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# $2^{nd}$ order system time response

ODE: $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 f(t)$, $x(0)=0$, $\dot{x}(0)=0$ (1)

$\omega_n$ = natural freq.

$\zeta$ = damping ratio

Laplace transform $\left\{ \begin{array}{l} x(t) \to X(s) \\ \dot{x}(t) \to sX(s) \\ \ddot{x}(t) \to s^2 X(s) \end{array} \right\}$ $f(t) \to F(s)$ (2)

(2)→(1): $s^2 X + 2\zeta\omega_n s X + \omega_n^2 X = \omega_n^2 F(s)$ —(3)

$$(s^2 + 2\zeta\omega_n s + \omega_n^2) X = \omega_n^2 F(s).$$

$$X = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} F(s). \quad (4)$$

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step response 


$\left\{ \begin{array}{l} f(t) = 1(t) \\ F(s) = \frac{1}{s} \end{array} \right.$ (1)

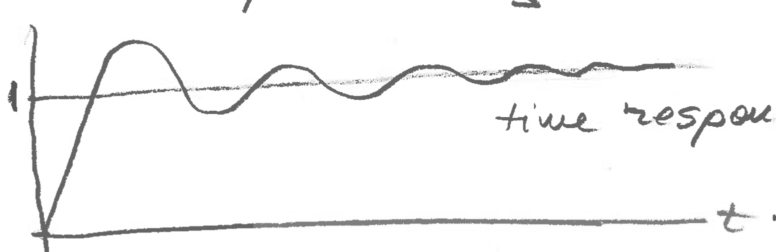
$$X(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \quad (2)$$

T2.1, #24:

$$x(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \varphi) \quad (3)$$

$\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\varphi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \sin^{-1}\sqrt{1-\zeta^2} \quad (4)$$



7] $x_c(t) = e^{-\zeta\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right)$ ; $A, B$

$= e^{-\zeta\omega_n t} C\sin(\omega_d t + \varphi)$ ; $C, \varphi$

$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 f(t)$ ; $\omega_n^2 f = \frac{f^*}{m} \rightarrow f = \frac{f^*}{k}$

Step input, $f(t) = 1(t)$


$x_p(t) = D$ , $\dot{x}_p = \ddot{x}_p = 0$

$\omega_n^2 D = \omega_n^2 \longrightarrow D = 1$

$x_p(t) = 1$

$x(t) = e^{-\overset{\alpha}{\zeta\omega_n} t} C\sin(\overset{\psi}{\overbrace{\omega_d t + \varphi}}) + 1$

$\dot{x}(t) = \left(-\alpha\sin\psi + \omega_d\cos\psi\right) C e^{-\alpha t}$

$x(0) = C\sin\varphi + 1 = 0$

$\dot{x}(0) = \left(-\alpha\sin\varphi + \omega_d\cos\varphi\right) = 0 \longrightarrow \tan\varphi = \frac{\omega_d}{\alpha}$

$\tan\varphi = \frac{\omega_n\sqrt{1-\zeta^2}}{\zeta\omega_n} = \frac{\sqrt{1-\zeta^2}}{\zeta} \rightarrow \varphi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$

$\sin^2\varphi = \frac{\tan^2\varphi}{1+\tan^2\varphi} = \frac{1-\zeta^2}{\zeta^2+1-\zeta^2} = 1-\zeta^2$

$\sin\varphi = \sqrt{1-\zeta^2}$ $\qquad$ $\varphi = \sin^{-1}\sqrt{1-\zeta^2}$

$C = -\frac{1}{\sin\varphi} = -\frac{1}{\sqrt{1-\zeta^2}}$

$x(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_d t + \varphi)$ unit response

for $f^* = k$

For $f^* = F_o$

$x^*(t) = \frac{F_o}{k}\left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_d t + \varphi)\right] = x_{st}\cdot x(t)$

