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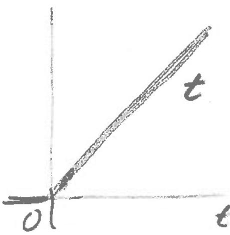
2. Step function $1(t)$

$1(t) \xrightarrow{LT} \frac{1}{s}$

$$\boxed{\begin{aligned} f(t) &= 1(t) \\ F(s) &= \frac{1}{s} \end{aligned}}$$

Proof: $\mathcal{L}\,1(t) = \int_0^\infty 1\cdot e^{-st}\,dt = \int_0^\infty e^{-st}\,dt = \frac{-1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s}$

3. Ramp function,



$t \xrightarrow{LT} \frac{1}{s^2}$

$$\boxed{\begin{aligned} f(t) &= t, \quad t>0 \\ F(s) &= \frac{1}{s^2} \end{aligned}}$$

Proof: $\mathcal{L}\,t = \int_0^\infty t\,e^{-st}\,dt$

Integration by parts:

$d[uv] = u\,dv + v\,du \longrightarrow v\,du = d[uv] - u\,dv$ | $\int_a^b v\,du = uv\Big|_a^b - \int_a^b u\,dv$.

$du = e^{-st}$; $u = \frac{1}{-s}e^{-st}$

$v = t \qquad dv = 1$

$$\int_0^\infty t\,e^{-st}\,dt = \frac{t}{-s}e^{-st}\Big|_0^\infty - \int_0^\infty \frac{1}{-s}e^{-st}\,dt = \frac{1}{s}\underbrace{\int_0^\infty e^{-st}\,dt}_{\mathcal{L}\,1(t) = \frac{1}{s}} = \frac{1}{s^2}$$

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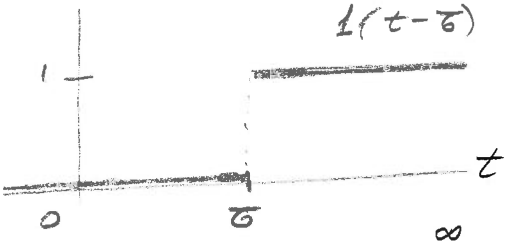
4. Shifted step function $1(t-\tau)$



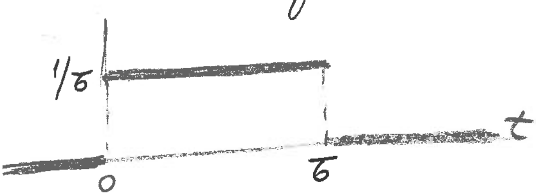
$$\begin{cases} f(t) = 1(t-\tau) \\ F(s) = e^{-\tau s}\dfrac{1}{s} \end{cases}$$

Proof $\mathcal{L}\,1(t-\tau) = \int_0^\infty 1(t-\tau)\, e^{-st}dt = \cancel{\int_0^\tau 0\cdot e^{-st}dt}^{=0} + \int_\tau^\infty e^{-st}dt$

change of variable $t^* = t-\tau$; $t = t^*+\tau$
$dt^* = dt$

$$= \int_0^\infty e^{-s(t^*+\tau)}dt^* = e^{-s\tau}\underbrace{\int_0^\infty e^{-st^*}dt^*}_{\mathcal{L}\,1(t)=\frac{1}{s}} = e^{-s\tau}\frac{1}{s}$$

5. Pulse function $p(t;\tau)$



$$\begin{cases} f(t) = p(t;\tau) \\ F(s) = \dfrac{1-e^{-s\tau}}{s\tau} \end{cases}$$

Proof: write the pulse as a step up followed by a step down at $t=\tau$ and scaled be $1/\tau$, i.e.,

$$p(t;\tau) = \underbrace{\frac{1}{\tau}1(t)}_{\downarrow} - \underbrace{\frac{1}{\tau}1(t-\tau)}_{\downarrow}$$

$$F(s) = \frac{1}{\tau}\frac{1}{s} - \frac{1}{\tau}e^{-s\tau}\frac{1}{s} = \frac{1-e^{-s\tau}}{s\tau}$$

