (deg): -6.59
System: G
Frequency (rad/sec): 0.99
Phase (deg): -88.6
System: G
Frequency (rad/sec): 3.23
Phase (deg): -165
Frequency (rad/sec)
Re-Im plot; Example 11.4 pp 641
Imag. G(iw)
Real G(iw)
$G(s) = \frac{1}{s^2+0.8s+1}$
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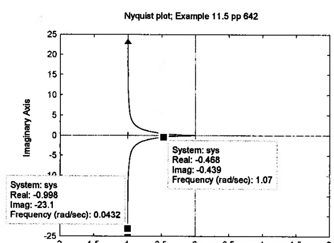

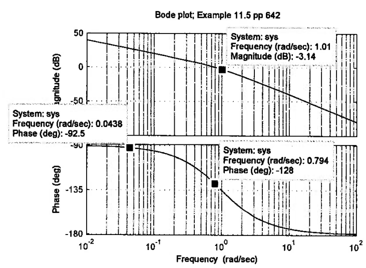

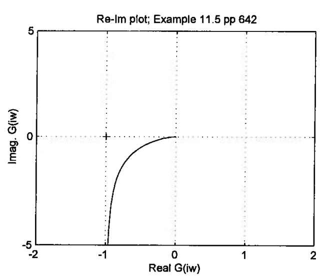

$$G(s) = \frac{1}{s(s+1)}.$$

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N2 **Nyquist plot trends**

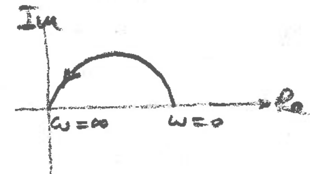
$$G(s) = \frac{K}{s^N} \frac{(T_a s+1)(T_b s+1)\cdots}{(T_1 s+1)(T_2 s+1)\cdots}$$

Type 0 systems $(N=0)$, $G(i\omega) = K \dfrac{(T_a i\omega+1)(T_b i\omega+1)\cdots}{(T_1 i\omega+1)(T_2 i\omega+1)\cdots}$ order m / order n

$\omega = 0$ $\quad G(i0) = K$

$\omega \to \infty$

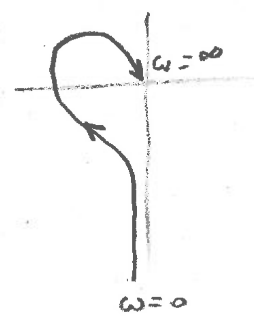
$$G(i\infty) = \begin{cases} 0 & \text{for } n > m \\ \text{const}, & n = m \\ \infty, & n < m \end{cases}$$



Type 1 systems $(N=1)$, $G(i\omega) = \dfrac{K}{i\omega} \cdot \dfrac{(\quad)(\quad)}{(\quad)(\quad)}$ $\begin{matrix} m \\ n \end{matrix}$

for $n=m$, $G(i\infty) = 0$

$G(i0) = \dfrac{1}{i0}$, $\angle G(i0) = -90^\circ$



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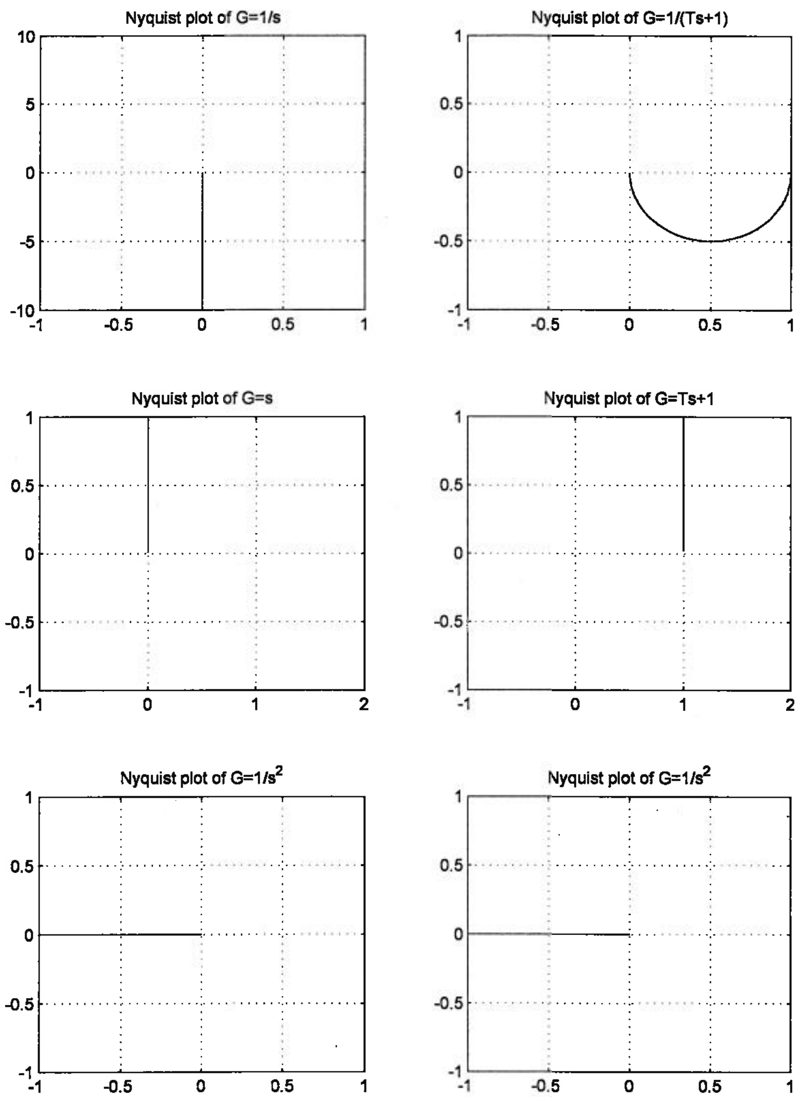
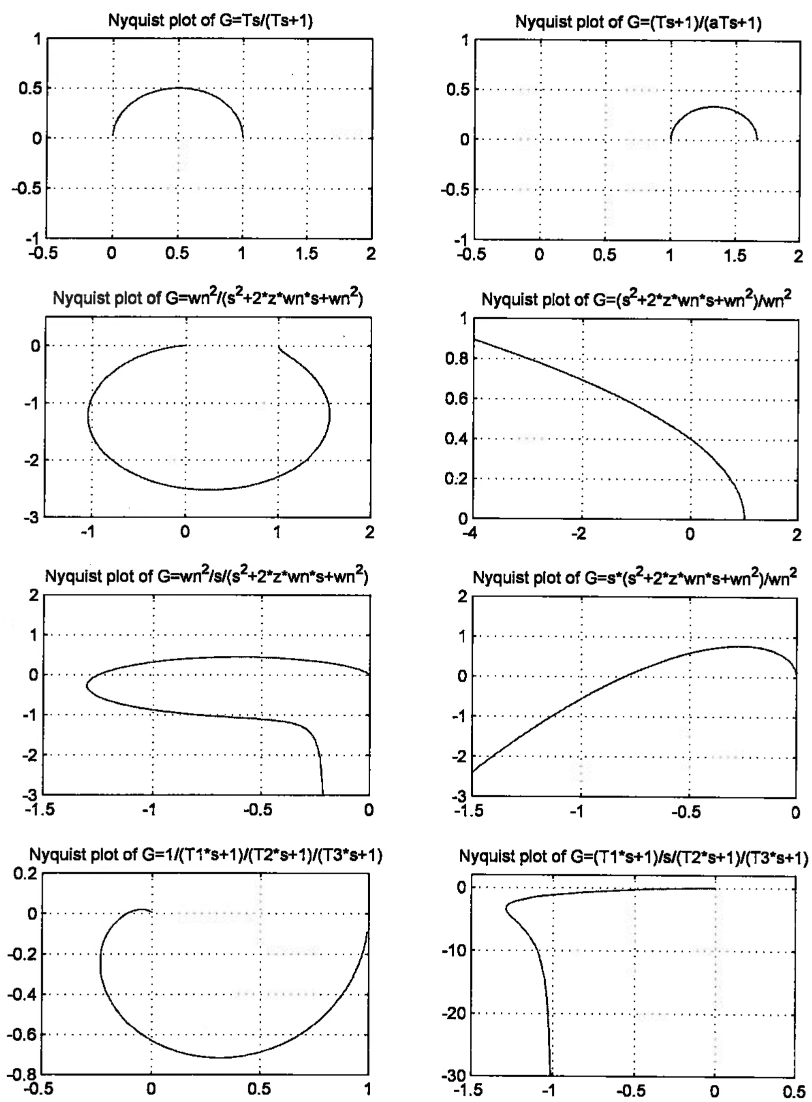

Nyquist plot of G=1/s
Nyquist plot of G=1/(Ts+1)
Nyquist plot of G=s
Nyquist plot of G=Ts+1
Nyquist plot of G=1/s2
Nyquist plot of G=1/s2
10
5
0
-5
-10
1
0.5
-0.5
-1
2


Nyquist plot of G=Ts/(Ts+1)
Nyquist plot of G=(Ts+1)/(aTs+1)
Nyquist plot of G=wn^2/(s^2+2*z*wn*s+wn^2)
Nyquist plot of G=(s^2+2*z*wn*s+wn^2)/wn^2
Nyquist plot of G=wn^2/s/(s^2+2*z*wn*s+wn^2)
Nyquist plot of G=s*(s^2+2*z*wn*s+wn^2)/wn^2
Nyquist plot of G=1/(T1*s+1)/(T2*s+1)/(T3*s+1)
Nyquist plot of G=(T1*s+1)/s/(T2*s+1)/(T3*s+1)

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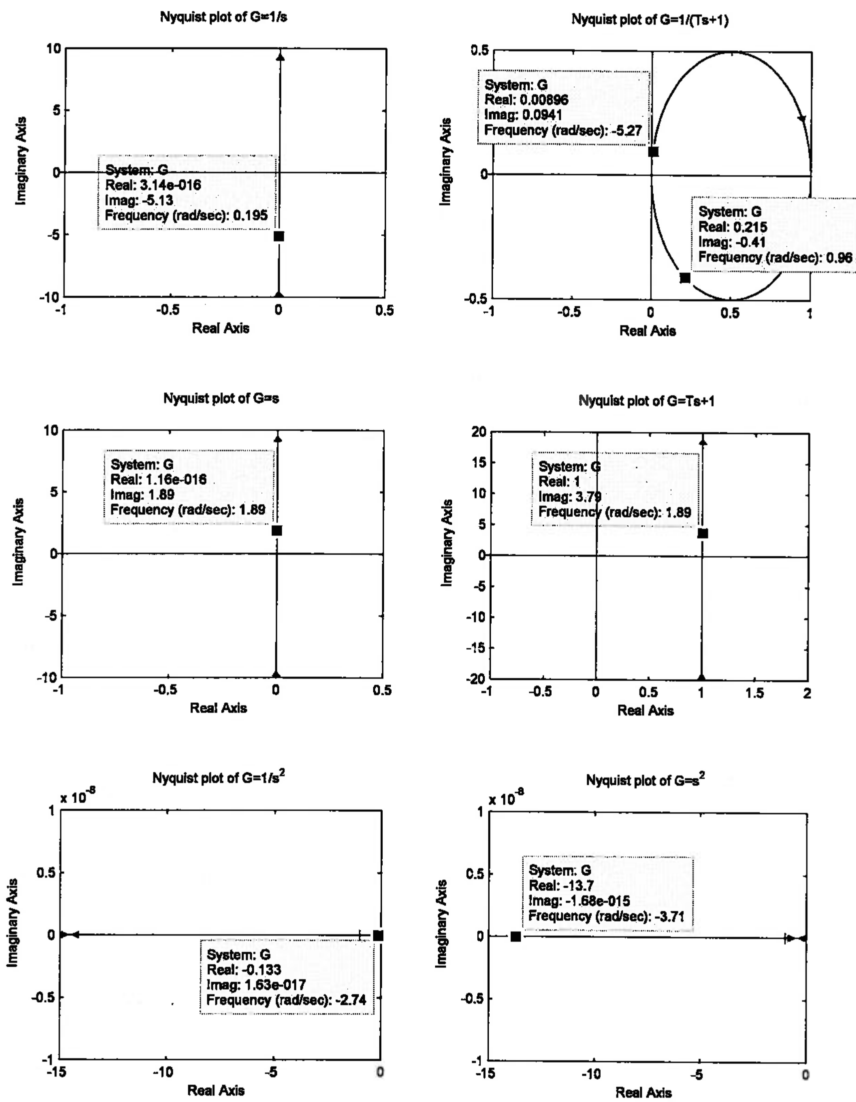
Nyquist plot of G=1/s
System: G
Real: 3.14e-016
Imag: -5.13
Frequency (rad/sec): 0.195
Nyquist plot of G=1/(Ts+1)
System: G
Real: 0.00896
Imag: 0.0941
Frequency (rad/sec): -5.27
System: G
Real: 0.215
Imag: -0.41
Frequency (rad/sec): 0.96
Nyquist plot of G=s
System: G
Real: 1.16e-016
Imag: 1.89
Frequency (rad/sec): 1.89
Nyquist plot of G=Ts+1
System: G
Real: 1
Imag: 3.79
Frequency (rad/sec): 1.89
Nyquist plot of G=1/s^2
System: G
Real: -0.133
Imag: 1.63e-017
Frequency (rad/sec): -2.74
Nyquist plot of G=s^2
System: G
Real: -13.7
Imag: -1.68e-015
Frequency (rad/sec): -3.71
Imaginary Axis
Real Axis

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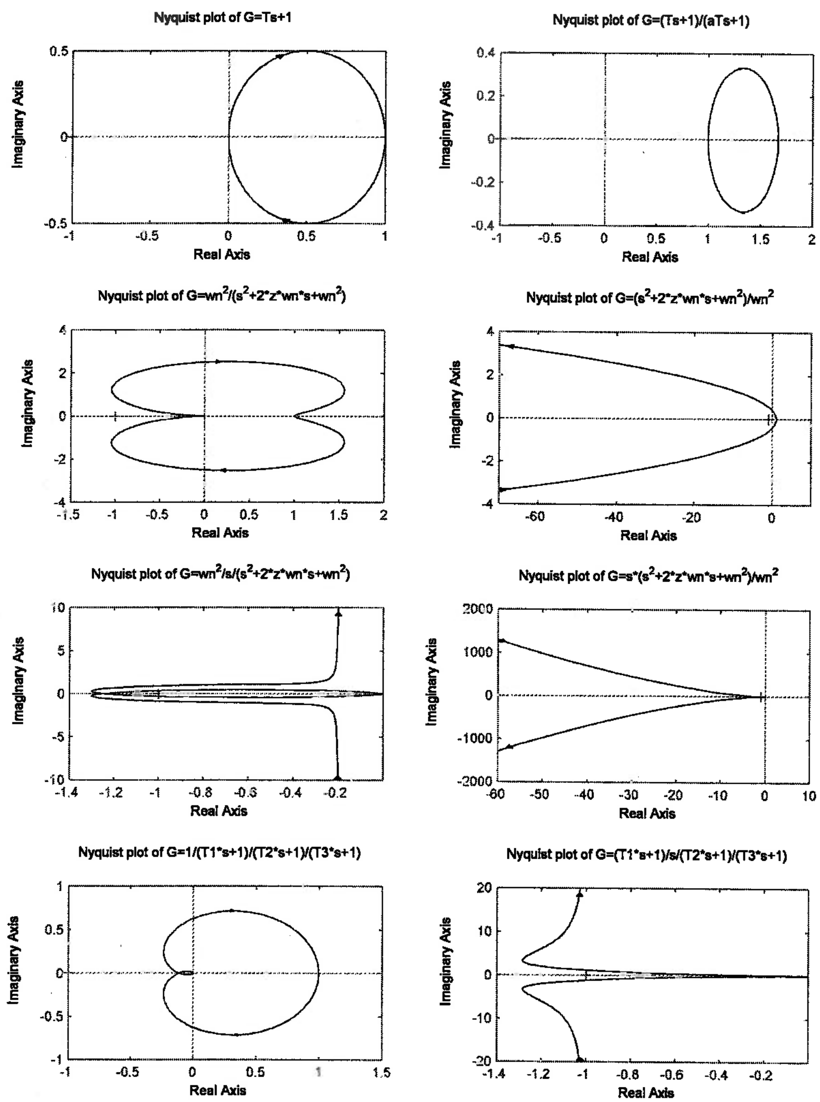
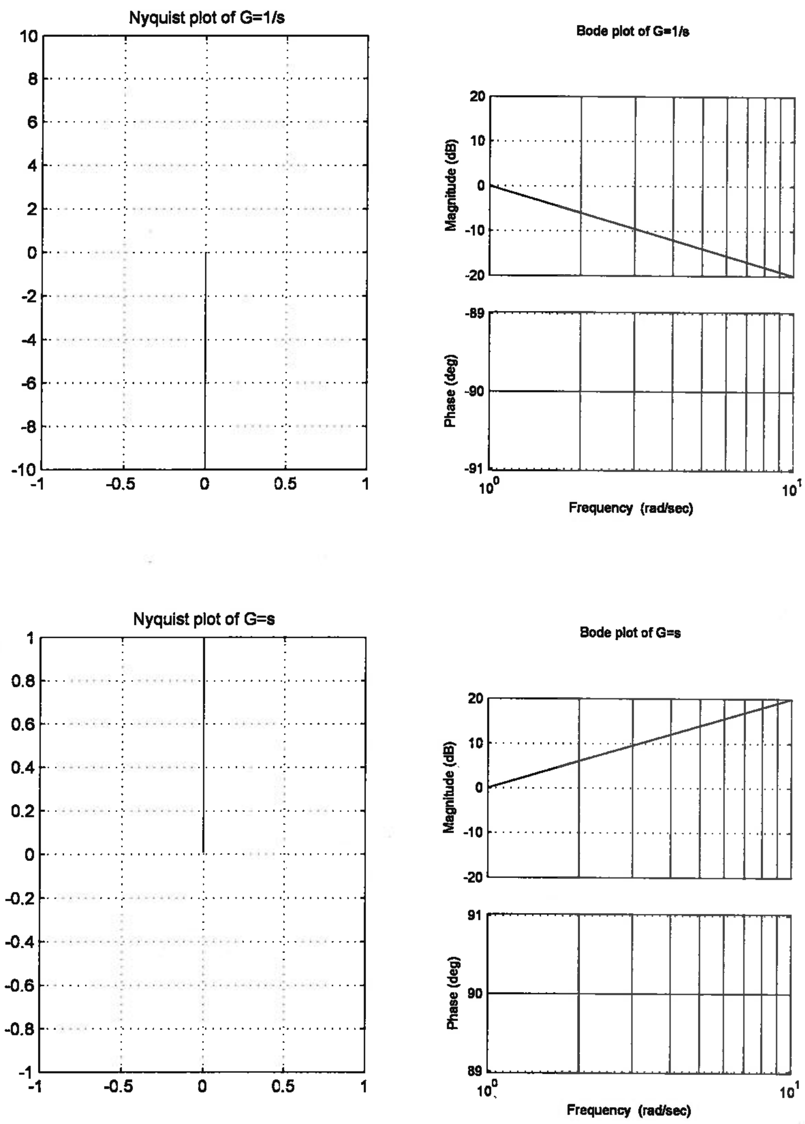

Nyquist plot of G=Ts+1
Nyquist plot of G=(Ts+1)/(aTs+1)
Nyquist plot of G=wn^2/(s^2+2*z*wn*s+wn^2)
Nyquist plot of G=(s^2+2*z*wn*s+wn^2)/wn^2
Nyquist plot of G=wn^2/s/(s^2+2*z*wn*s+wn^2)
Nyquist plot of G=s*(s^2+2*z*wn*s+wn^2)/wn^2
Nyquist plot of G=1/(T1*s+1)/(T2*s+1)/(T3*s+1)
Nyquist plot of G=(T1*s+1)/s/(T2*s+1)/(T3*s+1)
Imaginary Axis
Real Axis


Nyquist plot of G=1/s
Bode plot of G=1/s
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)
Nyquist plot of G=s
Bode plot of G=s
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)

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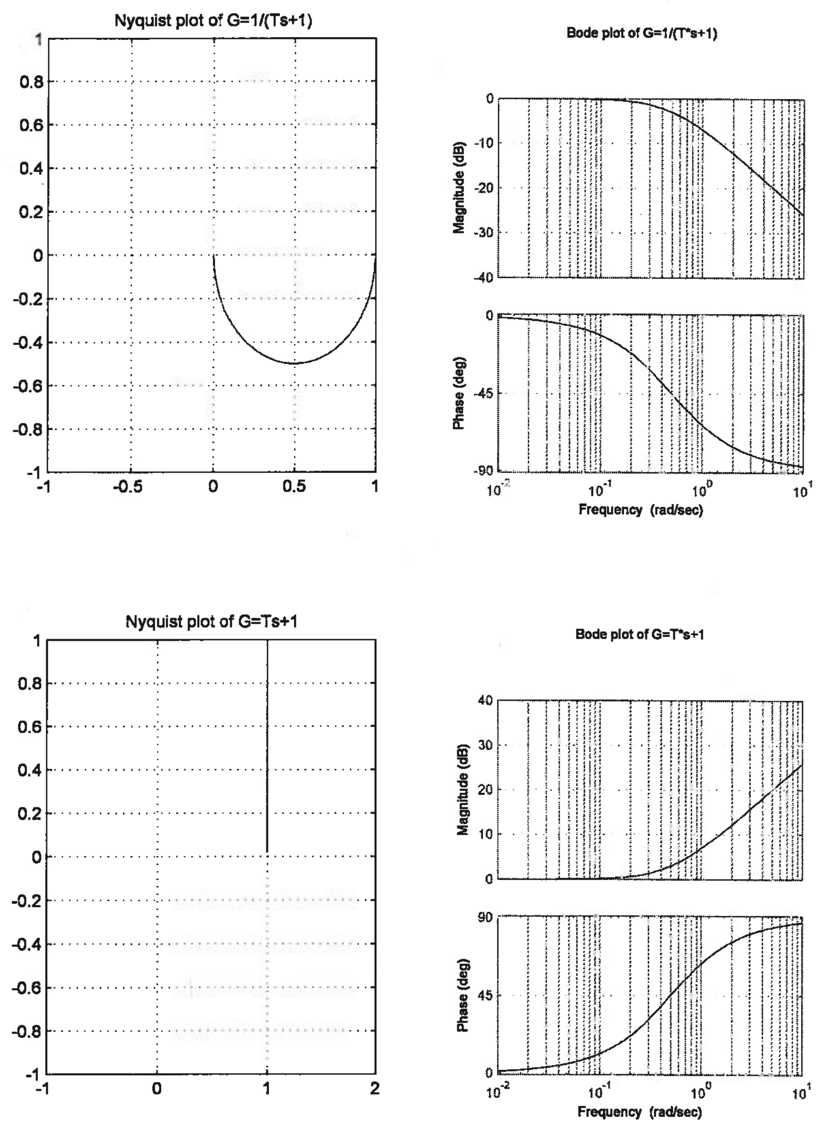
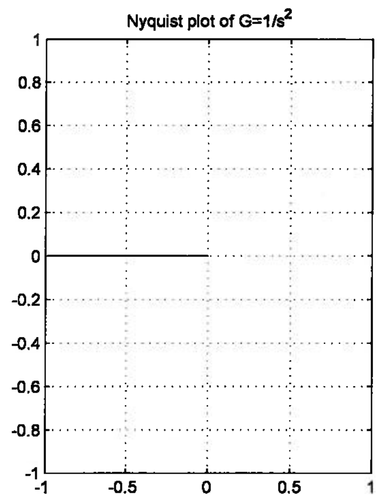
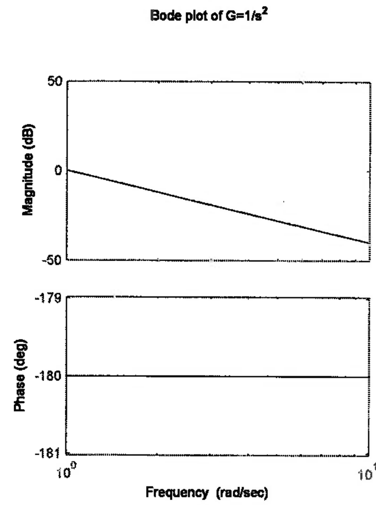
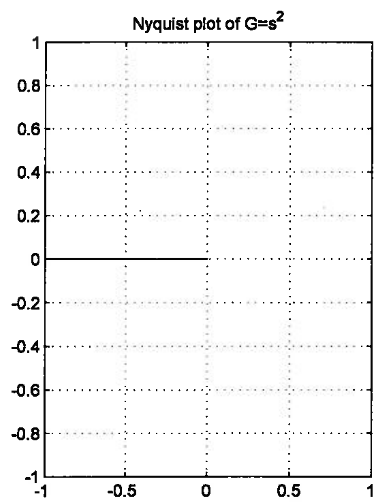
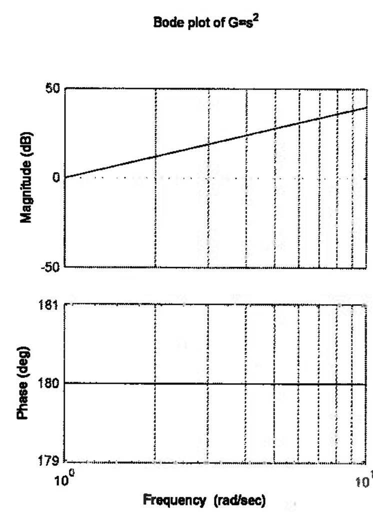

Nyquist plot of G=1/(Ts+1)
Bode plot of G=1/(T*s+1)
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)
Nyquist plot of G=Ts+1
Bode plot of G=T*s+1
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)


Nyquist plot of G=1/s^2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1


Bode plot of G=1/s^2
Magnitude (dB)
50
0
-50
Phase (deg)
-179
-180
-181
10^0
10^1
Frequency (rad/sec)


Nyquist plot of G=s^2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1


Bode plot of G=s^2
Magnitude (dB)
50
0
-50
Phase (deg)
181
180
179
10^0
10^1
Frequency (rad/sec)

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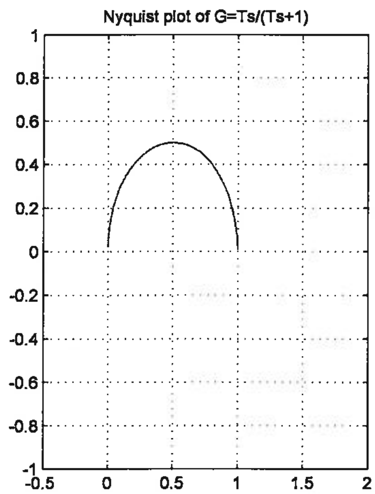
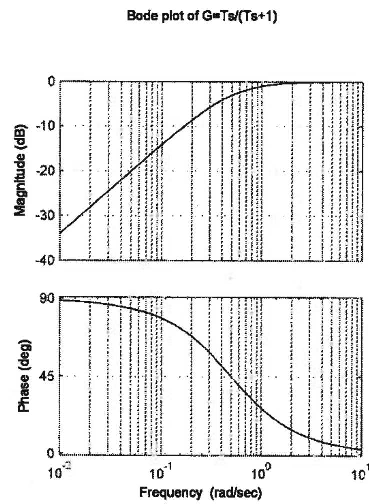
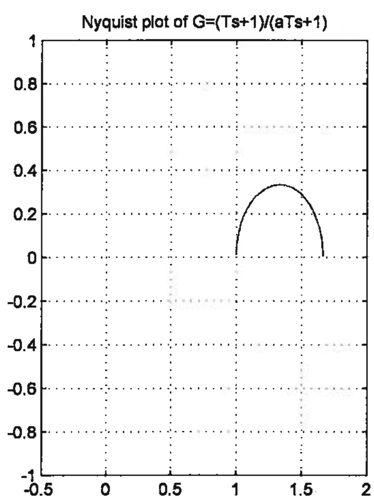
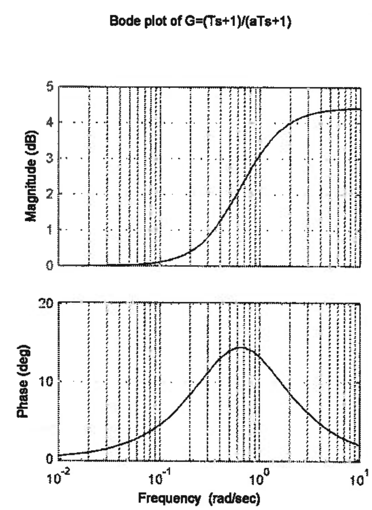
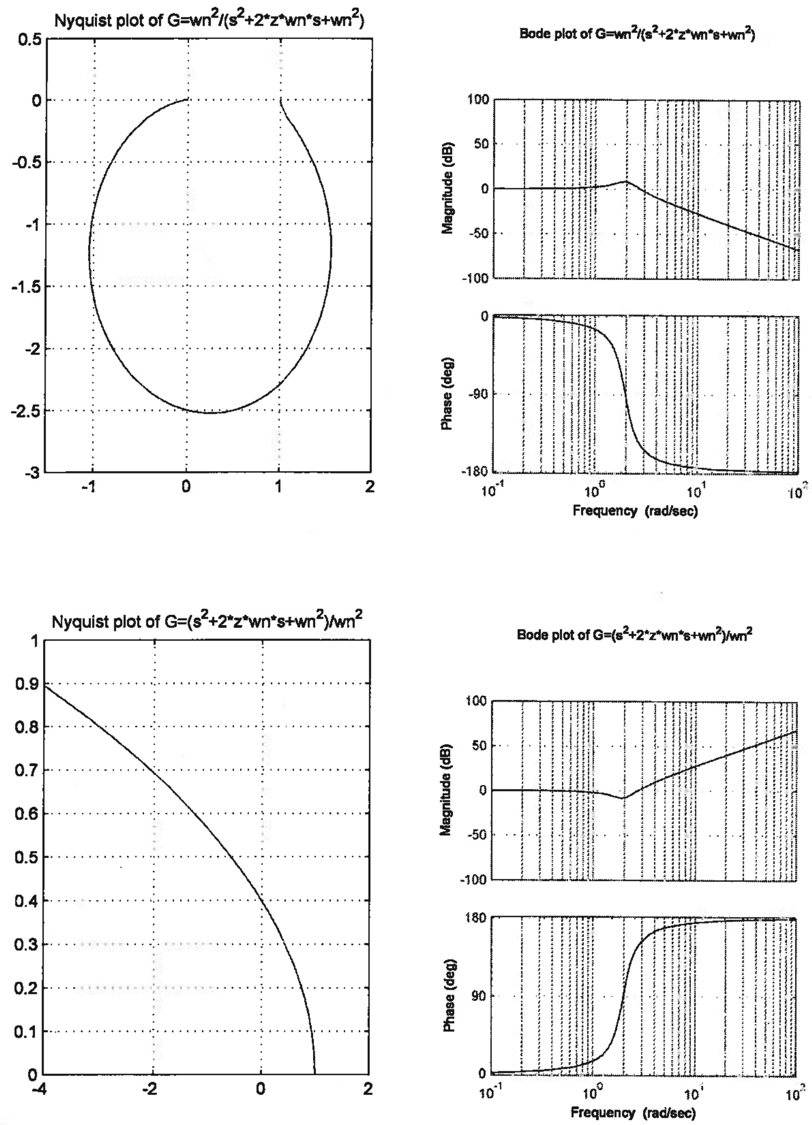
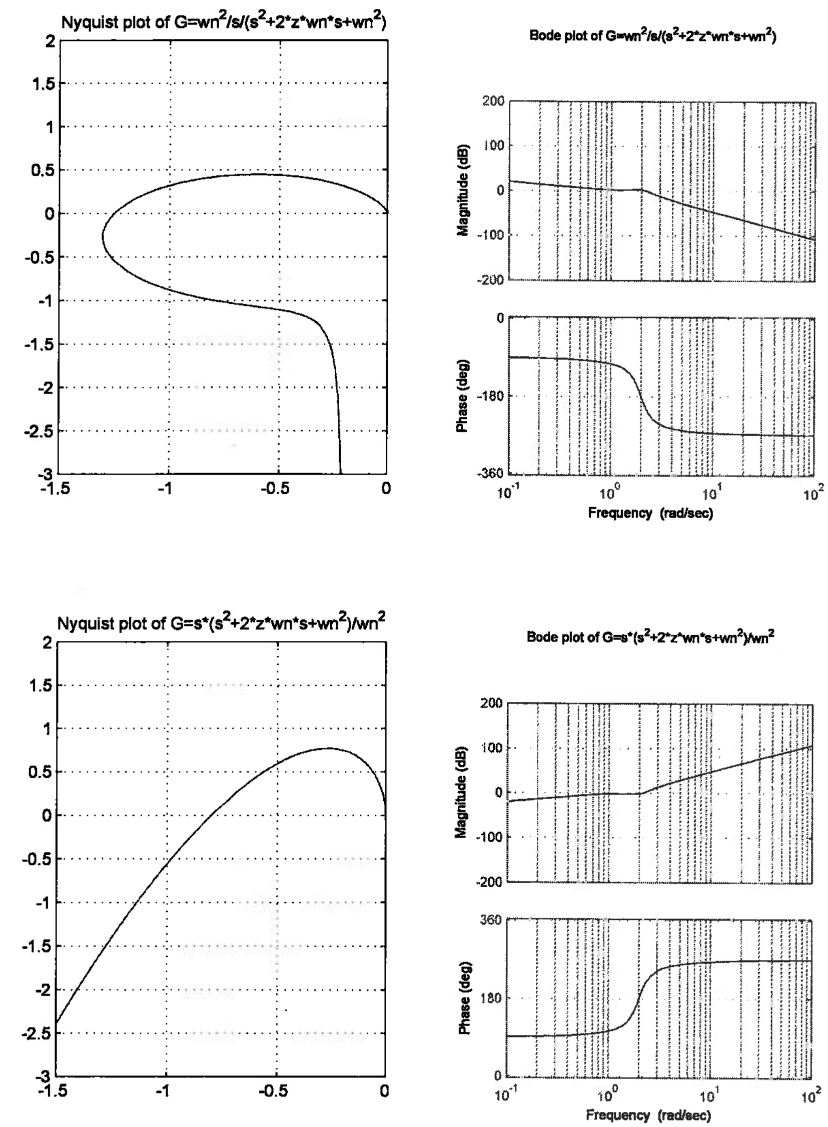

Nyquist plot of G=Ts/(Ts+1)

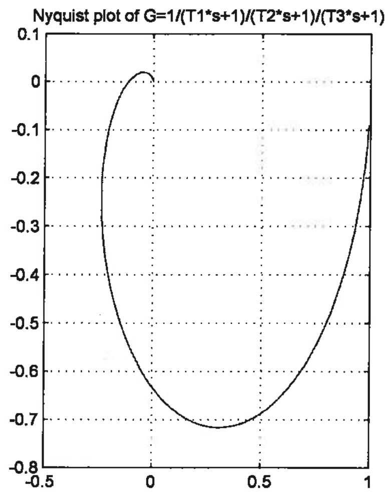

Bode plot of G=Ts/(Ts+1)
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)

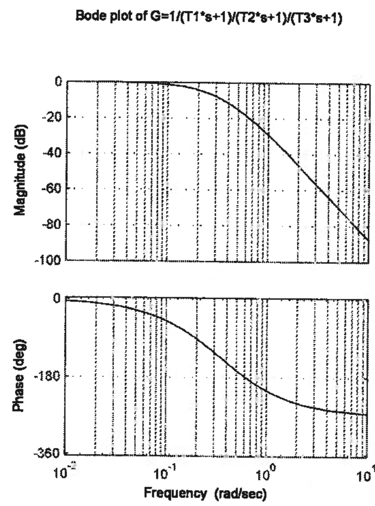

Nyquist plot of G=(Ts+1)/(aTs+1)

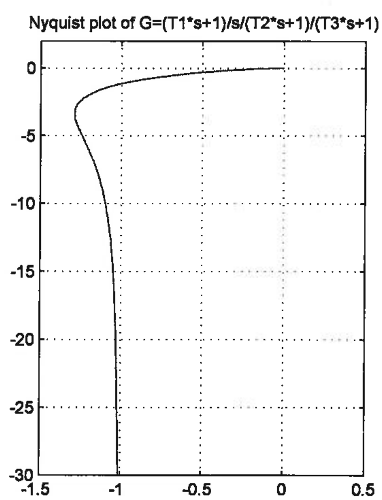

Bode plot of G=(Ts+1)/(aTs+1)
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)

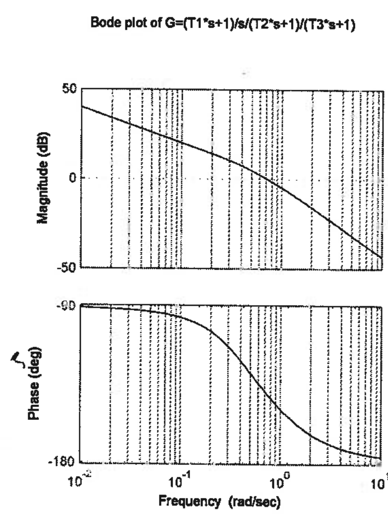

Nyquist plot of G=wn^2/(s^2+2*z*wn*s+wn^2)
Bode plot of G=wn^2/(s^2+2*z*wn*s+wn^2)
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)
Nyquist plot of G=(s^2+2*z*wn*s+wn^2)/wn^2
Bode plot of G=(s^2+2*z*wn*s+wn^2)/wn^2
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)


Nyquist plot of G=wn^2/s/(s^2+2*z*wn*s+wn^2)
Bode plot of G=wn^2/s/(s^2+2*z*wn*s+wn^2)
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)
Nyquist plot of G=s*(s^2+2*z*wn*s+wn^2)/wn^2
Bode plot of G=s*(s^2+2*z*wn*s+wn^2)/wn^2
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)


Nyquist plot of G=1/(T1*s+1)/(T2*s+1)/(T3*s+1)
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.5
0
0.5
1


Bode plot of G=1/(T1*s+1)/(T2*s+1)/(T3*s+1)
Magnitude (dB)
0
-20
-40
-60
-80
-100
Phase (deg)
0
-180
-360
10^{-2}
10^{-1}
10^{0}
10^{1}
Frequency (rad/sec)


Nyquist plot of G=(T1*s+1)/s/(T2*s+1)/(T3*s+1)
0
-5
-10
-15
-20
-25
-30
-1.5
-1
-0.5
0
0.5


Bode plot of G=(T1*s+1)/s/(T2*s+1)/(T3*s+1)
Magnitude (dB)
50
0
-50
Phase (deg)
-90
-180
10^{-2}
10^{-1}
10^{0}
10^{1}
Frequency (rad/sec)

8.6 Stability Margins

MIL-DTL-9490E defines margin requirements for aircraft flight control systems (FCS) with feedback:

GM = gain margin
PM = phase margin

TABLE III. Gain and phase margin requirements (dB, degrees).