$x_{st} = F_o/k$ static displacement

14) <u>Step response</u> $f(t)=1(t)$ , $F(s)=\frac{1}{s}$

ILT by partial fraction expansion

$$X(s)=\frac{\omega_n^2}{(s+\alpha)^2+\omega_d^2}\cdot\frac{1}{s}=\frac{A}{s}+\frac{Ds+E}{(s+\alpha)^2+\omega_d^2}$$

$$A(s+\alpha)^2+A\omega_d^2+Ds^2+Es=\text{wn^2}$$

$$A(s^2+2s\alpha+\alpha^2+\omega_d^2)+Ds^2+Es=1$$

$s^2$: $A+D=0$

$s^1$: $2\alpha A+E=0$

$s^0$: $A(\alpha^2+\omega_d^2)=\omega_n^2$ $\qquad A(\cancel{\zeta^2\omega_n^2}+(1-\cancel{\zeta^2})\omega_n^2)=\omega_n^2$

$A=1$

$D=-A=-1$

$E=-2\alpha A=-2\alpha$

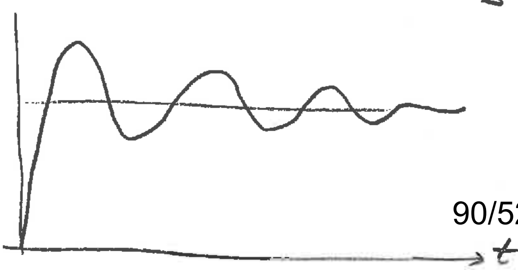
$$X(s)=\left[\frac{1}{s}-\frac{s+2\alpha}{(s+\alpha)^2+\omega_d^2}\right]=\left[\frac{1}{s}-\frac{\alpha+(s+\alpha)}{(s+\alpha)^2+\omega_d^2}\right]$$

$$x(t)=1-e^{-\alpha t}\left[\frac{\alpha}{\omega_d}\sin\omega_d t+\cos\omega_d t\right]$$

$$x(t)=1-e^{-\alpha t}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t+\cos\omega_d t\right)$$

$$x(t)=1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_d t+\varphi)\qquad \varphi=\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$x_{ss}=1$



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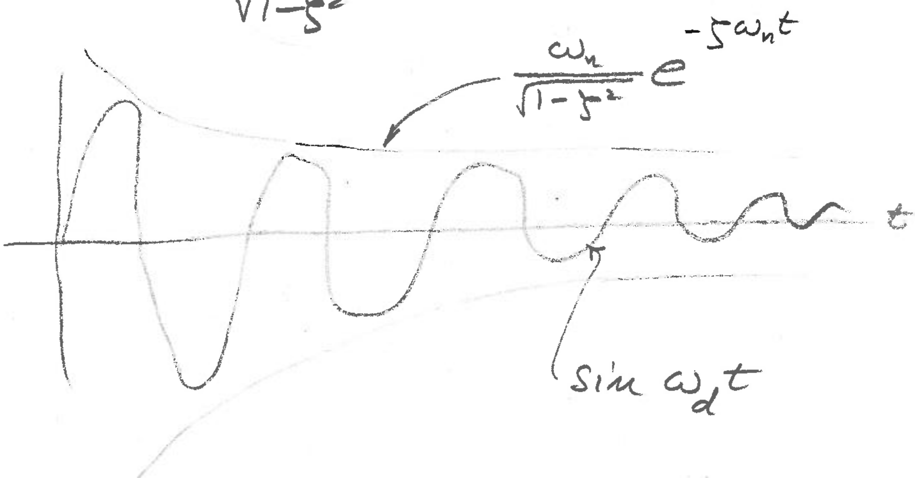
Impulse response


$$\begin{cases} f(t) = \delta(t) \\ F(s) = 1 \end{cases} \tag{1}$$

$$X = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{2}$$

T2.1, #22

$$x(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_d t \tag{3}$$



13a) Long hand solution

$$\frac{X}{\omega_n^2(s+p_1)(s+p_2)} = \frac{A}{s+p_1} + \frac{B}{s+p_2}$$

$$As + Ap_2 + Bs + Bp_1 = 1$$

$s^1$: $A + B = 0 \longrightarrow B = -A$

$s^0$: $Ap_2 + Bp_1 = 1 \longrightarrow A = \frac{1}{p_2 - p_1}, \quad B = \frac{1}{p_1 - p_2}$

$$\frac{1}{(s+p_1)(s+p_2)} = \frac{1}{p_2 - p_1}\left(\frac{1}{s+p_1} - \frac{1}{s+p_2}\right)$$

$$x(t) = \frac{1}{p_2 - p_1}\left(e^{-p_1 t} - e^{-p_2 t}\right)$$

$$p_{1,2} = \zeta\omega_n \mp i\omega_n\sqrt{1-\zeta^2} = \alpha \mp i\omega_d.$$

