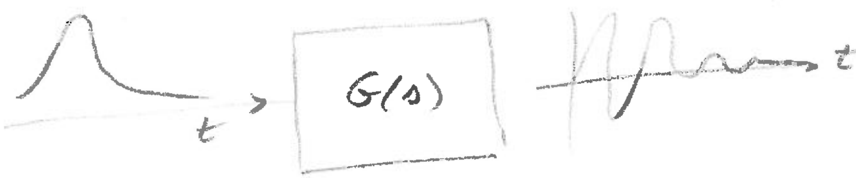
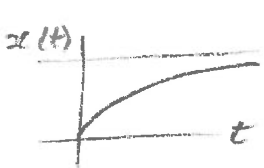
1
SR

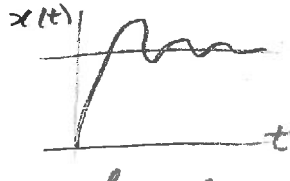
A system is stable if any stable input excitation produces a stable output response.



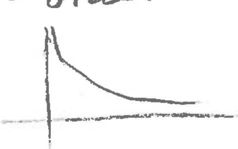
A response is stable if it remains bounded as $t \to \infty$

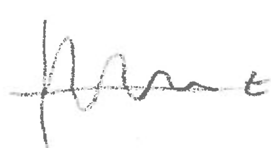


1st order



2nd order





Stability of 1st order Response

$$X(s) = \frac{K}{s-p}$$

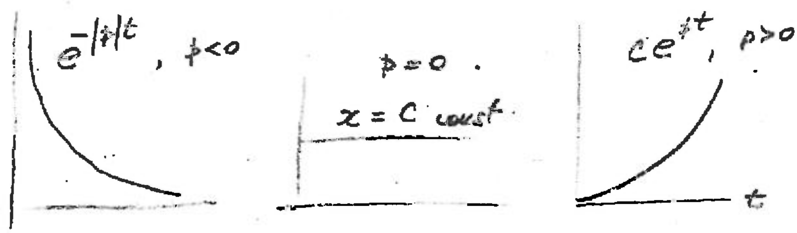
$$x(t) = K e^{pt}$$

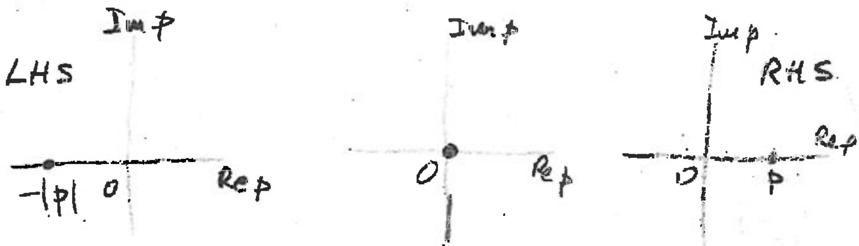
where p is the pole of $X(s)$.

The stability is dictated by the sign of p, i.e. its location in the complex p plane.

$p < 0$ — STABLE system : if a disturbance is applied, then the system returns to initial state.

p in LHS





$p<0$	$p=0$	$p>0$
Negative pole	Marginal	Positive pole
STABLE		UNSTABLE

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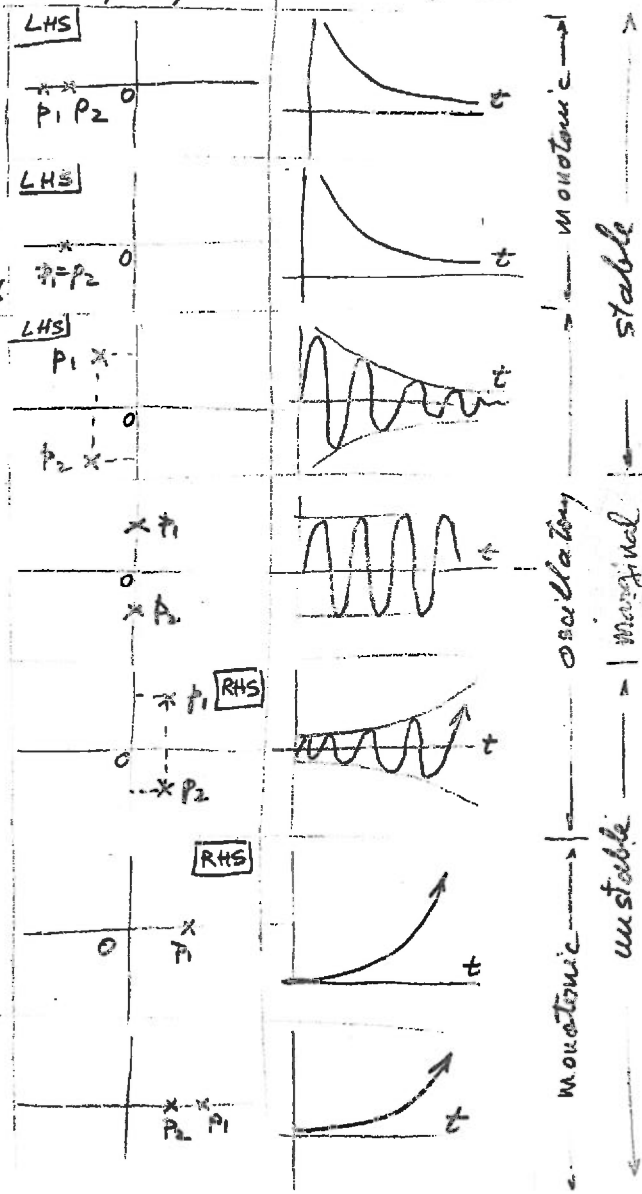
Stability of 2nd order Response

