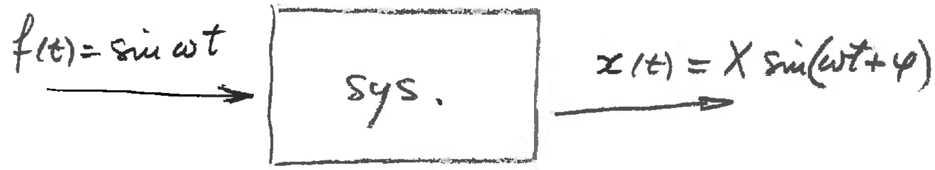
8 Frequency Analysis

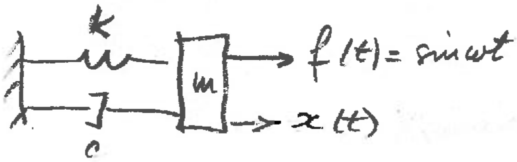
8.1 Time Response Under Harmonic Excitations

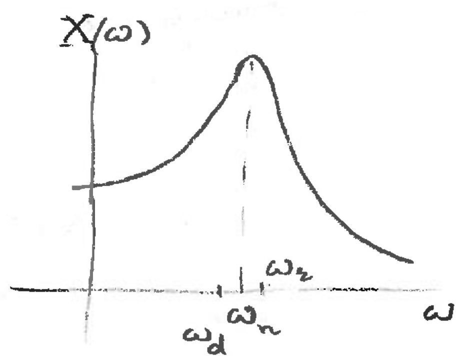
Time Response to harmonic excitation



One is interested in finding how the system responds to harmonic excitation of different frequency values, $\omega = 2\pi f$

Ex: For a 2^{nd} order system, one is concerned with avoiding resonance where the response may become very large



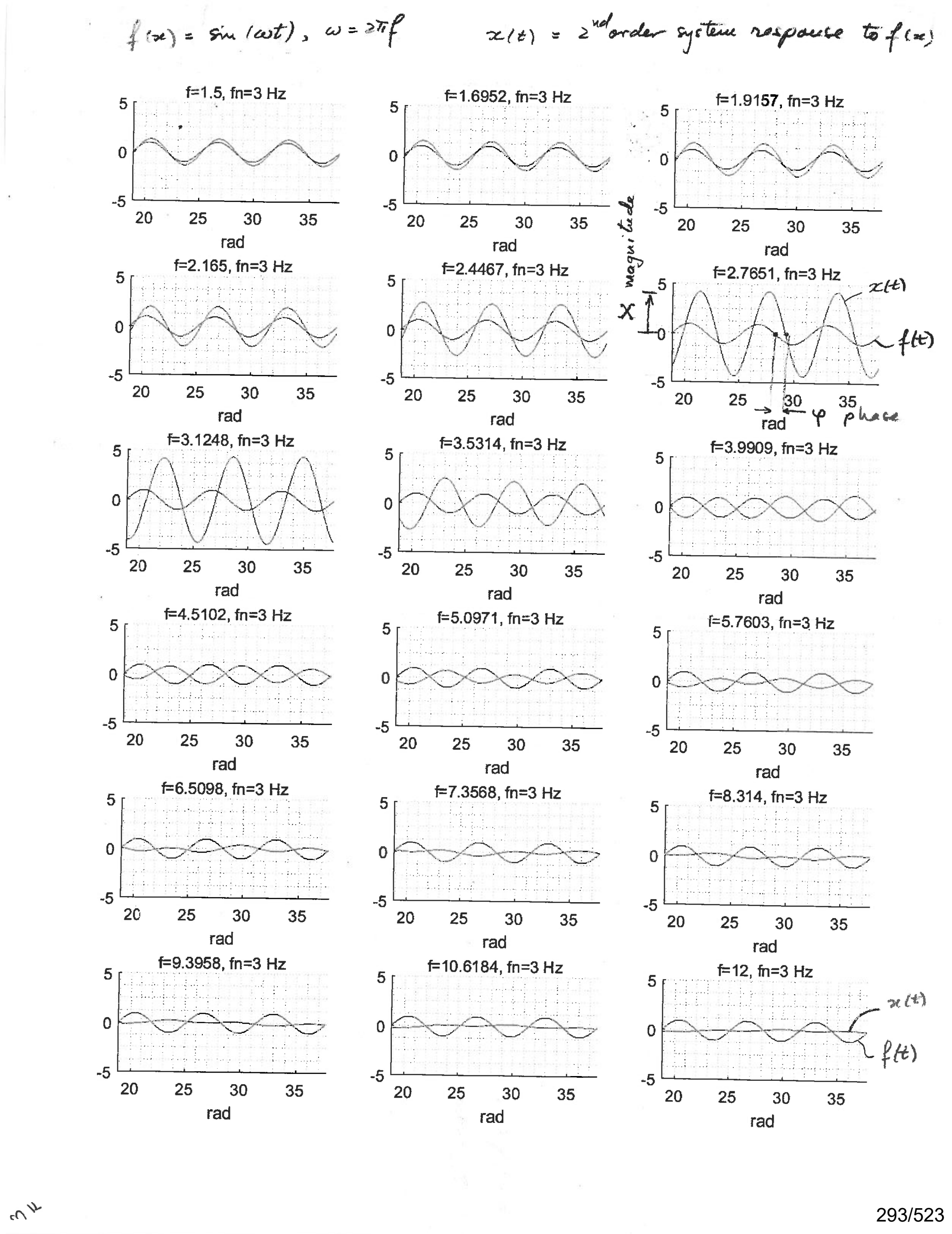


$$x(t) = X(\omega)\sin(\omega t + \varphi(\omega))$$

$$\zeta = \frac{c}{2\sqrt{km}}, \quad \omega_n = \sqrt{\frac{k}{m}}.$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

We notice that the amplitude and phase of the time response varies with excitation freq. ω


f(x) = sin(ωt), ω = 2πf
x(t) = 2nd order system response to f(x)
f=1.5, fn=3 Hz
f=1.6952, fn=3 Hz
f=1.9157, fn=3 Hz
f=2.165, fn=3 Hz
f=2.4467, fn=3 Hz
f=2.7651, fn=3 Hz
magnitude
X
x(t)
f(t)
φ phase
f=3.1248, fn=3 Hz
f=3.5314, fn=3 Hz
f=3.9909, fn=3 Hz
f=4.5102, fn=3 Hz
f=5.0971, fn=3 Hz
f=5.7603, fn=3 Hz
f=6.5098, fn=3 Hz
f=7.3568, fn=3 Hz
f=8.314, fn=3 Hz
f=9.3958, fn=3 Hz
f=10.6184, fn=3 Hz
f=12, fn=3 Hz
x(t)
f(t)
rad
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$$f(x) = \sin \omega t$$

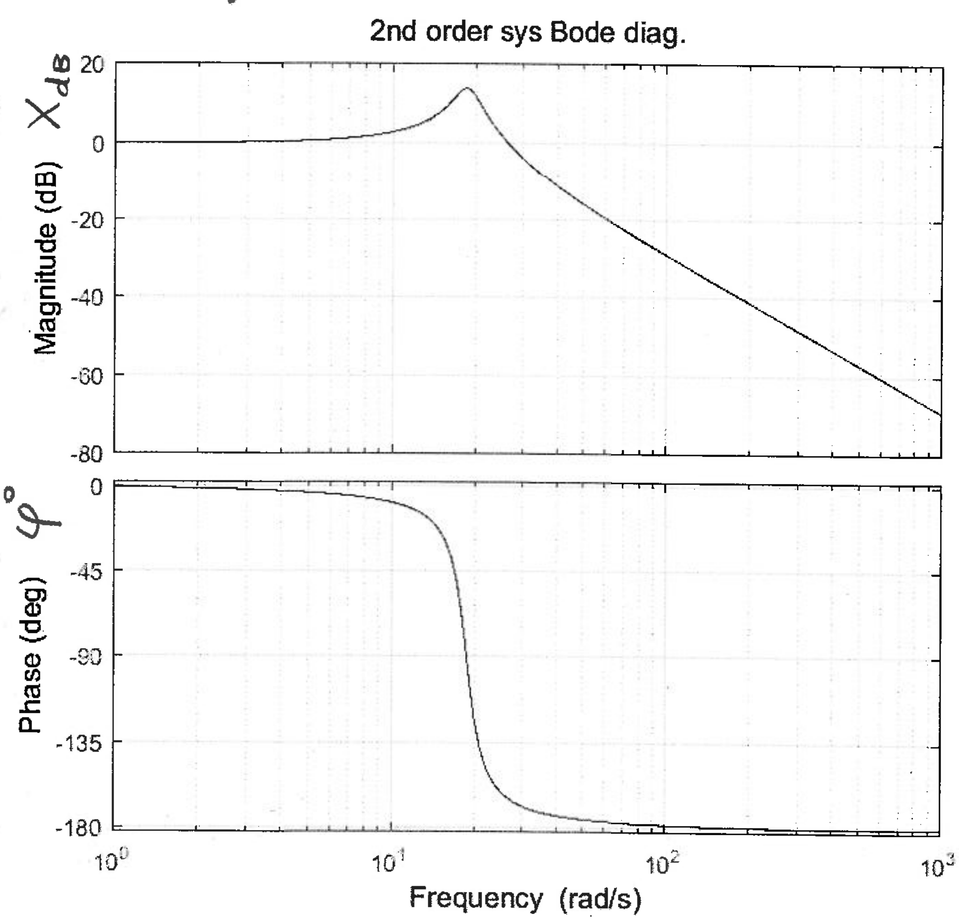
$$x(t) = X(\omega) \sin[\omega t + \varphi(\omega)]$$

$X(\omega)$ = magnitude of response

$\varphi(\omega)$ = phase of response

Magnitude X and phase φ vary with excitation freq. ω.

Bode diagram plots: variation of $X(\omega)$ & $\varphi(\omega)$



Note that φ is −ve, i.e. the response lags behind the excitation.

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```
% Amplitude_and_phase_2ndOrderSys.m
% AMPLITUDE AND PHASE IN FREQUENCY RESPONSE
clc
clear
close all
s=tf('s');
% f=input('f=');
figure (1);
%% 2nd order system
fn=3; z=10e-2; wn=2*pi*fn; G2=wn^2/(s^2+2*z*wn*s+wn^2);
M=6; N=3; Nplots=M*N;
fmin=fn/2; fmax=fn*4;
a=log10(fmin); b=log10(fmax); f=logspace(a,b,Nplots);
%% plotting setup
Na=1e3; amax=10*2*pi; da=amax/Na; angle=0:da:amax;
xmin=0.3*Na*da; xmax=0.6*Na*da;
%% plot respose at various frequencies
for i=1:Nplots
w=2*pi*f(i);    % excitation frequency
t=angle/w;      % time steps at this excitation freq.
A=1;  % forcing function amplitude
F=A*w/(s^2+w^2);    % Laplace transform of sine forcing function
fe=impulse(F,t);    % time response of forcing function
X2=G2*F;            % Laplace transform of 2nd order system response
subplot(M,N,i);
x2=impulse(X2,t);   % time response of 2nd order system
plot(angle,fe,angle,x2); hold  on
title(['f=' num2str(f(i)) ', fn=' num2str(fn) ' Hz' ],...
    'FontSize', 10,'FontWeight','normal')
xlabel('rad'); xlim([xmin xmax]); ylim([-5*A 5*A]);
% grid on
grid minor; box off
```

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C:\Mydata\1_USC\c...\FRF_2ndOrderSys_20161214.m Page 2

