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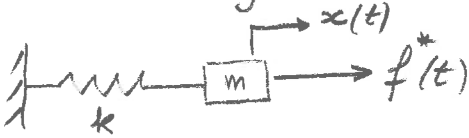
Spring-mass oscillator



Free body diagram (FBD)

Newton law of motion (NLM)

$$m\ddot{x} = -kx + f^*$$

or

$$m\ddot{x}(t) + kx(t) = f^*(t) \quad \text{Eq. of motion (EOM)} \qquad (1)$$

- Forced vibration equation.
- Inhomogeneous ODE in time, t.
- 2^{nd} order ODE

Solution of Eq. (1) consists of two parts, a complementary sol^n $x_c(t)$ and a particular sol^n $x_p(t)$, i.e.,

$$x(t) = x_c(t) + x_p(t) \qquad (2)$$

- $x_c(t)$ satisfies the homogeneous eq^n, i.e.,

$$m\ddot{x}_c + kx_c = 0 \qquad (3).$$

- $x_p(t)$ satisfies the complete Eq. (1).

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Homogeneous equation (free vibration)

The homogeneous eq^n is obtaine by setting the RHS to zero in Eq.(1), i.e.,

$$m\ddot{x} + kx = 0 \quad \text{(homogeneous equation)} \quad (4)$$

Divide by m to get

$\frac{1}{m}(4)$: $$\ddot{x} + \frac{k}{m}x = 0 \quad (5)$$

Denote

$$\omega_n^2 = \frac{k}{m}, \quad \omega_n = \sqrt{\frac{k}{m}} \quad \text{natural frequency} \quad (6)$$

rad/sec

$(6) \Rightarrow (5)$:

$$\ddot{x} + \omega_n^2 x = 0 \quad \text{homogeneous eq}^n \text{ in standard form} \quad (7)$$

To solve the homogeneous eq. (7), assume

$$x(t) = Ce^{pt} \quad (8)$$

Hence

$$\dot{x}(t) = pCe^{pt} = px$$
$$\ddot{x}(t) = p^2Ce^{pt} = p^2x \quad (9)$$

$(8),(9) \rightarrow (7)$:

$$p^2x + \omega_n^2 x = 0.$$

or $$\boxed{p^2 + \omega_n^2 = 0} \quad \text{characteristic equation} \quad (10)$$

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Free vibration response

Solution of Eq. (7) is

$$p^2 = -\omega_n^2 \qquad p_{1,2} = \pm i\omega_n \qquad (11)$$

(11) → (7) :

$$x_c(t) = C_1 e^{i\omega_n t} + C_2 e^{-i\omega_n t} \quad \text{complementary solution} \qquad (12)$$

C_1, C_2 = constants to be determined

Complementary solⁿ (12) can be written as

$$\boxed{x_c(t) = C \sin(\omega_n t + \varphi) \quad \text{free vibr. response} \quad (11)}$$

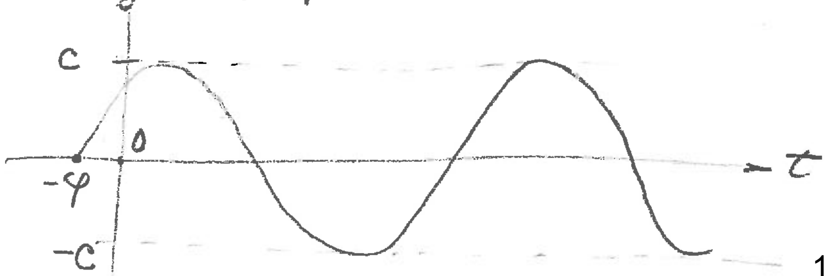
C = amplitude of osillation

φ = phase angle

Proof : use Euler identity $e^{i\alpha} = \cos\alpha + i\sin\alpha$

(see next page) →

Oscillatory motion:



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PROOF

$$C_1 e^{i\omega_n t} + C_2 e^{-i\omega_n t} \doteq C \sin(\omega_n t + \varphi)$$

Recall Euler's identity

$$e^{i\alpha} = \cos\alpha + i\sin\alpha .$$

$$\therefore\quad C_1 e^{i\omega_n t} = C_1 \cos\omega_n t + C_1 i \sin\omega_n t$$
$$C_2 e^{-i\omega_n t} = C_2 \cos\omega_n t - C_2 i \sin\omega_n t$$

$$C_1 e^{i\omega_n t} + C_2 e^{-i\omega_n t} = (C_1 + C_2)\cos\omega_n t + i(C_1 - C_2)\sin\omega_n t .$$

$$= A\cos\omega_n t + B\sin\omega_n t .$$

$$A = C_1 + C_2 \quad ; \quad B = i(C_1 - C_2)$$

$$A\cos\omega_n t + B\sin\omega_n t = C\sin(\omega_n t + \varphi).$$

Recall $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta .$

$$C\sin(\omega_n t + \varphi) = \underbrace{\boxed{C\cos\varphi}}_{B} \sin\omega_n t + \underbrace{\boxed{C\sin\varphi}}_{A} \cos\omega_n t .$$

$$A^2 + B^2 = C^2\sin^2\varphi + C^2\cos^2\varphi = C^2\underbrace{(\sin^2\varphi + \cos^2\varphi)}_{1} = C^2$$

$$C = \sqrt{A^2 + B^2}$$

$$\frac{A}{B} = \frac{C\sin\varphi}{C\cos\varphi} = \tan\varphi$$

$$\varphi = \tan^{-1}\frac{A}{B}.$$

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Standard form of inhomogeneous eq$^{\underline{n}}$

