TABLE 2–1 Laplace Transform Pairs

	$f(t)$	$F(s)$	
1	Unit impulse $\delta(t)$	1	
2	Unit step $1(t)$	$\frac{1}{s}$	
3	t	$\frac{1}{s^2}$	
4	$\frac{t^{n-1}}{(n-1)!} \quad (n = 1, 2, 3, \dots)$	$\frac{1}{s^n}$	
5	$t^n \quad (n = 1, 2, 3, \dots)$	$\frac{n!}{s^{n+1}}$	
6	e^{-at} | $\frac{1}{T}e^{-t/T}$ | e^{pt}	$\frac{1}{s+a}$	$\frac{1}{Ts+1}$ | $\frac{1}{s-p}$
7	te^{-at}	$\frac{1}{(s+a)^2}$	
8	$\frac{1}{(n-1)!}t^{n-1}e^{-at} \quad (n = 1, 2, 3, \dots)$	$\frac{1}{(s+a)^n}$	
9	$t^n e^{-at} \quad (n = 1, 2, 3, \dots)$	$\frac{n!}{(s+a)^{n+1}}$	
10	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	
11	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	
12	$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$	
13	$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$	
14	$\frac{1}{a}(1 - e^{-at})$ | $(1 - e^{-t/T})$	$\frac{1}{s(s+a)}$	$\frac{1}{s(Ts+1)}$
15	$\frac{1}{b-a}(e^{-at} - e^{-bt})$	$\frac{1}{(s+a)(s+b)}$	
16	$\frac{1}{b-a}(be^{-bt} - ae^{-at})$	$\frac{s}{(s+a)(s+b)}$	
17	$\frac{1}{ab}\left[1 + \frac{1}{a-b}(be^{-at} - ae^{-bt})\right]$	$\frac{1}{s(s+a)(s+b)}$	

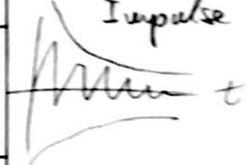
$t - T(1 - e^{-t/T})$

TABLE 2–1 *(continued)*

	$f(t)$	$F(s)$
18	$\frac{1}{a^2}(1 - e^{-at} - ate^{-at})$	$\frac{1}{s(s+a)^2}$
19	$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$
20	$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
21	$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
22	$\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_n\sqrt{1-\zeta^2}\,t$	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
23	$-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\,t - \phi)$ $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$	$\frac{s}{s^2+2\zeta\omega_n s+\omega_n^2}$
24	$1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\,t + \phi)$ $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \sin^{-1}\sqrt{1-\zeta^2}$	$\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$
25	$1 - \cos\omega t$	$\frac{\omega^2}{s(s^2+\omega^2)}$
26	$\omega t - \sin\omega t$	$\frac{\omega^3}{s^2(s^2+\omega^2)}$
27	$\sin\omega t - \omega t\cos\omega t$	$\frac{2\omega^3}{(s^2+\omega^2)^2}$
28	$\frac{1}{2\omega}t\sin\omega t$	$\frac{s}{(s^2+\omega^2)^2}$
29	$t\cos\omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
30	$\frac{1}{\omega_2^2-\omega_1^2}(\cos\omega_1 t - \cos\omega_2 t) \quad (\omega_1^2 \neq \omega_2^2)$	$\frac{s}{(s^2+\omega_1^2)(s^2+\omega_2^2)}$
31	$\frac{1}{2\omega}(\sin\omega t + \omega t\cos\omega t)$	$\frac{s^2}{(s^2+\omega^2)^2}$