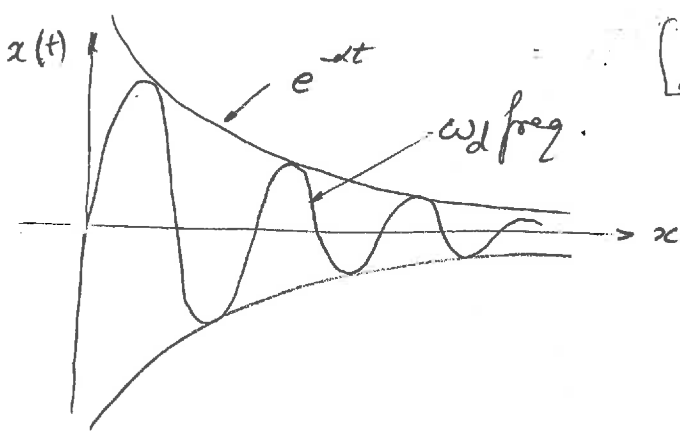
$$p_2 - p_1 = i2\omega_d$$

$$\frac{x(t)}{\omega_n^2} = \frac{1}{2i\omega_d}\left[e^{-(\alpha - i\omega_d)t} - e^{-(\alpha + i\omega_d)t}\right]$$

$$= \frac{1}{2i\omega_d} e^{-\alpha t}\left[e^{i\omega_d t} - e^{-i\omega_d t}\right]$$

$$= \frac{1}{\omega_d} e^{-\alpha t}\frac{e^{i\omega_d t} - e^{-i\omega_d t}}{2i} = \frac{e^{-\alpha t}}{\omega_d}\sin\omega_d t.$$

$$\begin{cases} \alpha = \zeta\omega_n \\ \omega_d = \omega_n\sqrt{1-\zeta^2} \end{cases}$$



Alternative way

$$\frac{1}{(s+p_1)(s+p_2)} \xrightarrow[\mathcal{L}^{-1}]{} \frac{1}{p_2-p_1}\left(e^{-p_1 t}-e^{-p_2 t}\right)$$

Another way

Residue Theorem

$$\frac{1}{(s+p_1)(s+p_2)} = \frac{a_1}{s+p_1} + \frac{a_2}{s+p_2}$$

$$a_1 = \left[\cancel{(s+p_1)}\frac{1}{\cancel{(s+p_1)}(s+p_2)}\right]_{s=-p_1} = \frac{1}{-p_1+p_2}$$

$$a_2 = \left[\cancel{(s+p_2)}\frac{1}{(s+p_1)\cancel{(s+p_2)}}\right]_{s=-p_2} = \frac{1}{p_1-p_2} = -a_1$$

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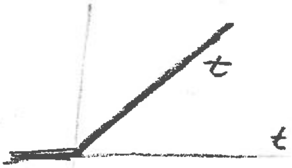
# Ramp response




$$\left.\begin{aligned} f(t) &= t \ , \quad t>0 \\ F(s) &= \frac{1}{s^2} \end{aligned}\right\} \qquad (1)$$