Divide Eq. (1) by m, i.e.,

$$\frac{1}{m}(1): \quad \ddot{x} + \frac{k}{m}x = \frac{1}{m}f^* \qquad (12)$$

Recall $\frac{k}{m} = \omega_n^2$ (13)

Define $f(t) = \frac{1}{k}f^*(t)$ normalized forcing function (14)

$$(14): \quad \frac{1}{m}f^*(t) \underset{(14)}{=} \frac{1}{m}kf(t) \underset{(13)}{=} \omega_n^2 f(t) \qquad (15)$$

$(13),(15) \longrightarrow (12)$:

$$\ddot{x} + \omega_n^2 x = \omega_n^2 f(t) \qquad (16)$$

2nd order inhomogeneous ODE in standard form

Forced vibration response

$$x(t) = x_c(t) + x_p(t) = C\sin(\omega_n t + \varphi) + x_p(t) \qquad (17)$$

$(17) \longrightarrow (16)$:

$$\ddot{x}_c + \ddot{x}_p + \omega_n^2(x_c + x_p) = \omega_n^2 f(t)$$

$$\underbrace{(\ddot{x}_c + \omega_n^2 x_c)}_{=0} + \ddot{x}_p + \omega_n^2 x_p = \omega_n^2 f(t)$$

$$\ddot{x}_p + \omega_n^2 x_p = \omega_n^2 f(t) \qquad (18)$$

Need to find $C, \varphi, x_p(t)$; not easy!

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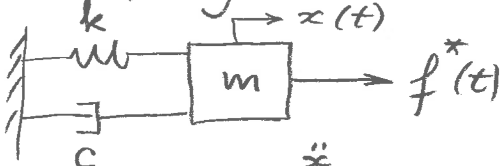
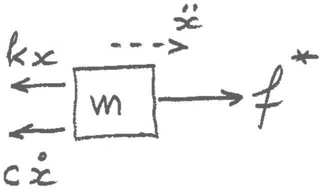
Spring-mass-damper oscillator



FBD 

NLM $\quad m\ddot{x} = -kx - c\dot{x} + f^*$

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f^*(t) \qquad \text{EOM} \qquad (1)$$

- Damped forced vibration eqn.
- 2nd order inhomogeneous ODE

Solution $x(t)$ of Eq.(1) consists of the sum of complementary soln $x_c(t)$ and of particular soln $x_p(t)$, where the complementary soln satisfies the homogeneous eqn while the particular solution satisfies the inhomogeneous eqn.

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2016/10/17 Damped free vibration (homogeneous eqⁿ)

Set to zero the RHS of Eq. (1) to get

$$m\ddot{x} + c\dot{x} + kx = 0 \quad \text{(damped free vibration eq}^n\text{)} \tag{2}$$

Assume $\quad x(t) = C e^{pt}$ (3)

Hence
$$\left.\begin{aligned} \dot{x} &= pCe^{pt} = px \\ \ddot{x} &= p^2 C e^{pt} = p^2 x \end{aligned}\right\} \tag{4}$$

(3), (4) → (2) :

$$mp^2 x + cpx + kx = 0$$

or
$$mp^2 + cp + k = 0 \quad \text{characteristic eq}^n \tag{5}$$

$$p_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

$$p_{1,2} = -\frac{c}{2m} \pm i\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} \tag{6}$$

Critical damping, c_{cr}

Define critical damping c_{cr} as the value of c to make zero the term under the square root sign, i.e.,
$$\frac{k}{m} - \left(\frac{c_{cr}}{2m}\right)^2 = 0$$

$$c_{cr}^2 = \frac{4m^2 k}{m} = 4mk \qquad c_{cr} = 2\sqrt{mk} \tag{7}$$

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10/16/2017

Damping ratio, ζ

Define damping ratio ζ as the ratio between damping c and critical damping c_{cr}, i.e.,

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{mk}} \quad \text{damping ratio} \qquad (8)$$

Natural frequency, ω_n

Recall the definition

$$\omega_n^2 = \frac{k}{m}, \qquad \omega_n = \sqrt{\frac{k}{m}} \qquad (9)$$

Damped frequency, ω_d

Calculate

$$\frac{c}{2m} \underset{(8)}{=} \frac{\zeta c_{cr}}{2m} \underset{(7)}{=} \zeta \frac{2\sqrt{mk}}{2m} = \zeta\sqrt{\frac{k}{m}} \underset{(9)}{=} \zeta\omega_n \qquad (10)$$