Air speed / Mode Frequency Hz	Below V_{oMIN}	V_{oMIN} to V_{oMAX}	At Limit Airspeed (V_L)	At 1.15 V_L
f_M < 0.06	GM= 6 dB (No Phase Requirement Below V_{oMIN})	GM = $\pm$ 4.5 PM = $\pm$ 30	GM = $\pm$ 3.0 PM = $\pm$ 20	GM=0 PM=0 (Stable at Nominal Phase and Gain)
0.06 ≤ f_M < First Aero-elastic Mode		GM = $\pm$ 6.0 PM = $\pm$ 45	GM = $\pm$ 4.5 PM = $\pm$ 30	
f_M > First Aero-Elastic Mode		GM = $\pm$ 8.0 PM = $\pm$ 60	GM = $\pm$ 6.0 PM = $\pm$ 45	

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Where:

V_L	=	Limit airspeed (MIL-A-8860).
V_{oMIN}	=	Minimum operational airspeed (MIL-F-8785)
V_{oMAX}	=	Maximum operational airspeed (MIL-F-8785)
Mode	=	A characteristics aeroelastic response of the aircraft as described by the an aeroelastic characteristic root of the coupled aircraft/FCS dynamic equation of motion.
GM (Gain Margin)	=	change The minimum chain in loop gain at normal phase, which results in instability beyond that allowed as a residual oscillation.
PM (Phase Margin)	=	The minimum change in phase at normal loop gain which results in instability.
f_M	=	Mode frequency in Hz
Nominal Phase and Gain	=	The contractor's best estimate or measurement of FCS and aircraft phase and gain characteristics available at the time of requirement verification.

3.1.3.6.2 Sensitivity analysis. Tolerances on feedback gain and phase shall be .established at the system level based on the anticipated range of gain and phase errors which will exist between nominal test values or predictions and in-service operation due to such factors as poorly defined nonlinear and higher order dynamics, anticipated manufacturing tolerances, aging, wear, maintenance and noncritical material failures. Gain and phase margins shall be defined, based on these tolerances, which will assure satisfactory operation in fleet usage. These gain and phase tolerances shall be established based on variations in system characteristics either anticipated or allowed by component or subsystem specification. The contractor shall establish, with the approval of the procuring agency, the range of variation to be considered based on a selected probability of exceedance for each type of variation. The contractor shall select the exceedance probability based on the criticality of the flight control function being provided. The stability requirements established through this sensitivity analysis shall not be less than 50 percent of the magnitude and phase requirements of 3.1.3.6.1.

3.1.3.7 Operation in turbulence. In Operational State I, while flying in the following applicable random and discrete turbulence environment, the FCS shall provide a safe level of operation and maintain mission accomplishment capabilities. For essential and flight phase

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commands by two aircrew members from causing any operation in opposing directions at the same time.

3.1.3.6 Stability. For FCS using feedback systems, the stability as specified in 3.1.3.6.1 shall be provided. Alternatively, when approved by the procuring activity, the stability defined by the contractor through the sensitivity analyses of 3.1.3.6.2 shall be provided. Where analysis is used to demonstrate compliance with these stability requirements, the effects of major system nonlinearities shall be included.

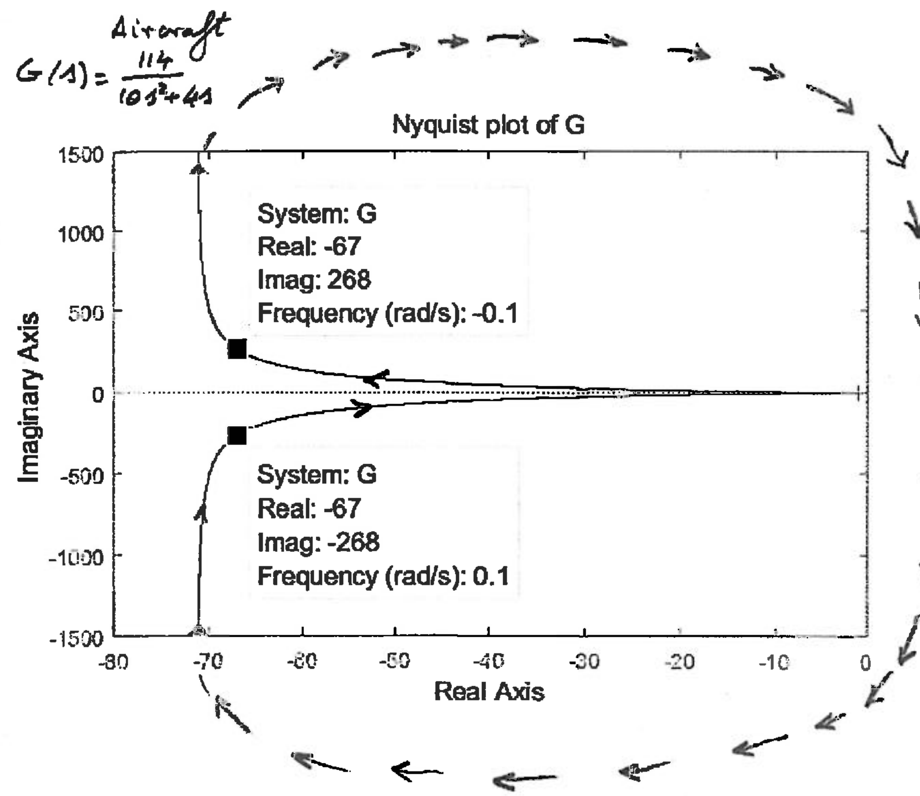
3.1.3.6.1 Stability margins. Required gain and phase margins about nominal are specified in table III for all aerodynamically closed loop FCS. With these gain or phase variations included, no oscillatory instabilities shall exist with amplitudes greater than those allowed for residual oscillations in 3.1.3.8, and any non oscillatory divergence of the aircraft shall remain within the applicable limits of MIL-F-8785 or MIL-F-83300. AFCS loops shall be stable with these gain or phase variations included for any amplitudes greater than those allowed for residual oscillations in 3.1.3.8. In multiple loop systems, variations shall be made with all gain and phase values in the feedback paths held at nominal values except for the path under investigation. A path is defined to include those elements connecting a sensor to a force or moment producer. For both aerodynamic and nonaerodynamic closed loops, at least 6 dB gain margin shall exist at zero airspeed. At the end of system wear tests, at least 4.5 dB gain margin shall exist for all loops at zero airspeed. The margins specified by table III shall be maintained under flight conditions of most adverse center-of-gravity, mass distribution, and external store configuration throughout the operational envelope and during ground operations.

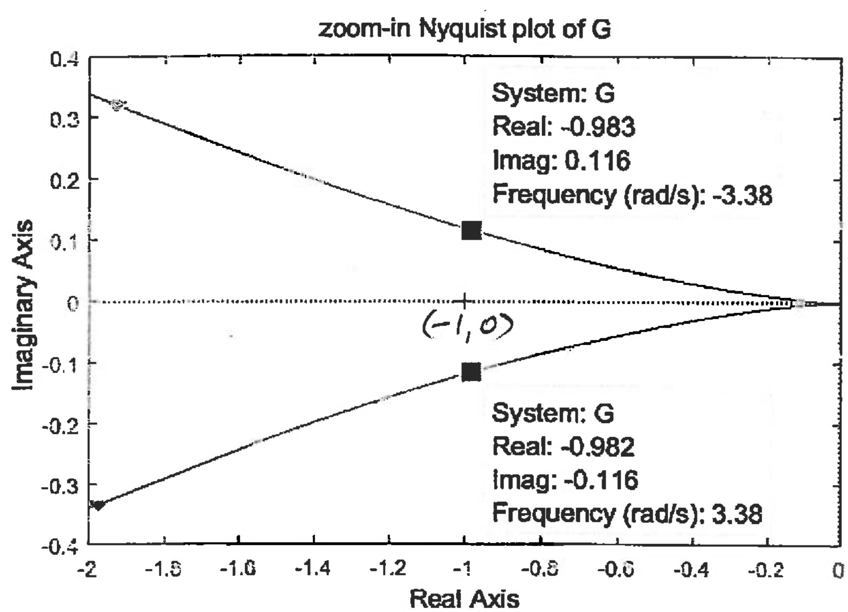
TABLE III. Gain and phase margin requirements (dB, degrees).

<table>
<tr><th>Air speed
Mode
Frequency
Hz</th><th>Below
V_{oMIN}</th><th>V_{oMIN}
to
V_{oMAX}</th><th>At
Limit Airspeed
(V_L)</th><th>At
1.15 V_L</th></tr>
<tr><td>f_M <0.06</td><td rowspan="3">GM= 6 dB
(No Phase Requirement Below V_{oMIN})</td><td>GM = $\pm$ 4.5
PM = $\pm$ 30</td><td>GM = $\pm$ 3.0
PM = $\pm$ 20</td><td rowspan="3">GM=0
PM=0
(Stable at Nominal Phase and Gain)</td></tr>
<tr><td>0.06 $\leq f_M$ <First Aero-elastic Mode</td><td>GM = $\pm$ 6.0
PM = $\pm$ 45</td><td>GM = $\pm$ 4.5
PM = $\pm$ 30</td></tr>
<tr><td>f_M > First Aero-Elastic Mode</td><td>GM =$\pm$ 8.0
PM = $\pm$ 60</td><td>GM =$\pm$ 6.0
PM = $\pm$ 45</td></tr>
</table>

4
M

Recall Nyquist plot





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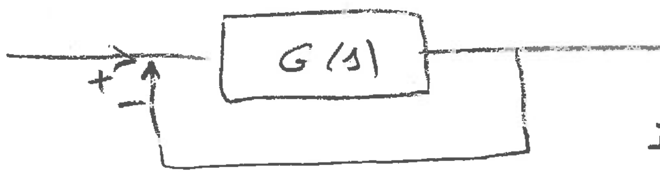
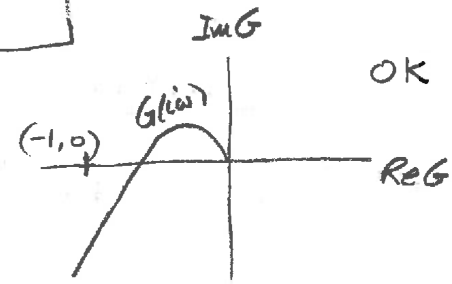
M

Margins analysis

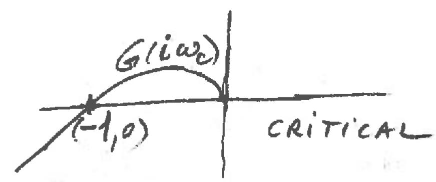
- Gain margin
- Phase margin





$$G_{CL}(s) = \frac{G(s)}{1+G(s)}.$$

$G_{CL}(i\omega) \longrightarrow \infty$
if $1+G(i\omega) \to 0$



CRITICAL CONDITION ω_c

$$\boxed{G(i\omega_c) = -1} < \begin{array}{l} |G(i\omega_c)| = 0 \text{ dB} \\ \angle G(i\omega_c) = -180^\circ \end{array}$$

$$-1 = e^{-i\pi}$$

Objective: stay away from (−1,0) point!

Method
(1) Plot Bode diagram
(2) Determine margins:
- Gain margin: distance from 0dB line
- Phase margin: distance from −180° line

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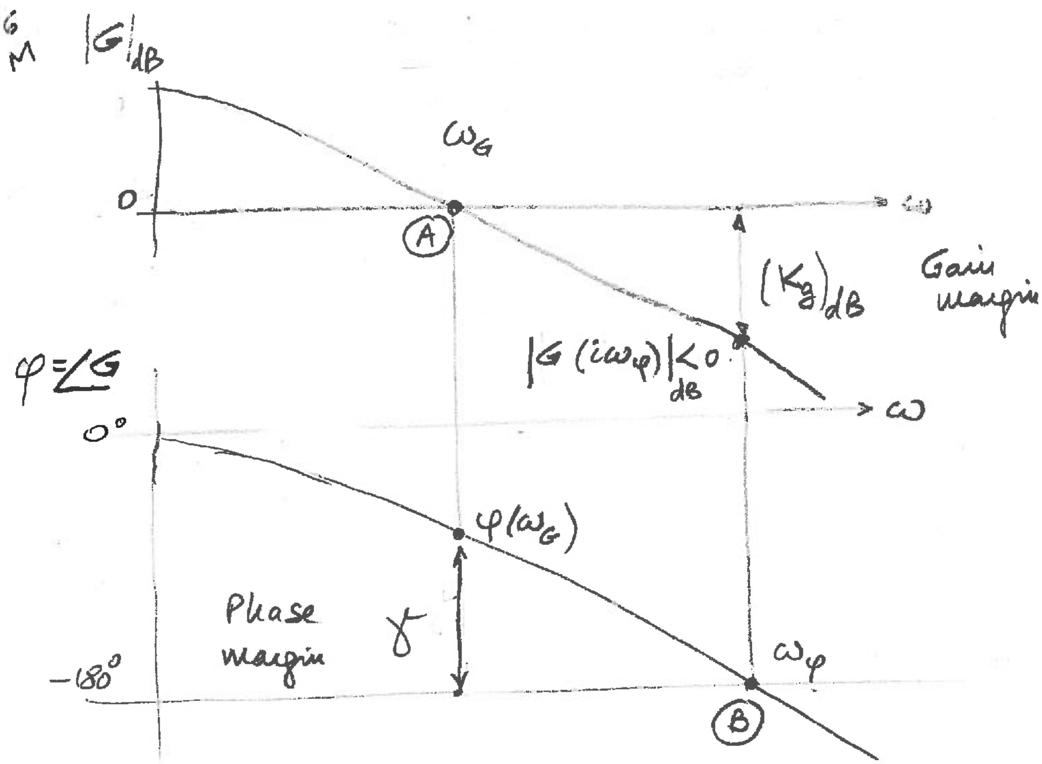
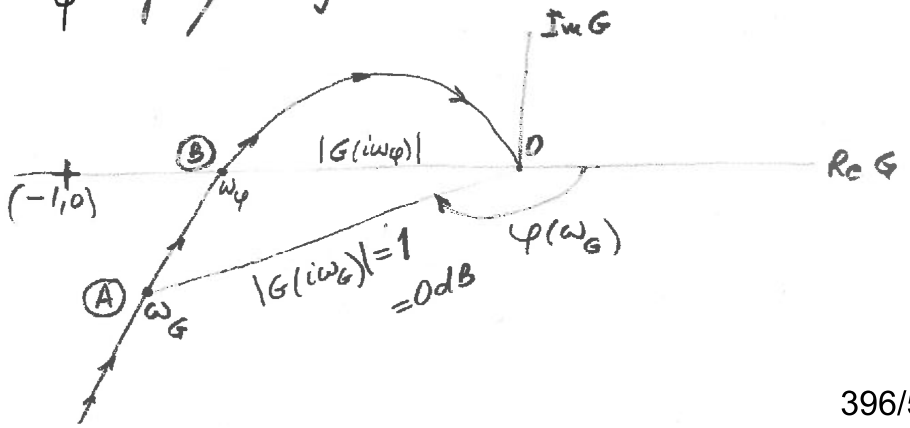


Gain margin $(K_g)_{dB} = 0 - |G(i\omega_\varphi)| = -|G(i\omega_\varphi)| > 0$

Phase margin $\gamma = \angle G(i\omega_G) - (-180^\circ) = 180^\circ + \angle G(i\omega_G) > 0$

ω_G = frequency when $|G(\omega_G)| = 0$ dB

ω_φ = frequency when $\varphi = -180^\circ$



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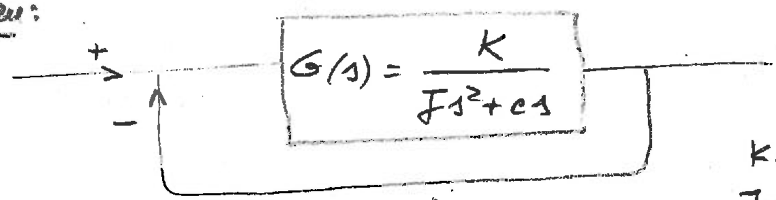
Example : aircraft roll model

Given:

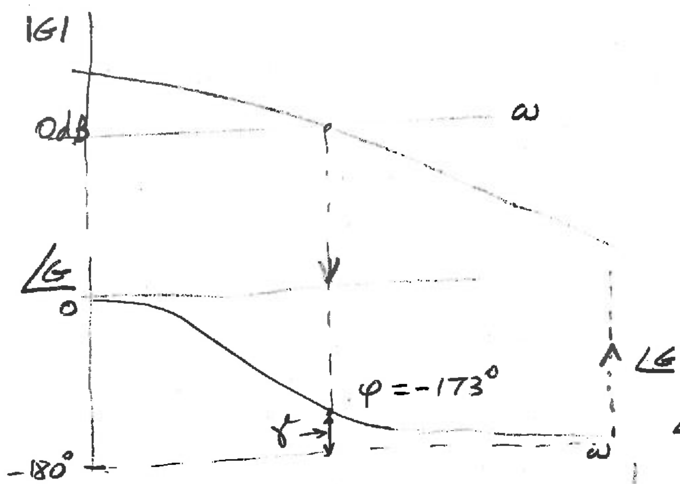


$K = 114$
$J = 10$
$c = 4$

Find

- gain margin, $(K_g)_{dB}$
- phase margin γ deg

Solution ; Bode plot (see MATLAB plot)(Fig. 1)



$|G|_{\angle G = -180^\circ} = -\infty$

∠G Never crosses −180° line

$\angle G \xrightarrow[\omega \to \infty]{} -180^\circ$

$\gamma = \varphi + 180^\circ = -173^\circ + 180^\circ = 7^\circ \ll PM$

Very small phase margin!

$K_g = \infty > GM$

OK

NEED TO IMPROVE PHASE MARGIN!

Plenty gain margin

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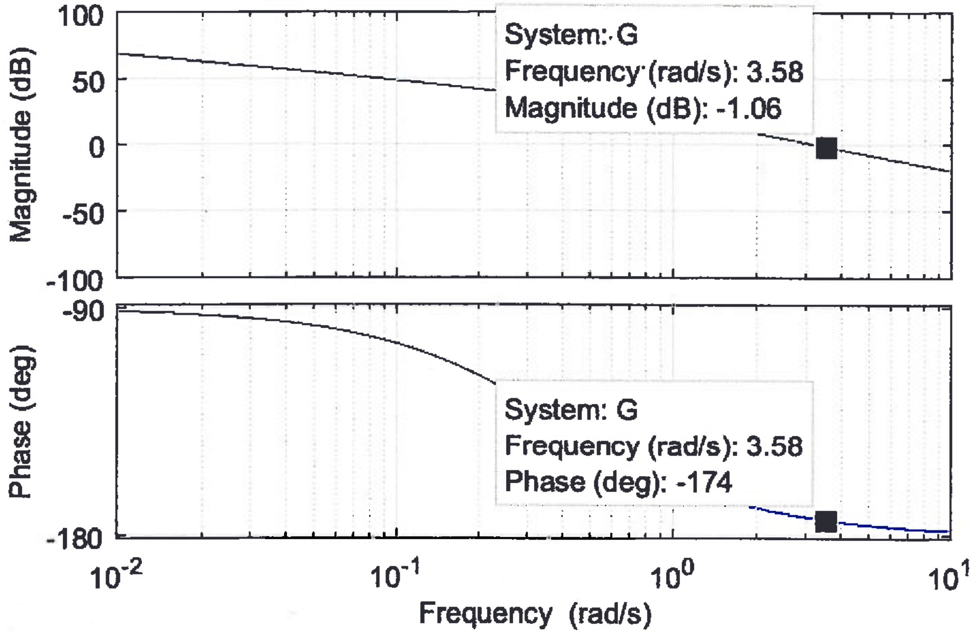
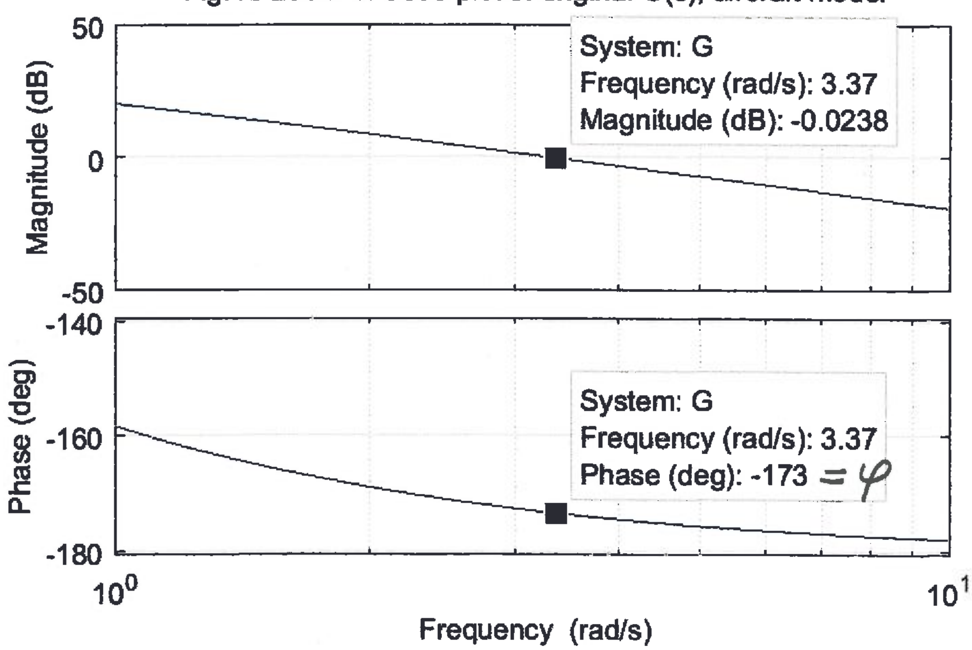
Aircraft roll model $G(s) = \frac{114}{10s^2+4s}$

Fig.1a Bode plot of original G(s); aircraft model



Phase margin $\gamma^{\circ} = 180^{\circ} - 173^{\circ} = 7^{\circ}$ $= 180^{\circ} - \varphi$; Gain margin $(K_g)_{dB} = \infty$

Fig.1b zoom-in Bode plot of original G(s); aircraft model



8
M

```
Phase margin: aircraft model
input data

    K   | J  |   c =

   114    10     4


G =

      114
  ------------
  10 s^2 + 4 s

Continuous-time transfer function.

========================

GM, dB | PM, deg =

    10    60

========================

phi = phase at |G|=0 dB point, deg =

  -173


gamma = phase margin, deg =

     7

========================
```

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margin

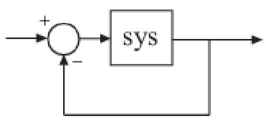
Gain margin, phase margin, and crossover frequencies

Syntax

```
[Gm,Pm,Wgm,Wpm] = margin(sys)
[Gm,Pm,Wgm,Wpm] = margin(mag,phase,w)
margin(sys)
```

Description

`margin` calculates the minimum gain margin, `Gm`, phase margin, `Pm`, and associated frequencies `Wgm` and `Wpm` of SISO open-loop models. The gain and phase margin of a system `sys` indicates the relative stability of the closed-loop system formed by applying unit negative feedback to `sys`, as in the following illustration.



The gain margin is the amount of gain increase or decrease required to make the loop gain unity at the frequency `Wgm` where the phase angle is –180° (modulo 360°). In other words, the gain margin is $1/g$ if g is the gain at the –180° phase frequency. Similarly, the phase margin is the difference between the phase of the response and –180° when the loop gain is 1.0. The frequency `Wpm` at which the magnitude is 1.0 is called the *unity-gain frequency* or *gain crossover frequency*. It is generally found that gain margins of three or more combined with phase margins between 30 and 60 degrees result in reasonable trade-offs between bandwidth and stability.

`[Gm,Pm,Wgm,Wpm] = margin(sys)` computes the gain margin `Gm`, the phase margin `Pm`, and the corresponding frequencies `Wgm` and `Wpm`, given the SISO open-loop dynamic system model `sys`. `Wgm` is the frequency where the gain margin is measured, which is a –180 degree phase crossing frequency. `Wpm` is the frequency where the phase margin is measured, which is a 0dB gain crossing frequency. These frequencies are expressed in radians/`TimeUnit`, where `TimeUnit` is the unit specified in the `TimeUnit` property of `sys`. When `sys` has several crossovers, `margin` returns the smallest gain and phase margins and corresponding frequencies.

The phase margin `Pm` is in degrees. The gain margin `Gm` is an absolute magnitude. You can compute the gain margin in dB by

```
Gm_dB = 20*log10(Gm)
```

`[Gm,Pm,Wgm,Wpm] = margin(mag,phase,w)` derives the gain and phase margins from Bode frequency response data (magnitude, phase, and frequency vector). `margin` interpolates between the frequency points to estimate the margin values. Provide the gain data `mag` in absolute units, and phase data `phase` in degrees. You can provide the frequency vector `w` in any units; `margin` returns `Wgm` and `Wpm` in the same units.

Note

When you use `margin(mag,phase,w)`, `margin` relies on interpolation to approximate the margins, which generally produces less accurate results. For example, if there is no 0 dB crossing within the `w` range, `margin` returns a phase margin of `Inf`. Therefore, if you have an analytical model `sys`, using `[Gm,Pm,Wgm,Wpm] = margin(sys)` is the most robust way to obtain the margins.

`margin(sys)`, without output arguments, plots the Bode response of `sys` on the screen and indicates the gain and phase margins on the plot. By default, gain margins are expressed in dB on the plot.

Examples

Gain and Phase Margins of Open-Loop Transfer Function

Create an open-loop discrete-time transfer function.

```
hd = tf([0.04798 0.0464],[1 -1.81 0.9048],0.1)
```

```
hd =

   0.04798 z + 0.0464
  ---------------------
  z^2 - 1.81 z + 0.9048

Sample time: 0.1 seconds
Discrete-time transfer function.
```

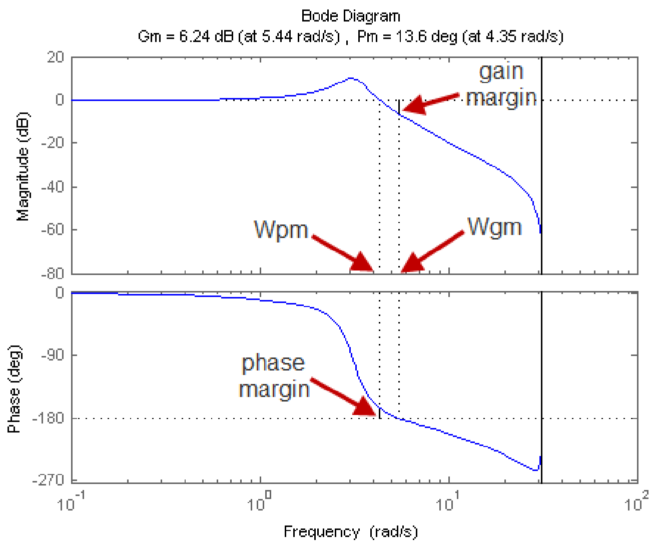
Compute the gain and phase margins.

```
[Gm,Pm,Wgm,Wpm] = margin(hd)
```

```
Gm =

    2.0517


Pm =

   13.5711


Wgm =

    5.4374


Wpm =

    4.3544
```

Display the gain and phase margins graphically.

```
margin(hd)
```



Solid vertical lines mark the gain margin and phase margin. The dashed vertical lines indicate the locations of `Wpm`, the frequency where the phase margin is measured, and `Wgm`, the frequency where the gain margin is measured.

Algorithms

The phase margin is computed using H_∞ theory, and the gain margin by solving $H(j\omega) = \overline{H(j\omega)}$ for the frequency ω.

See Also

Linear System Analyzer | `bode`

Introduced before R2006a

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```
margins_aircraft.m
%% initialization
clc %clear command window
clear %removes all variables from workspace; release memory
format compact
close all %closes all figures
s=tf('s');
%% original aircraft model
display('Phase margin: aircraft model')
K=114;                  % gain
J=10;                   % inertia
c=4;                    % damping
display('input data')
display([K J c],'    K    | J   |    c')
G=K/(J*s^2+c*s)         % G(s)
figure(1)
subplot(2,1,1)
bode(G)
grid
title('Fig.1a Bode plot of aircraft model')
subplot(2,1,2)
d1=0;d2=1;N=1e3; w=logspace(d1,d2,N);
bode(G,w)
grid
title('Fig.1b zoom-in Bode plot of aircraft model')
% READ ON PLOT: phase at |G|=0 dB point, deg
% phi=-173;
phi=input('Input phase read on Bode plot in deg, phi=');
gamma=phi-(-180);   % gamma = phase margin, deg
display([gamma],'gamma = phase margin, deg')
figure(2)
margin(G)
```

```
Phase margin: aircraft model
input data
    K    | J   |    c =
   114    10       4

G =

      114
  ------------
  10 s^2 + 4 s

Continuous-time transfer function.