$\frac{1}{s^2(Ts+1)}$

Impulse



$\frac{s^2}{s/(s^2+\zeta\omega s+\omega_n^2)}$

step

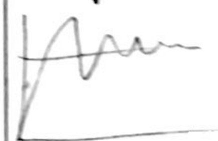


TABLE 2–2 Properties of Laplace Transforms

1	$\mathcal{L}[Af(t)] = AF(s)$
2	$\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$
3	$\mathcal{L}_\pm\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0\pm)$
4	$\mathcal{L}_\pm\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s) - sf(0\pm) - \dot{f}(0\pm)$
5	$\mathcal{L}_\pm\left[\frac{d^n}{dt^n}f(t)\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k} \overset{(k-1)}{f}(0\pm)$ where $\overset{(k-1)}{f}(t) = \frac{d^{k-1}}{dt^{k-1}}f(t)$
6	$\mathcal{L}_\pm\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{[\int f(t)\,dt]_{t=0\pm}}{s}$
7	$\mathcal{L}_\pm\left[\iint f(t)\,dt\,dt\right] = \frac{F(s)}{s^2} + \frac{[\int f(t)\,dt]_{t=0\pm}}{s^2} + \frac{[\iint f(t)\,dt\,dt]_{t=0\pm}}{s}$
8	$\mathcal{L}_\pm\left[\int \cdots \int f(t)(dt)^n\right] = \frac{F(s)}{s^n} + \sum_{k=1}^{n} \frac{1}{s^{n-k+1}}\left[\int \cdots \int f(t)(dt)^k\right]_{t=0\pm}$
9	$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$
10	$\int_0^\infty f(t)\,dt = \lim_{s\to 0} F(s) \qquad \text{if } \int_0^\infty f(t)\,dt \text{ exists}$
11	$\mathcal{L}[e^{-at}f(t)] = F(s+a)$
12	$\mathcal{L}[f(t-\alpha)1(t-\alpha)] = e^{-\alpha s}F(s) \qquad \alpha \geq 0$
13	$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$
14	$\mathcal{L}[t^2f(t)] = \frac{d^2}{ds^2}F(s)$
15	$\mathcal{L}[t^nf(t)] = (-1)^n\frac{d^n}{ds^n}F(s) \qquad n = 1, 2, 3, \ldots$
16	$\mathcal{L}\left[\frac{1}{t}f(t)\right] = \int_s^\infty F(s)\,ds \qquad \text{if } \lim_{t\to 0}\frac{1}{t}f(t) \text{ exists}$
17	$\mathcal{L}\left[f\left(\frac{t}{a}\right)\right] = aF(as)$

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LAPLACE TRANSFORM

$$F(s) = \mathcal{L} f(t) = \int_0^\infty f(t) e^{-st} dt \qquad 0 < t < \infty$$

$$\text{where } \int_0^\infty = \lim_{P \to \infty} \int_0^P \quad \text{exists !}$$

For now, $s \in \mathbb{R}$ (real). Later, we let $s \in \mathbb{C}$ (complex)

INVERSE LAPLACE TRANSFORM

- For a given $F(s)$, find the original $f(t)$.
- Use pairing method (Table 2.1):

LT pair : $f(t) \underset{ILT}{\overset{LT}{\rightleftarrows}} F(s)$

or

$$\text{LT pair} \begin{cases} F(s) = \mathcal{L} f(t) \\ f(t) = \mathcal{L}^{-1} F(s) \end{cases}$$

LT 1a

1. LT of exponential function $e^{p_0 t}$

$$\mathcal{L}\, e^{p_0 t} = \frac{1}{s-p_0} \qquad \begin{cases} f(t) = e^{p_0 t} \\ F(s) = \dfrac{1}{s-p_0} \end{cases}$$

Proof

$$\int_0^\infty f(t) e^{-st} dt = \int_0^\infty e^{p_0 t} e^{-st} dt = \int_0^\infty e^{-(s-p_0)t} dt$$

change of variable: $(s-p_0)t = t^*$

$$(s-p_0)dt = dt^*$$

$$dt = \frac{1}{s-p_0} dt^*.$$

$$\int_0^\infty e^{-(s-p_0)t} dt = \int_0^\infty e^{-t^*} \frac{1}{s-p_0} dt^*$$

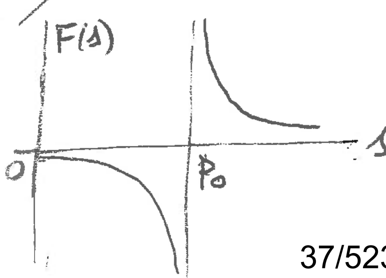
$$= \frac{1}{s-p_0} \int_0^\infty e^{-t^*} dt^* = \frac{1}{s-p_0} \left(-e^{-t^*}\right)\Big|_0^\infty$$

$$= \frac{-1}{s-p_0} \left(e^{-t^*}\right)\Big|_0^\infty = \frac{-1}{s-p_0} \Big[\underbrace{e^{-\infty}}_{\to 0} - \underbrace{e^{0}}_{=1} \Big]$$

$$= \frac{-1}{s-p_0}(-1) = \frac{1}{s-p_0}$$

Note: $F(s) \xrightarrow[s \to p_0]{} \infty$

p_0 = pole!



