# 5 Performance Indicators

Performance indicators are used to judge the quality of a control system.

## 5.1 1st-order System Generic Performance Indicators

Generic performance indicators are:

- steady-state value $x_{ss} = \lim_{t\to\infty} x(t)$
- steady-state error $e_{ss} = \lim_{t\to\infty} e(t)$ where $e(t) = f(t) - x(t)$

To expand on the error definition, $e(t) = f(t) - x(t)$, consider a step response where $x(\infty) = 1$ and $f(\infty) = 1$, therefore $e(\infty) = 0$.

For a 1st-order system, $x_{ss}$ and $e_{ss}$ can be found using the final value theorem and exist in both the time domain and the s-domain. Recall the final value theorem for $x_{ss}$ and $e_{ss}$ leads to

$$x_{ss} = \lim_{s\to 0} sX(s) \tag{1}$$

$$e_{ss} = \lim_{s\to 0} sE(s) \tag{2}$$

Therefore, starting at the transfer function of a 1st-order system

$$G(s) = \frac{1}{Ts+1} \tag{3}$$

we can expand on this to show

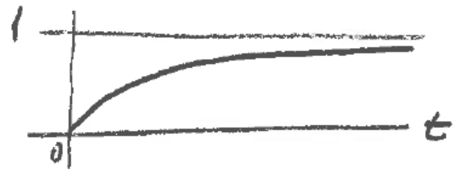
$$\begin{aligned} X(s) &= G(s)F(s) \qquad (4)\\ &= \frac{1}{Ts+1}F(s) \end{aligned}$$

next, the error is s-domain is shown to be

$$\begin{aligned} E(s) &= F(s) - X(s) \qquad (5)\\ &= F(s) - G(s)F(s) \qquad (6)\\ &= \big(1 - G(s)\big)F(s)\\ &= \frac{Ts}{Ts+1}F(s) \end{aligned}$$

Again, these are general terms for a 1st-order system.

### 5.1.1 Step response 1st-order system performance indicators




For a 1st-order system subjected to a step response, we want to find $x_{ss}$ and $e_{ss}$ and we know the transfer function is defined as

$$G(s) = \frac{1}{Ts+1} \tag{7}$$

while the equation of motion is

$$T\dot{x} + x = F(s) \tag{8}$$

Where the step function is defined as

$$f(t) = 1,\ t > 0 \tag{9}$$

therefore, the system response is

$$x(t) = 1 - e^{-t/T} \tag{10}$$

while the steady-state response is

$$\begin{aligned} x_{ss} &= \lim_{t\to\infty} \\ &= 1 \end{aligned} \tag{11}$$

Next, the error as a function of time is

$$\begin{aligned} e(t) &= 1 - (1 - e^{-t/T}) \\ &= e^{-t/T} \end{aligned} \tag{12}$$

while the steady-state error is

$$\begin{aligned} e_{ss} &= \lim_{t\to\infty} e(t) \\ &= \lim_{t\to\infty} e^{t/T} \\ &= 0 \end{aligned} \tag{13}$$

These same solutions can be found in the s-domain where the step function is $F(s) = \frac{1}{s}$. Consider the s-domain expression solved for $X(s)$,

$$X(s) = \frac{1}{Tx+1} \cdot \frac{1}{s} \tag{14}$$

the steady-state error is shown to be

$$\begin{aligned} x_{ss} &= \lim_{s\to 0} s\frac{1}{Ts+1}\cdot\frac{1}{s} \\ &= \lim_{s\to 0}\frac{1}{Ts+1} \\ &= 1 \end{aligned} \tag{15}$$

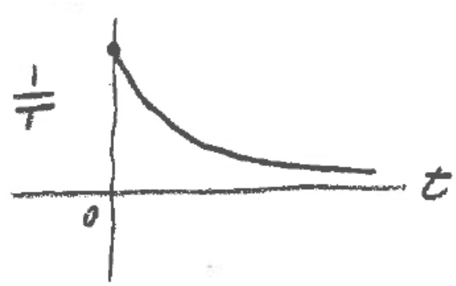
Next, we can build the s-domain representation of the error as

$$\begin{aligned} E(s) &= \frac{Ts}{Ts+1}\cdot\frac{1}{s} \\ &= \frac{T}{Ts+1} \end{aligned} \tag{16}$$

solving for the steady-state error results in

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} s\frac{T}{Ts+1} \\ &= 0 \end{aligned} \tag{17}$$

### 5.1.2 Impulse response 1st-order system performance indicators




For a 1st-order system subjected to a impulse response, we want to find $x_{ss}$ and $e_{ss}$. The impulse function is defined as

$$f(t) = \delta(t) \tag{18}$$

therefore, the response is

$$x(t) = \frac{1}{T}e^{-t/T} \tag{19}$$

where the steady-state response is

$$x_{ss} = 0 \tag{20}$$

The error is

$$e(t) = \delta(t) - \frac{1}{T}e^{-t/T} \tag{21}$$

Lastly, the steady-state error is

$$\begin{aligned} e_{ss} &= \lim_{t\to\infty} e(t) \\ &= 0 \end{aligned} \tag{22}$$

These same solutions can be found in the s-domain where the impulse function is $F(s) = 1$. Consider the s-domain expression solved for $X(s)$,

$$X(s) = \frac{1}{Ts+1} \tag{23}$$

the steady-state error is shown to be

$$\begin{aligned} x_{ss} &= \lim_{s \to 0} s \frac{1}{Ts+1} \\ &= 0 \end{aligned} \tag{24}$$