```
end
%% FRF Bode plots
figure(2)
bode(G2); grid on; title('2nd order sys Bode diag.' );
aw=log10(2*pi*fn/2); bw=log10(2*pi*fn*2); N=1e4; wBode=logspace(aw,bw,N);
figure(3)
bode(G2,wBode); grid on; title('zoom 2nd order sys Bode diag.' );
```

8.2 Frequency Response Function (FRF)

FRF

dB scale   deci bell
       1/10

dB scale is a $\log_{10}$ scale magnified by 20

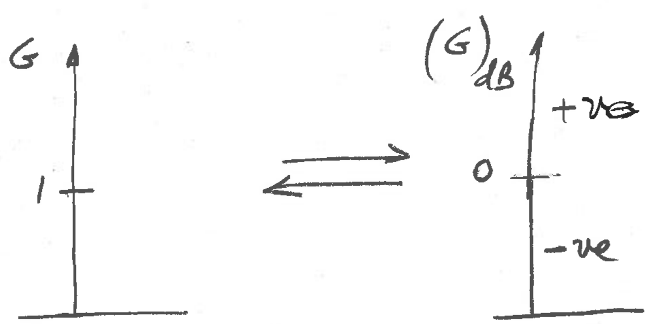
$$(G)_{dB} = 20 \log_{10} G$$

Properties

(1) Only +ve numbers have dB value
(cannot take log of −ve numbers!)

(2) $\left(\frac{1}{G}\right)_{dB} = -(G)_{dB}$ (reciprocal numbers)

(3)



$G > 1$	$(G)_{dB}$ is +ve
$G = 1$	$(G)_{dB} = 0$
$G < 1$	$(G)_{dB}$ is −ve

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2 FRF

(4) Scale up or scale down in physical values means shift up or shift down in dB

(a) double (octave up) = +6 dB

(b) "half" (octave down) = −6 dB

(c) "ten times" (decade) = +20 dB

octave = (A B C D E F G) A_1 8th musical note has double the frequency $f_{A_1} = 2 f_A$

Proof

(a) $\log_{10} 2 = 0.303$; $20 \log 2 \cong 6$

$G_2 = 2G_1$

$(G_2)_{dB} = 20 \log_{10}(2G_1) = 20 \log_{10} 2 + 20 \log_{10} G_1$

$= \underbrace{20 \times 0.303}_{\sim 6} + (G_1)_{dB}$

$(G_2)_{dB} = (G_1)_{dB} + 6\,dB$

>> mag2db(2)
ans 6.0206

(b) $G_2 = \frac{1}{2} G_1$

$(G_2)_{dB} = (G_1)_{dB} + 20 \log_{10}\left(\frac{1}{2}\right)$

$= (G_1)_{dB} + 20(-\log_{10} 2)$

$= (G_1)_{dB} - 6\,dB$

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3 FRF

(c) $G_2 = 10\, G_1$

$$(G_2)_{dB} = (G_1)_{dB} + 20 \underbrace{\log_{10}(10)}_{=1}$$

$$= (G_1)_{dB} + 20\,dB$$

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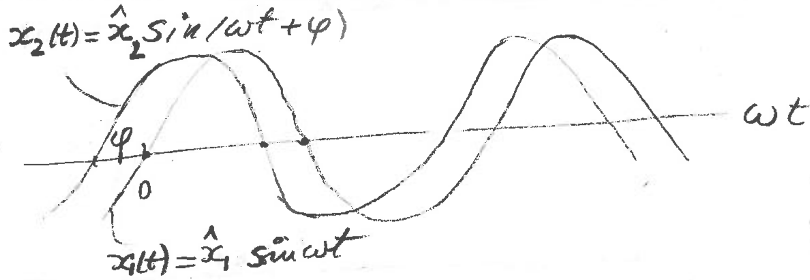
FRF

Phase

$x_1(t) = \hat{x}_1 \sin \omega t$ reference signal

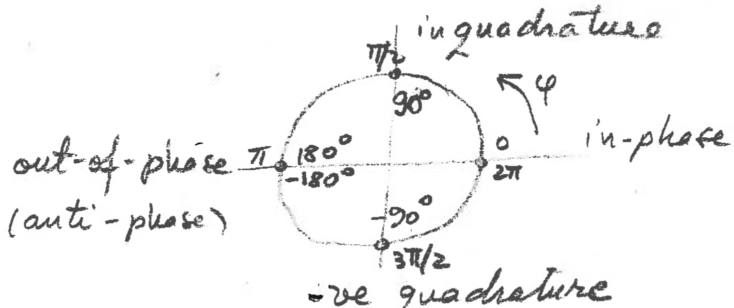
$x_2(t) = \hat{x}_2 \sin(\omega t + \varphi)$



$\psi_1 = \omega t$ phase of x_1 (reference)

$\psi_2 = \omega t + \varphi$ phase of x_2

$\varphi = \psi_2 - \psi_1$
- phase difference
- x_2 leads by φ



Periodic signals

$\left.\begin{matrix} 2\pi \\ 360^\circ \end{matrix}\right\}$ periodicity:
- $x(\omega t + 2\pi) = x(\omega t)$
- $x(\omega t + \pi) = x(\omega t - \pi)$

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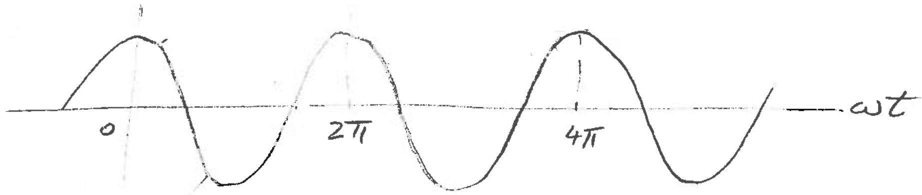
5 FRF

COMPLEX NUMBER REPRESENTATION OF HARMONIC SIGNALS



$\cos \omega t$

$$e^{i\omega t} = \cos \omega t + i \sin \omega t \quad \text{(Euler)}$$

$$\begin{cases} x(t) = \hat{x} \cos(\omega t + \varphi) \\ x(t) = \mathrm{Re}\left[\hat{x} e^{i(\omega t + \varphi)}\right] \end{cases}$$

$$\begin{cases} x(t) = \hat{x} \sin(\omega t + \varphi) \\ x(t) = \mathrm{Im}\left[\hat{x} e^{i(\omega t + \varphi)}\right] \end{cases}$$

In general $\quad x(t) = \hat{x} e^{i(\omega t + \varphi)}$

$$\hat{x} e^{i(\omega t + \varphi)} = \hat{x} e^{i\varphi} e^{i\omega t} = X e^{i\omega t}$$

$$X = \hat{x} e^{i\varphi} \quad \text{(phasor)}$$

$|X| = \hat{x}$

magnitude of $x(t)$

$|X| = \mathrm{abs}(X)$

$\angle X = \varphi$

phase of $x(t)$

$\angle X = \mathrm{angle}(X)$, rad

rad2deg$(\pi) = 180°$.

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6 FRF

Frequency response function (FRF)

- $G(s)$ TF: Transfer function in Laplace domain s
- $G(i\omega)$ FRF: Transfer function in frequency domain ω

$$f(t) = e^{i\omega t} \longrightarrow \boxed{G(i\omega)} \longrightarrow x(t) = G(i\omega)\, e^{i\omega t}$$

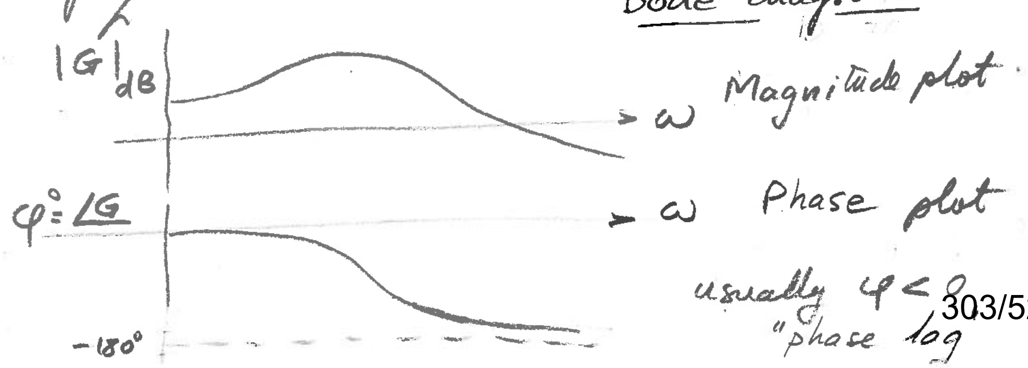
$G(i\omega)$ is a complex number

$|G(i\omega)|$ = magnitude, dB

$\angle G(i\omega)$ = phase, deg

Frequency response function (FRF) is $G(i\omega)$ measured over a range of frequencies ω

Bode diagram



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20181205

Harmonic Response via Laplace Transform

$$f(t) = e^{i\omega t} \xrightarrow{\mathcal{L}} F(s) = \frac{1}{s - i\omega}$$

$$X(s) = G(s)F(s) = G(s)\frac{1}{s-i\omega} \qquad (1)$$

Assume: $G(s) = K\dfrac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$ (2)

Poles in LHS $-p_n \cdots -p_1 < 0$

STABLE

(1) & (2) yields the PFE:

$$X(s) = \frac{a}{s-i\omega} + \frac{b_1}{s+p_1} + \frac{b_2}{s+p_2} + \cdots + \frac{b_n}{s+p_n} \qquad (3)$$

$$x(t) = a e^{i\omega t} + \underbrace{b_1 e^{-p_1 t} + b_2 e^{-p_2 t} + \cdots b_n e^{-p_n t}}_{\text{Transient response} \xrightarrow[t\to\infty]{} 0} \qquad (4)$$

steady state response.

$$x_{ss}(t) = a e^{i\omega t} \qquad (5)$$

To find a, multiply (3) by $s - i\omega$ and make $s = i\omega$ to get:

$$\Big[(s-i\omega)X(s)\Big]_{s=i\omega} = \left[a + \cancel{\left[\frac{b_1}{s+p_1} + \frac{b_2}{s+p_2} + \cdots \frac{b_n}{s+p_n}\right](s-i\omega)}\right]_{s=i\omega}$$

$$a = \Big[(s-i\omega)X(s)\Big]_{s=i\omega} \qquad (6)$$

$$= \left[\cancel{(s-i\omega)}\, G(s)\frac{1}{\cancel{(s-i\omega)}}\right]_{s=i\omega} = G(i\omega).$$

(6)→(5): $x_{ss}(t) = \boxed{G(i\omega)}e^{i\omega t}$.

complex amplitude

$$FRF = G(i\omega) = |G(i\omega)|e^{i\varphi}, \quad \varphi = \angle G(i\omega)$$

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FR14/

Harmonic Response via Laplace Transform

$$G(s) = \frac{1}{Ts+1}, \quad f(t) = e^{i\omega t} \longrightarrow F(s) = \frac{1}{s-i\omega}$$

$$X(s) = G(s)F(s) = \frac{1}{Ts+1}\cdot\frac{1}{s-i\omega} = \frac{1}{T}\cdot\frac{1}{s+\frac{1}{T}}\cdot\frac{1}{s-i\omega}$$

$s_1 = -\frac{1}{T} \quad s_2 = i\omega$

PFE, p 30, Eqⁿ (2.6), modified:

$$X(s) = \frac{a_1}{s-s_1} + \frac{a_2}{s-s_2} + \cdots + \frac{a_k}{s-s_k} + \cdots$$

$$a_k = \Big[(s-s_k)X(s)\Big]_{s=s_k}$$

$$a_1 = \left[\cancel{\left(s+\frac{1}{T}\right)}\frac{1}{T}\frac{1}{\cancel{s+\frac{1}{T}}}\frac{1}{s-i\omega}\right]_{s=-\frac{1}{T}} = \frac{1}{T}\frac{1}{-\frac{1}{T}-i\omega} = -\frac{1}{i\omega T+1}$$

$$a_2 = \left[\cancel{(s-i\omega)}\frac{1}{T}\frac{1}{s+\frac{1}{T}}\frac{1}{\cancel{s-i\omega}}\right]_{s=i\omega} = \frac{1}{T}\frac{1}{i\omega+\frac{1}{T}} = \frac{1}{i\omega T+1}$$

$$X(s) = -\frac{1}{i\omega T+1}\cdot\frac{1}{s+\frac{1}{T}} + \frac{1}{i\omega T+1}\cdot\frac{1}{s-i\omega}$$

$$x(t) = -\frac{1}{i\omega T+1}e^{-t/T} + \frac{1}{i\omega T+1}e^{i\omega t}$$

| transient | steady state |

$$x_{ss}(t) = \frac{1}{i\omega T+1}e^{i\omega t} = G(i\omega)e^{i\omega t}.$$

$$FRF = G(i\omega) = |G(i\omega)|e^{i\varphi}, \quad \varphi = \angle G(i\omega)$$

True for stable systems (i.e., transients vanish).

FR15 Harmonic Response via Laplace Transform for sine excitation

$$G(s) = \frac{1}{Ts+1}, \quad f(t) = \sin\omega t \rightarrow F(s) = \frac{\omega}{s^2+\omega^2}$$

$$X(s) = G(s)F(s) = \frac{1}{Ts+1} \cdot \frac{\omega}{s^2+\omega^2} = \frac{1}{T}\frac{1}{s+\frac{1}{T}} \cdot \frac{\omega}{(s-i\omega)(s+i\omega)}$$

Poles: $s_1 = -1/T \quad s_{2,3} = \pm i\omega$

PFE, p.30, Eq (2.6), modified:

$$X(s) = \frac{a_1}{s-s_1} + \frac{a_2}{s-s_2} + \dots + \frac{a_k}{s-s_k} + \dots$$

$$a_k = \Big[(s-s_k)X(s)\Big]_{s=s_k}$$

$$X(s) = \frac{a_1}{s+\frac{1}{T}} + \frac{a_2}{s-i\omega} + \frac{a_3}{s+i\omega}$$

$$a_1 = (s+\tfrac{1}{T})G(s)\Big|_{s=s_1} = \cancel{\left(s+\tfrac{1}{T}\right)}\frac{1/T}{\cancel{s+\frac{1}{T}}}\frac{\omega}{s^2+\omega^2}\Bigg|_{s=-\frac{1}{T}}$$

$$a_1 = \frac{1}{T}\frac{\omega}{\frac{1}{T^2}+\omega^2} = \frac{\omega T}{\omega^2T^2+1}$$

$$a_2 = (s-i\omega)\,G(s)\Big|_{s=i\omega} = \cancel{(s-i\omega)}\frac{1}{Ts+1}\frac{\omega}{\cancel{(s-i\omega)}(s+i\omega)}\Bigg|_{s=i\omega}$$

$$a_2 = \frac{1}{i\omega T+1} \cdot \frac{\omega}{i\omega+i\omega} = \frac{1}{i\omega T+1}\frac{1}{2i} = G(i\omega)\frac{1}{2i}$$

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FR16

$$a_3 = (s+i\omega)G(s)\Big|_{s=-i\omega} = \cancel{(s+i\omega)}\frac{1}{Ts+1}\frac{\omega}{(s-i\omega)\cancel{(s+i\omega)}}\Bigg|_{s=-i\omega}$$

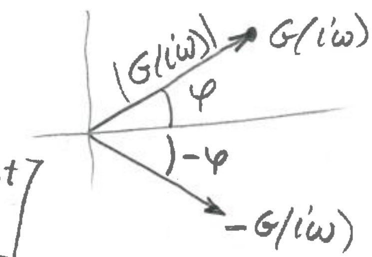
$$= \frac{1}{-i\omega T+1}\,\frac{1}{-2i} = -G(-i\omega)\frac{1}{2i}$$

$$X(s) = \underbrace{\frac{\omega T}{\omega^2T^2+1}\frac{1}{s+\frac{1}{T}}}_{\text{transient}} + \underbrace{\frac{1}{2i}\left[\frac{G(i\omega)}{s-i\omega} - \frac{G(-i\omega)}{s+i\omega}\right]}_{\text{steady state}}$$

$$x(t) = \frac{\omega T}{\omega^2T^2+1}e^{-t/T} + \frac{1}{2i}\left[G(i\omega)e^{i\omega t} - G(-i\omega)e^{-i\omega t}\right]$$

But $\quad G(i\omega) = |G(i\omega)|e^{i\varphi}, \quad \varphi = \angle G(i\omega)$

$\quad G(-i\omega) = |G(i\omega)|e^{-i\varphi}$



Hence

$$x_{ss}(t) = \frac{1}{2i}\left[G(i\omega)e^{i\omega t} - G(-i\omega)e^{-i\omega t}\right]$$

$$= \frac{1}{2i}|G(i\omega)|\left(e^{i\varphi}e^{i\omega t} - e^{-i\varphi}e^{-i\omega t}\right), \quad \varphi = \angle G(i\omega)$$

$$= |G(i\omega)|\frac{e^{i\psi} - e^{-i\psi}}{2i} \qquad \psi = \omega t + \varphi$$

$$= |G(i\omega)|\sin\psi$$

$$x_{ss}(t) = |G(i\omega)|\sin(\omega t+\varphi), \quad \varphi = \angle G(i\omega)$$

FR17

Lemma:

If $G(s)$ is polynomial (or fraction of)

and $G(i\omega) = |G(i\omega)| e^{i\varphi}$

Then $G(-i\omega) = |G(i\omega)| e^{-i\varphi}$.

Proof

(a) $G(s) = s + a$

$$G(i\omega) = i\omega + a = \sqrt{a^2+\omega^2}\, e^{i\varphi}, \quad \varphi = \tan^{-1}\frac{\omega}{a}.$$

$$G(-i\omega) = -i\omega + a = \sqrt{a^2+\omega^2}\, e^{i\varphi^*}, \quad \varphi^* = \tan^{-1}\frac{-\omega}{a} = -\varphi$$

(b) $G(s) = (s+a_1)(s+a_2)$

$$G(i\omega) = (i\omega + a_1)(i\omega + a_2)$$

$$= \sqrt{a_1^2+\omega^2}\sqrt{a_2^2+\omega^2}\, e^{i\varphi_1} e^{i\varphi_2}, \quad \varphi_1 = \tan^{-1}\frac{\omega}{a_1}, \quad \varphi_2 = \tan^{-1}\frac{\omega}{a_2}$$

$$G(-i\omega) = (-i\omega + a_1)(-i\omega + a_2)$$

$$= \sqrt{a_1^2+\omega^2}\sqrt{a_2^2+\omega^2}\, e^{i\varphi_1^*} e^{i\varphi_2^*} \qquad \varphi_1^* = \tan^{-1}\frac{-\omega}{a_1} = -\varphi_1, \quad \varphi_2^* = \tan^{-1}\frac{-\omega}{a_2} = -\varphi_2$$

etc.

1st order system FRF

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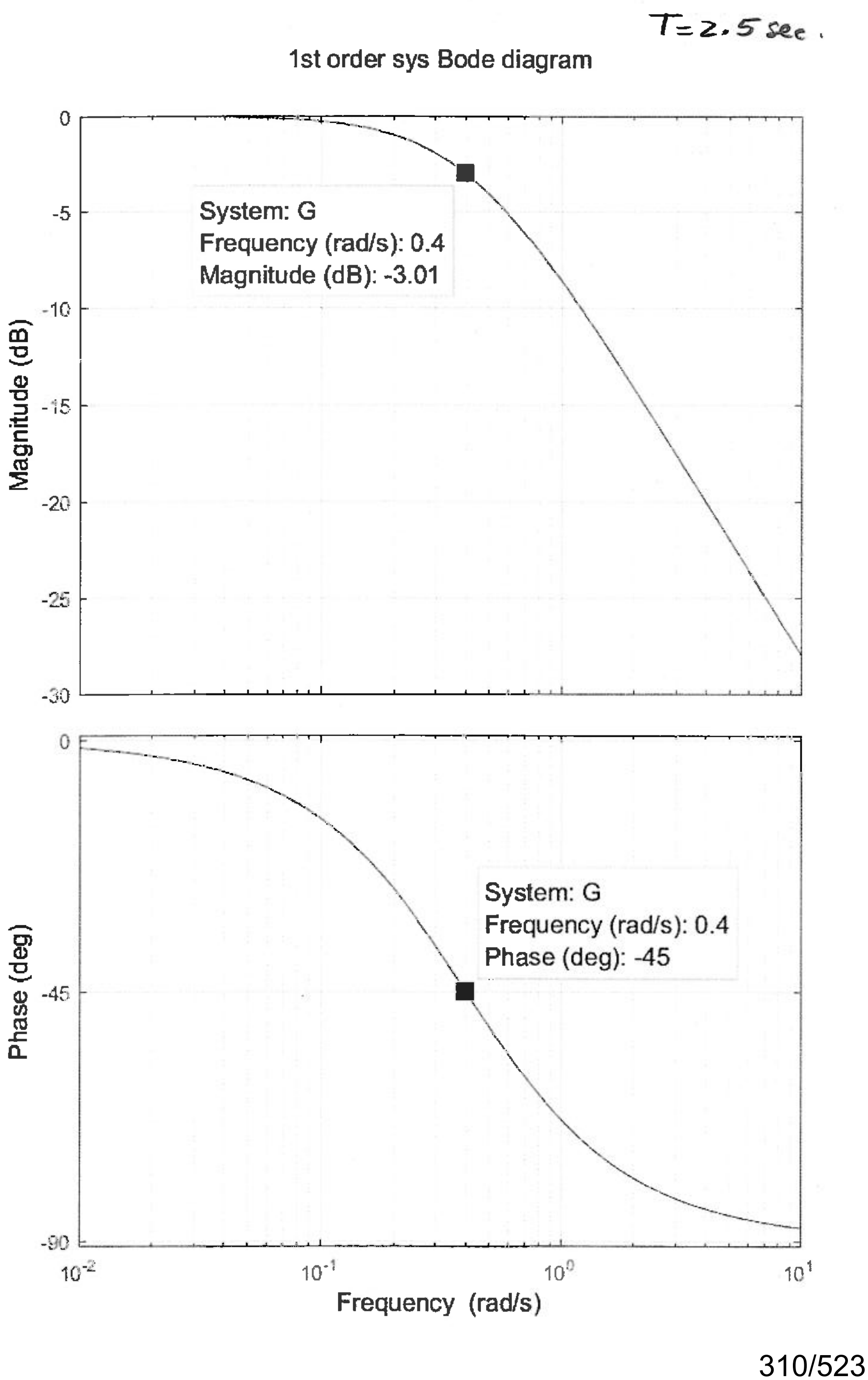
T=2.5 sec.
1st order sys Bode diagram
System: G
Frequency (rad/s): 0.4
Magnitude (dB): -3.01
Magnitude (dB)
0
-5
-10
-15
-20
-25
-30
System: G
Frequency (rad/s): 0.4
Phase (deg): -45
Phase (deg)
0
-45
-90
10^-2
10^-1
10^0
10^1
Frequency (rad/s)
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1^{st} Order System FRF

$$G(s) = \frac{1}{Ts+1}$$ transfer function (TF)

$$G(i\omega) = \frac{1}{i\omega T+1}$$ frequency response function (FRF)

Asymptotes

Define $\omega_c = 1/T$

Low freq. asymptote $\omega \ll \omega_c$, $\omega T \ll 1$ $G(i\omega) \to G_{LF}$

$$G(i\omega) = \frac{1}{i\omega T+1} \xrightarrow[\omega T \ll 1]{} \frac{1}{1} = 1$$

$\therefore$ $G_{LF}(i\omega) = 1$, $\left|G_{LF}\right|_{dB} = 0$ $\angle G_{LF} = 0^\circ$

High freq. asymptote $\omega \gg \omega_c$, $\omega T \gg 1$ $G(i\omega) \to G_{HF}$

$$G(i\omega) = \frac{1}{i\omega T+1} \xrightarrow[\omega T \gg 1]{} \frac{1}{i\omega T}$$

$\therefore$ $G_{HF}(i\omega) = \frac{1}{i\omega T} = \frac{1}{\omega T} e^{-i\frac{\pi}{2}}$

$$|G_{HF}| = \frac{1}{\omega T} \qquad \angle G_{HF} = -\frac{\pi}{2} = -90^\circ$$

Intersection of G_{LF} & G_{HF}

$1 = \frac{1}{\omega_c T}$ $\omega_c = \frac{1}{T}$ cutoff frequency, decay starts at ω_c

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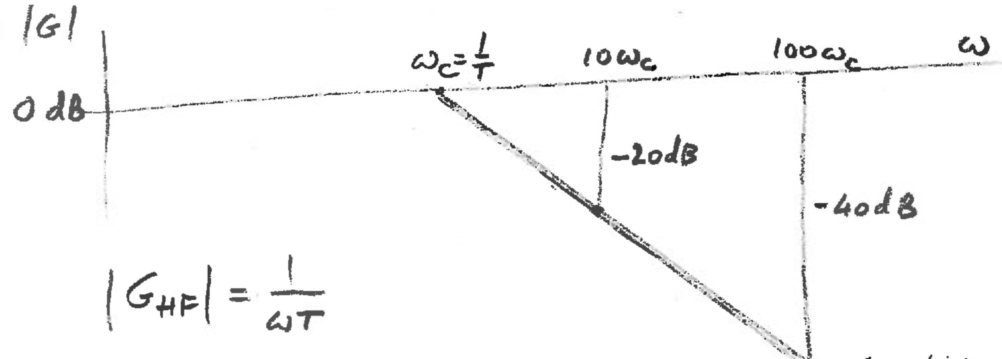
Slope of G_{HF}

$$|G_{HF}|_{dB} = -20 \log_{10}(\omega T)$$



$$|G_{HF}| = \frac{1}{\omega T}$$

$\omega_1 = \omega_c = \frac{1}{T}$, $\omega_1 T = 1$, $|G_{HF}(\omega_1 T)| = 0\,dB$

$\omega_2 = 10\omega_c$, $\omega_2 T = 10$, $|G_{HF}(\omega_2 T)| = \frac{1}{10} = -20\,dB$

$\omega_3 = 100\,\omega_c$, $\omega_3 T = 100$, $|G_{HP}(\omega_3 T)| = \frac{1}{100} = -40\,dB$

$|G_{HF}|$ drops $-20\,dB$/decade

For octave, take ω_c, $2\omega_c$, $4\omega_c$

ω_1	ω_2	ω_3
1	$\frac{1}{2}$	$\frac{1}{4}$
0dB	-6dB	-12dB.

$|G_{HF}|$ drops $-6\,dB$/octave

3
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Exact amplitude and phase at ω_c

$$\omega_c = \frac{1}{T}$$

$$G(i\omega_c) = \frac{1}{i\underbrace{\omega_c T}_{1}+1} = \frac{1}{1+i} = \frac{1}{\sqrt{2}} e^{-i\frac{\pi}{4}}$$

$$|G(i\omega_c)| = \frac{1}{\sqrt{2}} = -3dB$$

mag2db(1/sqrt(2))
3.0103

$$\angle G(i\omega_c) = -\frac{\pi}{4} = -45^\circ$$

Error of using asymptotes

Maximum error occurs at ω_c

Appox. value = 0 dB

Exact value = −3 dB

Error = 3 dB

Bandwidth ω_B

Bandwidth is defined as the frequency below which signal does not decrease more than 3 dB

For 1st order sys, $\omega_B = \omega_c = \frac{1}{T}$

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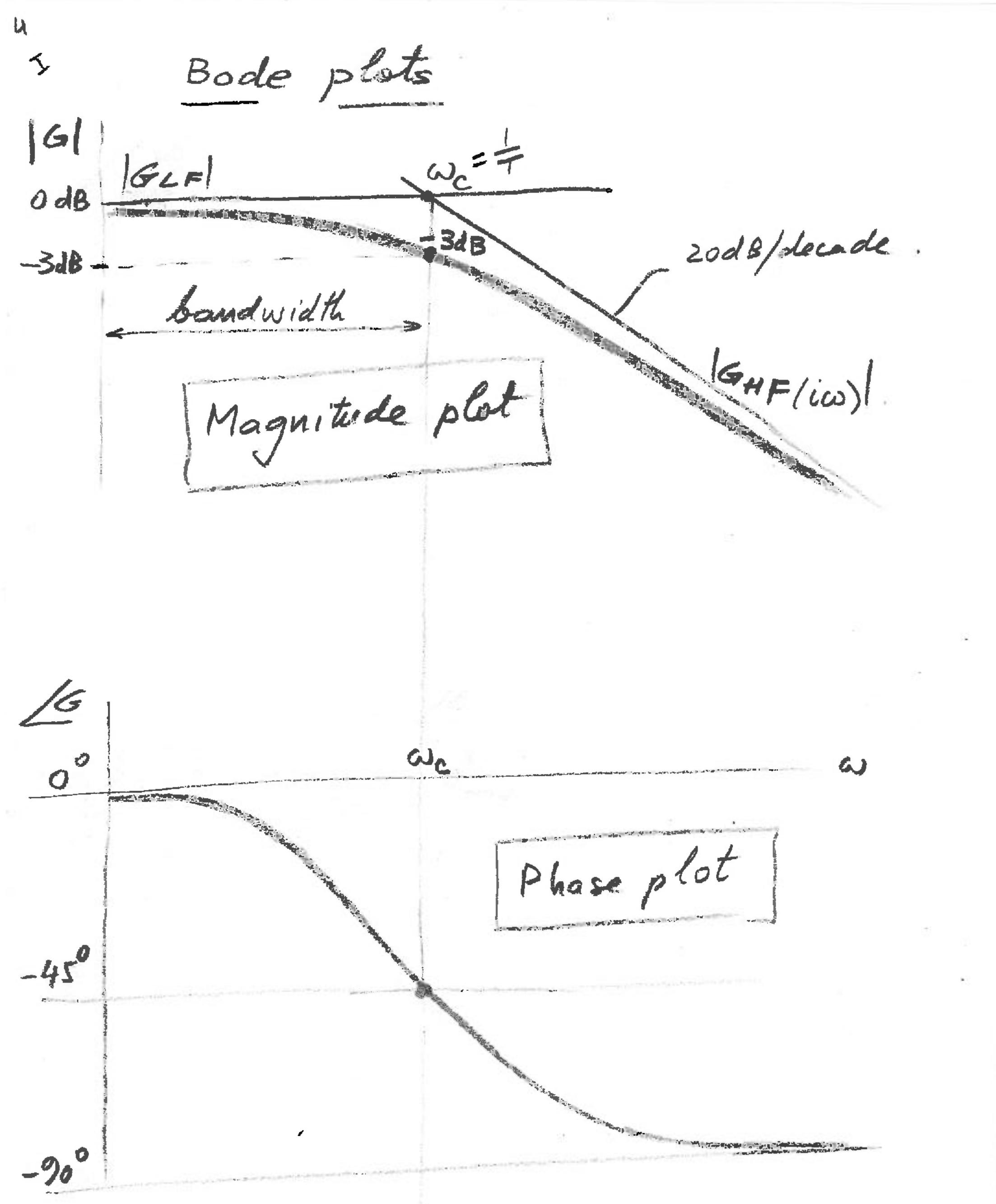
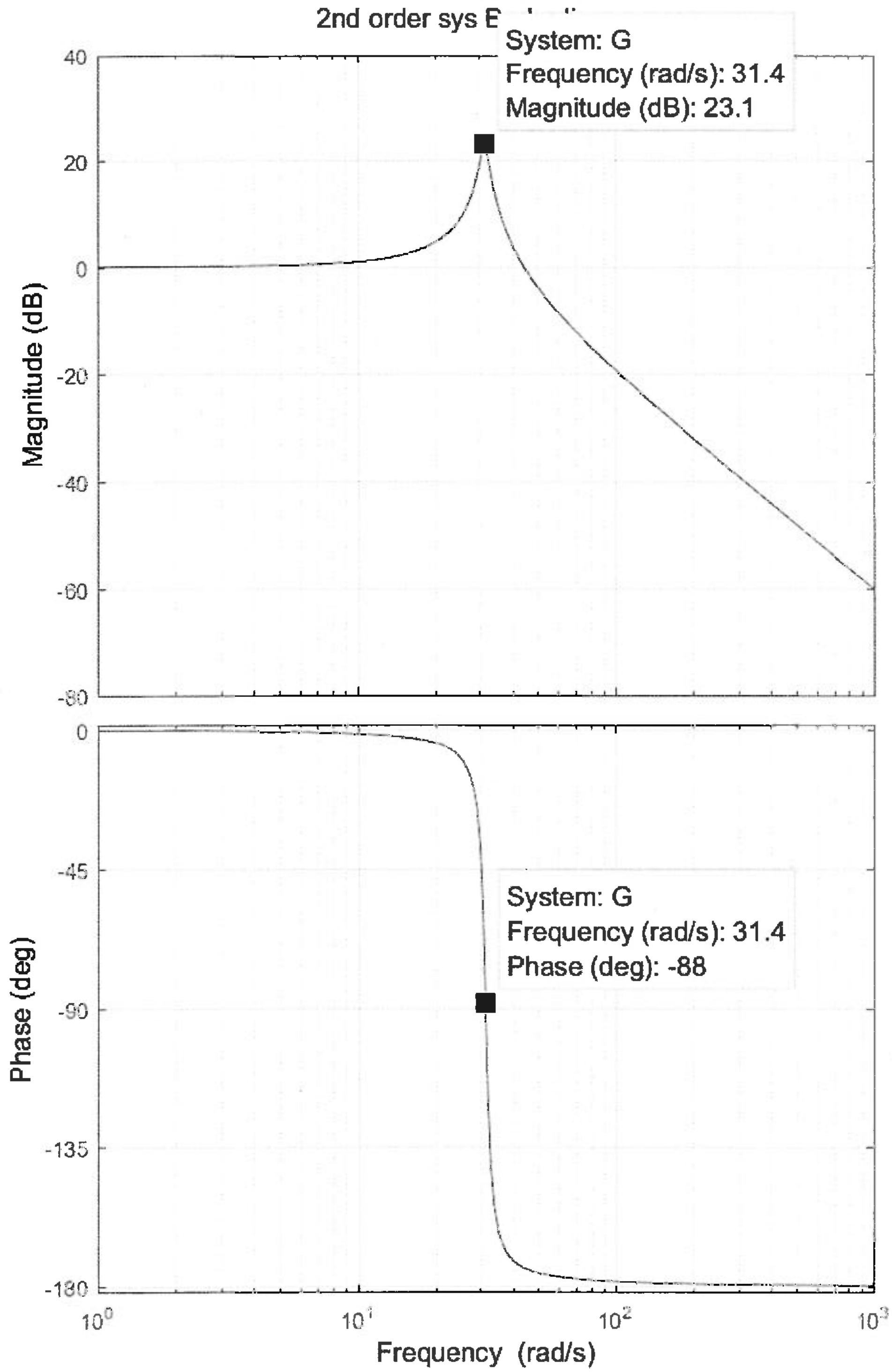

Bode plots
|G|
0 dB
-3dB
|GLF|
ωc = 1/T
-3dB
20dB/decade
bandwidth
|GHF(iω)|
Magnitude plot
∠G
0°
ωc
ω
Phase plot
-45°
-90°

314/523

2nd order system FRF

315/523

$f_n = 5\,Hz$
$\zeta = 3.5\%$



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2^{nd} order syst. FRF

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$G(i\omega) = \frac{\omega_n^2}{-\omega^2 + 2i\zeta\omega_n\omega + \omega_n^2}$$

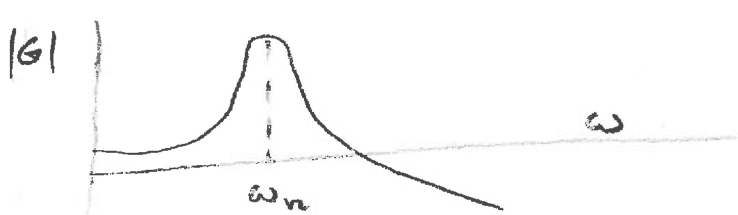
$$= \frac{\omega_n^2}{(-\omega^2 + \omega_n^2) + i\,2\zeta\omega_n\omega}$$

Phase resonance: $\omega = \omega_n$ (natural freq.)

$$G(i\omega_n) = \frac{\omega_n^2}{(\cancel{-\omega_n^2 + \omega_n^2}) + i\,2\zeta\omega_n\omega_n} = \frac{1}{i2\zeta}$$

$$|G(i\omega_n)| = \frac{1}{2\zeta}$$

$$\angle G(i\omega_n) = \angle \frac{1}{i2\zeta} = -90^\circ$$



2/II

LF asymptote

$$G(i\omega) \xrightarrow[\omega \ll \omega_n]{} G_{LF}(i\omega)$$

$$G(i\omega) = \frac{\omega_n^2}{(-\cancel{\omega^2}+\omega_n^2) + i 2\zeta\omega\omega_n} \xrightarrow[\omega \ll \omega_n]{} \frac{\omega_n^2}{\omega_n^2} = 1$$

$$G_{LF}(i\omega) = 1 \qquad |G_{LF}|_{dB} = 0\, dB$$

$$\angle G_{LF} = 0^\circ$$

HF asymptote

$$G(i\omega) \xrightarrow[\omega \gg \omega_n]{} G_{HF}(i\omega)$$

$$G(i\omega) = \frac{\omega_n^2}{(-\omega^2+\cancel{\omega_n^2}) + i 2\zeta\omega\omega_n} \xrightarrow[\omega \gg \omega_n]{} -\frac{\omega_n^2}{\omega^2}$$

$$G_{HF}(i\omega) = -\frac{\omega_n^2}{\omega^2} = \frac{\omega_n^2}{\omega^2} e^{-i\pi}$$

$$|G_{HF}(i\omega)| = \frac{\omega_n^2}{\omega^2} = \left(\frac{\omega_n}{\omega}\right)^2 = 1 \Big/ \left(\frac{\omega}{\omega_n}\right)^2$$

$$|G_{HF}(i\omega)|_{dB} = -40 \log_{10}\left(\frac{\omega}{\omega_n}\right)$$

$\omega_2 = 10\omega_1$

$\omega_2/\omega_1 = 10$; $\log_{10} \omega_2/\omega_1 = 1$ "-40 dB/decade"

$$\angle G_{HF}(i\omega) = -\pi = -180^\circ$$

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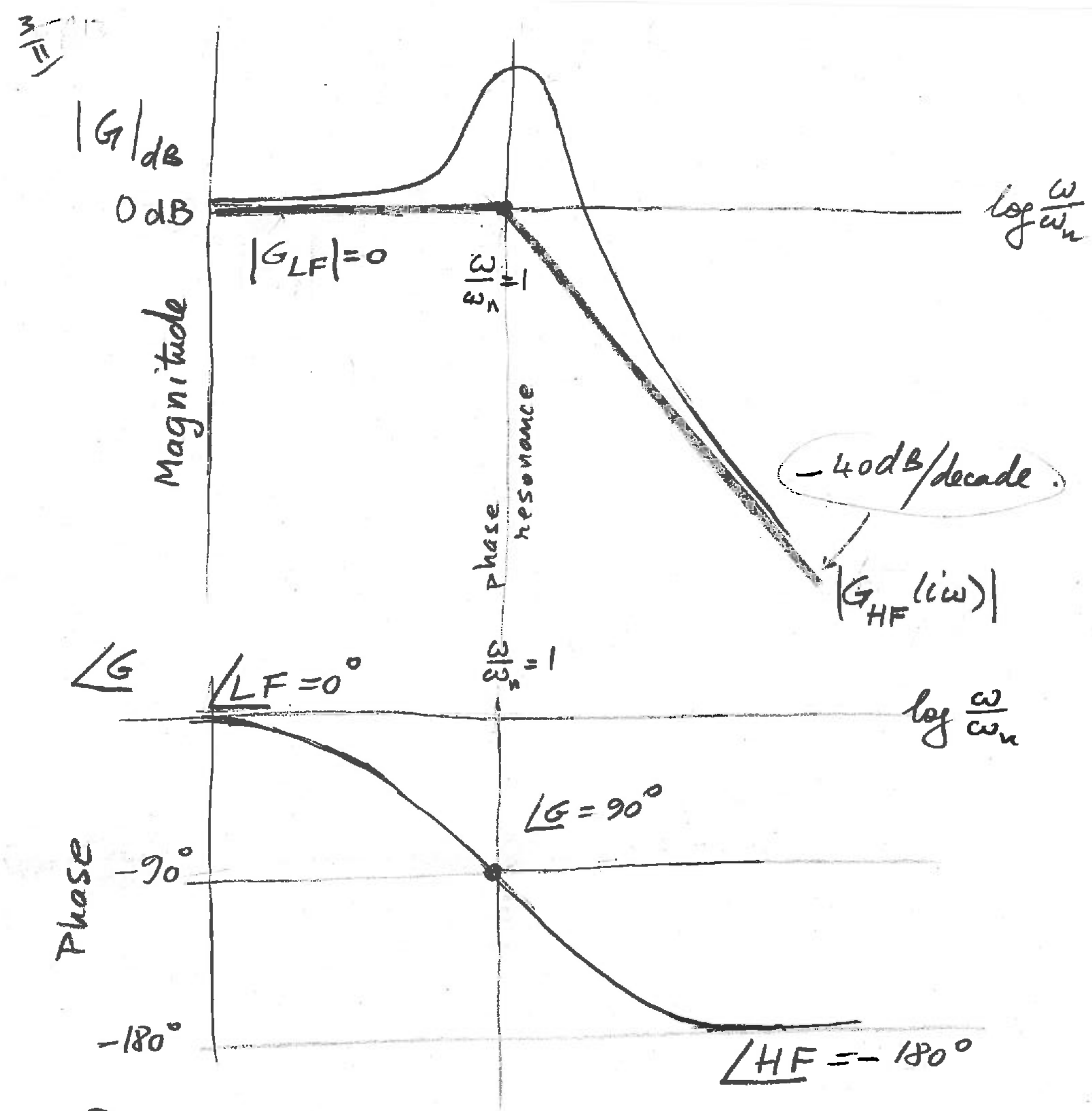


Phase

$$\angle G = -\tan^{-1}\left(\frac{2\zeta\omega\omega_n}{-\omega^2+\omega_n^2}\right) = -\tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega^2}}$$

$\frac{\omega}{\omega_n} \ll 1$, $\angle G = 0°$ (LF)

$\frac{\omega}{\omega_n} = 1$, $\angle G = -90°$ 'phase resonance'. ($\tan^{-1}(\infty) = 90°$)

$\frac{\omega}{\omega_n} \gg 1$, $\angle G = -180°$ (HF)

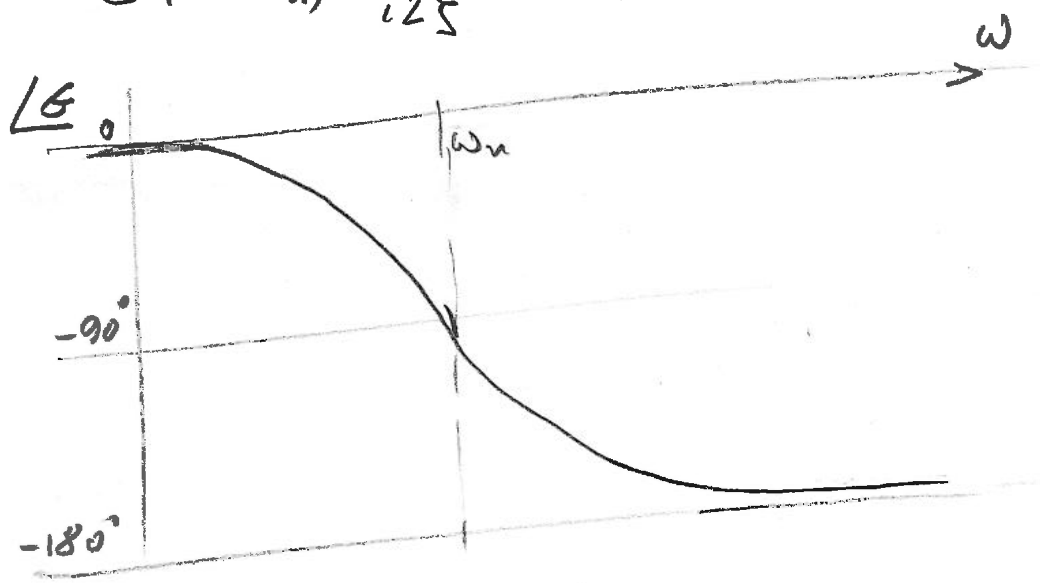
Phase diagram.

$$G(i\omega) = \frac{\omega_n^2}{(-\omega^2+\omega_n^2) + i2\zeta\omega\omega_n}$$

$$G_{LF}(\omega) = 1 \qquad \angle 1 = 0^\circ$$

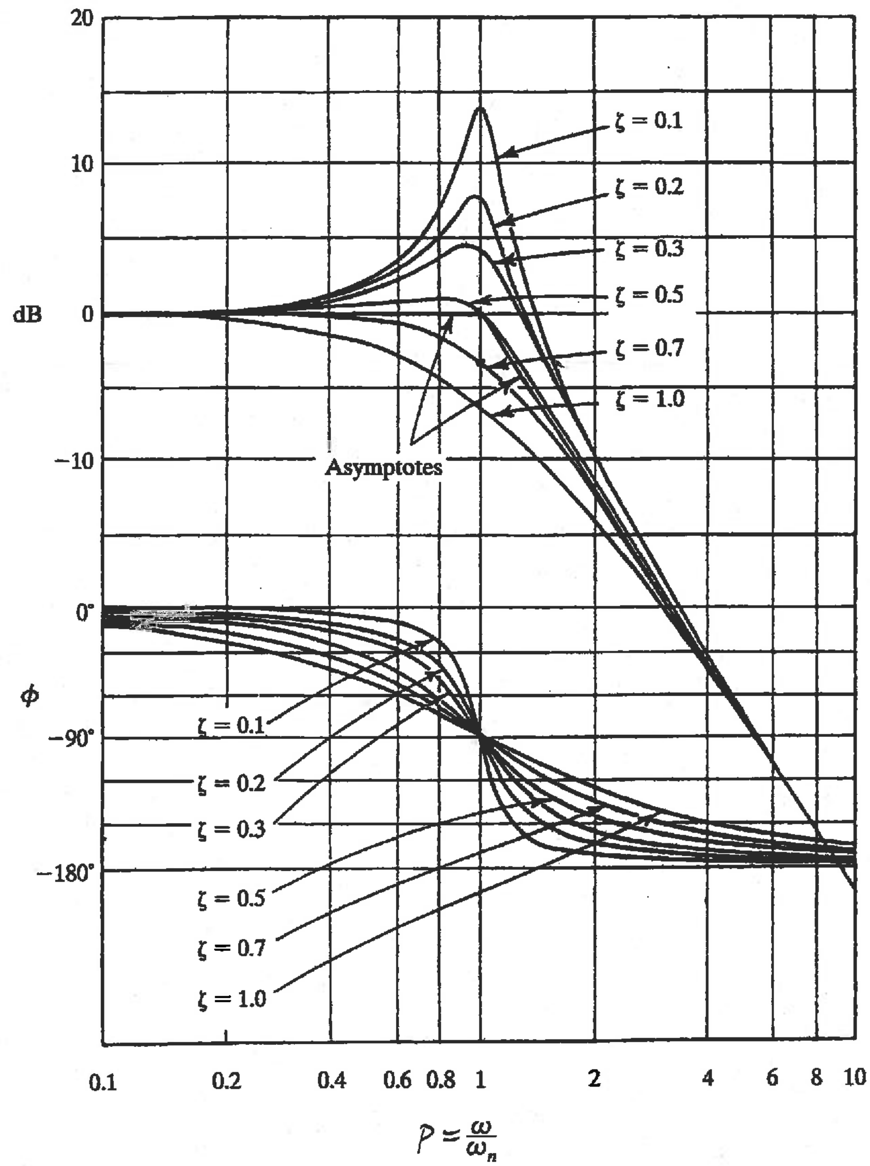
$$G_{HF}(\omega) = -\frac{\omega_n^2}{\omega^2} \qquad \angle -1 = -180^\circ$$

$$G(\omega=\omega_n) = \frac{1}{i2\zeta} \qquad \angle \frac{1}{i} = -90^\circ$$



4a
II

Bode Diagram Representation of the Frequency Response



Log-magnitude curves together with the asymptotes and phase-angle curves of the quadratic sinusoidal transfer function

MATLAB FD2

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8.3 Performance Indicators in Frequency Domains

PIF

Generic Performance
Indicators in Freq. Domain

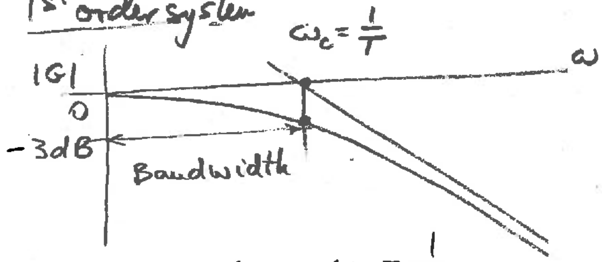
p18

Generic Performance Indicators in Freq. Domain

Bandwidth and cutoff frequency, ω_B

$|G(\omega_B)| = G(0) - 3dB$ has declined 3dB below LF value

1st order system

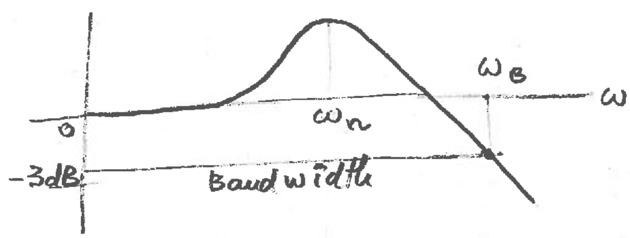


$$G(s) = \frac{1}{Ts+1}$$

$$\omega_B = \omega_c = \frac{1}{T}$$

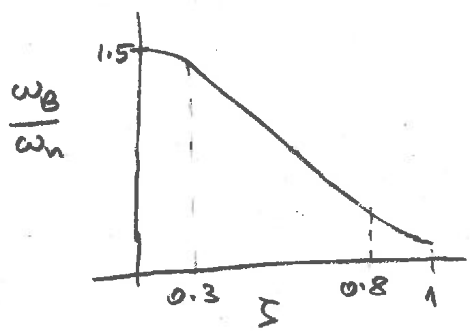
2nd order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



$$|G(i\omega)|_{dB} = -20 \log_{10} \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2} = -3dB$$

ω_B is determined graphically or solved numerically



$$\frac{\omega_B}{\omega_n} \approx -1.19\zeta + 1.85$$

$$0.3 < \zeta < 0.8$$

$\omega_B \uparrow$ as $\zeta \downarrow$

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3 PIF Ex. 11.2 : 1st order system performance

Given: two systems:

$$G_1 = \frac{1}{s+1} \quad ; \quad G_2 = \frac{1}{3s+1}$$

Find: (a) bandwidth

(b) frequency response

(c) step response

(d) ramp response

(e) discuss results

Solution:

$$G_1(i\omega) = \frac{1}{i\omega+1} = \frac{1}{i\omega T_1+1} \rightarrow T_1 = 1$$

$$G_2(i\omega) = \frac{1}{3i\omega+1} = \frac{1}{i\omega T_2+1} \longrightarrow T_2 = 3$$

(a) $\omega_B = \frac{1}{T}$ $\quad \omega_1^B = \frac{1}{1} = 1 \text{ rad/sec} \quad \omega_2^B = \frac{1}{3} \text{ rad/sec}$

(b), (c), (d): see MATLAB Ex. 11.2

(e) System 1 has bandwidth three times larger than system 2 ($\omega_1 = 1$ vs. $\omega_2 = 1/3$ rad/sec). Sys. 1 has faster step response and follows the ramp input much better / smaller ramp error)

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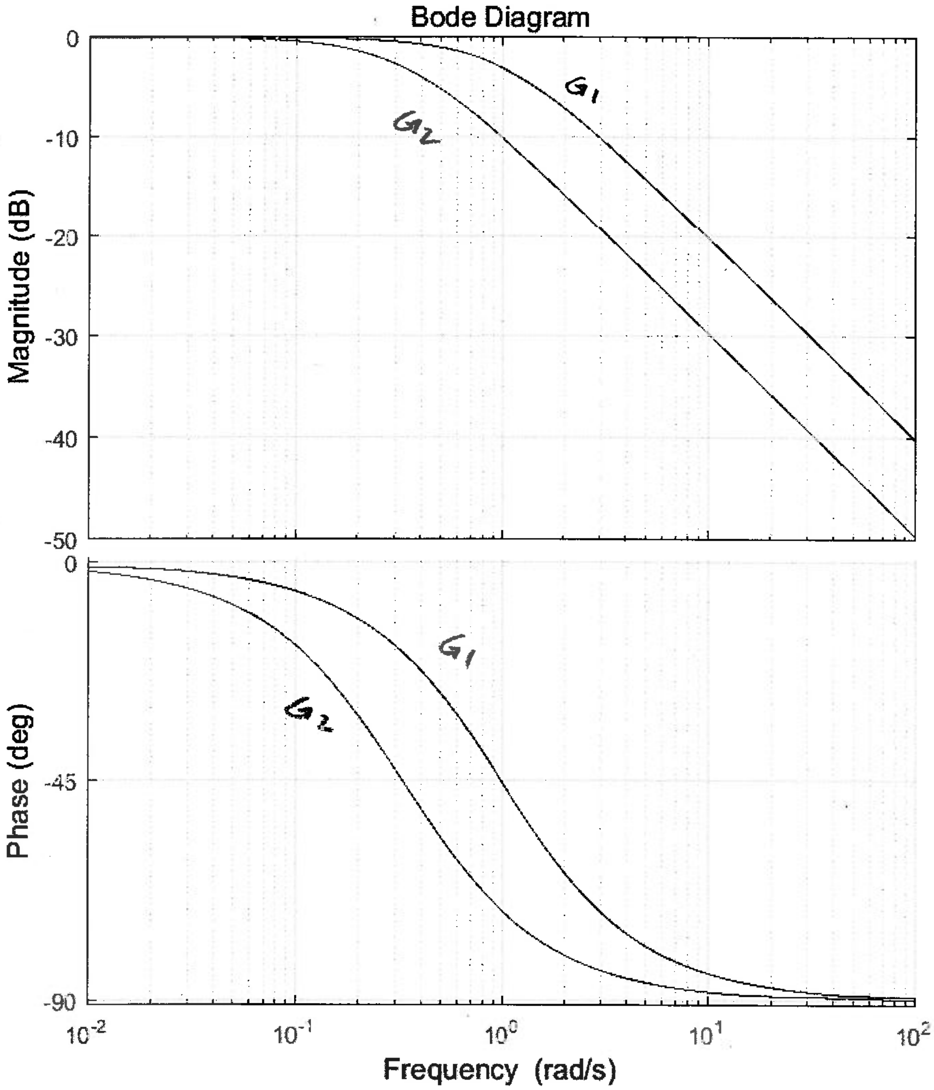
4
PIF


Bode Diagram
Magnitude (dB)
0
-10
-20
-30
-40
-50
G1
G2
Phase (deg)
0
-45
-90
G1
G2
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/s)

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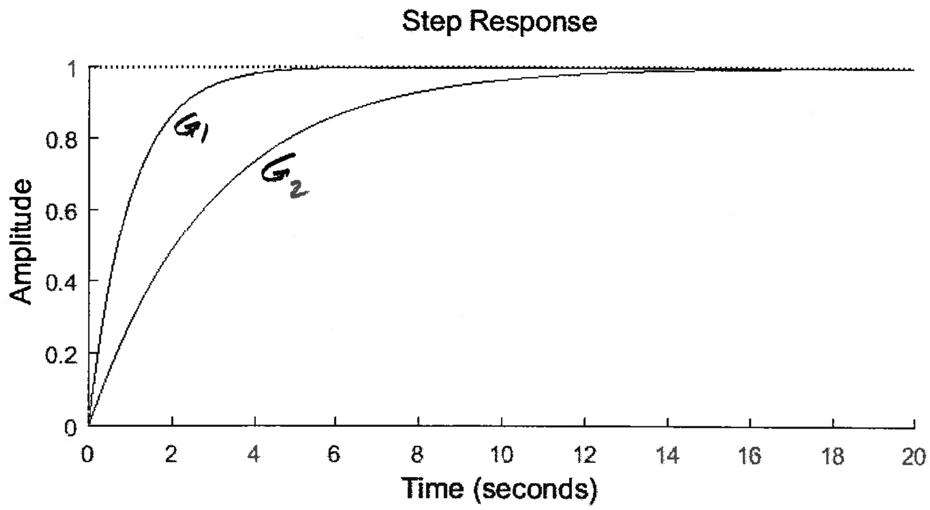
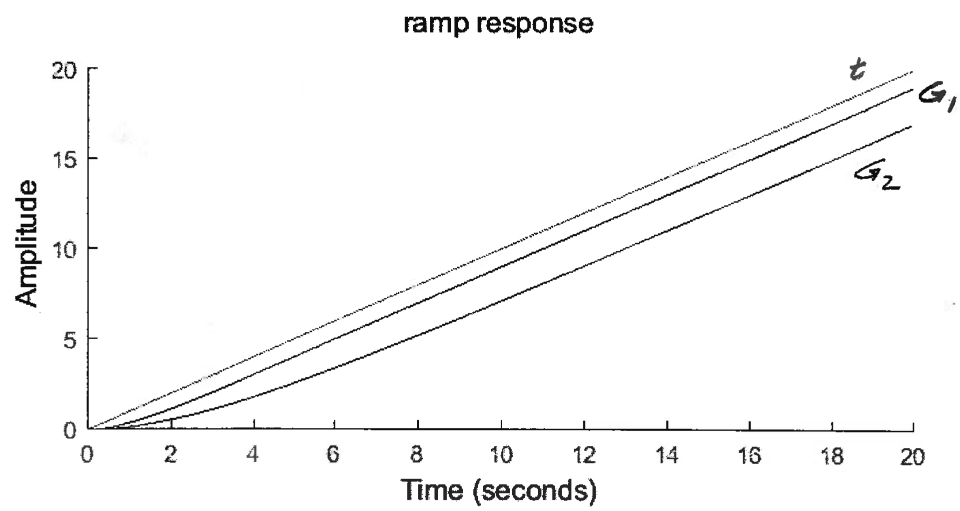
3
11.2


Step Response
Amplitude
Time (seconds)
G1
G2


ramp response
Amplitude
Time (seconds)
t
G1
G2

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C:\Mydata\1_USC...\Example11_2_p628_20161216.m Page 1

