FRF

dB scale   deci bell

$\frac{1}{10}$

dB scale is a $\log_{10}$ scale magnified by 20

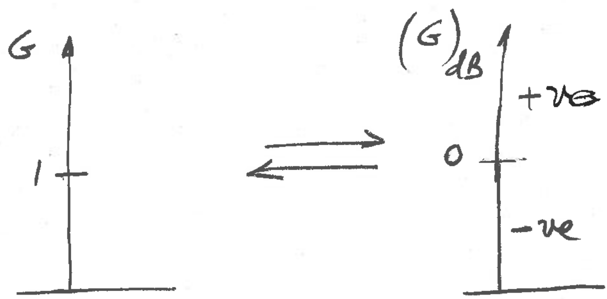
$$(G)_{dB} = 20 \log_{10} G$$

Properties

(1) Only +ve numbers have dB value (cannot take log of −ve numbers!)

(2) $\left(\frac{1}{G}\right)_{dB} = -(G)_{dB}$ (reciprocal numbers)

(3)



$G > 1$	$(G)_{dB}$ is +ve
$G = 1$	$(G)_{dB} = 0$
$G < 1$	$(G)_{dB}$ is −ve

2

FRF

(4) Scale up or scale down in physical values means shift up or shift down in dB

(a) double (octave up) = +6 dB

(b) "half" (octave down) = − 6 dB

(c) "ten times" (decade) = + 20 dB

octave = 8th musical note (A B C D E F G) A_1 has double the frequency $f_{A_1} = 2 f_A$

Proof

(a) $\log_{10} 2 = 0.303$; $20 \log 2 \cong 6$

$G_2 = 2G_1$

$(G_2)_{dB} = 20 \log_{10} (2G_1) = 20 \log_{10} 2 + 20 \log_{10} G_1$

$= \underbrace{20 \times 0.303}_{\sim 6} + (G_1)_{dB}$

$(G_2)_{dB} = (G_1)_{dB} + 6\,dB$

>> mag2db(2)

ans 6.0206

(b) $G_2 = \frac{1}{2} G_1$

$(G_2)_{dB} = (G_1)_{dB} + 20 \log_{10} \left(\frac{1}{2}\right)$

$= (G_1)_{dB} + 20 \left(-\log_{10} 2\right)$

$= (G_1)_{dB} - 6\,dB$

3
FRF

(c) $G_2 = 10\, G_1$

$$(G_2)_{dB} = (G_1)_{dB} + 20 \log_{10}\overbrace{(10)}^{=1}$$

$$= (G_1)_{dB} + 20\, dB$$

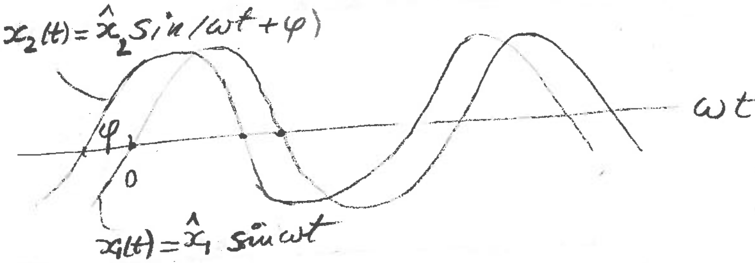
4

FRF

Phase

$x_1(t) = \hat{x}_1 \sin \omega t$ reference signal

$x_2(t) = \hat{x}_2 \sin(\omega t + \varphi)$

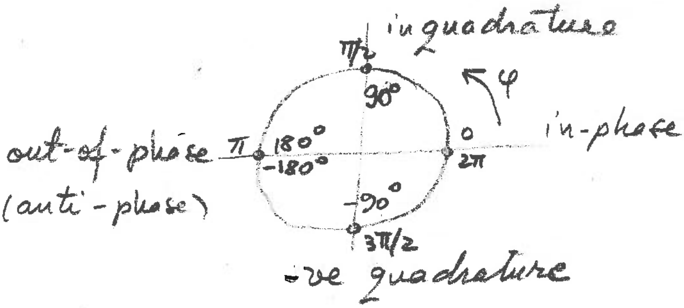


$\psi_1 = \omega t$ phase of x_1 (reference)

$\psi_2 = \omega t + \varphi$ phase of x_2

$\varphi = \psi_2 - \psi_1$

- phase difference
- x_2 leads by φ.

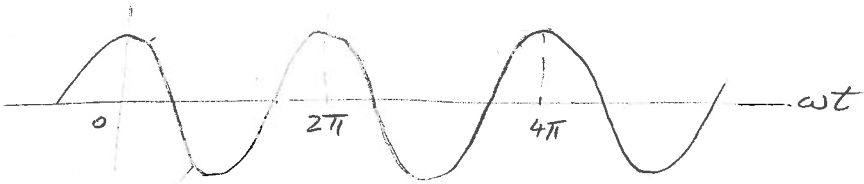


Periodic signals

$\left\{ \begin{matrix} 2\pi \\ 360° \end{matrix} \right.$ periodicity:

- $x(\omega t + 2\pi) = x(\omega t)$
- $x(\omega t + \pi) = x(\omega t - \pi)$

COMPLEX NUMBER REPRESENTATION OF HARMONIC SIGNALS



$\cos \omega t$

$$e^{i\omega t} = \cos\omega t + i\sin\omega t \quad \text{(Euler)}$$

$$\begin{cases} x(t) = \hat{x}\cos(\omega t + \varphi) \\ x(t) = \mathrm{Re}\left[\hat{x}e^{i(\omega t+\varphi)}\right] \end{cases}$$

$$\begin{cases} x(t) = \hat{x}\sin(\omega t + \varphi) \\ x(t) = \mathrm{Im}\left[\hat{x}e^{i(\omega t+\varphi)}\right] \end{cases}$$