LT/2

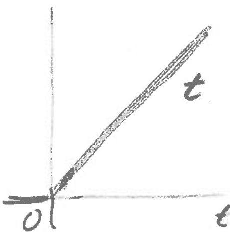
2. Step function $1(t)$

$$1(t) \xrightarrow{LT} \frac{1}{s}$$

$$\boxed{\begin{array}{l} f(t) = 1(t) \\ F(s) = \frac{1}{s} \end{array}}$$

Proof: $\mathcal{L}\,1(t) = \int_0^\infty 1\cdot e^{-st}\,dt = \int_0^\infty e^{-st}\,dt = \frac{-1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s}$

3. Ramp function,



$$t \xrightarrow{LT} \frac{1}{s^2}$$

$$\boxed{\begin{array}{l} f(t) = t, \quad t > 0 \\ F(s) = \frac{1}{s^2} \end{array}}$$

Proof: $\mathcal{L}\,t = \int_0^\infty t e^{-st}\,dt$

Integration by parts:

$d[uv] = u\,dv + v\,du \longrightarrow v\,du = d[uv] - u\,dv$ | $\int_a^b v\,du = uv\Big|_a^b - \int_a^b u\,dv$

$du = e^{-st}$; $u = \frac{1}{-s}e^{-st}$

$v = t \qquad dv = 1$

$$\int_0^\infty t e^{-st}\,dt = \frac{t}{-s}e^{-st}\Big|_0^\infty - \int_0^\infty \frac{1}{-s}e^{-st}\,dt = \frac{1}{s}\underbrace{\int_0^\infty e^{-st}\,dt}_{\mathcal{L}\,1(t) = \frac{1}{s}} = \frac{1}{s^2}$$

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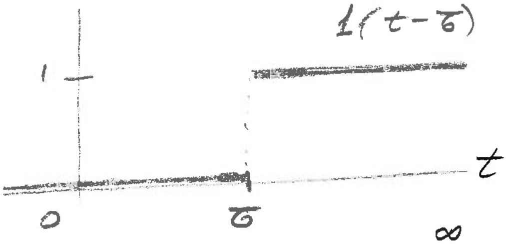
4. shifted step function $1(t-\tau)$



$$\begin{cases} f(t) = 1(t-\tau) \\ F(s) = e^{-\tau s}\dfrac{1}{s} \end{cases}$$

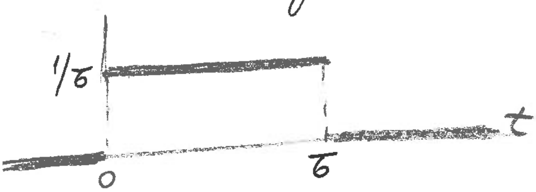
Proof $\mathcal{L}\,1(t-\tau) = \int_0^\infty 1(t-\tau)\, e^{-st}dt = \cancel{\int_0^\tau 0\cdot e^{-st}dt}^{=0} + \int_\tau^\infty e^{-st}dt$

change of variable $t^* = t-\tau$; $t = t^*+\tau$

$dt^* = dt$

$$= \int_0^\infty e^{-s(t^*+\tau)}dt^* = e^{-s\tau}\underbrace{\int_0^\infty e^{-st^*}dt^*}_{\mathcal{L}\,1(t) = \frac{1}{s}} = e^{-s\tau}\frac{1}{s}$$

5. Pulse function $p(t;\tau)$



$$\begin{cases} f(t) = p(t;\tau) \\ F(s) = \dfrac{1-e^{-s\tau}}{s\tau} \end{cases}$$

Proof : write the pulse as a step up followed by a step down at $t=\tau$ and scaled be $1/\tau$, i.e.,

$$p(t;\tau) = \underbrace{\frac{1}{\tau}1(t)}_{\downarrow} - \underbrace{\frac{1}{\tau}1(t-\tau)}_{\downarrow}$$

$$F(s) = \frac{1}{\tau}\frac{1}{s} - \frac{1}{\tau}e^{-s\tau}\frac{1}{s} = \frac{1-e^{-s\tau}}{s\tau}$$