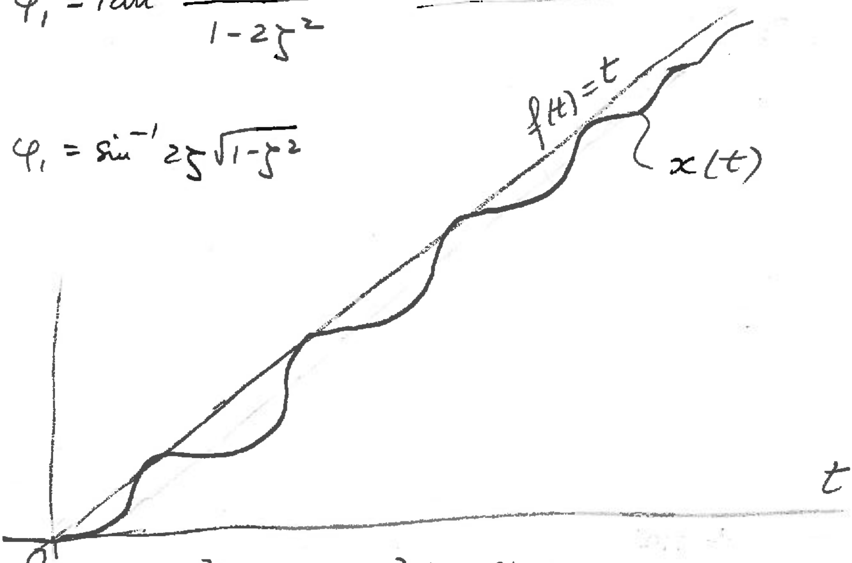
$$G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \quad ; \quad X(s)=G(s)F(s)$$

$$X(s) = \frac{\omega_n^2}{s^2(s^2+2\zeta\omega_n s+\omega_n^2)} \qquad (2)$$

$$x(t) = t - \frac{2\zeta}{\omega_n}\left[1+\frac{1}{\sin\varphi_1}e^{-\zeta\omega_n t}\sin(\omega_d t-\varphi_1)\right] \qquad (3)$$

$$\varphi_1 = \tan^{-1}\frac{2\zeta\sqrt{1-\zeta^2}}{1-2\zeta^2} \qquad (4)$$

$$\varphi_1 = \sin^{-1} 2\zeta\sqrt{1-\zeta^2}$$




$$\sin^2\varphi_1 = \frac{\tan^2\varphi_1}{1+\tan^2\varphi_1} = \frac{4\zeta^2(1-\zeta^2)}{(1-2\zeta^2)^2+4\zeta^2(1-\zeta^2)}$$

$$= \frac{4\zeta^2(1-\zeta^2)}{1-\cancel{4\zeta^2}+\cancel{4\zeta^4}+\cancel{4\zeta^2}-\cancel{4\zeta^4}} = 4\zeta^2(1-\zeta^2)$$

$$\sin\varphi_1 = 2\zeta\sqrt{1-\zeta^2}$$

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## PROOF of $2^{nd}$ order system Ramp response

ODE solution


$$x_p(t) = Dt + E$$

$$\dot{x}_p = D \;;\quad \ddot{x}_p = 0$$

$$\ddot{x}_p + 2\zeta\omega_n \dot{x}_p + \omega_n^2 x_p = \omega_n^2 f(t)$$

$$2\zeta\omega_n D + \omega_n^2 (Dt + E) = \omega_n^2 t$$

$$t^0: \; 2\zeta\omega_n D + \omega_n^2 E = 0 \longrightarrow E = -\frac{2\zeta}{\omega_n} D$$

$$t^1: \; \omega_n^2 D = \omega_n^2 \rightarrow D = 1 \longrightarrow E = \frac{2\zeta}{\omega_n}$$

$$x_p(t) = t - 2\zeta/\omega_n$$

$$x(t) = e^{-\underbrace{\zeta\omega_n}_{\alpha} t} C \sin(\omega_d t + \overbrace{\varphi_1}^{\varphi}) + t - \frac{2\zeta}{\omega_n} =$$

$$\dot{x}(t) = (-\alpha \sin\varphi + \omega_d \cos\varphi) C e^{-\alpha t} + 1$$

$$x(0) = 0: \quad C \sin\varphi_1 - \frac{2\zeta}{\omega_n} = 0 \longrightarrow \quad C\omega_n \sin\varphi_1 = 2\zeta \quad (a)$$

$$\dot{x}(0) = 0: \quad (-\alpha \sin\varphi_1 + \omega_d \cos\varphi_1) C + 1 = 0$$