In general $\quad x(t) = \hat{x}e^{i(\omega t+\varphi)}$

$$\hat{x}e^{i(\omega t+\varphi)} = \hat{x}e^{i\varphi}e^{i\omega t} = Xe^{i\omega t}$$

$$X = \hat{x}e^{i\varphi} \quad \text{(phasor)}$$

$|X| = \hat{x}$

magnitude of $x(t)$

$|X| = \mathrm{abs}(X)$

$\angle X = \varphi$

phase of $x(t)$

$\angle X = \mathrm{angle}(X)$, rad

rad2deg(π) = 180°

Frequency response function (FRF)

- $G(s)$ TF: Transfer function in Laplace domain s

- $G(i\omega)$ FRF: Transfer function in frequency domain ω

$$f(t)=e^{i\omega t} \longrightarrow \boxed{G(i\omega)} \longrightarrow x(t)=G(i\omega)\,e^{i\omega t}$$

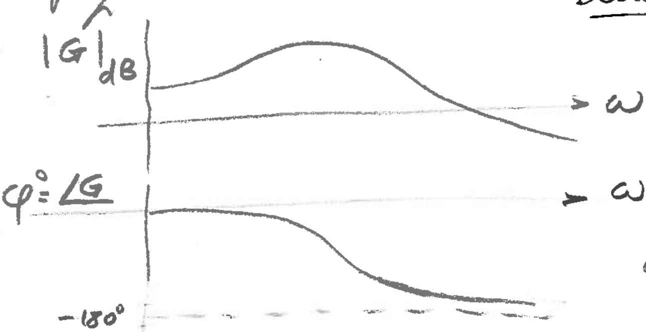
$G(i\omega)$ is a complex number

$|G(i\omega)|$ = magnitude, dB

$\angle G(i\omega)$ = phase, deg

Frequency response function (FRF) is $G(i\omega)$ measured over a range of frequencies ω

Bode diagram



Magnitude plot

Phase plot

usually $\varphi < 0$
"phase lag"

20181205

Harmonic Response via Laplace Transform

$$f(t) = e^{i\omega t} \xrightarrow{\mathcal{L}} F(s) = \frac{1}{s-i\omega}$$

$$X(s) = G(s)\,F(s) = G(s)\,\frac{1}{s-i\omega} \qquad (1)$$

Assume: $G(s) = K\,\dfrac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$ (2)

Poles in LHS $-p_n \cdots -p_1 < 0$

STABLE

(1) & (2) yields the PFE:

$$X(s) = \frac{a}{s-i\omega} + \frac{b_1}{s+p_1} + \frac{b_2}{s+p_2} + \cdots + \frac{b_n}{s+p_n} \qquad (3)$$

$$x(t) = \underset{\substack{\uparrow \\ \text{steady state} \\ \text{response}}}{a e^{i\omega t}} + \underbrace{b_1 e^{-p_1 t} + b_2 e^{-p_2 t} + \cdots b_n e^{-p_n t}}_{\text{Transient response} \xrightarrow[t\to\infty]{} 0} \qquad (4)$$

$$x_{ss}(t) = a e^{i\omega t} \qquad (5)$$

To find a, multiply (3) by $s-i\omega$ and make $s = i\omega$ to get:

$$\Big[(s-i\omega)X(s)\Big]_{s=i\omega} = \left[a + \cancel{\left[\frac{b_1}{s+p_1} + \frac{b_2}{s+p_2} + \cdots \frac{b_n}{s+p_n}\right](s-i\omega)}\right]_{s=i\omega}$$

$$a = \Big[(s-i\omega)X(s)\Big]_{s=i\omega} \qquad (6)$$

$$= \left[\cancel{(s-i\omega)}\,G(s)\,\frac{1}{\cancel{(s-i\omega)}}\right]_{s=i\omega} = G(i\omega).$$

(6) → (5): $\quad x_{ss}(t) = \boxed{G(i\omega)}\,e^{i\omega t}.$

complex amplitude

$$\text{FRF} = G(i\omega) = |G(i\omega)|\,e^{i\varphi}, \qquad \varphi = \angle G(i\omega)$$

Harmonic Response via Laplace Transform

$$G(s) = \frac{1}{Ts+1}, \quad f(t) = e^{i\omega t} \longrightarrow F(s) = \frac{1}{s-i\omega}$$

$$X(s) = G(s)F(s) = \frac{1}{Ts+1} \cdot \frac{1}{s-i\omega} = \frac{1}{T} \cdot \frac{1}{s+\frac{1}{T}} \cdot \frac{1}{s-i\omega}$$

$$s_1 = -\frac{1}{T} \quad s_2 = i\omega$$

PFE, p 30, Eq^n (2.6), modified:

$$X(s) = \frac{a_1}{s-s_1} + \frac{a_2}{s-s_2} + \cdots + \frac{a_k}{s-s_k} + \cdots$$

$$a_k = \Big[(s-s_k)X(s)\Big]_{s=s_k}$$

$$a_1 = \left[\cancel{\left(s+\frac{1}{T}\right)} \frac{1}{T} \frac{1}{\cancel{s+\frac{1}{T}}} \frac{1}{s-i\omega}\right]_{s=-\frac{1}{T}} = \frac{1}{T} \frac{1}{-\frac{1}{T}-i\omega} = -\frac{1}{i\omega T+1}$$

$$a_2 = \left[\cancel{(s-i\omega)} \frac{1}{T} \frac{1}{s+\frac{1}{T}} \frac{1}{\cancel{s-i\omega}}\right]_{s=i\omega} = \frac{1}{T} \frac{1}{i\omega+\frac{1}{T}} = \frac{1}{i\omega T+1}$$

$$X(s) = -\frac{1}{i\omega T+1} \cdot \frac{1}{s+\frac{1}{T}} + \frac{1}{i\omega T+1} \cdot \frac{1}{s-i\omega}$$

$$x(t) = -\frac{1}{i\omega T+1} e^{-t/T} + \frac{1}{i\omega T+1} e^{i\omega t}$$

| transient | steady state |

$$x_{ss}(t) = \frac{1}{i\omega T+1} e^{i\omega t} = G(i\omega)\, e^{i\omega t}$$

$$FRF = G(i\omega) = |G(i\omega)|\, e^{i\varphi}, \quad \varphi = \angle G(i\omega)$$

True for stable systems (i.e., transients vanish).

