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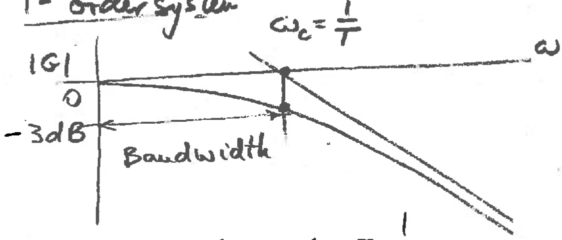
Generic Performance Indicators in Freq. Domain p18

Generic Performance Indicators in Freq. Domain

Bandwidth and cutoff frequency, ω_B

$$|G(\omega_B)| = G(0) - 3dB$$ has declined 3dB below LF value

1st order system

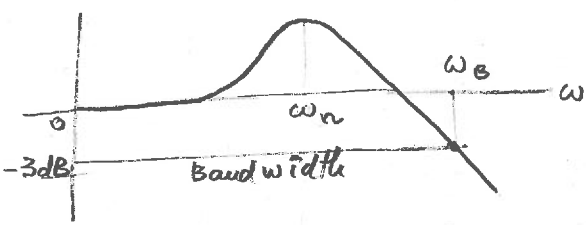


$$G(s) = \frac{1}{Ts+1}$$

$$\omega_B = \omega_c = \frac{1}{T}$$

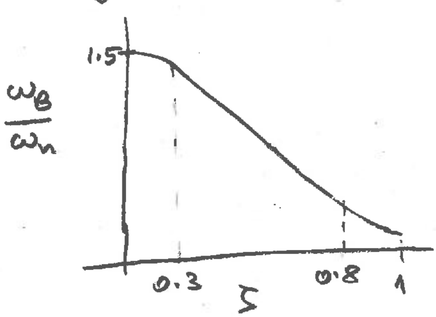
2nd order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



$$|G(i\omega)|_{dB} = -20 \log_{10} \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2} = -3dB$$

ω_B is determined graphically or solved numerically



$$\frac{\omega_B}{\omega_n} \approx -1.19\zeta + 1.85$$

$$0.3 < \zeta < 0.8$$

$\omega_B \uparrow$ as $\zeta \downarrow$

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Ex. 11.2

1st order system performance

Given: two systems:

$$G_1 = \frac{1}{s+1} \quad ; \quad G_2 = \frac{1}{3s+1}$$

Find: (a) bandwidth

(b) frequency response

(c) step response

(d) ramp response

(e) discuss results

Solution:

$$G_1(i\omega) = \frac{1}{i\omega+1} = \frac{1}{i\omega T_1+1} \rightarrow T_1 = 1$$

$$G_2(i\omega) = \frac{1}{3i\omega+1} = \frac{1}{i\omega T_2+1} \longrightarrow T_2 = 3$$

(a)

$$\omega_B = \frac{1}{T} \qquad \omega_1^B = \frac{1}{1} = 1 \text{ rad/sec} \qquad \omega_2^B = \frac{1}{3} \text{ rad/sec}$$

(b), (c), (d): see MATLAB Ex. 11.2

(e) System 1 has bandwidth three times larger than system 2 ($\omega_1 = 1$ vs. $\omega_2 = 1/3$ rad/sec). Sys. 1 has faster step response and follows the ramp input much better (smaller ramp error)