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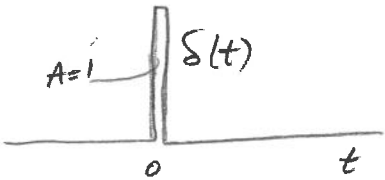
6. Impulse function $\delta(t)$



$$\boxed{\begin{array}{l} f(t) = \delta(t) \\ F(s) = 1 \end{array}}$$

(A) Proof : Consider $\delta(t)$ as the limit of $p(t;\tau)$ as $\tau \to 0$ and take LT.

$$\delta(t) = \lim_{\tau \to 0} p(t;\tau)$$

$$\mathcal{L}\delta(t) = \lim_{\tau\to 0} \mathcal{L} p(t;\tau) = \lim_{\tau \to 0} \frac{1-e^{-s\tau}}{s\tau}$$

The limit gives $\frac{1-1}{0} = \frac{0}{0}$. Apply l'Hospital rule,

$$\lim_{\tau\to 0} \frac{\frac{\partial}{\partial\tau}\left(1-e^{-s\tau}\right)}{\frac{\partial}{\partial \tau}(s\tau)} = \lim_{\tau\to 0}\frac{-(-\cancel{s}e^{-s\tau})}{\cancel{s}} = \lim_{\tau\to 0} e^{-s\tau} = 1$$

QED

(B) Proof : use localization property of the delta function, i.e.:

Recall $\int_{-\infty}^{\infty} \delta(t) g(t)\, dt = g(0)$

Then:

$$\mathcal{L}\delta(t) = \int_0^\infty \delta(t) \underbrace{e^{-st}}_{g(t)} dt = e^{-st}\Big|_{t=0} = e^0 = 1$$

QED

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Differentiation Property

$$\mathcal{L} f'(t) = s F(s) \quad \text{if } f(0)=0 \qquad (1)$$

$$\mathcal{L} f''(t) = s^2 F(s) \quad \text{if } f'(0) = f(0) = 0 \qquad (2)$$

$\vdots$

$$\mathcal{L} f^{(n)}(t) = s^n F(s) \quad \text{if } f^{(n-1)}(0) = \dots = f'(0) = f(0) = 0 \qquad (n)$$

Proof

(1): $\mathcal{L} f'(t) = \int_0^\infty f'(t) e^{-st} dt = f e^{-st}\Big|_0^\infty - \int_0^\infty f(-s) e^{-st} dt$

Integration by parts $\quad u\,dv = d(uv) - v\,du$

$u = e^{-st}$, $dv = f' dt$, $v = f$, $du = -se^{-st}$

$$= \underbrace{\cancel{f(\infty) e^{-\infty}}}_{=0} - f(0) + s \underbrace{\int_0^\infty f e^{-st} dt}_{F(s)} = \underbrace{s F(s)}_{} - \underbrace{\cancel{f(0)}}_{=0}$$

QED

(2): denote $g(t) = f'(t)$

$G(s) = \mathcal{L} g(t) = \mathcal{L} f'(t) = s F(s)$

$\mathcal{L} f''(t) = \mathcal{L} g'(t) = s\, G(s) = s^2 F(s)$

QED

(n): by induction ...

Bottom line: "to differentiate, multiply by s"

provided $f(0) = 0$, $f'(0) = 0$, etc.

Else: need to subtract them $\quad \mathcal{L} f'(t) = s F(s) - f(0)$ etc

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Integration Property

$$\mathcal{L}\left(\int_0^t f(t^*)\,dt^*\right) = \frac{1}{s} F(s)$$

Proof Denote $g(t) = \int_0^t f(t^*)\,dt^*$

Then $g'(t) = f(t)$, $g(0) = 0$

$\mathcal{L}$ LHS: $\mathcal{L}\, g'(t) = s\mathcal{L}\, g(t) = s\,\mathcal{L}\left(\int_0^t f(t^*)\,dt^*\right)$

$\mathcal{L}$ RHS: $\mathcal{L} f(t) = F(s)$

$\mathcal{L}$ LHS = $\mathcal{L}$ RHS: $\quad s\,\mathcal{L}\left(\int_0^t f(t^*)\,dt^*\right) = F(s)$

divide by s to get $\quad \mathcal{L}\int_0^t f(t^*)\,dt^* = \frac{1}{s} F(s)$ QED