FR15

Harmonic Response via Laplace Transform for sine excitation

$$G(s) = \frac{1}{Ts+1}, \quad f(t) = \sin \omega t \rightarrow F(s) = \frac{\omega}{s^2+\omega^2}$$

$$X(s) = G(s)F(s) = \frac{1}{\underbrace{Ts+1}} \cdot \frac{\omega}{s^2+\omega^2} = \frac{1}{T} \frac{1}{s+\frac{1}{T}} \cdot \frac{\omega}{(s-i\omega)(s+i\omega)}$$

Poles: $\quad s_1 = -1/T \quad s_{2,3} = \pm i\omega$

PFE, p.30, Eq. (2.6), modified:

$$X(s) = \frac{a_1}{s-s_1} + \frac{a_2}{s-s_2} + \dots + \frac{a_k}{s-s_k} + \dots$$

$$a_k = \Big[(s-s_k)X(s)\Big]_{s=s_k}$$

$$X(s) = \frac{a_1}{s+\frac{1}{T}} + \frac{a_2}{s-i\omega} + \frac{a_3}{s+i\omega}$$

$$a_1 = (s+\tfrac{1}{T})G(s)\Big|_{s=s_1} = \left.\cancel{\left(s+\frac{1}{T}\right)} \frac{1/T}{\cancel{s+\frac{1}{T}}} \frac{\omega}{s^2+\omega^2}\right|_{s=-\frac{1}{T}}$$

$$a_1 = \frac{1}{T} \frac{\omega}{\frac{1}{T^2}+\omega^2} = \frac{\omega T}{\omega^2 T^2+1}$$

$$a_2 = (s-i\omega)\,G(s)\Big|_{s=i\omega} = \left.\cancel{(s-i\omega)} \frac{1}{Ts+1} \frac{\omega}{\cancel{(s-i\omega)}(s+i\omega)}\right|_{s=i\omega}$$

$$a_2 = \frac{1}{i\omega T+1} \cdot \frac{\omega}{i\omega+i\omega} = \frac{1}{i\omega T+1} \frac{1}{2i} = G(i\omega)\frac{1}{2i}$$

$$a_3 = (s+i\omega)G(s)\Big|_{s=-i\omega} = \cancel{(s+i\omega)}\,\frac{1}{Ts+1}\,\frac{\omega}{(s-i\omega)\cancel{(s+i\omega)}}\Bigg|_{s=-i\omega}$$

$$= \frac{1}{-i\omega T+1}\,\frac{1}{-2i} = -G(-i\omega)\,\frac{1}{2i}$$

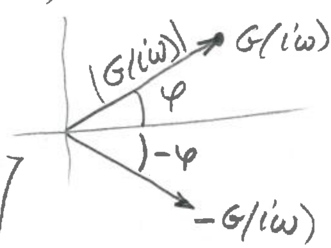
$$X(s) = \underbrace{\frac{\omega T}{\omega^2T^2+1}\,\frac{1}{s+\frac{1}{T}}}_{\text{transient}} + \underbrace{\frac{1}{2i}\left[\frac{G(i\omega)}{s-i\omega} - \frac{G(-i\omega)}{s+i\omega}\right]}_{\text{steady state}}$$

$$x(t) = \frac{\omega T}{\omega^2T^2+1}\,e^{-t/T} + \frac{1}{2i}\left[G(i\omega)e^{i\omega t} - G(-i\omega)e^{-i\omega t}\right]$$

But

$$G(i\omega) = |G(i\omega)|e^{i\varphi}, \quad \varphi = \angle G(i\omega)$$

$$G(-i\omega) = |G(i\omega)|e^{-i\varphi}$$



Hence

$$x_{ss}(t) = \frac{1}{2i}\left[G(i\omega)e^{i\omega t} - G(-i\omega)e^{-i\omega t}\right]$$

$$= \frac{1}{2i}|G(i\omega)|\left(e^{i\varphi}e^{i\omega t} - e^{-i\varphi}e^{-i\omega t}\right), \quad \varphi = \angle G(i\omega)$$

$$= |G(i\omega)|\,\frac{e^{i\psi} - e^{-i\psi}}{2i} \qquad \psi = \omega t + \varphi$$

$$= |G(i\omega)|\sin\psi$$

$$x_{ss}(t) = |G(i\omega)|\sin(\omega t+\varphi), \quad \varphi = \angle G(i\omega)$$

FR17

Lemma:

If $G(s)$ is polynomial (or fraction of)

and $G(i\omega) = |G(i\omega)| e^{i\varphi}$

Then $G(-i\omega) = |G(i\omega)| e^{-i\varphi}$.

Proof

(a) $G(s) = s + a$

$G(i\omega) = i\omega + a = \sqrt{a^2+\omega^2}\, e^{i\varphi}$, $\varphi = \tan^{-1}\frac{\omega}{a}$.