Input phase read on Bode plot in deg, phi=-173
gamma = phase margin, deg =
     7
```

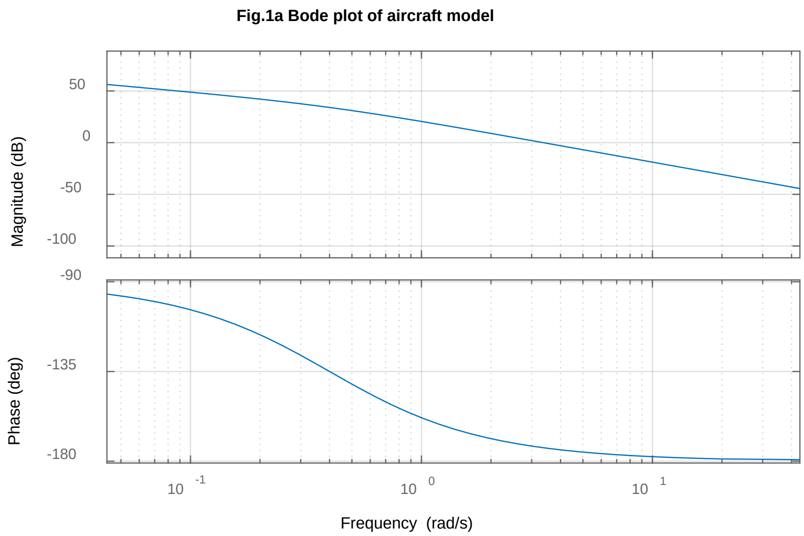


Fig.1a Bode plot of aircraft model
Magnitude (dB)
50
0
-50
-100
Phase (deg)
-90
-135
-180
10^-1
10^0
10^1
Frequency (rad/s)

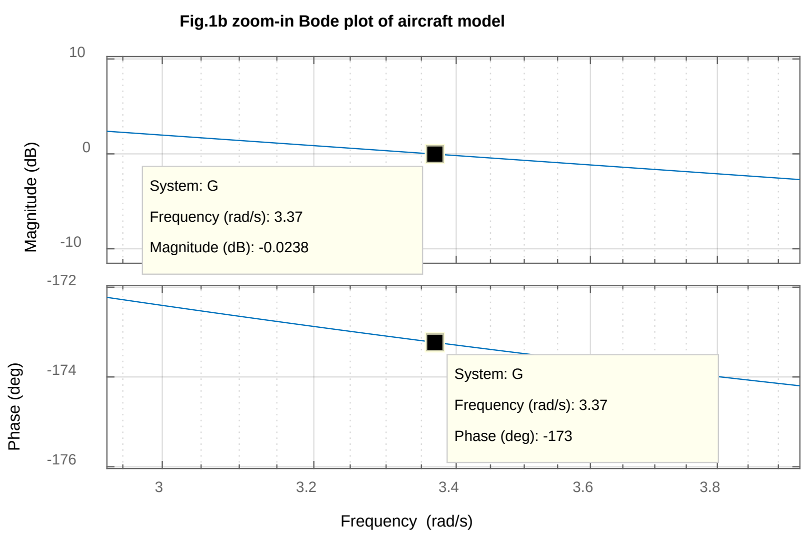
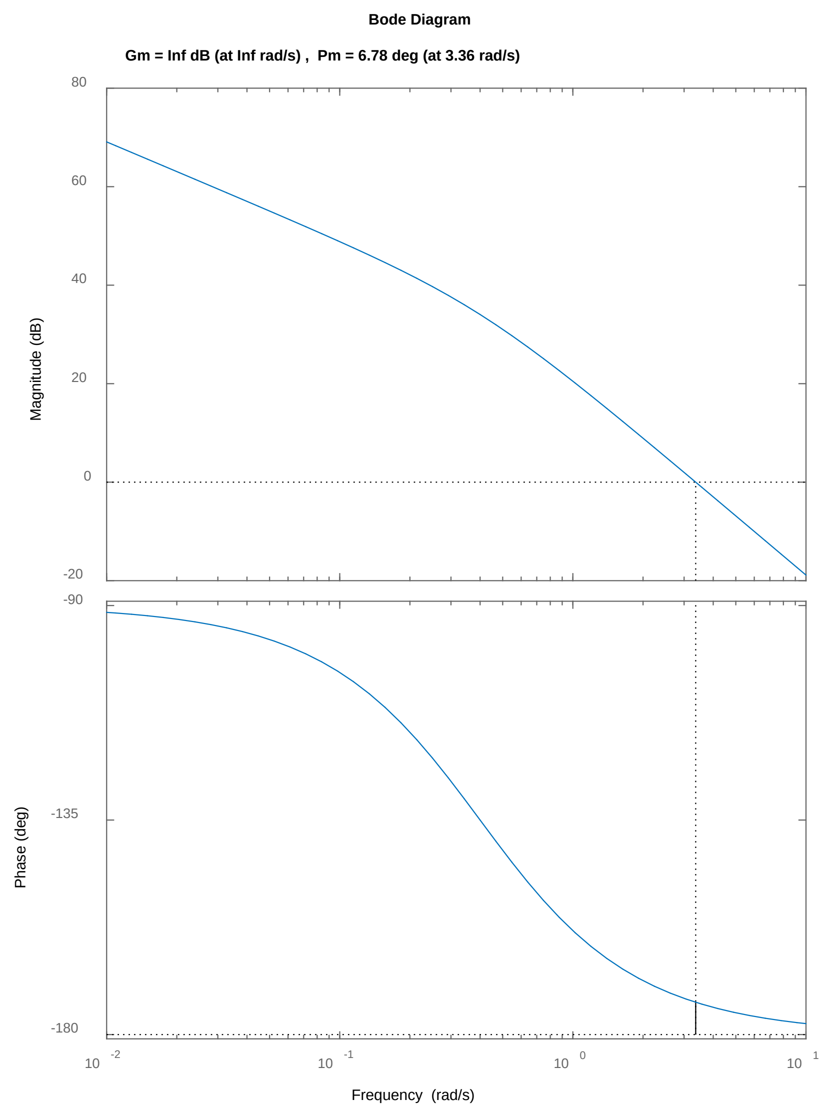


Fig.1b zoom-in Bode plot of aircraft model
Magnitude (dB)
10
0
-10
System: G
Frequency (rad/s): 3.37
Magnitude (dB): -0.0238
Phase (deg)
-172
-174
-176
System: G
Frequency (rad/s): 3.37
Phase (deg): -173
3
3.2
3.4
3.6
3.8
Frequency (rad/s)


Bode Diagram
Gm = Inf dB (at Inf rad/s) , Pm = 6.78 deg (at 3.36 rad/s)
80
60
40
20
0
-20
Magnitude (dB)
-90
-135
-180
Phase (deg)
10 -2
10 -1
10 0
10 1
Frequency (rad/s)

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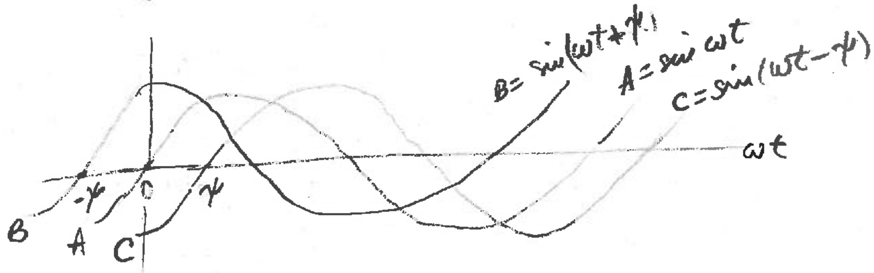
8.7 Phase Compensators

C1 PHASE COMPENSATORS

Recall signal phase definition:



A : $\sin(\omega t) = 0$ @ $t = 0$

B: $\sin(\omega t + \psi) = 0$ @ $\omega t = -\psi$ "ahead of time" it leads

C: $\sin(\omega t - \psi) = 0$ @ $\omega t = \psi$ "delayed" it lags

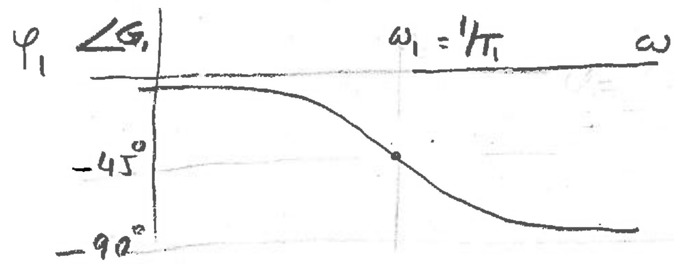
Phase compensator

$G_c(i\omega) = |G_c(i\omega)| e^{i\varphi_c}$, $\varphi_c = \angle G_c(i\omega)$

if $\varphi_c > 0$, then "lead compensator"

$\varphi_c < 0$ "lag compensator"

Example 1: 1^{st} order system $G_1(s) = \dfrac{1}{sT_1 + 1}$



$\varphi_1 < 0$

"Lag"

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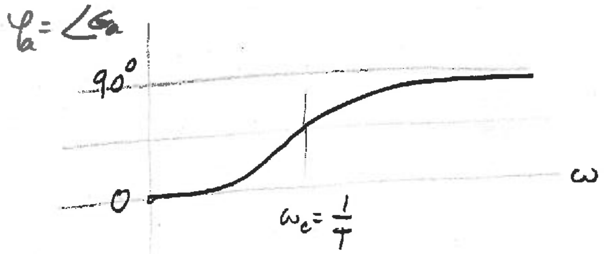
C2

Example 2: $G_a = sT_a + 1$

$$G_a(i\omega) = i\omega T_a + 1; \quad \varphi_a = \angle G_a = \tan^{-1} \omega T_a > 0$$



$\varphi_a > 0$

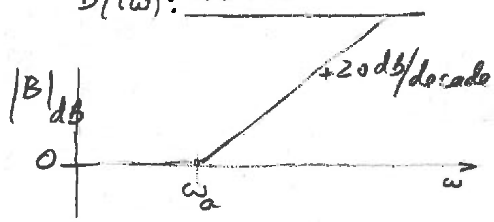
"Lead"

Example 3 $G_c = \frac{sT_a + 1}{sT_1 + 1} = \frac{B(s)}{A(s)}$

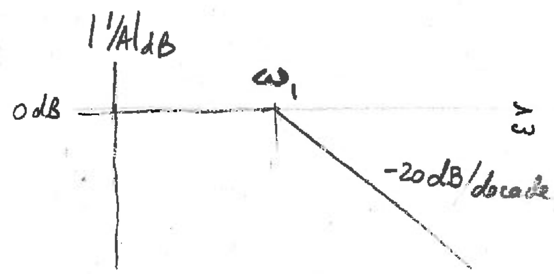
$$G_c(i\omega) = \frac{i\omega T_a + 1}{i\omega T_1 + 1} = \frac{B(i\omega)}{A(i\omega)}$$

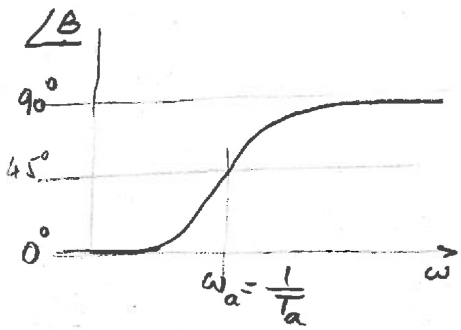
$$|G_c|_{dB} = |B|_{dB} - |A|_{dB} \quad ; \quad \angle G_c = \angle B - \angle A$$

$B(i\omega)$: NUMERATOR

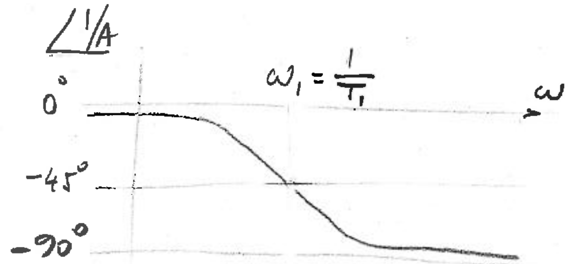


$A(i\omega)$: DENOMINATOR





"Lead"



"Lag"

C2a

LEAD COMPENSATOR

FOR PHASE MARGIN IMPROVEMENT

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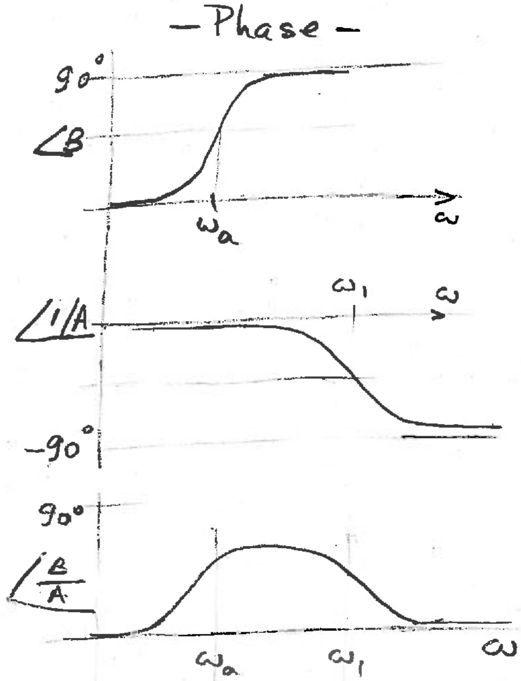
C3

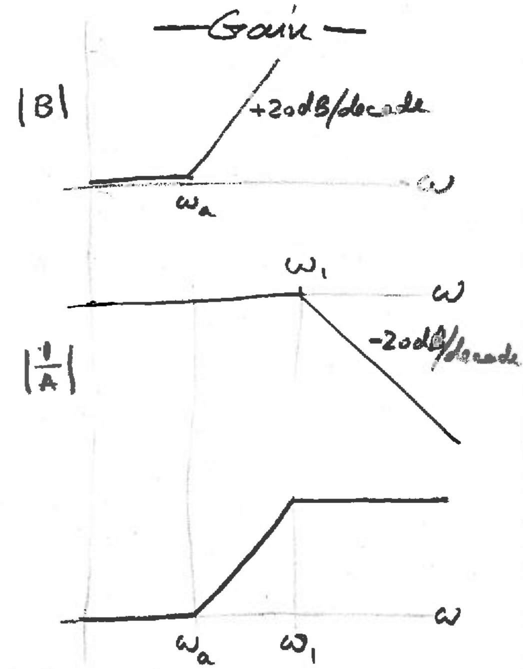
LEAD COMPENSATOR

$\omega_a < \omega_1$, $T_1 < T_a$

$G_c = \dfrac{sT_a+1}{sT_1+1}$

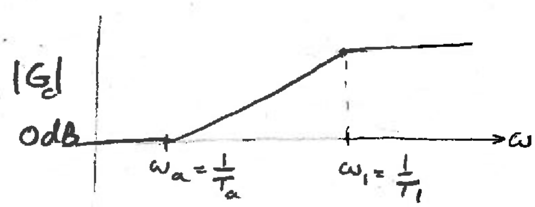
$\omega_a = \dfrac{1}{T_a}$, $\omega_1 = \dfrac{1}{T_1}$





"phase lead" between ω_a & ω_1

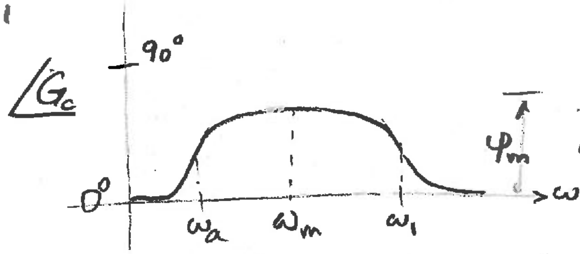
High pass filter
(higher frequencies are amplified)



HP filter

$G_c(s) = \dfrac{sT_a+1}{sT_1+1}$

$T_1 < T_a$



maximum phase lead.

C3a

Lead Compensator Design

Use lead compensator to move phase plot upward and improve phase margin

$$G_c = \frac{sT_a+1}{sT_1+1} = \frac{sT+1}{s\alpha T+1} \quad \text{lead compensator} \quad (1)$$

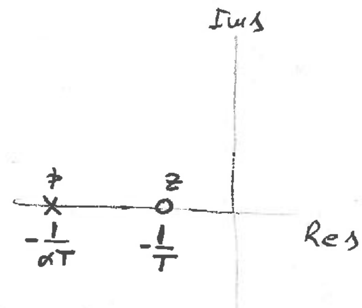
$$\omega_a < \omega_1 \qquad 0 < \alpha < 1$$

$$T_a > T_1$$

Poles & zeros :

$$sT+1=0 \longrightarrow z = -\frac{1}{T}$$

$$s\alpha T+1=0 \longrightarrow p = -\frac{1}{\alpha T}$$

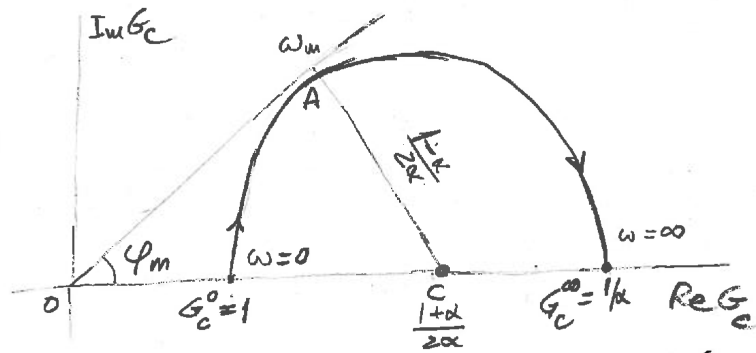


Nyquist plot

$$G_c(i\omega) = \frac{i\omega T+1}{i\omega\alpha T+1}$$

$$G_c^0(i0) = 1$$

$$G_c^\infty(i\infty) = \frac{1}{\alpha}$$



$$\sin\varphi_m = \frac{1-\alpha}{1+\alpha} \qquad (2)$$

maximum phase lead $\boxed{\varphi_m = \sin^{-1}\frac{1-\alpha}{1+\alpha}}$

Proof : $OC = \frac{1}{2}\left(1+\frac{1}{\alpha}\right) = \frac{1+\alpha}{2\alpha}$

$$\text{Radius} = \frac{1+\alpha}{2\alpha} - 1 = \frac{1-\alpha}{2\alpha} = AC$$

$$\sin\varphi_m = \frac{AC}{OC} = \frac{1-\alpha}{1+\alpha} \qquad \text{QED}.$$

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C36 Calculation of α

Eq (2) can be used to determine the value of α in Eq. (1), i.e.,

$$\sin\varphi_m = \frac{1-\alpha}{1+\alpha}$$

$$\frac{1+\sin\varphi_m}{1-\sin\varphi_m} = \frac{1-\alpha+1+\alpha}{1+\alpha-(1-\alpha)} = \frac{2}{2\alpha} = \frac{1}{\alpha}$$

$$\alpha = \frac{1-\sin\varphi_m}{1+\sin\varphi_m} \qquad (3)$$

Calculation of T

- ω_m is the geometric mean of the corner frequencies

$$\omega_m = \sqrt{\omega_a\omega_1} = \sqrt{\frac{1}{T_a}\cdot\frac{1}{T_1}}\Bigg|_{\substack{T_a=T \\ T_1=\alpha T}} = \sqrt{\frac{1}{T}\cdot\frac{1}{\alpha T}} = \frac{1}{T}\frac{1}{\sqrt{\alpha}}$$

$$T = \frac{1}{\omega_m\sqrt{\alpha}}, \quad \omega_m T = \frac{1}{\sqrt{\alpha}} \qquad (4)$$

- $$|G_c(\omega_m)| = \left|\frac{i\omega_m T+1}{i\omega_m\alpha T+1}\right| = \left|\frac{i\frac{1}{\sqrt{\alpha}}+1}{i\alpha\frac{1}{\sqrt{\alpha}}+1}\right| = \left|\frac{i+\sqrt{\alpha}}{i\sqrt{\alpha}+1}\right|\frac{1}{\sqrt{\alpha}}$$

$$= \frac{\sqrt{1+\alpha}}{\sqrt{1+\alpha}}\frac{1}{\sqrt{\alpha}} = \frac{1}{\sqrt{\alpha}}$$

$$|G_c(\omega_m)| = \frac{1}{\sqrt{\alpha}}, \quad 0<\alpha<1 \qquad (5)$$

The phase compensator will produce a gain $\frac{1}{\sqrt{\alpha}}$.

To find ω_m, examine the Bode diagram of the original system G, and identify the freq. at which $|G(\omega_m)| = \sqrt{\alpha}$, such that

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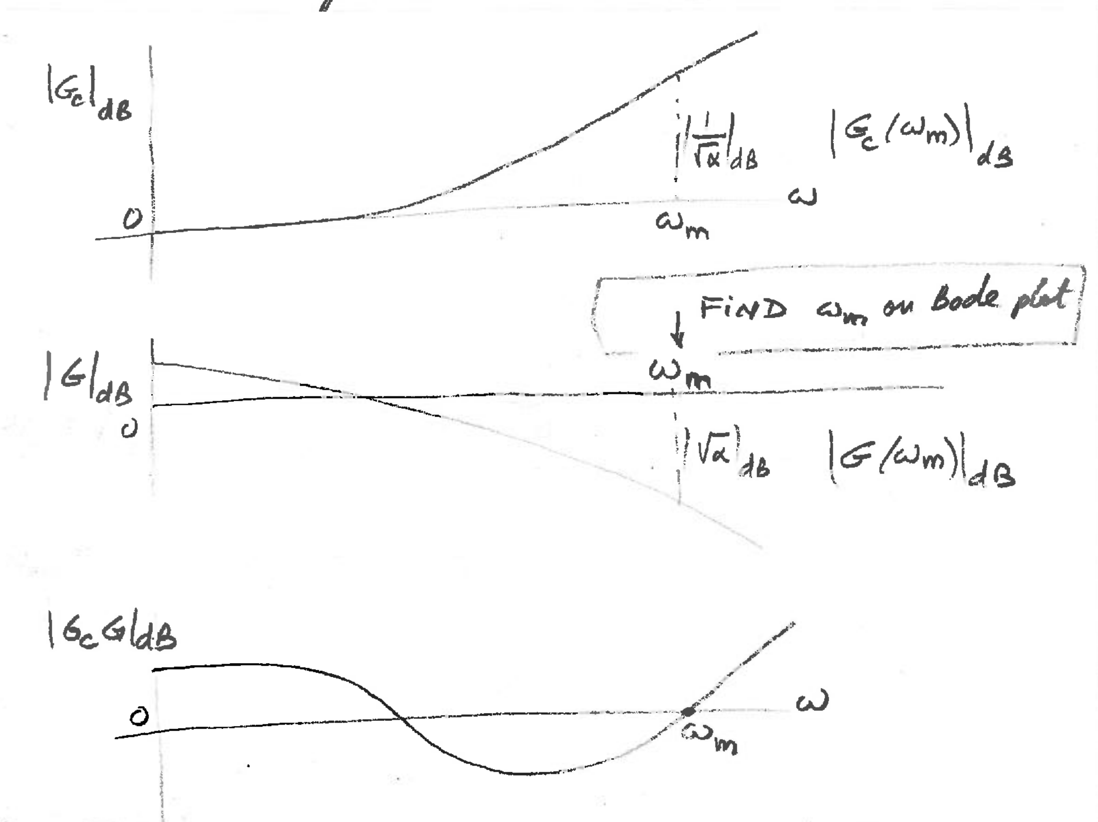
C3c

ω_m is selected at a value at which the gain drop $|G(\omega_n)|$ of the original system balances the gain addition $|G_c(\omega_m)|$ due to the compensator (balance condition)



- After determining ω_m graphically, calculate T with Eq.(4) i.e.,

$$T = \frac{1}{\omega_n} \cdot \frac{1}{\sqrt{\alpha}} \qquad (6)$$

Thus:

$$G_c = \frac{sT+1}{s\alpha T+1} \qquad (7)$$

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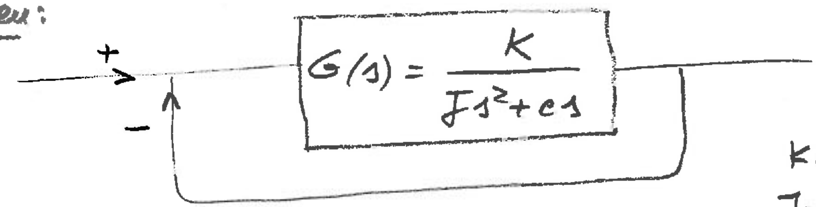
E Example : aircraft roll model with lead comp.

Given:



$K = 114$
$J = 10$
$c = 4$

Find

(a) - gain margin, $(K_g)_{dB}$

- phase margin γ deg

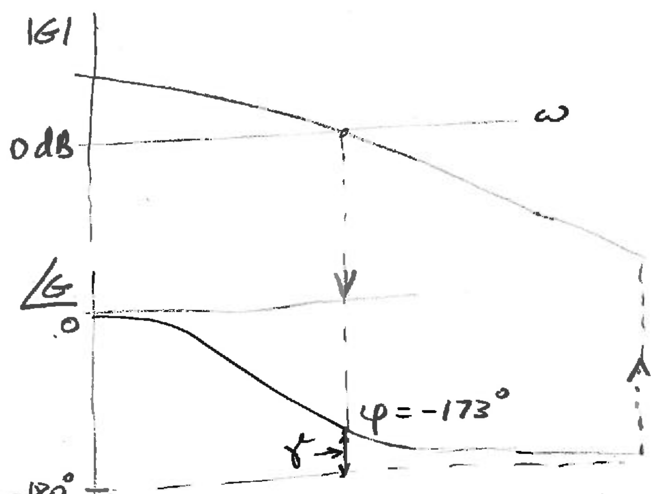
(b) design compensator to achieve

GM = 10 dB (1)

PM = 60 deg

(c) plot time response

Solution (a) Bode plot (see MATLAB plot, Fig. 1)



$|G|_{\angle G = -180°} = -\infty$

∠G never crosses −180° line

$\angle G \underset{\omega \to \infty}{\longrightarrow} -180°$

$\gamma = \varphi + 180° = -173° + 180° = 7°$

Very small phase margin!

$K_g = \infty > GM$

OK

NEED TO IMPROVE PHASE MARGIN!

Plenty gain margin

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2E

```
Phase margin: aircraft model
input data

    K   | J  |   c =

   114    10     4


G =

     114
  ------------
  10 s^2 + 4 s

Continuous-time transfer function.

========================

GM, dB | PM, deg =

    10    60

========================

phi = phase at |G|=0 dB point, deg =

  -173


gamma = phase margin, deg =

     7

========================
```

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ω

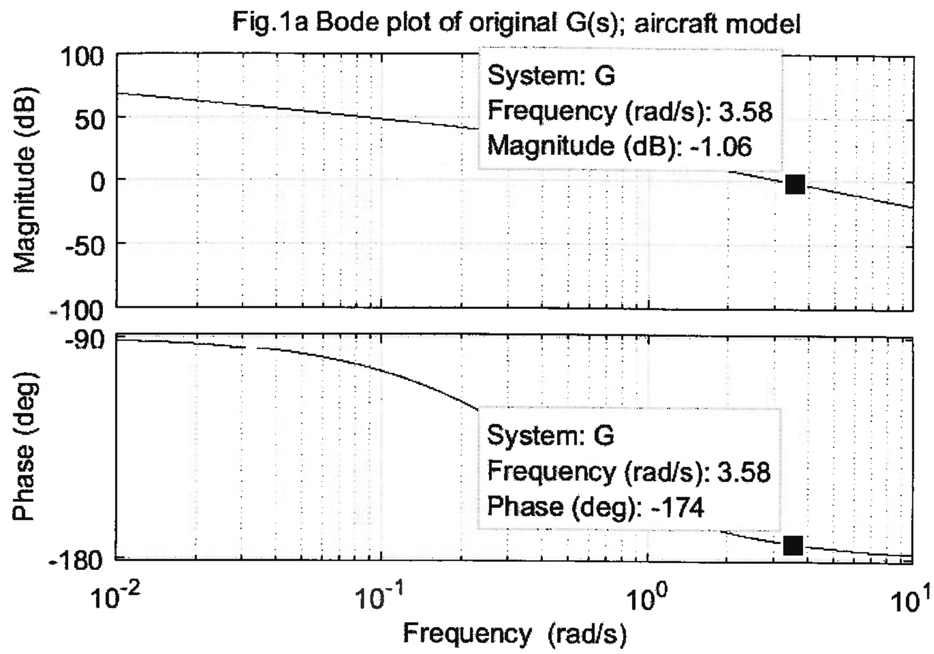
Fig.1a Bode plot of original G(s); aircraft model
Magnitude (dB)
100
50
0
-50
-100
System: G
Frequency (rad/s): 3.58
Magnitude (dB): -1.06
Phase (deg)
-90
-180
System: G
Frequency (rad/s): 3.58
Phase (deg): -174
10^-2
10^-1
10^0
10^1
Frequency (rad/s)

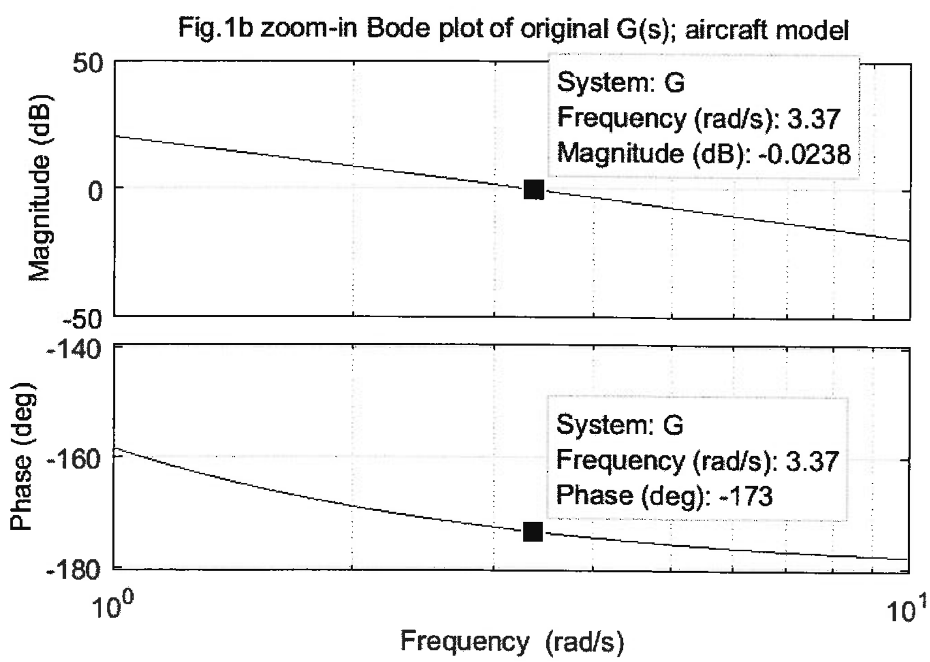
Fig.1b zoom-in Bode plot of original G(s); aircraft model
Magnitude (dB)
50
0
-50
System: G
Frequency (rad/s): 3.37
Magnitude (dB): -0.0238
Phase (deg)
-140
-160
-180
System: G
Frequency (rad/s): 3.37
Phase (deg): -173
10^0
10^1
Frequency (rad/s)

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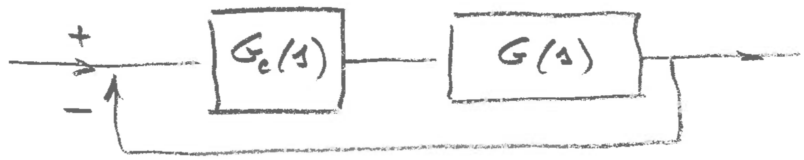
4E (b) Design phase compensator to improve phase margin



$$G_c(s) = \frac{Ts+1}{\alpha Ts+1} \quad \text{lead compensator} \qquad (2)$$

(b1) We need to improve phase margin from $\gamma = 7°$ to $PM = 60°$, i.e., the phase compensator must add $\varphi_m = 53°$.

- To calculate α, recall

$$\alpha = \left.\frac{1-\sin\varphi_m}{1+\sin\varphi_m}\right|_{\varphi_m = 53°} = 0.1120 \qquad (3)$$

- To calculate T, recall $T = \frac{1}{\omega_m\sqrt{\alpha}}$

We need ω_n. We find ω_m from the balance condition, i.e.,

$$|G_c(\omega_m)| = \frac{1}{\sqrt{\alpha}} = 2.9887 = 9.5096\,dB \qquad (4)$$

$$|G(\omega_m)| = -|G_c(\omega_m)| = -9.5096\,dB \qquad (5)$$

From Bode plot Fig. 2, we find $\omega_n = 5.83$ rad/sec.

Hence $T = 0.5126$ sec. (6)

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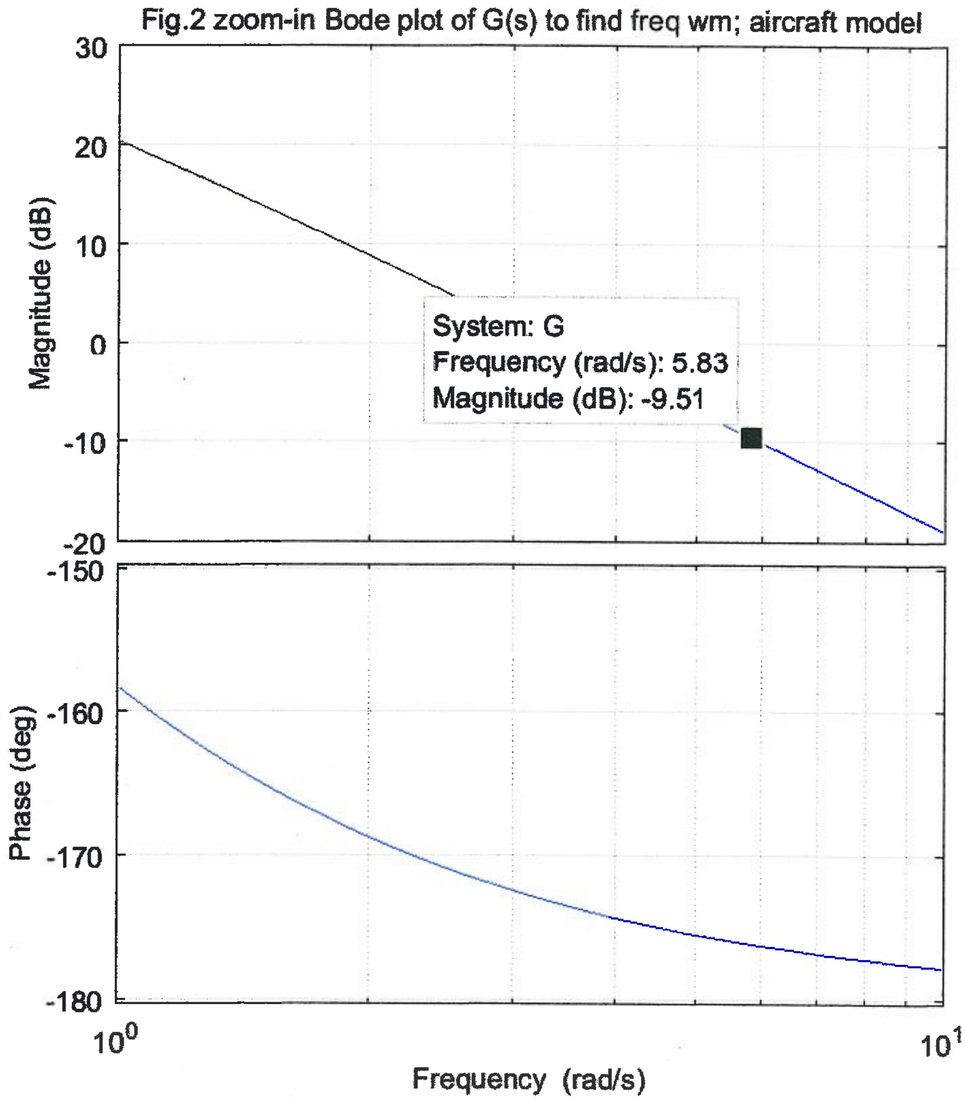
5⃝



$\omega_m = 5.83$ rad/s

$|G(\omega_m)|_{dB} = -9.51$ dB.

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$\overset{6}{E}$ (b2): Test compensated system G_cG

Fig. 3 shows the performance of the compensated system G_cG in comparison to the original system G. The compensated system G_cG has

$\varphi_c = -123^\circ$ @ $\omega_c = 5.83$ rad/s where $|G_cG| = 0\,dB$

The phase margin of the compensated system is

$$\gamma_c = \varphi_c + 180^\circ = -123^\circ + 180^\circ = 57^\circ < \underset{PM}{60^\circ}$$

The system has improved, but the phase margin is still less than PM.

(b3). Need to adjust the compensator to add a little more phase shift;

$$\Delta\varphi = PM - \gamma_c = 60^\circ - 57^\circ = 3^\circ \quad (7)$$

The new φ_m is

$$\varphi_{m_1} = \varphi_m + \Delta\varphi = 53^\circ + 3^\circ = 56^\circ \quad (8)$$

The new α is

$$\alpha_1 = \frac{1 - \sin\varphi_{m_1}}{1 + \sin\varphi_{m_1}} = 0.0935 \quad (9)$$

We keep same time constant, $T_1 = T$ (10)

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7
ε

(b1)

```
========================
Phase compensator design: aircraft model
phi_m =
    53
alpha =
    0.1120
1/sqrt(alpha) =
    2.9887
Gc_wm, dB =
    9.5096
wm =
    5.8300
T =
    0.5126

Gc =

  0.5126 s + 1
  -------------
  0.05739 s + 1

Continuous-time transfer function.

phi_c = phase at |GcG|=0 dB point, deg =
  -123
gamma_c = phase margin of GcG, deg =
    57
========================
```

8E

(6.2)

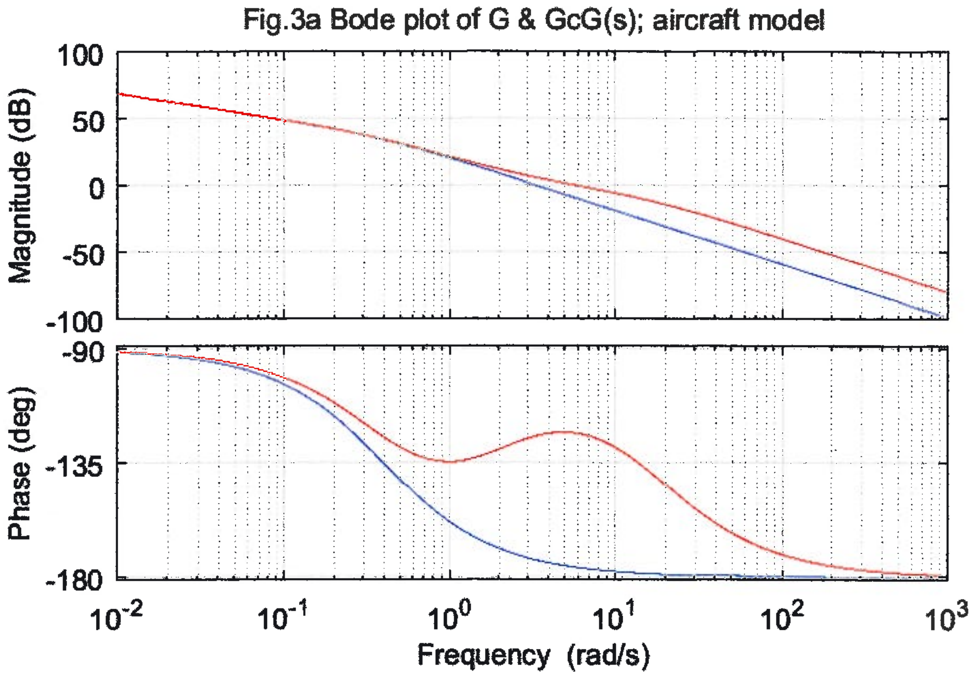
Fig.3a Bode plot of G & GcG(s); aircraft model
Magnitude (dB)
100
50
0
-50
-100
Phase (deg)
-90
-135
-180
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/s)

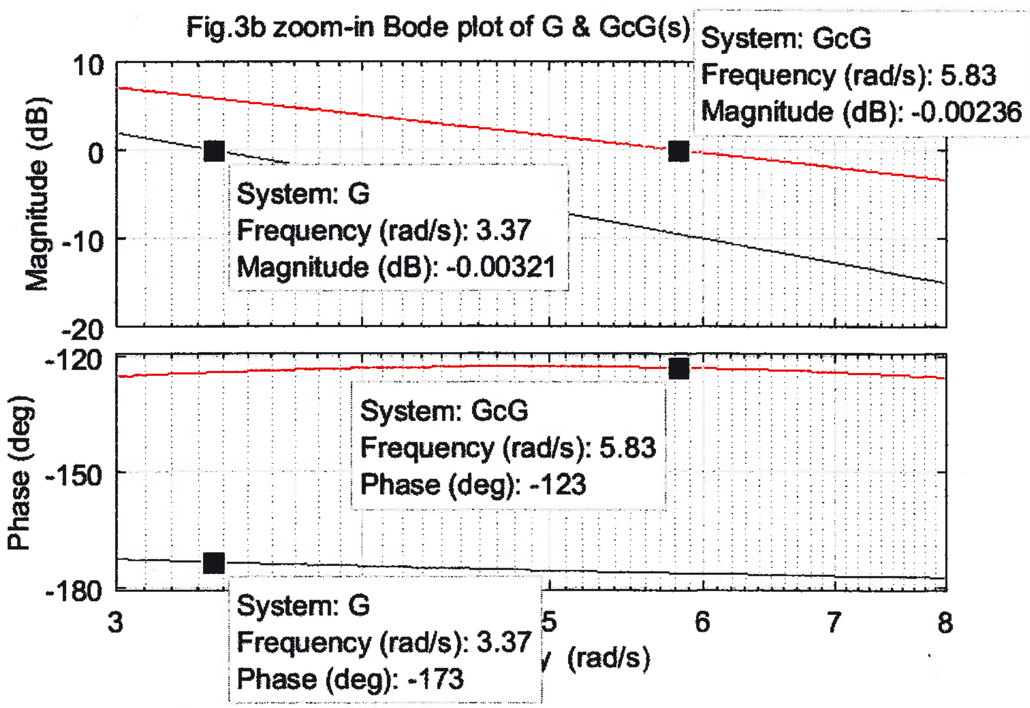
Fig.3b zoom-in Bode plot of G & GcG(s)
System: GcG
Frequency (rad/s): 5.83
Magnitude (dB): -0.00236
System: G
Frequency (rad/s): 3.37
Magnitude (dB): -0.00321
System: GcG
Frequency (rad/s): 5.83
Phase (deg): -123
System: G
Frequency (rad/s): 3.37
Phase (deg): -173
Magnitude (dB)
10
0
-10
-20
Phase (deg)
-120
-150
-180
3
5
6
7
8
(rad/s)

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(63)

```
========================
Adjustment of phase compensator: aircraft model
dphi =
     3
phi_m1 =
    56
alpha1 =
    0.0935
T1 =
    0.5126