```
%{
 EXAMPLE 11.2
 1st Order System analysis
%}
%% initialization
clc
clear all
% close all
s=tf('s');
%% system definition
G1=1/(s+1); G2=1/(3*s+1);
%% Bode plots
figure(1)
bode(G1,G2)
box off
grid on
%% time response
figure(2)
Tfinal=20;
dt=0.1; t=0:dt:Tfinal; u=1.*t;
subplot(2,1,1)
step(G1,G2,Tfinal)
box off
subplot(2,1,2)
lsim(G1,G2,u,t)
axis([0 Tfinal 0 Tfinal]); title( 'ramp response' )
box off
```

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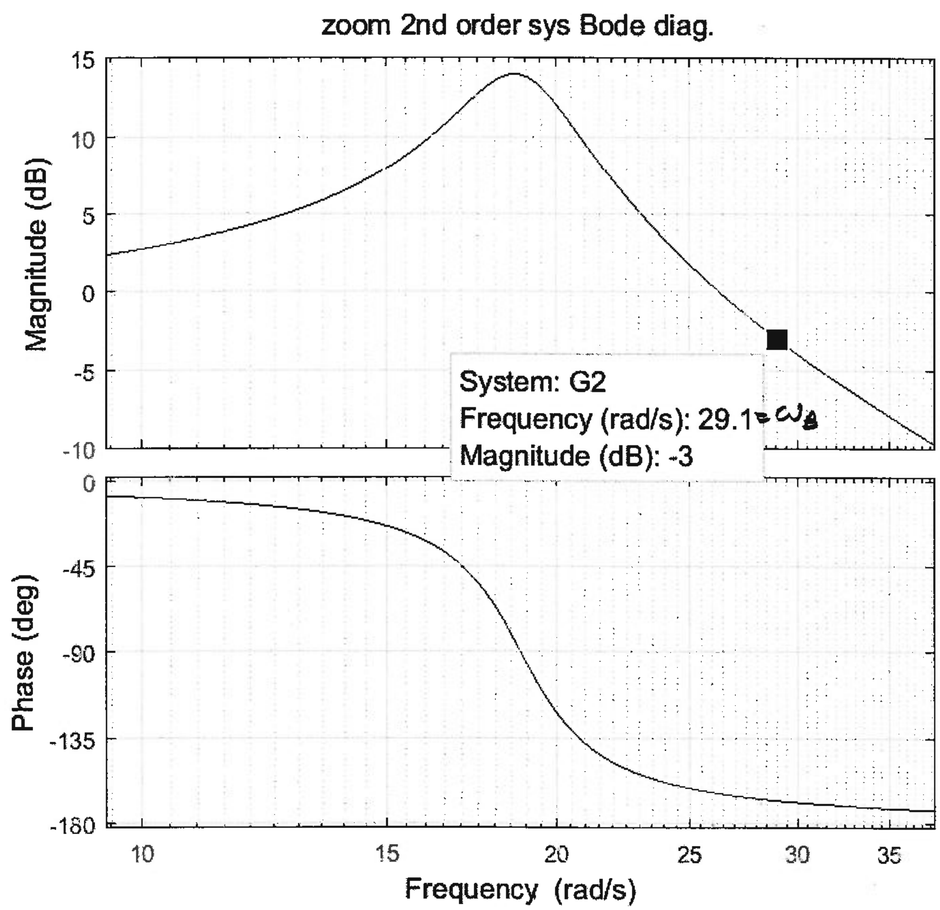
[6] PIF

Ex: 2nd order Sys. Performance

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$\omega_n = 2\pi f_n$
$f_n = 3\ Hz$
$\zeta = 10\%$

$\omega_B = 29.1\ Hz.$



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7
PIF

Specific Performance Indicators
in Freq. Domain for
2^{nd} Order Systems

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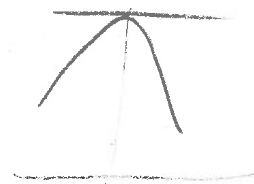
8
PIF

Resonance: peak response
max. response

$$|G(i\omega)|^2 = \frac{1}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}.$$



$$\frac{d}{d\omega}|G(i\omega)|^2 = 0 \quad \text{for peak}$$

$$\frac{d}{d\omega}\left[\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2\right] = 0.$$

use auxiliary variable $p = \frac{\omega}{\omega_n}$ and write

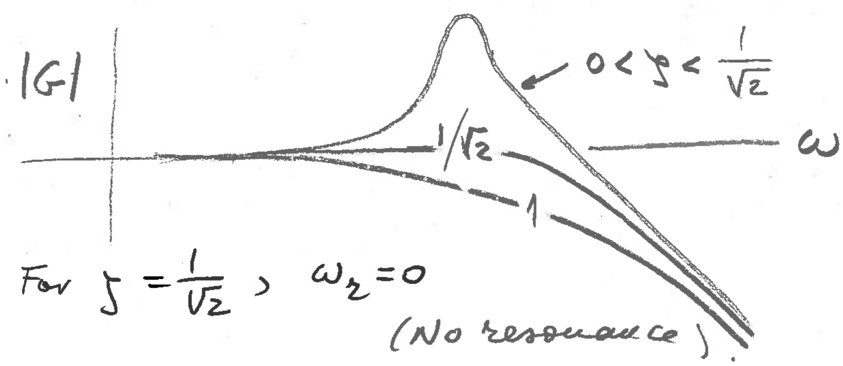
$$\frac{d}{dp}\left[(1-p^2)^2 + (2\zeta p)^2\right] = 0 .$$

$$2(-2p)(1-p^2) + 2(2\zeta)(2\zeta p) = 0 .$$

$$(1-p^2)\cancel{p} = 2\zeta^2\cancel{p}$$

$$p^2 = 1 - 2\zeta^2 \longrightarrow p_r = \sqrt{1-2\zeta^2}$$

$$\omega_r = \omega_n\sqrt{1-2\zeta^2}$$



For $\zeta = \frac{1}{\sqrt{2}}$, $\omega_r = 0$
(No resonance).

Peak exist only for $0 < \zeta < \frac{1}{\sqrt{2}}$

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9 PIF

Amplitude at resonance

$$M_r^2 = \left|G(i\omega_r)\right|^2 = \frac{1}{\left[1-(1-2\zeta^2)\right]^2 + (2\zeta)^2(1-2\zeta^2)}$$

$$= \frac{1}{(\cancel{1}-\cancel{1}+2\zeta^2)^2 + 4\zeta^2(1-2\zeta^2)}$$

$$= \frac{1}{4\zeta^4+4\zeta^2-8\zeta^4} = \frac{1}{4\zeta^2(1-\zeta^2)}$$

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

$$\zeta = \frac{1}{\sqrt{2}}$$

$$M_r = \frac{1}{2\frac{1}{\sqrt{2}}\sqrt{1-\frac{1}{2}}} = 1$$

Phase at resonance, φ_r

$$G(i\omega) = \frac{\omega_n^2}{(i\omega)^2 + 2i\zeta\omega\omega_n + \omega_n^2} = \frac{1}{(1-p^2)+2i\zeta p}$$

$$p_r = \sqrt{1-2\zeta^2}, \quad 1-p_r^2 = 1-(1-2\zeta^2) = 2\zeta^2$$

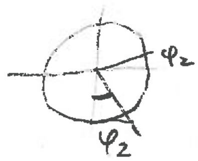
$$G(i\omega_r) = \frac{1}{2\zeta^2 + 2i\zeta\sqrt{1-2\zeta^2}}$$

$$\varphi_r = \angle G(i\omega_r) = -\tan^{-1}\frac{\sqrt{1-2\zeta^2}}{\zeta}$$

$$\varphi_r = -90^\circ + \underbrace{\sin^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}}}_{=\varphi_2} = -90^\circ + \varphi_2$$

Proof

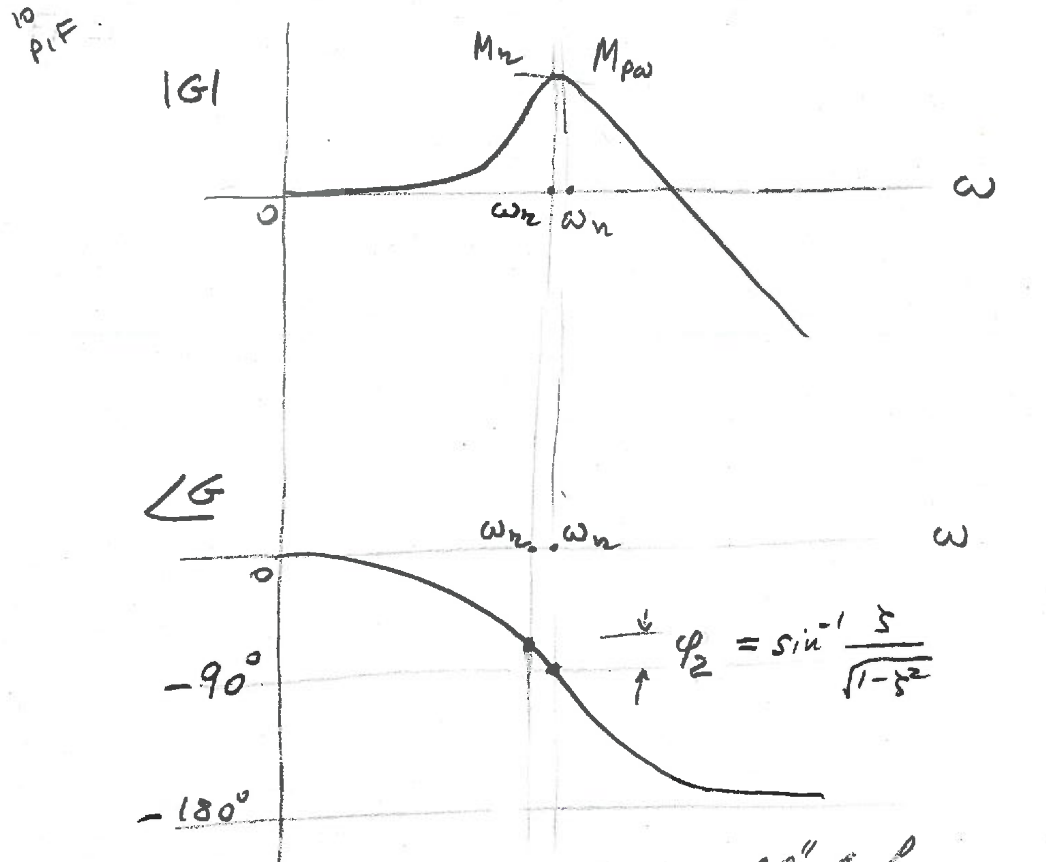
$$\tan(-90^\circ+\varphi_2) = -\operatorname{cotan}\varphi_2$$



$$\sin\varphi_2 = \frac{\zeta}{\sqrt{1-\zeta^2}}$$

$$\cos^2\varphi_2 = 1-\sin^2\varphi_2 = \frac{1-2\zeta^2}{1-\zeta^2}$$

$$\operatorname{cotan}\varphi_2 = \frac{\cos\varphi_2}{\sin\varphi_2} = \frac{\sqrt{1-2\zeta^2}/\sqrt{1-\zeta^2}}{\zeta/\sqrt{1-\zeta^2}} = -\tan\varphi_r$$



Resonance happens "slightly" before ω_n

Two definitions of "resonance":

(1) 90° phase @ ω_n, $M_{p\omega} = \frac{1}{2\zeta}$
(phase resonance)

(2) peak value @ $\omega_r = \omega_n\sqrt{1-2\zeta^2}$

$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$ "RESONANCE"

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PERFORMANCE INDICATORS vs. ζ

ζ	0.01	0.1	0.4	$1/\sqrt{2}$	0.8	0.99	1
$M_{p\omega} = \frac{1}{2\zeta}$	50	5	1.25	0.7	0.625	0.505	0.5
$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$	50	5.025	1.36	1	—	—	—
$1+M_p = 1+e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$	1.97	1.73	1.25	1.04	1.02	1.00	1
$\frac{\omega_r}{\omega_n} = \sqrt{1-2\zeta^2}$	1	0.99	0.825	0	—	—	—
$\frac{\omega_d}{\omega_n} = \sqrt{1-\zeta^2}$	1	0.995	0.917	$1/\sqrt{2}$	0.6	0.141	0
$\frac{t_s}{\tau_n} = \frac{2}{\pi\zeta}$	63	6.37	1.59	0.9	0.8	0.64	0.64
$\varphi_r = -\tan^{-1}\frac{\sqrt{1-2\zeta^2}}{\zeta}$	-89°	-84°	-64°	0°	—	—	—

$$t_s = \frac{4}{\zeta\omega_n}$$

$$\tau_n = \frac{1}{f_n} = \frac{2\pi}{\omega_n} \text{ (osc. period)}$$

$$\frac{t_s}{\tau_n} = \frac{4}{\zeta}\,\frac{1}{2\pi} = \frac{2}{\pi\zeta}$$

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PIF

Specific PIs vs ζ 2nd order system

performance indicators comparison, 2nd order sys

z =	0.0100	0.1000	0.4000	0.7071	0.8000	0.9900	1.0000
Mpw =	50.0000	5.0000	1.2500	0.7071	0.6250	0.5051	0.5000
Mr =	50.0025	5.0252	1.3639	1.0000	0	0	NaN
1+Mp =	1.9691	1.7292	1.2538	1.0432	1.0152	1.0000	1.0000
wr/wn =	0.9999	0.9899	0.8246	0.0000	0	0	0
wd_wn =	0.9999	0.9950	0.9165	0.7071	0.6000	0.1411	0
ts_tn =	63.6620	6.3662	1.5915	0.9003	0.7958	0.6431	0.6366
phi_r =	-89.4270	-84.2318	-64.1233	0.0000	0	0	0

$M_{p\omega} = \frac{1}{2\zeta}$ magnitude at phase res.

$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$ resonance peak

$1+M_p = 1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}}$ step response peak

$\frac{\omega_r}{\omega_n} = \sqrt{1-2\zeta^2}$ resonance freq. ratio

$\frac{\omega_d}{\omega_n} = \sqrt{1-\zeta^2}$ damped freq / natural freq

$t_s/\tau_n = \frac{2}{\pi\zeta}$ settling time / osc. period

$\varphi_r = -\tan^{-1}\frac{\sqrt{1-2\zeta^2}}{\zeta}$ phase at resonance

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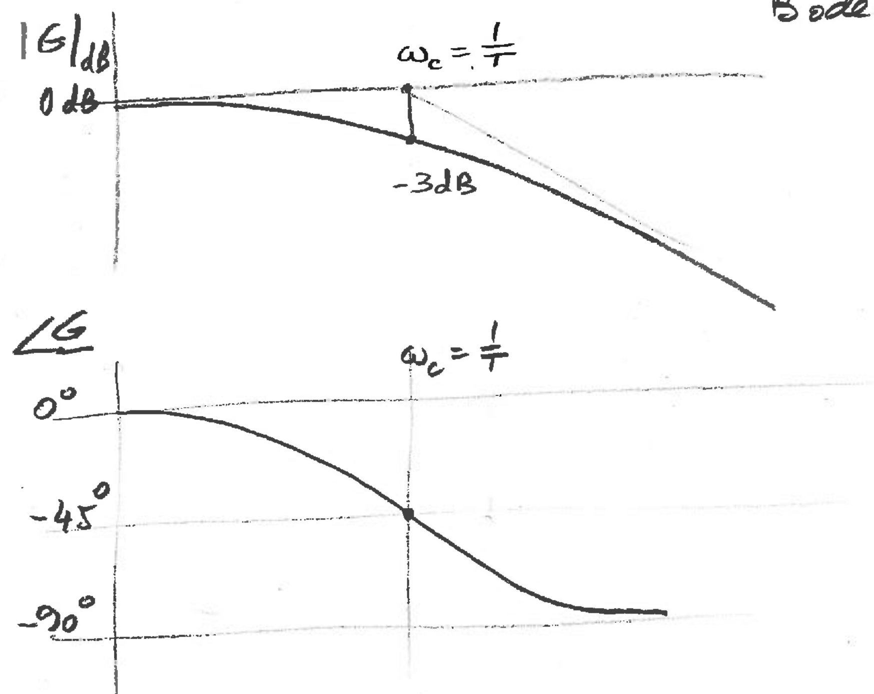
8.4 System Identification in Frequency Domains

IDF

1st order sys ID in Freq Domain

$$G(i\omega) = \frac{1}{i\omega T + 1}$$

Bode plots



Given : Bode plots

Find : T

Sol^n : Read: -3dB point on $|G|_{dB}$ plot

45° point on $\angle G$ plot

Estimate ω_c

Calulate $T = \frac{1}{\omega_c}$

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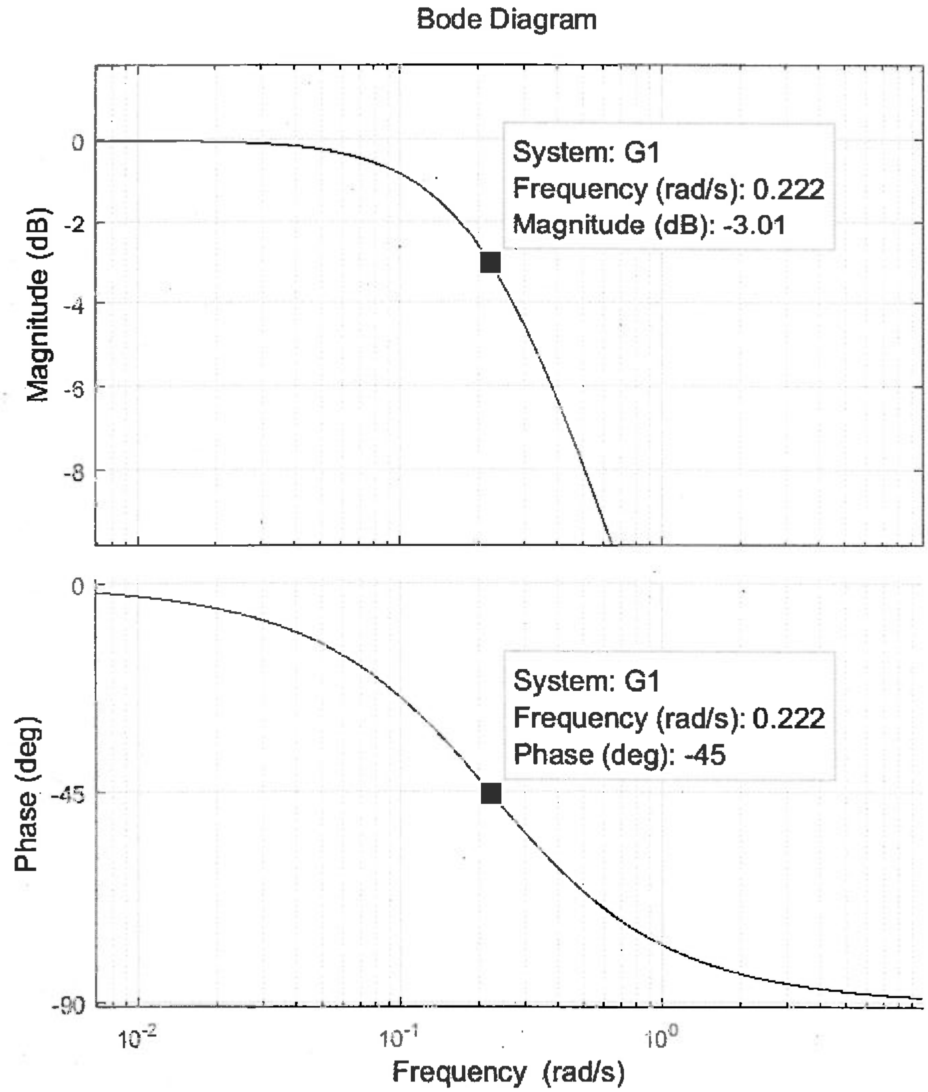
2
1DF

Ex: ID 1st order sys. $G(s) = \frac{1}{4.5s+1}$



$\omega_c = 0.222$

$T = \frac{1}{\omega_c} = 4.5045$

error : $-0.1\ \%$

In practice, error may be larger due to noise.

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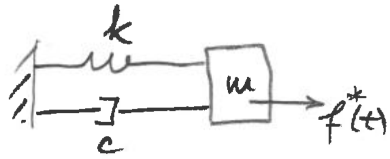
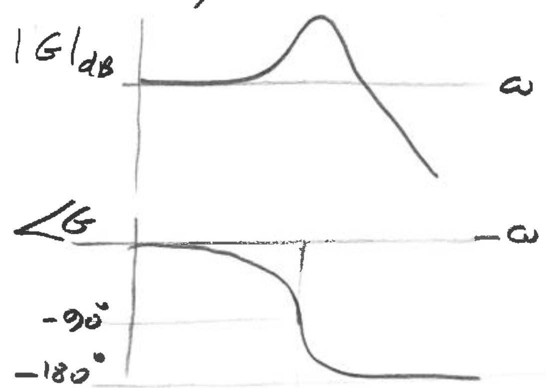
3 IDF Ex: ID 2nd order sys





Given: Bode diagram FRF

Find: m, c, k.

Solution: recall FBD, EOM

$$m\ddot{x} + c\dot{x} + kx = f^*(t)$$

$$\mathcal{L}T \quad (ms^2 + cs + k)X(s) = F^*(s)$$

$$G(s) = \frac{X(s)}{F^*(s)} = \frac{1}{ms^2 + cs + k}$$

$$G(i\omega) = \frac{1}{-m\omega^2 + i\omega c + k} \quad ; \quad G(i0) = \frac{1}{k}$$

$$\omega_n^2 = k/m \qquad G(i\omega_n) = \frac{1}{i\omega_n c} \,;\, |G(i\omega_n)| = \frac{1}{\omega_n c}$$

Numerical example: $m = 2\,kg$, $c = 4\,\frac{N}{m/s}$, $k = 20\,\frac{N}{m}$

Run MATLAB SYS_ID_2ndOrder_Ex11_1

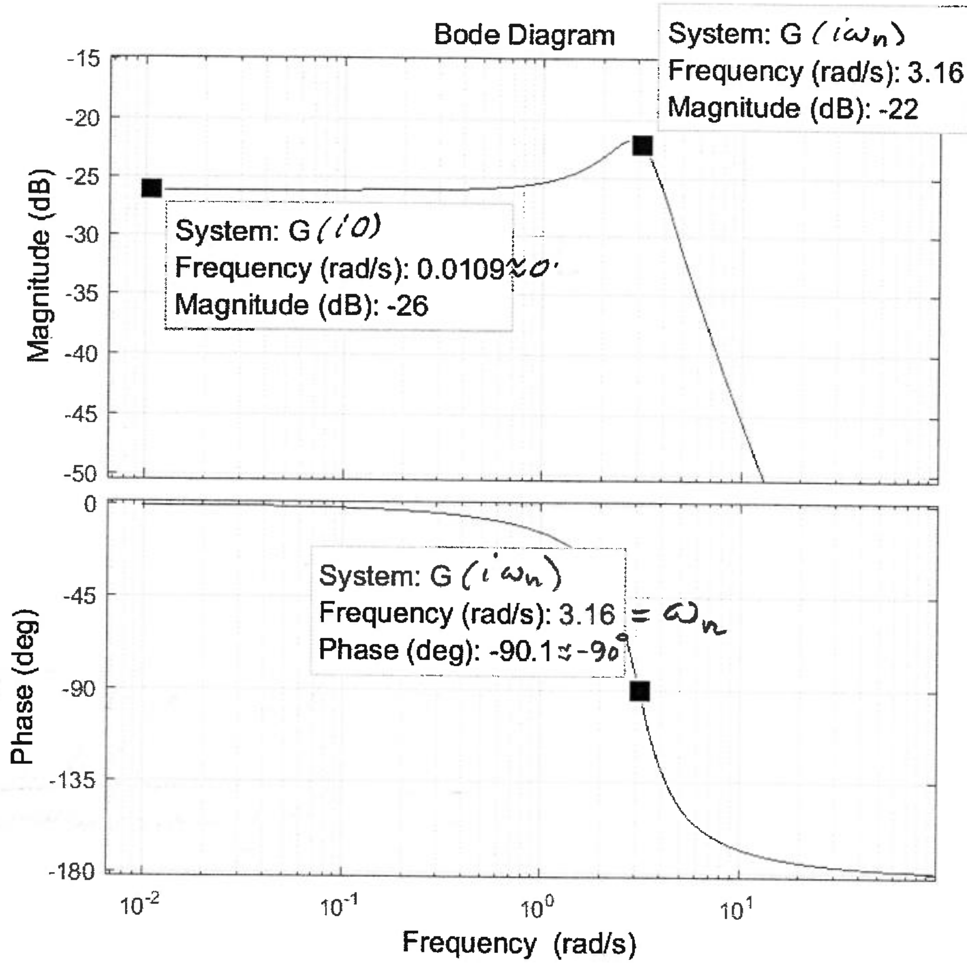
Read on Bode diagram:

$$|G(i0)|_{dB} = -26\,dB$$

$$\omega_n = 3.16\ rad/sec$$

$$|G(i\omega_n)|_{dB} = -22\,dB$$

4
IDF



$$(M_{p\omega})_{dB} = |G(i\omega_n)|_{dB} - |G(i0)|_{dB}$$

$$= -22dB - (-26dB) = +4dB$$

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5
1DF calculate: k_1, ω_1, c_1

$$k_1 = \frac{1}{G(i0)} = 1/db2mag(G(i0)_{dB})$$

$$= 19.9526 \ N/m \approx 20 \ N/m \quad \checkmark$$

$\omega_n^2 = k/m$

$$m_1 = \frac{k_1}{\omega_n^2} = 1.9981 \ kg \approx 2 \ kg \quad \checkmark$$

$$c_1 = \frac{1}{\omega_n |G(i\omega_n)|} = 1/\omega_n/db2mag(G(i\omega_n)_{dB}) \doteq 3.9839$$

or

$$\approx 4 \frac{N}{m/s}$$

$$(M_{p\omega})_{dB} = |G(i\omega_n)|_{dB} - |G(i0)|_{dB}$$

$$M_{p\omega} = db2mag(|G(i\omega_n)|_{dB} - |G(i0)|_{dB}|)$$

$$M_{p\omega} = \frac{1}{2\zeta} \longrightarrow \zeta = \frac{1}{2M_{p\omega}}$$

$$c_2 = 2\zeta\omega_n m_1 = 3.9920 \ \frac{N}{m/s} \approx 4 \frac{N}{m/s} \quad \checkmark$$

All three parameters of the system, k, m, c have been recovered with quite acceptable error.

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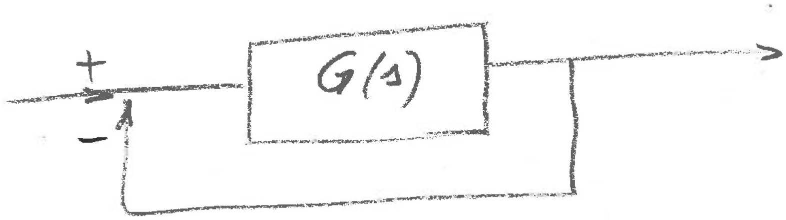
8.5 Frequency Domain Analysis of Feedback System Stability

STABILITY ANALYSIS IN FREQ. DOMAIN

Evaluate stability of CL system by analyzing OL system in freq. domain



Two methods:

- Nyquist criterion
- Gain & phase margins

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NYQUIST ANALYSIS OF FEEDBACK STABILITY

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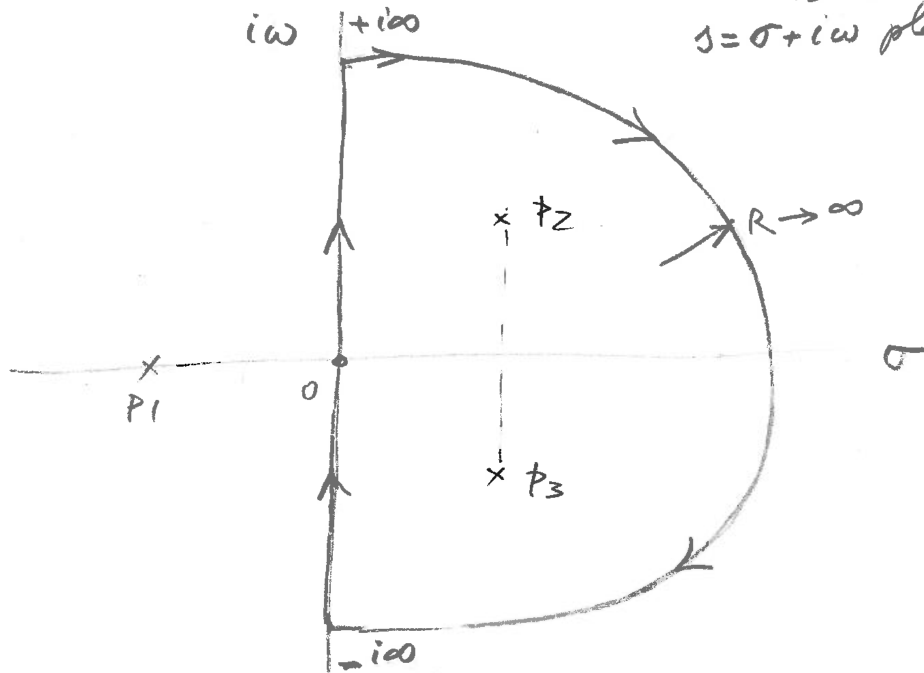
Nyquist circuit

$$G(s) = \frac{1}{(s-p_1)(s-p_2)(s-p_3)}$$

Assume
$p_1 \in \mathbb{R}$, $p_1 < 0$
$p_2, p_3 \in \mathbb{C}$ in RHS
$p_3 = \overline{p_2}$
$s = \sigma + i\omega$ plane

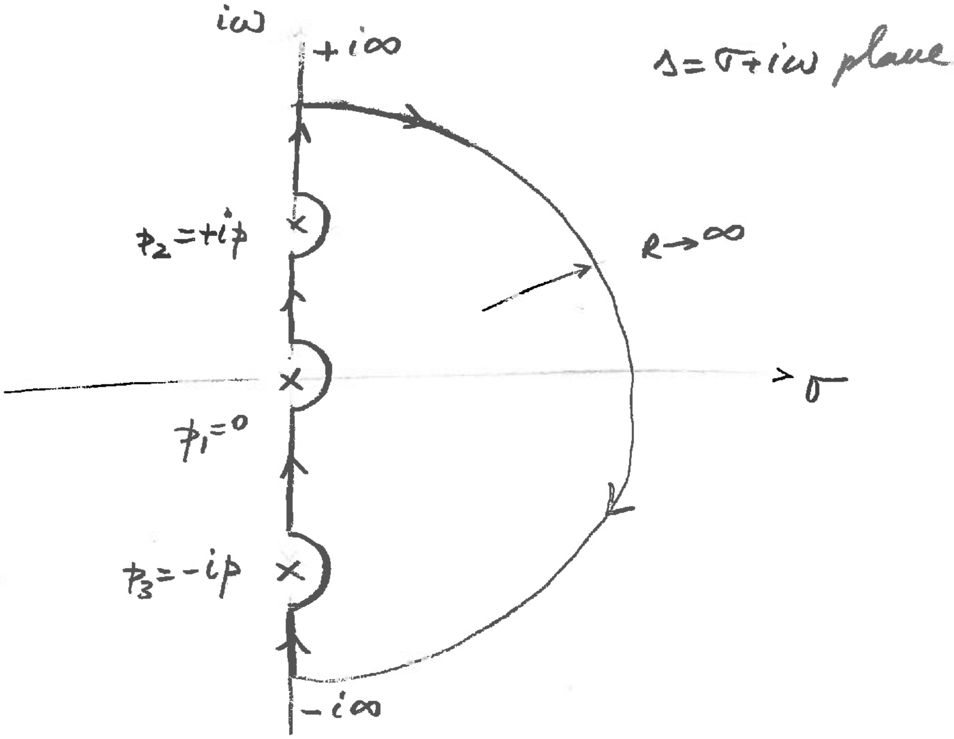


Nyquist circuit is:

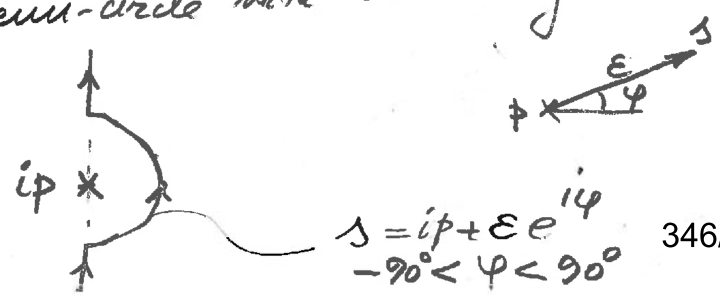
- semi-circle path with $R \to \infty$
- in the RHS of s-plane
- along vertical axis from $-i\infty$ to $+i\infty$
- traveled clockwise (CW)

Nyquist circuit with poles on the vertical axis

$$G(s) = \frac{1}{s\,(s-ip)(s+ip)}$$



The poles on the vertical axis must be excluded. We travel around them in the RHS. To do so, we follow a small semi-circle with vanishing radius ε

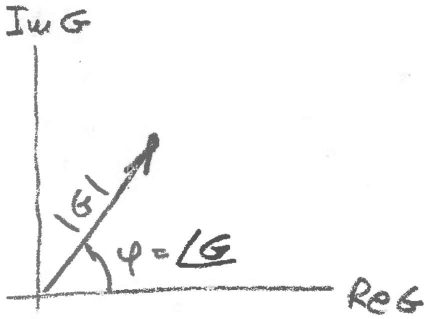


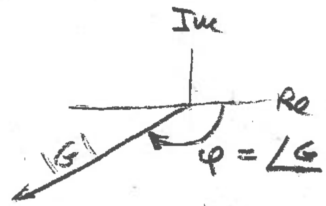
$$s = ip + \varepsilon e^{i\varphi}$$
$$-90^\circ < \varphi < 90^\circ$$

N1

POLAR PLOTS

$$G(i\omega) = \mathrm{Re}\, G(i\omega) + i\, \mathrm{Im}\, G(i\omega)$$



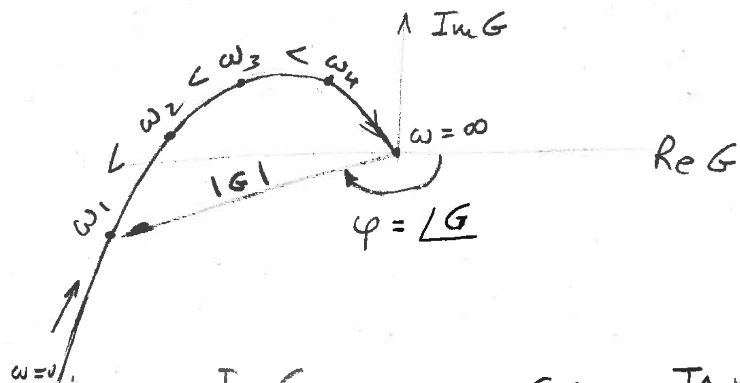


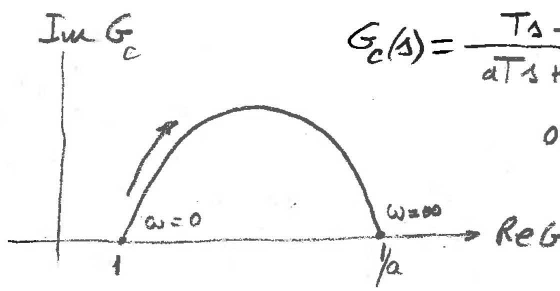
-ve value

$-\pi < \text{angle}(G) < \pi$

$-180°$ $\quad$ $180°$

(see MATLAB Help)





$$G_c(s) = \frac{Ts+1}{\alpha Ts+1}$$

$$0 < \alpha < 1$$

Run MATLAB examples.

NPP

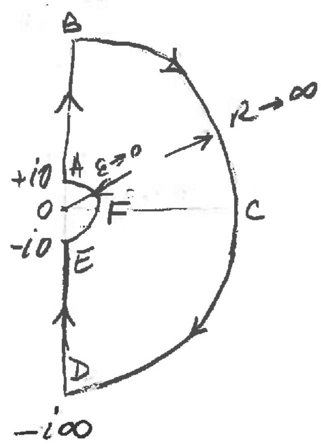
NYQUIST POLAR PLOTS

"s follows N-circuit ; G(s) follows N-polar plot"

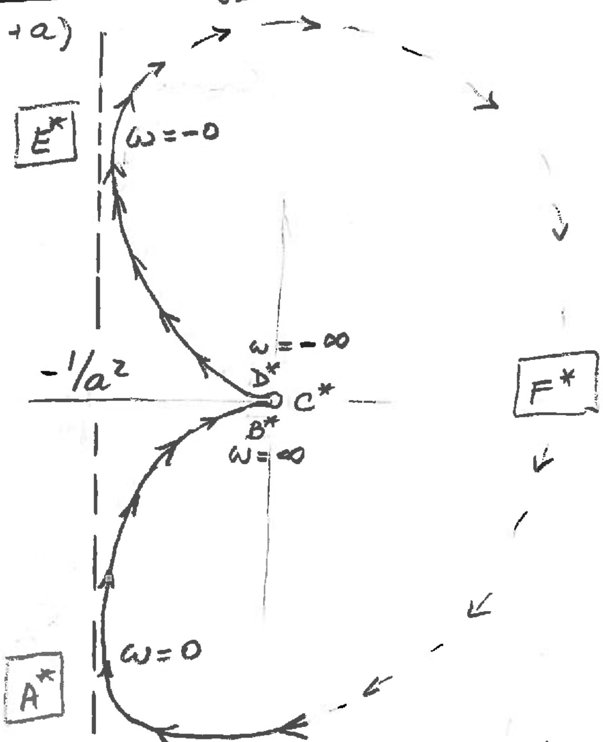
Example: $G(s) = \frac{1}{s(s+a)}$

2 poles: $p_1 = 0$, $p_2 = -a$

Nyquist circuit



Nyquist polar plot



- The variable s follows the Nyquist circuit ABCDEFA clockwise in the s-plane
- The function G(s) follows the resulting circuit in the G-plane

In this example, we distinguish 4 segments to be analyzed individually and then assembled in one continuous circuit

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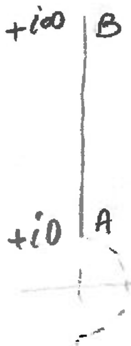
2 NPP

Segment $(\overset{A}{+i\varepsilon}, \overset{B}{+iR})$
$\varepsilon \to 0 \qquad R \to \infty$



$$G(s) = \frac{1}{s(s+a)} = \frac{1}{as} - \frac{1}{a(s+a)}$$

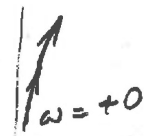
$$\boxed{A}\quad G(i\varepsilon) = \frac{1}{ai\varepsilon} - \frac{1}{a(\cancel{i\varepsilon}+a)} \qquad \varepsilon \ll a$$

$$G(i\varepsilon) \simeq -\frac{1}{a^2} - i\frac{1}{a\varepsilon}$$

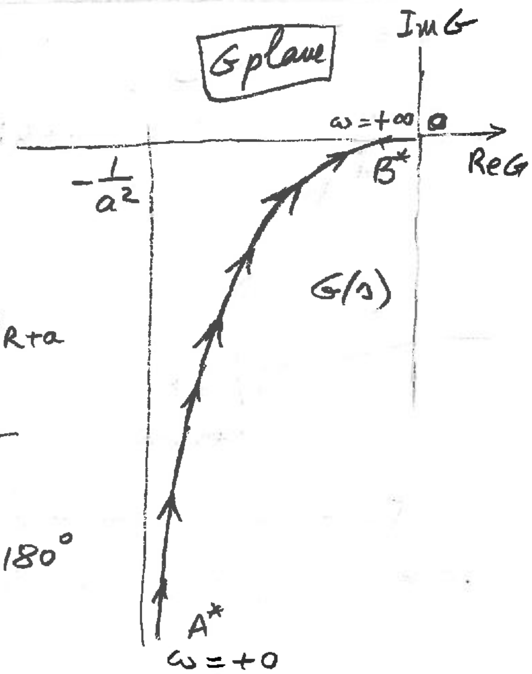