$G(-i\omega) = -i\omega + a = \sqrt{a^2+\omega^2}\, e^{i\varphi^*}$, $\varphi^* = \tan^{-1}\frac{-\omega}{a} = -\varphi$

(b) $G(s) = (s+a_1)(s+a_2)$

$G(i\omega) = (i\omega + a_1)(i\omega + a_2)$

$= \sqrt{a_1^2+\omega^2}\sqrt{a_2^2+\omega^2}\, e^{i\varphi_1} e^{i\varphi_2}$, $\varphi_1 = \tan^{-1}\frac{\omega}{a_1}$

$\varphi_2 = \tan^{-1}\frac{\omega}{a_2}$

$G(-i\omega) = (-i\omega + a_1)(-i\omega + a_2)$

$= \sqrt{a_1^2+\omega^2}\sqrt{a_2^2+\omega^2}\, e^{i\varphi_1^*} e^{i\varphi_2^*}$

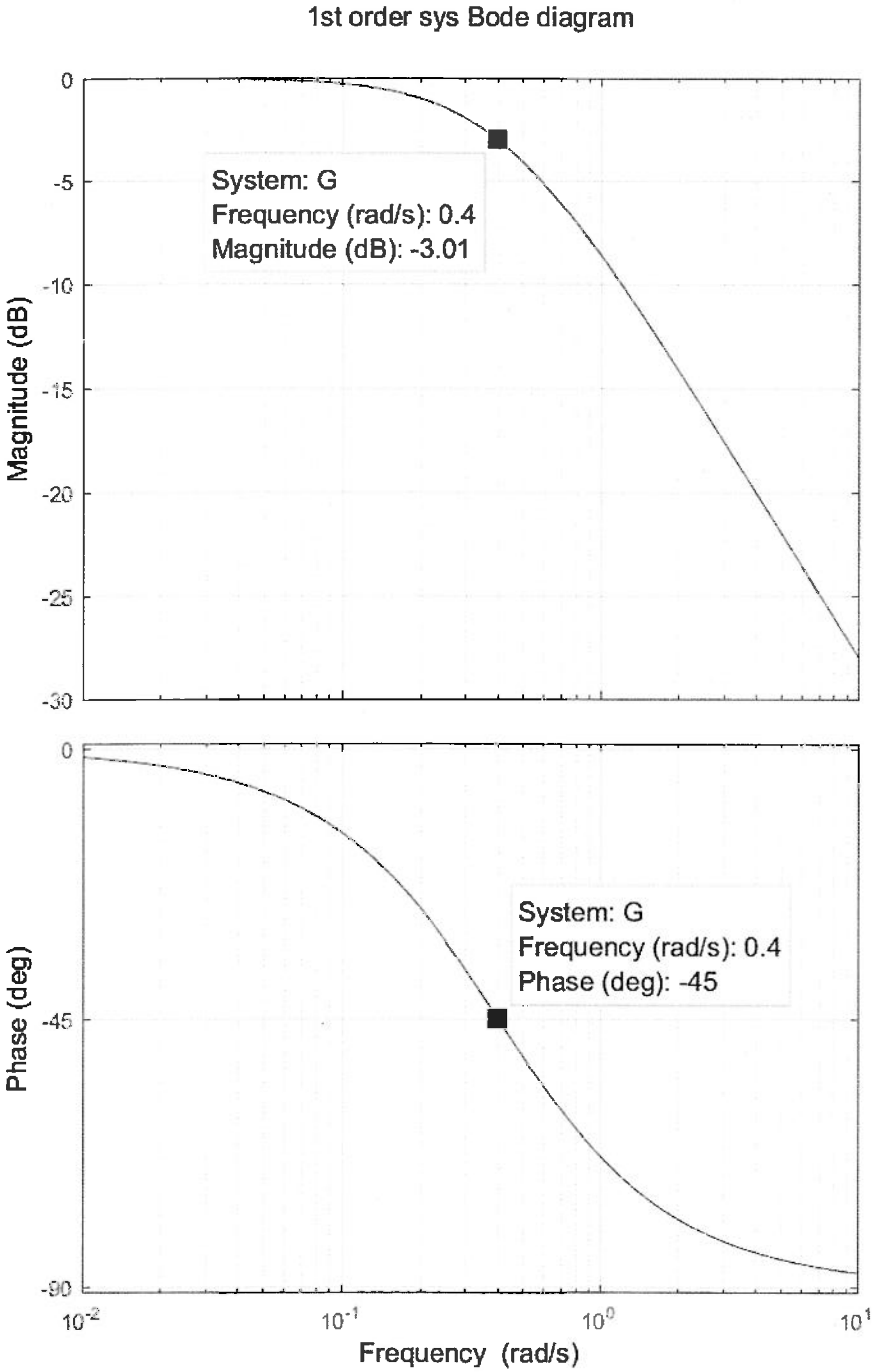
$\varphi_1^* = \tan^{-1}\frac{-\omega}{a_1} = -\varphi_1$

$\varphi_2^* = \tan^{-1}\frac{-\omega}{a_2} = -\varphi_2$

etc.

1st order system FRF

$T = 2.5$ sec.


1st order sys Bode diagram
System: G
Frequency (rad/s): 0.4
Magnitude (dB): -3.01
System: G
Frequency (rad/s): 0.4
Phase (deg): -45
Magnitude (dB)
Phase (deg)
Frequency (rad/s)
0
-5
-10
-15
-20
-25
-30
0
-45
-90
10^{-2}
10^{-1}
10^{0}
10^{1}

I

1^{st} Order System FRF

$$G(s) = \frac{1}{Ts+1}$$ transfer function (TF)

$$G(i\omega) = \frac{1}{i\omega T+1}$$ frequency response function (FRF)

Asymptotes

Define $\omega_c = 1/T$

Low freq. asymptote $\omega \ll \omega_c$, $\omega T \ll 1$ $\quad G(i\omega) \to G_{LF}$

$$G(i\omega) = \frac{1}{i\omega T+1} \xrightarrow[\omega T \ll 1]{} \frac{1}{1} = 1$$