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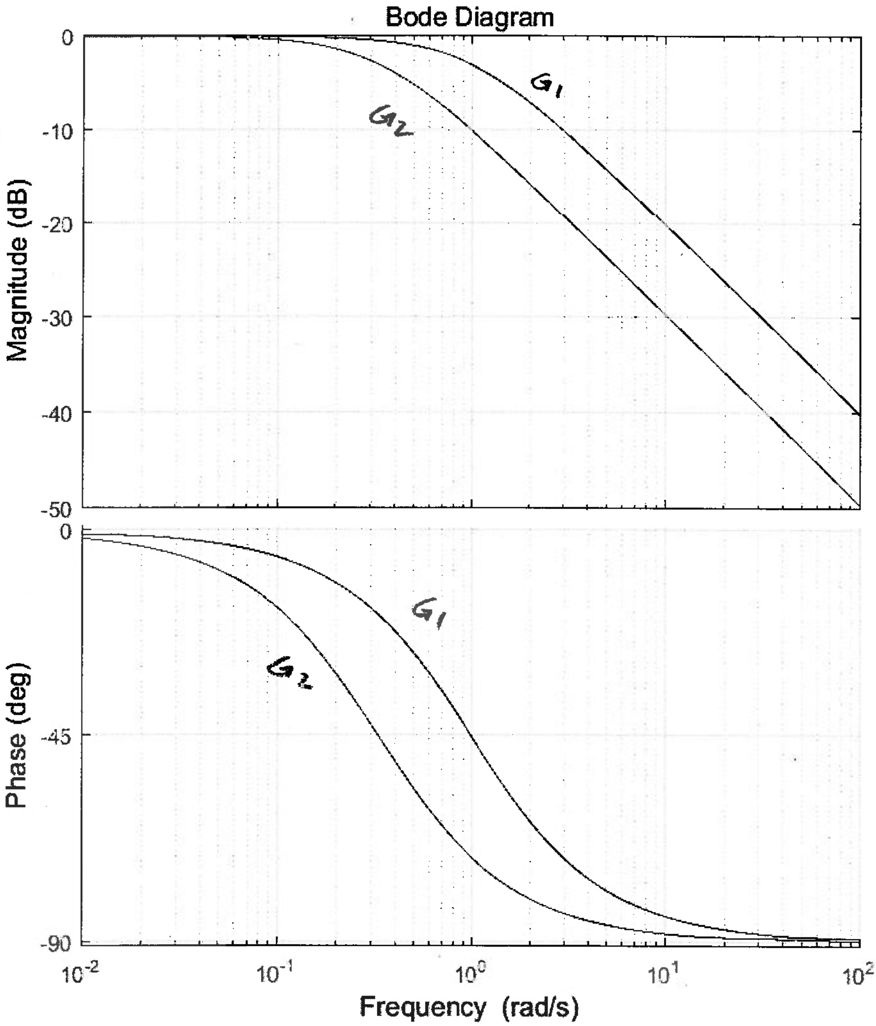

Bode Diagram
Magnitude (dB)
0
-10
-20
-30
-40
-50
G1
G2
Phase (deg)
0
-45
-90
G1
G2
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/s)