$$G(+i0) = \lim_{\varepsilon \to 0} G(i\varepsilon) = -\frac{1}{a^2} - i\infty$$

$$|G(+i0)| = \infty$$

$$\angle G(+i0) = \angle -i = -90^\circ$$



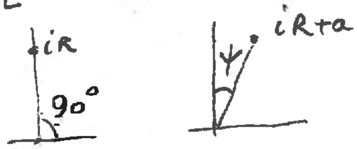
$$\boxed{B}\quad G(iR) = \frac{1}{iR(iR+a)}$$



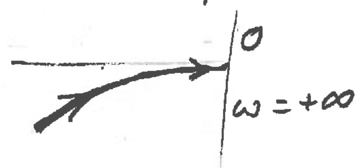
$$|G(iR)| \underset{R \gg a}{\approx} \frac{1}{R^2} \xrightarrow[R \to \infty]{} 0$$

$$\angle G(iR) = -\left[\angle iR + \angle iR + a\right]$$

$$\varphi = \tan^{-1}\frac{a}{R}$$

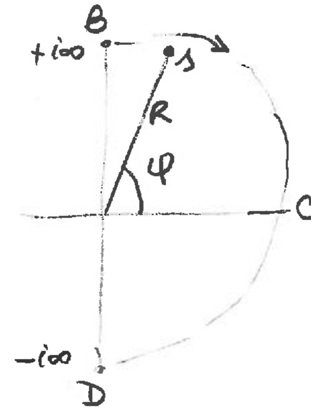


$$\angle G(iR) = -(90^\circ + 90^\circ - \psi)$$
$$= -180^\circ + \psi \xrightarrow[\substack{R \to \infty \\ \psi \to 0}]{} -180^\circ$$



3 NPP Segment $(\overset{B}{iR}, \overset{C}{R}, \overset{D}{-iR})$ Big circle
$R \to \infty$

$$s = Re^{i\varphi}, \quad \varphi \in (90^\circ \;\; 0 \;\; -90^\circ)$$

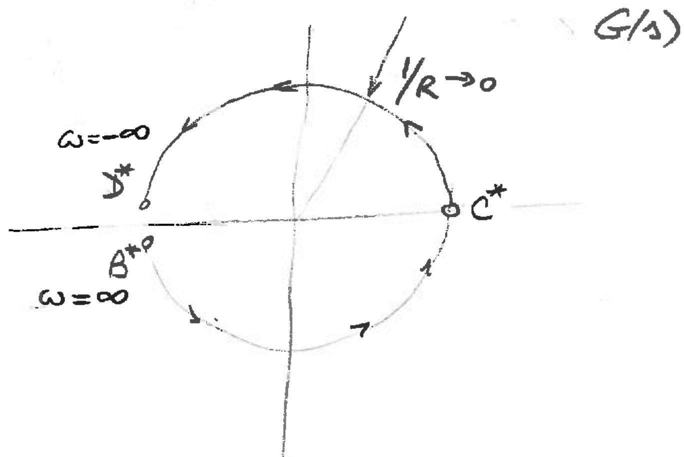


$$G(s) = \frac{1}{Re^{i\varphi}(Re^{i\varphi}+a)}$$

$$\underset{R\to\infty}{\approx} \frac{1}{R^2 e^{2i\varphi}} = \frac{1}{R^2} e^{-i2\varphi}$$

$$\angle G(s) = -2\varphi$$

$$\angle G(s) \in (\underset{B^*}{-180^\circ} \;\; \underset{C^*}{0^\circ} \;\; \underset{D^*}{180^\circ})$$



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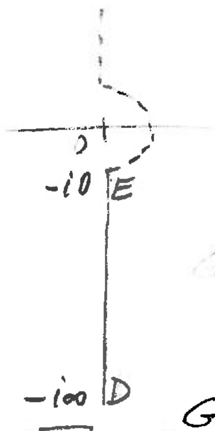
Segment $\left(\overset{D}{-iR}, \overset{E}{-i\varepsilon}\right)$ $\quad R \to \infty \quad \varepsilon \to 0$

s-plane



$$G(s) = \frac{1}{s(s+a)} = \frac{1}{as} - \frac{1}{a(s+a)}$$

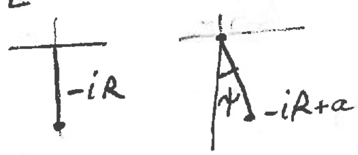
$\boxed{D}$

$$G(s)\Big|_{s=-iR} = \frac{1}{-iR(-iR+a)}$$

$$|G(s)| = \frac{1}{R^2} \xrightarrow[R\to\infty]{} 0$$

$$\angle G(s) = -\left(\angle -iR + \angle -iR+a\right)$$

$$= -\left[-90^\circ + (-90^\circ + \psi)\right] = 180^\circ - \psi \xrightarrow[\substack{R\to\infty \\ \psi \to 0}]{} 180^\circ$$



$$\psi = \tan^{-1}\frac{a}{R}$$

$\boxed{E}$

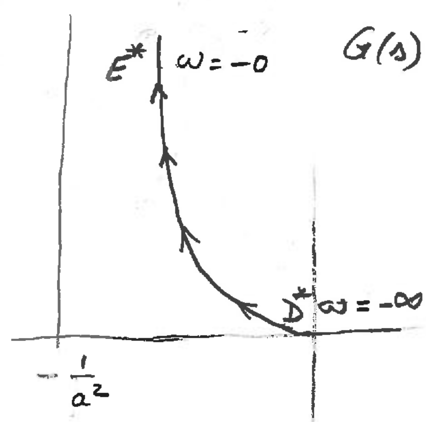
$$G(-i\varepsilon) = -\frac{1}{a(-i\varepsilon)} - \frac{1}{a(-i\varepsilon+a)} = \frac{i}{a\varepsilon} - \frac{1}{a^2}$$

$$G(-i0) = \lim_{\varepsilon\to 0} G(-i\varepsilon)$$

$$G(-i0) = -\frac{1}{a^2} + \frac{i}{a\varepsilon}\Big|_{\varepsilon\to 0} = -\frac{1}{a^2} + i\infty$$

$$|G(-i0)| = \infty$$

$$\angle G(-i0) = 90^\circ$$



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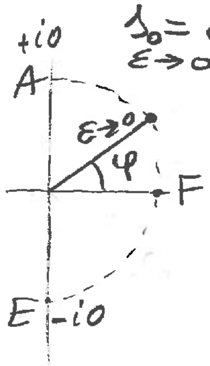
5 N^{PP}

Segment $(\overset{E}{-i\varepsilon}, \overset{F}{\varepsilon}, \overset{A}{i\varepsilon})$ small circle

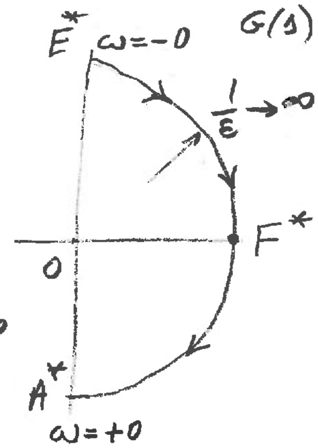
$\varepsilon \to 0$



$$s_0 = \underset{\varepsilon \to 0}{\varepsilon} e^{i\varphi}, \quad \varphi \in (\underset{D}{-90^\circ}, \underset{E}{0}, \underset{F}{90^\circ})$$

$$G(s) = \frac{1}{\varepsilon e^{i\varphi}(\varepsilon e^{i\varphi} + a)}$$

$$\underset{\varepsilon \to 0}{\approx} \frac{1}{a\varepsilon e^{i\varphi}} = \frac{1}{a\varepsilon} e^{-i\varphi}$$



$$|G(s_0)| = \frac{1}{a\varepsilon} \xrightarrow[\varepsilon \to 0]{} \infty$$

$$\angle G(s_0) = -\varphi \in (\underset{E^*}{90^\circ} \quad \underset{F^*}{0^\circ} \quad \underset{A^*}{-90^\circ})$$

$\varphi = -90^\circ$ $\varphi = 0^\circ$ $\varphi = +90^\circ$

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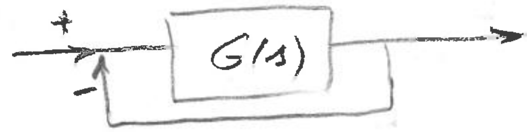
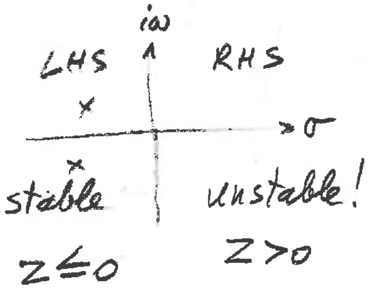
NYQUIST STABILITY CRITERION



Find stability of $G_{CL} = \frac{G}{1+G}$ by analyzing the polar plot of $G(s)$ as s follow the Nyquist circuit.

$$G_{CL}(s) = \frac{G(s)}{1+G(s)}$$

$\boxed{\text{Poles } G_{CL}(s)}$



Poles $G_{CL}(s) \Rightarrow$ zeros of $1+G(s)$

Nyquist Criterion:

$$\boxed{Z = P + N \leq 0}$$

P = number of $G(s)$ poles in RHS

$$G(s) = \frac{B(s)}{A(s)} \rightarrow p_1, p_2, \dots$$

N = number of clockwise encirclements of the $(-1,0)$ point as s follows the Nyquist path (circuit)

Z = number of zeros of $1+G(s)$ in RHS

STABLE if $Z \leq 0$

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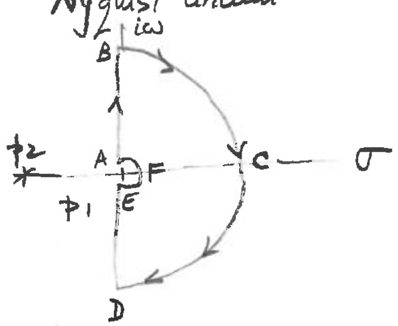
Example : aircraft roll model

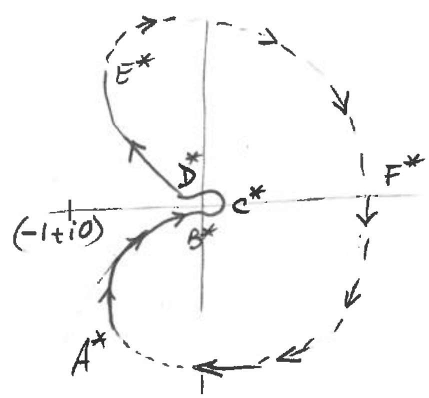
$$G(s) = \frac{114}{10s^2 + 4s} = \frac{114}{s(10s+4)}$$

$$p_1 = 0 \quad , \quad p_2 = -2/5 \; : \; P = 0$$

Recall $G(s) = \dfrac{1}{s(s+a)} = \dfrac{1}{s^2 + sa}$ no poles in RHS

Nyquist circuit





$P = 0$ no poles in RHS

$N = 0$ no encirclements of $(-1+i0)$ point

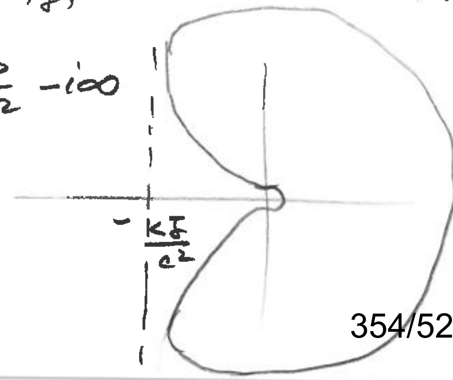
$Z = N + P = 0$ No CL poles in RHS

Conclusion : system is unconditionally stable

Note : $G = \dfrac{K}{Js^2 + cs} = \dfrac{K}{J} \dfrac{1}{s(s + c/J)} = B \dfrac{1}{s(s+a)}$ $\quad B = K/J \quad a = c/J$

$$\underset{\varepsilon \to 0}{G(i\varepsilon)} = B\left(-\frac{1}{a^2} - i\infty\right) = -\frac{B}{a^2} - i\infty$$

$$\frac{B}{a^2} = \frac{K}{J} \frac{J^2}{c^2} = \frac{KJ}{c^2}$$



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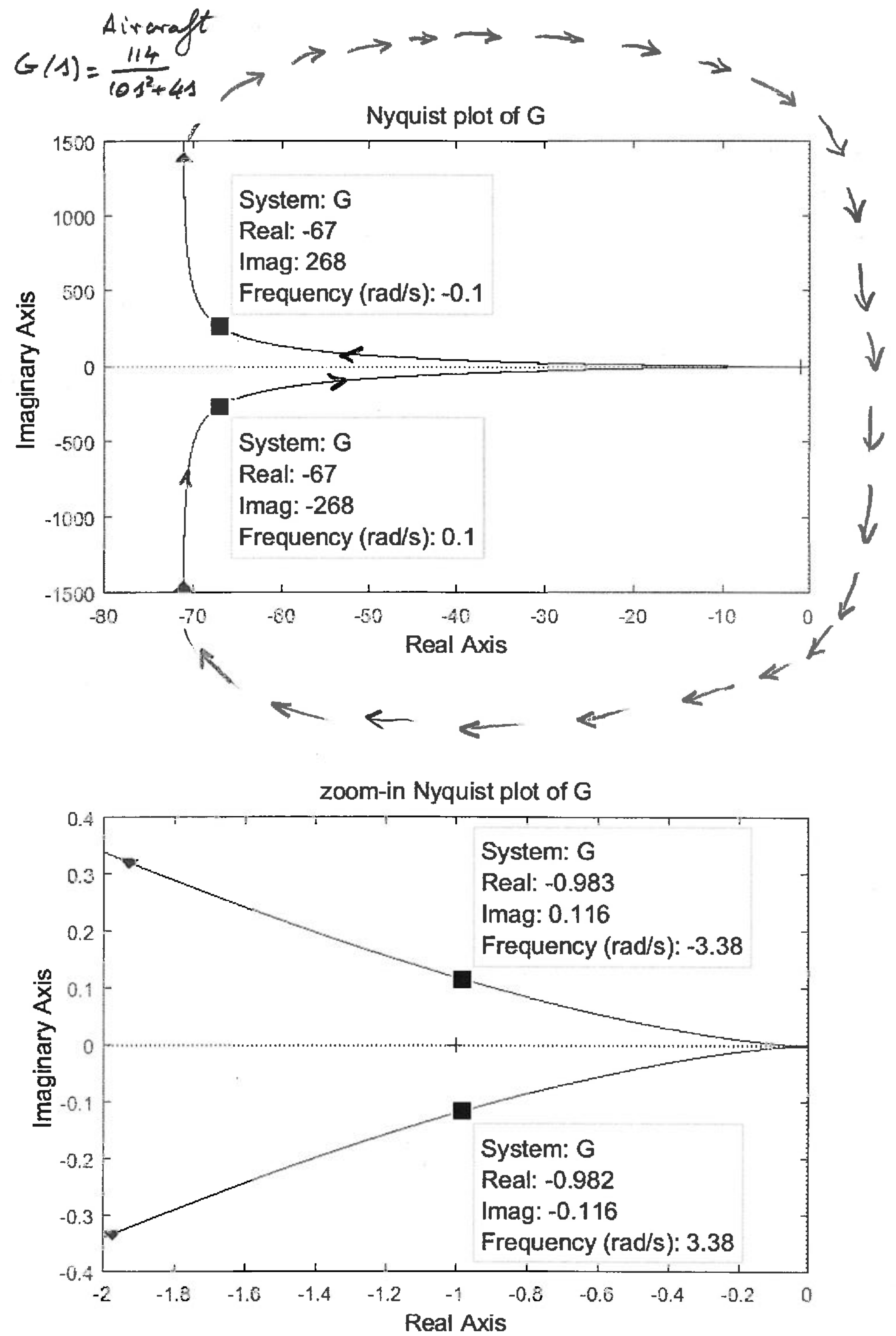

Aircraft
G(s) = 114 / (10s²+4s)
Nyquist plot of G
System: G
Real: -67
Imag: 268
Frequency (rad/s): -0.1
System: G
Real: -67
Imag: -268
Frequency (rad/s): 0.1
Imaginary Axis
Real Axis
1500
1000
500
0
-500
-1000
-1500
-80
-70
-60
-50
-40
-30
-20
-10
0
zoom-in Nyquist plot of G
System: G
Real: -0.983
Imag: 0.116
Frequency (rad/s): -3.38
System: G
Real: -0.982
Imag: -0.116
Frequency (rad/s): 3.38
Imaginary Axis
Real Axis
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-2
-1.8
-1.6
-1.4
-1.2
-1
-0.8
-0.6
-0.4
-0.2
0

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Example AM.8

Given: $G = K\dfrac{1}{s-1}$

Note: pole in RHS, $s=1$

Find: critical value of K for stability using Nyquist criterion

Solution: do Nyquist plot of $G(s)$

$$G(0) = -\frac{K}{1} = -K$$

$$G(i\infty) = \lim_{\omega\to\infty} \frac{K}{i\omega-1} \simeq \frac{K}{i\omega}, \quad |G(i\omega)| = 0$$

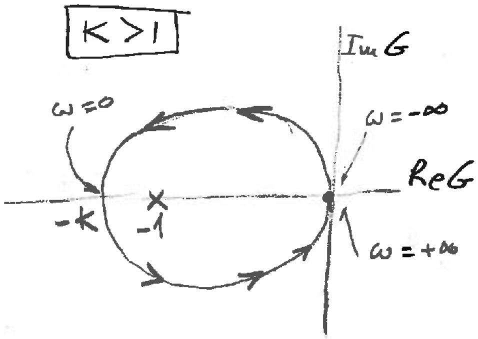
$$\angle G(i\omega) = -90^\circ$$

$$G(-i\infty) \longrightarrow |G(-i\omega)| = 0$$

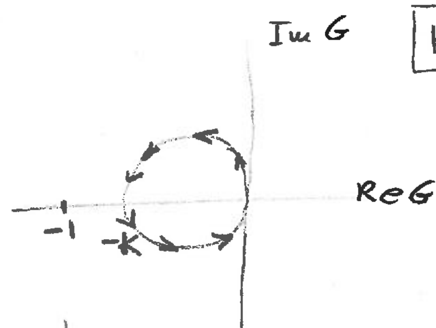
$$\angle G(-i\omega) = 90^\circ$$

Plot in MATLAB

$\boxed{K>1}$



$\boxed{K<1}$



$N=-1$ (counterclockwise)

$P=1$ ($s=1$ pole of G in RHS)

$Z = -1+1 = 0$

STABLE for $K>1$

$N=0$

$P=1$

$Z=1$

UNSTABLE for $K<1$

$\boxed{\text{CRITICAL VALUE: } K=1}$

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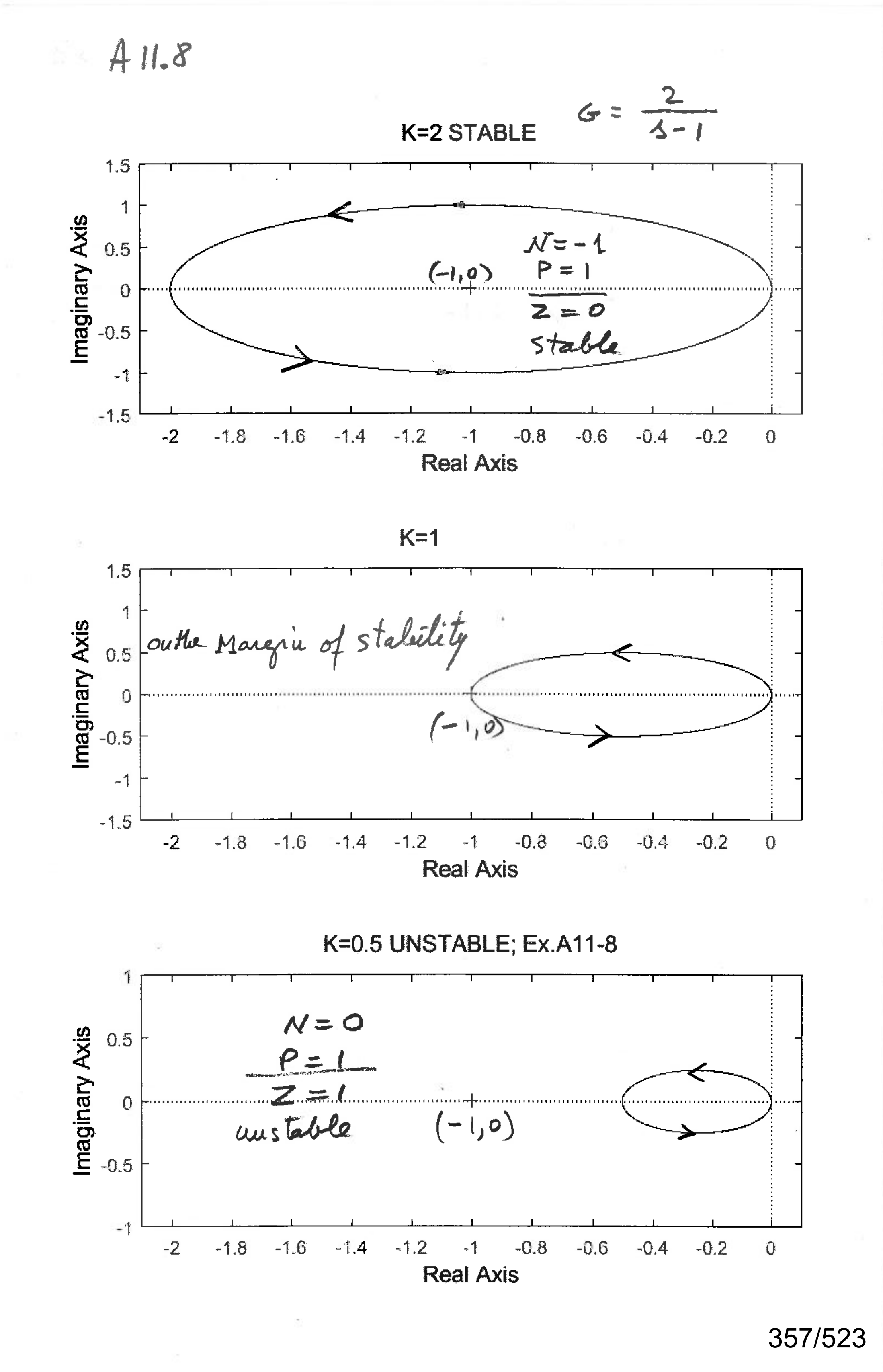
A 11.8
$G = \frac{2}{s-1}$
K=2 STABLE
$N = -1$
$P = 1$
$Z = 0$
stable
(-1,0)
Imaginary Axis
Real Axis
K=1
on the Margin of stability
(-1,0)
Imaginary Axis
Real Axis
K=0.5 UNSTABLE; Ex.A11-8
$N = 0$
$P = 1$
$Z = 1$
unstable
(-1,0)
Imaginary Axis
Real Axis
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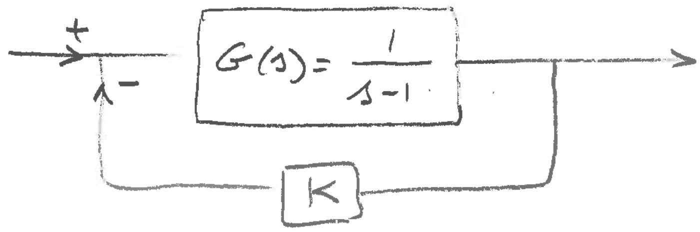
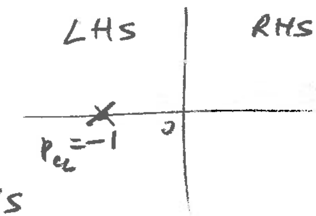
A11.8



$$G_{CL} = \frac{G}{1+GH} = \frac{1}{s-1+K} = \frac{1}{s-(1-K)}$$

$$p_{CL} = 1-K$$

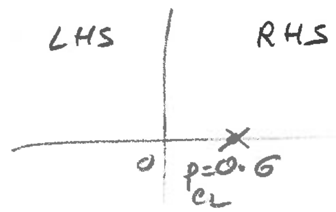
$K = 2$, $p_{CL} = -1$



STABLE

$p_{CL} \in LHS$

$k = 0.4$, $p_{CL} = 0.6$

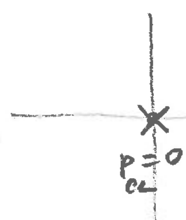


UNSTABLE

$p_{CL} \in RHS$

$p_{CL} = 1 - K_{cr} = 0$

$K_{cr} = 1$



critical K value

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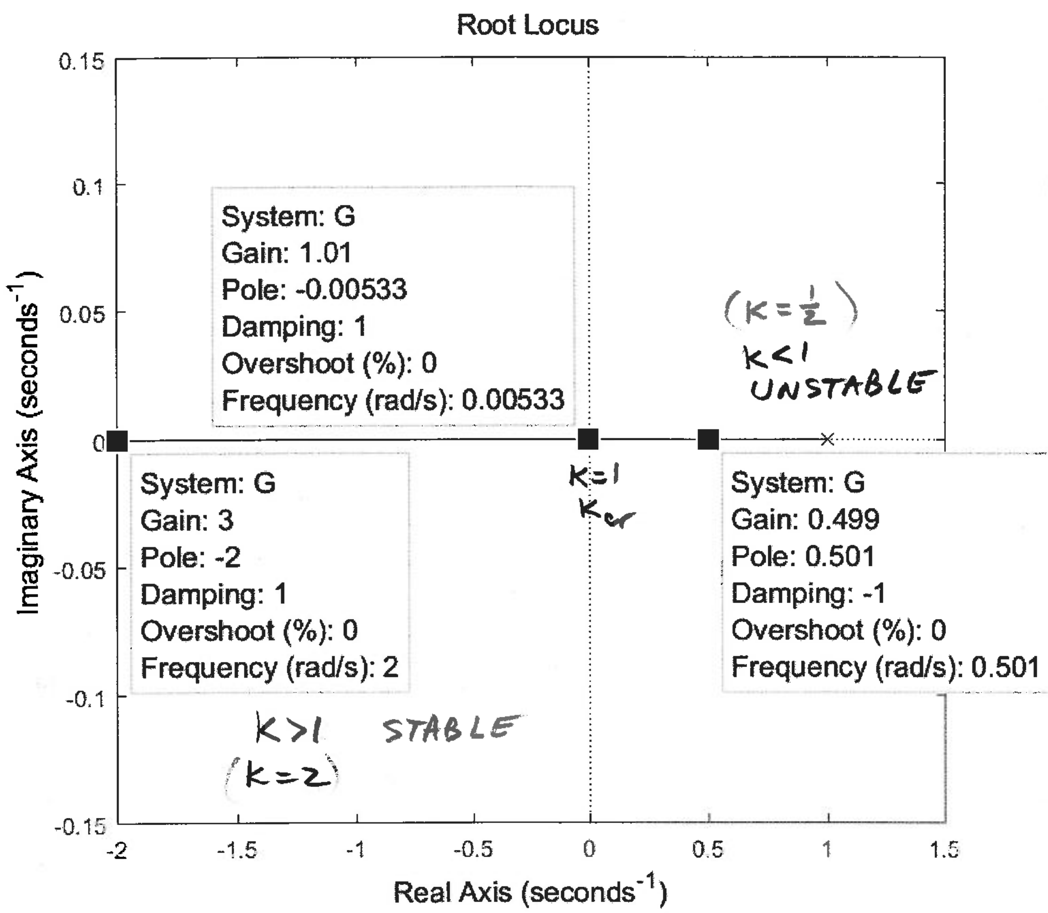
A11.8


Root Locus
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-2
-1.5
-1
-0.5
0
0.5
1
1.5
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
System: G
Gain: 1.01
Pole: -0.00533
Damping: 1
Overshoot (%): 0
Frequency (rad/s): 0.00533
System: G
Gain: 3
Pole: -2
Damping: 1
Overshoot (%): 0
Frequency (rad/s): 2
System: G
Gain: 0.499
Pole: 0.501
Damping: -1
Overshoot (%): 0
Frequency (rad/s): 0.501
$(K=\frac{1}{2})$
$K<1$
UNSTABLE
$K=1$
K_{cr}
$K>1$ STABLE
$(K=2)$

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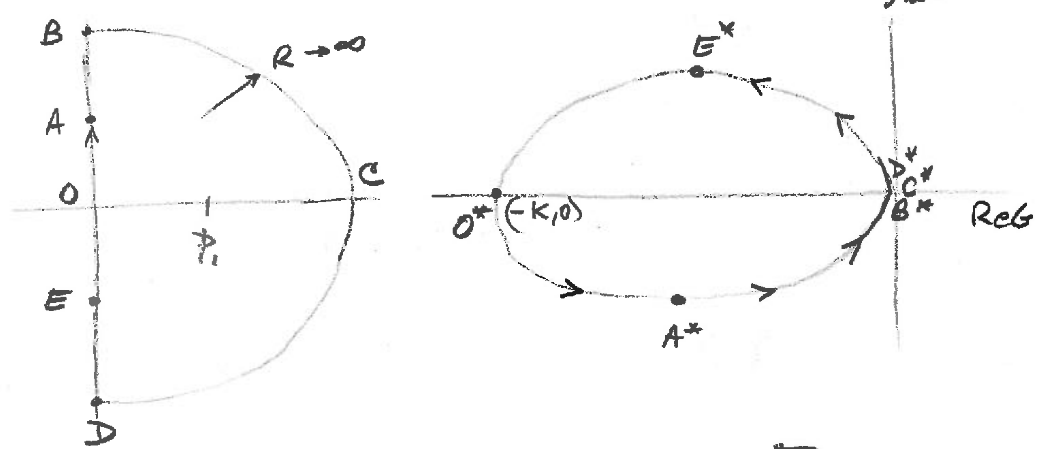
All 8a MANUAL PLOT.

$$G = k\frac{1}{s-1} \qquad p_1 = 1$$



$O:$ $$G(0) = K\frac{1}{0-1} = -K = Ke^{i\pi}$$

$A = i$ $$G(A) = K\frac{1}{i-1} = -K\frac{1+i}{2} = K\frac{\sqrt{2}}{2}e^{i(\pi+\frac{\pi}{4})}$$

$$\angle G(A) = \frac{5\pi}{4}$$

$B = iR$ $$G(B) = K\frac{1}{iR-1} = -K\frac{1+iR}{1+R^2} = \frac{K}{\sqrt{1+R^2}}e^{i(\pi+\varphi)}$$

$$\varphi = \tan^{-1}R \xrightarrow[R\to\infty]{} \frac{\pi}{2}$$

$$|G(B)| \xrightarrow[R\to\infty]{} 0$$

$$\angle G(B) \xrightarrow[R\to\infty]{} \frac{3\pi}{2}$$

$C = R$ $$G(C) = K\frac{1}{R-1} \xrightarrow[R\to\infty]{} 0$$

$D = -iR$ $$G(D) = K\frac{1}{-iR-1} = -K\frac{1-iR}{1+R^2} = \frac{K}{\sqrt{1+R^2}}e^{i(\pi-\varphi)}$$

$$\varphi = \tan^{-1}R \xrightarrow[R\to\infty]{} \frac{\pi}{2}$$

$$|G(D)| \xrightarrow[R\to\infty]{} 0$$

$$\angle G(D) \xrightarrow[R\to\infty]{} \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

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A11.8b

$\varepsilon = -i \qquad G(\varepsilon) = K\frac{1}{-i-1} = -K\,\frac{1-i}{2} = K\frac{\sqrt{2}}{2}e^{i(\pi-\frac{\pi}{4})}$

$\angle G(\varepsilon) = \frac{3\pi}{4}$

Nyquist criterion gives $K < 1$ UNSTABLE

$K > 1$ STABLE

To verify, calculate the poles of G_{CL}

$$G_{CL} = \frac{G}{1+G} = \frac{K}{s-1+K} = \frac{K}{s-(1-K)}$$

$p_{CL} = 1-K$

for $K<1$, $p_{CL} > 0$, in RHS, UNSTABLE

$K > 1$, $p_{CL} < 0$, in LHS, STABLE

QED

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p634

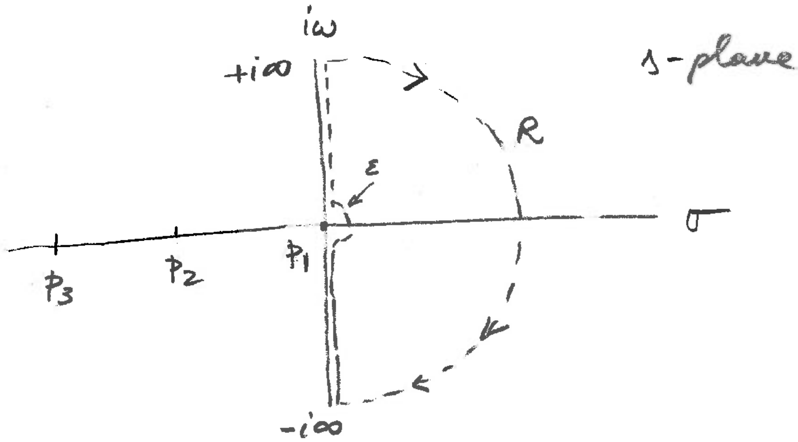
Example p634

$$G(s) = K \frac{1}{s(T_1 s+1)(T_2 s+1)} \qquad T_2 > T_1$$

Poles: $p_1 = 0$

$p_2 = -1/T_1$

$p_3 = -1/T_2$



Nyquist circuit will have to go around the origin on a small circle of radius $\varepsilon \to 0$. We study four segments:

- positive iω axis $s \in (+i0, +i\infty)$
- big circle $R \to \infty$
- negative iω axis $s \in (-i\infty, -i0)$
- small circle $\varepsilon \to 0$

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$+i\omega$ axis.

$s = (+i\varepsilon \ , \ iR)$

$\varepsilon \to 0$
$R \to \infty$

Proof

$$\frac{K}{i\varepsilon(i\varepsilon T_1+1)(i\varepsilon T_2+1)} = \frac{K}{i\varepsilon\left[i^2\varepsilon^2 T_1T_2 + i\varepsilon(T_1+T_2)+1\right]} =$$

$$= \frac{K}{i\varepsilon}\,\frac{1}{i\varepsilon(T_1+T_2)+1}$$

$$= \frac{K}{i\varepsilon}\,\frac{-i\varepsilon(T_1+T_2)+1}{\varepsilon^2\cancel{(T_1+T_2)}+1}$$

$$= -K(T_1+T_2) - i\frac{K}{\varepsilon}$$

$$G(+i\varepsilon) = -K(T_1+T_2) - i\frac{K}{\varepsilon}$$

$$G(+i\varepsilon) \xrightarrow[\varepsilon\to 0]{} -K(T_1+T_2) - i\infty$$

$$|G(i\varepsilon)| \xrightarrow[\varepsilon\to 0]{} \infty$$

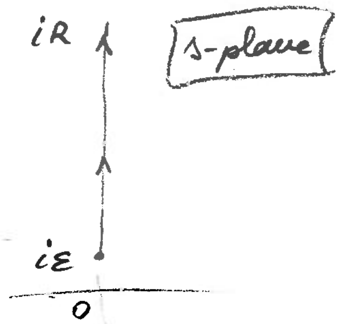
$$\angle G(i\varepsilon) \xrightarrow[\varepsilon\to 0]{} -90^\circ$$

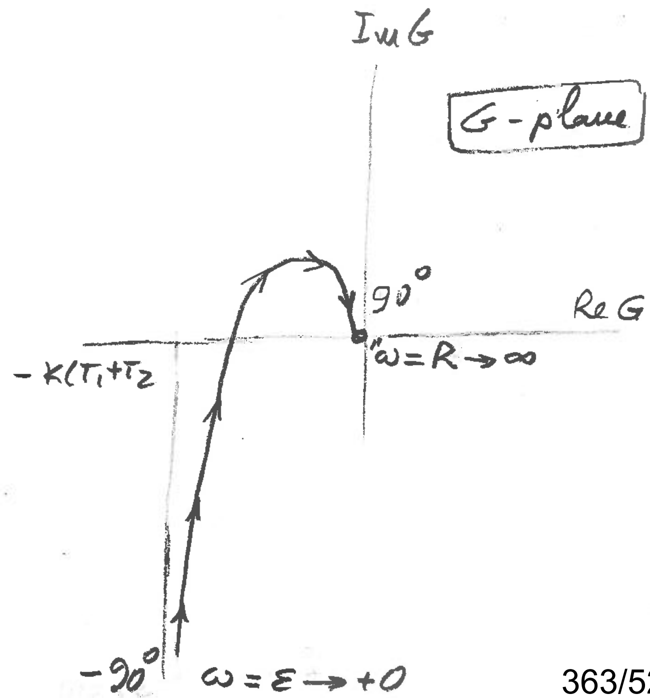
$$G(iR) = \frac{K}{iR(T_1 iR+1)(T_2 iR+1)} \approx \frac{K}{T_1T_2R^3 i^3} = \frac{K}{T_1T_2R^3}e^{i\frac{\pi}{2}}$$

$$\frac{1}{i^3} = \frac{i^4}{i^3} = i = e^{i\frac{\pi}{2}}$$