$\ast$ $G_{LF}(i\omega) = 1$, $\left|G_{LF}\right|_{dB} = 0$ $\quad \angle G_{LF} = 0^\circ$

High freq. asymptote $\omega \gg \omega_c$, $\omega T \gg 1$ $\quad G(i\omega) \to G_{HF}$

$$G(i\omega) = \frac{1}{i\omega T+1} \xrightarrow[\omega T \gg 1]{} \frac{1}{i\omega T}$$

$\ast$ $$G_{HF}(i\omega) = \frac{1}{i\omega T} = \frac{1}{\omega T} e^{-i\frac{\pi}{2}}$$

$$|G_{HF}| = \frac{1}{\omega T} \qquad \angle G_{HF} = -\frac{\pi}{2} = -90^\circ$$

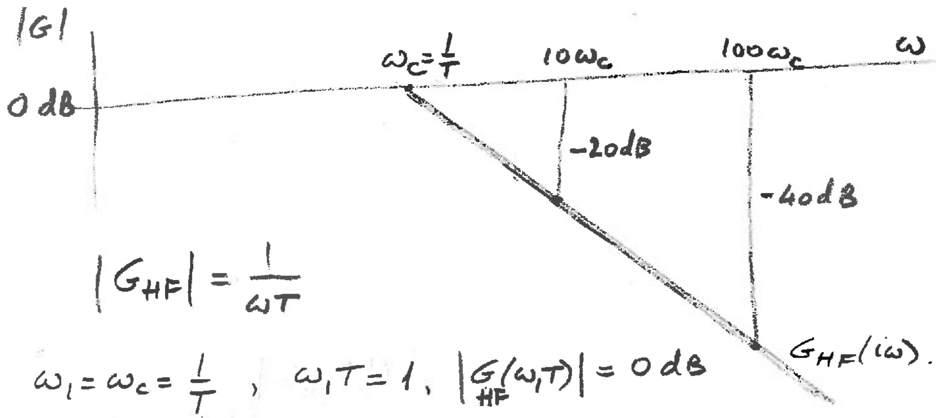
Intersection of G_{LF} & G_{HF}

$$1 = \frac{1}{\omega_c T} \qquad \omega_c = \frac{1}{T}$$ cutoff frequency

decay starts at ω_c

Slope of G_{HF}

$$|G_{HF}|_{dB} = -20 \log_{10}(\omega T)$$



$$|G_{HF}| = \frac{1}{\omega T}$$

$\omega_1 = \omega_c = \frac{1}{T}$, $\omega_1 T = 1$, $|G_{HF}(\omega_1 T)| = 0\,dB$

$\omega_2 = 10\omega_c$, $\omega_2 T = 10$, $|G_{HF}(\omega_2 T)| = \frac{1}{10} = -20\,dB$

$\omega_3 = 100\,\omega_c$, $\omega_3 T = 100$, $|G_{HF}(\omega_3 T)| = \frac{1}{100} = -40\,dB$

$|G_{HF}|$ drops $-20\,dB$/decade

For octave, take ω_c, $2\omega_c$, $4\omega_c$

ω_1	ω_2	ω_3
1	$\frac{1}{2}$	$\frac{1}{4}$
0dB	-6dB	-12dB.

$|G_{HF}|$ drops $-6\,dB$/octave

3
I

Exact amplitude and phase at ω_c

$$\omega_c = \frac{1}{T}$$

$$G(i\omega_c) = \frac{1}{\underbrace{i\omega_c T}_{1}+1} = \frac{1}{1+i} = \frac{1}{\sqrt{2}} e^{-i\frac{\pi}{4}}$$

$$|G(i\omega_c)| = \frac{1}{\sqrt{2}} = -3dB$$

$\boxed{\begin{array}{c} \text{mag2db(1/sqrt(2))} \\ 3.0103 \end{array}}$

$$\angle G(i\omega_c) = -\frac{\pi}{4} = -45^\circ$$

Error of using asymptotes

Maximum error occurs at ω_c

Appox. value = 0 dB

Exact value = −3 dB

Error = 3 dB

Bandwidth ω_B

Bandwidth is defined as the frequency below which signal does not decrease more than 3dB