Gc1 =

  0.5126 s + 1
  -------------
  0.04792 s + 1

Continuous-time transfer function.
```

^{10}E (b4) Test adjusted compensator
Fig. 4 shows the performance of the system with adjusted compensator $G_{c1}G$ compared to the original system G.
The new phase value at $|G_{c1}G| = 0\,dB$ is:

$$\varphi_{c_1} = -120^\circ \text{ @ } 5.91\,rad/s \text{ where } |G_{c_1}G| = 0\,dB.$$

The new phase margin is

$$\gamma_{c_1} = \varphi_{c_1} + 180^\circ = -120^\circ + 180^\circ = 60^\circ = PM \quad (11).$$

The adjusted system meets the phase margin specification $PM = 60^\circ$.

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\\ e

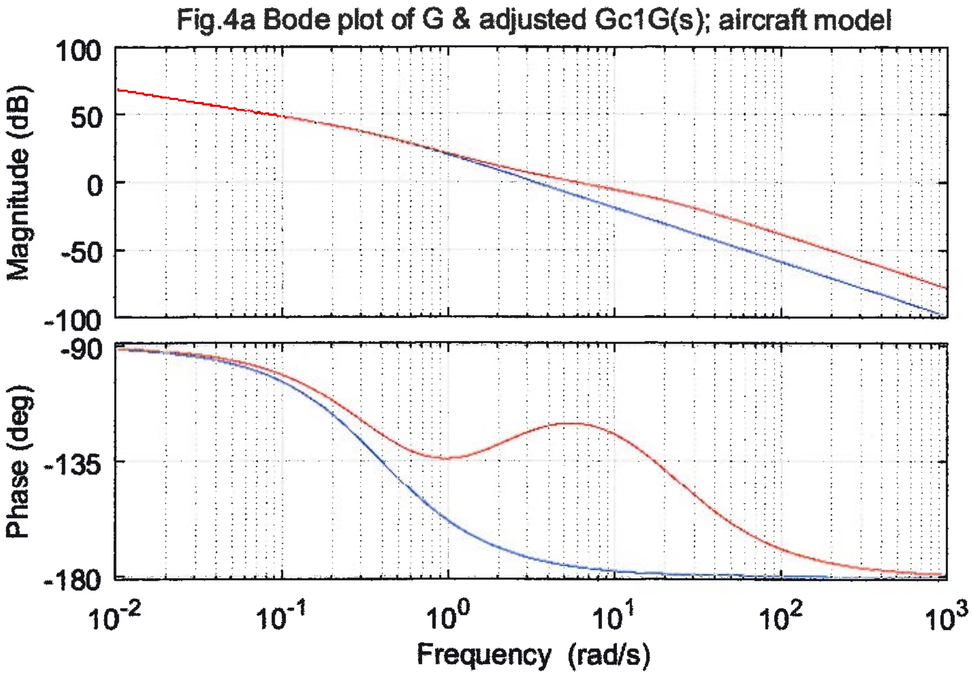
Fig.4a Bode plot of G & adjusted Gc1G(s); aircraft model
Magnitude (dB)
100
50
0
-50
-100
Phase (deg)
-90
-135
-180
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/s)

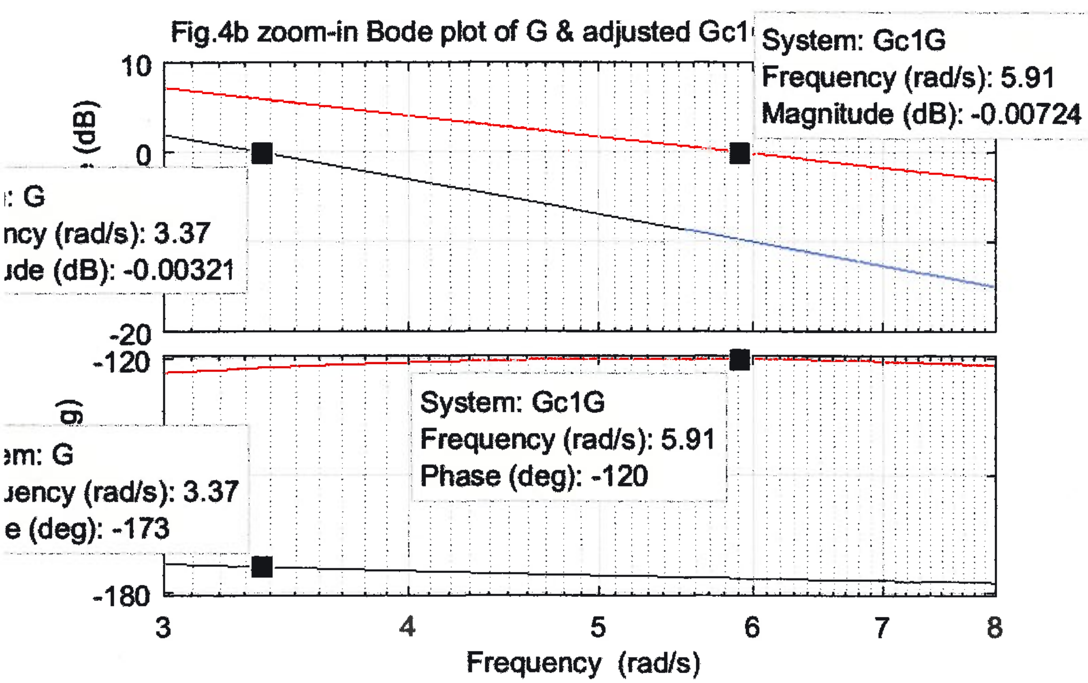
Fig.4b zoom-in Bode plot of G & adjusted Gc1
System: Gc1G
Frequency (rad/s): 5.91
Magnitude (dB): -0.00724
10
0
-20
: G
ncy (rad/s): 3.37
ude (dB): -0.00321
-120
-180
System: Gc1G
Frequency (rad/s): 5.91
Phase (deg): -120
em: G
uency (rad/s): 3.37
e (deg): -173
3
4
5
6
7
8
Frequency (rad/s)

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12
E

(c) Time response behavior of the compensated system.

Fig 5 displays the time response of the compensated system $(G_{c1}G)_{CL}$ compared with the response of the original system G_{CL}. It can be observed that the compensated system has a fast rise time and a small overshoot

$t_r = 0.296$ sec

$M_p = 17\%$

(12)

The original system had a longer rise time, a much higher overshoot, and took a long time to settle.

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Fig.5a Step response aircraft model G_{CL}, $Gc1G_{CL}$

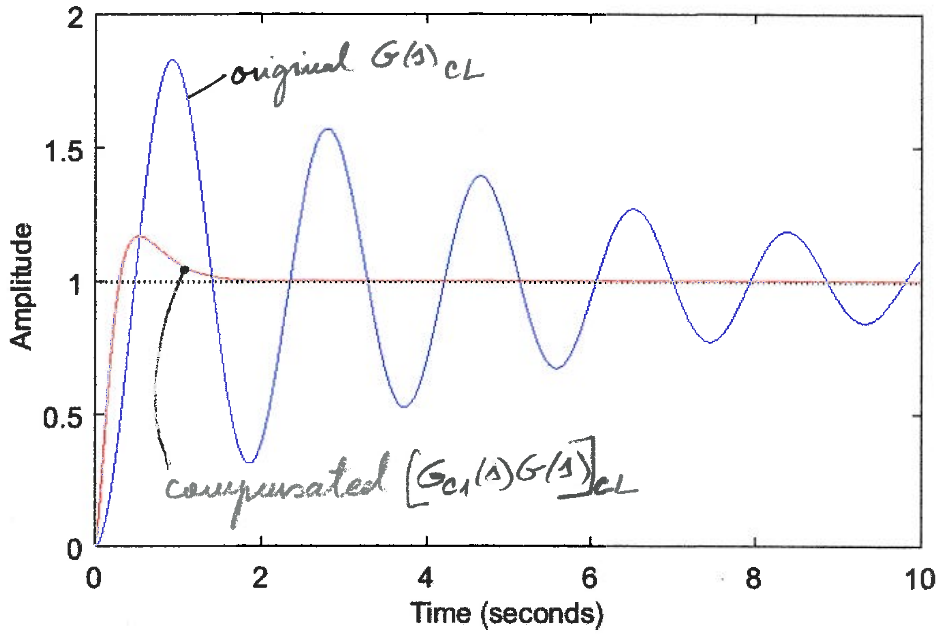
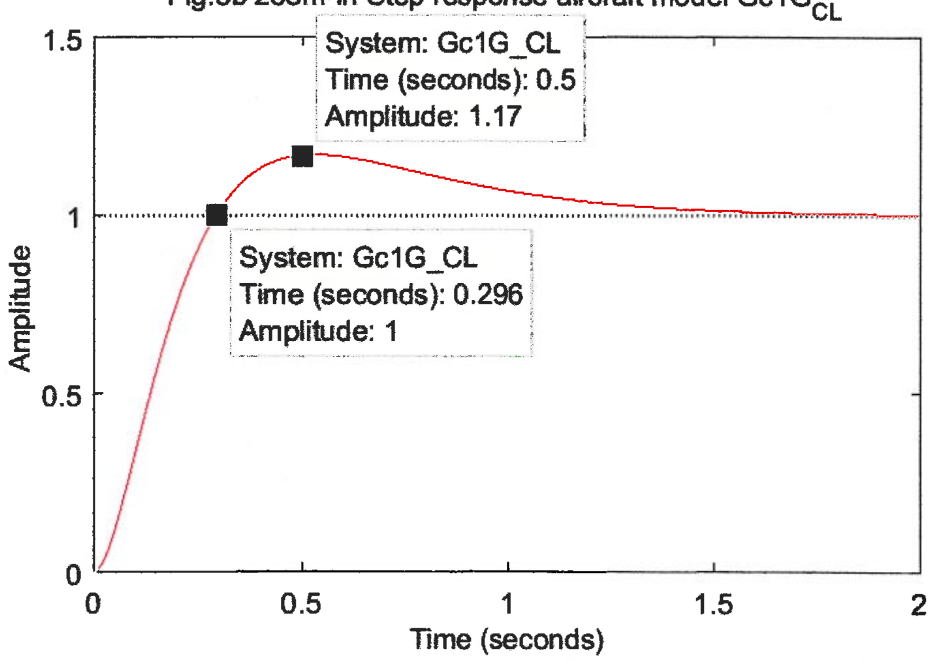


Fig.5b zoom-in Step response aircraft model $Gc1G_{CL}$



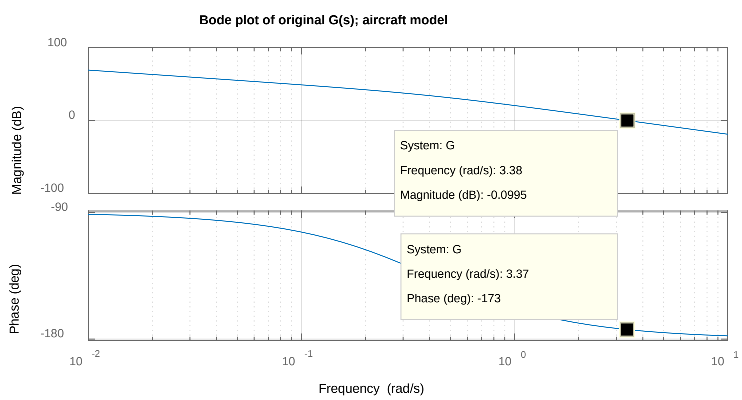
Lead_compensator_aircraft_20180510.m

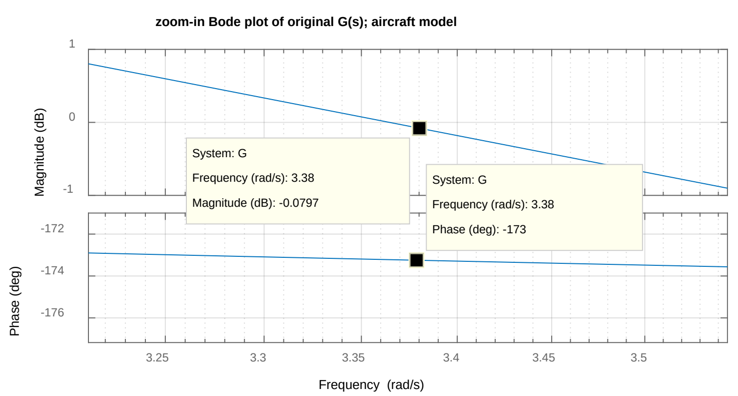
```
% Phase compensator design: aircraft model
%% initialization
clc %clear command window
clear %removes all variables from workspace; release memory
format compact
close all %closes all figures
s=tf('s');
%% original aircraft model
display('Phase margin: aircraft model')
K=114;              % gain
J=10;               % inertia
c=4;                % damping
display('input data')
display([K J c],'    K    | J  |    c')
G=K/(J*s^2+c*s)     % G(s)
display('=========================')
%% specs: MIL-DTL-9490E margin requirements
GM=10;   % gain margin spec, dB
PM=60;   % phase margin spec, deg
display([GM PM],'GM, dB | PM, deg')
figure(1)
subplot(2,1,1)
bode(G)
grid
title('Bode plot of original G(s); aircraft model')
subplot(2,1,2)
d1=0;d2=1;N=1e3; w=logspace(d1,d2,N);
bode(G,w)
grid
title('zoom-in Bode plot of original G(s); aircraft model')
display('=========================')
% READ ON PLOT: phase in deg for |G|=0 dB
% phi=-173;
phi=input('Input phase in deg when |G|=0 dB, phi=');
gamma=phi-(-180);   % gamma = phase margin, deg
% display([phi],'phi = phase at |G|=0 dB point, deg')
display([gamma],'gamma = phase margin, deg')
display('=========================')
%% add compensator Gc(s)
display('Phase compensator design: aircraft model')
phi_m=PM-gamma % maximum compensator phase that need to be obtained
alpha=(1-sind(phi_m))/(1+sind(phi_m)) % compensator attenuation factor alpha
display(1/sqrt(alpha),'1/sqrt(alpha)') % expected gain rise from compensator
Gc_wm=mag2db(1/sqrt(alpha)); % expected gain rise from compensator, dB
G_wm=-Gc_wm;
display(G_wm,'G_wm, dB')
figure(2) % Bode plot to find the freq wm
d1=log10(3);d2=log10(12);N=1e3; w=logspace(d1,d2,N);
bode(G,w)
grid
title('zoom-in Bode plot of |G| to find freq wm; aircraft model')
% READ ON PLOT: frequecy for which |G|=G_wm
% wm=5.83
wm=input('Input frequency in rad/s when |G|=G_wm, wm=');
T=1/wm/sqrt(alpha) % time constant of the compensator
```

```
%% Test compensated system
figure(3)
subplot(2,1,1)
Gc=(s*T+1)/(s*alpha*T+1)
GcG=Gc*G;
bode(G,GcG)
grid
title('Bode plot of G & GcG(s); aircraft model')
subplot(2,1,2)
d1=log10(3);d2=log10(8);N=1e3; w=logspace(d1,d2,N);
bode(G,GcG,w)
grid
title('zoom-in Bode plot of G & GcG(s); aircraft model')
% READ ON PLOT: phase in deg for |GcG|=0 dB
% phi_c=-123;
phi_c=input('Input phase in deg when |GcG|=0 dB, phi_c=');
gamma_c=phi_c-(-180);    % gamma_c = phase margin of GcG, deg
display([phi_c],'phi_c = phase at |GcG|=0 dB point, deg')
display([gamma_c],'gamma_c = phase margin of GcG, deg')
display('========================')
%% Adjust compensator to reach PM requirements
display('Adjustment of phase compensator: aircraft model')
dphi=PM-gamma_c  % additional phase shift needed
phi_m1=phi_m+dphi   % adjusted phase shift
alpha1=(1-sind(phi_m1))/(1+sind(phi_m1)) % adjusted alpha
Gc1_wm=mag2db(1/sqrt(alpha1)); % expected gain rise from compensator, dB
G_wm1=-Gc1_wm;
display(G_wm1,'G_wm1, dB')
figure(4) % Bode plot to find the freq wm for |G_wm|dB=-|Gc_wm|dB
d1=0;d2=1;N=1e3; w=logspace(d1,d2,N);
bode(G,w)
grid
title('zoom-in Bode plot of G(s) to find new freq wm1; aircraft model')
% READ ON PLOT: frequecy for which |G|=-Gc_wm1, dB
% wm1=6.1;
wm1=input('Input frequency rad/s when |G|=G_wm1, wm1=');
T1=1/wm1/sqrt(alpha1) % updated time constant of the compensator
%% Final compensator design
Gc1=(s*T1+1)/(s*alpha1*T1+1)
Gc1G=Gc1*G;
figure(5)
subplot(2,1,1)
bode(G,Gc1G)
grid
title('Bode plot of G & adjusted Gc1G(s); aircraft model')
subplot(2,1,2)
d1=log10(3);d2=log10(8);N=1e3; w=logspace(d1,d2,N);
bode(G,Gc1G,w)
grid
title('zoom-in Bode plot of G & adjusted Gc1G(s); aircraft model')
phi_c1=input('Input phase in deg for |Gc1G|=0 dB, phi_c1=');
gamma_c1=phi_c1-(-180);    % gamma_c = phase margin of GcG, deg
display([phi_c1],'phi_c1 = phase at |Gc1G|=0 dB point, deg')
display([gamma_c1],'gamma_c = phase margin of GcG, deg')
%% Plot step response response of the original and compensated systems
figure(6)
Tfinal=10; dt=0.01; t=0:dt:Tfinal;
G_CL=feedback(G,1);
Gc1G_CL=feedback(Gc1G,1);
subplot(2,1,1)
step(G_CL,Gc1G_CL,t)
title('step response aircraft model G_C_L, Gc1G_C_L')
subplot(2,1,2)
Tzoom=2; Nt=1e3; dt=Tzoom/Nt; tz=0:dt:Tzoom;
step(Gc1G_CL,tz,'r')
title('zoom-in step response aircraft model Gc1G_C_L')
ylim([0 1.5])
```

428/523

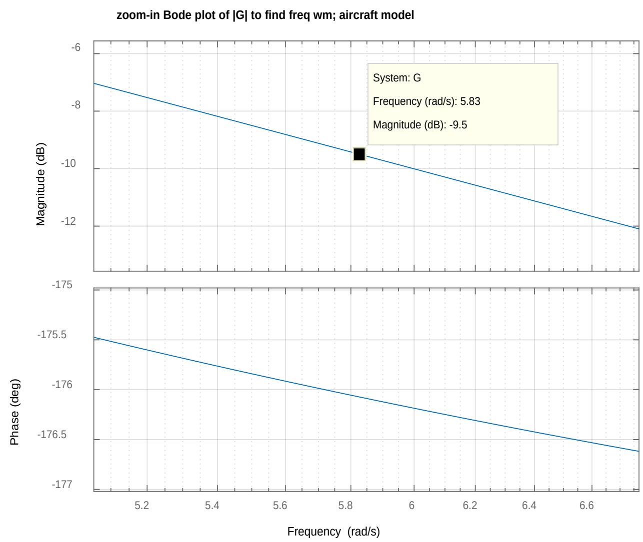
```
Phase margin: aircraft model
input data
    K    | J  |    c =
   114    10      4

G =

      114
  ------------
  10 s^2 + 4 s

Continuous-time transfer function.

=========================
GM, dB | PM, deg =
    10     60
=========================
```






```
Input phase in deg when |G|=0 dB, phi=-173
gamma = phase margin, deg =
     7
```

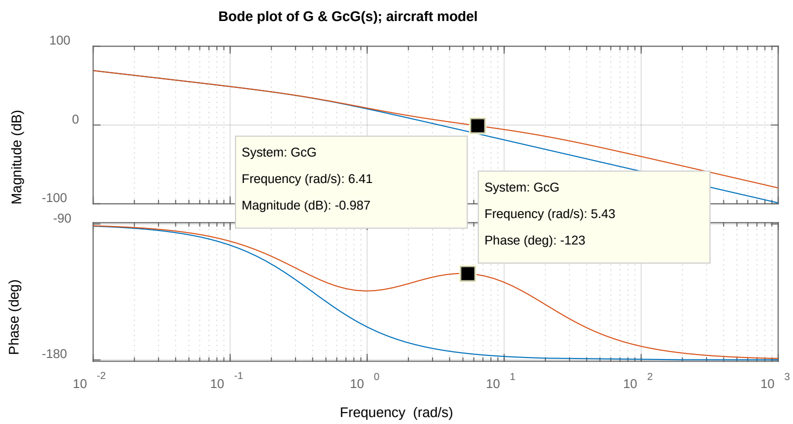
```
Phase compensator design: aircraft model
phi_m =
    53
alpha =
    0.1120
1/sqrt(alpha) =
    2.9887
G_wm, dB =
   -9.5096
```

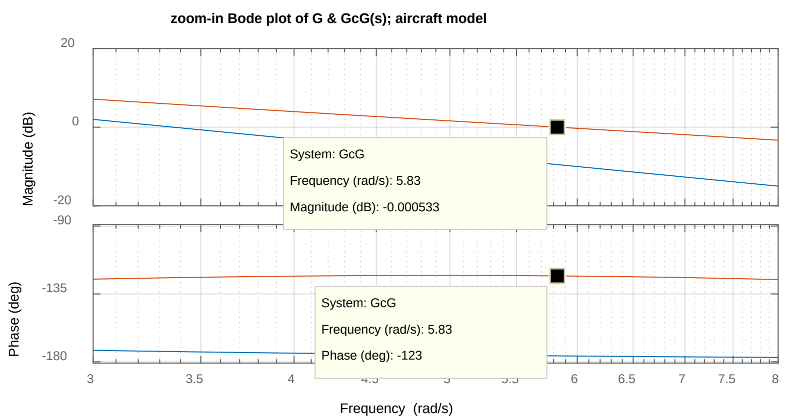



```
Input frequency in rad/s when |G|=G_wm, wm=5.83
T =
    0.5126
```

430/523

```
Gc =

  0.5126 s + 1
  -------------
  0.05739 s + 1
```

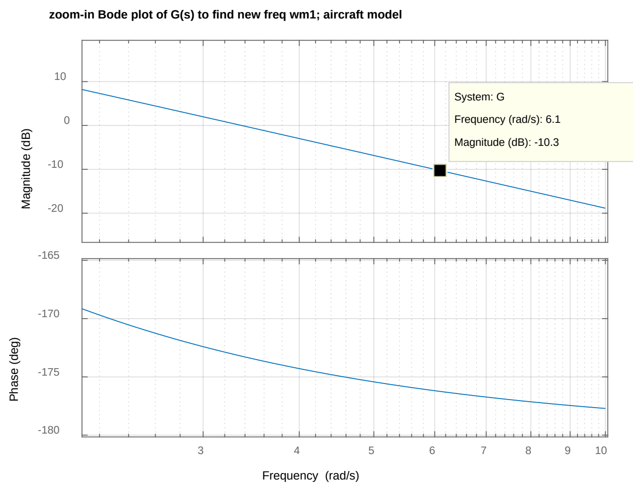





```
Input phase in deg when |GcG|=0 dB, phi_c=-123
phi_c = phase at |GcG|=0 dB point, deg =
  -123
gamma_c = phase margin of GcG, deg =
    57
-------------------------
```

431/523

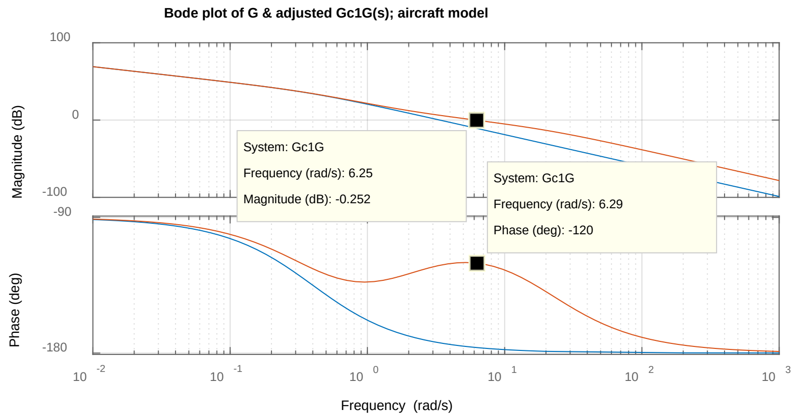
```
Adjustment of phase compensator: aircraft model
dphi =
     3
phi_m1 =
    56
alpha1 =
    0.0935
G_wm1, dB =
  -10.2932
```

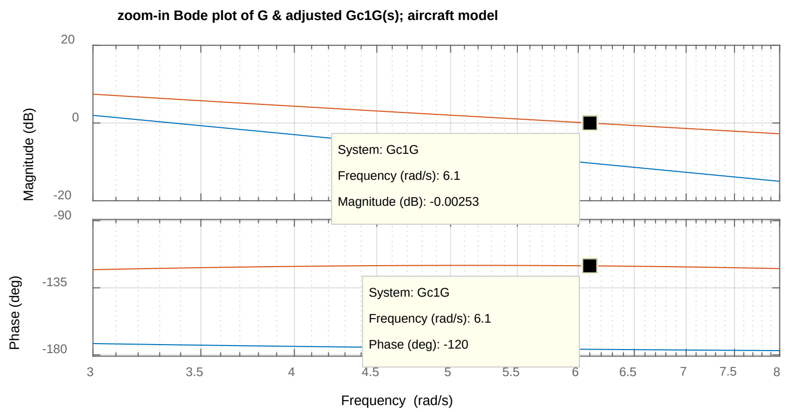



```
Input frequency rad/s when |G|=G_wm1, wm1=6.1
T1 =
    0.5362
```

432/523

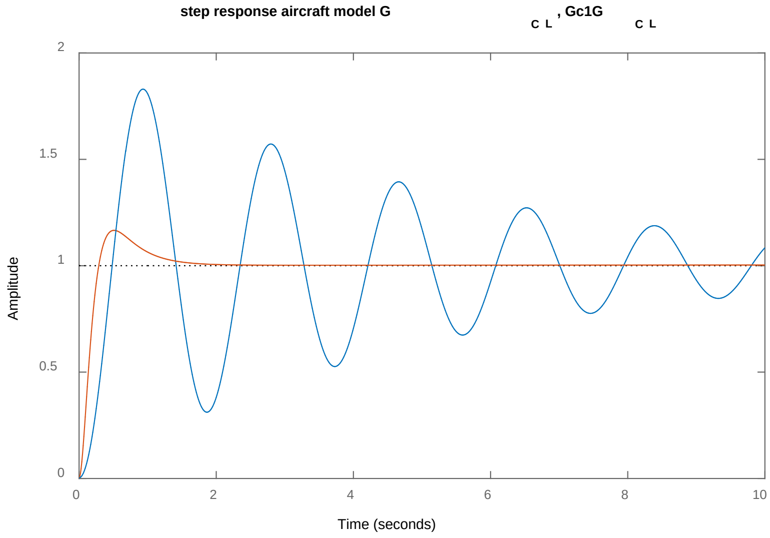
```
Gc1 =

  0.5362 s + 1
  -------------
  0.05012 s + 1
```

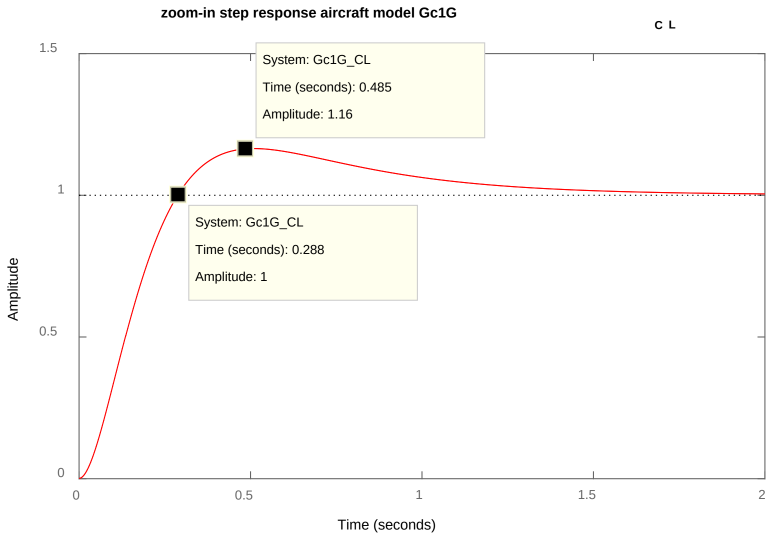





```
Input phase in deg for |Gc1G|=0 dB, phi_c1=-120
phi_c1 = phase at |Gc1G|=0 dB point, deg =
  -120
gamma_c = phase margin of GcG, deg =
    60
```

433/523

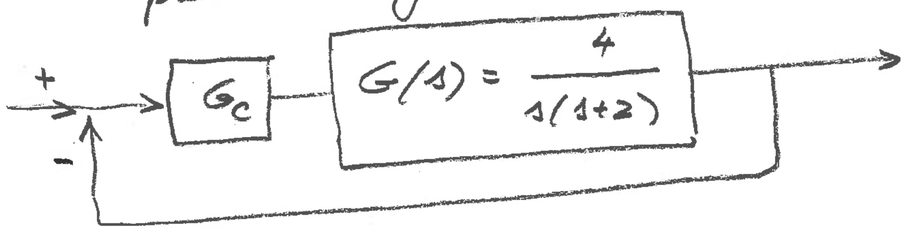

step response aircraft model G
C L
, Gc1G
C L
2
1.5
1
0.5
0
Amplitude
0
2
4
6
8
10
Time (seconds)


zoom-in step response aircraft model Gc1G
C L
System: Gc1G_CL
Time (seconds): 0.485
Amplitude: 1.16
System: Gc1G_CL
Time (seconds): 0.288
Amplitude: 1
1.5
1
0.5
0
Amplitude
0
0.5
1
1.5
2
Time (seconds)

C7

Ex. 11.6

Design a lead compensator to reduce servomotor ramp error and meet phase and gain margin requirements.



<u>Given</u> : $G(s) = \dfrac{4}{s(s+2)}$

<u>Find</u> :

(a) static velocity error const. K_V

phase margin, γ

gain margine, $(K_g)_{dB}$

(b) design compensator to achieve:

$K_V \geq 20 \,/sec$

$\gamma \geq 50^\circ$ $(PM = 50^\circ)$

$(K_g)_{dB} \geq 10 dB$ $(GM = 10 dB)$

<u>Solution</u>

(a) characterize current system

- $K_V = \lim_{s \to 0} s\, G(s) = \lim_{s \to 0} s \dfrac{4}{s(s+2)} = \dfrac{4}{2} = 2/sec.$
- $\varphi(G = 0dB) = -128^\circ ; \gamma = -128^\circ + 180^\circ = 52^\circ$
- $(K_g)_{dB} = \infty$ because $\angle G$ never crosses -180° line

C7a

Fig.1a Bode plot of original G(s); Example 11.6

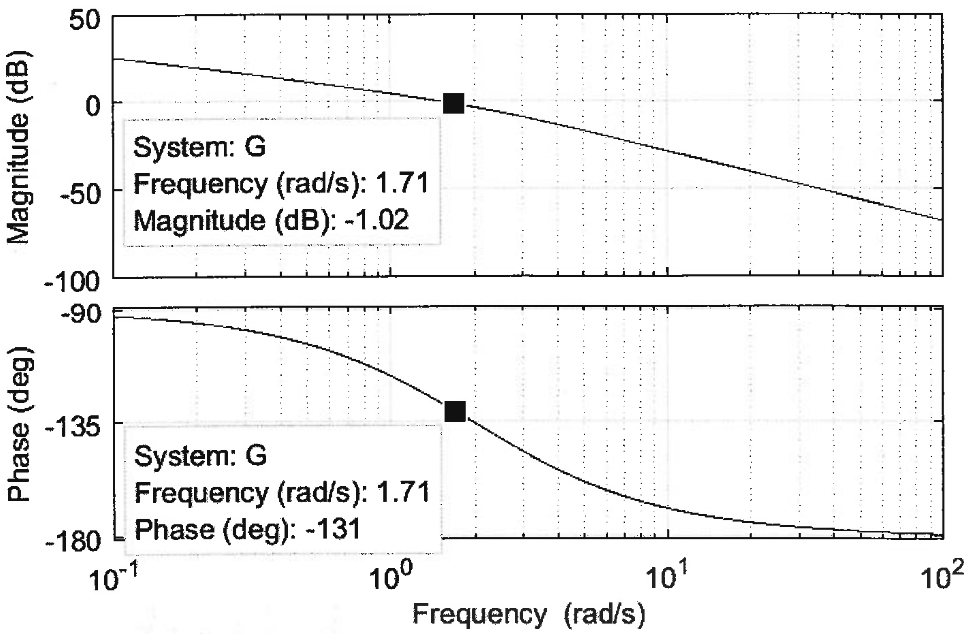
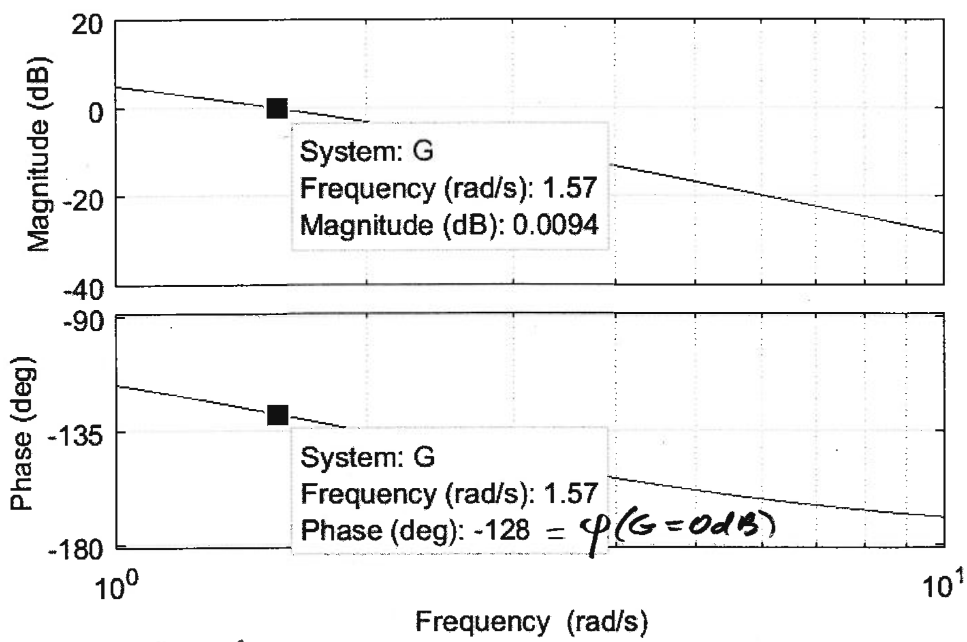


Fig.1b zoom-in Bode plot of original G(s); Example 11.6



$\gamma = -128 + 180 = 52^\circ$

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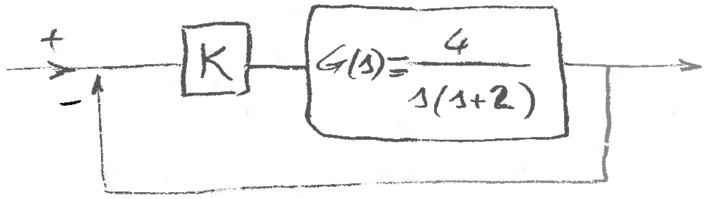
C8/ (b) Design P-controller to improve K_v



Roadmap

(1) add K to improve K_V to $K_V^* = 20/\text{sec}$

(2) Notice that adding K shifts magnitude plot upward and spoils the phase margin.

(3) add compensator G_c to improve phase and bring phase back to $\gamma^* \cong 50°$

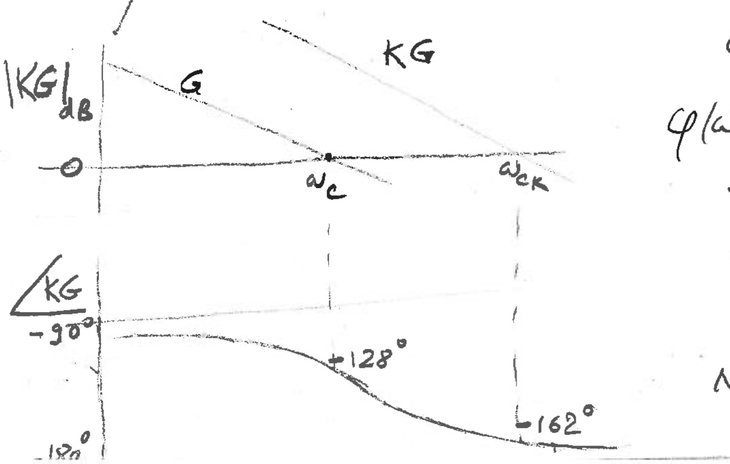
Design

D1. Calculate K needed to bring K_v to value $K_v^* = 20/s$

$$K_v^* = \lim_{s \to 0} s\, G^*(s) = \lim_{s \to 0} K \cdot \frac{4}{s(s+2)} = 2K = 20$$

$\rightarrow$ $\boxed{K = 10}$ $K_{dB} = 20\,dB$

D2. Draw Bode diagram for new system. We find that gain crossover freq ω_c has moved to the right and the phase margin has decreased



$\omega_{cK} = 6.16$ rad/sec.

$\varphi(\omega_{cK}) = -162°$

$\gamma_K = 18° < 50°$

$(-162° + 180° = 18°)$

Need to add 32° !

C8a

Fig.2a Bode plot of G & KG(s);

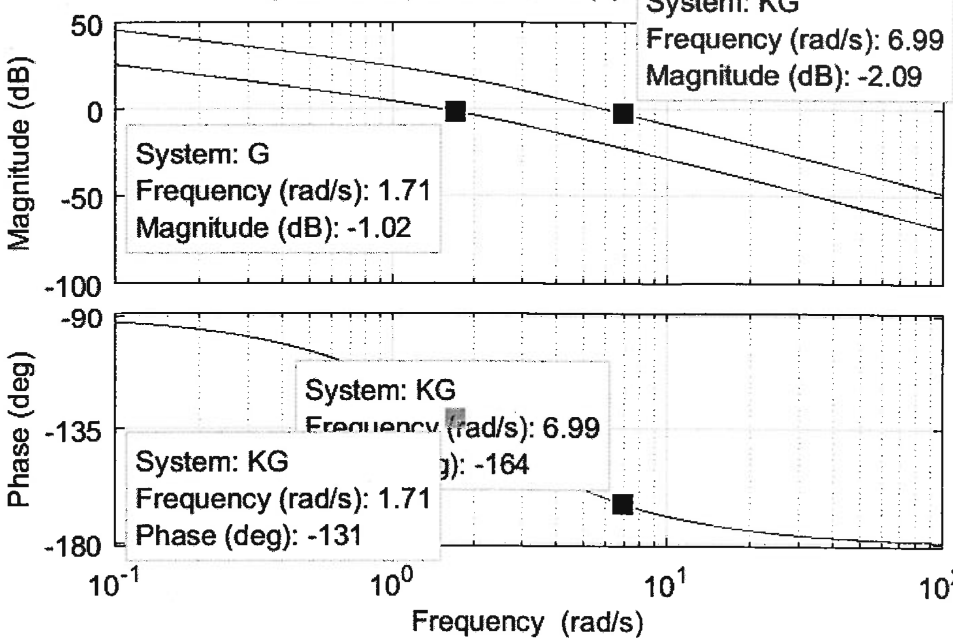
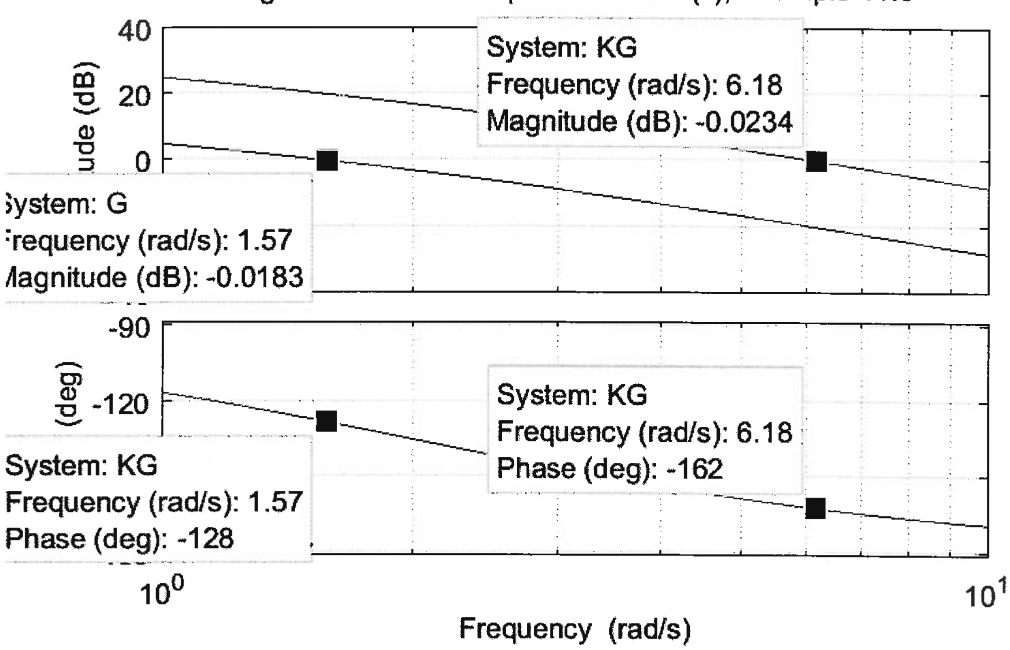


Fig.2b zoom-in Bode plot of G & KG(s); Example 11.6



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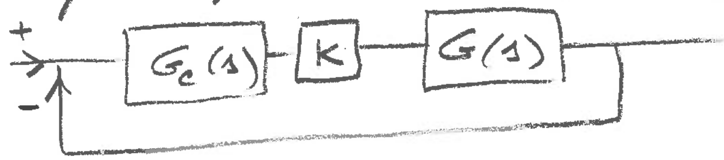
C9 Design compensator to improve γ.



We need lead compensator

$$G_c(s) = \frac{Ts+1}{\alpha Ts+1}$$

We need to improve the phase margin from $\gamma = 18^\circ$ to $\gamma_M = 50^\circ$, i.e., we need the phase compensator to add $\varphi_m = 32^\circ$

Recall $\alpha = \left.\frac{1-\sin\varphi_m}{1+\sin\varphi_m}\right|_{\varphi_m = 32^\circ} = 0.3073$

To calculate ω_m, recall

$$|G_c(\omega_m)| = \frac{1}{\sqrt{\alpha}} = 1.8040 = 5.125\,dB$$

We identify ω_n as the point on the Bode plot of $KG(\omega)$ where

$$|KG(\omega_m)|_{dB} = -|G_c(\omega_n)|_{dB} = -5.125\,dB.$$

From Bode plot, we read

$\omega_m = 8.38$ rad/sec for $|KG|_{dB} = -5.13\,dB$

Hence $T = \frac{1}{\omega_m\sqrt{\alpha}} = 0.2153$ sec.

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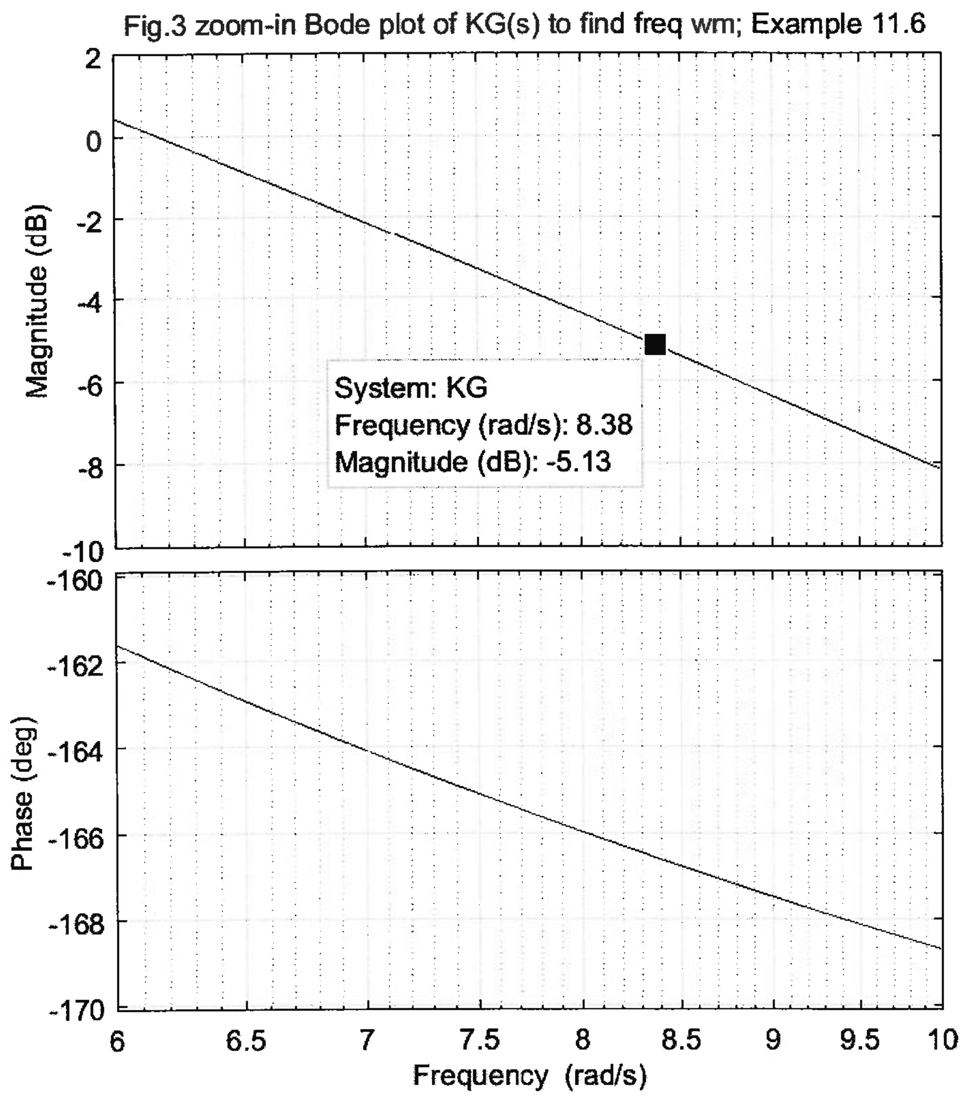
C9a



$\omega_m = 8.38$ rad/sec for $|KG|_{dB} = -5.13$dB

440/523

C96