$$|G(iR)| = \frac{K}{T_1T_2R^3} \xrightarrow[R\to\infty]{} 0$$

$$\angle G(iR) = \frac{\pi}{2} = 90^\circ$$





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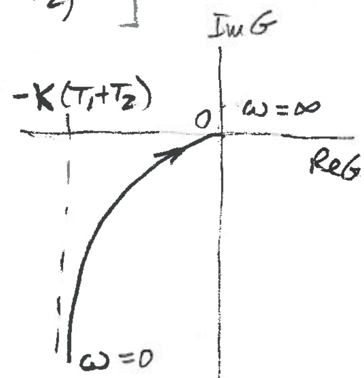
$$G(i\varepsilon) = \frac{K}{i\varepsilon(i\varepsilon T_1+1)(i\omega T_2+1)}$$

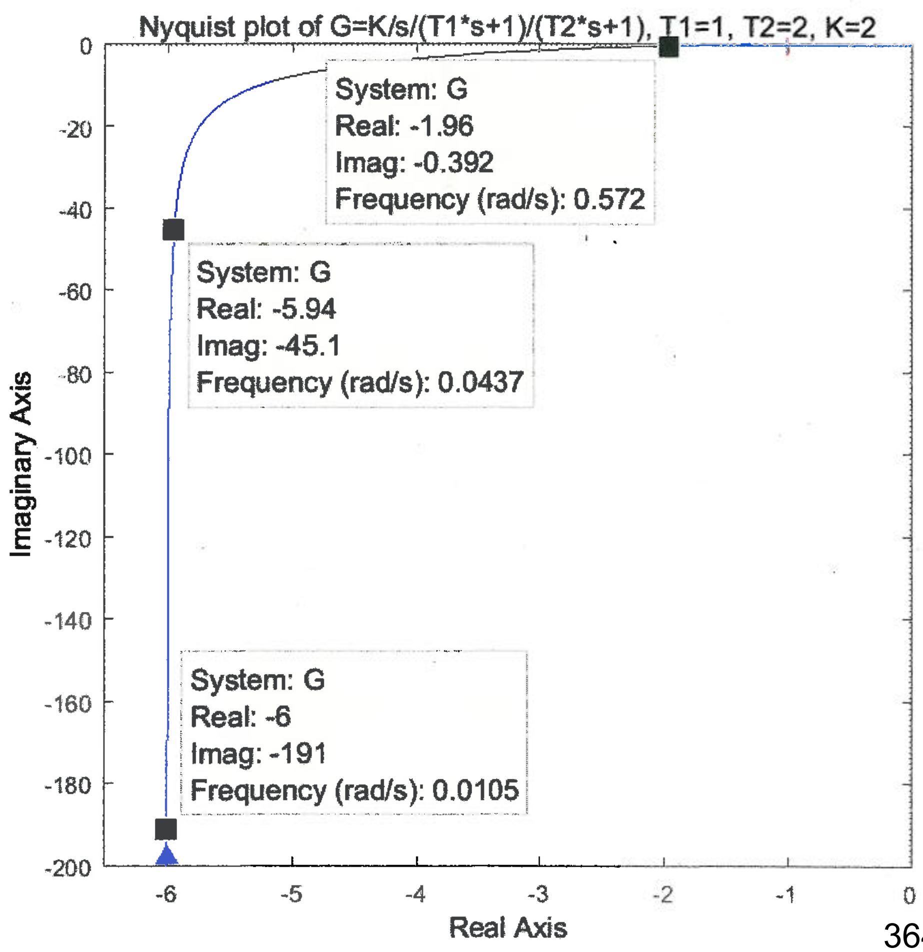
$$= \frac{K}{i\varepsilon\left(\cancel{i^2\varepsilon^2 T_1 T_2} + i\varepsilon(T_1+T_2)+1\right)} = \frac{K}{\varepsilon}\,\frac{1}{-\varepsilon(T_1+T_2)+i}$$

$$= \frac{K}{\varepsilon}\,\frac{-\varepsilon(T_1+T_2)-i}{\cancel{\varepsilon^2(T_1+T_2)^2}+1} = -\frac{K}{\varepsilon}\left[\varepsilon(T_1+T_2)+i\right]$$

$$G(i\varepsilon) = -K(T_1+T_2) - i\frac{K}{\varepsilon}$$

$$\xrightarrow[\varepsilon\to 0]{} \underbrace{-K(T_1+T_2)}_{ReG = Const} - i\infty$$





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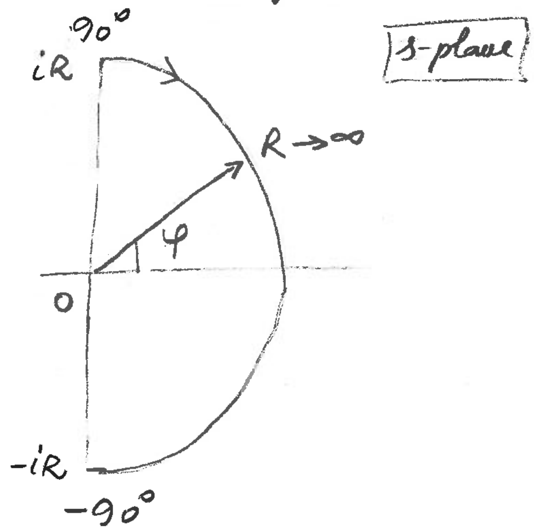
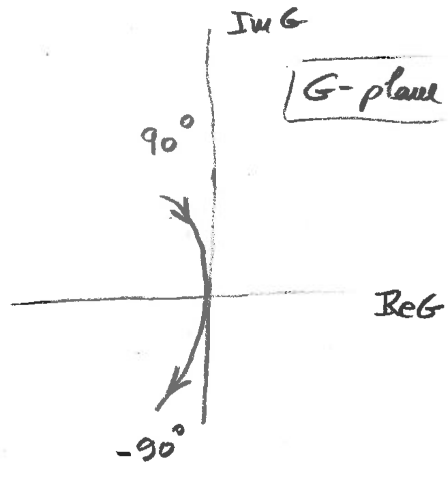
Big circle





$$G(s) = \frac{K}{Re^{i\varphi}(T_1 Re^{i\varphi}+1)(T_2 Re^{i\varphi}+1)}$$

$$G(s) \underset{R>>1}{\approx} \frac{K}{T_1 T_2 R^3 e^{3i\varphi}} = \frac{K}{T_1 T_2 R^3} e^{-3i\varphi}$$

$$|G(s)| = \frac{K}{T_1 T_2 R^3} \xrightarrow[R\to\infty]{} 0$$

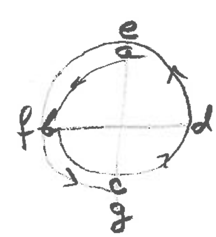
$$\angle G(s) = -3\varphi = (-270^\circ \quad +270^\circ)$$

$$\varphi = 90 \qquad -90$$

$(-270+360=90^\circ;\ 270-360=-90)$

$$\angle G(s) = (90^\circ, -90^\circ)$$

$\varphi =$	90°	60°	30°	0°	-30°	-60°	-90°
$\angle G$	-270	-180	-90	0	90	180	270
	90	180	-90	0	90	180	-90
	a	b	c	d	e	f	g



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G(s) goes round origin 1.5 times

s p634

$-i\omega$ axis

$$s = (-iR \quad -i\varepsilon)$$

$\varepsilon \to 0$
$R \to \infty$

$$G(-i\varepsilon) \cong -K(T_1+T_2) + i\frac{K}{\varepsilon}$$

$$G(-i\epsilon) \xrightarrow[\varepsilon \to 0]{} -K(T_1+T_2) + i\infty$$

$$|G(-i\epsilon)| \xrightarrow[\varepsilon \to 0]{} \infty$$

$$\angle G(-i\varepsilon) \xrightarrow[\varepsilon \to 0]{} 90^\circ$$

Proof

$$\frac{K}{-i\varepsilon(-i\varepsilon T_1+1)(-i\varepsilon T_2+1)} = \frac{K}{-i\varepsilon[\cancel{i^2\varepsilon^2 T_1 T_2} - i\varepsilon(T_1+T_2)+1]}$$

$$= i\frac{K}{\varepsilon}\,\frac{i\varepsilon(T_1+T_2)+1}{\cancel{\varepsilon^2(T_1+T_2)^2}+1}$$

$$= i\frac{K}{\varepsilon}\left[i\varepsilon(T_1+T_2)+1\right]$$

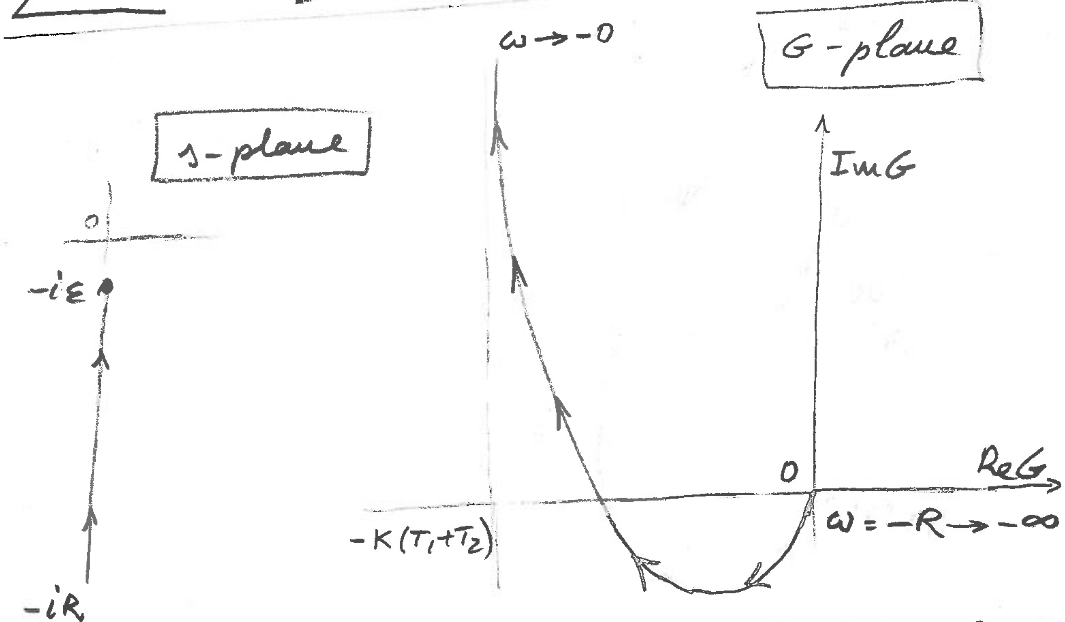
$$= -K(T_1+T_2) + i\,K/\varepsilon$$

$$G(-iR) = \frac{K}{-iR(-T_1 iR+1)(-T_2 iR+1)} \cong \frac{K}{-T_1T_2R^3 i^3} = \frac{K}{T_1T_2R^3}e^{-i\frac{\pi}{2}}$$

$$\frac{-1}{i^3} = \frac{i^2}{i^3} = \frac{1}{i} = -i$$

$$|G(-iR)| = \frac{K}{T_1T_2R^3} \xrightarrow[R \to \infty]{} 0$$

$$\angle G(iR) = -\frac{\pi}{2} = -90^\circ$$



exists 0 at -90°

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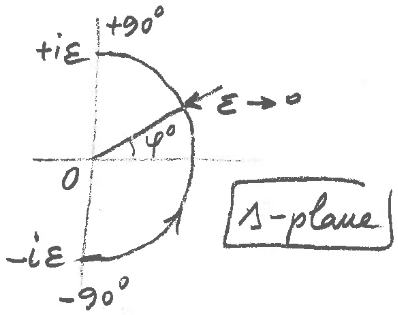
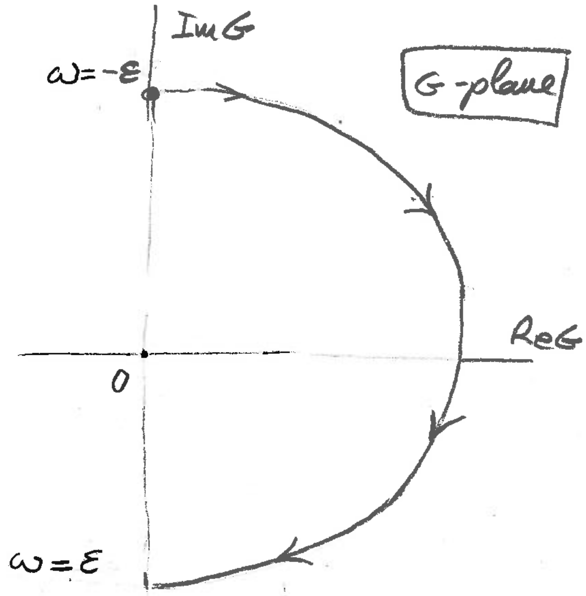
Small circle

$s_0 = \varepsilon e^{i\varphi_0}$

$\varphi_0 = (-90^\circ, 90^\circ)$





$$\varphi_0 = (-90^\circ, 90^\circ)$$

$$G(s_0) = \frac{K}{\varepsilon e^{i\varphi_0}(T_1 \varepsilon e^{i\varphi_0} + 1)(T_2 \varepsilon e^{i\varphi_0} + 1)}$$

$$G(s_0) \approx \frac{K}{\varepsilon e^{i\varphi_0}} = \frac{K}{\varepsilon} e^{-i\varphi_0}$$

$$|G(s_0)| = \frac{K}{\varepsilon} \xrightarrow[\varepsilon \to 0]{} \infty$$

$$\angle G(s_0) = -\varphi_0 = (90^\circ, -90^\circ)$$

$$\varphi_0 = \underset{-i\varepsilon}{-90^\circ}, \underset{i\varepsilon}{90^\circ})$$

$G(s_0)$ travels a big semicircle from 90° to -90°.

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8 p364 Finally, assemble the four segments:

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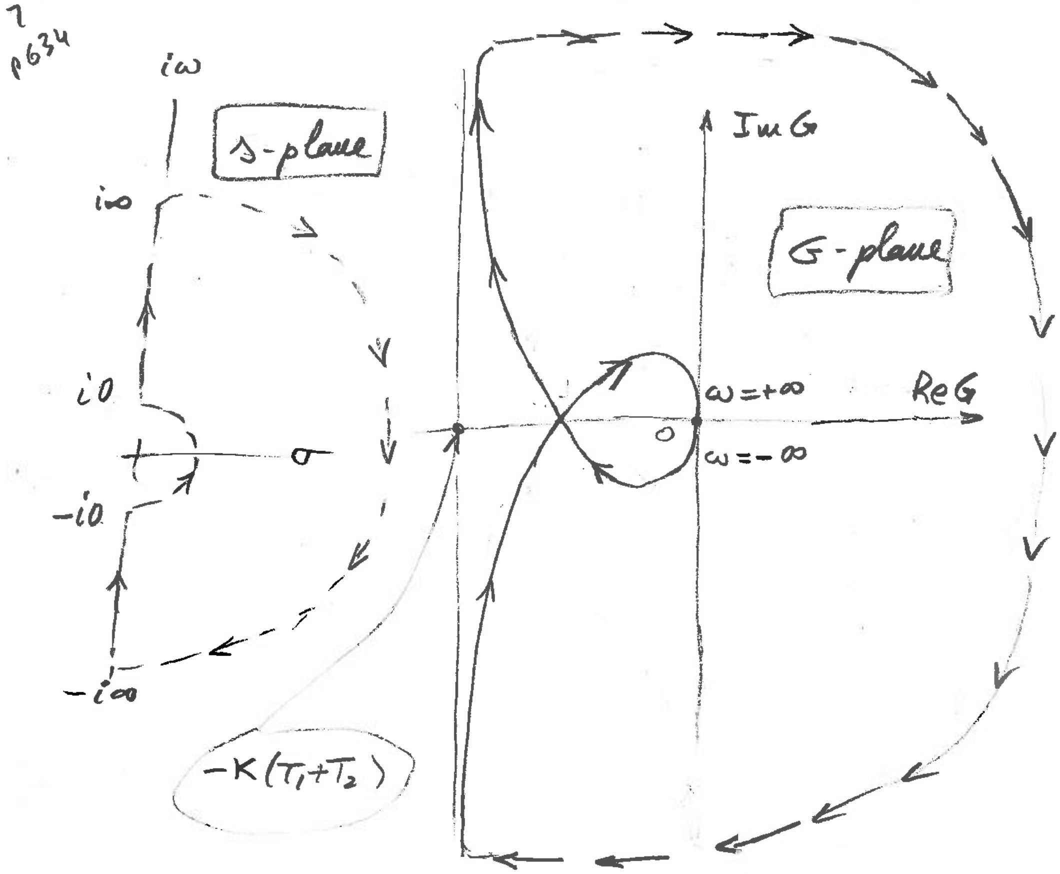
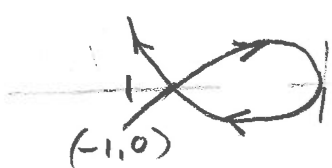


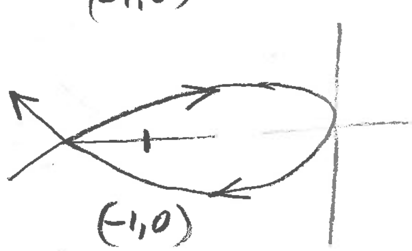
Note that $G(s)$ goes around two times!

Depending on K, it may or may not enclose the point $(-1,0)$.



STABLE

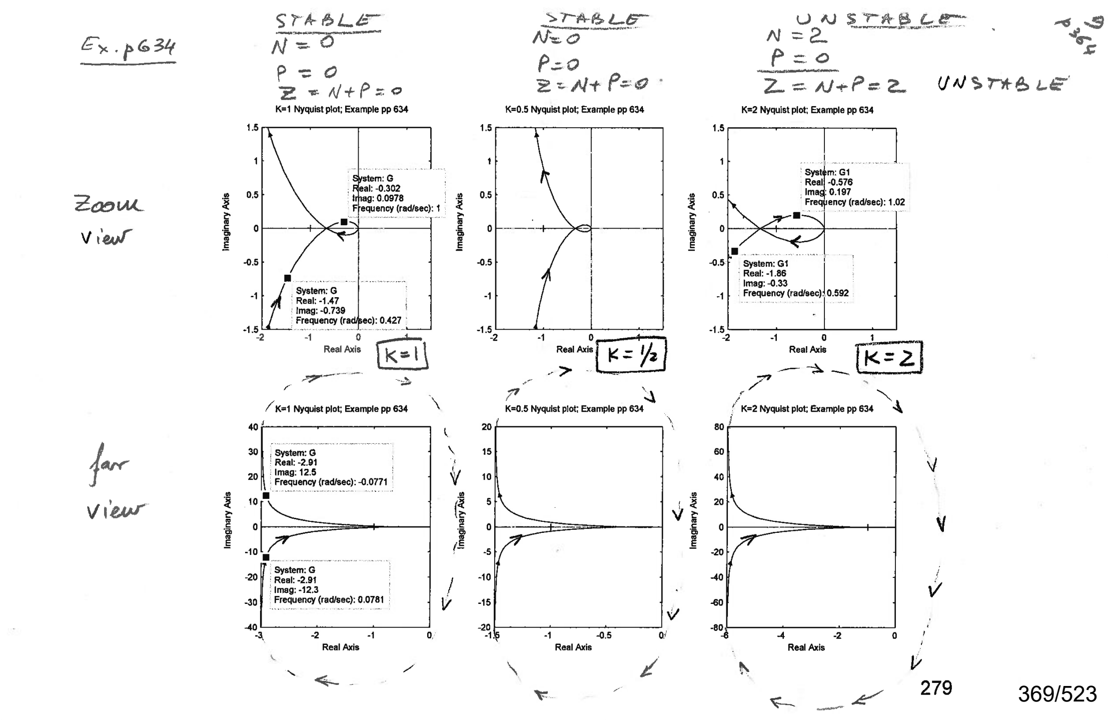
$$\begin{array}{r} N=0 \\ P=0 \\ \hline Z=0 \end{array}$$



UNSTABLE

$$\begin{array}{r} N=2 \\ P=0 \\ \hline Z=2 \end{array}$$

(see MATLAB plot next page) 368/523



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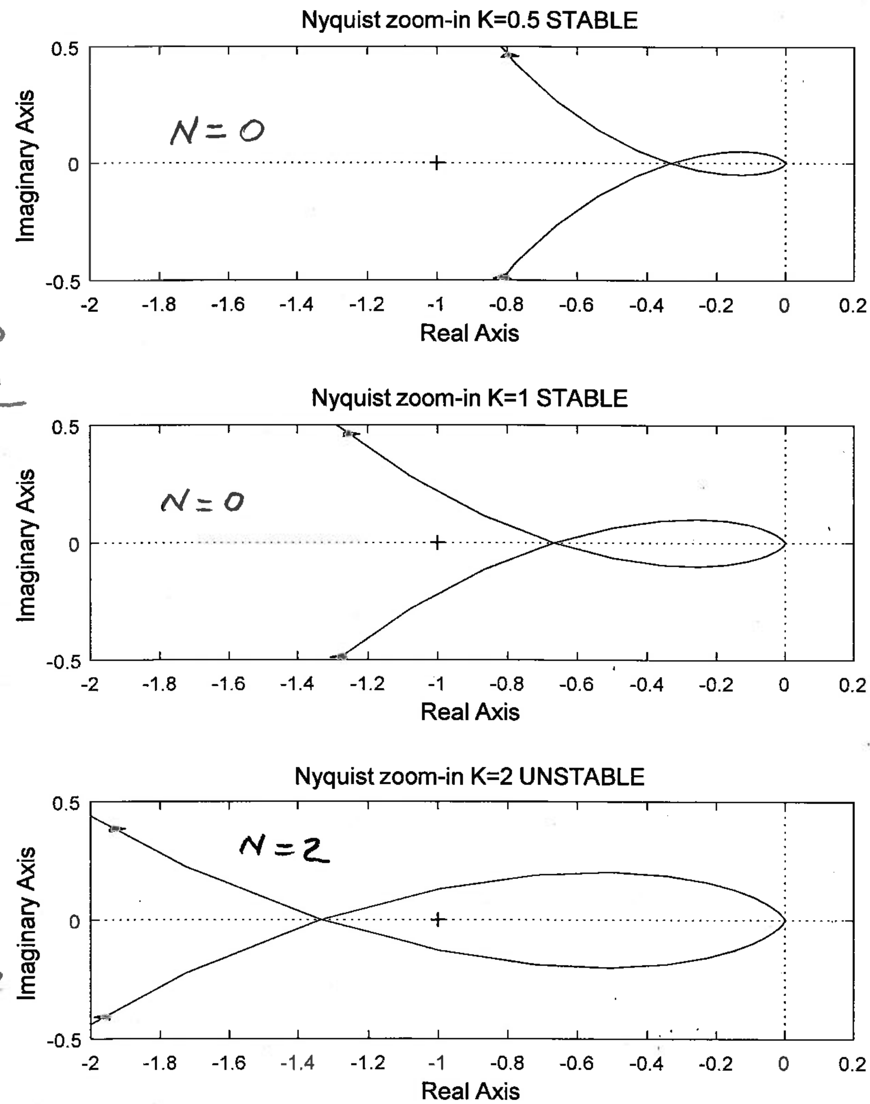
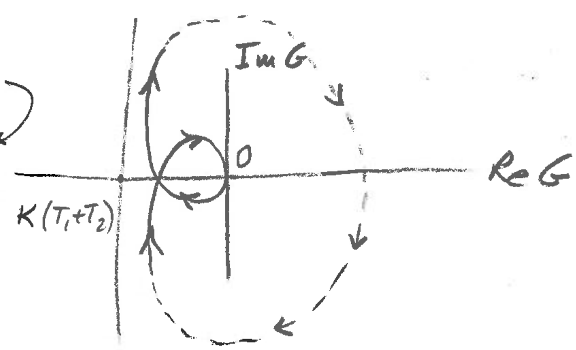


N=0
P=0
Z=0
stable

N=2
P=0
Z=2
unstable

two times around



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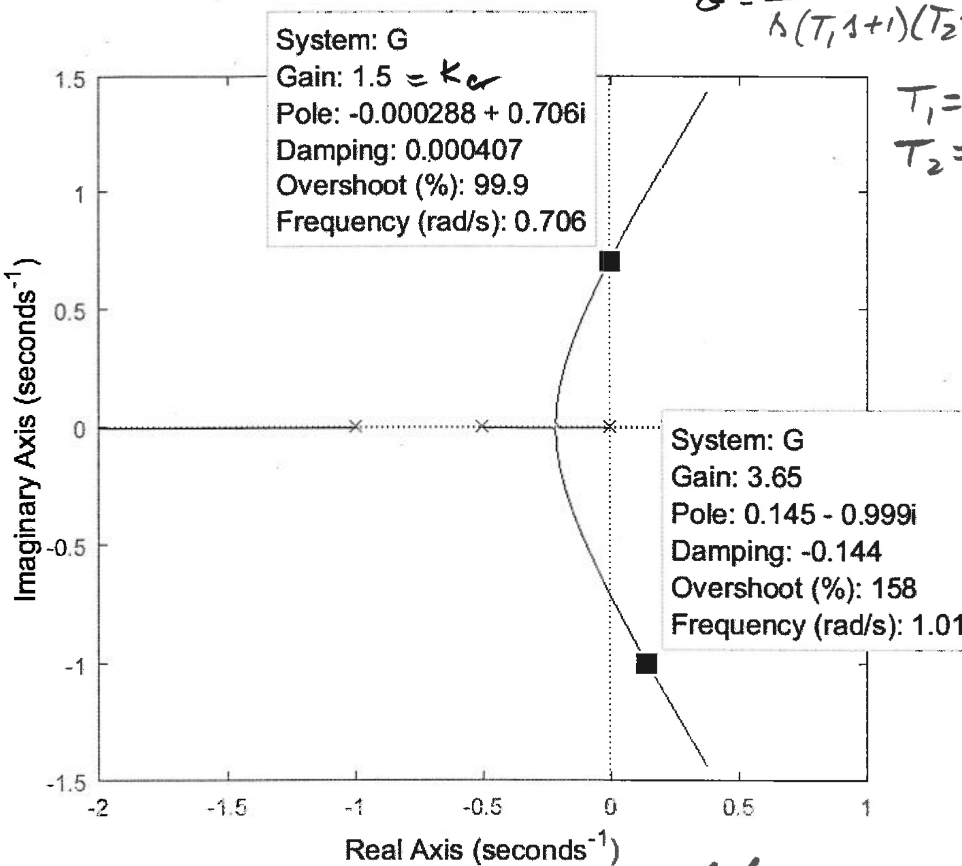
ROOT LOCUS

$$G = \frac{K}{s(T_1 s+1)(T_2 s+1)}$$

$T_1 = 1$

$T_2 = 2$



$K_{cr} = 1.5$ according to r locus plot

verify using $G_{CL} = \dfrac{G}{1+G} = \dfrac{K}{s(T_1 s+1)(T_2 s+1)+K}$

$$s(T_1 s+1)(T_2 s+1)+K\Big|_{\substack{T_1=1\\T_2=2}} = s(s+1)(2s+1)+K$$

$$s(2s^2+3s+1)+K = 2s^3+3s^2+s+K$$

Routh criterion

s^3	2	1
s^2	3	K
s^1	$\frac{3-2K}{3}$	
s^0	K	

For sign change:

$3-2K=0$

$K_{cr} = \frac{3}{2} = 1.5$

QED.

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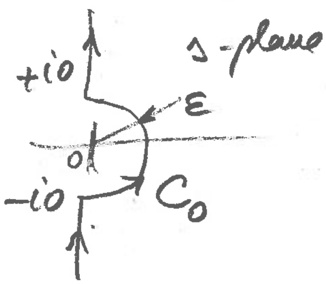
N3d

Tips on Nyquist Plot



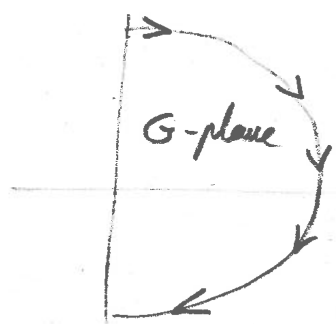
$$s_0 = \varepsilon e^{i\varphi}$$

$$-90^\circ < \varphi < 90^\circ.$$

$G = \frac{1}{s}$, $G(s_0) = \frac{1}{\varepsilon e^{i\varphi}} = \frac{1}{\varepsilon} e^{-i\varphi}$

(Type 1 system)

$$+90^\circ > \angle G(s_0) > -90^\circ$$

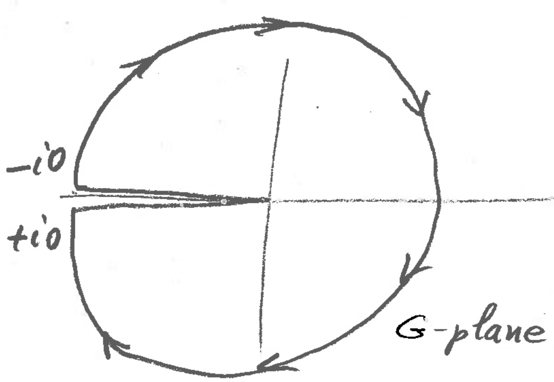


Half-circle, clockwise.

$G = \frac{1}{s^2}$, $G(s_0) = \frac{1}{(\varepsilon e^{i\varphi})^2} = \frac{1}{\varepsilon^2} e^{-i2\varphi}$.

(Type 2 system)

$$+180^\circ > \angle G > -180^\circ.$$



Full circle, clockwise.

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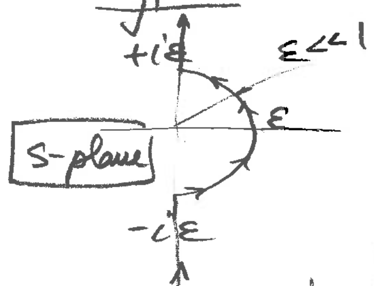
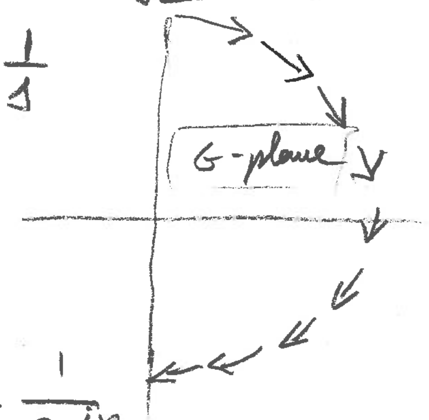
Details on behavior of common functions on the Nyquist circuit

Type 1

$G(s) = \frac{1}{s}$





$s_0 = \varepsilon e^{i\varphi}$

$-90^\circ < \varphi < 90^\circ$

$G(s_0) = \frac{1}{\varepsilon e^{i\varphi}} = \frac{1}{\varepsilon} e^{-i\varphi}$

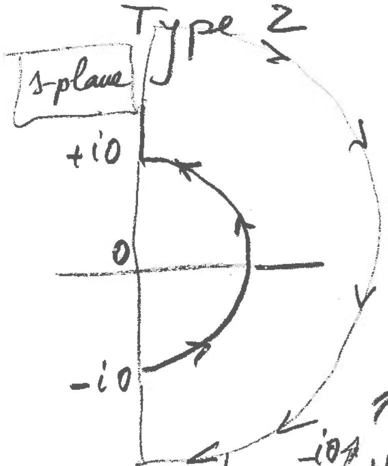
$90^\circ > \angle G(s_0) > -90^\circ$

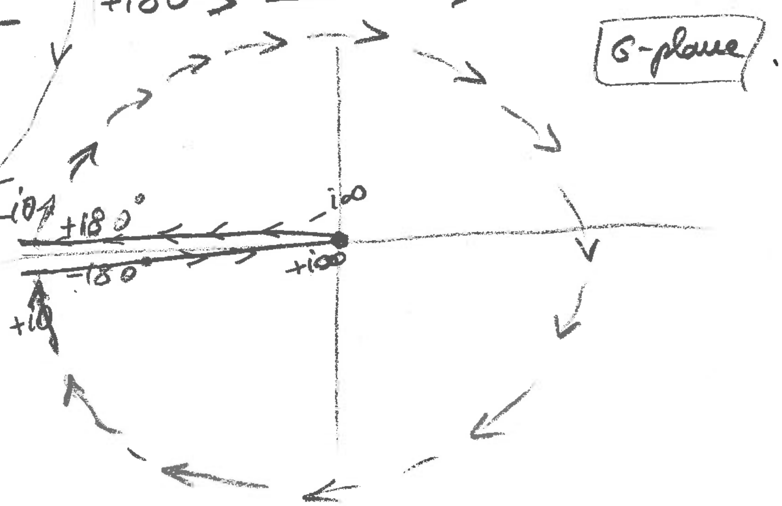
Type 2

$G(s) = \frac{1}{s^2}$

$G(s_0) = \frac{1}{\varepsilon^2} e^{-i2\varphi}, \quad -90^\circ < \varphi < +90^\circ$

$+180^\circ > \angle G(s_0) > -180^\circ$





$G(-i0) = \frac{1}{(-i0)^2}$

$G(+i0) = \frac{1}{(i0)^2}$

N1a

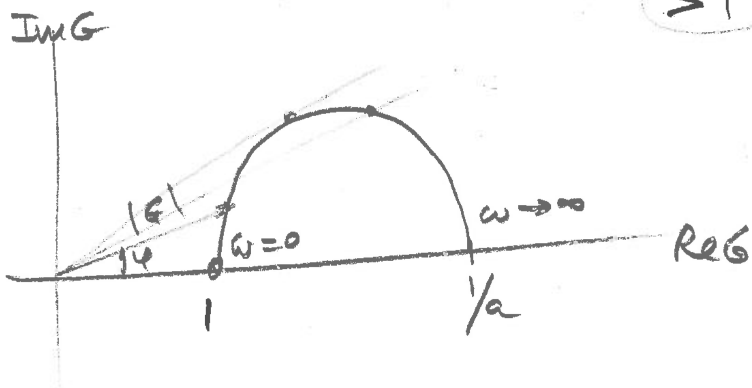
$$G(s) = \frac{Ts+1}{aTs+1} \qquad 0<a<1$$

$$G(i\omega) = \frac{T\cdot(i\omega)+1}{a\cdot T\cdot(i\omega)+1}$$

$\omega \to 0$, $G(i0) = 1$

$\omega \to \infty$, $G(i\infty) = \lim_{\omega\to\infty} \frac{T(i\omega)+1}{a T\cdot(i\omega)+1} = \frac{T}{aT} = \frac{1}{a}$

>1



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NIG.

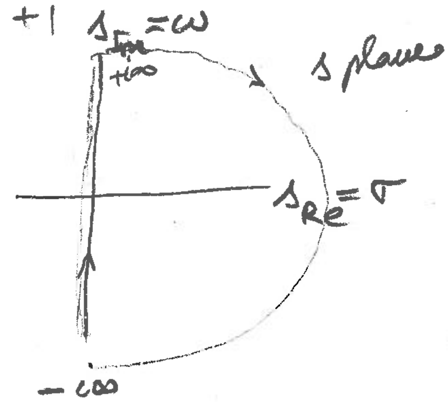
Ex 11.4, p 641

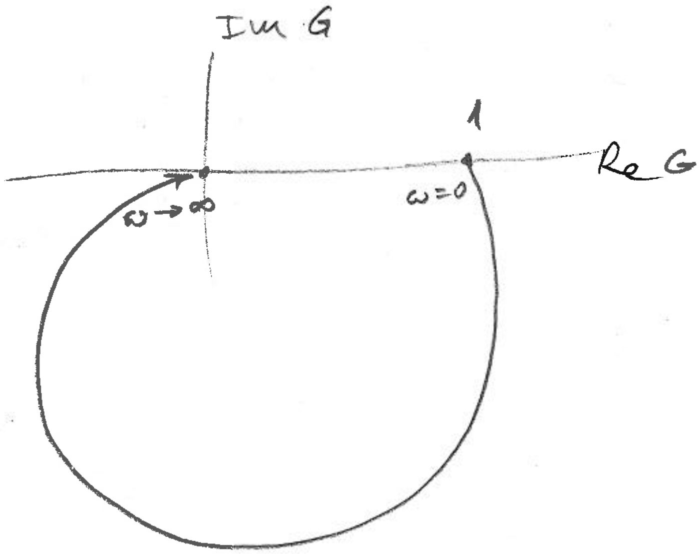
$$G(s) = \frac{1}{s^2 + 0.8s + 1}$$

G = 1/- - -

nyquist(G)

Nyquist circuit for ω





Ex. 11.5 p 642

$$G(s) = \frac{1}{s(s+1)}.$$

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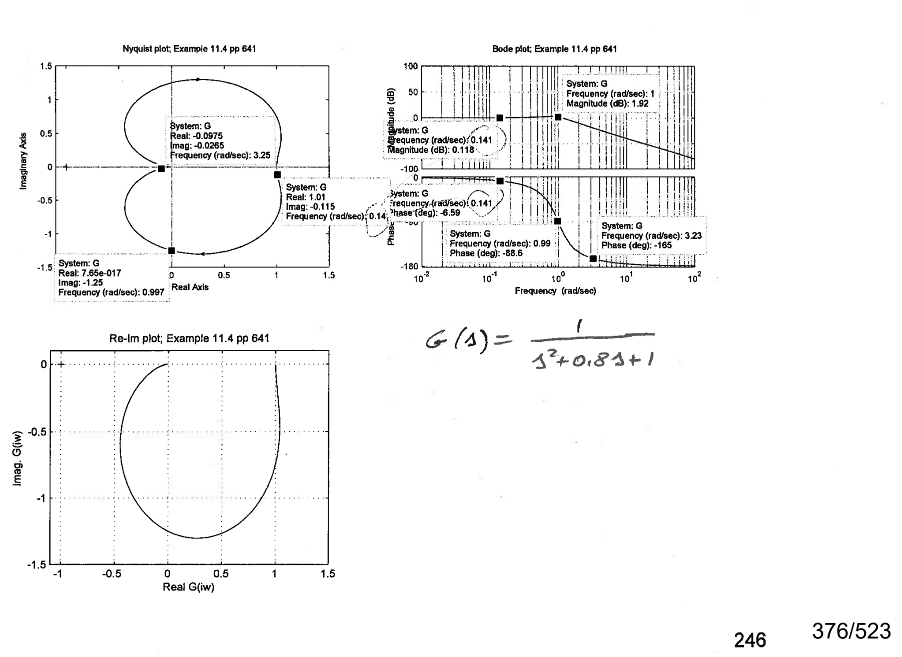
Nyquist plot; Example 11.4 pp 641
Imaginary Axis
Real Axis
System: G
Real: -0.0975
Imag: -0.0265
Frequency (rad/sec): 3.25
System: G
Real: 1.01
Imag: -0.115
System: G
Real: 7.65e-017
Imag: -1.25
Frequency (rad/sec): 0.997
Bode plot; Example 11.4 pp 641
System: G
Frequency (rad/sec): 1
Magnitude (dB): 1.92
System: G
Frequency (rad/sec): 0.141
Magnitude (dB): 0.118
System: G
Frequency (rad/sec): 0.99
Phase (deg): -88.6
System: G
Frequency (rad/sec): 3.23
Phase (deg): -165
Frequency (rad/sec)
Re-Im plot; Example 11.4 pp 641
Imag. G(iw)
Real G(iw)
$G(s) = \frac{1}{s^2+0.8s+1}$
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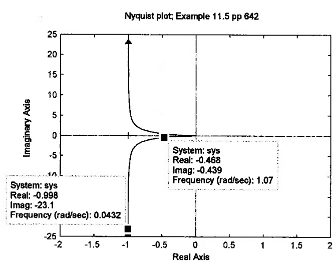

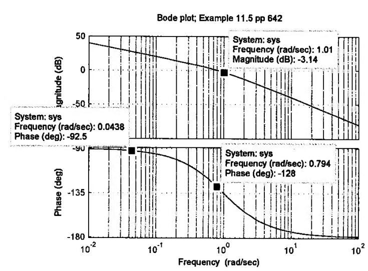

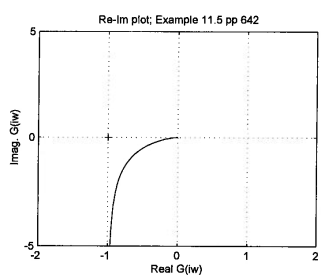

$$G(s) = \frac{1}{s(s+1)}.$$

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N2

Nyquist plot trends

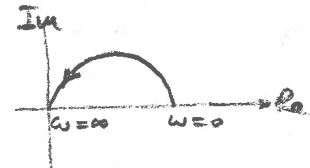
$$G(s) = \frac{K}{s^N}\frac{(T_a s+1)(T_b s+1)\dots}{(T_1 s+1)(T_2 s+1)\dots}$$

Type 0 systems $(N=0)$, $G(i\omega) = K\dfrac{(T_a i\omega+1)(T_b i\omega+1)\dots}{(T_1 i\omega+1)(T_2 i\omega+1)\dots}$ order m / order n

$\omega = 0$ $\quad G(i0) = K$

$\omega \to \infty$

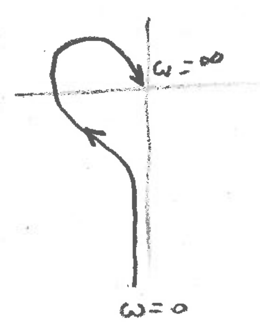
$$G(i\infty) = \begin{cases} 0 & \text{for } n>m \\ \text{const}, & n=m \\ \infty, & n<m \end{cases}$$



Type 1 systems $(N=1)$, $G(i\omega) = \dfrac{K}{i\omega}\cdot\dfrac{(\quad)(\quad)}{(\quad)(\quad)}$ $\begin{matrix} m \\ n \end{matrix}$

for $n=m$, $G(i\infty) = 0$

$G(i0) = \dfrac{1}{i0}$, $\angle G(i0) = -90^\circ$



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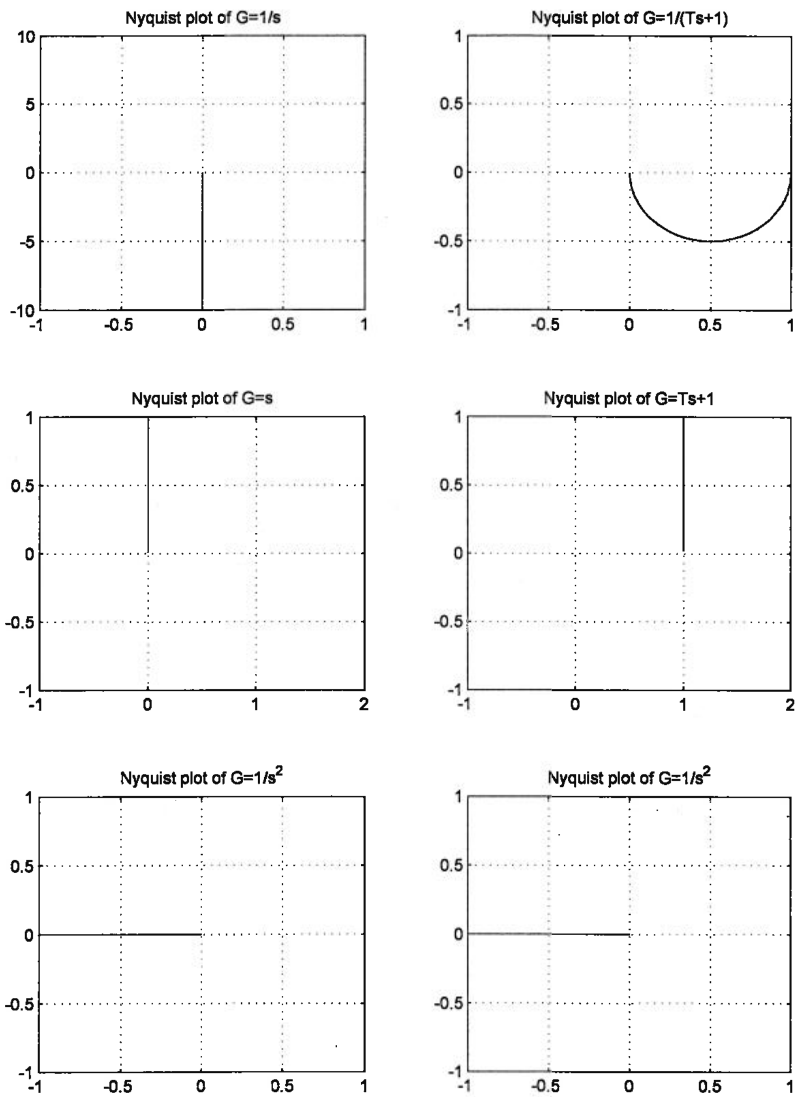
Nyquist plot of G=1/s
Nyquist plot of G=1/(Ts+1)
Nyquist plot of G=s
Nyquist plot of G=Ts+1
Nyquist plot of G=1/s2
Nyquist plot of G=1/s2
10
5
0
-5
-10
1
0.5
0
-0.5
-1
-0.5
0
0.5
1
2

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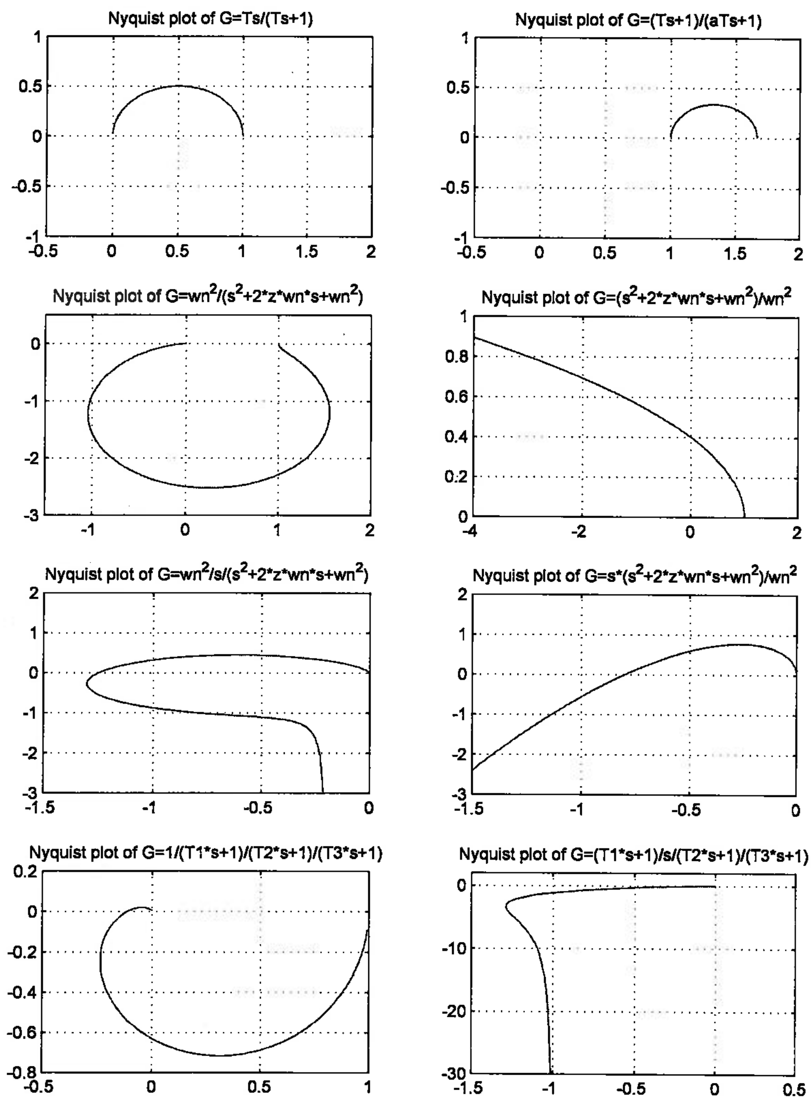
Nyquist plot of G=Ts/(Ts+1)
Nyquist plot of G=(Ts+1)/(aTs+1)
Nyquist plot of G=wn^2/(s^2+2*z*wn*s+wn^2)
Nyquist plot of G=(s^2+2*z*wn*s+wn^2)/wn^2
Nyquist plot of G=wn^2/s/(s^2+2*z*wn*s+wn^2)
Nyquist plot of G=s*(s^2+2*z*wn*s+wn^2)/wn^2
Nyquist plot of G=1/(T1*s+1)/(T2*s+1)/(T3*s+1)
Nyquist plot of G=(T1*s+1)/s/(T2*s+1)/(T3*s+1)

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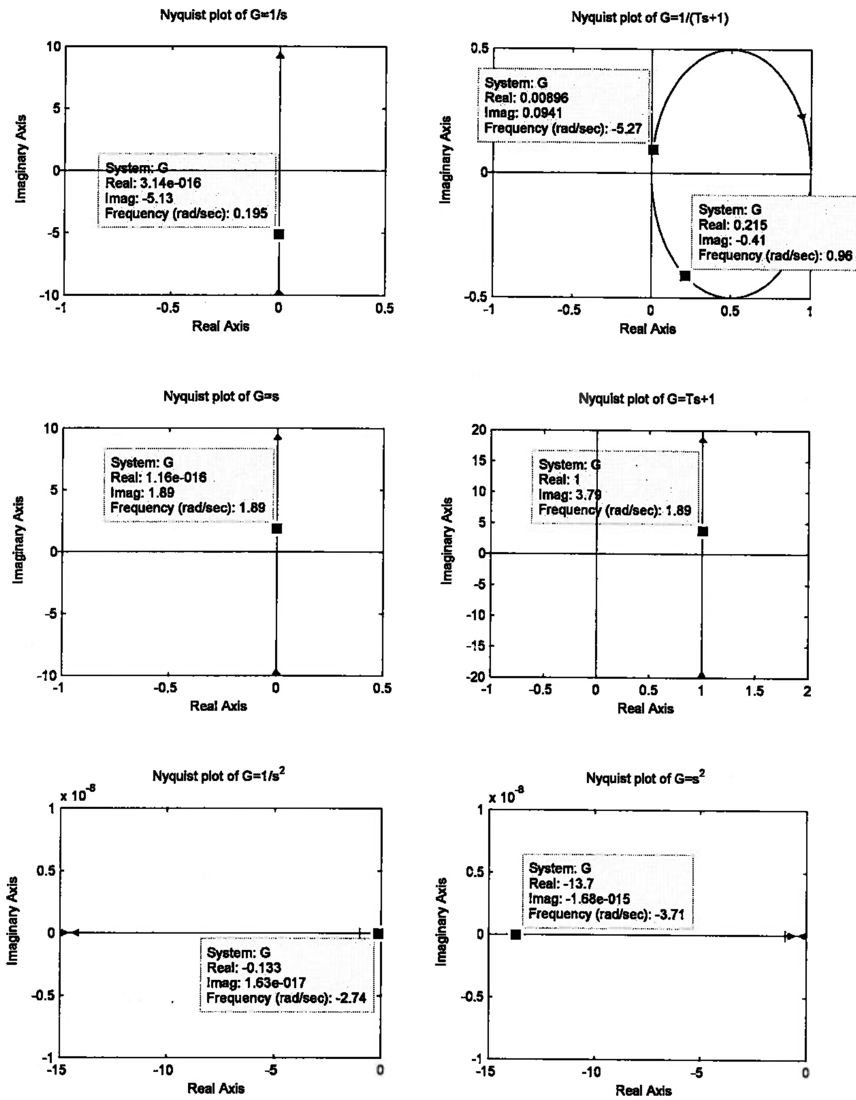
Nyquist plot of G=1/s
System: G
Real: 3.14e-016
Imag: -5.13
Frequency (rad/sec): 0.195
Imaginary Axis
Real Axis
Nyquist plot of G=1/(Ts+1)
System: G
Real: 0.00896
Imag: 0.0941
Frequency (rad/sec): -5.27
System: G
Real: 0.215
Imag: -0.41
Frequency (rad/sec): 0.96
Imaginary Axis
Real Axis
Nyquist plot of G=s
System: G
Real: 1.16e-016
Imag: 1.89
Frequency (rad/sec): 1.89
Imaginary Axis
Real Axis
Nyquist plot of G=Ts+1
System: G
Real: 1
Imag: 3.79
Frequency (rad/sec): 1.89
Imaginary Axis
Real Axis
Nyquist plot of G=1/s^2
x 10^-8
System: G
Real: -0.133
Imag: 1.63e-017
Frequency (rad/sec): -2.74
Imaginary Axis
Real Axis
Nyquist plot of G=s^2
x 10^-8
System: G
Real: -13.7
Imag: -1.68e-015
Frequency (rad/sec): -3.71
Imaginary Axis
Real Axis

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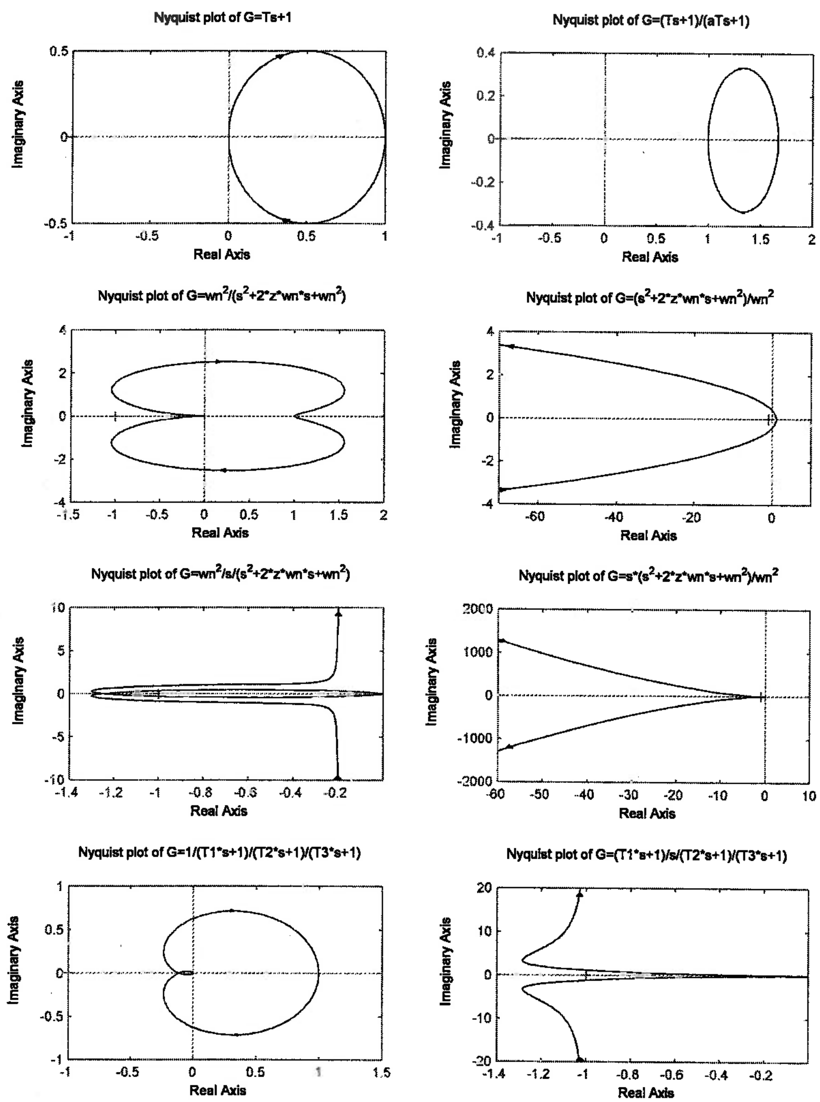
Nyquist plot of G=Ts+1
Nyquist plot of G=(Ts+1)/(aTs+1)
Nyquist plot of G=wn^2/(s^2+2*z*wn*s+wn^2)
Nyquist plot of G=(s^2+2*z*wn*s+wn^2)/wn^2
Nyquist plot of G=wn^2/s/(s^2+2*z*wn*s+wn^2)
Nyquist plot of G=s*(s^2+2*z*wn*s+wn^2)/wn^2
Nyquist plot of G=1/(T1*s+1)/(T2*s+1)/(T3*s+1)
Nyquist plot of G=(T1*s+1)/s/(T2*s+1)/(T3*s+1)
Imaginary Axis
Real Axis

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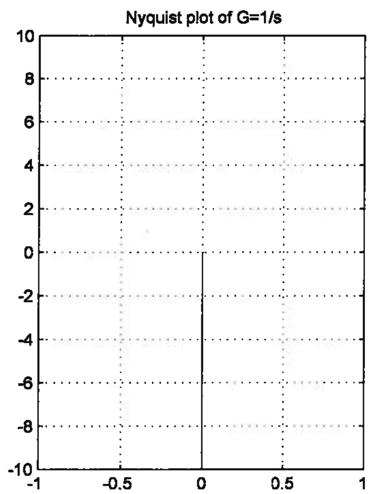