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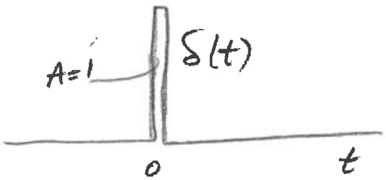
6. Impulse function $\delta(t)$



$$\boxed{\begin{array}{l} f(t) = \delta(t) \\ F(s) = 1 \end{array}}$$

(A) Proof : Consider $\delta(t)$ as the limit of $p(t;\tau)$ as $\tau \to 0$ and take LT.

$$\delta(t) = \lim_{\tau \to 0} p(t;\tau)$$

$$\mathcal{L}\delta(t) = \lim_{\tau \to 0} \mathcal{L}\, p(t;\tau) = \lim_{\tau \to 0} \frac{1-e^{-s\tau}}{s\tau}$$

The limit gives $\frac{1-1}{0} = \frac{0}{0}$. Apply l'Hospital rule,

$$\lim_{\tau \to 0} \frac{\frac{\partial}{\partial \tau}\left(1-e^{-s\tau}\right)}{\frac{\partial}{\partial \tau}(s\tau)} = \lim_{\tau \to 0} \frac{-(-\cancel{s}e^{-s\tau})}{\cancel{s}} = \lim_{\tau \to 0} e^{-s\tau} = 1$$

QED

(B) Proof : use localization property of the delta function, i.e.:

Recall $$\int_{-\infty}^{\infty} \delta(t)\, g(t)\, dt = g(0)$$

Then :

$$\mathcal{L}\delta(t) = \int_0^{\infty} \delta(t) \underbrace{e^{-st}}_{g(t)} dt = e^{-st}\Big|_{t=0} = e^{0} = 1$$

QED

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Differentiation Property

$$\mathcal{L} f'(t) = s F(s) \quad \text{if } f(0) = 0 \qquad (1)$$

$$\mathcal{L} f''(t) = s^2 F(s) \quad \text{if } f'(0) = f(0) = 0 \qquad (2)$$

$$\vdots$$

$$\mathcal{L} f^{(n)}(t) = s^n F(s) \quad \text{if } f^{(n-1)}(0) = \dots = f'(0) = f(0) = 0 \qquad (n)$$

Proof

(1): $\mathcal{L} f'(t) = \int_0^\infty f'(t) e^{-st} dt = f e^{-st}\Big|_0^\infty - \int_0^\infty f(-s) e^{-st} dt$

Integration by parts $\quad \underbrace{u}_{e^{-st}}\, \underbrace{dv}_{f'dt} = d(uv) - \underbrace{v}_{f}\, \underbrace{du}_{-se^{-st}}$

$$= f(\infty)\underbrace{e^{-\infty}}_{=0} - f(0) + s \underbrace{\int_0^\infty f e^{-st} dt}_{F(s)} = \underbrace{s F(s)}\, - \underbrace{\cancel{f(0)}}_{=0}$$

QED

(2): denote $g(t) = f'(t)$

$G(s) = \mathcal{L} g(t) = \mathcal{L} f'(t) = s F(s)$

$\mathcal{L} f''(t) = \mathcal{L} g'(t) = s\, G(s) = s^2 F(s)$

QED

(n): by induction ...

Bottom line: "to differentiate, multiply by s"

provided $f(0) = 0$, $f'(0) = 0$, etc.

Else: need to subtract them $\quad \mathcal{L} f'(t) = s F(s) - f(0)$ etc

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Integration Property

$$\mathcal{L}\left(\int_0^t f(t^*)\,dt^*\right) = \frac{1}{s} F(s)$$

<u>Proof</u> Denote $g(t) = \int_0^t f(t^*)\,dt^*$

Then $g'(t) = f(t)$, $g(0) = 0$

$\mathcal{L}$ LHS: $\mathcal{L}\,g'(t) = s\mathcal{L}\,g(t) = s\mathcal{L}\left(\int_0^t f(t^*)\,dt^*\right)$