For 1st order sys, $\omega_B = \omega_c = \frac{1}{T}$

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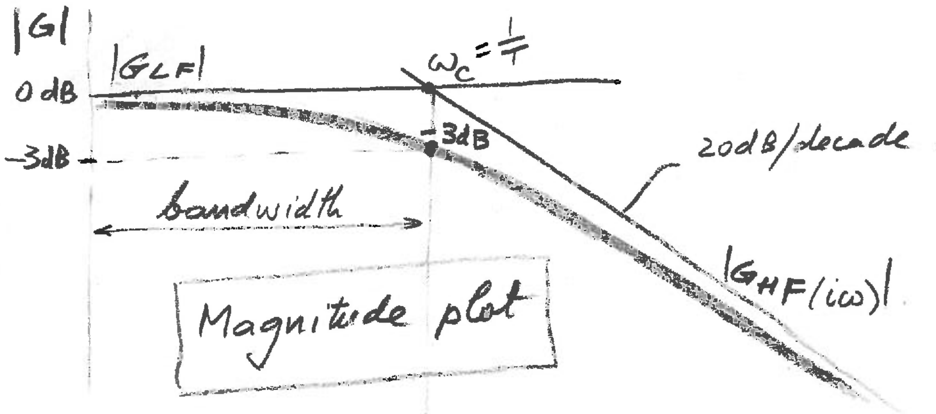
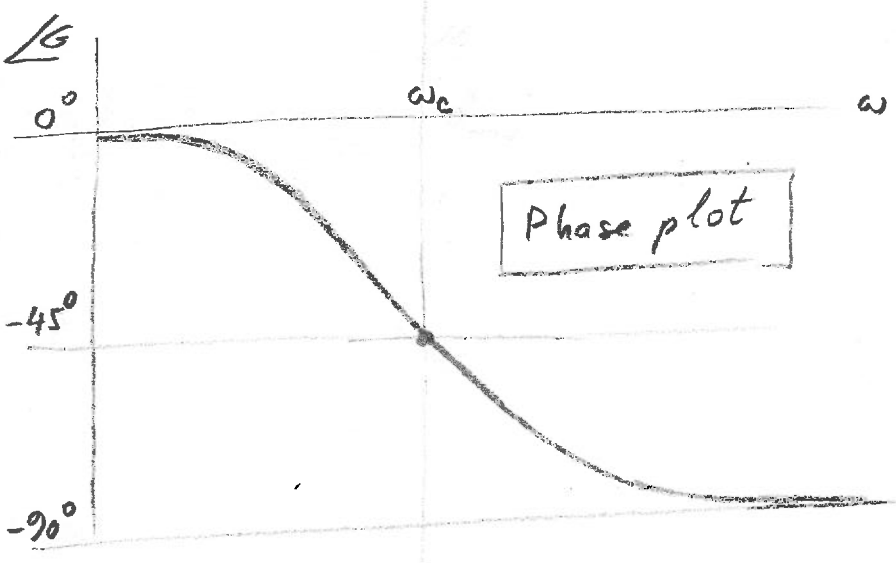
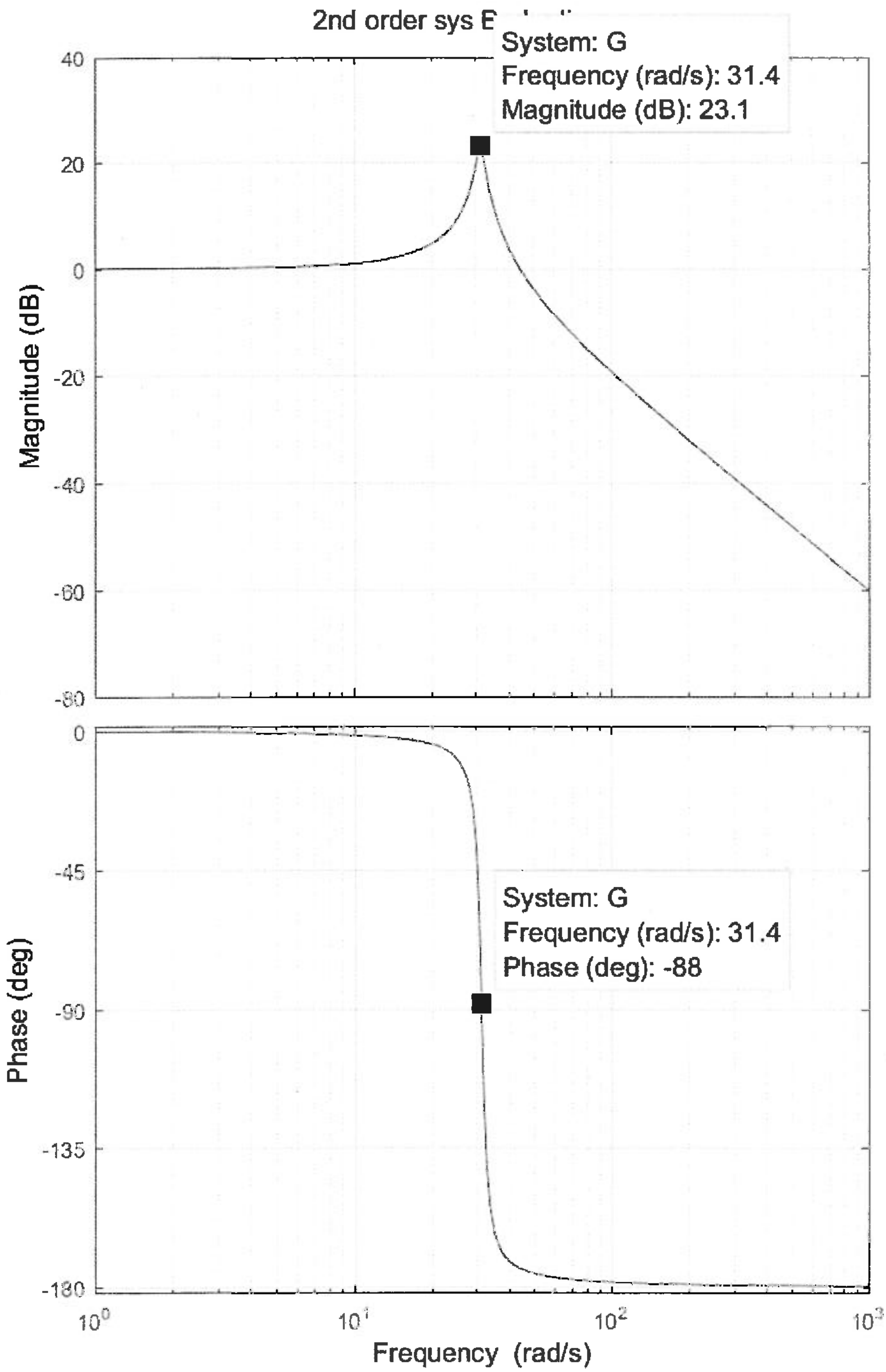
Bode plots





2nd order system FRF

$f_n = 5\,Hz$
$\zeta = 3.5\%$



2^{nd} order syst. FRF

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$G(i\omega) = \frac{\omega_n^2}{-\omega^2 + 2i\zeta\omega_n\omega + \omega_n^2}$$

$$= \frac{\omega_n^2}{(-\omega^2+\omega_n^2) + i\,2\zeta\omega_n\omega}$$

Phase resonance: $\omega = \omega_n$ (natural freq.)