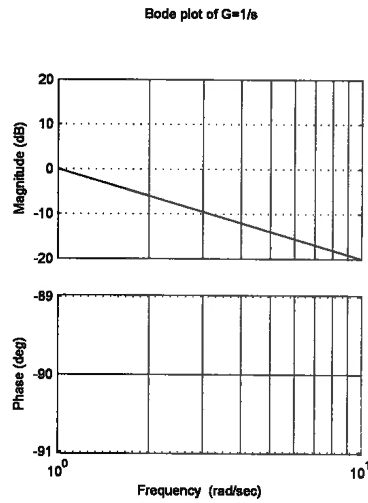
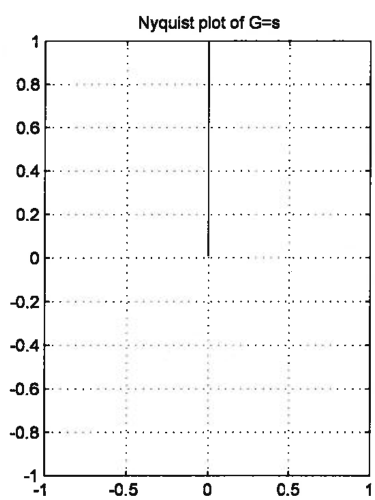
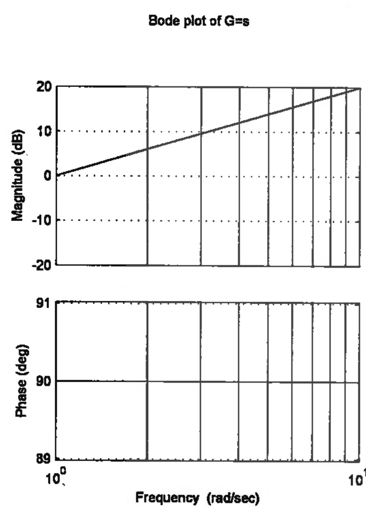

Nyquist plot of G=1/s
10
8
6
4
2
0
-2
-4
-6
-8
-10
-1
-0.5
0
0.5
1


Bode plot of G=1/s
Magnitude (dB)
20
10
0
-10
-20
Phase (deg)
-89
-90
-91
10^0
10^1
Frequency (rad/sec)


Nyquist plot of G=s
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1


Bode plot of G=s
Magnitude (dB)
20
10
0
-10
-20
Phase (deg)
91
90
89
10^0
10^1
Frequency (rad/sec)

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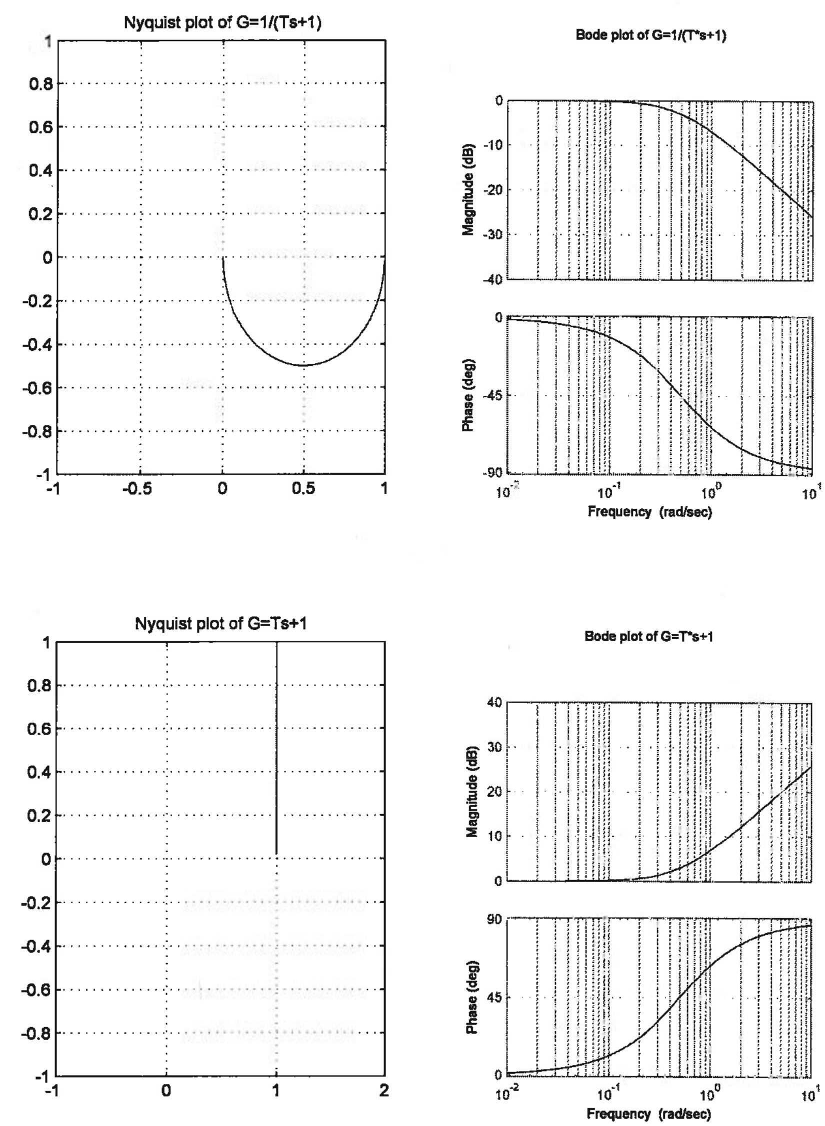
Nyquist plot of G=1/(Ts+1)
Bode plot of G=1/(T*s+1)
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)
Nyquist plot of G=Ts+1
Bode plot of G=T*s+1
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)

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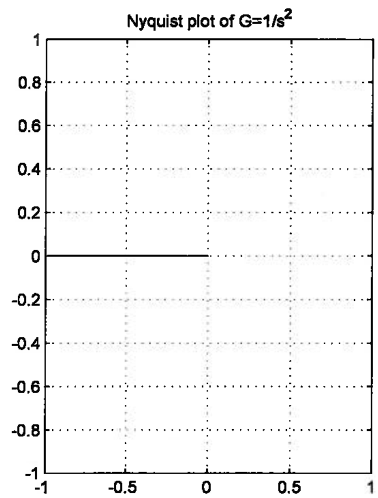
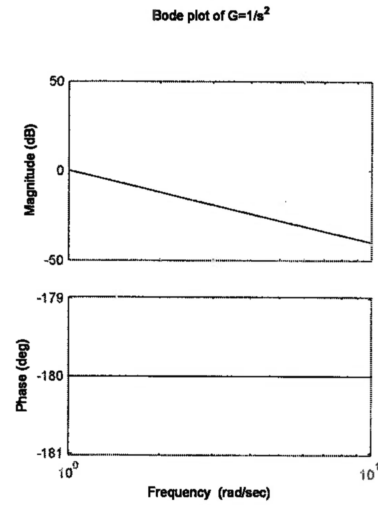
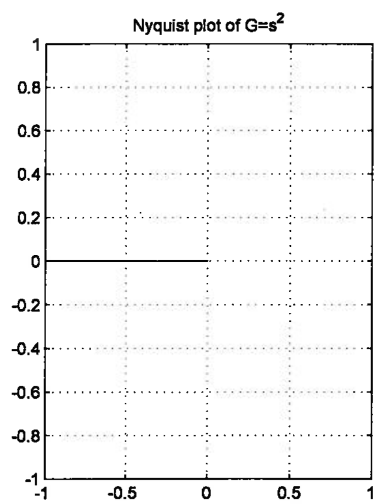
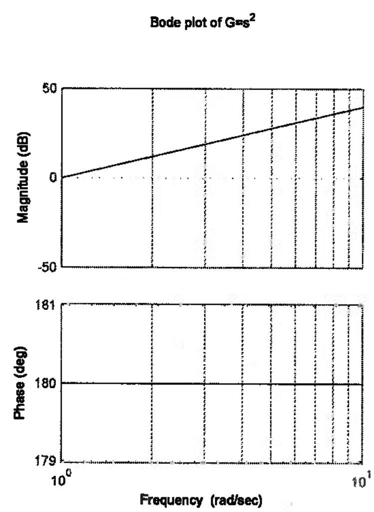

Nyquist plot of G=1/s^2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1


Bode plot of G=1/s^2
Magnitude (dB)
50
0
-50
Phase (deg)
-179
-180
-181
10^0
10^1
Frequency (rad/sec)


Nyquist plot of G=s^2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1


Bode plot of G=s^2
Magnitude (dB)
50
0
-50
Phase (deg)
181
180
179
10^0
10^1
Frequency (rad/sec)

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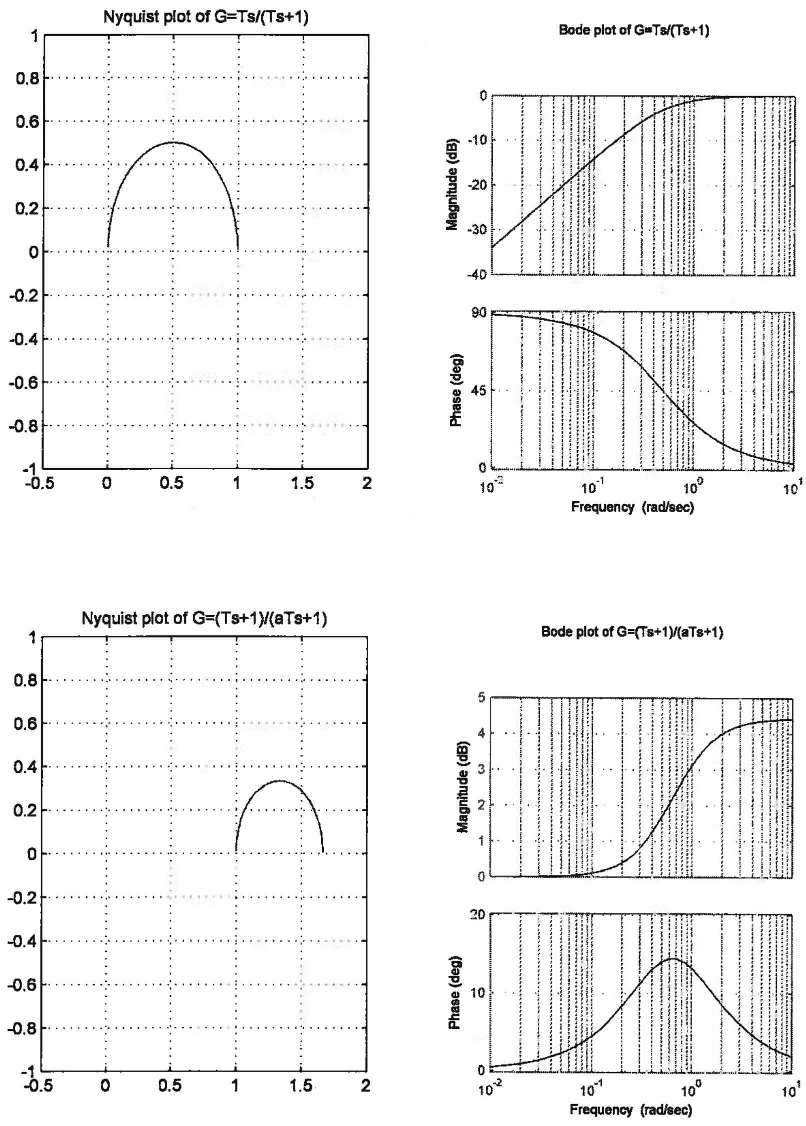
Nyquist plot of G=Ts/(Ts+1)
Bode plot of G=Ts/(Ts+1)
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)
Nyquist plot of G=(Ts+1)/(aTs+1)
Bode plot of G=(Ts+1)/(aTs+1)
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)

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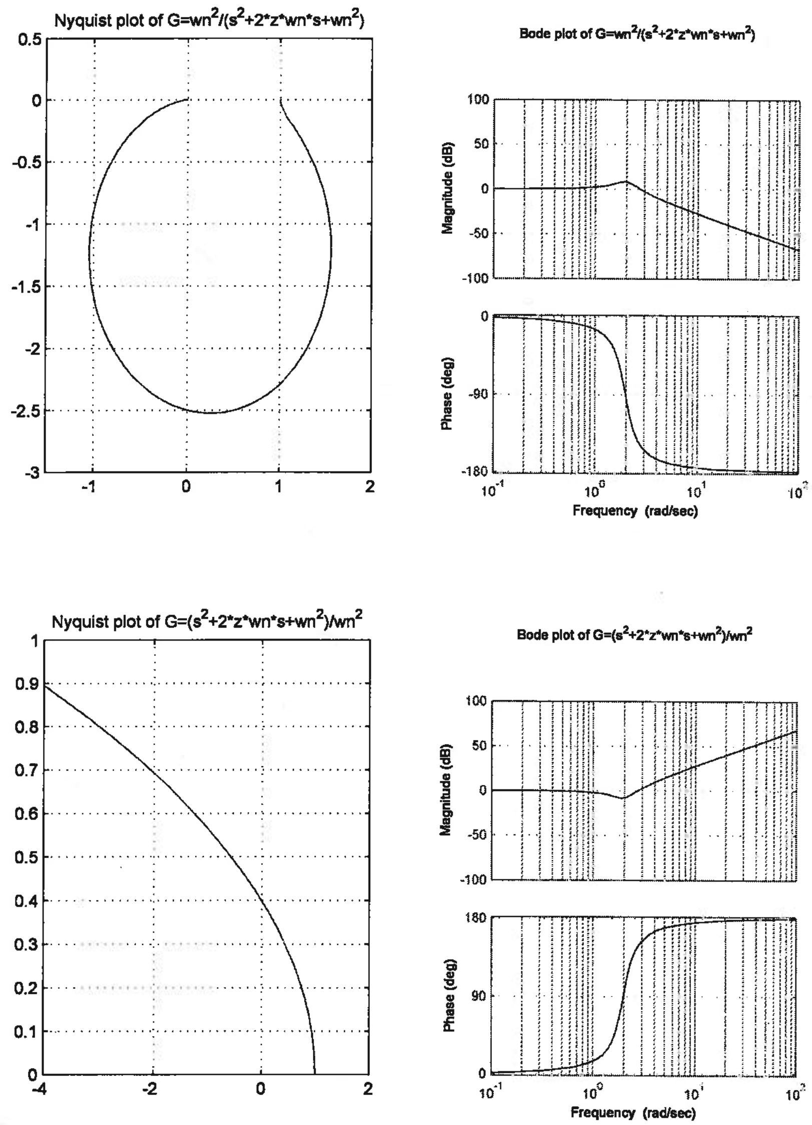
Nyquist plot of G=wn^2/(s^2+2*z*wn*s+wn^2)
Bode plot of G=wn^2/(s^2+2*z*wn*s+wn^2)
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)
Nyquist plot of G=(s^2+2*z*wn*s+wn^2)/wn^2
Bode plot of G=(s^2+2*z*wn*s+wn^2)/wn^2
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)

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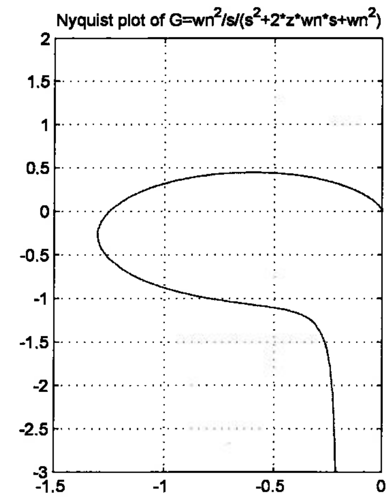
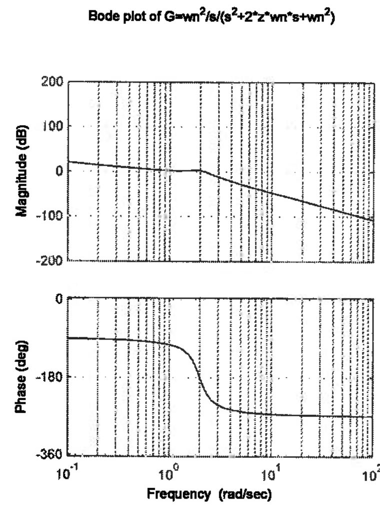
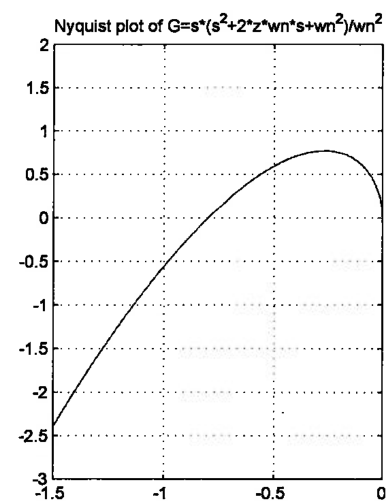
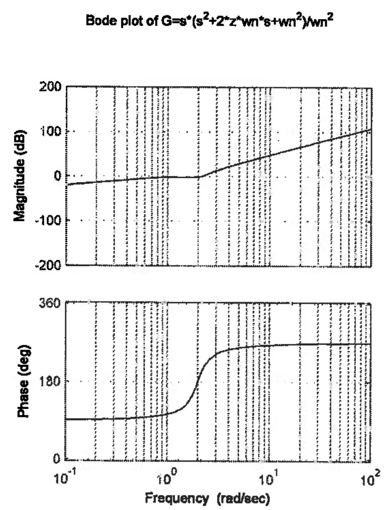
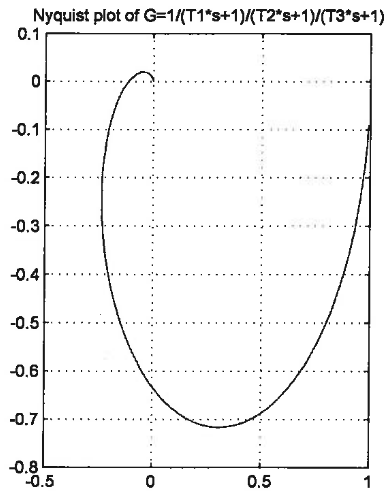
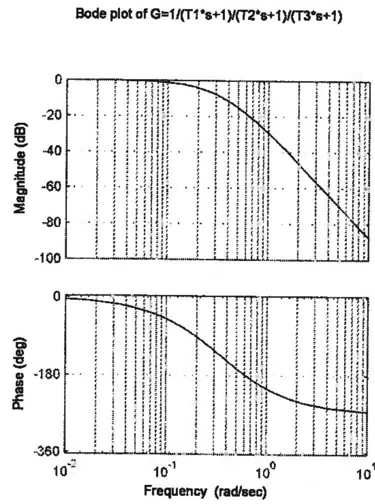

Nyquist plot of G=wn^2/s/(s^2+2*z*wn*s+wn^2)

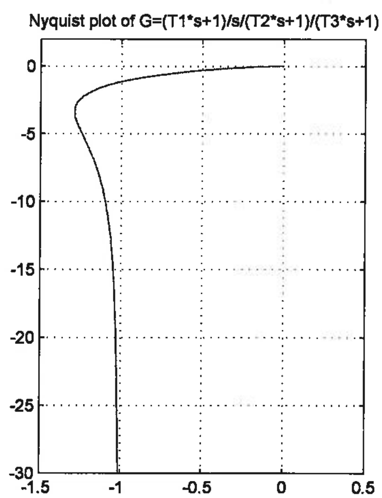

Bode plot of G=wn^2/s/(s^2+2*z*wn*s+wn^2)
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)

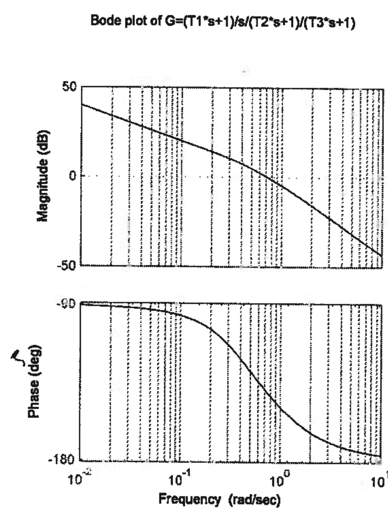

Nyquist plot of G=s*(s^2+2*z*wn*s+wn^2)/wn^2


Bode plot of G=s*(s^2+2*z*wn*s+wn^2)/wn^2
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)


Nyquist plot of G=1/(T1*s+1)/(T2*s+1)/(T3*s+1)


Bode plot of G=1/(T1*s+1)/(T2*s+1)/(T3*s+1)
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)


Nyquist plot of G=(T1*s+1)/s/(T2*s+1)/(T3*s+1)


Bode plot of G=(T1*s+1)/s/(T2*s+1)/(T3*s+1)
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)

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8.6 Stability Margins

MIL-DTL-9490E defines margin requirements for aircraft flight control systems (FCS) with feedback:

GM = gain margin
PM = phase margin

TABLE III. Gain and phase margin requirements (dB, degrees).

Mode Frequency Hz \ Air speed	Below V_{oMIN}	V_{oMIN} to V_{oMAX}	At Limit Airspeed (V_L)	At 1.15 V_L
$f_M < 0.06$	GM= 6 dB (No Phase Requirement Below V_{oMIN})	GM = $\pm$ 4.5 PM = $\pm$ 30	GM = $\pm$ 3.0 PM = $\pm$ 20	GM=0 PM=0 (Stable at Nominal Phase and Gain)
$0.06 \le f_M <$ First Aero-elastic Mode		GM = $\pm$ 6.0 PM = $\pm$ 45	GM = $\pm$ 4.5 PM = $\pm$ 30	
$f_M >$ First Aero-Elastic Mode		GM = $\pm$ 8.0 PM = $\pm$ 60	GM = $\pm$ 6.0 PM = $\pm$ 45	

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Where:

V_L	=	Limit airspeed (MIL-A-8860).
V_{oMIN}	=	Minimum operational airspeed (MIL-F-8785)
V_{oMAX}	=	Maximum operational airspeed (MIL-F-8785)
Mode	=	A characteristics aeroelastic response of the aircraft as described by the an aeroelastic characteristic root of the coupled aircraft/FCS dynamic equation of motion.
GM (Gain Margin)	=	The minimum change chain in loop gain at normal phase, which results in instability beyond that allowed as a residual oscillation.
PM (Phase Margin)	=	The minimum change in phase at normal loop gain which results in instability.
f_M	=	Mode frequency in Hz
Nominal Phase and Gain	=	The contractor's best estimate or measurement of FCS and aircraft phase and gain characteristics available at the time of requirement verification.

3.1.3.6.2 Sensitivity analysis. Tolerances on feedback gain and phase shall be .established at the system level based on the anticipated range of gain and phase errors which will exist between nominal test values or predictions and in-service operation due to such factors as poorly defined nonlinear and higher order dynamics, anticipated manufacturing tolerances, aging, wear, maintenance and noncritical material failures. Gain and phase margins shall be defined, based on these tolerances, which will assure satisfactory operation in fleet usage. These gain and phase tolerances shall be established based on variations in system characteristics either anticipated or allowed by component or subsystem specification. The contractor shall establish, with the approval of the procuring agency, the range of variation to be considered based on a selected probability of exceedance for each type of variation. The contractor shall select the exceedance probability based on the criticality of the flight control function being provided. The stability requirements established through this sensitivity analysis shall not be less than 50 percent of the magnitude and phase requirements of 3.1.3.6.1.

3.1.3.7 Operation in turbulence. In Operational State I, while flying in the following applicable random and discrete turbulence environment, the FCS shall provide a safe level of operation and maintain mission accomplishment capabilities. For essential and flight phase commands by two aircrew members from causing any operation in opposing directions at the same time.

3.1.3.6 Stability. For FCS using feedback systems, the stability as specified in 3.1.3.6.1 shall be provided. Alternatively, when approved by the procuring activity, the stability defined by the contractor through the sensitivity analyses of 3.1.3.6.2 shall be provided. Where analysis is used to demonstrate compliance with these stability requirements, the effects of major system nonlinearities shall be included.

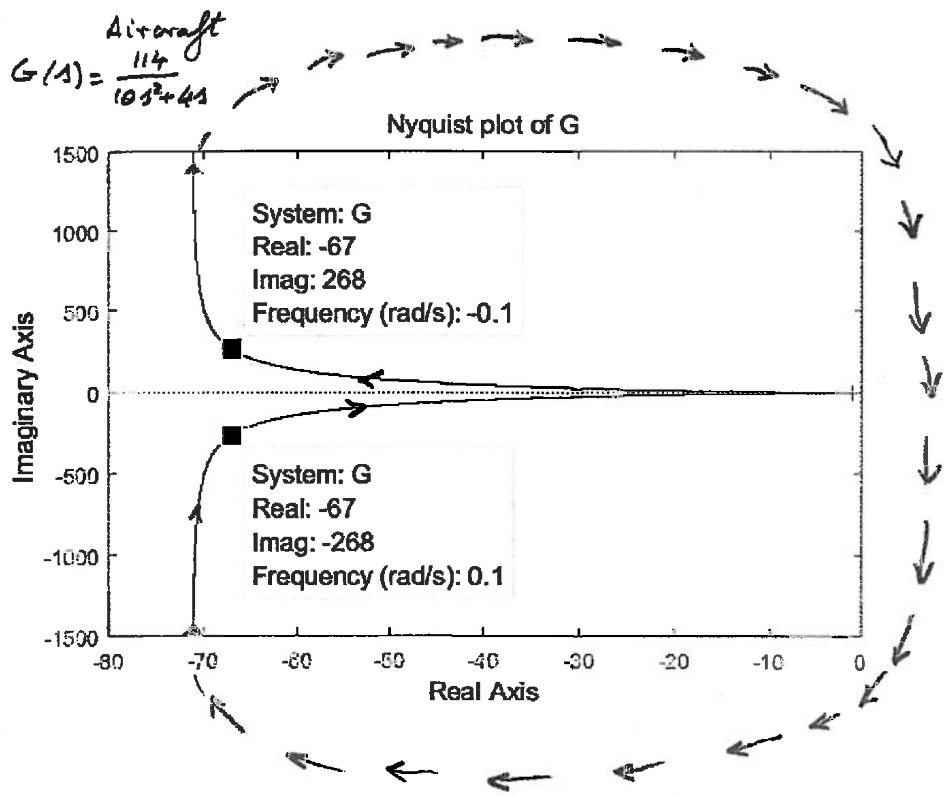
3.1.3.6.1 Stability margins. Required gain and phase margins about nominal are specified in table III for all aerodynamically closed loop FCS. With these gain or phase variations included, no oscillatory instabilities shall exist with amplitudes greater than those allowed for residual oscillations in 3.1.3.8, and any non oscillatory divergence of the aircraft shall remain within the applicable limits of MIL-F-8785 or MIL-F-83300. AFCS loops shall be stable with these gain or phase variations included for any amplitudes greater than those allowed for residual oscillations in 3.1.3.8. In multiple loop systems, variations shall be made with all gain and phase values in the feedback paths held at nominal values except for the path under investigation. A path is defined to include those elements connecting a sensor to a force or moment producer. For both aerodynamic and nonaerodynamic closed loops, at least 6 dB gain margin shall exist at zero airspeed. At the end of system wear tests, at least 4.5 dB gain margin shall exist for all loops at zero airspeed. The margins specified by table III shall be maintained under flight conditions of most adverse center-of-gravity, mass distribution, and external store configuration throughout the operational envelope and during ground operations.

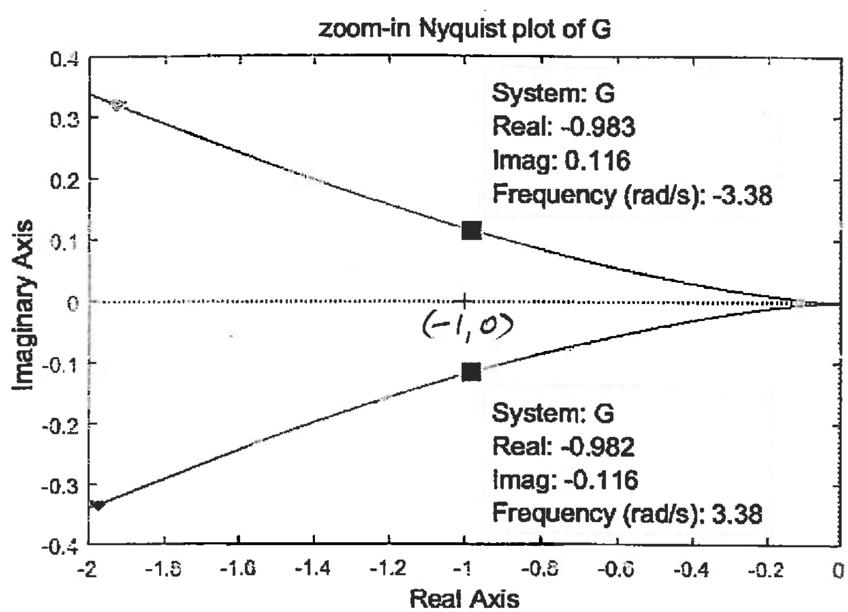
TABLE III. Gain and phase margin requirements (dB, degrees).

Air speed / Mode Frequency Hz	Below V_{oMIN}	V_{oMIN} to V_{oMAX}	At Limit Airspeed (V_L)	At 1.15 V_L
f_M <0.06	GM= 6 dB (No Phase Requirement Below V_{oMIN})	GM = ± 4.5 PM = ± 30	GM = ± 3.0 PM = ± 20	GM=0 PM=0 (Stable at Nominal Phase and Gain)
0.06≤ f_M <First Aero-elastic Mode		GM = ± 6.0 PM = ± 45	GM = ± 4.5 PM = ± 30	
f_M > First Aero-Elastic Mode		GM =± 8.0 PM = ± 60	GM =± 6.0 PM = ± 45	

4
M

Recall Nyquist plot





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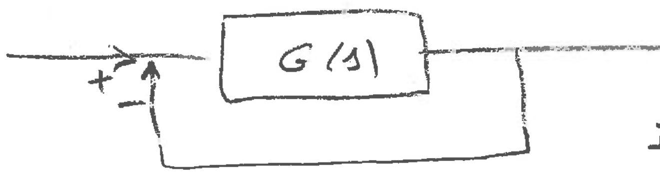
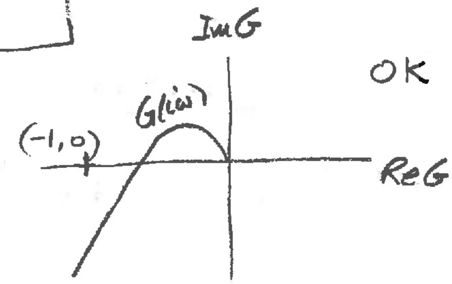
5 M

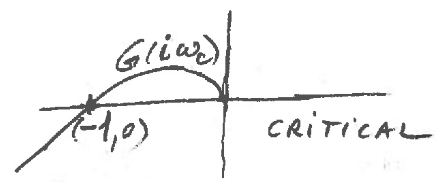
Margins analysis

- Gain margin
- Phase margin





$$G_{CL}(s) = \frac{G(s)}{1+G(s)}.$$



$G_{CL}(i\omega) \longrightarrow \infty$ if $1+G(i\omega)\to 0$

CRITICAL CONDITION ω_c

$$\boxed{G(i\omega_c) = -1} < \begin{cases} |G(i\omega_c)| = 0 \text{ dB} \\ \angle G(i\omega_c) = -180^\circ \end{cases}$$

$$-1 = e^{-i\pi}$$

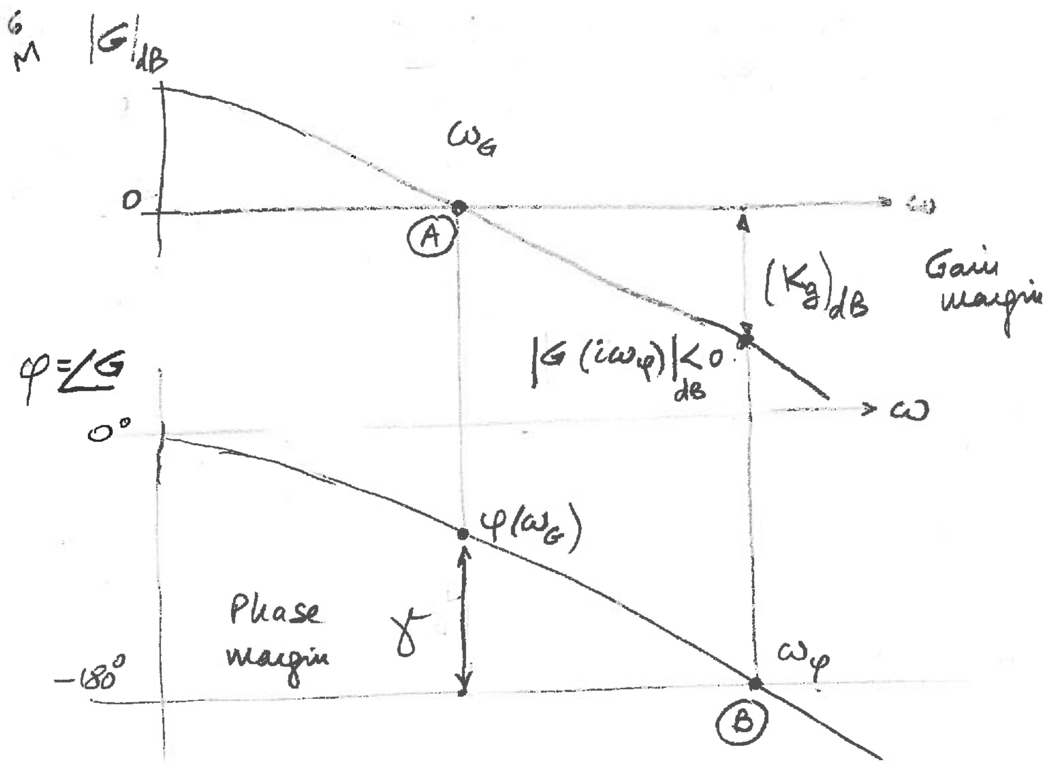
Objective: stay away from $(-1,0)$ point!

Method

(1) Plot Bode diagram

(2) Determine margins:

- Gain margin: distance from 0dB line
- Phase margin: distance from -180° line

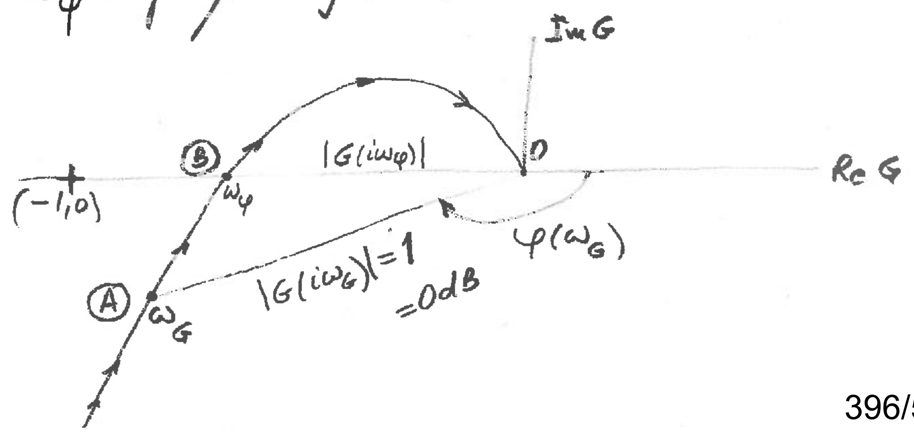


Gain margin $(K_g)_{dB} = 0 - |G(i\omega_\varphi)| = -|G(i\omega_\varphi)| > 0$

Phase margin $\gamma = \angle G(i\omega_G) - (-180^\circ) = 180^\circ + \angle G(i\omega_G) > 0$

$\omega_G =$ frequency when $|G(\omega_G)| = 0$ dB

$\omega_\varphi =$ frequency when $\varphi = -180^\circ$



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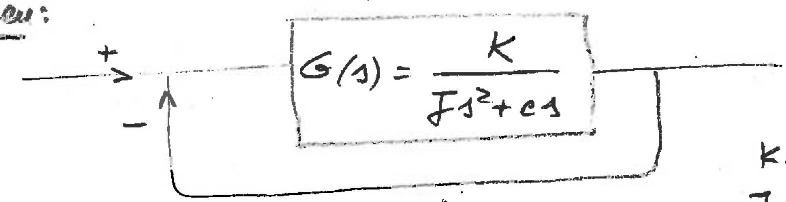
Example: aircraft roll model

Given:



$K = 114$

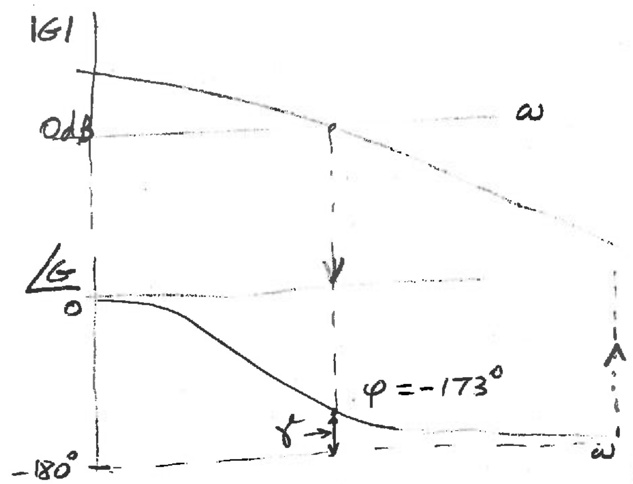
$J = 10$

$c = 4$

Find

- gain margin, $(K_g)_{dB}$
- phase margin γ deg

Solution: Bode plot (see MATLAB plot)(Fig. 1)



$|G|_{\angle G = -180°} = -\infty$

∠G Never crosses −180° line

$\angle G \xrightarrow[\omega\to\infty]{} -180°$

$\gamma = \varphi + 180° = -173° + 180° = 7° \ll PM$

Very small phase margin!

$K_g = \infty > GM$

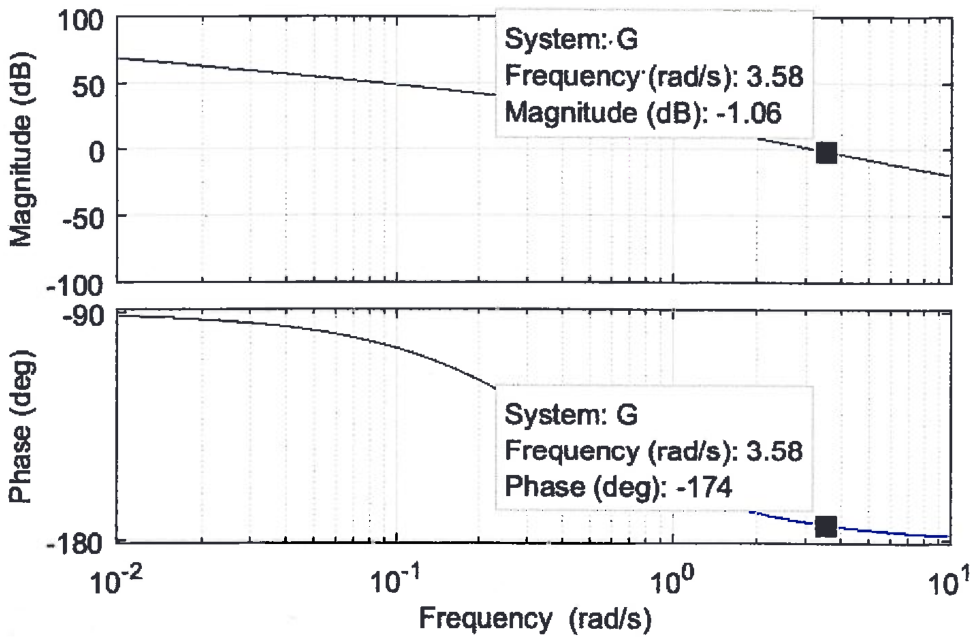
OK

Plenty gain margin

NEED TO IMPROVE PHASE MARGIN!

Aircraft roll model $G(s) = \dfrac{114}{10s^2+4s}$

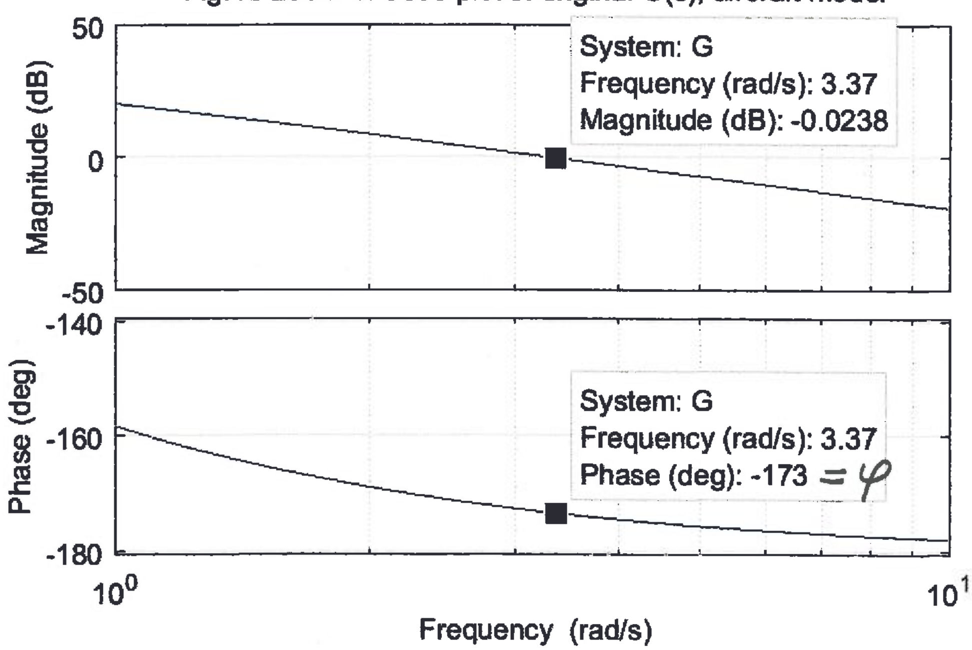
Fig.1a Bode plot of original G(s); aircraft model



Phase margin $\gamma^o = 180^o - 173^o = 7^o$ $\Big\{$ Gain margin $(K_g)_{dB} = \infty$

$= 180^o - \varphi$

Fig.1b zoom-in Bode plot of original G(s); aircraft model



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8
M