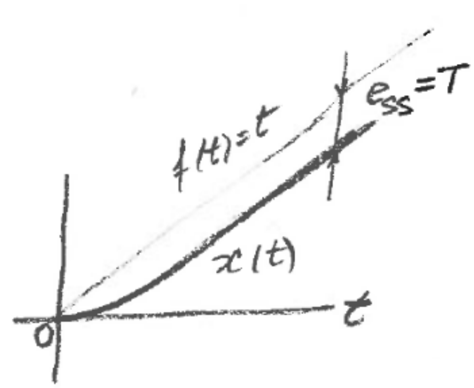
The s-domain error is expressed as

$$E(s) = \frac{T}{Ts+1} \tag{25}$$

which leads to the steady-state error value

$$\begin{aligned} e_{ss} &= \lim_{s \to 0} s \frac{T}{Ts+1} \\ &= 0 \end{aligned} \tag{26}$$

### 5.1.3 Ramp response 1st-order system performance indicators




For a 1st-order system subjected to a ramp response, we want to find $x_{ss}$ and $e_{ss}$. The impulse function is defined as

$$f(t) = t \tag{27}$$

therefore, the response is

$$x(t) = t - T(1 - e^{-t/T}) \tag{28}$$

which leads to the steady-state response

$$x_{ss} = \infty \tag{29}$$

which means that there is no steady-state value. The error is

$$e(t) = T(1 - e^{-t/T}) \tag{30}$$

Lastly, the steady-state error is

$$\begin{aligned} e_{ss} &= \lim_{t\to\infty} e(t) \\ &= T - \lim_{t\to\infty} e^{-t/T} \\ &= T \end{aligned} \tag{31}$$

as $\lim_{t\to\infty} e^{-t/T} = 0$.

These same solutions can be found in the s-domain where the ramp function is $F(s) = \frac{1}{s^2}$. Consider the s-domain expression solved for $X(s)$,

$$X(s) = \frac{1}{Ts+1} \cdot \frac{1}{s^2} \tag{32}$$

the steady-state error is shown to be

$$\begin{aligned} x_{ss} &= \lim_{s\to 0} s \frac{1}{Ts+1} \cdot \frac{1}{s^2} \\ &= \lim_{s\to 0} \frac{1}{Ts+1} \cdot \frac{1}{s} \\ &= \infty \end{aligned} \tag{33}$$

therefore, there is no steady-state value. The s-domain error is expressed as

$$\begin{aligned} E(s) &= \frac{Ts}{Ts+1} \cdot \frac{1}{s^2} \\ &= \frac{T}{Ts+1} \cdot \frac{1}{s} \end{aligned} \tag{34}$$

which leads to the steady-state error value

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} s \frac{T}{Ts+1} \cdot \frac{1}{s} \\ &= \lim_{s\to 0} \frac{T}{Ts+1} \\ &= T \end{aligned} \tag{35}$$

## 5.2 1st-order System Specific Performance Indicators

Specific performance indicators exist. They depend on system order and excitation type. Examples are:

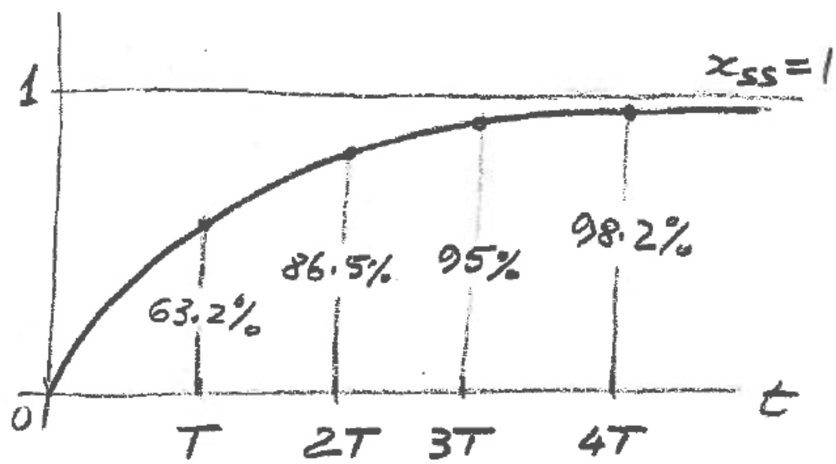
- rise-time $\rightarrow t_r$
- delay time $\rightarrow t_d$
- settling time $\rightarrow t_s$

- decay time / half-time $\rightarrow t_{1/2}$

For a step response, consider the system displacement

$$x(t) = 1 - e^{-t/T} \tag{36}$$

that is plotted as




The rise-time ($t_r$) of a system

- to rise 63.2% of $x_s s$ is $t_{r\_63.2\%} = T$
- to rise 86.5% of $x_s s$ is $t_{r\_86.5\%} = 2T$
- to rise 95% of $x_s s$ is $t_{r\_95\%} = 3T$
- to rise 98.2% of $x_s s$ is $t_{r\_98.2\%} = 4T$

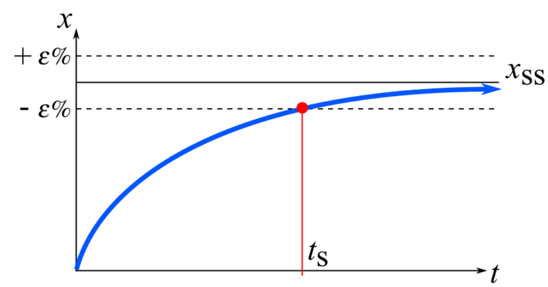
in general, $1 - e^{-t/T} = x \rightarrow t = -T\ln(1-x)$
The delay time ($t_d$) is the time it takes to rise to 50% of $x_{ss}$,

$$\begin{aligned} x(t) &= 1 - e^{-t/T} \\ &= 0.5 \end{aligned} \tag{37}$$

therefore, solving for $t = t_d$,

$$\begin{aligned} e^{-t_d/T} &= 0.5 \\ \frac{-t_d}{T} &= \ln(0.5) \\ t_d &= -T\ln(0.5) \\ t_d &= 0.693T \\ t_d &\approx 0.7T \end{aligned} \tag{38}$$