$$G(i\omega_n) = \frac{\omega_n^2}{\cancel{(-\omega_n^2+\omega_n^2)} + i\,2\zeta\omega_n\omega_n} = \frac{1}{i2\zeta}$$

$$|G(i\omega_n)| = \frac{1}{2\zeta}$$

$$\angle G(i\omega_n) = \angle\frac{1}{i2\zeta} = -90^\circ$$

2/II

LF asymptote

$$G(i\omega) \xrightarrow[\omega \ll \omega_n]{} G_{LF}(i\omega)$$

$$G(i\omega) = \frac{\omega_n^2}{\cancel{(-\omega^2} + \omega_n^2) + \cancel{i 2 \zeta \omega \omega_n}} \xrightarrow[\omega \ll \omega_n]{} \frac{\omega_n^2}{\omega_n^2} = 1$$

$$G_{LF}(i\omega) = 1 \qquad |G_{LF}|_{dB} = 0\,dB$$

$$\angle G_{LF} = 0^\circ$$

HF asymptote

$$G(i\omega) \xrightarrow[\omega \gg \omega_n]{} G_{HF}(i\omega)$$

$$G(i\omega) = \frac{\omega_n^2}{(-\omega^2 + \cancel{\omega_n^2}) + \cancel{i 2 \zeta \omega \omega_n}} \xrightarrow[\omega \gg \omega_n]{} -\frac{\omega_n^2}{\omega^2}$$

$$G_{HF}(i\omega) = -\frac{\omega_n^2}{\omega^2} = \frac{\omega_n^2}{\omega^2} e^{-i\pi}$$

$$|G_{HF}(i\omega)| = \frac{\omega_n^2}{\omega^2} = \left(\frac{\omega_n}{\omega}\right)^2 = 1 \Big/ \left(\frac{\omega}{\omega_n}\right)^2$$

$$|G_{HF}(i\omega)|_{dB} = -40 \log_{10} \left(\frac{\omega}{\omega_n}\right)$$

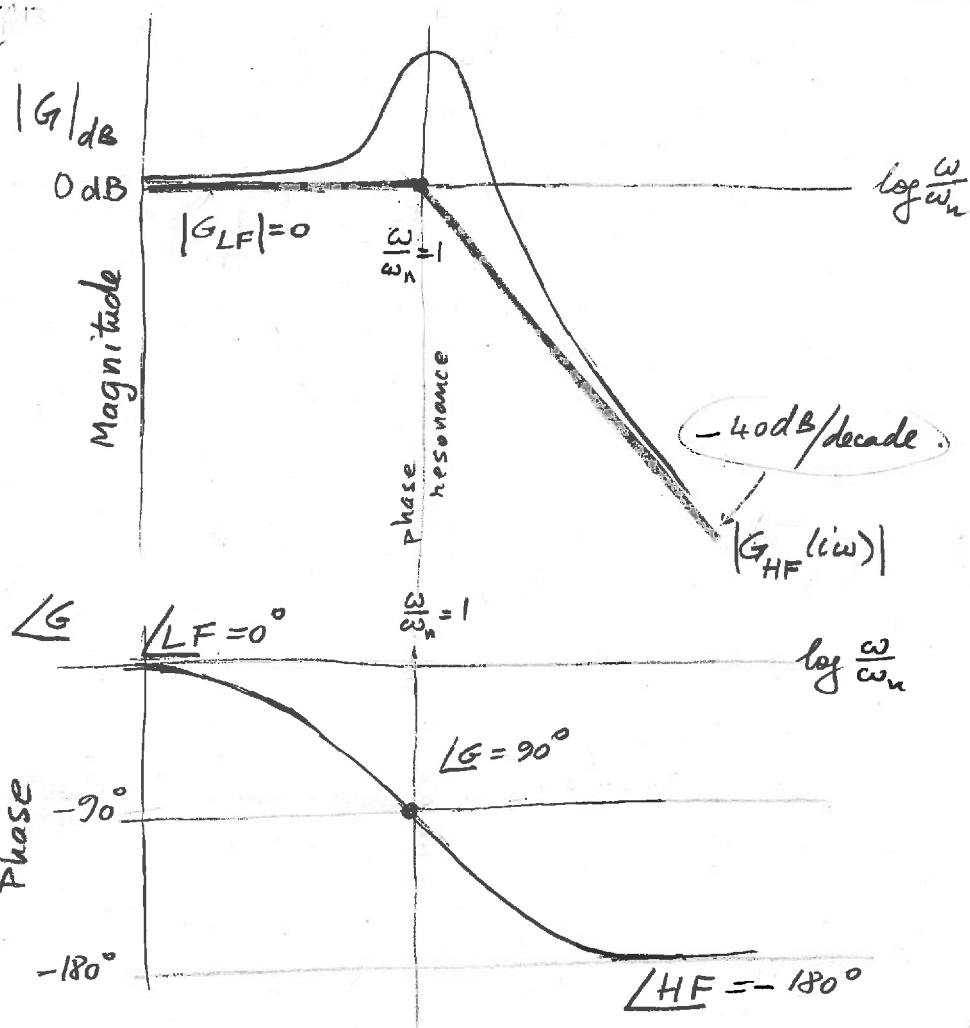
$\omega_2 = 10\omega_1$

$\omega_2/\omega_1 = 10 \; ; \log_{10} \omega_2/\omega_1 = 1$

"$-$ 40 dB/decade"

$$\angle G_{HF}(i\omega) = -\pi = -180^\circ$$

3/II



Phase

$$\angle G = -\tan^{-1}\left(\frac{2\zeta\omega\omega_n}{-\omega^2+\omega_n^2}\right) = -\tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}$$