```
Phase margin: aircraft model
input data

    K   | J  |   c =

   114    10     4


G =

      114
  ------------
  10 s^2 + 4 s

Continuous-time transfer function.

=========================

GM, dB | PM, deg =

    10    60

=========================

phi = phase at |G|=0 dB point, deg =

  -173


gamma = phase margin, deg =

     7

=========================
```

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margin

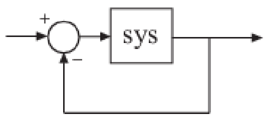
Gain margin, phase margin, and crossover frequencies

Syntax

```
[Gm,Pm,Wgm,Wpm] = margin(sys)
[Gm,Pm,Wgm,Wpm] = margin(mag,phase,w)
margin(sys)
```

Description

`margin` calculates the minimum gain margin, `Gm`, phase margin, `Pm`, and associated frequencies `Wgm` and `Wpm` of SISO open-loop models. The gain and phase margin of a system `sys` indicates the relative stability of the closed-loop system formed by applying unit negative feedback to `sys`, as in the following illustration.



The gain margin is the amount of gain increase or decrease required to make the loop gain unity at the frequency `Wgm` where the phase angle is –180° (modulo 360°). In other words, the gain margin is $1/g$ if g is the gain at the –180° phase frequency. Similarly, the phase margin is the difference between the phase of the response and –180° when the loop gain is 1.0. The frequency `Wpm` at which the magnitude is 1.0 is called the *unity-gain frequency* or *gain crossover frequency*. It is generally found that gain margins of three or more combined with phase margins between 30 and 60 degrees result in reasonable trade-offs between bandwidth and stability.

`[Gm,Pm,Wgm,Wpm] = margin(sys)` computes the gain margin `Gm`, the phase margin `Pm`, and the corresponding frequencies `Wgm` and `Wpm`, given the SISO open-loop dynamic system model `sys`. `Wgm` is the frequency where the gain margin is measured, which is a –180 degree phase crossing frequency. `Wpm` is the frequency where the phase margin is measured, which is a 0dB gain crossing frequency. These frequencies are expressed in radians/`TimeUnit`, where `TimeUnit` is the unit specified in the `TimeUnit` property of `sys`. When `sys` has several crossovers, `margin` returns the smallest gain and phase margins and corresponding frequencies.

The phase margin `Pm` is in degrees. The gain margin `Gm` is an absolute magnitude. You can compute the gain margin in dB by

```
Gm_dB = 20*log10(Gm)
```

`[Gm,Pm,Wgm,Wpm] = margin(mag,phase,w)` derives the gain and phase margins from Bode frequency response data (magnitude, phase, and frequency vector). `margin` interpolates between the frequency points to estimate the margin values. Provide the gain data `mag` in absolute units, and phase data `phase` in degrees. You can provide the frequency vector `w` in any units; `margin` returns `Wgm` and `Wpm` in the same units.

Note

When you use `margin(mag,phase,w)`, `margin` relies on interpolation to approximate the margins, which generally produces less accurate results. For example, if there is no 0 dB crossing within the `w` range, `margin` returns a phase margin of `Inf`. Therefore, if you have an analytical model `sys`, using `[Gm,Pm,Wgm,Wpm] = margin(sys)` is the most robust way to obtain the margins.

`margin(sys)`, without output arguments, plots the Bode response of `sys` on the screen and indicates the gain and phase margins on the plot. By default, gain margins are expressed in dB on the plot.

Examples

Gain and Phase Margins of Open-Loop Transfer Function

Create an open-loop discrete-time transfer function.

```
hd = tf([0.04798 0.0464],[1 -1.81 0.9048],0.1)
```

```
hd =

   0.04798 z + 0.0464
  ---------------------
  z^2 - 1.81 z + 0.9048

Sample time: 0.1 seconds
Discrete-time transfer function.
```

Compute the gain and phase margins.

```
[Gm,Pm,Wgm,Wpm] = margin(hd)
```

```
Gm =

    2.0517


Pm =

   13.5711


Wgm =

    5.4374


Wpm =

    4.3544
```

Display the gain and phase margins graphically.

```
margin(hd)
```

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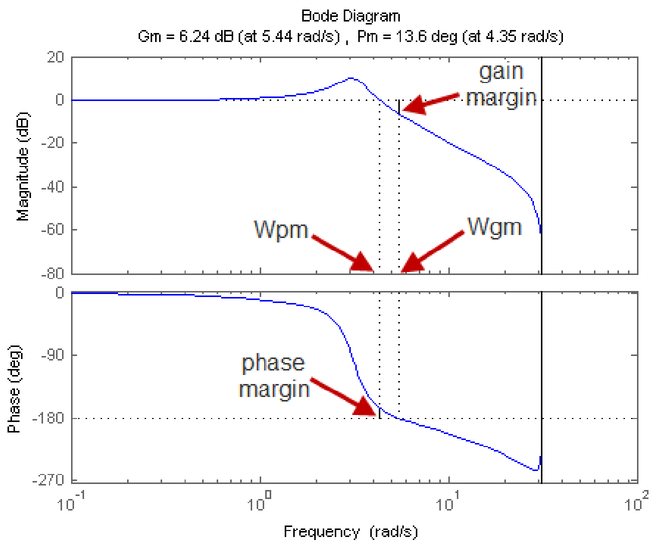

Solid vertical lines mark the gain margin and phase margin. The dashed vertical lines indicate the locations of `Wpm`, the frequency where the phase margin is measured, and `Wgm`, the frequency where the gain margin is measured.

Algorithms

The phase margin is computed using H_∞ theory, and the gain margin by solving $H(j\omega) = \overline{H(j\omega)}$ for the frequency ω.

See Also

Linear System Analyzer | bode

Introduced before R2006a

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margins_aircraft.m

```
%% initialization
clc %clear command window
clear %removes all variables from workspace; release memory
format compact
close all %closes all figures
s=tf('s');
%% original aircraft model
display('Phase margin: aircraft model')
K=114;              % gain
J=10;               % inertia
c=4;                % damping
display('input data')
display([K J c],'    K    | J  |    c')
G=K/(J*s^2+c*s)     % G(s)
figure(1)
subplot(2,1,1)
bode(G)
grid
title('Fig.1a Bode plot of aircraft model')
subplot(2,1,2)
d1=0;d2=1;N=1e3; w=logspace(d1,d2,N);
bode(G,w)
grid
title('Fig.1b zoom-in Bode plot of aircraft model')
% READ ON PLOT: phase at |G|=0 dB point, deg
% phi=-173;
phi=input('Input phase read on Bode plot in deg, phi=');
gamma=phi-(-180);   % gamma = phase margin, deg
display([gamma],'gamma = phase margin, deg')
figure(2)
margin(G)
```

```
Phase margin: aircraft model
input data
    K    | J  |    c =
   114    10     4


G =

 
      114
  ------------
  10 s^2 + 4 s
 
Continuous-time transfer function.