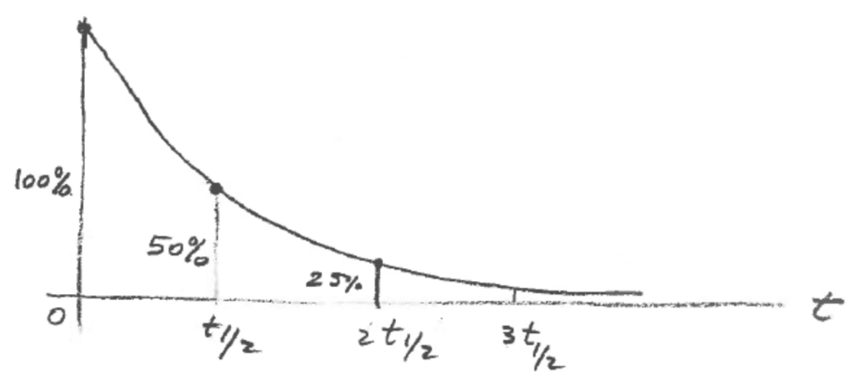
The settling time ($t_s$) is the time its takes $x$ to get within $\varepsilon\%$ of $x_{ss}$.




A more precise estimator is the rise time to any selected $x$ which can be defined as

$$\begin{aligned} x_{ss}(1-e^{-t/T}) &= x \\ e^{(-t/T)} &= 1-\frac{x}{x_{ss}} \\ -t/T &= \log\left[1-\frac{x}{x_{ss}}\right] \\ t &= -T\log\left[1-\frac{x}{x_{ss}}\right] \end{aligned} \tag{39}$$

### 5.2.1 Impulse Response Specific Performance Indicators




The half-cycle decay time ($t_{1/2}$) is an impulse response specific performance indicator. Given that the system response to an impulse is

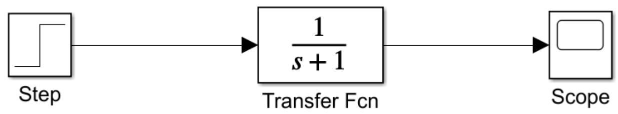
$$x(t) = \frac{1}{T}e^{-t/T} \tag{40}$$

the half-life decay time is where $e^{-t/T} = \frac{1}{2}$, therefore

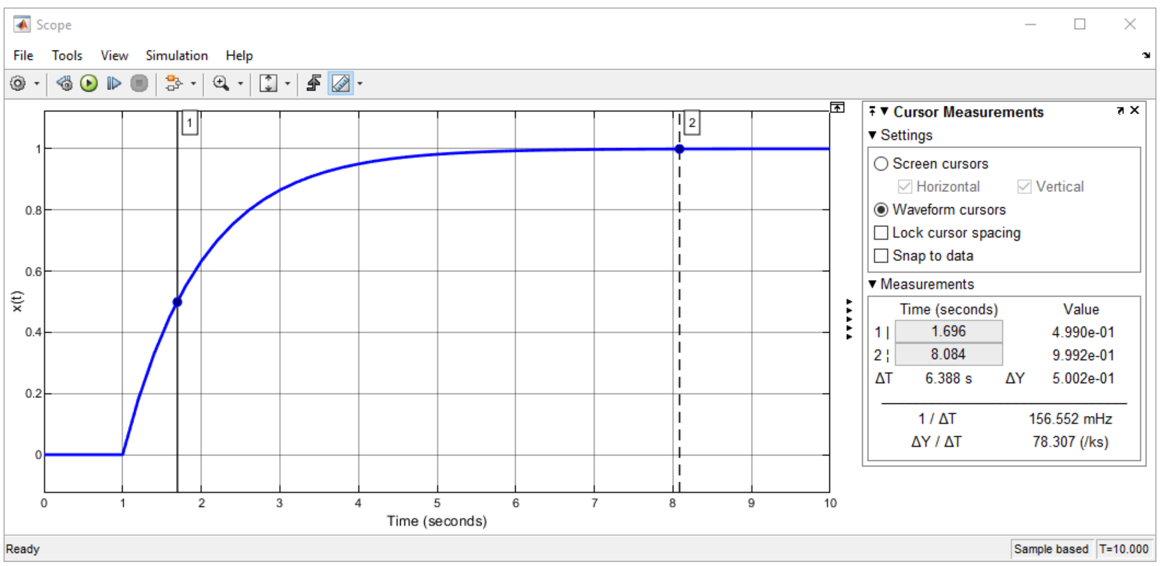
$$\begin{aligned} t_{1/2} &= -\ln\left(\frac{1}{2}\right)T \\ &\approxeq 0.693T \end{aligned} \tag{41}$$

Note that the signal continues to decay by half of its value after each additional $t_{1/2}$.

**Example 5.1 SIMULINK Tutorial on Measuring Performance Indicators**
Build the simple 1st-order system as shown below




Open the scope and press the 'Cursor Measurements' button to activate the cursors.