$$X(s) = \frac{K(s - z_1)}{(s - p_1)(s - p_2)}$$

pole location in complex plane | time response

1. $p_1, p_2 < 0$ — negative real poles in LHS
2. $p_1 = p_2 < 0$ — negative real double pole in LHS
3. $p_{1,2} = \sigma \pm i\omega$, $\sigma < 0$ — complex poles in LHS
4. $p_{1,2} = \pm i\omega$ — imaginary poles ($\sigma = 0$)
5. $p_{1,2} = \sigma \pm i\omega$, $\sigma > 0$ — complex poles in RHS
6. $p_1 = p_2 > 0$ — positive real double pole in RHS
7. $p_1, p_2 > 0$ — positive real poles in RHS



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STABILITY OF HIGHER ORDER RESPONSE

$$X(s) = \frac{B(s)}{A(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0}{a_n s^n + a_{m-1} s^{m-1} + \dots + a_0}, \quad m < n$$

Partial fraction expansion:

$$X(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \dots\dots + \frac{r_n}{s-p_n}$$

$p_1, p_2, \dots$ poles i.e. roots of $A(s) = 0$

$r_1, r_2, \dots$ residues

MATLAB: [r, p, k] = residue (B, A).

Note: poles can be either real or complex

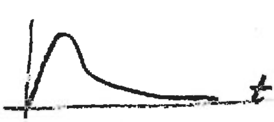
REAL POLES

- simple pole: $\frac{r}{s-p} \xrightarrow{\mathcal{L}^{-1}} r e^{pt}$
- double poles: $\frac{r}{(s-p)^2} \xrightarrow{\mathcal{L}^{-1}} r t e^{pt}$
- multiple poles: $\frac{r}{(s-p)^j} \xrightarrow{\mathcal{L}^{-1}} \frac{1}{(j-1)!} t^{j-1} e^{pt}$

Stable (LHS)

$p < 0$

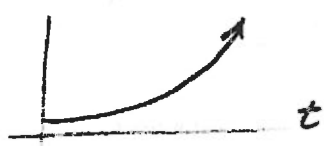
$x(t) \sim e^{pt} = e^{-|p|t}$



Unstable (RHS).

$p > 0$

$x(t) \sim e^{pt}$



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COMPLEX POLES

$$p_{1,2} = \sigma \pm i\omega$$

complex poles always in conjugate pairs!

$$\frac{1}{s-p_1} = \frac{1}{s-(\sigma+i\omega)} \xrightarrow{\mathcal{L}^{-1}} e^{(\sigma+i\omega)t} = e^{\sigma t} e^{i\omega t}$$

$$\frac{1}{s-p_2} = \frac{1}{s-(\sigma-i\omega)} \xrightarrow{\mathcal{L}^{-1}} e^{(\sigma-i\omega)t} = e^{\sigma t} e^{-i\omega t}$$

$$x(t) \sim e^{\sigma t}\left(e^{i\omega t} - e^{-i\omega t}\right) \sim e^{\sigma t} \sin\omega t$$

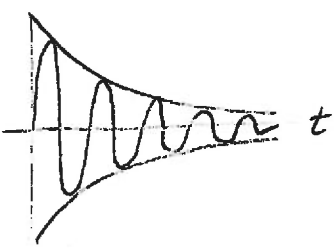
Euler's formulae.

Stable (LHS)

$\sigma < 0$

$$x(t) \sim e^{\sigma t} \sin\omega t$$

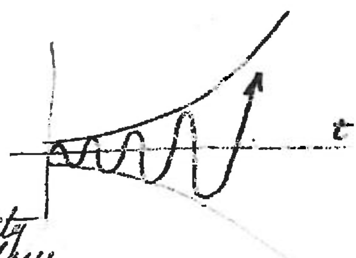
$$x(t) \sim e^{-|\sigma| t} \sin\omega t$$



Unstable (RHS)

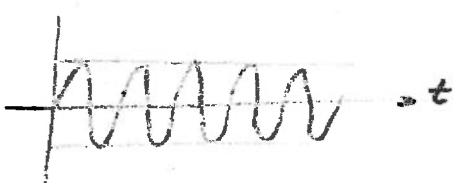
$\sigma > 0$

$$x(t) \sim e^{\sigma t} \sin\omega t$$



stability boundary

$\sigma = 0$, $p_{1,2} = \pm i\omega$



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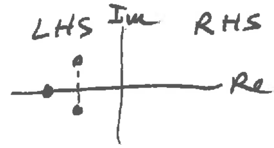
SR

Absolute stability

A necessary and sufficient condition for a system to be stable is that its poles are placed in the LHS of the complex plane



Marginal stability

If some poles are purely imaginary (i.e., placed on imaginary axis) then the system has *marginal stability*

- Bounded impulse response OK!
- Unbounded response for other inputs

Ex: $G(s)=\frac{1}{s^2+1}$, $x(t)=t\cos t$ for $f(x)=\sin t$ NOT OK!

Relative stability

Would a stable system still be stable if its parameters are slightly changed?
What margin of safety is there?