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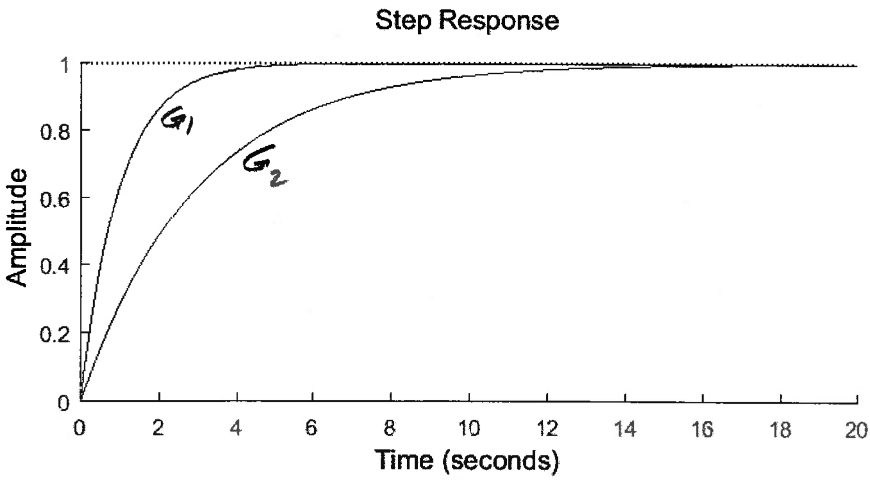
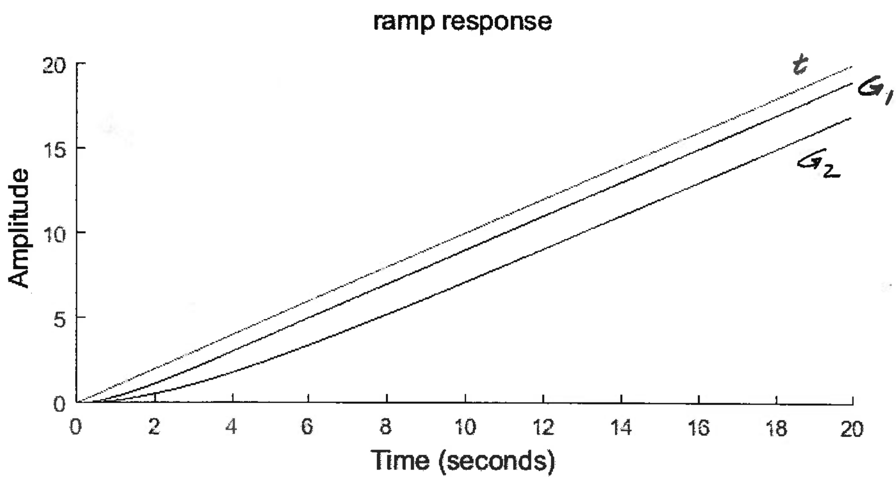
11.2


Step Response
1
0.8
0.6
0.4
0.2
0
Amplitude
G1
G2
0
2
4
6
8
10
12
14
16
18
20
Time (seconds)


ramp response
20
15
10
5
0
Amplitude
t
G1
G2
0
2
4
6
8
10
12
14
16
18
20
Time (seconds)

```
%{
 EXAMPLE 11.2
 1st Order System analysis
%}
%% initialization
clc
clear all
% close all
s=tf('s');
%% system definition
G1=1/(s+1); G2=1/(3*s+1);
%% Bode plots
figure(1)
bode(G1,G2)
box off
grid on
%% time response
figure(2)
Tfinal=20;
dt=0.1; t=0:dt:Tfinal; u=1.*t;
subplot(2,1,1)
step(G1,G2,Tfinal)
box off
subplot(2,1,2)
lsim(G1,G2,u,t)
axis([0 Tfinal 0 Tfinal]); title( 'ramp response')
box off
```

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Ex: 2nd order Sys. Performance