(9), (10) → (6) :

$$p_{1,2} = -\zeta\omega_n \pm i\sqrt{\omega_n^2 - \zeta^2\omega_n^2}$$

$$= -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$$

$$p_{1,2} = -\zeta\omega_n \pm i\omega_d \qquad (11)$$

where

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \quad \text{damped frequency} \qquad (12)$$

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2016/10/19

Damped free vibration response

(11) → (3):

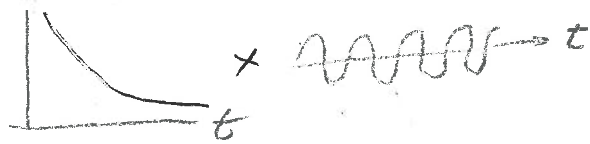
$$x(t) = C_1 e^{(-\zeta\omega_n + i\omega_d)t} + C_2 e^{(-\zeta\omega_n - i\omega_d)t}$$

$$= e^{-\zeta\omega_n t}\left(C_1 e^{i\omega_d t} + C_2 e^{-i\omega_d t}\right)$$

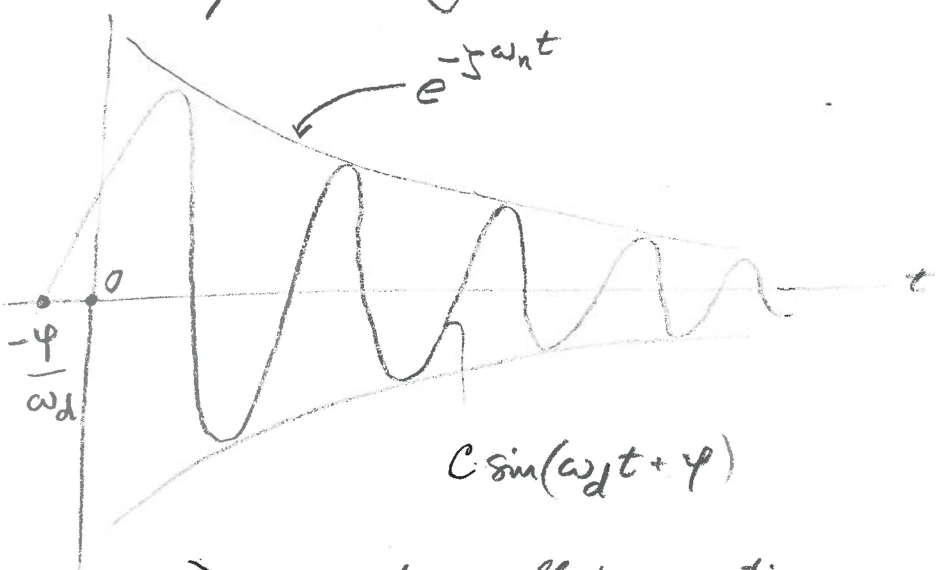
or

$$x_c(t) = C e^{-\zeta\omega_n t} \sin(\omega_d t + \varphi) \qquad (13)$$

natural freq. ω_n ↑ damped freq. ω_d



- Free damped vibration response
- Complementary sol^{n} $x_c(t)$



Damped oscillatory motion

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Standard form of damped forced vibration equation

Divide Eq.(1) by m, i.e.

$\frac{1}{m}(1)$: $$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{1}{m}f^*(t) \qquad (14)$$

Calulate

$$\frac{c}{m} \underset{(8)}{=} \frac{\zeta\, c_{cr}}{m} \underset{(7)}{=} \frac{\zeta\, 2\sqrt{mk}}{m} \underset{(9)}{=} 2\zeta\omega_n \qquad (15)$$

Define $f(t) = \frac{1}{k}f^*(t)$ normalized forcing function (16)

Calculate

$$\frac{1}{m}f^*(t) \underset{(16)}{=} \frac{1}{m}kf(t) \underset{(9)}{=} \omega_n^2 f(t) \qquad (17)$$

(15), (17) → (14) :

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 f(t) \qquad (18)$$

- Damped force vibration eq$^{\underline{n}}$ in standard form
- 2nd order ODE in standard form.

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Damped forced vibration response

$$x(t) = x_c(t) + x_p(t)$$
$$= e^{-\zeta\omega_n t} C \sin(\omega_d t + \varphi) + x_p(t) \quad (19)$$

(19)→(18):

$$\underbrace{\left(\ddot{x}_c + 2\zeta\omega_n \dot{x}_c + \omega_n^2 x_c\right)}_{=0}$$
$$+ \ddot{x}_p + 2\zeta\omega_n \dot{x}_p + \omega_n^2 x_p = \omega_n^2 f(t) \quad (20)$$

Need to find $C, \varphi, x_p(t)$

NOT easy!