$\mathcal{L}$ RHS: $\mathcal{L}f(t) = F(s)$

$\mathcal{L}$ LHS = $\mathcal{L}$ RHS: $s\mathcal{L}\left(\int_0^t f(t^*)\,dt^*\right) = F(s)$

divide by s to get $\mathcal{L}\int_0^t f(t^*)\,dt^* = \frac{1}{s} F(s)$ QED

Bottom line: "To integrate, divide by s"

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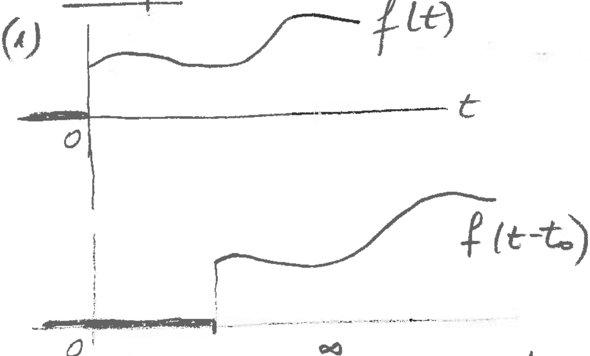
Shift Properties

Shift in t: $\quad \mathcal{L} f(t-t_0) = e^{-t_0 s} F(s) \qquad (1)$

Shift in s: $\quad \mathcal{L} e^{s_0 t} f(t) = F(s-s_0) \qquad (2)$

Proof

(1)



Function $f(t)$ is zero for -ve argument

Function $f(t-t_0)$ is zero for $t<t_0$.

$$\mathcal{L} f(t-t_0) = \int_0^\infty f(t-t_0) e^{-st} dt = \cancel{\int_0^{t_0} 0 \cdot e^{-st} dt} + \int_{t_0}^\infty f(t-t_0) e^{-st} dt$$

$$= \int_{t_0}^\infty f(t-t_0) e^{-st} dt$$

change of variable $\left| \begin{array}{l} t^* = t - t_0 \rightarrow t = t^* + t_0 \\ dt^* = dt \end{array} \right.$

$$= \int_0^\infty f(t^*) e^{-s(t^*+t_0)} dt^* = e^{-st_0} \underbrace{\int_0^\infty f(t^*) e^{-st^*} dt^*}_{F(s)}.$$

QED

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(2) $$\mathcal{L} e^{s_0 t} f(t) = \int_0^\infty e^{s_0 t} f(t) e^{-st} dt = \underbrace{\int_0^\infty f(t) e^{-(s-s_0)t} dt}_{F(s-s_0)}$$

QED

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Final Value Theorem

steady-state Response.

$$x_{ss} = x(\infty) = \lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s).$$

steady state response

final value

$$\boxed{x_{ss} = \lim_{s\to 0} sX(s)}$$

Proof Start with LT of the derivative $\dot{x}(t)$, i.e.,

$$\mathcal{L}\,\dot{x}(t) = sX(s), \quad x(0)=0 \qquad (1)$$

But, by definition

$$\mathcal{L}\,\dot{x}(t) = \int_0^\infty \dot{x}(t)e^{-st}dt \qquad (2).$$

(1) ≡ (2):

$$\int_0^\infty \dot{x}(t)e^{-st}dt = sX(s) \qquad (3)$$

Take limit $s\to 0$ of Eq. (3), i.e.,

$$\lim_{s\to 0}\left[\int_0^\infty \dot{x}(t)e^{-st}dt\right] = \lim_{s\to 0} sX(s) \qquad (4)$$

but

$$\lim_{s\to 0} e^{-st} = e^0 = 1 \qquad (5)$$

(5) → (4) LHS:

$$\int_0^\infty \dot{x}(t)dt = x(t)\Big|_0^\infty = x(\infty) = x_{ss}$$

LHS = RHS:

$$x_{ss} = \lim_{s\to 0} sF(s)$$

QED

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INVERSE LAPLACE TRANSFORM

Assume $$X(s) = \frac{B(s)}{A(s)} \quad (1)$$

$A(s), B(s)$ polynomials in s:

$A(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0$: degree n

$B(s) = b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0$: degree $m < n$

Then $X(s)$ can be expanded in partial fractions, i.e.,

$$X(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \cdots \frac{r_n}{s-p_n} = \sum_{i=1}^{n} \frac{r_i}{s-p_i} \quad (2)$$

Recall $\mathcal{L}\, e^{p_0 t} = \frac{1}{s-p_0}$ (pole p_0)

Hence, the ILT of (2) is a sum, i.e.,

$$x(t) = r_1 e^{p_1 t} + r_2 e^{p_2 t} + \ldots r_n e^{p_n t} = \sum_{i=1}^{n} r_i e^{p_i t} \quad (3)$$

The values $p_1, p_2, \ldots, p_n$ are the roots of the denominator $A(s)$. They are called "poles". They may be complex numbers.