$$C(\alpha \sin\varphi_1 - \omega_d \cos\varphi_1) = 1$$

$$C\zeta\omega_n \sin\varphi_1 - C\omega_d \cos\varphi_1 = 1$$

$$(a): \quad 2\zeta^2 - C\omega_d \cos\varphi_1 = 1$$

$$C\omega_n \sqrt{1-\zeta^2} \cos\varphi_1 = 2\zeta^2 - 1$$

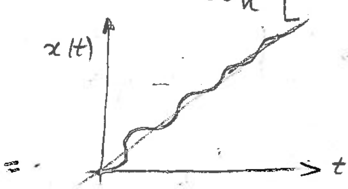
$$C\omega_n \cos\varphi_1 = -\frac{1-2\zeta^2}{\sqrt{1-\zeta^2}} \quad (b)$$

$$\frac{(a)}{(b)} = \tan\varphi_1 = -\frac{2\zeta\sqrt{1-\zeta^2}}{1-2\zeta^2}$$

Note: $\varphi_1$ is different from $\varphi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$ of step response

$$\varphi_1 = -\tan^{-1}\frac{2\zeta\sqrt{1-\zeta^2}}{1-2\zeta^2}, \quad C = \frac{2\zeta}{\omega_n} \cdot \frac{1}{\sin\varphi_1}$$

$$x(t) = t - \frac{2\zeta}{\omega_n}\left[1 - \frac{1}{\sin\varphi_1} e^{-\zeta\omega_n t} \sin(\omega_d t + \varphi_1)\right]$$



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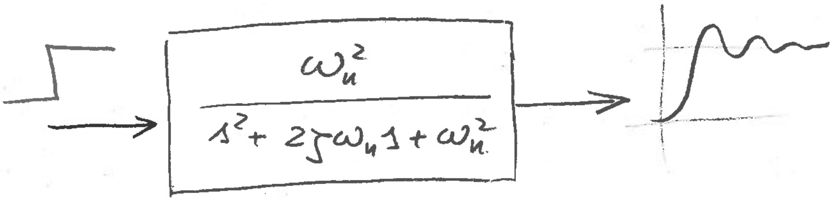
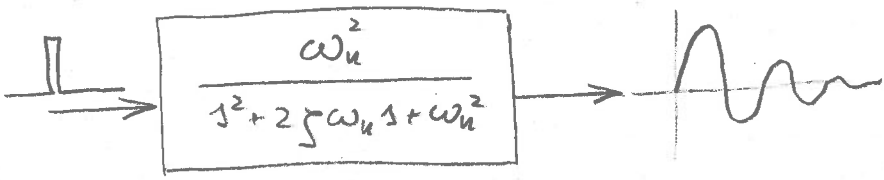
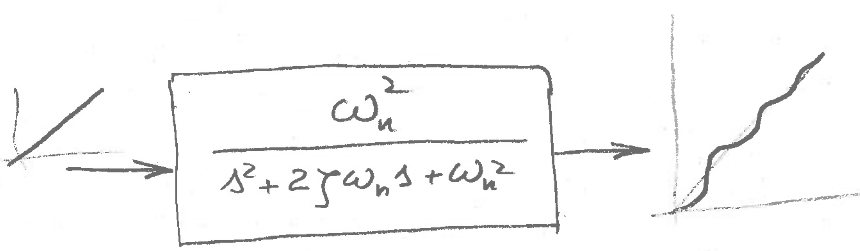
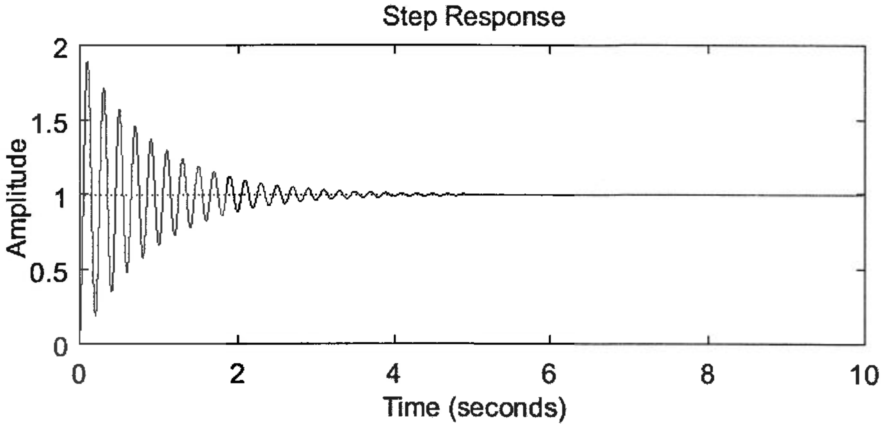
# $2^{nd}$ order system response