```
Phase compensator design: Exampe 11.6

phi =

  -128


gamma =

    52


alpha =

    0.3073


1/sqrt(alpha =

    1.8040


Gc_wm, dB =

    5.1250


T =

    0.2153


Gc =

  0.2153 s + 1
  -------------
  0.06615 s + 1
```

441/523

cgc

Fig.4a Bode plot of KG & GcKG(s); Example 11.6

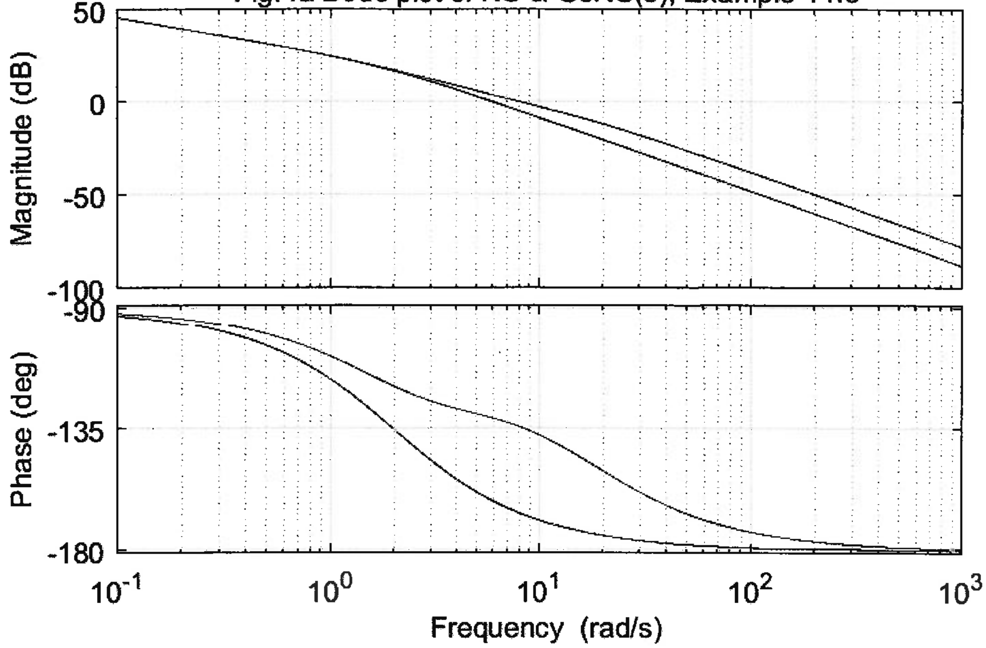
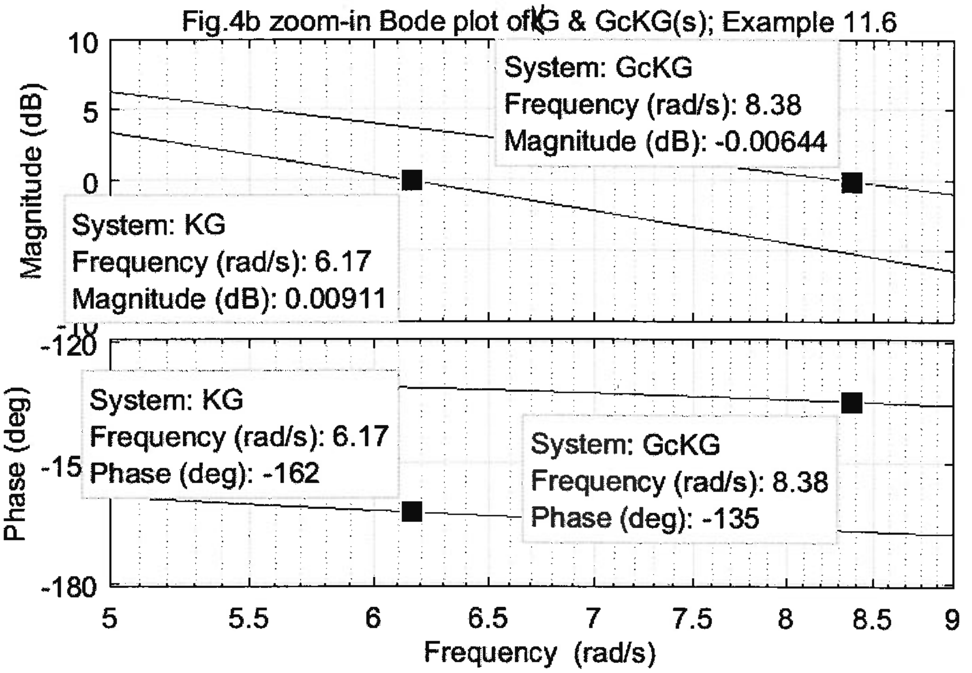


Fig.4b zoom-in Bode plot of KG & GcKG(s); Example 11.6



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C10 Test the compensator

Fig 4 shows that the compensator is not sufficient because the compensated phase margin is

$$\gamma_c = -135 + 180 = 45^\circ$$

We need 50°; thus, we need to adjust the compensator to get to 50°. We need an additional 5° of compensation, i.e., $\Delta\varphi = 5^\circ$. The new φ_m is

$$\varphi_{m_1} = \varphi_m + \Delta\varphi = 32 + 5 = 37^\circ$$

The new α is $\alpha_1 = \dfrac{1 - \sin\varphi_{m_1}}{1 + \sin\varphi_{m_1}} = 0.2486$

We keep $T_1 = T$ and calculate new compensator

$$G_{c_1} = \frac{T_1 s + 1}{\alpha_1 T_1 s + 1}$$

With this new compensator, the Bode plot is as shown in Fig. 5. The phase at $G_{c_1}KG = 0\,dB$ is -130°

The margin is $\gamma_{c_1} = -130^\circ + 180^\circ = 50^\circ = \gamma_M$

We have met the specification!

C10a