The values $r_1, r_2, \ldots, r_n$ are called "residues"

$$r_i = \lim_{s \to p_i} \left[(s-p_i) X(s) \right] \quad (4)$$

Use MATLAB function "residue" to find p_i & r_i

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iLT with complex poles.

When complex poles appear, they are in conjugate pairs, i.e.,

$$p_{1,2} = \sigma \pm i\omega_d \qquad (1)$$

PFE: $$X(s) = X_1(s) + X_2(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} \qquad (2)$$

It can be shown that $r_1 = -ir_0$, $r_2 = ir_0$ (3)

$$X_1(t) = \frac{-ir_0}{1-p_1}, \quad p_1 = \sigma + i\omega_d \qquad (4)$$

$$x_1(t) = \mathcal{L}^{-1} X_1(s) = \mathcal{L}^{-1}\frac{-ir_0}{s-p_1} = -ir_0 e^{p_1 t} \qquad (5)$$

$$= -ir_0 e^{(\sigma+i\omega_d)t} = -ir_0 e^{\sigma t} e^{i\omega_d t}$$

$$= -ir_0 e^{\sigma t}(\cos\omega_d t + i\sin\omega_d t) \qquad (6)$$

$$= r_0 e^{\sigma t}(-i\cos\omega_d t + \sin\omega_d t)$$

Similarly

$$X_2(s) = \frac{ir_0}{s-p_2}, \quad p_2 = \sigma - i\omega_d \qquad (7)$$

$$x_2(t) = ir_0 e^{\sigma t} e^{-i\omega_d t} = ir_0 e^{\sigma t}(\cos\omega_d t - i\sin\omega_d t) \qquad (8)$$

$$= r_0 e^{\sigma t}(i\cos\omega_d t + \sin\omega_d t)$$

$x_1(t) + x_2(t) =$

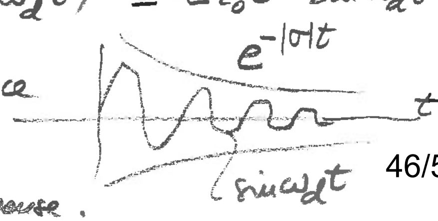
$$r_0 e^{\sigma t}(-i\cos\omega_d t + \sin\omega_d t)$$

$$+ r_0 e^{\sigma t}(i\cos\omega_d t + \sin\omega_d t) = 2r_0 e^{\sigma t}\sin\omega_d t$$

We expect $\sigma < 0$, hence

$$\sigma = -|\sigma|$$

Poles in LHS for stable response.



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MATLAB instruction residue

$$X(s) = \frac{B(s)}{A(s)}$$

$$B(s) = b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0$$

$$B = [b_m \ \ b_{m-1} \ \cdots \ b_0]$$

$$A(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0$$

$$A = [a_n \ \ a_{n-1} \ \cdots \ a_0]$$

$[r, p, k]$ = residue (B, A)

where : p = vector of poles (roots of $A(s)$)

r = vector of residues at the poles

k = vector of direct term coeff.