To measure the delay time $t_d$, place the first cursor around the point where 'Value' measurement is closest to 0.5. $x(t) = 0.5$. Read the 'Time' value. This is estimate for $t_d$ . It gives the value $t_d = 0.70$ sec, as the step function happens at 1 sec.

The second cursor can be used to get the settling time $t_s$ . We are going to use the 1% definition of $t_d$ . This means that the response should be around 0.99, or 9.9e-1. Reading the corresponding time value, we get $t_s \approx 7.0$ sec.

## 5.3 2nd-order System Generic Performance Indicators

Again, starting at the equation of motion

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 f(t) \tag{42}$$

and transfer function of a 2nd-order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{43}$$

we can expand on this to show

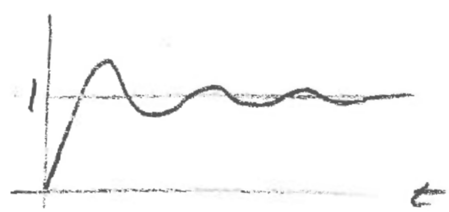
$$\begin{aligned} X(s) &= G(s)F(s) \\ &= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} F(s) \end{aligned} \tag{44}$$

next, the error in the s-domain is shown to be

$$\begin{aligned} E(s) &= F(s) - X(s) \\ &= F(s) - G(s)F(s) \\ &= \big(1 - G(s)\big)F(s) \\ &= \frac{s^2 + 2\zeta\omega_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2} F(s) \end{aligned} \tag{45}$$

Again, these are general terms for a $2^{nd}$-order system. Note that the definitions for the steady-state value $x_{ss}$ and steady-state error $e_{ss}$ remain largely unchanged from the $1^{st}$-order system.

### 5.3.1 Step response $2^{nd}$-order system performance indicators




For a $2^{nd}$-order system subjected to a step response, we want to find $x_{ss}$ and $e_{ss}$. Using the transfer function method to solve for the response, we know the system response is

$$x(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) \tag{46}$$

where

$$\omega_d = \omega_n \sqrt{1-\zeta^2} \tag{47}$$

and

$$\phi = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta} = \sin^{-1} \sqrt{1-\zeta^2} \tag{48}$$

next, the steady-state displacement can be solved for as

$$\begin{aligned} x_{ss} &= \lim_{t\to\infty} x(t) \\ &= 1 - \frac{\sqrt{1-\zeta^2}}{\zeta} \lim_{t\to\infty} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) \\ &= 1 \end{aligned} \tag{49}$$

as $e^{-\zeta\omega_n t}$ goes to 0. Next, the error as a function of time is

$$\begin{aligned} e(t) &= f(t) - x(t) \\ &= \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) \end{aligned} \tag{50}$$

while the steady-state error is

$$\begin{aligned} e_{ss} &= \lim_{t\to\infty} e(t) \\ &= \frac{\sqrt{1-\zeta^2}}{\zeta} \lim_{t\to\infty} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) \\ &= 0 \end{aligned} \tag{51}$$

as $e^{-\zeta\omega_n t}$ goes to 0.

These same solutions can be found in the s-domain where the step function is $F(s) = \frac{1}{s}$. Consider the s-domain expression solved for $X(s)$,

$$X(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s} \tag{52}$$

the steady-state displacement is shown to be

$$\begin{aligned} x_{ss} &= \lim_{s\to 0} s \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s} \\ &= \frac{\omega_n^2}{\omega_n^2} \\ &= 1 \end{aligned} \tag{53}$$

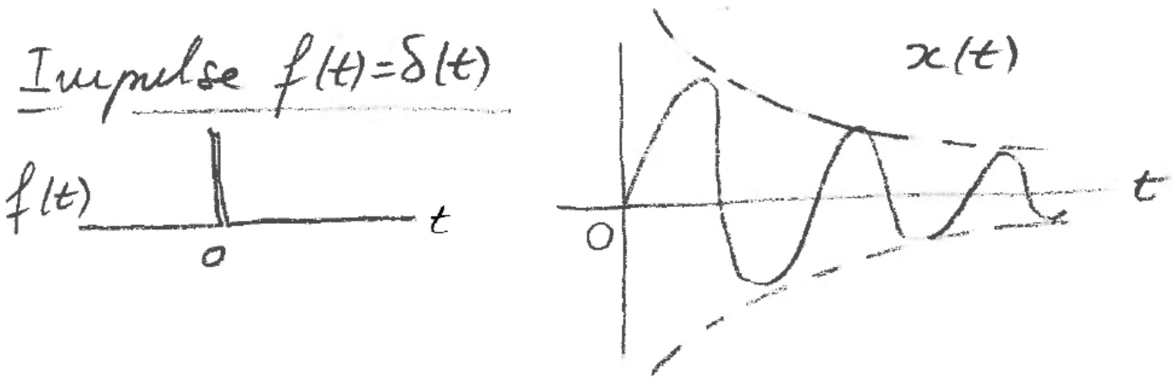
Next, we can build the s-domain representation of the error as

$$E(s) = \frac{s^2 + 2\zeta\omega_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s} \tag{54}$$

solving for the steady-state error results in

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} s \frac{s^2 + 2\zeta\omega_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s} \\ &= \frac{0}{\omega_n} \\ &= 0 \end{aligned} \tag{55}$$