```
dphi =

     5


phi_m1 =

    37


alpha1 =

    0.2486


T1 =

    0.2153


Gc1 =

  0.2153 s + 1
  -------------
  0.05352 s + 1
```

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C10b

Fig.5a Bode plot of KG & adjusted Gc1KG(s); Example 11.6

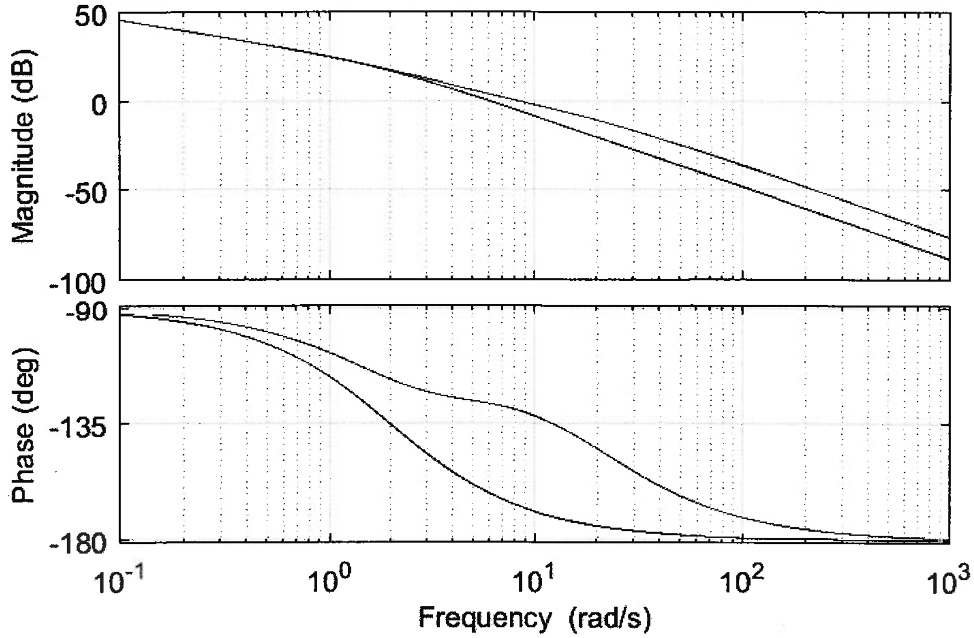
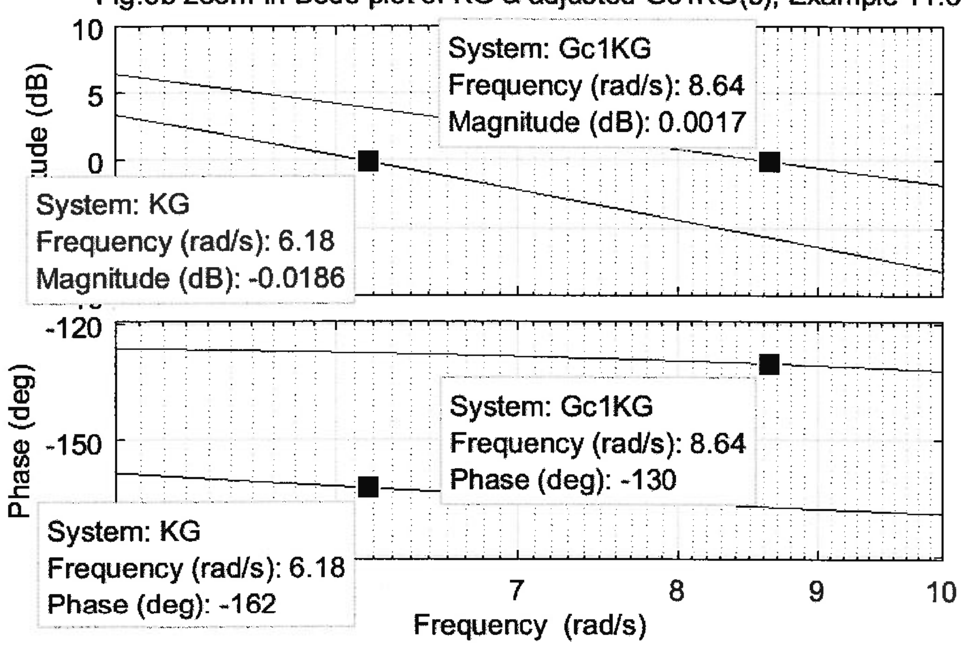


Fig.5b zoom-in Bode plot of KG & adjusted Gc1KG(s); Example 11.6



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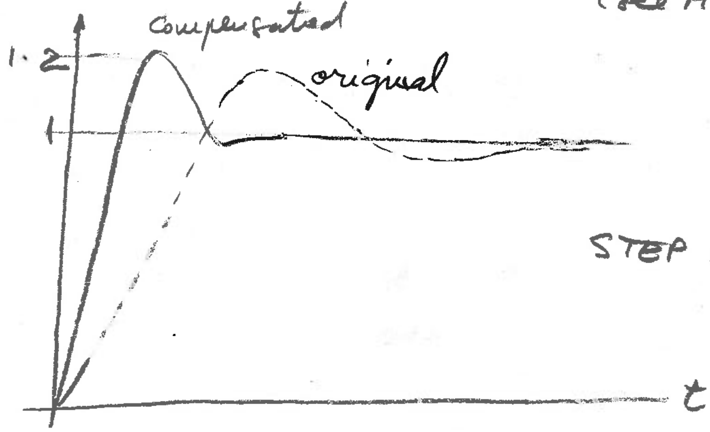
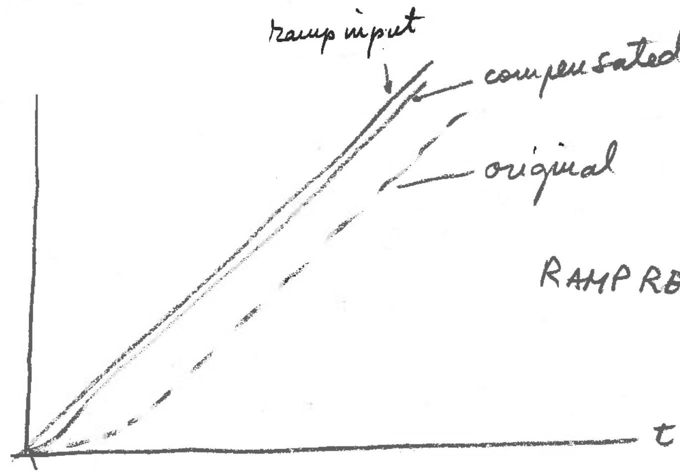
C11) Plot step response and ramp response of the original and compensated systems

(see MATLAB)



STEP RESPONSE



RAMP RESPONSE

Discussion

Ramp response has improved dramatically

Step response has a faster rise time and a faster settling time, but it has a slightly higher overshoot e11a

Fig.6a Step response Example 11.6

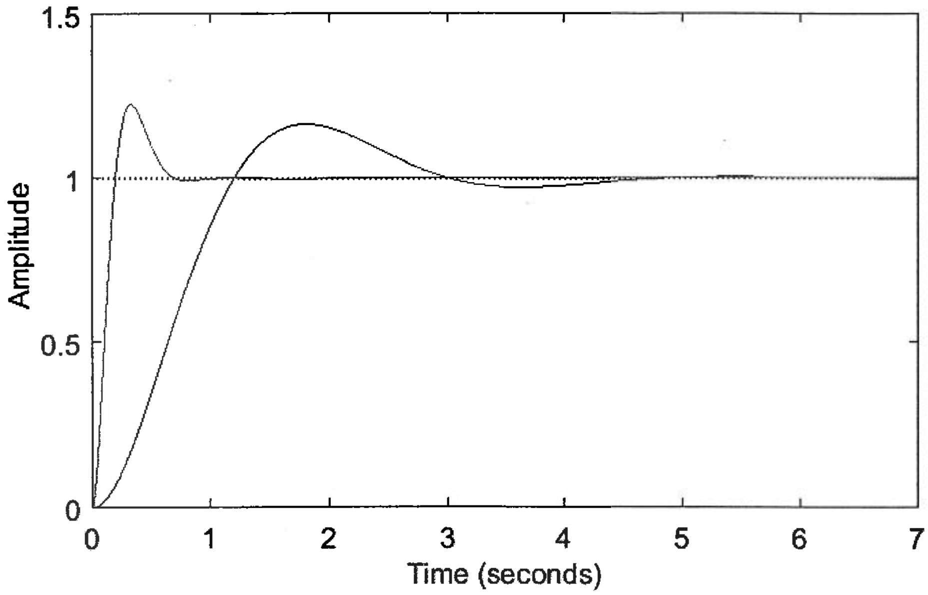
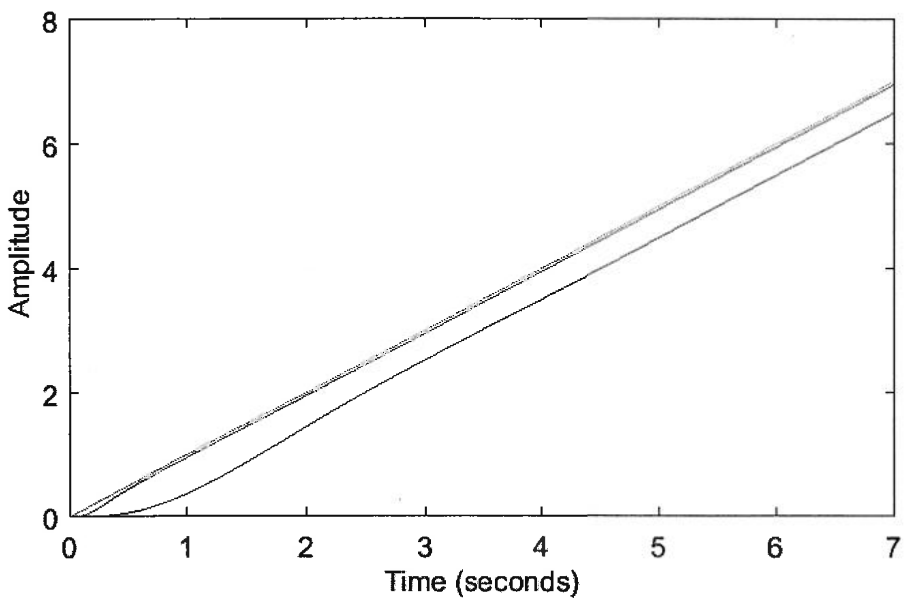


Fig.6b Ramp response Example 11.6



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8.8 Lag Compensators

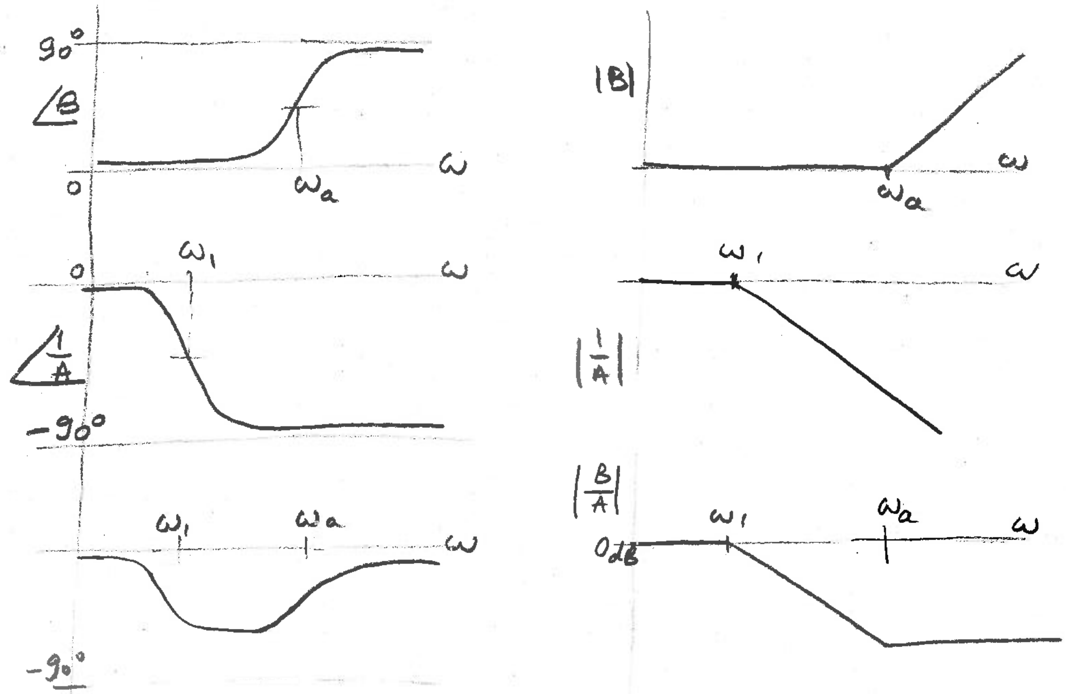
C4

LAG COMPENSATOR

$$G_c = \frac{sT_a+1}{sT_1+1}$$

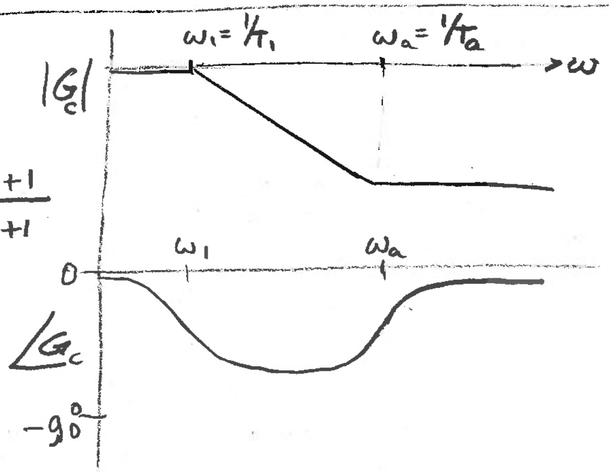
$\omega_1 < \omega_a$, $T_a < T_1$

— Phase — — Gain —



Adds "lag" between ω_1 & ω_a

Low pass filter
(higher freq. are reduced)



$$G_c(s) = \frac{sT_a+1}{sT_1+1}$$

$T_a < T_1$

LP filter

Phase lag

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c5

LAG-LEAD COMPENSATOR
(NOTCH FILTER)

$$G_c = \frac{sT_a+1}{sT_1+1}\cdot\frac{sT_b+1}{sT_2+1}$$

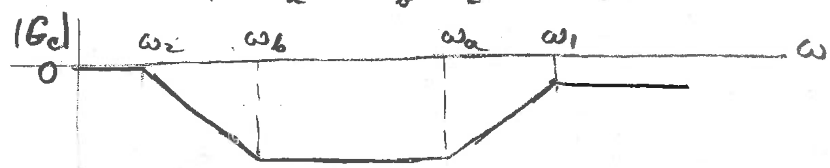
lead lag

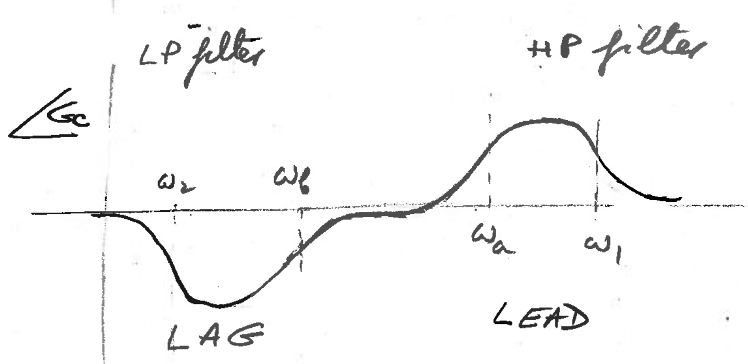
$G_c(i0) = 1$

$G_c(i\infty) = \frac{T_a}{T_1}\cdot\frac{T_b}{T_2}$

$T_1 < T_a < T_b < T_2$

$\omega_1 > \omega_a > \omega_b > \omega_2$

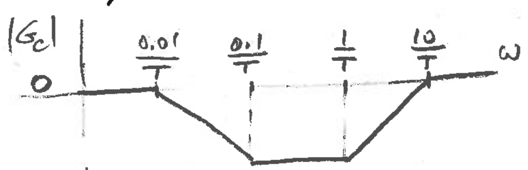




Example (Fig 11.35, p 648). $G_c = \frac{s+1}{0.1Ts+1}\cdot\frac{10Ts+1}{100Ts+1}$

$T: \quad 0.1T < T < 10T < 100T$

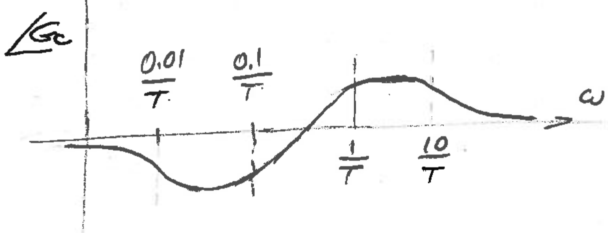
$\omega: \quad \frac{10}{T} > \frac{1}{T} > \frac{0.1}{T} > \frac{0.01}{T}$



NOTCH FILTER

$G(i0) = 1 \quad (0\,dB)$

$G(i\infty) = 1 \quad (0\,dB)$



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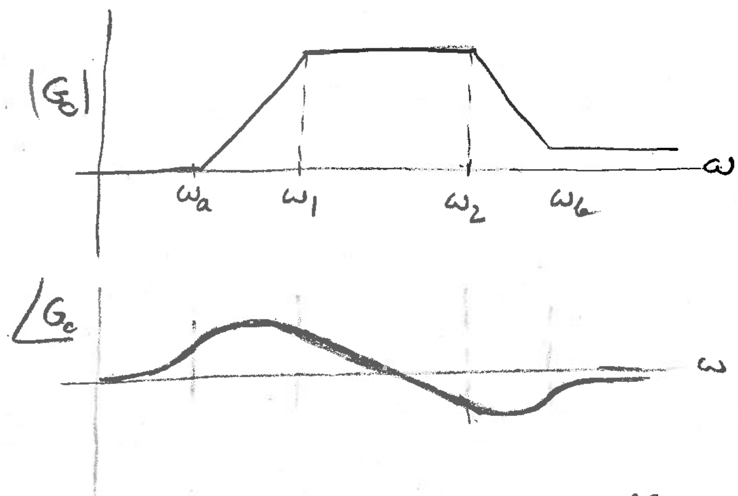
C6

LEAD–LAG COMPENSATOR
(BAND PASS FILTER)

$$G_c = \frac{sT_a + 1}{sT_1 + 1} \cdot \frac{sT_b + 1}{sT_2 + 1}$$

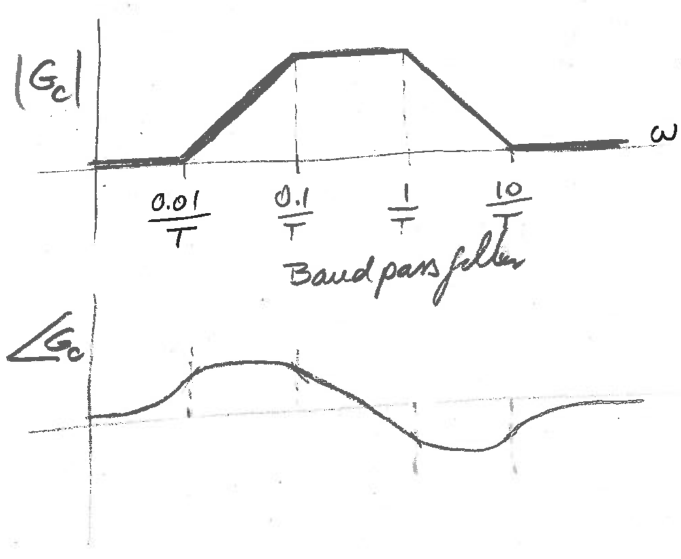
$$\omega_a < \omega_1 < \omega_2 < \omega_b$$

$$T_a > T_1 > T_2 > T_b$$



Example : Band pass filter

$$G_c = \frac{100Ts + 1}{10Ts + 1} \cdot \frac{0.1Ts + 1}{Ts + 1}$$



C6a Notch filter

Example

$$G_c = \frac{Ts+1}{0.1Ts+1} \cdot \frac{10Ts+1}{100Ts+1}$$

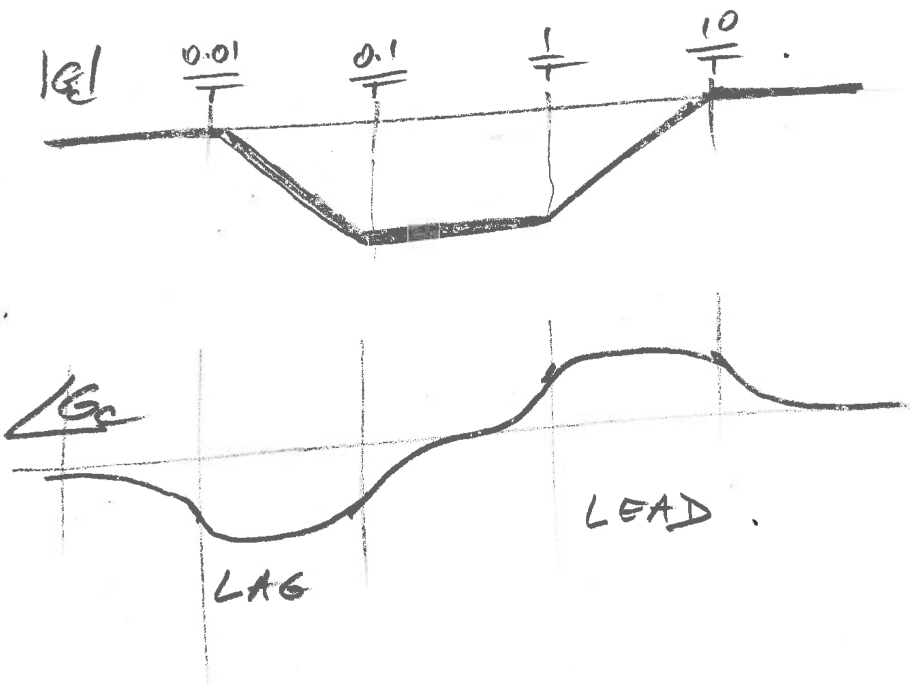
$$\overset{T_1}{0.1T} < \overset{T_a}{T} < \overset{T_b}{10T} < \overset{T_2}{100T}$$

$$\omega_1 > \omega_a > \omega_b > \omega_2$$

$$\frac{10}{T} > \frac{1}{T} > \frac{0.1}{T} > \frac{0.01}{T}$$

$$\omega_2 < \omega_b < \omega_a < \omega_1$$

$$\frac{0.01}{T} \quad \frac{0.1}{T} \quad \frac{1}{T} \quad \frac{10}{T}$$



9 Additional Topics

9.1 Combined Analysis

Combined analysis

Purpose : use all the tools to evaluate the system stability

Time domain: · Root locus of G
· Step response of G_{CL}

Frequency domain:
- Nyquist plot of G
- Bode plots of G: margins < gain K_g, phase γ

Given: $G(s)$

Find: combined analysis : Quad plot

Nyquist plot	Margins on Bode plot
Root locus for Gain=1	Step response

Examples:

open MATLAB codes

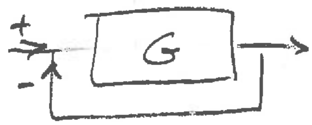
aircraft roll mode
- Nyquist plot : STABLE
- root locus — " —
- step response — " —

Bode plot : INSUFFICIENT phase margin

B11.9 – STABLE : N-plot
r locus
step response
– INSUFFICIENT phase margin

B11.8 UNSTABLE

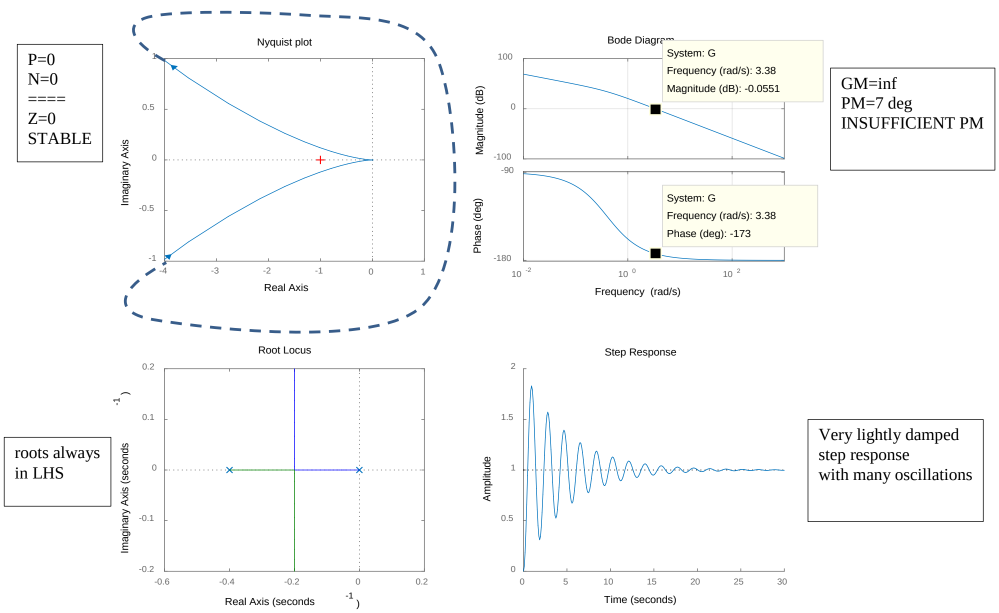
Combined analysis of aircraft roll model.



$$G(s) = \frac{K}{Js^2 + cs} = \frac{114}{10s^2 + 4s}$$

$$p_1 = 0$$
$$p_2 = -0.4.$$

See plot


P=0
N=0
====
Z=0
STABLE
Nyquist plot
Real Axis
Imaginary Axis
Bode Diagram
System: G
Frequency (rad/s): 3.38
Magnitude (dB): -0.0551
System: G
Frequency (rad/s): 3.38
Phase (deg): -173
Magnitude (dB)
Phase (deg)
Frequency (rad/s)
GM=inf
PM=7 deg
INSUFFICIENT PM
roots always
in LHS
Root Locus
Real Axis (seconds -1)
Imaginary Axis (seconds -1)
Step Response
Amplitude
Time (seconds)
Very lightly damped
step response
with many oscillations

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20170417

Aircraft roll model combined analysis (cont.)

Nyquist.

✓ stable

$Z=0$.

Bode

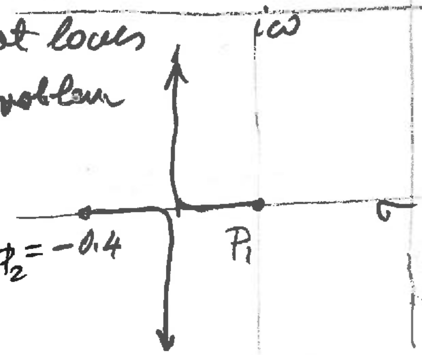
Margins

$(K_g)_{dB}=\infty$ ✓

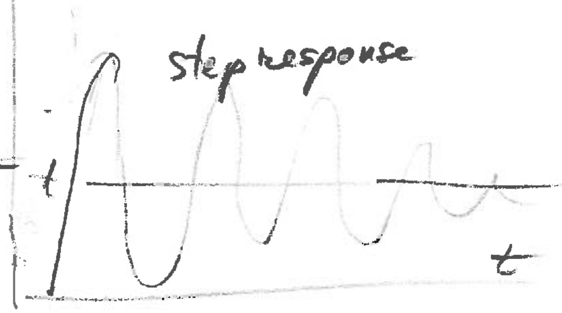
$\gamma^\circ = 7^\circ$ Not sufficient

Root locus

No problem



step response



<u>Conclusion</u>

- Nyquist and root locus indicate stable system.
- Margin analysis warns about low phase margin; → need compensator
- step response — oscillatory

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Problem B11.9

$$G(s) = \frac{20(s^2+s+0.5)}{s(s-1)(s+10)}$$

$$\left.\begin{array}{l} p_1 = 0 \\ p_2 = +1 \\ p_3 = -10 \end{array}\right\} \quad P = 1$$

Nyquist	Bode
root locus	step response

see plot

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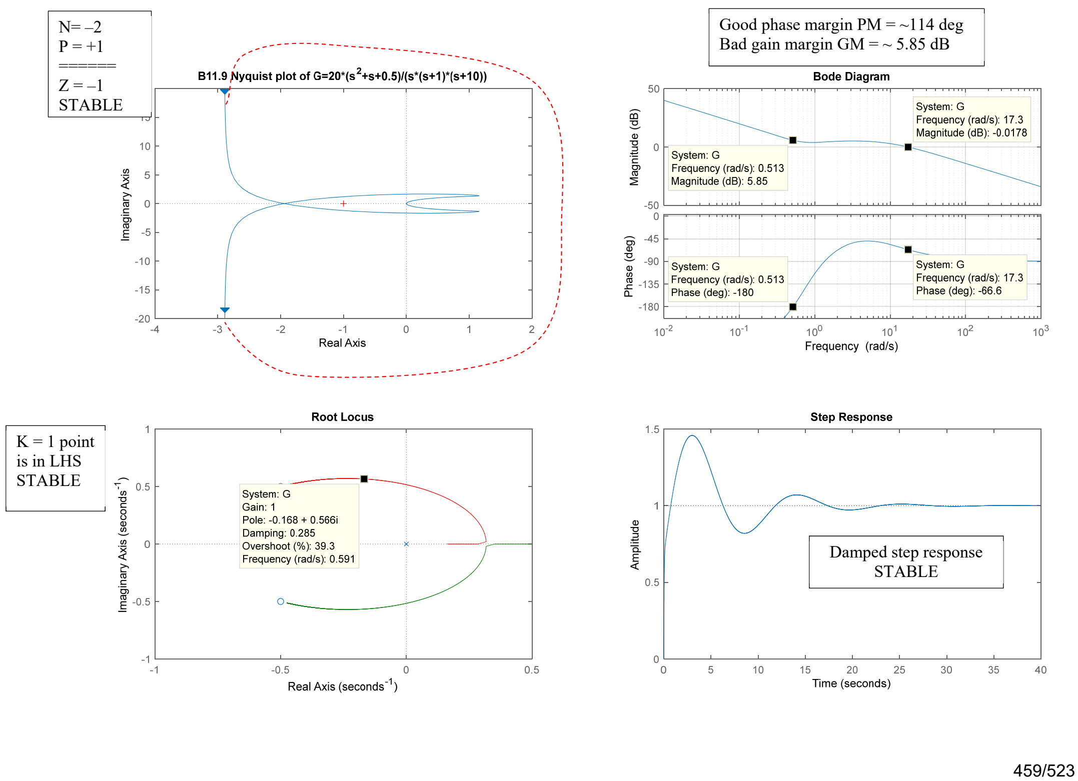
N= –2
P = +1
======
Z = –1
STABLE
B11.9 Nyquist plot of G=20*(s^2+s+0.5)/(s*(s+1)*(s+10))
Imaginary Axis
Real Axis
Good phase margin PM = ~114 deg
Bad gain margin GM = ~ 5.85 dB
Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/s)
System: G
Frequency (rad/s): 0.513
Magnitude (dB): 5.85
System: G
Frequency (rad/s): 17.3
Magnitude (dB): -0.0178
System: G
Frequency (rad/s): 0.513
Phase (deg): -180
System: G
Frequency (rad/s): 17.3
Phase (deg): -66.6
K = 1 point
is in LHS
STABLE
Root Locus
Imaginary Axis (seconds^-1)
Real Axis (seconds^-1)
System: G
Gain: 1
Pole: -0.168 + 0.566i
Damping: 0.285
Overshoot (%): 39.3
Frequency (rad/s): 0.591
Step Response
Amplitude
Time (seconds)
Damped step response
STABLE
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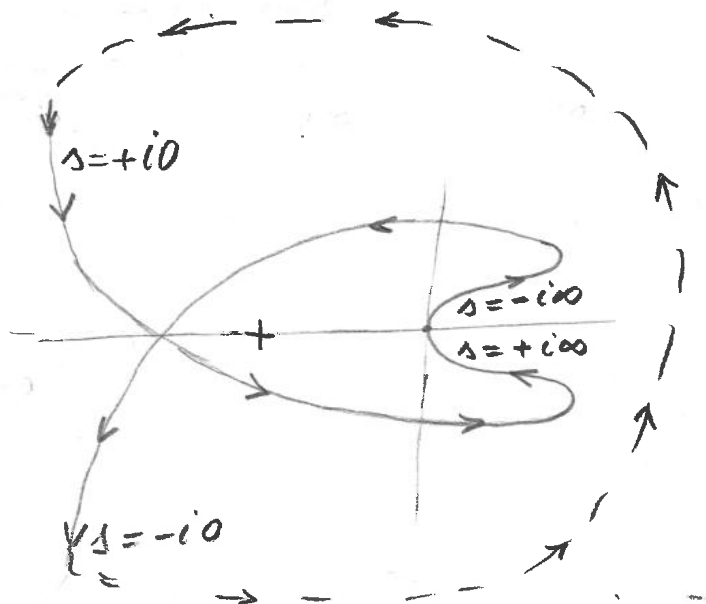
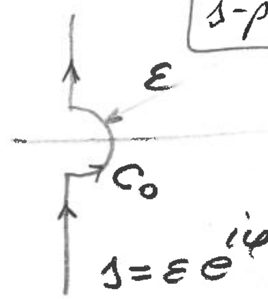
Problem B11.9 (cont.)

Nyquist plot





$$G^o = \frac{2o}{-s} = -2\frac{1}{\varepsilon}e^{-i\varphi}$$

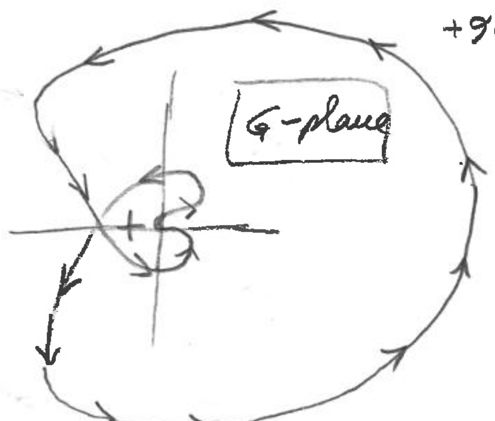
$$s = \varepsilon e^{i\varphi}$$

$$G^o = \frac{2}{\varepsilon}e^{i(-180^\circ-\varphi)}$$

$$-90^\circ < \varphi < 90^\circ \qquad -90^\circ < \varphi < 90^\circ$$

$$-90^\circ < \angle G^o < \underline{-270^\circ}$$

$$+90^\circ$$



$N = -2$

$P = 1$

$Z = -1$

$\boxed{\text{Stable}}$

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(B11.9 cont.)

Bode plot

Good phase margin: $\gamma \approx 114^\circ$ @ $16.6 \frac{rad}{sec}$

$\boxed{\text{Bad}}$ gain margin at low freq.: $K_g = -5.3$ dB @ $0.55 \frac{rad}{sec}$

Good gain margin at higher $\omega > 16 \frac{rad}{sec}$, $\angle G$

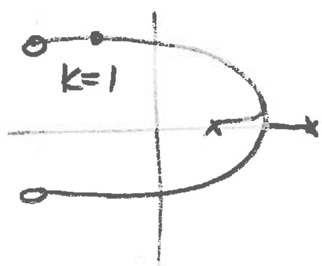
(φ does not cross -180°)

There could be problems at low ω ($\omega \sim 0.5$ rad/sec) due to imperfections

Root locus

K=1 in LHS

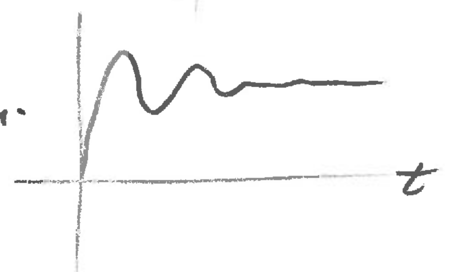
Stable



Step response

damped oscillation.

STABLE



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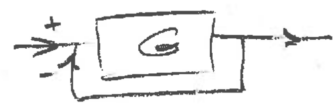
Problem B11.8



$$G(s) = \frac{320\,(s+2)}{s(s+1)(s^2+8s+44)}$$

4 Poles: 0
-1
$-4 \pm i5.3$

all in LHS

N	B
R	S

See plot

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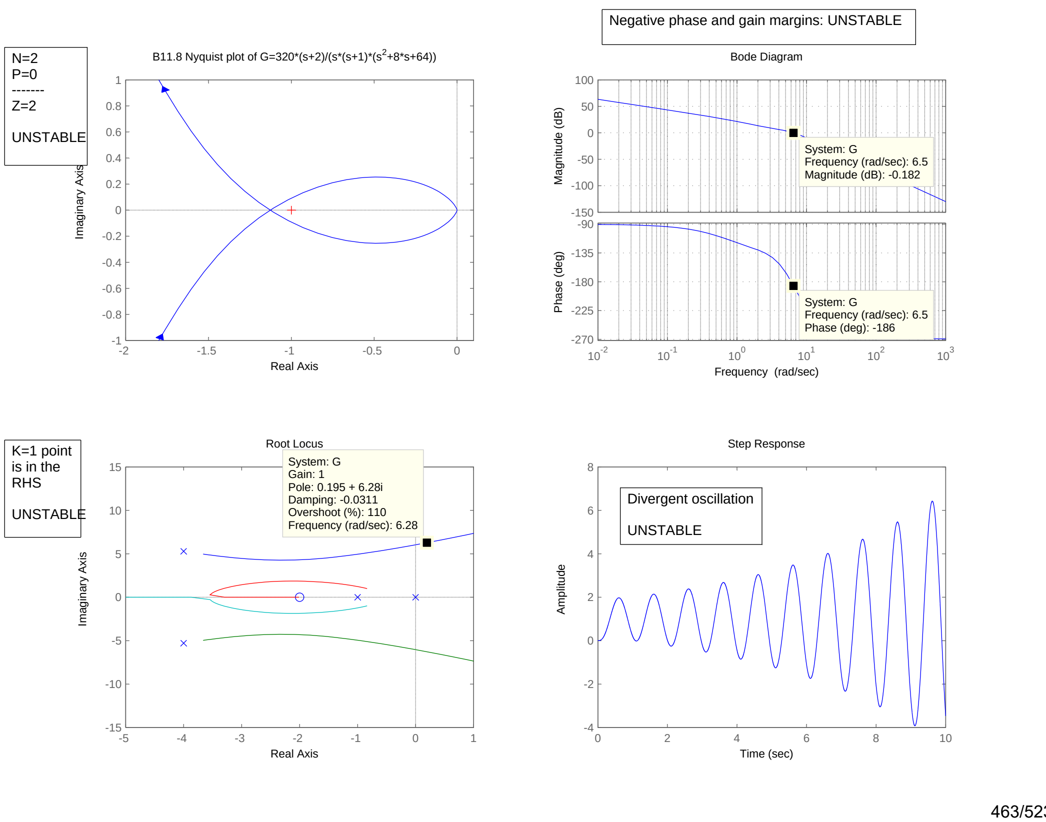
N=2
P=0

Z=2
UNSTABLE
B11.8 Nyquist plot of G=320*(s+2)/(s*(s+1)*(s^2+8*s+64))
Imaginary Axis
Real Axis
Negative phase and gain margins: UNSTABLE
Bode Diagram
Magnitude (dB)
Phase (deg)
System: G
Frequency (rad/sec): 6.5
Magnitude (dB): -0.182
System: G
Frequency (rad/sec): 6.5
Phase (deg): -186
Frequency (rad/sec)
K=1 point
is in the
RHS
UNSTABLE
Root Locus
System: G
Gain: 1
Pole: 0.195 + 6.28i
Damping: -0.0311
Overshoot (%): 110
Frequency (rad/sec): 6.28
Imaginary Axis
Real Axis
Step Response
Divergent oscillation
UNSTABLE
Amplitude
Time (sec)
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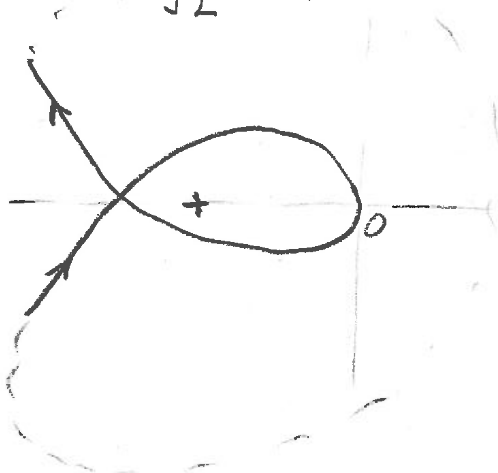
Problem B 11.8 (cont.)

Nyquist plot



$P = 0$ no poles in RHS

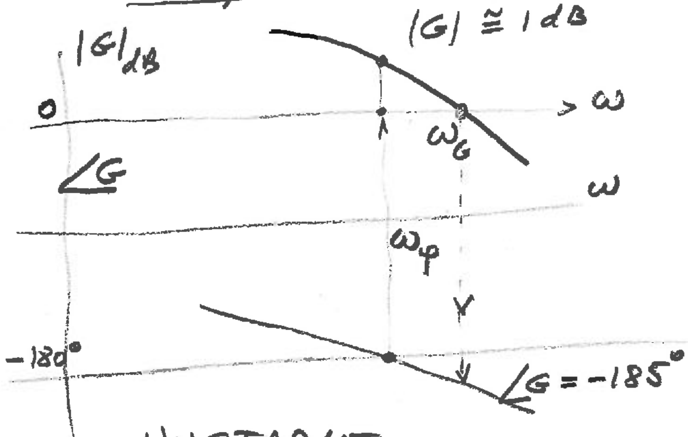
$N = 2$

$Z = P + N = 2$ zeros in RHS of $1 + G(s)$

UNSTABLE because

$$\boxed{G_{CL} = \frac{G(s)}{1 + G(s)}}$$

Bode plot : margins



$K_g = -1\,dB < 0\,dB$

$\gamma = -5° < 0$

UNSTABLE

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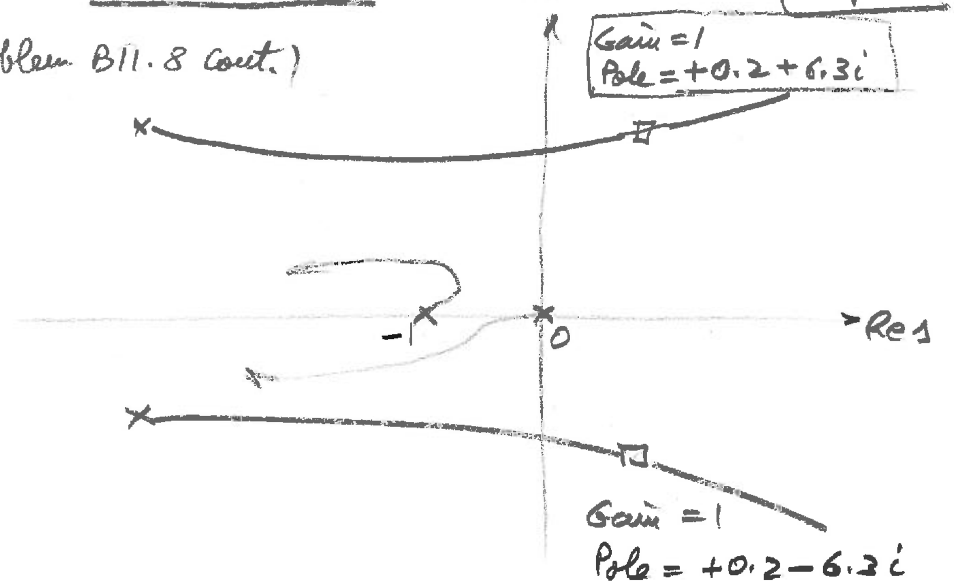
Root locus

(Problem B11.8 cont.)

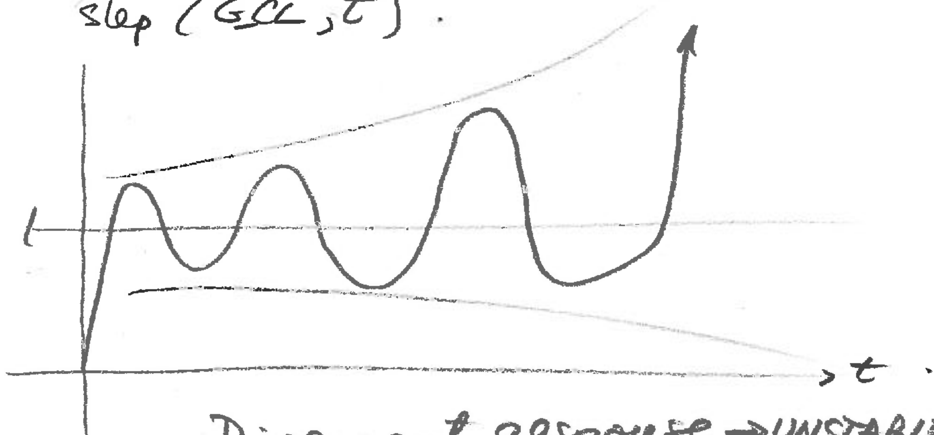


Roots in RHS → UNSTABLE.

Step response

G_{CL} = feedback(G,1).

step (GCL, t).



Divergent response → UNSTABLE

All four criteria predicted an UNSTABLE closed loop response.

ACTION: DO NOT close the loop!

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9.2 Control Systems Designer (MATLAB SISO Tool)

CSD

Control System Designer (CSD) tool is an interactive GUI environment for analyzing and modifying feedback control systems.

- CSD was introduced in R2015a
- previous, we used "SisoTool"

To start CSD, type in Command Window:

>> controlSystemDesigner

The legacy siso command is also recognized:

>> sisotool

Before starting CSD, run a MATLAB program to create the basic system G(s).

- Single input, single output (SISO)
- Graphical User Interface (GUI)
- Requires separate definition of plant transfer function G(s) through an m file

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v

CSD

Control System Designer

Design single-input, single-output (SISO) controllers

Description

The **Control System Designer** app lets you design single-input, single-output (SISO) controllers for feedback systems modeled in MATLAB or Simulink (requires Simulink Control Design™ software).

Using this app, you can:

- Design controllers using:
  - Interactive Bode, root locus, and Nichols graphical editors for adding, modifying, and removing controller poles, zeros, and gains.
  - Automated PID, LQG, or IMC tuning.
  - Optimization-based tuning (requires Simulink Design Optimization™ software).
  - Automated loop shaping (requires Robust Control Toolbox™ software).
- Tune compensators for single-loop or multiloop control architectures.
- Analyze control system designs using time-domain and frequency-domain responses, such as step responses and pole-zero maps.
- Compare response plots for multiple control system designs.
- Design controllers for multimodel control applications.

Open the Control System Designer App

- MATLAB Toolstrip: On the **Apps** tab, under **Control System Design and Analysis**, click the app icon.
- MATLAB command prompt: Enter `controlSystemDesigner`.
- Simulink model editor: Select **Analysis** > **Control Design** > **Control System Designer**.

Examples

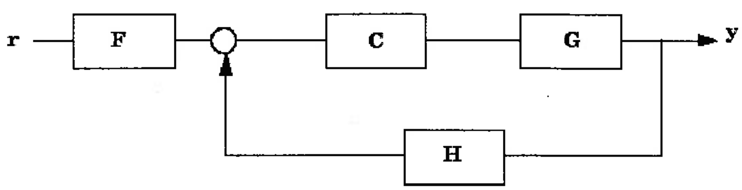
- "Control System Designer Tuning Methods"

3
CSD

- "Bode Diagram Design"
- "Root Locus Design"
- "Design Compensator Using Automated Tuning Methods"
- "Design Multiloop Control System"
- "Analyze Designs Using Response Plots"
- "Compare Performance of Multiple Designs"
- "Multimodel Control Design"

Programmatic Use

`controlSystemDesigner` opens the **Control System Designer** app using the following default control architecture:



The architecture consists of the LTI objects:

- G — Plant model
- C — Compensator
- H — Sensor model
- F — Prefilter

By default, the app configures each of these models as a unit gain.

`controlSystemDesigner(plant)` initializes the plant, G, to `plant`. `plant` can be any SISO LTI model created with `ss`, `tf`, `zpk` or `frd`, or an array of such models.

`controlSystemDesigner(plant,comp)` initializes the compensator, C, to the SISO LTI model `comp`.

u
CSD

`controlSystemDesigner(plant,comp,sensor)` initializes the sensor model, *H*, to `sensor`. `sensor` can be any SISO LTI model or an array of such models. If you specify both `plant` and `sensor` as LTI model arrays, the lengths of the arrays must match.

`controlSystemDesigner(plant,comp,sensor,prefilt)` initializes the prefilter model, *F*, to the SISO LTI model `prefilt`.

`controlSystemDesigner(views)` opens the app and specifies the initial graphical editor configuration. `views` can be any of the following character vectors, or a cell array of multiple character vectors.

- `'rlocus'` — Root locus editor
- `'bode'` — Open-loop Bode Editor
- `'nichols'` — Open-loop Nichols Editor
- `'filter'` — Bode Editor for the closed-loop response from prefilter input to the plant output

In addition to opening the specified graphical editors, the app plots the closed-loop, input-output step response.

`controlSystemDesigner(views,plant,comp,sensor,prefilt)` specifies the initial plot configuration and initializes the plant, compensator, sensor, and prefilter using the specified models. If a model is omitted, the app uses the default value.

`controlSystemDesigner(initData)` opens the app and initializes the system configuration using the initialization data structure `initdata`. To create `initdata`, use `sisoinit`.

`controlSystemDesigner(sessionFile)` opens the app and loads a previously saved session. `sessionFile` is the name of a session data file on the MATLAB path. This data includes the current system architecture and plot configuration, and any designs and responses saved in the **Data Browser**.

To save a session, in the **Control System Designer** app, on the **Control System** tab, click **Save Session**.

See Also

Apps
Control System Tuner

5
CSD

Functions
pidTuner | sisoinit

Introduced in R2015a

‘Control System Designer’ MATLAB Tool

Aircraft Roll Motion Autopilot Development
PD Example

Table of Contents:

1 CONTROL SYSTEM DESIGN OBJECTIVES

Consider the aircraft roll transfer function $G(s)=\dfrac{K}{Js^2+cs}=\dfrac{114}{10s^2+4s}$

Design a feedback control system to control the aircraft roll motion with the following control design objectives:

Design Objective 1: Control the unconstraint aircraft motion resulting from an aileron input. Have a autopilot system that can maintain the aircraft at a constant bank angle

We will achieve this objective through feedback (FB)

Design Objective 2: Achieve a reasonable aircraft roll response.

Design Objective 3: Ensure safe and stable operation of the feedback control system

We define 'reasonable response' using two control design specifications:

- DS1: Fast response time as measured by rise time
  $t_r \leq 1.5$ sec
- DS2: maximum percentage overshoot for step input less than 20%
  $M_p \leq 20\%$

We ensue safe and stable operation of the feedback control system by meeting a third design specification based on stability margins:

- DS3: $GM = 10\,\text{dB}$, $PM = 60°$

2 CREATE PLANT TRANSFER FUNCTION G

Run m-file `“aircraft_roll_model.m”` to create the plant transfer function G in the Workspace (Figure 1)

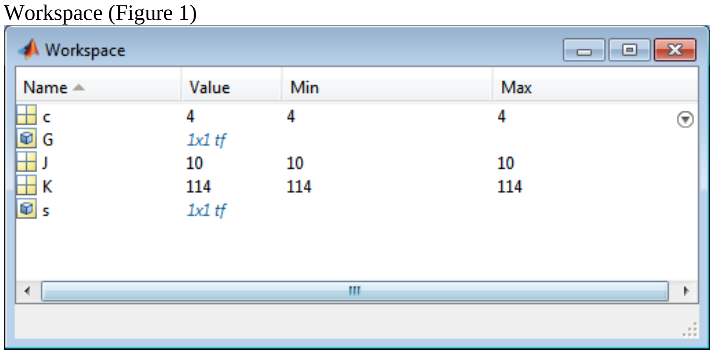


Figure 1

3 CREATE 'CONTROL SYSTEM DESIGNER' MODEL

3.1 OPEN 'CONTROL SYSTEM DESIGNER' SOFTWARE TOOL

In the Command Window, type "`controlSystemDesigner`"
`>>controlSystemDesigner`
(legacy name "`sisotool`", also works, i.e., `>>sisotool`)

The Control System Designer GUI opens as shown in Figure 2.

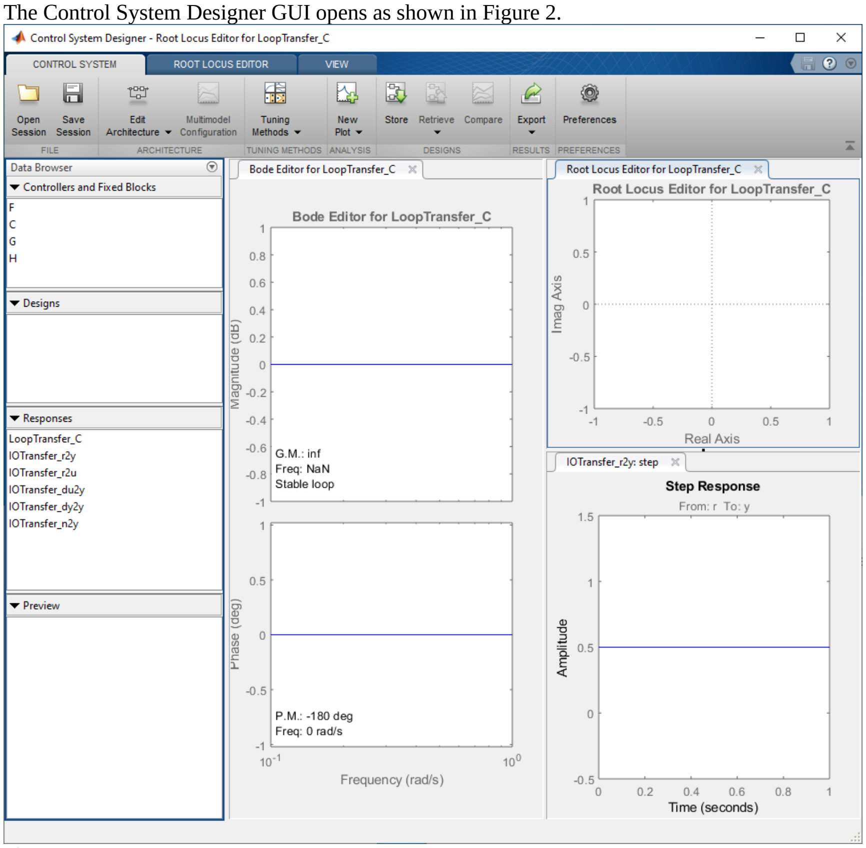


Figure 2

The default configuration has four blocks (F, C, G, H) as indicated in the LHS pane. All the blocks in this configuration are set to be unitary transfer functions by default (the plant block G, which is selected in the upper LHS pane, is shown in the lower LHS pane with Value: 1).

3.2 Import Plant Transfer Function G

Press “Edit Architecture” button to import the system G data from the Workspace (Figure 3).

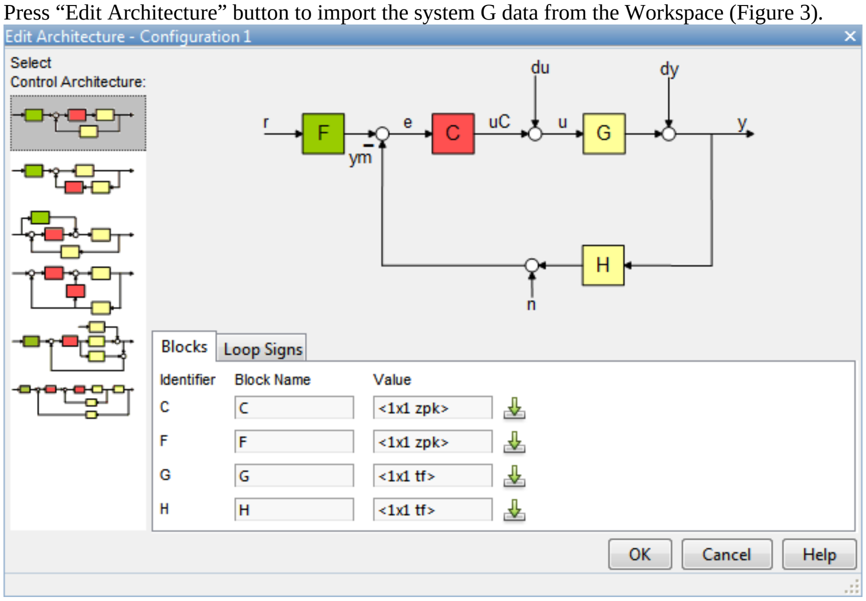


Figure 3

Note that the first control system configuration is selected by default (this is the configuration we work with, no need to change it.)

Press the down arrow icon at the RHS end of the G row to open the “Import Data for G” dialog box and select ‘G’ from the Base Workspace (Figure 4).

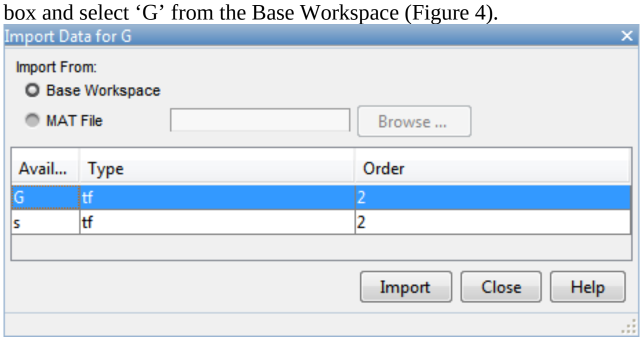


Figure 4

Press ‘Import’ to return to ‘Edit Architecture’ dialog box and press OK to return to the main view.

Now, our system $G(s)=\dfrac{114}{10s^2+2s}$ is shown in the lower LHS pane of the model (Figure 5)

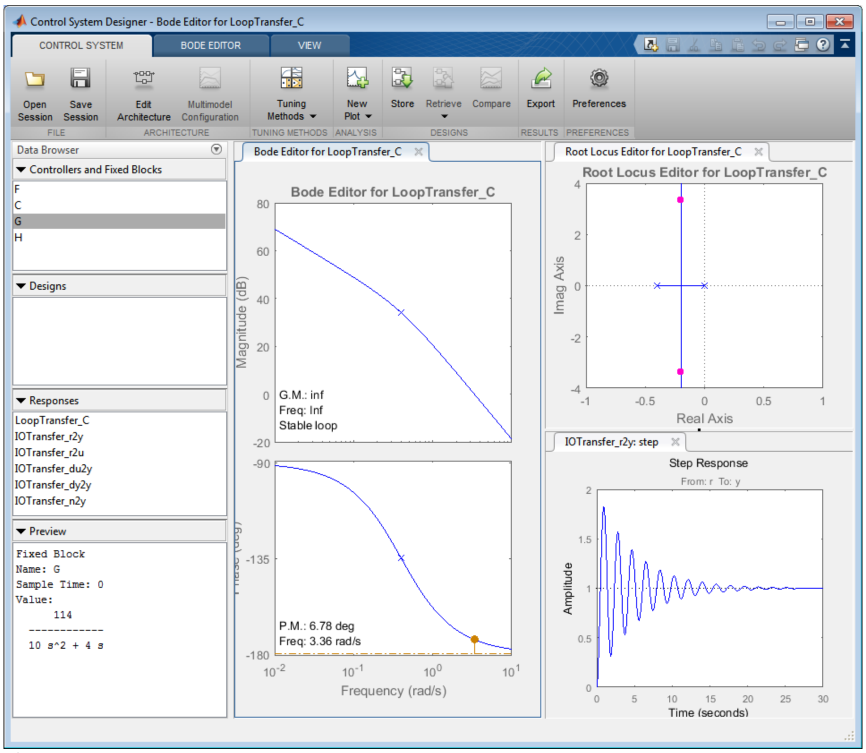


Figure 5

3.3 ADD DESIGN SPECIFICATIONS

Right-click on the Bode Editor window and select Design Requirements → New. The following window shown in Figure 6 appears:

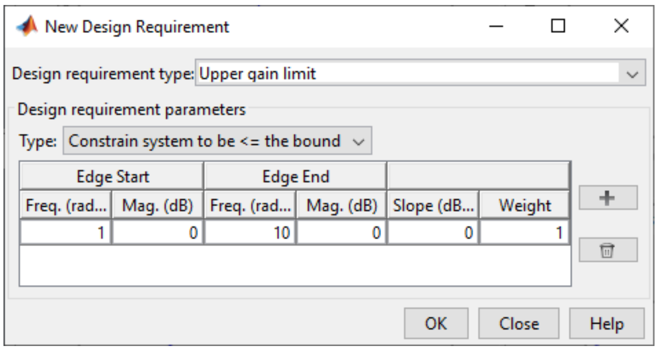


Figure 6

From the pull-down menu "Design requirement type" select "Gain & Phase margins" and enter the required DS3 values 10dB and 60deg to make the window look like shown in Figure 7.

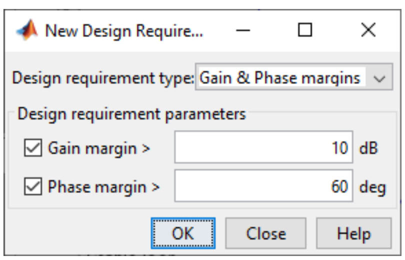


Figure 7

Press OK. Your screen should look like Figure 8.

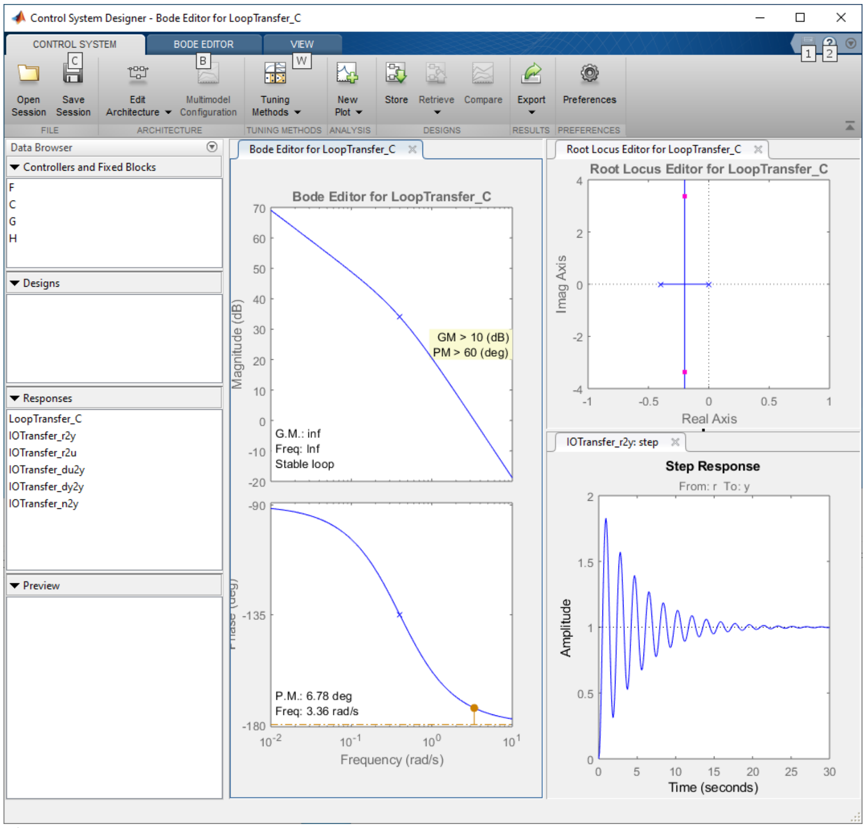


Figure 8

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3.4 Store the Baseline Design as G

Press ‘Store’ button to store this design; it will shown as ‘Design1’ in the second LHS pane. Save the CSD Sesion: press ‘Save Session’ button and save your session with the name ‘CSD_aircraft_Session_20200116’. The extension of the file, if shown, should be ‘.mat’. Your screen should look like Figure 9.

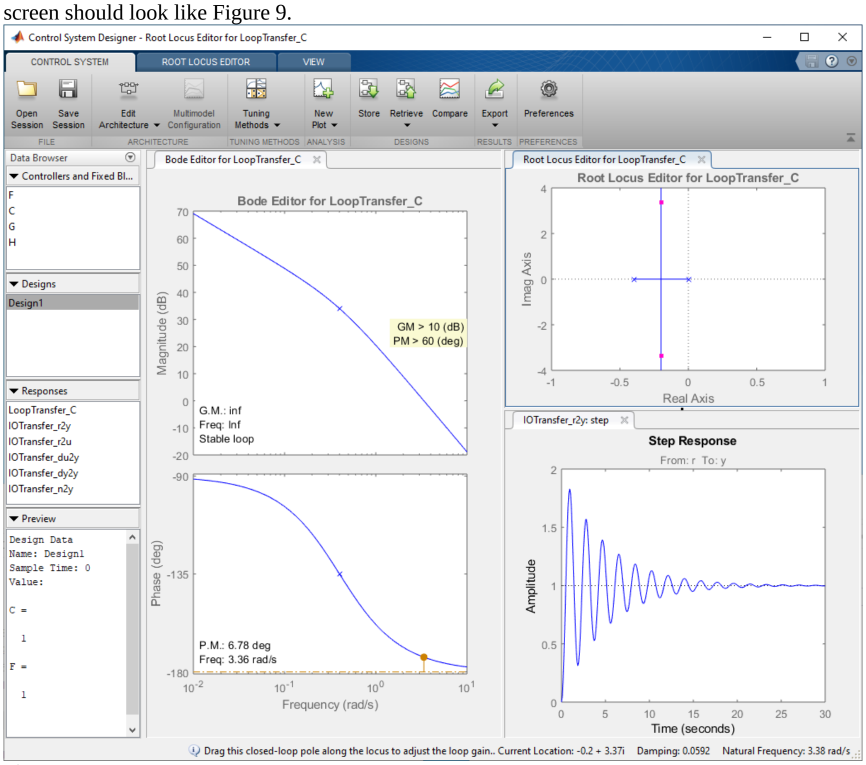


Figure 9

3.5 Record the Baseline Performance and Stability Reading of G Design

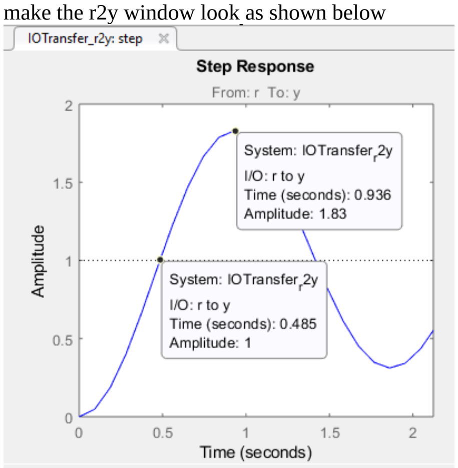
Record the following readings from this CSD windows:

- GM: inf
- PM: 6.78 deg at 3.36 rad/s
- CL poles $-0.2+3.37i$ and $-0.2-3.37i$ shown as the two red dots in the Root Locus Editor. Click on them and read the values at the bottom of the window.
- A lightly damped step response 'r2y: step' (i.e., from input r to output y) with a large overshoot

It is apparent that the response is unsatisfactory because:

- many oscillations until it settles down to $x_{ss}=1$
- large overshoot
- insufficient phase margin

Zoom into the 'r2y: step' window and use data tips to mark the rise time t_r and overshoot M_p and make the r2y window look as shown below



3.6 Verify Design Specifications

It is apparent from the reading taken so far that:

DS1: t_r=0.485 sec < 1.5 sec indicating that DS1 is satisfied
DS2: M_p=83% > 20% indicating that DS2 is NOT satisfied
DS3: GM= inf; PM=6.78 deg indicating that DS3 is NOT satisfied

A controller must be design to improve system performance and satisfy all the design specifications.

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4 PID CONTROLLER USING AUTOMATIC PID TUNING TO GENERATE DESIGN1

In this section we explain how a PD controller can be added and tuned to give a better system performance that satisfies all the design specifications.

Press 'Tuning Methods' and select 'PID' from the pull-down menu. The 'PID Tuning' window opens up (Figure 10).

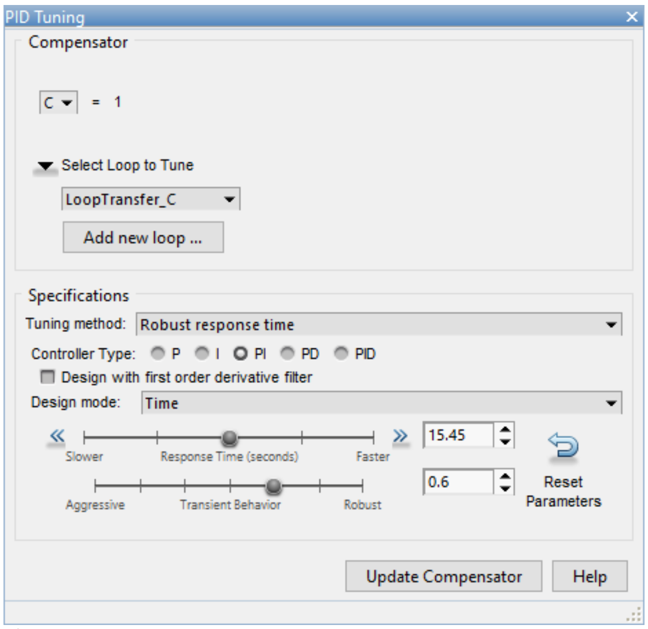

Figure 10

4.1 Choose PD Controller and Adjust 'Response Time' Range

Select 'PD' on 'Controller Type' line. Press the >> arrow on the RHS of the 'Response Time' bar to see '1.545' in the RHS window. Use the up/down scroll (or just type in the value) until it shows the value 1.5 (see Figure 11).

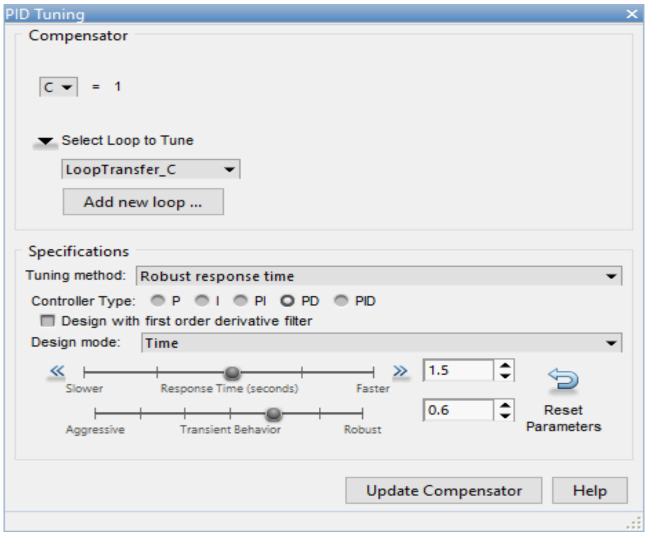

Figure 11

Press ‘Update Compensator’button. The ‘PID Tuning’ window looks as shown in Figure 12.

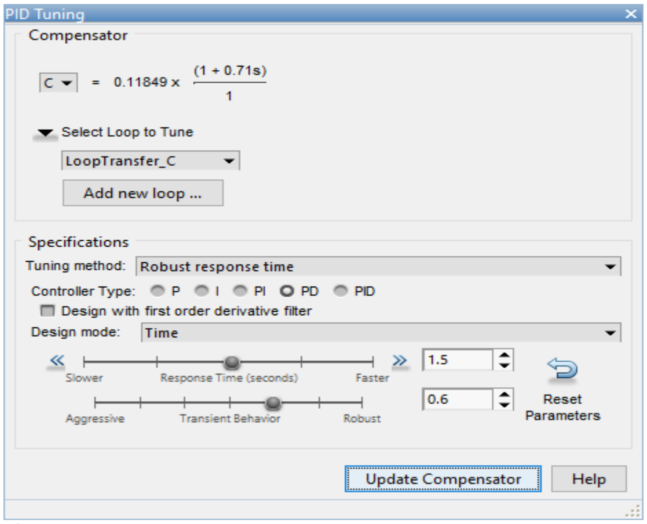

Figure 12

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5 EVALUATE DESIGN1 MODEL

Close 'PID Tuning' widow and return to the 'Control System Designer' main window. Press 'Store'. A new name appears in the 'Designs' LHS window; the new name is 'Design2'. Restore the original view in the step response window using the zoom -out button. Now, the main window looks as shown in Figure 13.

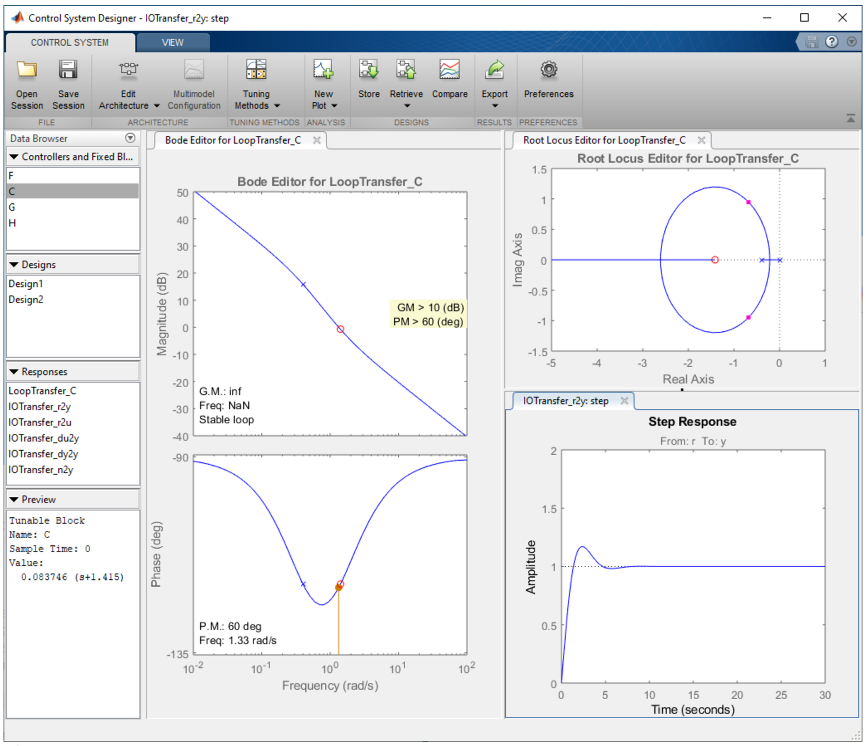

Figure 13

Note that the controller formula appear as