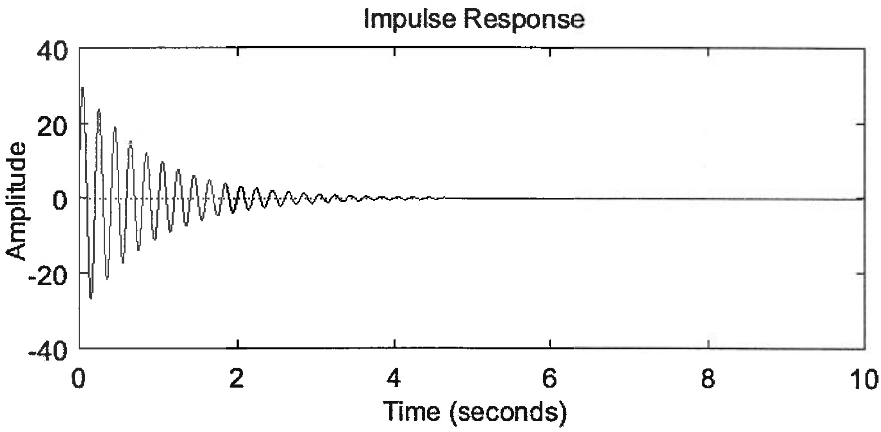
## SUMMARY

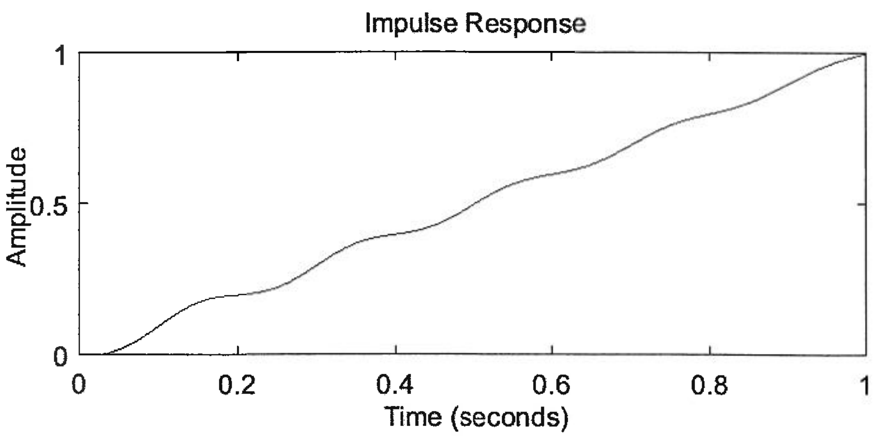



Step Response
Amplitude
Time (seconds)
2
1.5
1
0.5
0
0
2
4
6
8
10


Impulse Response
Amplitude
Time (seconds)
40
20
0
-20
-40
0
2
4
6
8
10


Impulse Response
Amplitude
Time (seconds)
1
0.5
0
0
0.2
0.4
0.6
0.8
1

```
%{
% This program studies time response of 2nd order systems
%}
%% Initialization
clc
clear
% close all
format compact
%% Given data
fn=5; wn=2*pi()*fn   % natural frequency fn, Hz and wn, rad/sec
z=3.5e-2             % damping
%% time range setup
Tmax=10;
dt=Tmax*1e-4; t=0:dt:Tmax;  % time range
%% Define system
B=[wn^2]; A=[1 2*z*wn wn^2];  G=tf(B,A)
%% Step response
figure(1)
subplot(3,1,1)
step(G,t)
%% Impulse response
subplot(3,1,2)
impulse(G,t)
%% Ramp response
F_ramp=tf([1],[1 0 0])
subplot(3,1,3)
impulse(G*F_ramp,t)
xlim([0 1])
```