### 5.3.2 Impulse response 2nd-order system performance indicators




For a 2nd-order system subjected to a impulse response, we want to find $x_{ss}$ and $e_{ss}$. The impulse function is defined as

$$f(t) = \delta(t) \tag{56}$$

therefore, the response is

$$x(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_d t) \tag{57}$$

where the steady-state response is

$$\begin{aligned} x_{ss} &= \lim_{t\to\infty} x(t) \\ &= \frac{\omega_n}{\sqrt{1-\zeta^2}} \lim_{t\to\infty} e^{-\zeta\omega_n t}\sin(\omega_d t+\phi) \\ &= 0 \end{aligned} \tag{58}$$

as $e^{-\zeta\omega_n t}$ goes to 0. Next, the error as a function of time is

$$\begin{aligned} e(t) &= f(t) - x(t) \\ &= \delta(t) - \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_d t+\phi) \end{aligned} \tag{59}$$

Lastly, the steady-state error is

$$\begin{aligned} e_{ss} &= \lim_{t\to\infty} e(t) \\ &= \lim_{t\to\infty} \delta(t) - \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_d t+\phi) \\ &= 0 \end{aligned} \tag{60}$$

as $\delta(t)$ and $e^{-\zeta\omega_n t}$ both go to 0.

These same solutions can be found in the s-domain where the impulse function is $F(s) = 1$. Consider the s-domain expression solved for $X(s)$,

$$X(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{61}$$

the steady-state error is shown to be

$$x_{ss} = \lim_{s\to 0} s\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \\ = 0 \tag{62}$$

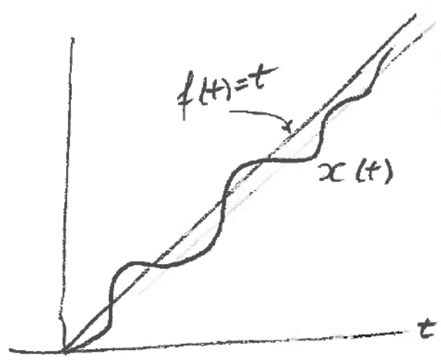
The s-domain error is expressed as

$$E(s) = \frac{s^2+2\zeta\omega_n s}{s^2+2\zeta\omega_n s+\omega_n^2} \tag{63}$$

which leads to the steady-state error value

$$e_{ss} = \lim_{s\to 0} s\frac{s^2+2\zeta\omega_n s}{s^2+2\zeta\omega_n s+\omega_n^2} \\ = 0 \tag{64}$$

### 5.3.3 Ramp response 2nd-order system performance indicators




For a 2nd-order system subjected to a ramp response, we want to find $x_{ss}$ and $e_{ss}$. The impulse function is defined as

$$f(t) = t \tag{65}$$

therefore, the response can be found through the transfer function approach as

$$x(t) = t - \frac{2\zeta}{\omega_n}\left[1+\frac{1}{\sin\phi_1}e^{-\zeta\omega_n t}\sin(\omega_d t+\phi_1)\right] \tag{66}$$

where

$$\phi_1 = \tan^{-1}\frac{2\zeta\sqrt{1-\zeta^2}}{1-2\zeta^2} \tag{67}$$

which leads to the response

$$x_{ss} = \lim_{t\to\infty} x(t) \\ = t - \frac{2\zeta}{\omega_n} \\ = \infty \tag{68}$$

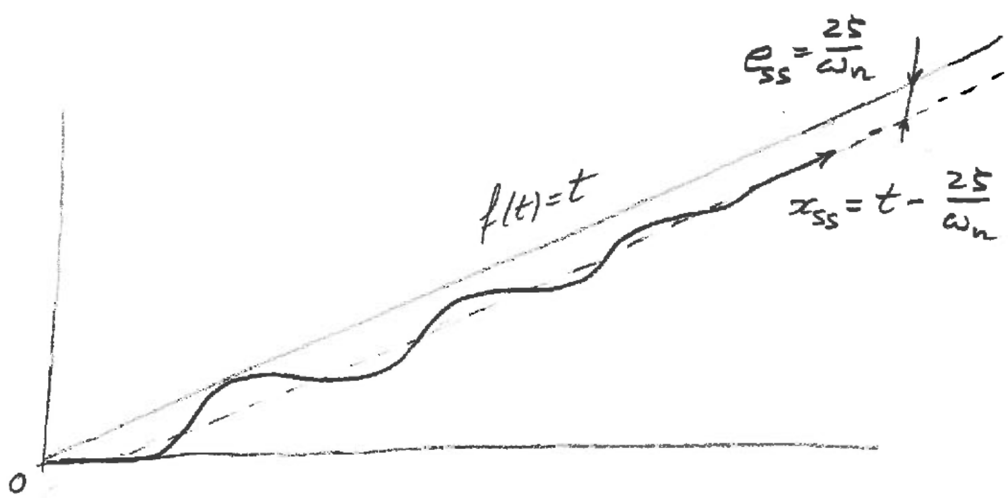
which means that there is no steady-state value. Next, the error as a function of time is

$$
\begin{aligned}
e(t) &= f(t) - x(t) \\
&= t - \left[ t - \frac{2\zeta}{\omega_n} \left[ 1 + \frac{1}{\sin\phi_1} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi_1) \right] \right] \\
&= \frac{2\zeta}{\omega_n} \left[ 1 + \frac{1}{\sin\phi_1} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi_1) \right]
\end{aligned}
\tag{69}
$$

Lastly, the steady-state error is

$$
\begin{aligned}
e_{ss} &= \lim_{t\to\infty} e(t) \\
&= \frac{2\zeta}{\omega_n} \lim_{t\to\infty} \left[ 1 + \frac{1}{\sin\phi_1} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi_1) \right] \\
&= \frac{2\zeta}{\omega_n}
\end{aligned}
\tag{70}
$$

as the term in the brackets goes to 1.