Bottom line: "To integrate, divide by s"

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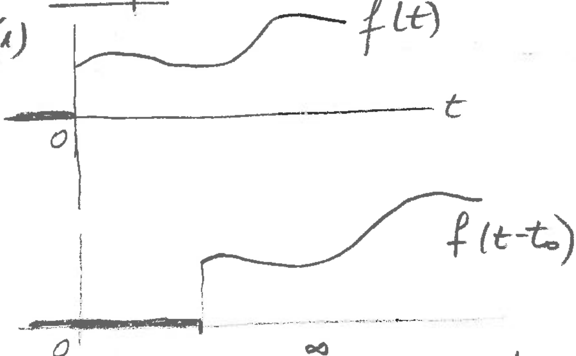
Shift Properties

Shift in t: $\mathcal{L} f(t-t_0) = e^{-t_0 s} F(s)$ (1)

shift in s: $\mathcal{L} e^{s_0 t} f(t) = F(s-s_0)$ (2)

Proof

(1)



Function $f(t)$ is zero for $-$ve argument

Function $f(t-t_0)$ is zero for $t<t_0$.

$$\mathcal{L} f(t-t_0) = \int_0^\infty f(t-t_0) e^{-st} dt = \cancel{\int_0^{t_0} 0 \cdot e^{-st} dt}^{\,0} + \int_{t_0}^\infty f(t-t_0) e^{-st} dt$$

$$= \int_{t_0}^\infty f(t-t_0) e^{-st} dt$$

change of variable $\left| \begin{array}{l} t^* = t - t_0 \rightarrow t = t^* + t_0 \\ dt^* = dt \end{array} \right.$

$$= \int_0^\infty f(t^*) e^{-s(t^*+t_0)} dt^* = e^{-st_0} \underbrace{\int_0^\infty f(t^*) e^{-st^*} dt^*}_{F(s)} .$$

QED

(2) $$\mathcal{L} e^{s_0 t} f(t) = \int_0^\infty e^{s_0 t} f(t) e^{-st} dt = \underbrace{\int_0^\infty f(t) e^{-(s-s_0)t} dt}_{F(s-s_0)}$$

QED

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Final Value Theorem

steady-state Response.

$$x_{ss} = x(\infty) = \lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s).$$

steady state response   final value

$$\boxed{x_{ss} = \lim_{s \to 0} sX(s)}$$

Proof Start with LT of the derivative $\dot{x}(t)$, i.e.,

$$\mathcal{L}\,\dot{x}(t) = sX(s) \quad , \quad x(0)=0 \qquad (1)$$

But, by definition

$$\mathcal{L}\,\dot{x}(t) = \int_0^\infty \dot{x}(t)e^{-st}dt \qquad (2).$$

$$(1) \equiv (2): \quad \int_0^\infty \dot{x}(t)e^{-st}dt = sX(s) \qquad (3).$$

Take limit $s \to 0$ of Eq. (3), i.e.,

$$\lim_{s \to 0}\left[\int_0^\infty \dot{x}(t)e^{-st}dt\right] = \lim_{s \to 0} sX(s) \qquad (4)$$

$$\text{but } \lim_{s \to 0} e^{-st} = e^0 = 1 \qquad (5)$$

$$(5) \to (4)\,LHS: \int_0^\infty \dot{x}(t)dt = x(t)\Big|_0^\infty = x(\infty) = x_{ss}$$

$$LHS = RHS: \qquad x_{ss} = \lim_{s \to 0} sF(s) \qquad QED$$

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INVERSE LAPLACE TRANSFORM

Assume $$X(s) = \frac{B(s)}{A(s)} \qquad (1)$$

$A(s), B(s)$ polynomials in s:

$A(s) = a_n s^n + a_{n-1} s^{n-1} + --- + a_0$: degree n

$B(s) = b_m s^m + b_{m-1} s^{m-1} + --- + b_0$: degree $m < n$

Then $X(s)$ can be expanded in partial fractions, i.e.,

$$X(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + --- \frac{r_n}{s-p_n} = \sum_{i=1}^{n} \frac{r_i}{s-p_i} \qquad (2)$$

Recall $\mathcal{L} e^{p_0 t} = \frac{1}{s-p_0}$ (pole p_0)

Hence, the ILT of (2) is a sum, i.e.,

$$x(t) = r_1 e^{p_1 t} + r_2 e^{p_2 t} + \dots r_n e^{p_n t} = \sum_{i=1}^{n} r_i e^{p_i t} \qquad (3)$$

The values $p_1, p_2, \dots, p_n$ are the roots of the denominator $A(s)$. They are called "poles". They may be complex numbers.