```
Tunable Block
Name: C
Sample Time: 0
Value:
  0.083746 (s+1.415)
```

5.1 Performance Indicators of Design2 Model

Next, verify the status of DS1 and DS2 conditions. Recall

- DS1: Fast response time as measured by rise time
  $t_r \le 1.5$ sec
- DS2: maximum percentage overshoot for step input less than 20%
  $M_p \le 20\%$

Use datatips to read t_r and M_p on the 'Step Response' plot for the closed loop system 'IOTransfer_r2y: step' (Figure 14).

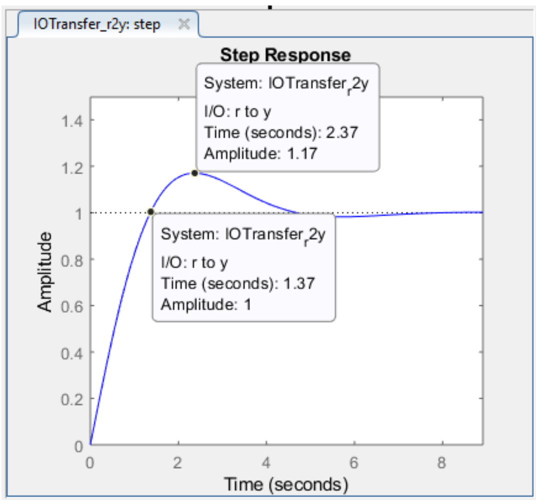

Figure 14

It is apparent that:

- t_r=1.37 sec < 1.5 sec which means that DS1 is satisfied
- M_p=17% < 20% which means that DS2 is satisfied.

5.2 Stability Margins of Design2 Model

The stability margins can be read in the Bode plot of Figure 13; they are much better than in the original design:

- GM: inf which is better than $GM = 10\,\text{dB}$ required by DS3
- PM: 60 deg, which meets the value $PM = 60°$ required by DS3

The stability margin criteria are satisfied which means that DS3 is satisfied.

We state that all three design specification DS1, DS2, DS3, have been met and the control system design process has completed successfully.

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6 COMPARE THE STEP RESPONSE OF THE INITIAL AND FINAL DESIGNS

To compare the initial and final designs, do the following:

Press ‘Compare’ button and a ‘Compare Designs’ window pops up (Figure 15).

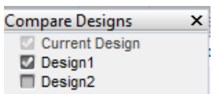


Figure 15

Check box in front of ‘Design1’. The step response plot contains the two responses overlapped (Figure 16).

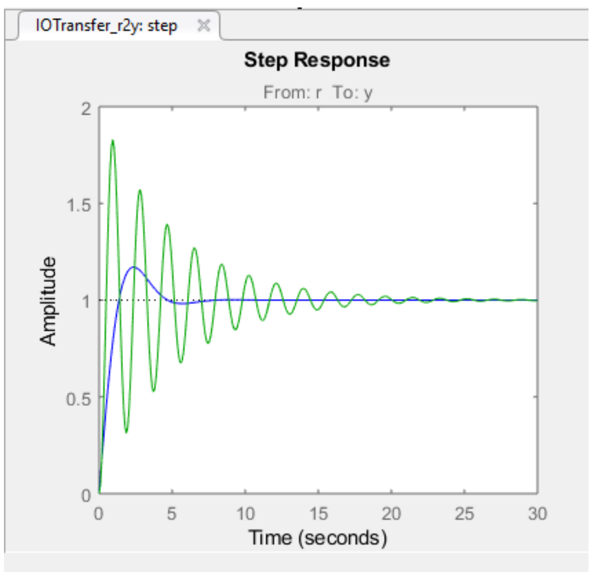
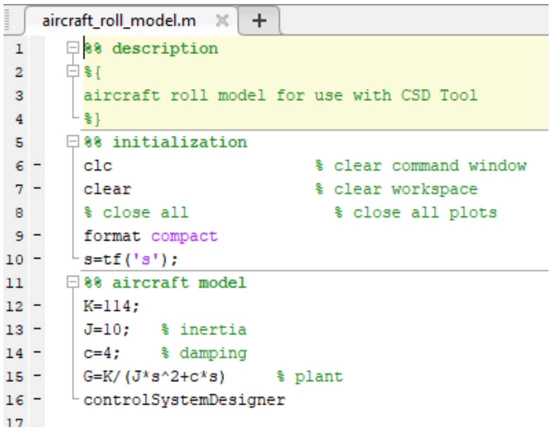


Figure 16

The initial design ‘Design1’ is shown in green, whereas the current design is shown in blue.

The response improvements achieved through this controller design process are quite apparent.



aircraft_roll_model.m

```
%% description
%{
aircraft roll model for use with CSD Tool
%}
%% initialization
clc                         % clear command window
clear                       % clear workspace
% close all                   % close all plots
format compact
s=tf('s');
%% aircraft model
K=114;
J=10;    % inertia
c=4;     % damping
G=K/(J*s^2+c*s)       % plant
controlSystemDesigner
```

‘Control System Designer’ MATLAB Tool

Aircraft Roll Motion Autopilot Development
PID Example

Table of Contents:

1 CONTROL SYSTEM DESIGN OBJECTIVES

Consider the aircraft roll transfer function $G(s) = \dfrac{K}{Js^2 + cs} = \dfrac{114}{10s^2 + 4s}$

Design a feedback control system to control the aircraft roll motion with the following control design objectives:

Design Objective 1: Control the unconstraint aircraft motion resulting from an aileron input. Have a autopilot system that can maintain the aircraft at a constant bank angle

We will achieve this objective through feedback (FB)

Design Objective 2: Achieve a reasonable aircraft roll response.

Design Objective 3: Ensure safe and stable operation of the feedback control system

We define 'reasonable response' using two control design specifications:

- DS1: Fast response time as measured by rise time
  $t_r \le 1.5$ sec
- DS2: maximum percentage overshoot for step input less than 20%
  $M_p \le 20\%$

We ensue safe and stable operation of the feedback control system by meeting a third design specification based on stability margins:

- DS3: $GM = 10\,\text{dB}$, $PM = 60°$

2 CREATE PLANT TRANSFER FUNCTION G

Run m-file `"aircraft_roll_model.m"` to create the plant transfer function G in the Workspace (Figure 1)

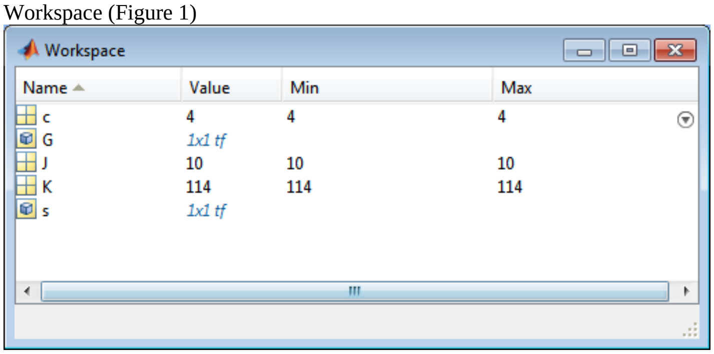


Figure 1

3 CREATE 'CONTROL SYSTEM DESIGNER' MODEL

3.1 OPEN 'CONTROL SYSTEM DESIGNER' SOFTWARE TOOL

In the Command Window, type "`controlSystemDesigner`"
`>>controlSystemDesigner`
(legacy name "`sisotool`", also works, i.e., `>>sisotool`)

The Control System Designer GUI opens as shown in Figure 2.

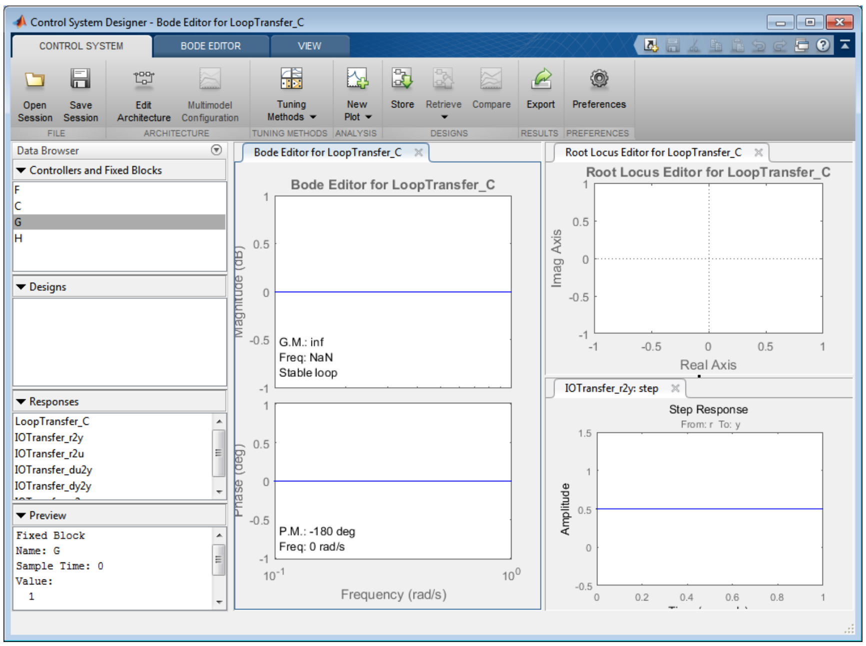


Figure 2

The default configuration has four blocks (F, C, G, H) as indicated in the LHS pane. All the blocks in this configuration are set to be unitary transfer functions by default (the plant block G, which is selected in the upper LHS pane, is shown in the lower LHS pane with Value: 1).

3.2 Import Plant Transfer Function G

Press “Edit Architecture” button to import the system G data from the Workspace (Figure 3).

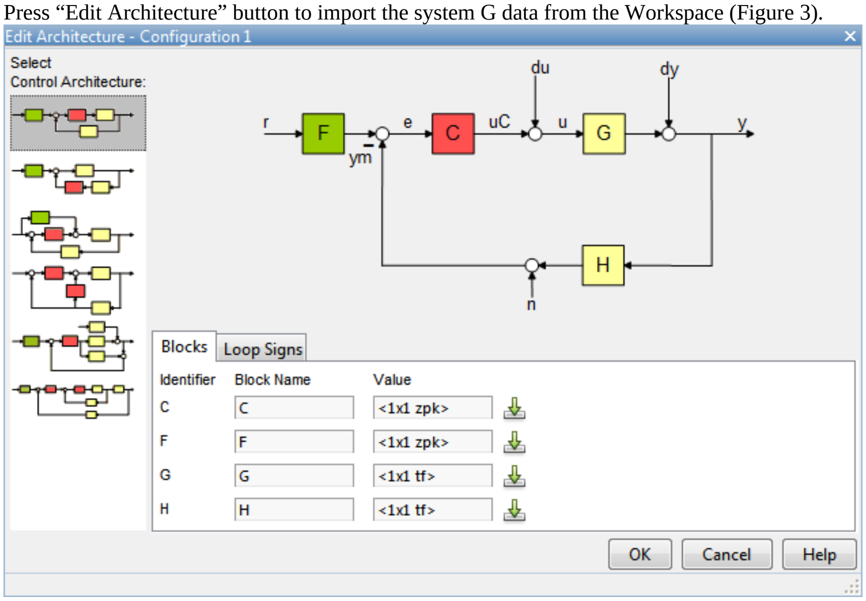


Figure 3

Note that the first control system configuration is selected by default (this is the configuration we work with, no need to change it.)
Press the down arrow icon at the RHS end of the row G to open the “Import Data for G” dialog box and select ‘G’ from the Base Workspace (Figure 4).

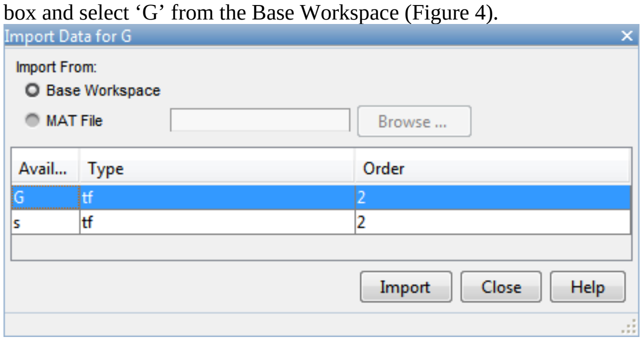


Figure 4

Press ‘Import’ to return to ‘Edit Architecture’ dialog box and press OK to return to the main view. Now, our system $G(s) = \dfrac{114}{10s^2 + 2s}$ is shown in the lower LHS pane of the model (Figure 5)

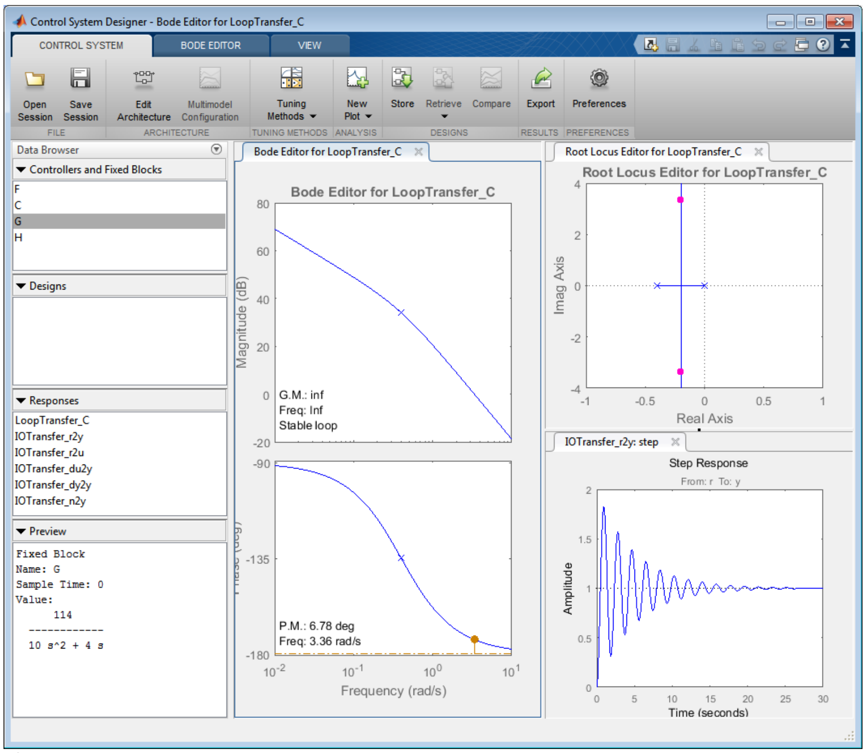


Figure 5

3.3 Store the Baseline Design as G

Press ‘Store’ button to store this design; it will shown as ‘Design1’ in the second LHS pane; double click on it to allow name edit and rename it as ‘G’. Your screen should look like Figure 6.

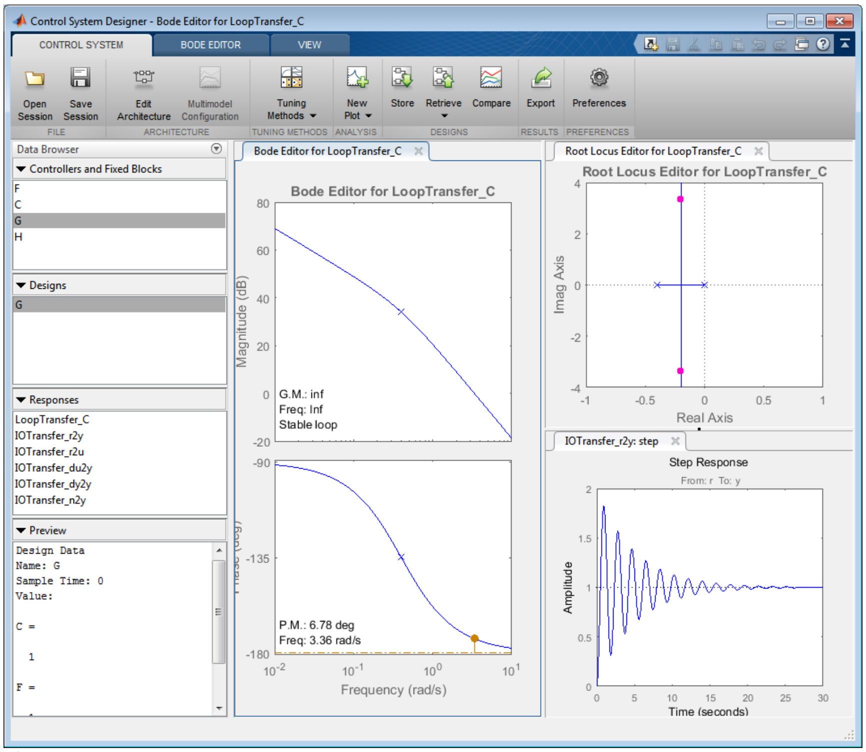


Figure 6

3.4 Save the CSD Sesion

Press ‘Save Session’ button and save your session with the name ‘CSD_aircraft_Session_20161219’. The extension of the file, if shown, should be ‘.mat’.

3.5 Record the Baseline Performance and Stability Reading of G Design

Record the following readings from this windown:

- GM: inf
- PM: 6.78 deg at 3.36 rad/s
- CL poles $-0.2+3.37i$ and $-0.2-3.37i$ shown as the two red dots in the Root Locus Editor
- A lightly damped step response 'r2y' (i.e., from input r to output y) with a large overshoot

It is apparent that the response is unsatisfactory because:

- many oscillations until it settles down to $x_{ss}=1$
- large overshoot ($x_p \approx 1.8$, $M_p \approx 80\%$)
- insufficient phase margin

4 PID CONTROLLER USING AUTOMATIC PID TUNING TO GENERATE DESIGN1

Press 'Tuning Methods' and select 'PID' from the pull-down menu. The 'PID Tuning' window opens up (Figure 7).

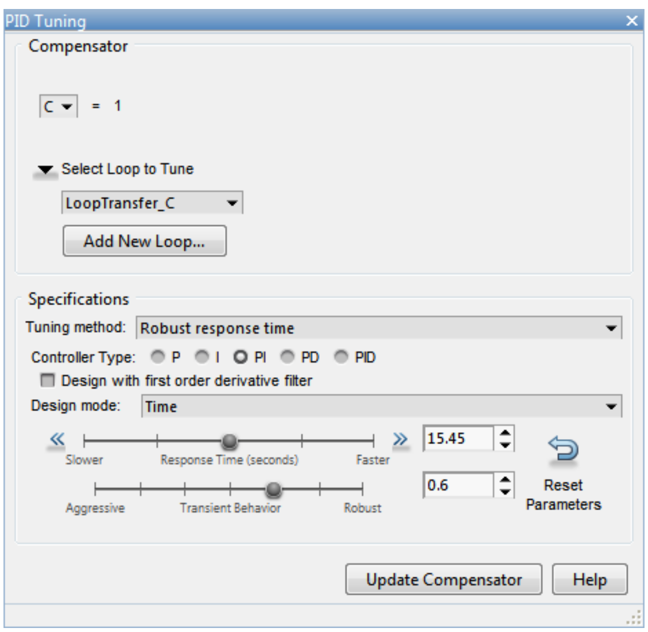


Figure 7

4.1 ADJUST 'RESPONSE TIME' RANGE

Select 'PID' on 'Controller Type' line. Press the >> arrow on the RHS of the 'Response Time' bar to see '1.545' in the RHS window (Figure 8).

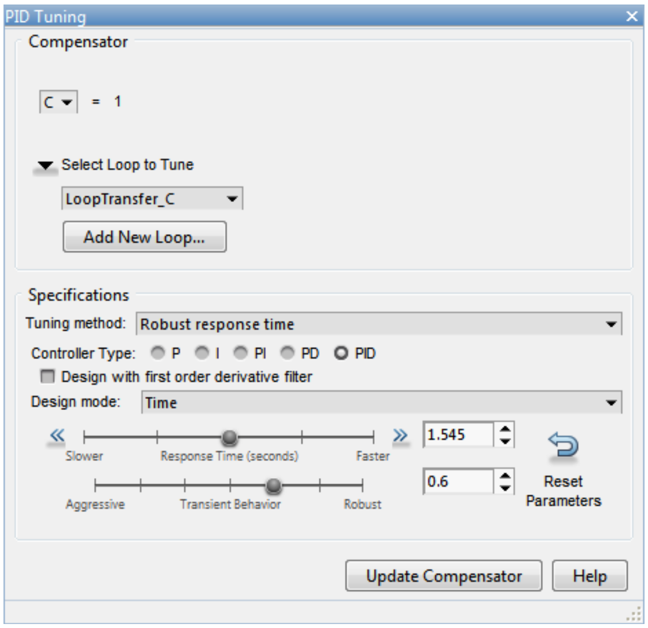


Figure 8

4.2 Adjust the 'Time Response' and 'Transient Behavior'

Use the up/down arrows to make the 'Time Response' reading in the RHS window '1.422' and the 'Transient Behavior' reading in the RHS window be '0.6'.

Press 'Update Compensator'button. The 'PID Tuning' window looks as shown in Figure 9.

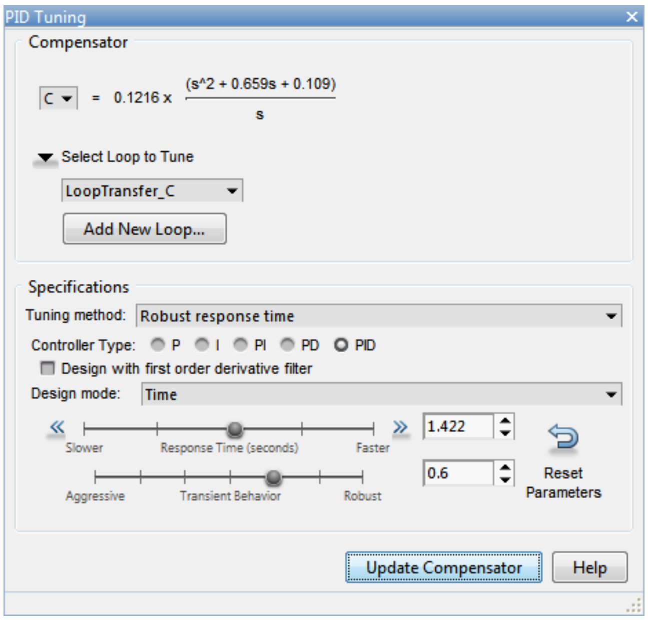


Figure 9

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5 EVALUATE DESIGN1 MODEL

Close ‘PID Tuning’ widow and return to the ‘Control System Designer’ main window. Press ‘Store’. A new name appears in the ‘Designs’ LHS window; the new name is ‘Design1’. Now, the main window looks as shown in Figure 10.

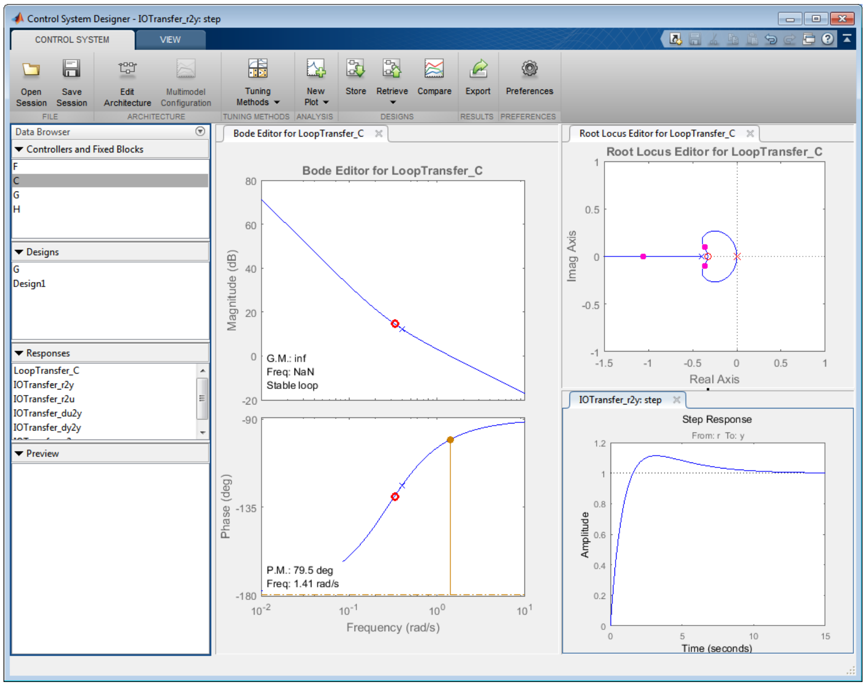

Figure 10

Note that the compensator formula appear as

```
C =

  0.11236 (s+0.3118)^2
  --------------------
           s
```

5.1 STABILITY MARGINS OF DESIGN1 MODEL

The stability margins can be read in the Bode plot; they are much better:

- GM: inf which is better than $GM = 10\,\text{dB}$ required by DS3
- PM: 79.5 deg, which is better tan , $PM = 60°$ required by DS3

The stability margin criteria are satisfied and hence we can say that DS3 condition has been met.

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5.2 PERFORMANCE INDICATORS OF DESIGN 1 MODEL

Next, verify the status of DS1 and DS2 conditions. Recall

- DS1: Fast response time as measured by rise time
  $t_r \leq 1.5$ sec
- DS2: maximum percentage overshoot for step input less than 20%
  $M_p \leq 20\%$

We use datatips to read t_r and M_p on the 'Step Response' plot for the closed loop system 'IOTransfer_r2y' (Figure 11).

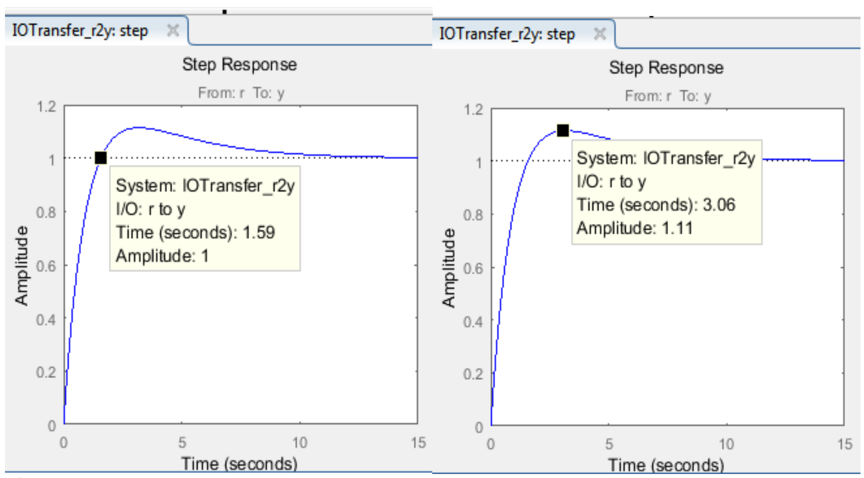

Figure 11

It is apparent that $t_r = 1.59\,\text{sec}$, $M_p = 11\%$

It seems that DS2 has been also satisfied byt DS1 is not yet fully satisfied (though it is very close to it).

6 MANUAL ADJUSTMENT OF THE CONTROLLER – DESIGN2 MODEL

To further improve the system performance and meet the design specifications, we perform manual tuning in the Bode diagram.

6.1 GRAB AND MOVE COMPENSATOR ZERO TO IMPROVE DESIGN

Grab the compensator Zero shown by a red circle on th Bode Phase plot (Figure 12).

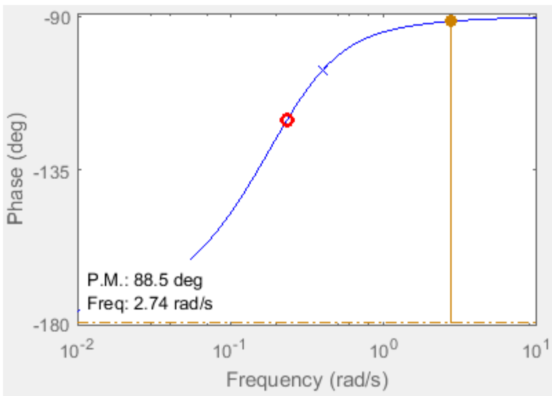

Figure 12

Move this red circle slightly to improve the response time t_r while maintaing GM: inf and a large PM.

This manual adjustment operation must be performed with great delicacy and in very small steps. Continous monitoring of the changes in stability margin should be done continously. If stability is lost, than it should be restored immediately through a backward step and fine adjustment should be continued with delicacy. Note that the gain margin GM is prone of jumping from 'inf', which is OK, to a large negative value which is not OK!

When a satisfactory situation has been met, press 'Store'; a new design named 'Design2' appears in the 'Designs' LHS pane. The overall picture is shown in Figure 13.

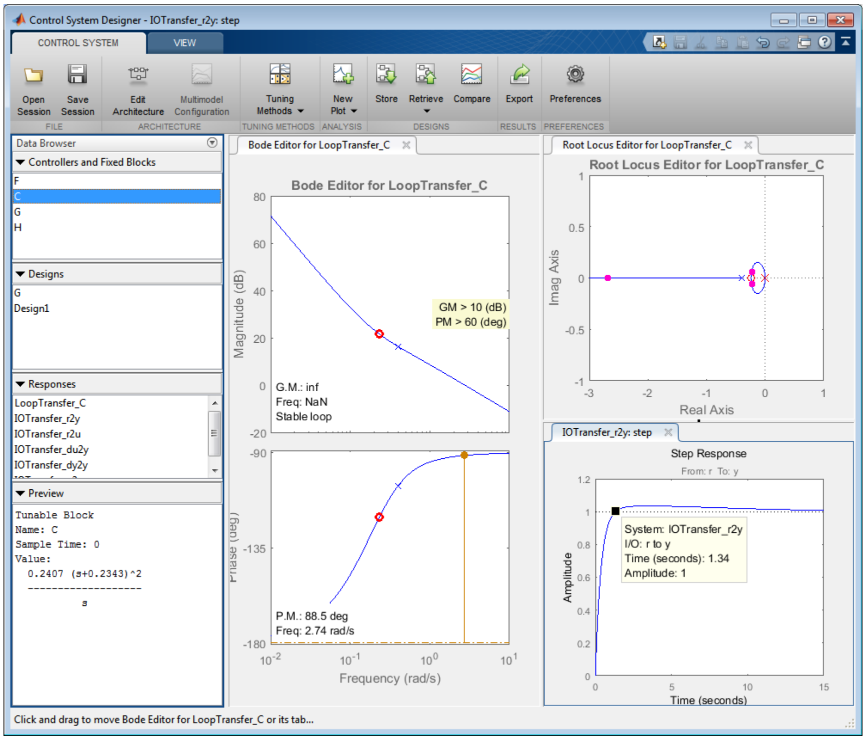

Figure 13

6.2 Adjust Compensator in Compensator Editor

One can also adjust the compensator directly by entering values for its properties. Right click on ‘C’ in the upper LHS pane and choose ‘Open Selection’ from the pulldown menu. The ‘Compensator Editor’ window opens as shown in Figure 14. The compensator properties, e.g., the ‘Real Part’ of the Complex Zero can be entered here.

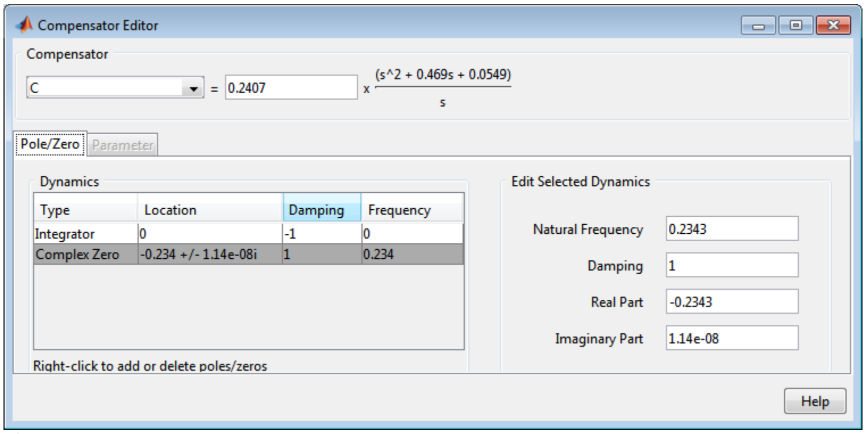

Figure 14

6.3 Save the CSD Sesion

To save the latest designs, press the 'Save Session' button and save your session with the name 'CSD_aircraft_Session_20161219'. The file extension, if shown, should be '.mat'.

7 EVALUATE DESIGN2 MODEL

7.1 Stability Margins of Design2 Model

The stability margins can be read on the Bode plots of Figure 13; they are very good:

- GM: inf which is better than $GM = 10\,\text{dB}$ required by DS3
- PM: 88.5 deg, which is better than $PM = 60°$ required by DS3

The stability margin criteria are satisfied and DS3 condition has been met.

7.2 Performance Indicators of Design2 Model

Figure 13 indicates that the step response has a faster response time and a smaller overshoot than before (Figure 15).

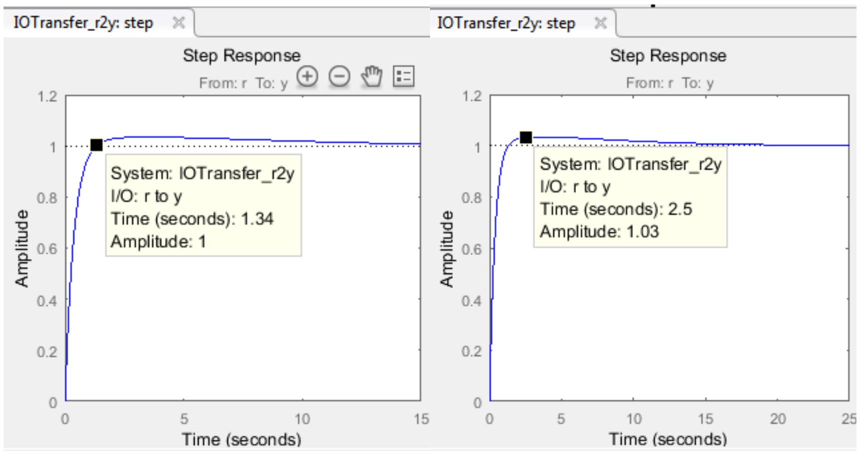

Figure 15

From Figure 15, we read: $t_r = 1.34\,\text{sec}$, $M_p = 3\%$.

It seems that DS1 and DS2 have been met.

The only drawback this this Design2 situation could be that the system seems to take longer than before to settle down to the steady state value.

Further tweaking of the controller could be attempted to overcome this aspect if considered important.

However, for now, we state that all three design specification DS1, DS2, DS3 have been met and the control system design process can be considered complete.

8 COMPARE THE STEP RESPONSE OF THE INITIAL AND FINAL DESIGNS

To compare the initial and final designs, do the following:

Press ‘Compare’ button and a ‘Compare Designs’ window pops up (Figure 16).

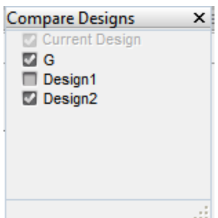

Figure 16

Check boxes in front of ‘G’ and ‘Design2’. The step response plot contains the two responses overlapped (Figure 17).

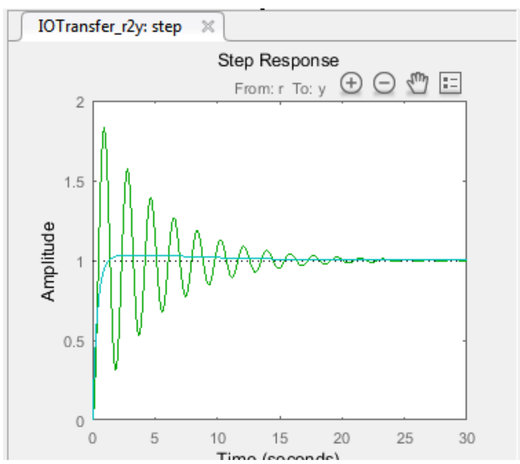

Figure 17

The initial design ‘G’ is shown in green, whereas the final design ‘Design2’ is shown in blue.

The response improvements achieved through this controller design process are quite apparent.

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9.3 State-Space Representations

ss1dof

STATE-SPACE REPRESENTATION

State-space representation is a time domain model of the form:

$$\left.\begin{aligned} \frac{d}{dt}z &= Az + Bu \quad \text{(dynamic system)} \quad (1a) \\ y &= Cz + Du \quad \text{(output)} \quad (1b) \end{aligned}\right\} (1)$$

The variables involved in Eq.(1) are:

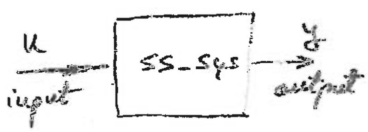
z = column of state variables, length N_z

u = column of input variables, — N_u

y = column of output variables — N_y

A, B, C, D = state space matrices

$$\begin{array}{cc} & N_z \\ N_z & \begin{bmatrix} A \end{bmatrix} \end{array} \qquad \begin{array}{cc} & N_u \\ N_z & \begin{bmatrix} B \end{bmatrix} \end{array}$$



$$\begin{array}{cc} & N_z \\ N_y & \begin{bmatrix} C \end{bmatrix} \end{array} \qquad \begin{array}{cc} & N_u \\ N_y & \begin{bmatrix} D \end{bmatrix} \end{array}$$

The SS representation is a time-domain representation, whereas the TF representation was a Laplace-domain representation.

MATLAB accepts both TF and SS representations.

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1a
SS

1st order system
State-space representation

Recall

$$T\dot{x}(t) + x(t) = f(t) \qquad (1)$$

Rewrite (1) as:

$$\dot{x} = -\frac{1}{T}x + f$$

or

$$\underset{\dot{z}}{\frac{d}{dt}x} = \underset{A}{-\frac{1}{T}}\,\underset{z}{x} + \underset{B}{\frac{1}{T}}\,\underset{f}{f} \qquad (2)$$

$$\begin{cases} \frac{d}{dt}z = Az + Bu \\ y = Cz + Du \end{cases}
\qquad
\begin{array}{l} z = x \\ u = f \\ y = x \\ A = -1/T \\ B = 1/T \\ C = 1 \\ D = 0 \end{array}$$

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2nd Order System

state-space representation

$\overset{2}{\text{ss}}$ 1dof To derive the SS representation of a 1-dof system, recall its 2^{nd} order ordinary differential equation in standard form, i.e.,

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \omega_n^2 f(t) \qquad (2).$$

Write (2) in terms of x and $\dot{x}$ only, i.e.,

$$\frac{d}{dt}\dot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \omega_n^2 f(t) \qquad (3)$$

Add the identity $\frac{d}{dt}x = \dot{x}$ and write (3) as an extended 1^{st} order system, i.e.,

$$\begin{cases} \frac{d}{dt}x = \dot{x} \\ \frac{d}{dt}\dot{x} = -2\zeta\omega_n\dot{x} - \omega_n^2 x + \omega_n^2 f(t) \end{cases} \qquad (4)$$

Express (4) in matrix form, i.e.,

$$\left.\begin{aligned} &\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} f(t)}_{\text{(dynamic system)}} \\ &y = [x] \quad \text{(output)} \end{aligned}\right\} \quad (5)$$

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ss1dof Define :

$$z = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}, \quad u = [f(t)] \quad y = [x]$$

$$N_z = 2 \qquad N_u = 1 \qquad N_y = 1$$

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} \tag{6}$$

$$C = [1 \quad 0] \qquad D = [0]$$

(6) → (5) :

$$\begin{cases} \dfrac{d}{dt} z = Az + Bu \\ y = Cz + Du. \end{cases} \tag{7}$$

Eqs. (6), (7) form the SS representation of the dynamic system (2). The TF representation of the same system is

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The TF and SS representations are interchangeable in MATLAB

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4
ss1dof