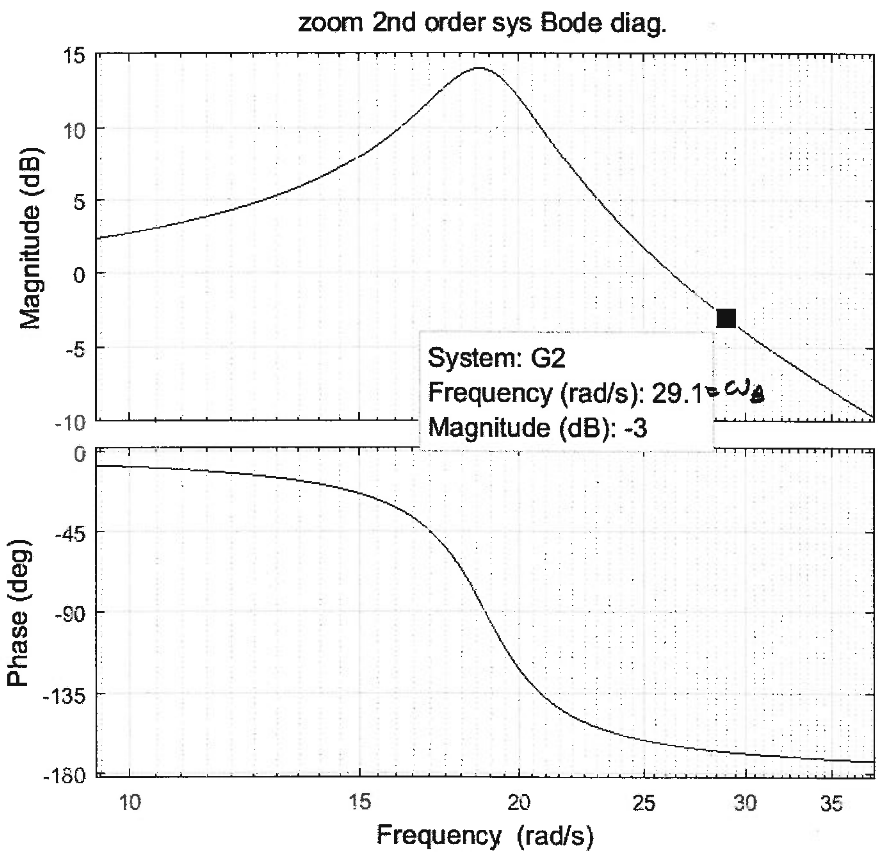
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$\omega_n = 2\pi f_n$

$f_n = 3\,Hz$

$\zeta = 10\%$

$\omega_B = 29.1\ Hz$



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Specific Performance Indicators in Freq Domain for 2nd Order Systems

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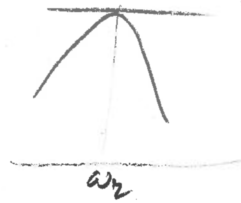
Resonance: peak response
max. response

$$|G(i\omega)|^2 = \frac{1}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}.$$



$\frac{d}{d\omega}|G(i\omega)|^2 = 0$ for peak

$$\frac{d}{d\omega}\left[\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2\right] = 0.$$

use auxiliary variable $p = \frac{\omega}{\omega_n}$ and write

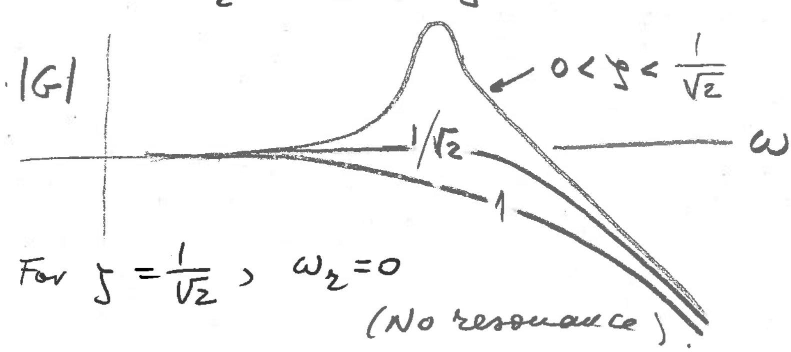
$$\frac{d}{dp}\left[(1-p^2)^2 + (2\zeta p)^2\right] = 0.$$

$$2(-2p)(1-p^2) + 2(2\zeta)(2\zeta p) = 0.$$

$$(1-p^2)\cancel{p} = 2\zeta^2\cancel{p}$$

$$p^2 = 1-2\zeta^2 \longrightarrow p_r = \sqrt{1-2\zeta^2}$$

$$\omega_r = \omega_n\sqrt{1-2\zeta^2}$$



For $\zeta = \frac{1}{\sqrt{2}}$, $\omega_r = 0$
(No resonance)