The values $r_1, r_2, \dots, r_n$ are called "residues"

$$r_i = \lim_{s \to p_i} \left[(s - p_i) X(s) \right] \qquad (4)$$

Use MATLAB function "residue" to find p_i & r_i

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iLT with complex poles.

When complex poles appear, they are in conjugate pairs, i.e.,

$$p_{1,2} = \sigma \pm i\omega_d \qquad (1)$$

PFE : $X(s) = X_1(s) + X_2(s) = \dfrac{r_1}{s-p_1} + \dfrac{r_2}{s-p_2}$ (2)

It can be shown that $r_1 = -ir_0$, $r_2 = ir_0$ (3)

$$X_1(t) = \frac{-ir_0}{1-p_1}, \quad p_1 = \sigma + i\omega_d \qquad (4)$$

$$x_1(t) = \mathcal{L}^{-1} X_1(s) = \mathcal{L}^{-1} \frac{-ir_0}{s-p_1} = -ir_0 e^{p_1 t} \qquad (5)$$

$$= -ir_0 e^{(\sigma + i\omega_d)t} = -ir_0 e^{\sigma t} e^{i\omega_d t}$$

$$= -ir_0 e^{\sigma t}(\cos\omega_d t + i\sin\omega_d t) \qquad (6)$$

$$= r_0 e^{\sigma t}(-i\cos\omega_d t + \sin\omega_d t)$$

Similarly

$$X_2(s) = \frac{ir_0}{s-p_2}, \quad p_2 = \sigma - i\omega_d \qquad (7)$$

$$x_2(t) = ir_0 e^{\sigma t} e^{-i\omega_d t} = ir_0 e^{\sigma t}(\cos\omega_d t - i\sin\omega_d t) \qquad (8)$$

$$= r_0 e^{\sigma t}(i\cos\omega_d t + \sin\omega_d t)$$

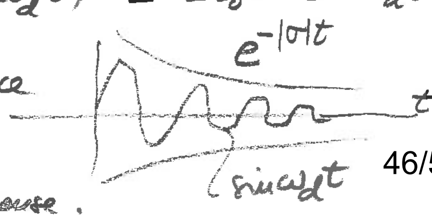
$x_1(t) + x_2(t) =$

$$r_0 e^{\sigma t}(\cancel{-i\cos\omega_d t} + \sin\omega_d t)$$

$$+ r_0 e^{\sigma t}(\cancel{i\cos\omega_d t} + \sin\omega_d t) = 2r_0 e^{\sigma t}\sin\omega_d t$$

We expect $\sigma < 0$, hence

$$\sigma = -|\sigma|$$



Poles in LHS for stable response.

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MATLAB instruction residue

$$X(s) = \frac{B(s)}{A(s)}$$

$$B(s) = b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0$$

$$B = [b_m \ \ b_{m-1} \ \ \ldots \ b_0]$$

$$A(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0$$

$$A = [a_n \ \ a_{n-1} \ \ \ldots \ a_0]$$

$[r, p, k]$ = residue (B, A)

where: p = vector of poles (roots of $A(s)$)

r = vector of residues at the poles

k = vector of direct term coeff.

In our work, $m < n$, then $k = [\,]$ void matrix

Partial fraction expansion (PFE):

$$X(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \cdots \qquad (1)$$

Note: If $m \geq n$, then define direct term as

$$k_1 s^q + k_2 s^{q-1} + \cdots + k_{q+1}$$

and add it to the end of (1) to get $X(s)$

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Example

$$X(s) = \frac{1}{s+2} + \frac{2}{s+1} = \frac{(s+1) + 2(s+2)}{(s+1)(s+2)}$$

$$= \frac{3s+5}{s^2+3s+2} = \frac{B(s)}{A(s)}$$

(a) Define:

$B = [3 \;\; 5]$; $A = [1 \;\; 3 \;\; 2]$

$[r, p, k] = \text{residue}(B, A)$

$r_1 = 1$ $\quad p_1 = -2$ $\quad k = \text{void}$.