```
Comparison of TF and SS representations
initial data: fn, Hz;    z% =
     2     2
TF poles =
  -0.2513 +12.5639i
  -0.2513 -12.5639i
f_TF, Hz;    z_TF% =
    1.9996    2.0000
    1.9996    2.0000
SS poles =
  -0.2513 +12.5639i
  -0.2513 -12.5639i
f_SS, Hz;    z_SS% =
    1.9996    2.0000
    1.9996    2.0000
```

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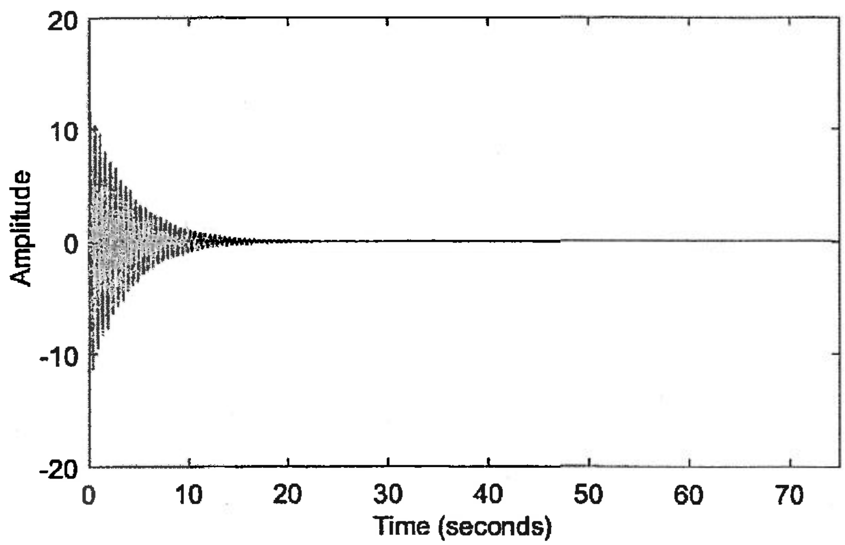
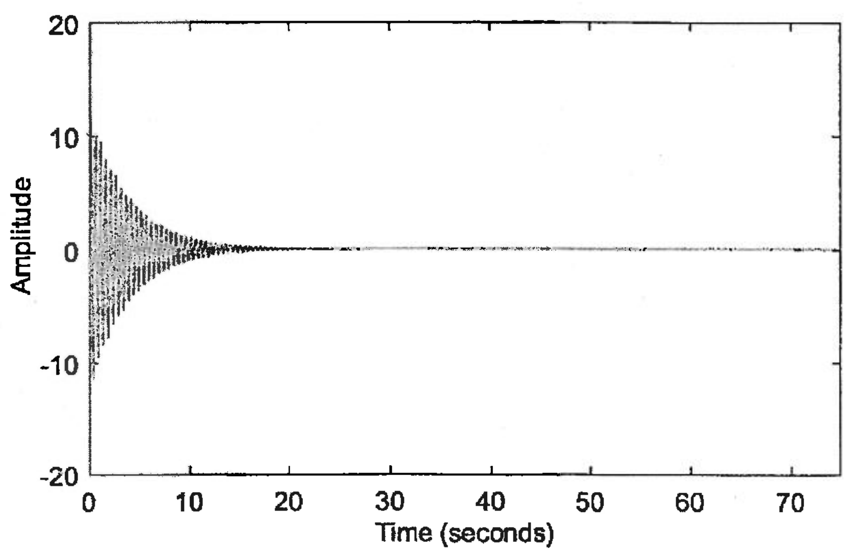
5
ss idof

impulse response of TF system: z TF= 2%



impulse response of SS system: z SS= 2%



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G:\EMCH516\EMCH516_2...\single_dof_time_response_TF_SS.m Page 1

```
%% DESCRIPTION
%{
SISO system time response
Comparison of TF and SS representations
%}
%% initialization
opengl hardwarebasic    % switch to basic hardware graphics functions
clc                     % clear command window
clear                   % clear workspace
% close all                % close all plots
format compact
tol=1e-10;  % tolerance for discarding machine zero
%% DEFINE PARAMETERS
% ----- data for class -----
fn=2; wn=2*pi*fn; % plunge frequency, Hz
z=2e-2;      % plunge damping ratio
%% DEFINE TIME RANGE
T=1/fn; % time scale
% dt=1e-3;
% tmax=50*T;
tmax=150*T;
Nt=1000; dt=tmax/Nt;
t=0:dt:tmax; % time range
%% CALCULATE SISO TRANSFER MATRIX
G=tf([wn^2],[1 2*z*wn wn^2]);
%% CALCULATE POLES OF G
[~,poles_TF,~]=zpkdata(G); s_TF=poles_TF{1,1};     % poles
% display(s_TF,'poles of G')
f_TF=abs(imag(s_TF)/(2*pi));                            % frequencies
zz=-real(s_TF)./abs(s_TF); z_TF=zz.*(abs(zz)>tol); % damping
%% CALCULATE IMPULSE RESPONSE OF TF SYSTEM
figure(1);
subplot(2,1,1); impulse(G,t);
title(['impulse response of TF system: z TF= 'num2str(z_TF(1)*100) '%'])
%% DISPLAY TF RESULTS
display('Comparison of TF and SS representations')
display([fn z*100], 'initial data: fn, Hz;   z%');
display(s_TF, 'TF poles');
display([f_TF z_TF*100], 'f_TF, Hz;   z_TF%');
%% CALCULATE SISO STATE SPACE MODEL
%---------------- state space matrices -----------
A=[ 0          1  ;
    -wn^2   -2*z*wn];
B=[  0 ;
    wn^2];
C=[1 0];
D=[0];
ss_sys=ss(A,B,C,D);
%% EXTRACT POLES, FREQUENCY, DAMPING
[~,zz,poles_SS]=damp(ss_sys);
```

G:\EMCH516\EMCH516_2...\single_dof_time_response_TF_SS.m Page 2

```
% display(ss,'poles of ss_sys')
s_SS=poles_SS;                         % poles
f_SS=abs(imag(poles_SS))/(2*pi);       % frequencies
z_SS=zz.*(abs(zz)>tol);                % damping
%% DISPLAY SS RESULTS
% display(' ');
display(s_SS, 'SS poles');
display([f_SS z_SS*100], 'f_SS, Hz;   z_SS%');
%% CALCULATE IMPULSE RESPONSE OF SS SYSTEM
subplot(2,1,2); impulse(ss_sys,t);
title(['impulse response of SS system: z SS= 'num2str(z_SS(1)*100) '%'])
```

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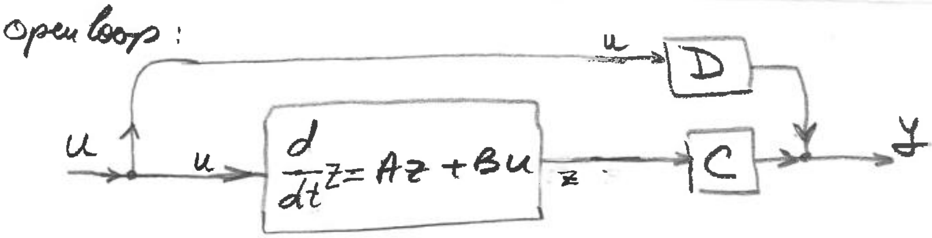
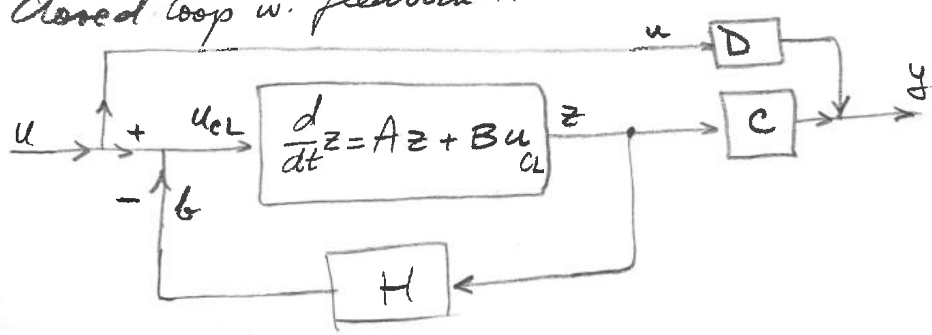
FB of SS system

$$\frac{d}{dz} z = Az + Bu \qquad (1a)$$
$$y = Cz + Du \qquad (1b)$$

open loop:



closed loop w. feedback H:



$$u_{CL} = u - b = u - Hz$$

$$\frac{d}{dt} z = Az + Bu_{CL} = Az + Bu - BHz$$

$$\frac{d}{dt} z = (A - BH)z + Bu$$

$$\boxed{A_{CL} = A - BH}$$ same A_{CL} as for simplified model

$$\left\{ \begin{aligned} \frac{d}{dt} z &= A_{CL} z + Bu \\ y &= Cz + Du. \end{aligned} \right.$$ SS sys w. feedback H

2
20160801

$\underline{H}$ for velocity feedback

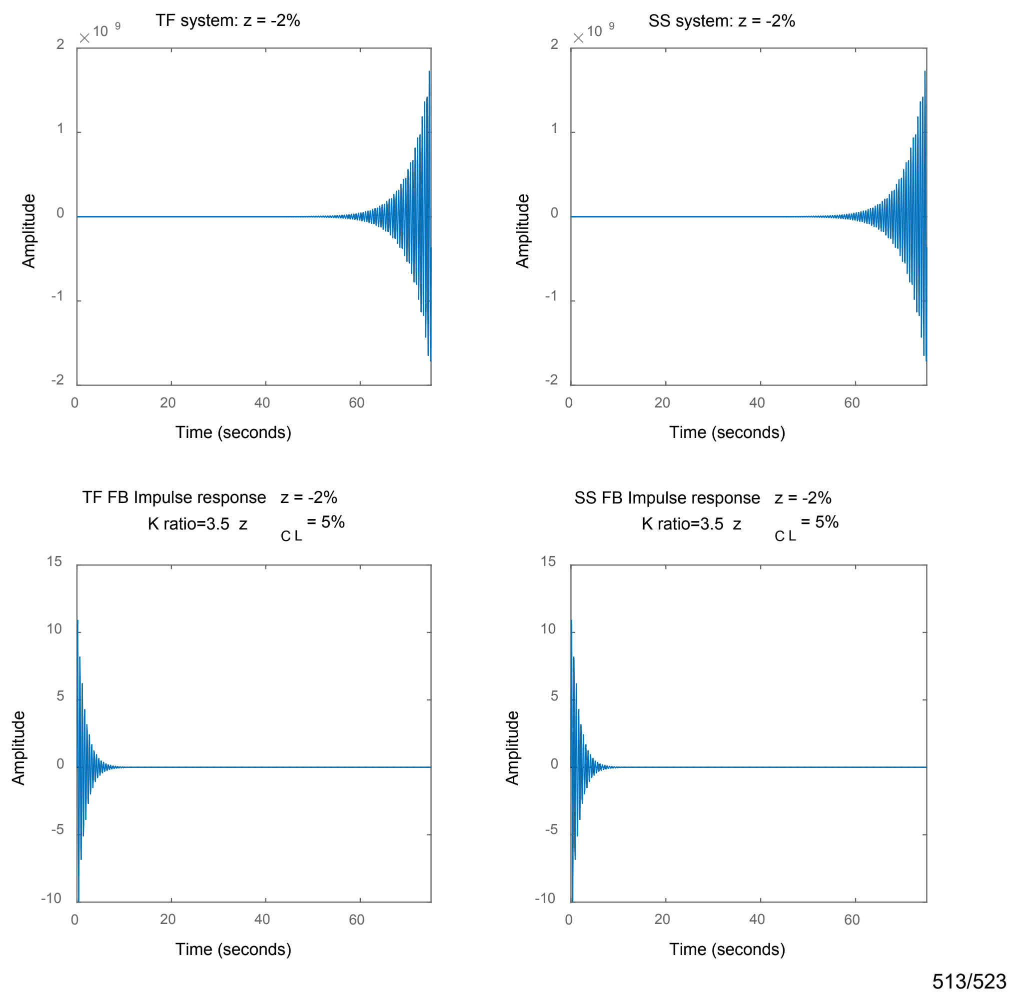
Recall $z = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$ (6)

For velocity feedback use

$$H = \begin{bmatrix} 0 & K \end{bmatrix} \quad (7)$$

The $b(t) = Hz(t) = \begin{bmatrix} 0 & K \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = K\dot{x}(t)$ — (8)

feedback signal $b(t)$
is proportional to velocity $\dot{x}(t)$.



TF system: z = -2%
SS system: z = -2%
TF FB Impulse response z = -2%
K ratio=3.5 z CL = 5%
SS FB Impulse response z = -2%
K ratio=3.5 z CL = 5%
Amplitude
Time (seconds)
×10 9
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G:\EMCH516\EMCH51...\FB_single_dof_time_response_TF_SS.m Page 1

```
%% DESCRIPTION
%{
SISO system time response with feedback
Comparison of TF and SS representations
use feedback function in TF
use direct definition of A_CL in SS
%}
%% initialization
opengl hardwarebasic    % switch to basic hardware graphics functions
clc                     % clear command window
clear                   % clear workspace
% close all                % close all plots
format compact;
tol=1e-10;  % tolerance for discarding machine zero
%% DEFINE PARAMETERS
% ----- data for example -----
% ----- data for HW -----
m=2;                % mass, kg
fn=2; wn=2*pi*fn; % plunge frequency, Hz
z=-2e-2;      % plunge damping ratio
z_ratio=2.5;        % desired z_ratio defined as zCL/|z|
display('HW06a')
display([fn z*100], 'initial data: fn, Hz;   z%');
%% CALCULATE critical FB gain
K_cr=2*abs(z)/wn; % critical FB gain
%% DEFINE TIME RANGE
T=1/fn; % time scale
% dt=1e-3;
% tmax=50*T;
tmax=150*T;
Nt=1000; dt=tmax/Nt;
t=0:dt:tmax; % time range
%% CALCULATE SISO TRANSFER MATRIX
G=tf(wn^2,[1 2*z*wn wn^2]);% transfer function G
%% EXTRACT TF POLES, FREQUENCY, DAMPING of G
[~,poles_TF,~]=zpkdata(G); s_TF=poles_TF{1,1};     % poles
% display(s_TF,'poles of G')
f_TF=abs(imag(s_TF)/(2*pi));                           % frequencies
zz=-real(s_TF)./abs(s_TF); z_TF=zz.*(abs(zz)>tol); % damping
%% CALCULATE IMPULSE RESPONSE OF TF SYSTEM
figure(1);
subplot(2,2,1); impulse(G,t);
title(['TF system: z = ' num2str(z_TF(1)*100) '%'])
%% CALCULATE SISO STATE SPACE MODEL
%---------------- state space matrices -----------
AA=[ 0          1  ;
    -wn^2   -2*z*wn];
BB=[  0 ;
     wn^2];
CC=[1 0];
```

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G:\EMCH516\EMCH51...\FB_single_dof_time_response_TF_SS.m Page 2

```
DD=[0];
ss_sys=ss(AA,BB,CC,DD);
%% EXTRACT SS POLES, FREQUENCY, DAMPING
[~,zz,poles_SS]=damp(ss_sys);
% display(ss,'poles of ss_sys')
s_SS=poles_SS;                        % poles
f_SS=abs(imag(poles_SS))/(2*pi);      % frequencies
z_SS=zz.*(abs(zz)>tol);               % damping
%% CALCULATE IMPULSE RESPONSE OF SS SYSTEM
subplot(2,2,2); impulse(ss_sys,t);
title(['SS system: z = ' num2str(z_SS(1)*100) '%'])
%% ADD VELOCITY FEEDBACK TO IMPROVE DAMPING
K_ratio=1+z_ratio;
K=K_ratio*K_cr; % FB gain
display([z_ratio K_ratio K],'z_ratio    K_ratio   K')
H=K*tf([1 0],1);
%% TF SYSTEM WITH FB
G_CL=feedback(G,H);
[~,p_CL_TF,~]=zpkdata(G_CL); s_CL_TF=p_CL_TF{1,1};
f_CL_TF=imag(s_CL_TF)/(2*pi);    % frequencies
zz=-real(s_CL_TF)./abs(s_CL_TF); z_CL_TF=zz.*(abs(zz)>tol);% damping
%% CALCULATE TF IMPULSE RESPONSE WITH FB
subplot(2,2,3); impulse(G_CL,t)
T1=['TF FB Impulse response'];
T2=['   z = ' num2str(z*100) '%'];
T3=['K ratio=' num2str(K_ratio)];
T4=['  z_C_L= ' num2str(z_CL_TF(1)*100) '%'];
line1=[T1 T2];
line2=[T3 T4];
title({line1; line2})
%% SS SYSTEM WITH FB
% ---- SS velocity feedback MATRIX-----
H=[0 K]; % FB matrix
% ================= state space matrices with FB =================
AA_CL=AA-BB*H;  % state matrix AA with FB
% AA_CL=feedback(ss_sys,H); % this does not work; H should be also ss
CC_CL=CC-DD*H;  % state matrix CC with FB
ss_sys_CL=ss(AA_CL, BB, CC_CL, DD);
%% EXTRACT POLES, FREQUENCY, DAMPING
[~,zz_CL_SS,poles_CL_SS]=damp(ss_sys_CL);
%ss_CL=eig(AA0);
% display(ss_CL,'poles of G_CL')
s_CL_SS=poles_CL_SS;                    % poles
f_CL_SS=abs(imag(poles_CL_SS))/(2*pi);    % frequencies
z_CL_SS=zz_CL_SS.*(abs(zz_CL_SS)>tol);       % damping
%% CALCULATE IMPULSE RESPONSE OF SS SYSTEM WITH FB
subplot(2,2,4); impulse(ss_sys_CL,t);
title(['impulse response of SS FB system: zCL SS= 'num2str(z_CL_SS(1)*100) '%'])
T1_SS=['SS FB Impulse response'];
T2_SS=['   z = ' num2str(z*100) '%'];
```

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G:\EMCH516\EMCH51...\FB_single_dof_time_response_TF_SS.m Page 3

```
T3_SS=['K ratio=' num2str(K_ratio)];
T4_SS=['  z_C_L= ' num2str(z_CL_TF(1)*100) '%'];
line1_SS=[T1_SS T2_SS];
line2_SS=[T3_SS T4_SS];
title({line1_SS; line2_SS})
%% DISPLAY TF RESULTS
display(s_TF, 'TF poles');
display([f_TF z_TF*100],'f_TF, Hz;   z_TF%');
%% DISPLAY TF FB RESULTS
display(s_CL_TF, 'TF FB poles');
display([f_CL_TF z_CL_TF*100],'f_CL, Hz;   zh_CL %');
%% DISPLAY SS RESULTS
% display(' ');
display(s_SS, 'SS poles');
display([f_SS z_SS*100],'f_SS, Hz;   z_SS%');
%% DISPLAY SS FB RESULTS
% display(' ');
display(s_CL_SS, 'SS FB poles');
display([f_CL_SS z_CL_SS*100],'f_CL_SS, Hz;   z_CL_SS%');
```

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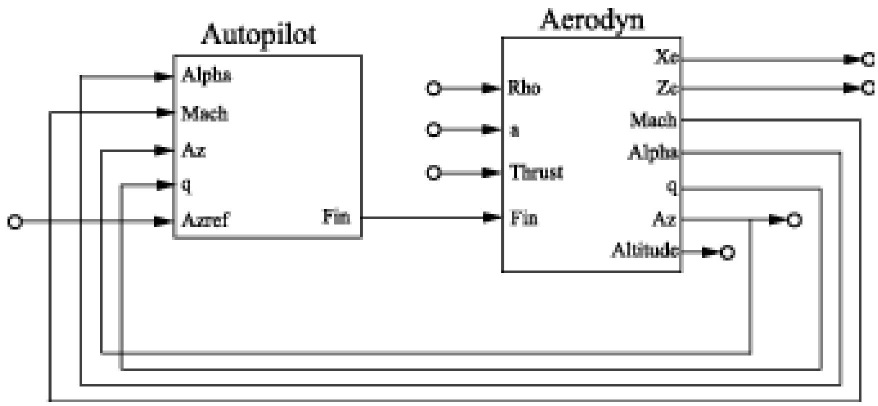
MIMO Feedback Loop

This example shows how to obtain the closed-loop response of a MIMO feedback loop in three different ways.

In this example, you obtain the response from `Azref` to `Az` of the MIMO feedback loop of the following block diagram.



You can compute the closed-loop response using one of the following three approaches:

- Name-based interconnection with `connect`
- Name-based interconnection with `feedback`
- Index-based interconnection with `feedback`

You can use whichever of these approaches is most convenient for your application.

Load the plant `Aerodyn` and the controller `Autopilot` into the MATLAB® workspace. These models are stored in the datafile `MIMOfeedback.mat`.

```
load(fullfile(matlabroot,'examples','control','MIMOfeedback.mat'))
```

`Aerodyn` is a 4-input, 7-output state-space (`ss`) model. `Autopilot` is a 5-input, 1-output `ss` model. The inputs and outputs of both models names appear as shown in the block diagram.

Compute the closed-loop response from `Azref` to `Az` using `connect`.

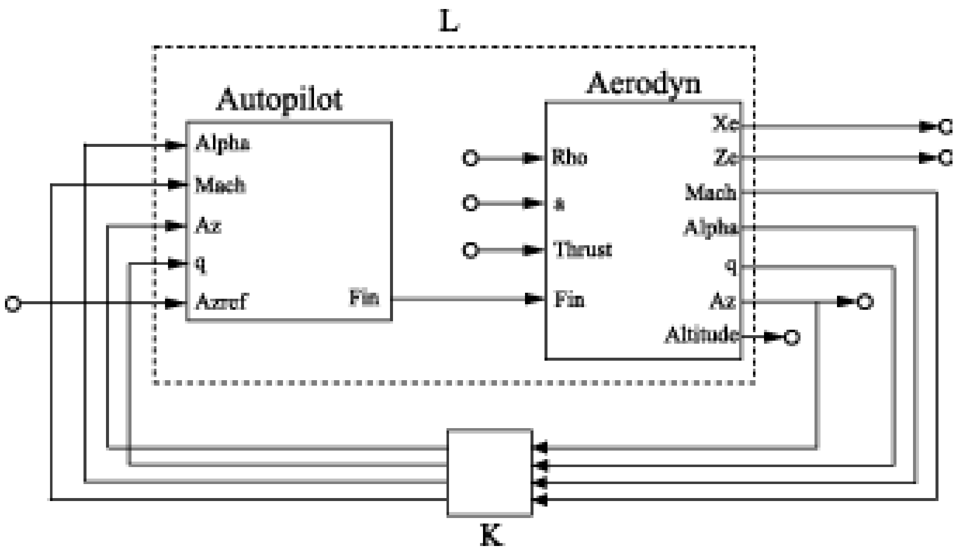
```
T1 = connect(Autopilot,Aerodyn,'Azref','Az');
```

```
Warning: The following block inputs are not used: Rho,a,Thrust.
Warning: The following block outputs are not used: Xe,Ze,Altitude.
```

The `connect` function combines the models by joining the inputs and outputs that have matching names. The last two arguments to `connect` specify the input and output signals of the resulting model. Therefore, `T1` is a state-space model with input `Azref` and output `Az`. The `connect` function ignores the other inputs and outputs in `Autopilot` and `Aerodyn`.

Compute the closed-loop response from `Azref` to `Az` using name-based interconnection with the `feedback` command. Use the model input and output names to specify the interconnections between `Aerodyn` and `Autopilot`.

When you use the `feedback` function, think of the closed-loop system as a feedback interconnection between an open-loop plant-controller combination `L` and a diagonal unity-gain feedback element `K`. The following block diagram shows this interconnection.




```
L = series(Autopilot,Aerodyn,'Fin');

FeedbackChannels = {'Alpha','Mach','Az','q'};
K = ss(eye(4),'InputName',FeedbackChannels,...
              'OutputName',FeedbackChannels);

T2 = feedback(L,K,'name',+1);
```

The closed-loop model `T2` represents the positive feedback interconnection of `L` and `K`. The `'name'` option causes `feedback` to connect `L` and `K` by matching their input and output names.

`T2` is a 5-input, 7-output state-space model. The closed-loop response from `Azref` to `Az` is `T2('Az','Azref')`.

Compute the closed-loop response from Azref to Az using feedback, using indices to specify the interconnections between Aerodyn and Autopilot.

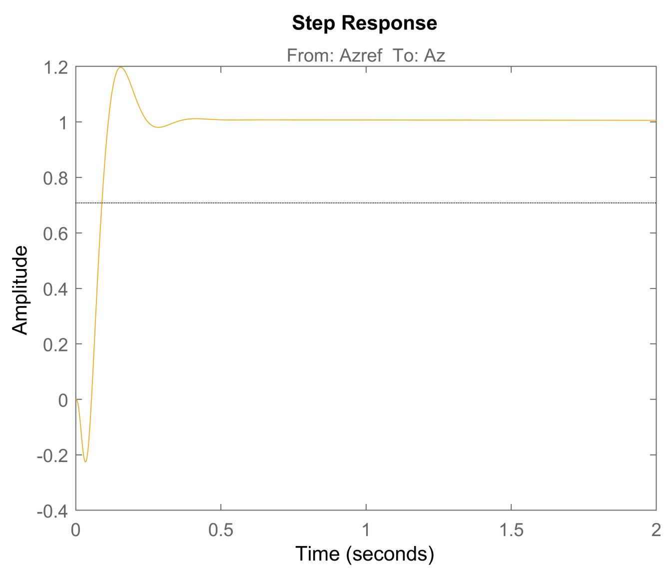
```
L = series(Autopilot,Aerodyn,1,4);
K = ss(eye(4));
T3 = feedback(L,K,[1 2 3 4],[4 3 6 5],+1);
```

The vectors [1 2 3 4] and [4 3 6 5] specify which inputs and outputs, respectively, complete the feedback interconnection. For example, feedback uses output 4 and input 1 of L to create the first feedback interconnection. The function uses output 3 and input 2 to create the second interconnection, and so on.

T3 is a 5-input, 7-output state-space model. The closed-loop response from Azref to Az is T3(6,5).

Compare the step response from Azref to Az to confirm that the three approaches yield the same results.

```
step(T1,T2('Az','Azref'),T3(6,5),2)
```



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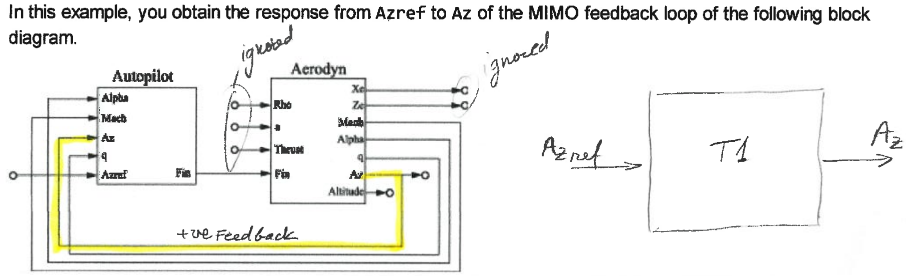
MIMO Feedback Loop - MATLAB & Simulink http://www.mathworks.com/help/control/ug/build-a-model-of-a-multi-i...

MIMO Feedback Loop

This example shows how to obtain the closed-loop response of a MIMO feedback loop in three different ways. Open This Example

In this example, you obtain the response from `Azref` to `Az` of the MIMO feedback loop of the following block diagram.



Compute the closed-loop response from `Azref` to `Az` using `connect`.

T1 Input output

```
T1 = connect(Autopilot,Aerodyn,'Azref','Az');
```

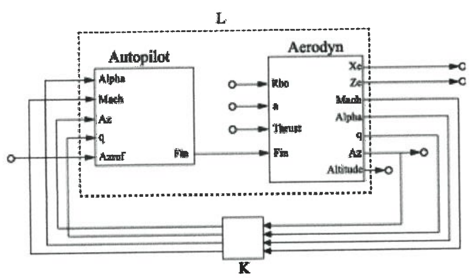
```
Warning: The following block inputs are not used: Rho,a,Thrust.
Warning: The following block outputs are not used: Xe,Ze,Altitude.
```

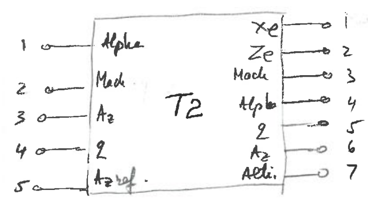
The connect function combines the models by joining the inputs and outputs that have matching names. The last two arguments to `connect` specify the input and output signals of the resulting model. Therefore, `T1` is a 520/523

T2

Compute the closed-loop response from `Azref` to `Az` using name-based interconnection with the `feedback` command. Use the model input and output names to specify the interconnections between `Aerodyn` and `Autopilot`.

When you use the `feedback` function, think of the closed-loop system as a feedback interconnection between an open-loop plant-controller combination `L` and a diagonal unity-gain feedback element `K`. The following block diagram shows this interconnection.






```
L = series(Autopilot,Aerodyn,'Fin');     series construct using variable Fin

FeedbackChannels = {'Alpha','Mach','Az','q'};
K = ss(eye(4),'InputName',FeedbackChannels,...     unit ss system from Alpha, Mach, Az, q
             'OutputName',FeedbackChannels);             to Alpha, Mach, Az, q

                          match names
T2 = feedback(L,K,'name',+1);     +ve feedback
```

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T3

Compute the closed-loop response from Azref to Az using feedback, using indices to specify the interconnections between Aerodyn and Autopilot.

```
L = series(Autopilot,Aerodyn,1,4);
K = ss(eye(4));
T3 = feedback(L,K,[1 2 3 4],[4 3 6 5],+1);
```

Fin out Fin in

Alpha Mach Az q

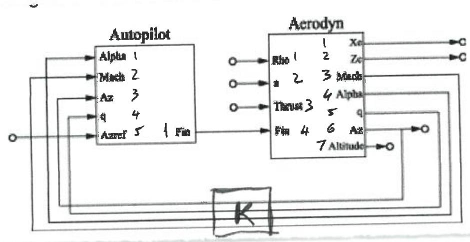
Alpha Mach Az q

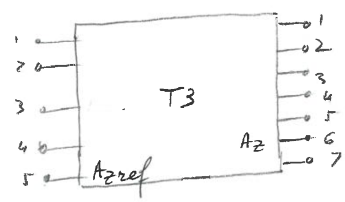
+ve feedback

feedback (G, H, input, output, ±1)

The vectors [1 2 3 4] and [4 3 6 5] specify which inputs and outputs, respectively, complete the feedback interconnection. For example, feedback uses output 4 and input 1 of L to create the first feedback interconnection. The function uses output 3 and input 2 to create the second interconnection, and so on.

T3 is a 5-input, 7-output state-space model. The closed-loop response from Azref to Az is T3(6,5).





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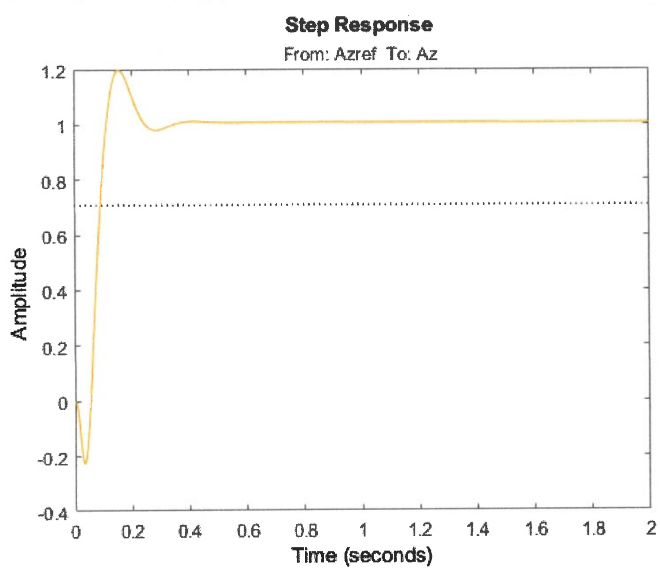
Compare the step response from `Azref` to `Az` to confirm that the three approaches yield the same results.

```
step(T1,T2('Az','Azref'),T3(6,5),2)
```



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