These same solutions can be found in the s-domain where the ramp function is $F(s) = \frac{1}{s^2}$. Consider the s-domain expression solved for $X(s)$,

$$
X(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s^2} \tag{71}
$$

the steady-state error is shown to be

$$\begin{aligned} x_{ss} &= \lim_{s\to 0} s\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\cdot\frac{1}{s^2} \\ &= \lim_{s\to 0}\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\cdot\frac{1}{s} \\ &= \lim_{s\to 0}\frac{1}{s} \\ &= \infty \end{aligned} \tag{72}$$

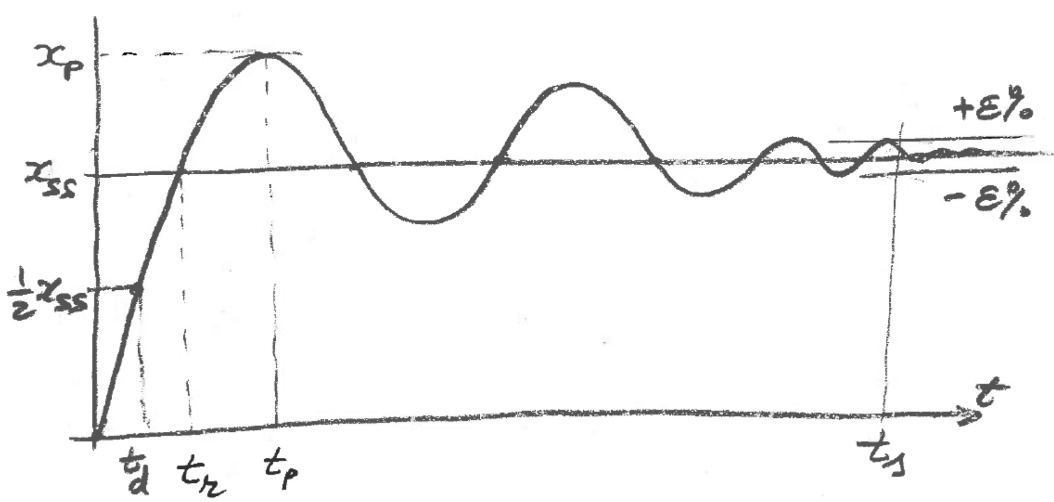
therefore, there is no steady-state value. The s-domain error is expressed as

$$E(s) = \frac{s^2+2\zeta\omega_n s}{s^2+2\zeta\omega_n s+\omega_n^2}\cdot\frac{1}{s^2} \tag{73}$$

which leads to the steady-state error value

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} s\frac{s^2+2\zeta\omega_n s}{s^2+2\zeta\omega_n s+\omega_n^2}\cdot\frac{1}{s^2} \\ &= \lim_{s\to 0}\frac{s^2(s+2\zeta\omega_n)}{s^2+2\zeta\omega_n s+\omega_n^2}\cdot\frac{1}{s^2} \\ &= \frac{2\zeta\omega_n}{\omega_n^2} \\ &= \frac{2\zeta}{\omega_n} \end{aligned} \tag{74}$$

## 5.4 2^nd^-order System Specific Performance Indicators




Specific performance indicators exist. They depend on system order and excitation type. Examples are:

- rise time $\rightarrow t_r$
- peak time $\rightarrow t_p$
- peak value $\rightarrow x_p$
- settling time $\rightarrow t_s$
- delay time $\rightarrow t_d$
- max percentage overshoot $\rightarrow M_p$

Many of these are the same as a 1st-order system or taken directly from the system response. The max percentage overshoot ($M_p$) is defined as

$$M_p = \left(\frac{x_p}{x_{ss}} - 1\right) \cdot 100 \tag{75}$$

### 5.4.1 Procedure for a Step Response

Given the 2nd-order system, with the transfer function

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{76}$$

where $\omega_n$ is the natural frequency and $\zeta$ is the critical damping ratio. Considering that the system is subjected to a step response, we know the system response is

$$x(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) \tag{77}$$

where we can calculate:

$$\omega_d = \omega_n \sqrt{1-\zeta^2} \tag{78}$$

and

$$\phi = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta} = \sin^{-1} \sqrt{1-\zeta^2} \tag{79}$$

Next, compute the values for the performance indicators.

First the rise time is calculated by the fact that $x(0) = 0$ and $x(t_r) = 1$. Form equation 77, the mean height of the system is shown to be when $\sin(\omega_d t + \phi) = 0$, therefore, as $\sin(\pi) = 0$, we can show that $\omega_d t + \phi = \pi$, rearranging this yields

$$t_r = \frac{\pi - \phi}{\omega_d} \tag{80}$$

To find the peak time ($t_p$), we need to find the peak, which is defined as $\frac{dx}{dt}\big|_{t=t_p} = 0$, or drawn as


where we consider only the decaying part of the signal caused by the step function, or $x = e^{-\zeta\omega_n t}\sin(\omega_d t+\phi)$. Therefore

$$\begin{aligned}0 &= \frac{dx}{dt} \\ &= \frac{d}{dt}\left[e^{-\zeta\omega_n t}\sin(\omega_d t+\phi)\right] \\ &= -\zeta\omega_n e^{-\zeta\omega_n t}\sin(\omega_d t+\phi)+\omega_d e^{-\zeta\omega_n t}\cos(\omega_d t+\phi) \\ \zeta\omega_n\sin(\omega_d t+\phi) &= \omega_d\cos(\omega_d t+\phi) \\ \frac{\sin(\omega_d t+\phi)}{\cos(\omega_d t+\phi)} &= \frac{\omega_d}{\zeta\omega_n}\end{aligned} \tag{81}$$