$\frac{\omega}{\omega_n} \ll 1$, $\angle G = 0°$ (LF)

$\frac{\omega}{\omega_n} = 1$, $\angle G = -90°$ 'phase resonance'. $(\tan^{-1}\infty = 90°)$

$\frac{\omega}{\omega_n} \gg 1$, $\angle G = -180°$ (HF)

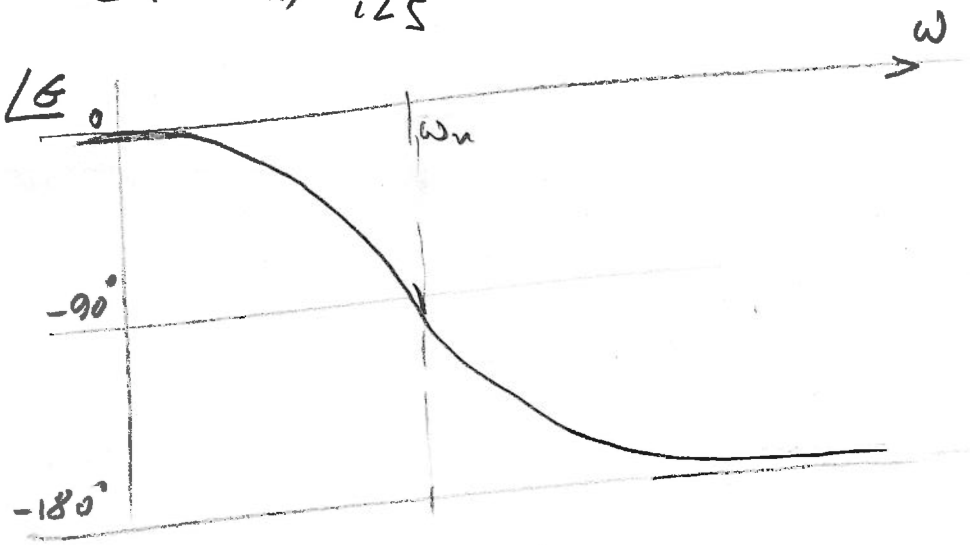
Phase diagram.

$$G(i\omega) = \frac{\omega_n^2}{(-\omega^2+\omega_n^2) + i\,2\zeta\omega\omega_n}$$

$$G_{LF}(\omega) = 1 \qquad \angle 1 = 0^\circ$$

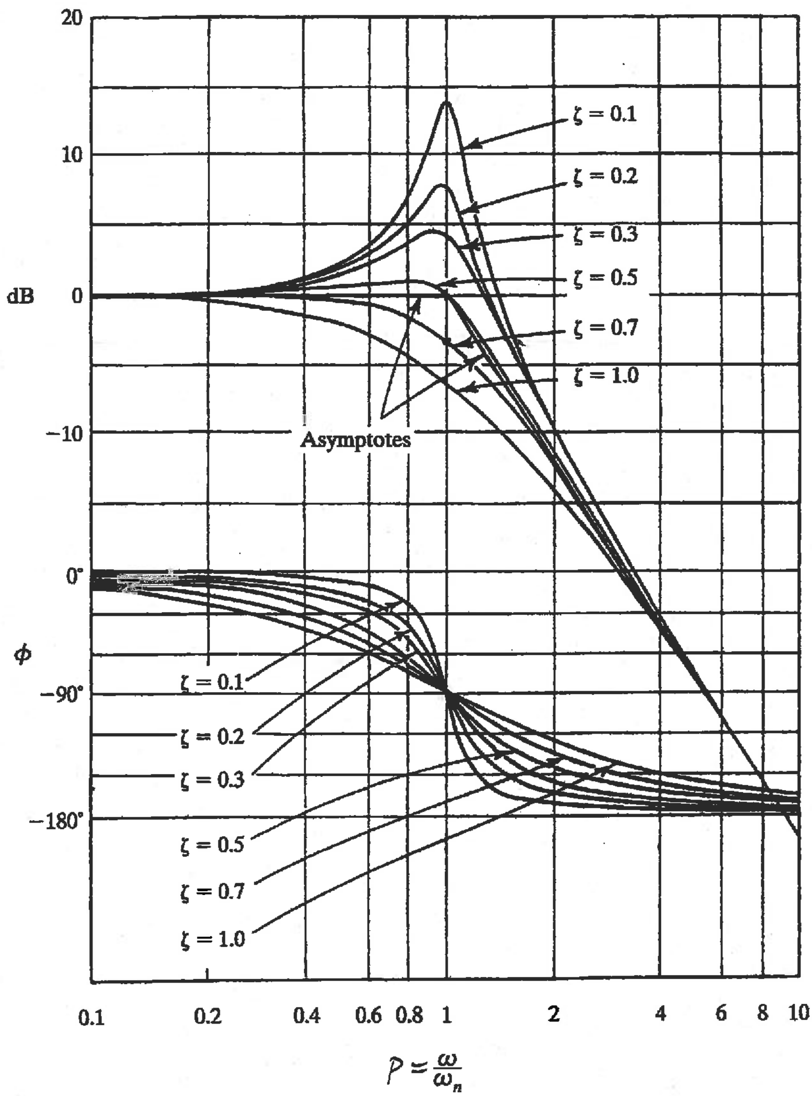
$$G_{HF}(\omega) = -\frac{\omega_n^2}{\omega^2} \qquad \angle -1 = -180^\circ$$

$$G(\omega=\omega_n) = \frac{1}{i2\zeta} \qquad \angle \frac{1}{i} = -90^\circ$$



4a
II

Bode Diagram Representation of the Frequency Response



Log-magnitude curves together with the asymptotes and phase-angle curves of the quadratic sinusoidal transfer function

MATLAB FD2