Peak exist only for $0<\zeta<\frac{1}{\sqrt{2}}$

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Amplitude at resonance

$$M_r^2 = |G(i\omega_r)|^2 = \frac{1}{[1-(1-2\zeta^2)]^2 + (2\zeta)^2(1-2\zeta^2)}$$

$$= \frac{1}{(\cancel{1}-\cancel{1}+2\zeta^2)^2 + 4\zeta^2(1-2\zeta^2)}$$

$$= \frac{1}{4\zeta^4 + 4\zeta^2 - 8\zeta^4} = \frac{1}{4\zeta^2(1-\zeta^2)}$$

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

$\zeta = \frac{1}{\sqrt{2}}$

$M_r = \frac{1}{2\frac{1}{\sqrt{2}}\sqrt{1-\frac{1}{2}}} = 1$

Phase at resonance, φ_r

$$G(i\omega) = \frac{\omega_n^2}{(i\omega)^2 + 2i\zeta\omega\omega_n + \omega_n^2} = \frac{1}{(1-p^2) + 2i\zeta p}$$

$$p_r = \sqrt{1-2\zeta^2}, \quad 1-p_r^2 = 1-(1-2\zeta^2) = 2\zeta^2$$

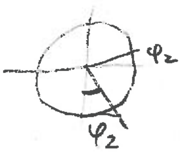
$$G(i\omega_r) = \frac{1}{2\zeta^2 + 2i\zeta\sqrt{1-2\zeta^2}}$$

$$\varphi_r = \angle G(i\omega_r) = -\tan^{-1}\frac{\sqrt{1-2\zeta^2}}{\zeta}$$

$$\varphi_r = -90^\circ + \underbrace{\sin^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}}}_{=\varphi_2} = -90^\circ + \varphi_2$$

Proof

$\tan(-90^\circ + \varphi_2) = -\operatorname{cotan}\varphi_2$



$$\sin\varphi_2 = \frac{\zeta}{\sqrt{1-\zeta^2}}$$

$$\cos^2\varphi_2 = 1 - \sin^2\varphi_1 = \frac{1-2\zeta^2}{1-\zeta^2}$$

$$\operatorname{cotan}\varphi_2 = \frac{\cos\varphi_2}{\sin\varphi_2} = \frac{\sqrt{1-2\zeta^2}/\cancel{\sqrt{1-\zeta^2}}}{\zeta/\cancel{\sqrt{1-\zeta^2}}} = -\tan\varphi_r$$

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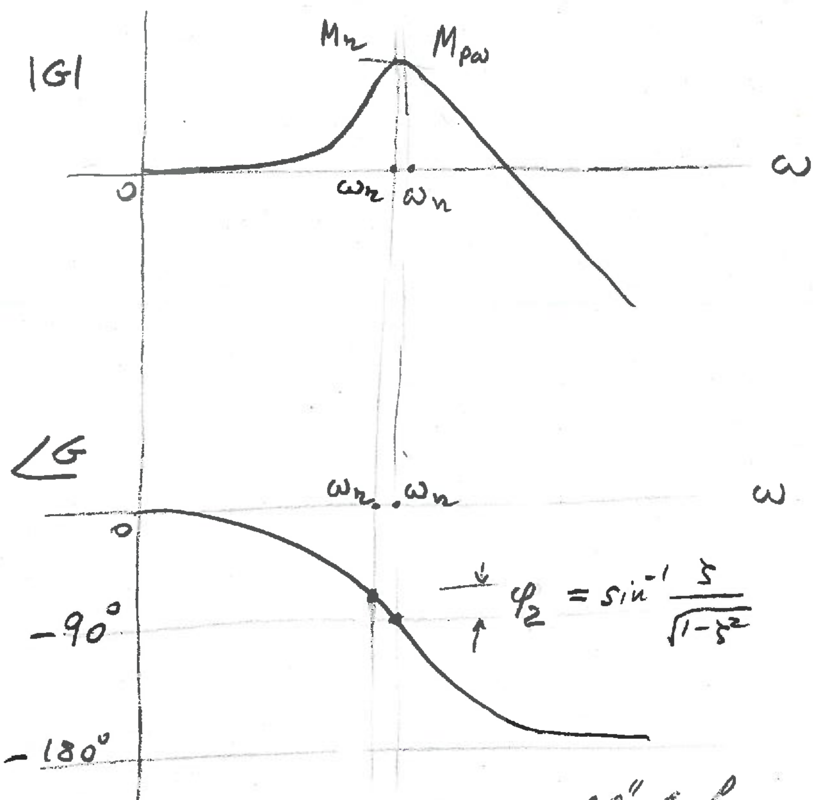