$r_2 = 2$ $\quad p_2 = -1$

$$\frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} = \frac{1}{s-(-2)} + \frac{2}{s-(-1)} = \frac{1}{s+2} + \frac{2}{s+1}$$

checks!

(b) Define:

$r = [1 \;\; 2]$; $p = [-2 \;\; -1]$; $k = 0$

$[b, a] = \text{residue}(r, p, k)$

$b = [3 \;\; 5]$

$a = [1 \; 3 \; 2]$ checks!

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Convolution property of Laplace Transform

$$\mathcal{L}^{-1} G(s)\cdot F(s) = \mathcal{L}^{-1} F(s)\, G(s)$$

$$= \int_0^t f(\tau)\, g(t-\tau)\, d\tau \qquad (1)$$

excitation function — $f(\tau)$

impulse response of the system — $g(t-\tau)$

Convolution expresses the system response $x(t)$ to a complicated excitation $f(\tau)$ as an integral using the impulse response $g(t)$ shifted by τ to $(t-\tau)$, i.e.,

$$x(t) = (f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau \qquad (2)$$

The integral in Eq. (2) is not easily computed in time domain; however, Laplace transform makes it easy because:

1. calulate $F(s) = \mathcal{L} f(t)$, $G(s) = \mathcal{L} g(t)$
2. multiply $F(s)\, G(s)$
3. take inverse Laplace transform to get $x(t)$

$$\mathcal{L}^{-1} F(s)\, G(s) = (f * g)(t) = \underline{x(t)}$$

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DOMINANT POLES

Given:

$$X(s) = \frac{B(s)}{A(s)} = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \cdots + \frac{r_k}{s-p_k} + \cdots \quad (1)$$

$p_k \in \mathbb{C}$ complex number

$$p_k = \sigma_k + i\omega_k, \text{ conjugate pairs} \quad (2a)$$

or

$$p_k \in \mathbb{R} \qquad p_k = \sigma_k \quad (2b)$$

Perform $\mathcal{L}^{-1}$ Eq.(1):

$$x(t) = r_1 e^{p_1 t} + r_2 e^{p_2 t} + \cdots + r_k e^{p_k t} + \cdots \quad (3)$$

We are interested in the long term behavior of $x(t)$, i.e., to find

$$x_{ss}(t) = \lim_{t\to\infty} x(t) \quad \text{steady state response}$$

Note that every term in the expansion will have the form

$$r_k e^{p_k t} = \underbrace{r_k e^{\sigma_k t}}_{\text{exponential function}} \cdot \underbrace{e^{i\omega_k t}}_{\text{harmonic oscillation.}}$$

We distinguish the following possible cases:

<u>A.</u> If at least one σ_k is +ve ($\sigma_k > 0$) then

$$e^{\sigma_k t} \longrightarrow \infty, \text{ UNSTABLE system.}$$

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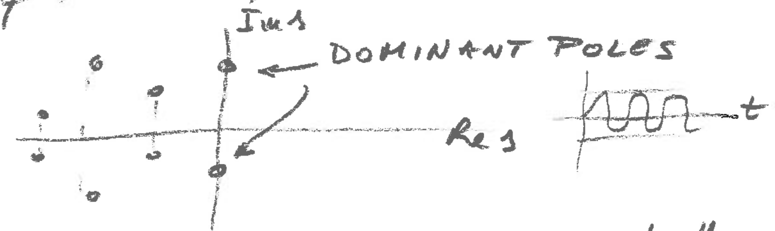
B. If system is STABLE, i.e., all $\sigma_k \leq 0$, (all poles in LHS) then:

B1. If terms with $\sigma_k = 0$ exist, then these will survive as sustained oscillations while the rest have died out. This means that poles situated on the imaginary axis, if the exist, are dominant poles



B2. If terms with $\sigma_k = 0$ do NOT exist, then the dominant poles are the poles closest to the imaginary axis because the die hardest having small σ_k values (small damping