By converting the left has side of this equation to $\tan(\omega_d t_p+\phi)$ yields

$$\begin{aligned}\tan(\omega_d t_p+\phi) &= \frac{\omega_d}{\zeta\omega_n} \\ &= \frac{\sqrt{1-\zeta^2}}{\zeta} \\ &= \tan(\phi)\end{aligned} \tag{82}$$

when considering the definition of tan provided by equation 79. We must find $t_p$ such that $\tan(\omega_d t_p+\phi) = \tan(\phi)$. Given that the function $\tan(\alpha)$ repeats itself after $\pi$, $2\pi$, $3\pi$, $\cdots$ as shown in figure 5.1, we know $\tan(\alpha) = \tan(\alpha+\pi) = \tan(\alpha+2\pi), \cdots$.

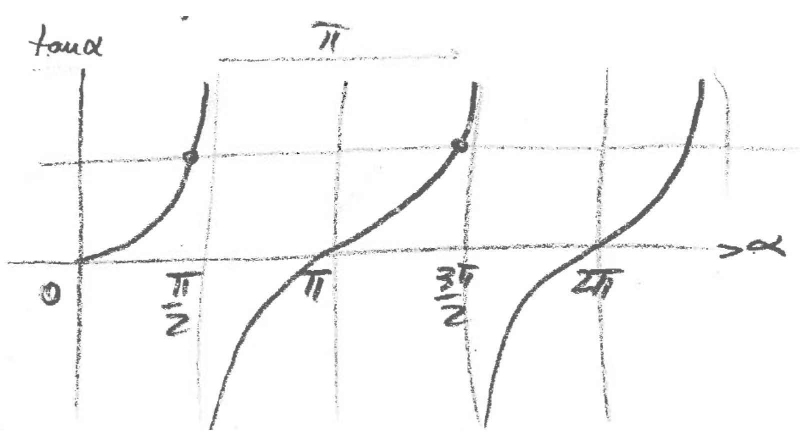



Figure 5.1: Plotting the $\tan(\alpha)$

Hence, picking back up from equation 82 lead to

$$\begin{aligned}\omega_d t_p+\phi &= \phi+\pi \\ \omega_d t_p &= \pi\end{aligned} \tag{83}$$

which results in

$$t_p = \frac{\pi}{\omega_d} \tag{84}$$

The peak value ($x_p$) can be found as

$$\begin{aligned} x_p &= x(t_p) \\ &= 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta \omega_n \frac{\pi}{\omega_d}} \sin\left(\omega_d ({}^{\pi}/{}_{\omega_d}) + \phi\right) \\ &= 1 + \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta \omega_n \frac{\pi}{\omega_d}} \sin(\phi) \\ &= 1 + \frac{1}{\sqrt{1-\zeta^2}} \left(e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}\right) \sqrt{1-\zeta^2} \\ &= 1 + e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi} \end{aligned} \tag{85}$$

when considering that $\sin(\phi+\pi) = -\sin(\phi)$ and $\sin(\phi) = \sqrt{(1-\zeta^2)}$.

The max overshoot ($M_p$) can be found as

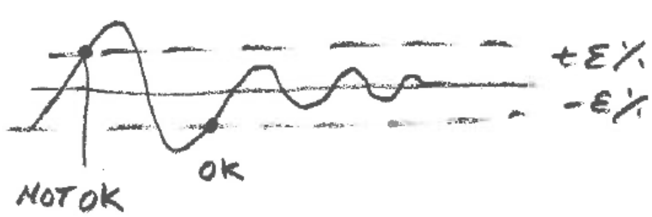
$$\begin{aligned} M_p &= \frac{x_p - x_{ss}}{x_{ss}} \\ &= \frac{1 + e^{\frac{\zeta}{\sqrt{1-\zeta^2}}\pi} - 1}{1} \\ &= e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi} \end{aligned} \tag{86}$$

for a few select damping ratios, the overshoot percentages are shown in Table 1. However, the typical range of damping is $0.4 < \zeta < 0.8$ and therefore the typical max overshoot is $0.25\% < M_p < 1.5\%$.

Table 1: Overshoot percentages for select damping ratios.

| $\zeta$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1 |
|---|---|---|---|---|---|---|
| $M_p$ | 100.00% | 52.68% | 25.40% | 9.49% | 1.52% | 0.00% |

The definition of settling time ($t_s$) is to “get withing $\pm\varepsilon\%$ of $x_{ss}$ and stay so”.




The settling time is defined as the time when both

$$|x_{ss} - x(t_s)| < \Delta \tag{87}$$

and

$$|x_{ss} - x(t > t_s)| < \Delta \tag{88}$$

are true where $\Delta = \varepsilon \cdot x_{ss}$. This is shown in figure 5.2.

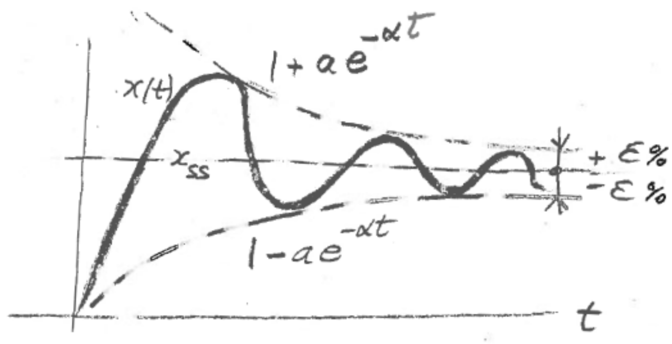



Figure 5.2: Settling time for a 2nd-order system.