Resonance happens "slightly" before ω_n

Two definitions of "resonance":

(1) 90° phase @ ω_n, $M_{p\omega} = \frac{1}{2\zeta}$

(phase resonance)

(2) peak value @ $\omega_r = \omega_n\sqrt{1-2\zeta^2}$

$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$ "RESONANCE"

"
PIF

PERFORMANCE INDICATORS vs. ζ

ζ	0.01	0.1	0.4	$1/\sqrt{2}$	0.8	0.99	1
$M_{p\omega} = \frac{1}{2\zeta}$	50	5	1.25	0.7	0.625	0.505	0.5
$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$	50	5.025	1.36	1	—	—	—
$1+M_p = 1+e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$	1.97	1.73	1.25	1.04	1.02	1.00	1
$\frac{\omega_r}{\omega_n} = \sqrt{1-2\zeta^2}$	1	0.99	0.825	0	—	—	—
$\frac{\omega_d}{\omega_n} = \sqrt{1-\zeta^2}$	1	0.995	0.917	$1/\sqrt{2}$	0.6	0.141	0
$\frac{t_s}{\tau_n} = \frac{2}{\pi\zeta}$	63	6.37	1.59	0.9	0.8	0.64	0.64
$\varphi_r = -\tan^{-1}\frac{\sqrt{1-2\zeta^2}}{\zeta}$	$-89°$	$-84°$	$-64°$	$0°$	—	—	—

$$t_s = \frac{4}{\zeta\omega_n}$$

$$\tau_n = \frac{1}{f_n} = \frac{2\pi}{\omega_n} \quad \text{(osc. period)}$$

$$\frac{t_s}{\tau_n} = \frac{4}{\zeta}\,\frac{1}{2\pi} = \frac{2}{\pi\zeta}$$

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Specific PIs vs ζ — 2nd order system

```
performance indicators comparison, 2nd order sys
z =
   0.0100    0.1000    0.4000    0.7071    0.8000    0.9900    1.0000
Mpw =
  50.0000    5.0000    1.2500    0.7071    0.6250    0.5051    0.5000
Mr =
  50.0025    5.0252    1.3639    1.0000         0         0       NaN
1+Mp =
   1.9691    1.7292    1.2538    1.0432    1.0152    1.0000    1.0000
wr/wn =
   0.9999    0.9899    0.8246    0.0000         0         0         0
wd_wn =
   0.9999    0.9950    0.9165    0.7071    0.6000    0.1411         0
ts_tn =
  63.6620    6.3662    1.5915    0.9003    0.7958    0.6431    0.6366
phi_r =
 -89.4270  -84.2318  -64.1233    0.0000         0         0         0
```

$$M_{p\omega} = \frac{1}{2\zeta}$$ magnitude at phase res.

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$ resonance peak

$$1+M_p = 1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$ step response peak

$$\frac{\omega_r}{\omega_n} = \sqrt{1-2\zeta^2}$$ resonance freq. ratio

$$\frac{\omega_d}{\omega_n} = \sqrt{1-\zeta^2}$$ damped freq / natural freq

$$t_s/\tau_n = \frac{2}{\pi\zeta}$$ settling time / osc. period

$$\varphi_r = -\tan^{-1}\frac{\sqrt{1-2\zeta^2}}{\zeta}$$ phase at resonance