In our work, $m < n$, then $k = [\ ]$ void matrix

Partial fraction expansion (PFE):

$$X(s) = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + \cdots \qquad (1)$$

Note : If $m \geq n$, then define direct term as

$$k_1 s^q + k_2 s^{q-1} + \cdots + k_{q+1}$$

and add it to the end of (1) to get $X(s)$

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Example

$$X(s) = \frac{1}{s+2} + \frac{2}{s+1} = \frac{(s+1) + 2(s+2)}{(s+1)(s+2)}$$

$$= \frac{3s+5}{s^2+3s+2} = \frac{B(s)}{A(s)}$$

(a) Define:

$B = [3 \ \ 5]$; $A = [1 \ \ 3 \ \ 2]$

$[r, p, k] = \text{residue}(B, A)$

$r_1 = 1 \qquad p_1 = -2 \qquad k = \text{void}.$

$r_2 = 2 \qquad p_2 = -1$

$$\frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} = \frac{1}{s-(-2)} + \frac{2}{s-(-1)} = \frac{1}{s+2} + \frac{2}{s+1}$$

checks!

(b) Define:

$r = [1 \ \ 2]$; $p = [-2 \ \ -1]$; $k = 0$

$[b, a] = \text{residue}(r, p, k)$

$b = [3 \ \ 5]$

$a = [1 \ 3 \ 2]$ checks!

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Convolution property of Laplace Transform

$$\mathcal{L}^{-1} G(s) \cdot F(s) = \mathcal{L}^{-1} F(s)\, G(s)$$

$$= \int_0^t f(\tau)\, g(t-\tau)\, d\tau \qquad (1)$$

excitation function

impulse response of the system

Convolution expresses the system response $x(t)$ to a complicated excitation $f(\tau)$ as an integral using the impulse response $g(t)$ shifted by τ to $(t-\tau)$, i.e.,

$$x(t) = (f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau \qquad (2)$$

The integral in Eq. (2) is not easily computed in time domain; however, Laplace transform makes it easy because:

1. calculate $F(s) = \mathcal{L} f(t)$, $G(s) = \mathcal{L} g(t)$
2. multiply $F(s)\, G(s)$
3. take inverse Laplace transform to get $x(t)$

$$\mathcal{L}^{-1} F(s)\, G(s) = (f * g)(t) = x(t)$$

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DOMINANT POLES

Given:

$$X(s) = \frac{B(s)}{A(s)} = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \dots + \frac{r_k}{s-p_k} + \dots \quad (1)$$

$p_k \in \mathbb{C}$ complex number

$p_k = \sigma_k + i\omega_k$, conjugate pairs (2a)

or

$p_k \in \mathbb{R}$ $\quad p_k = \sigma_k$ (2b)

Perform $\mathcal{L}^{-1}$ Eq.(1):

$$x(t) = r_1 e^{p_1 t} + r_2 e^{p_2 t} + \dots + r_k e^{p_k t} + \dots \quad (3)$$

We are interested in the long term behavior of $x(t)$, i.e., to find

$$x_{ss}(t) = \lim_{t\to\infty} x(t) \quad \text{steady state response}$$

Note that every term in the expansion will have the form

$$r_k e^{p_k t} = \underbrace{r_k e^{\sigma_k t}}_{\text{exponential function}} \cdot \underbrace{e^{i\omega_k t}}_{\text{harmonic oscillation}}.$$

We distinguish the following possible cases:

A. If at least one σ_k is +ve ($\sigma_k > 0$) then

$e^{\sigma_k t} \rightarrow \infty$, UNSTABLE system.

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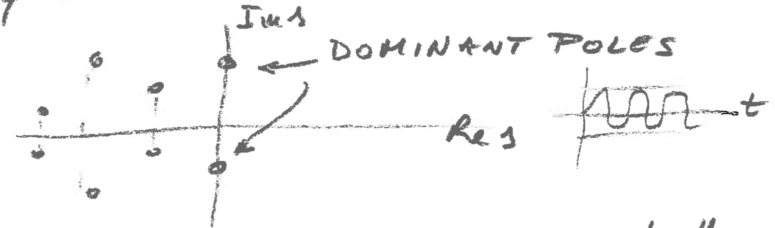
B. If system is STABLE, i.e., all $\sigma_K \leq 0$, (all poles in LHS) then:

B1. If terms with $\sigma_K = 0$ exist, then these will survive as sustained oscillations while the rest have died out. This means that poles situated on the imaginary axis, if the exist, are <u>dominant poles</u>



B2. If terms with $\sigma_K = 0$ do NOT exist, then the dominant poles are the poles closest to the imaginary axis because the die hardest having small σ_K values (small damping