We can find a generalized expression for $t_s$ if we consider the system response for $x_{ss} = 1$ as

$$x(t) = 1 - \left[\frac{1}{\sqrt{1-\zeta^2}}\right] e^{-[\zeta\omega_n]t} \sin(\omega_d t + \phi) \tag{89}$$

redefining the items in the brackets as $a$ and $\alpha$, respectively, the system response can be simplified to

$$x(t) = 1 - ae^{-\alpha t} \sin(\omega_d t + \phi) \tag{90}$$

Therefore, the envelop of the system response is $1 \pm ae^{-\alpha t}$ while the settling condition is $ae^{-\alpha t} = \Delta$. Considering that the final peak aboove the error range will happen when $a \approxeq 1$, we make the approximate calculation

$$a = \left.\frac{1}{\sqrt{1-\zeta^2}}\right|_{\zeta<<1} \approxeq 1 \tag{91}$$

and considering that $t_s$ is when the system decays under the value $\varepsilon$, we need to find the $t$ value when

$$e^{-\alpha t} = \varepsilon \tag{92}$$

therefore, setting $\varepsilon = 2\%$, we can find

$$e^{-\alpha t} = 0.02 \tag{93}$$

$$\begin{aligned} -\alpha t &= \log(0.02) \\ &= -3.9 \\ &\approxeq -4 \end{aligned} \tag{94}$$

Therefore, knowing that $\alpha = \zeta\omega_n$, we can deduce

$$\begin{aligned} -\alpha t_s &\approxeq -4 \\ -\zeta\omega_n t_s &\approxeq -4 \\ t_s &\approxeq \frac{4}{\zeta\omega_n} \end{aligned} \tag{95}$$

when $\zeta << 1$ this simplified expression is within $\pm 2\%$.

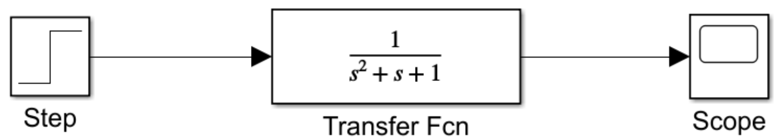
Importantly, one should always consider the effects of $\zeta$ and $\omega_n$ on performance.

- An increase in $\omega_n$ shortens rise time ($t_r$), peak time ($t_p$), and settling time ($t_s$).
- An increase in $\zeta$ reduces max overshoot ($M_p$) and shortens settling time ($t_s$).

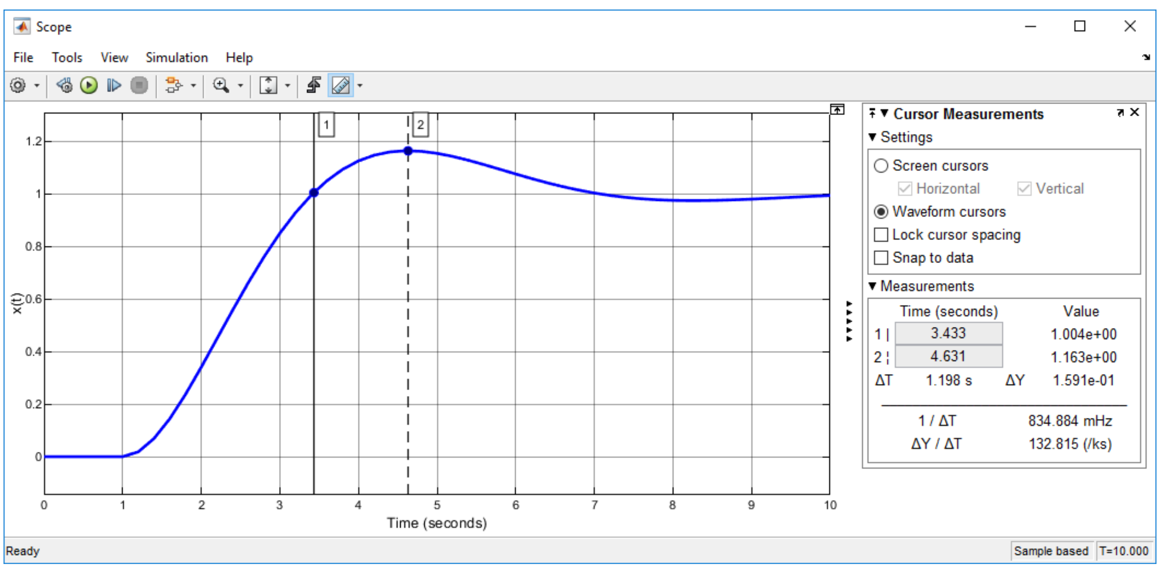
As before, the definition of time delay ($t_d$) for a 2nd-order system is the time it takes to rise to 50% of $x_{ss}$ the first time. Similarly, the max percentage overshoot ($M_p$) for a 2nd-order system is calculated in the same way as a 1st-order system.

**Example 5.2 SIMULINK Tutorial on Measuring Performance Indicators**

Build the simple 2nd-order system as shown below




Open the scope and press the 'Cursor Measurements' button to activate the cursors.




Use the first cursor to find the first crossing of $x_{ss} = 1$ . Read the time as $t_r = 2.433$ sec as the step function starts at 1 sec. Place the second cursor at peak value. Read the peak time $t_r = 3.631$ sec and peak amplitude $x_p = 1.163$ . Calculate $M_p = 16.3\%$.