Input phase read on Bode plot in deg, phi=-173
gamma = phase margin, deg =
     7
```

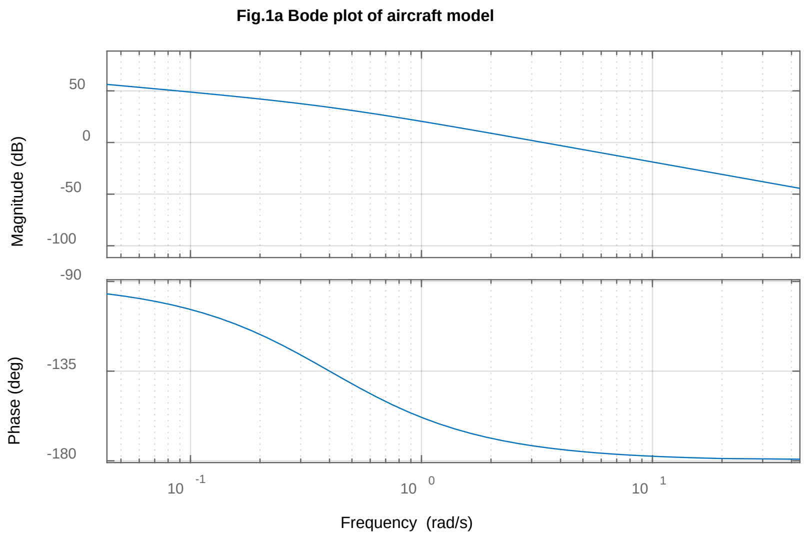


Fig.1a Bode plot of aircraft model
Magnitude (dB)
50
0
-50
-100
Phase (deg)
-90
-135
-180
10 -1
10 0
10 1
Frequency (rad/s)

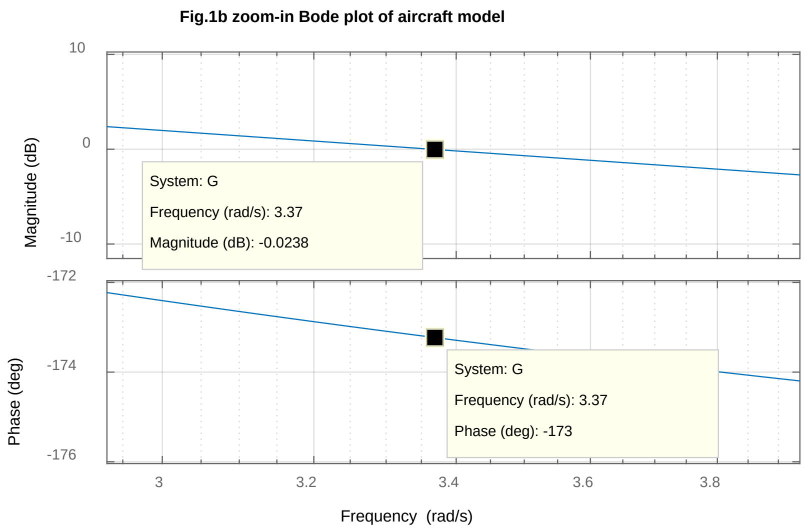
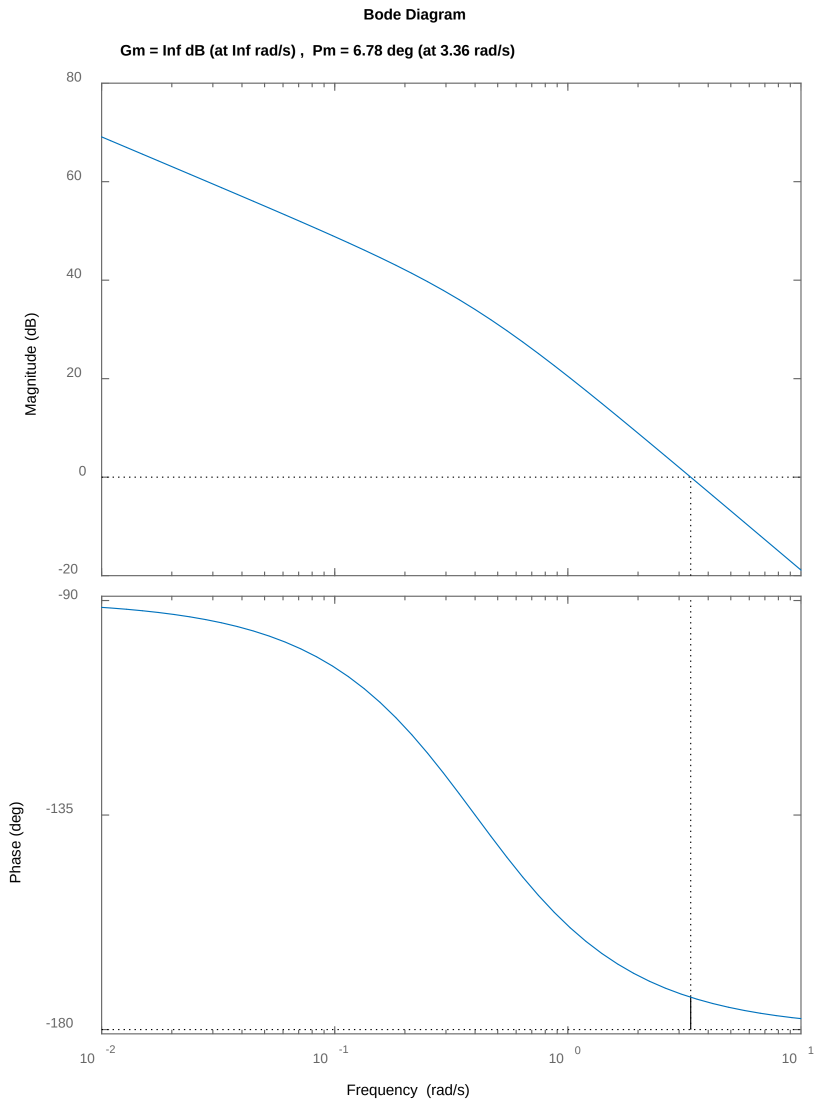


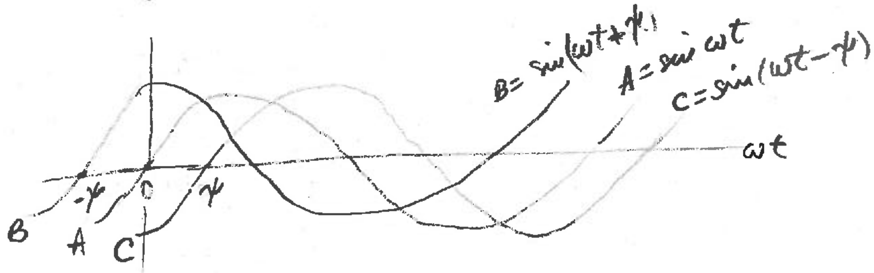
Fig.1b zoom-in Bode plot of aircraft model
Magnitude (dB)
10
0
-10
System: G
Frequency (rad/s): 3.37
Magnitude (dB): -0.0238
Phase (deg)
-172
-174
-176
System: G
Frequency (rad/s): 3.37
Phase (deg): -173
3
3.2
3.4
3.6
3.8
Frequency (rad/s)


Bode Diagram
Gm = Inf dB (at Inf rad/s) , Pm = 6.78 deg (at 3.36 rad/s)
Magnitude (dB)
80
60
40
20
0
-20
Phase (deg)
-90
-135
-180
10 -2
10 -1
10 0
10 1
Frequency (rad/s)

8.7 Phase Compensators

C1 PHASE COMPENSATORS

Recall signal phase definition:



A : $\sin(\omega t) = 0$ @ $t = 0$

B : $\sin(\omega t + \psi) = 0$ @ $\omega t = -\psi$ "ahead of time" it leads

C : $\sin(\omega t - \psi) = 0$ @ $\omega t = \psi$ "delayed" it lags

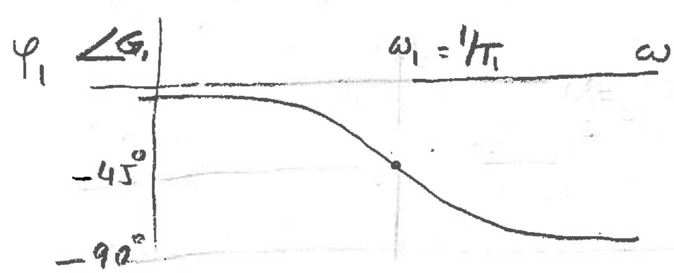
Phase compensator

$G_c(i\omega) = |G_c(i\omega)| e^{i\varphi_c}$, $\varphi_c = \angle G_c(i\omega)$

if $\varphi_c > 0$, then "lead compensator"

$\varphi_c < 0$ "lag compensator"

Example 1: 1st order system $G_1(s) = \frac{1}{sT_1 + 1}$



$\varphi_1 < 0$ "Lag"

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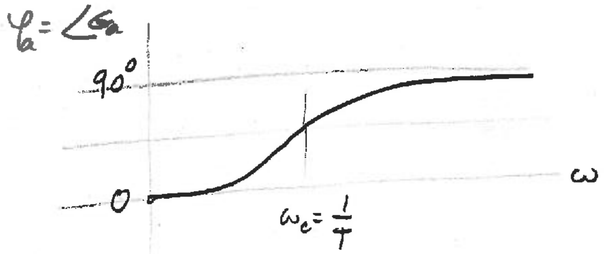
C2

Example 2 : $G_a = sT_a + 1$

$G_a(i\omega) = i\omega T_a + 1 \; ; \quad \varphi_a = \angle G_a = \tan^{-1} \omega T_a > 0$



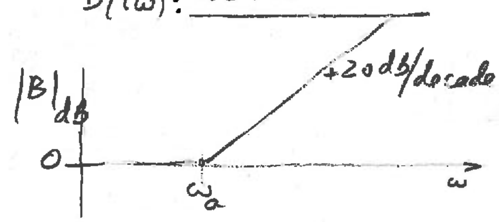
$\varphi_a > 0$
"Lead"

Example 3 $G_c = \frac{sT_a + 1}{sT_1 + 1} = \frac{B(s)}{A(s)}$

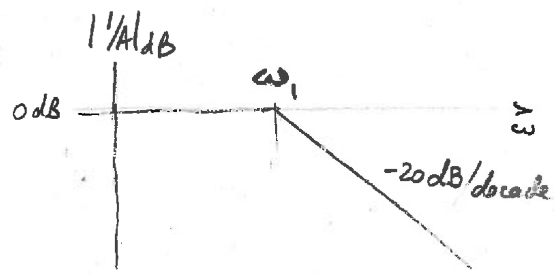
$G_c(i\omega) = \frac{i\omega T_a + 1}{i\omega T_1 + 1} = \frac{B(i\omega)}{A(i\omega)}$.

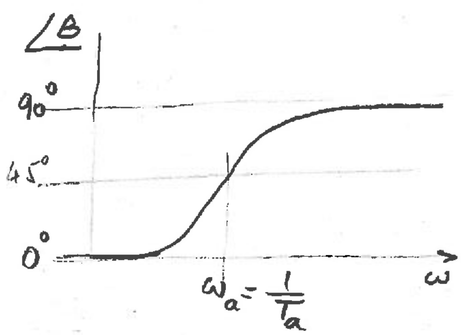
$|G_c|_{dB} = |B|_{dB} - |A|_{dB} \quad ; \quad \angle G_c = \angle B - \angle A$

$B(i\omega)$: NUMERATOR

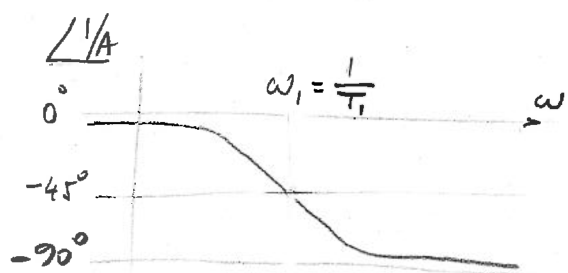


$A(i\omega)$: DENOMINATOR





"Lead"



"Lag"

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C2a

LEAD COMPENSATOR

FOR PHASE MARGIN IMPROVEMENT

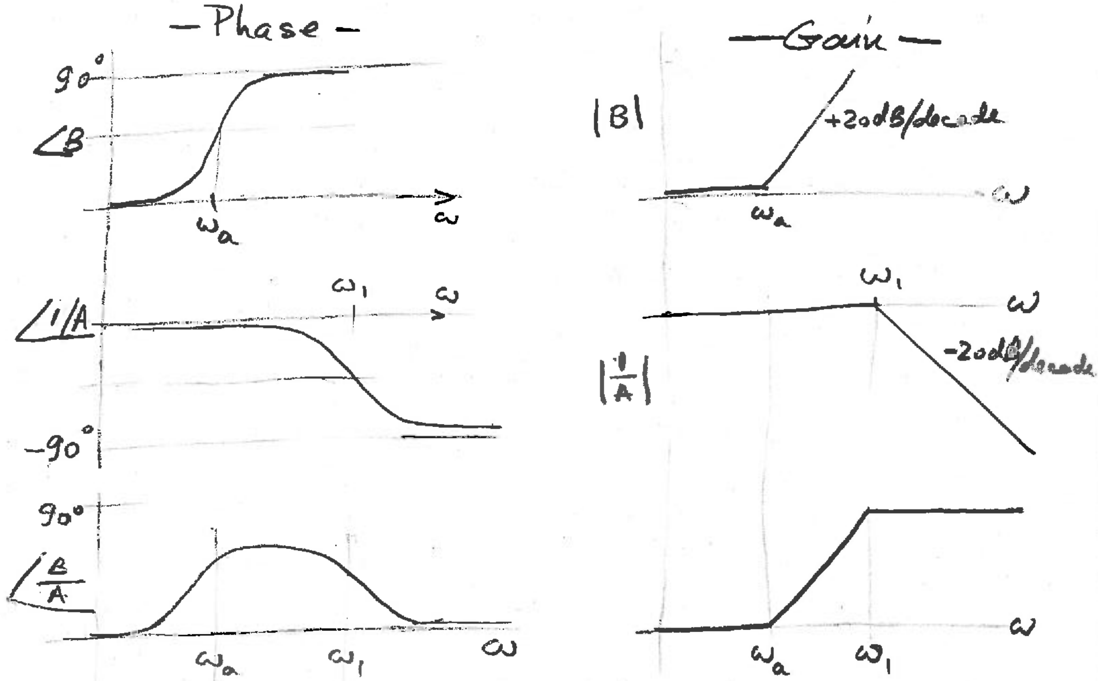
409/523

C3

LEAD COMPENSATOR

$$G_c = \frac{sT_a+1}{sT_1+1}$$

$\omega_a < \omega_1$, $T_1 < T_a$ $\qquad \omega_a = \frac{1}{T_a}$, $\omega_1 = \frac{1}{T_1}$

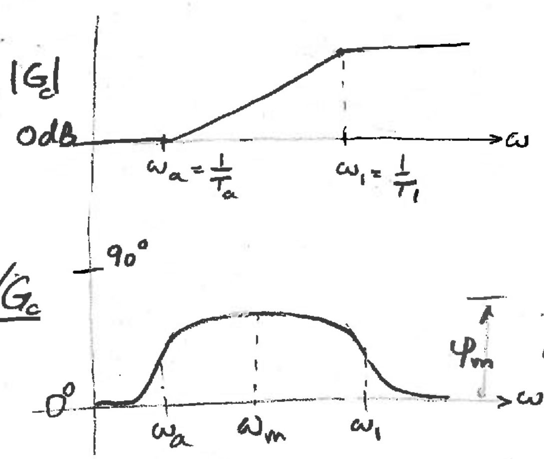


"phase lead" between ω_a & ω_1

High pass filter
(higher frequencies are amplified)

$G_c(s) = \frac{sT_a+1}{sT_1+1}$

$T_1 < T_a$



HP filter

maximum phase lead

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C3a

Lead Compensator Design

Use lead compensator to move phase plot upward and improve phase margin

$$G_c = \frac{sT_a+1}{sT_1+1} = \frac{sT+1}{s\alpha T+1} \quad \text{lead compensator} \quad (1)$$

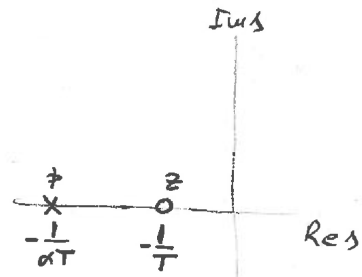
$\omega_a < \omega_1 \qquad 0 < \alpha < 1$

$T_a > T_1$

Poles & zeros :

$sT+1=0 \longrightarrow z = -\frac{1}{T}$

$s\alpha T+1=0 \longrightarrow p = -\frac{1}{\alpha T}$

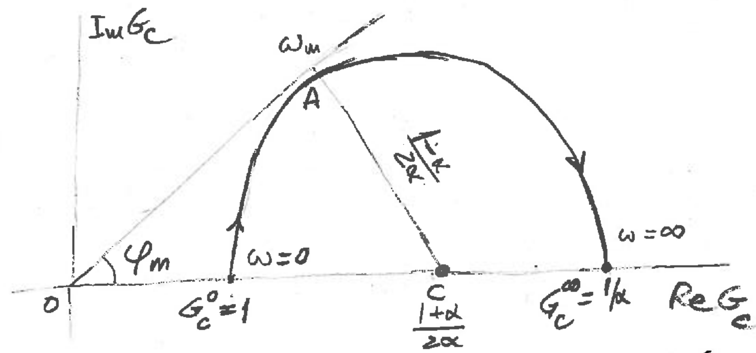


Nyquist plot

$G_c(i\omega) = \frac{i\omega T+1}{i\omega\alpha T+1}$

$G_c^0(i0) = 1$

$G_c^\infty(i\infty) = \frac{1}{\alpha}$



$$\sin\varphi_m = \frac{1-\alpha}{1+\alpha} \qquad (2)$$

maximum phase lead $\boxed{\varphi_m = \sin^{-1}\frac{1-\alpha}{1+\alpha}}$

Proof : $OC = \frac{1}{2}\left(1+\frac{1}{\alpha}\right) = \frac{1+\alpha}{2\alpha}$

Radius $= \frac{1+\alpha}{2\alpha} - 1 = \frac{1-\alpha}{2\alpha} = AC$

$\sin\varphi_m = \frac{AC}{OC} = \frac{1-\alpha}{1+\alpha}$ QED .

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C36 Calculation of α

Eq (2) can be used to determine the value of α in Eq. (1), i.e.,

$$\sin\varphi_m = \frac{1-\alpha}{1+\alpha}$$

$$\frac{1+\sin\varphi_m}{1-\sin\varphi_m} = \frac{1-\alpha+1+\alpha}{1+\alpha-(1-\alpha)} = \frac{2}{2\alpha} = \frac{1}{\alpha}$$

$$\alpha = \frac{1-\sin\varphi_m}{1+\sin\varphi_m} \qquad (3)$$

Calculation of T

- ω_m is the geometric mean of the corner frequencies

$$\omega_m = \sqrt{\omega_a \omega_1} = \sqrt{\frac{1}{T_a}\cdot\frac{1}{T_1}}\Bigg|_{\substack{T_a = T \\ T_1 = \alpha T}} = \sqrt{\frac{1}{T}\cdot\frac{1}{\alpha T}} = \frac{1}{T}\frac{1}{\sqrt{\alpha}}$$

$$T = \frac{1}{\omega_m\sqrt{\alpha}}, \quad \omega_m T = \frac{1}{\sqrt{\alpha}} \qquad (4)$$

- $$|G_c(\omega_m)| = \left|\frac{i\omega_m T+1}{i\omega_m\alpha T+1}\right| = \left|\frac{i\frac{1}{\sqrt{\alpha}}+1}{i\alpha\frac{1}{\sqrt{\alpha}}+1}\right| = \left|\frac{i+\sqrt{\alpha}}{i\sqrt{\alpha}+1}\right|\frac{1}{\sqrt{\alpha}}$$

$$= \frac{\sqrt{1+\alpha}}{\sqrt{1+\alpha}}\frac{1}{\sqrt{\alpha}} = \frac{1}{\sqrt{\alpha}}$$

$$|G_c(\omega_m)| = \frac{1}{\sqrt{\alpha}}, \quad 0<\alpha<1 \qquad (5)$$

The phase compensator will produce a gain $\frac{1}{\sqrt{\alpha}}$.

To find ω_m, examine the Bode diagram of the original system G, and identify the freq. at which $|G(\omega_m)| = \sqrt{\alpha}$, such that

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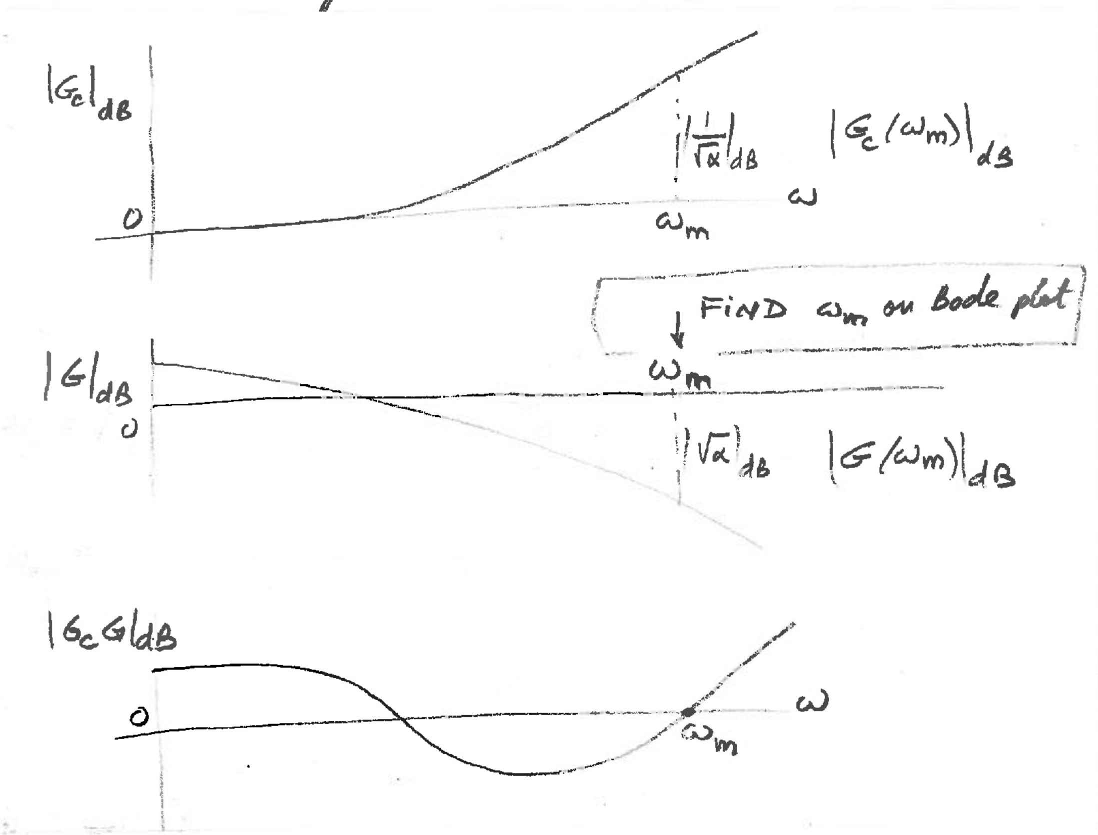
C3c

ω_m is selected at a value at which the gain drop $|G(\omega_m)|$ of the original system balances the gain addition $|G_c(\omega_m)|$ due to the compensator (balance condition)



- After determining ω_m graphically, calculate T with Eq.(4) i.e.,

$$T = \frac{1}{\omega_m} \cdot \frac{1}{\sqrt{\alpha}} \tag{6}$$

Thus:

$$G_c = \frac{sT+1}{s\alpha T+1} \tag{7}$$

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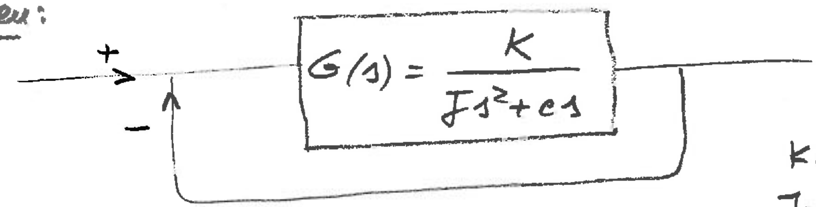
E

Example : aircraft roll model with lead comp.

Given:



$K = 114$
$J = 10$
$c = 4$

Find

(a) - gain margin, $(K_g)_{dB}$

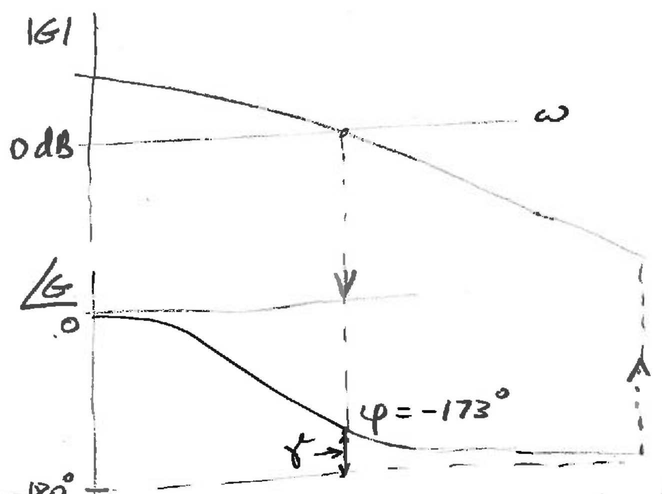
- phase margin γ deg

(b) design compensator to achieve

$GM = 10$ dB
$PM = 60$ deg (1)

(c) plot time response

Solution (a) Bode plot (see MATLAB plot, Fig. 1)



$|G|_{\angle G = -180^\circ} = -\infty$

∠G never crosses −180° line

$\angle G \xrightarrow[\omega \to \infty]{} -180^\circ$

$\gamma = \varphi + 180^\circ = -173^\circ + 180^\circ = 7^\circ$

Very small phase margin!

$K_g = \infty > GM$

OK

NEED TO IMPROVE PHASE MARGIN!

Plenty gain margin

3E

```
Phase margin: aircraft model
input data

    K   | J  |   c =

   114    10     4


G =

      114
  ------------
  10 s^2 + 4 s

Continuous-time transfer function.

=========================

GM, dB | PM, deg =

    10    60

=========================

phi = phase at |G|=0 dB point, deg =

  -173


gamma = phase margin, deg =

     7

=========================
```

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Fig.1a Bode plot of original G(s); aircraft model

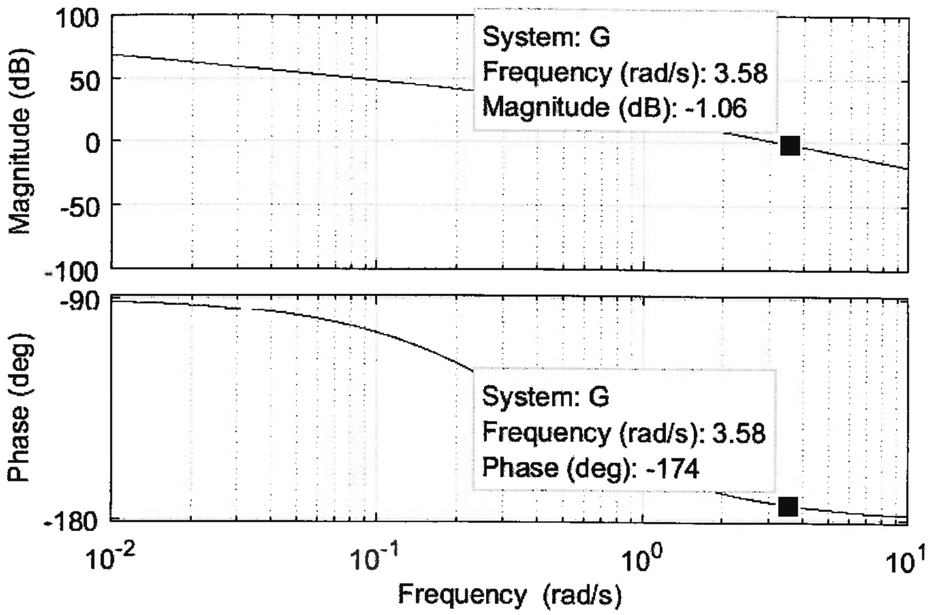
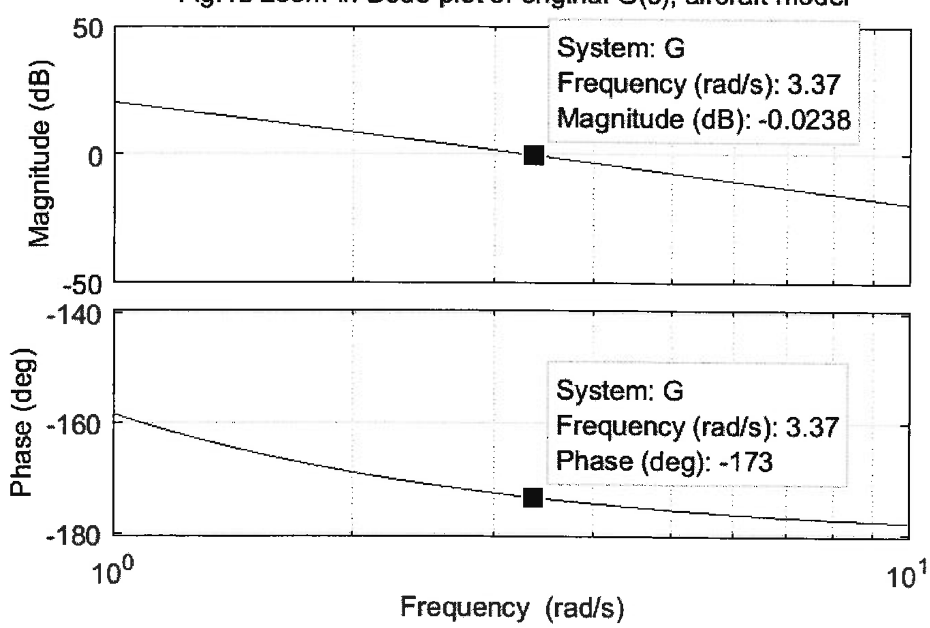


Fig.1b zoom-in Bode plot of original G(s); aircraft model



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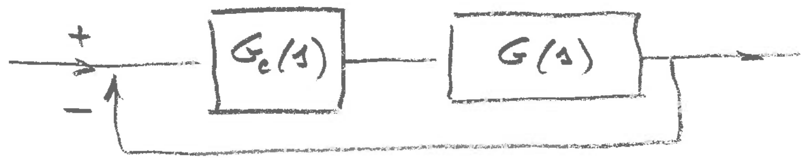
4E (b) Design phase compensator to improve phase margin



$$G_c(s) = \frac{Ts+1}{\alpha T s+1} \quad \text{lead compensator} \qquad (2)$$

(b1)
We need to improve phase margin from $\gamma = 7^\circ$ to $PM = 60^\circ$, i.e., the phase compensator must add $\varphi_m = 53^\circ$.

- To calculate α, recall

$$\alpha = \left.\frac{1-\sin\varphi_m}{1+\sin\varphi_m}\right|_{\varphi_m = 53^\circ} = 0.1120 \qquad (3)$$

- To calculate T, recall $T = \dfrac{1}{\omega_m\sqrt{\alpha}}$

We need ω_n. We find ω_m from the balance condition, i.e.,

$$|G_c(\omega_m)| = \frac{1}{\sqrt{\alpha}} = 2.9887 = 9.5096\,dB \qquad (4)$$

$$|G(\omega_m)| = -|G_c(\omega_m)| = -9.5096\ dB \qquad (5)$$

From Bode plot Fig. 2, we find $\omega_n = 5.83$ rad/sec.

Hence $T = 0.5126$ sec. (6)

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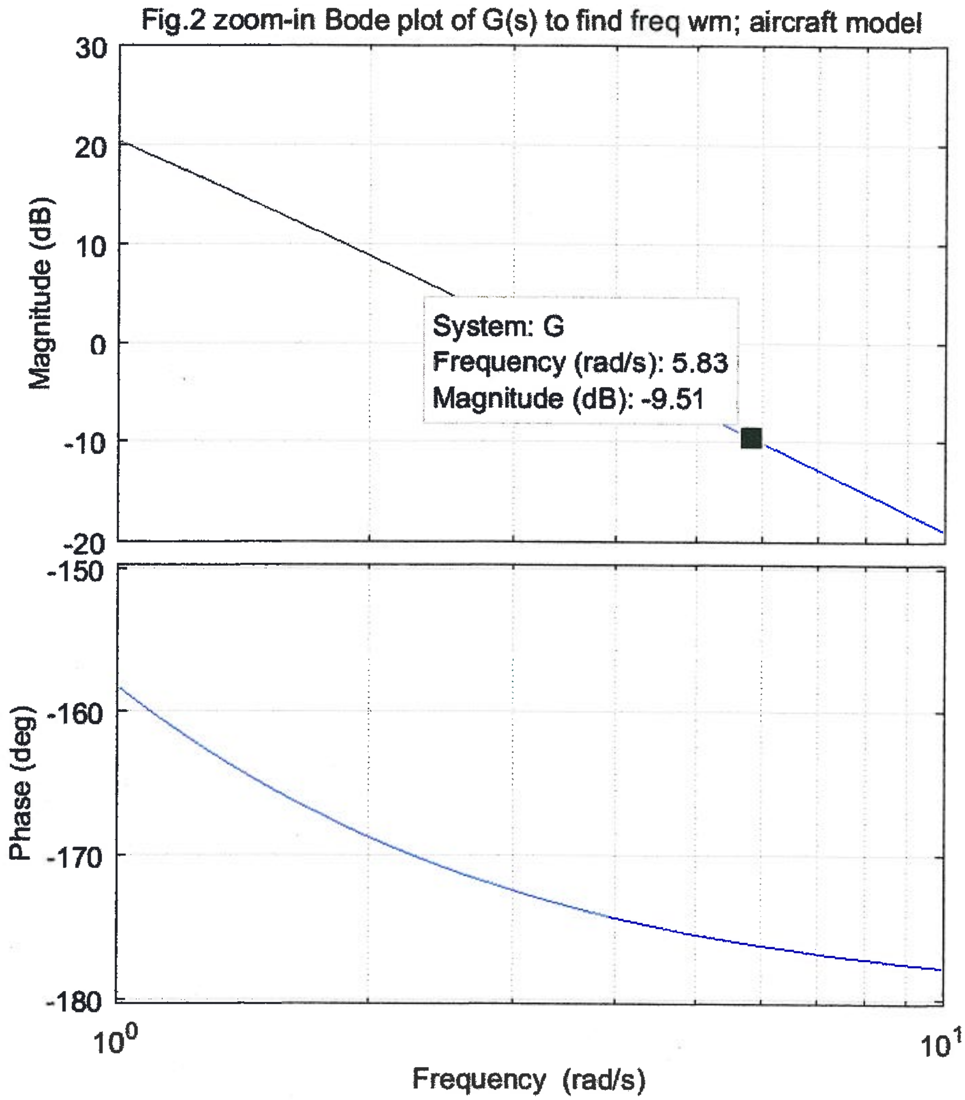
5 Ⓔ



$\omega_m = 5.83$ rad/s

$|G(\omega_m)|_{dB} = -9.51$ dB.

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6
E (b2): Test compensated system G_cG

Fig. 3 shows the performance of the compensated system G_cG in comparison to the original system G. The compensated system G_cG has

$\varphi_c = -123^\circ$ @ $\omega_c = 5.83$ rad/s where $|G_cG| = 0\,dB$

The phase margin of the compensated system is

$$\gamma_c = \varphi_c + 180^\circ = -123^\circ + 180^\circ = 57^\circ < \underset{PM}{60^\circ}$$

The system has improved, but the phase margin is still less than PM.

(b3). Need to adjust the compensator to add a little more phase shift,

$$\Delta\varphi = PM - \gamma_c = 60^\circ - 57^\circ = 3^\circ \qquad (7)$$

The new φ_m is

$$\varphi_{m_1} = \varphi_m + \Delta\varphi = 53^\circ + 3^\circ = 56^\circ \qquad (8)$$

The new α is

$$\alpha_1 = \frac{1 - \sin\varphi_{m_1}}{1 + \sin\varphi_{m_1}} = 0.0935 \qquad (9)$$

We keep same time constant, $T_1 = T$ (10)

7
ε

(b1)

```
=========================
Phase compensator design: aircraft model
phi_m =
    53
alpha =
    0.1120
1/sqrt(alpha) =
    2.9887
Gc_wm, dB =
    9.5096
wm =
    5.8300
T =
    0.5126

Gc =

  0.5126 s + 1
  ------------
  0.05739 s + 1

Continuous-time transfer function.

phi_c = phase at |GcG|=0 dB point, deg =
  -123
gamma_c = phase margin of GcG, deg =
    57
=========================
```

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8E

(62)

Fig.3a Bode plot of G & GcG(s); aircraft model

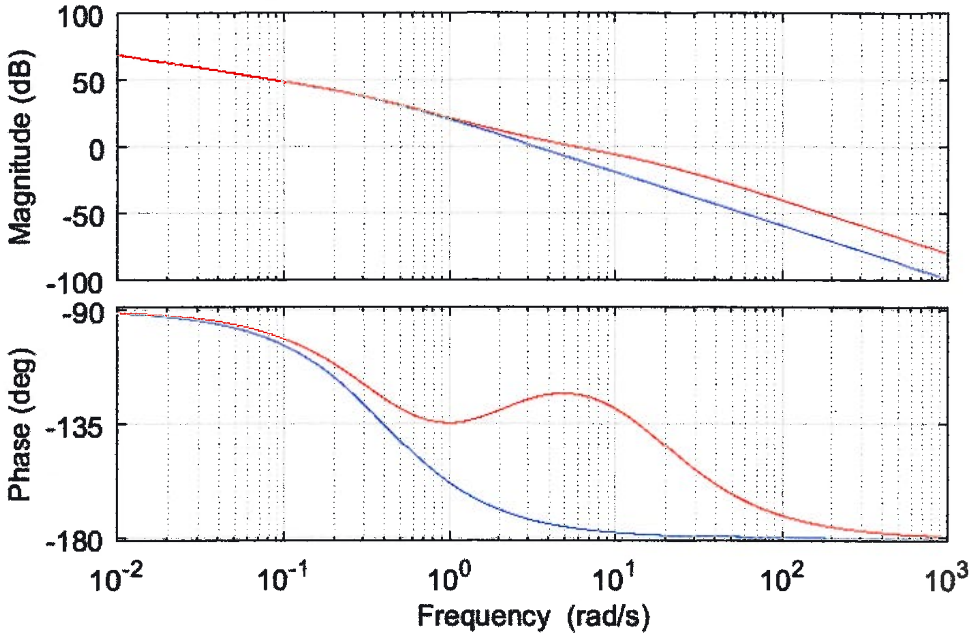
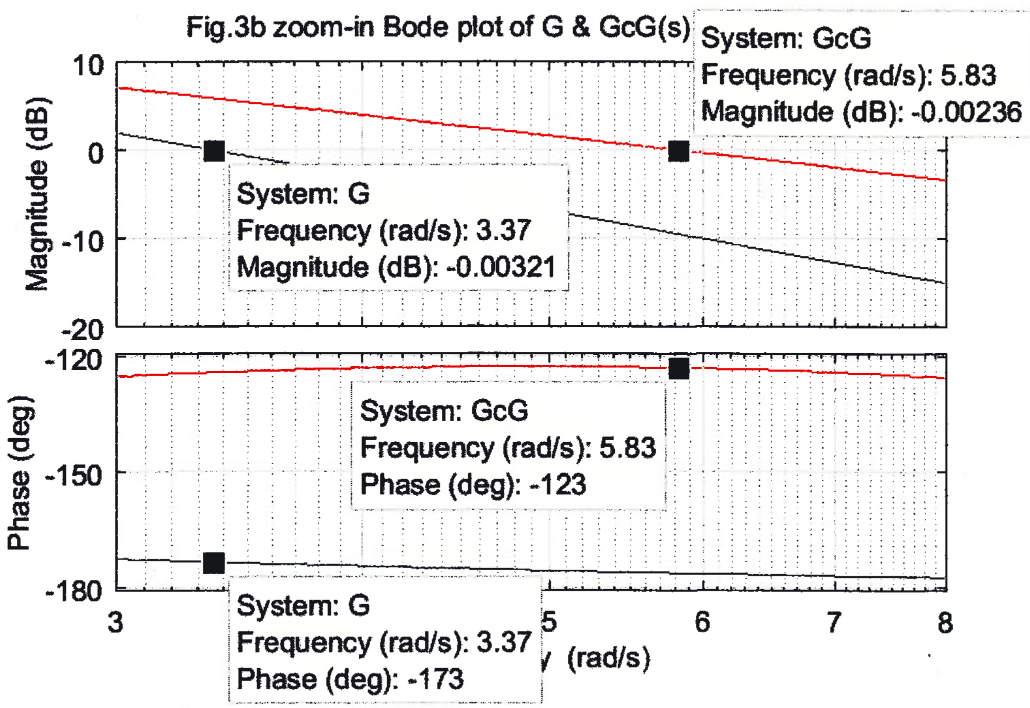


Fig.3b zoom-in Bode plot of G & GcG(s)



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9
E

(63)

```
========================
Adjustment of phase compensator: aircraft model
dphi =
     3
phi_m1 =
    56
alpha1 =
    0.0935
T1 =
    0.5126

Gc1 =

  0.5126 s + 1
  -------------
  0.04792 s + 1

Continuous-time transfer function.
```

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E[10] (b4) Test adjusted compensator

Fig. 4 shows the performance of the system with adjusted compensator $G_{c1}G$ compared to the original system G.

The new phase value at $|G_{c1}G| = 0dB$ is:

$\varphi_{c_1} = -120^\circ$ @ 5.91rad/s where $|G_{c_1}G| = 0\,dB$.

The new phase margin is

$$\gamma_{c_1} = \varphi_{c_1} + 180^\circ = -120^\circ + 180^\circ = 60^\circ = PM \quad (11).$$

The adjusted system meets the phase margin specification $PM = 60^\circ$.

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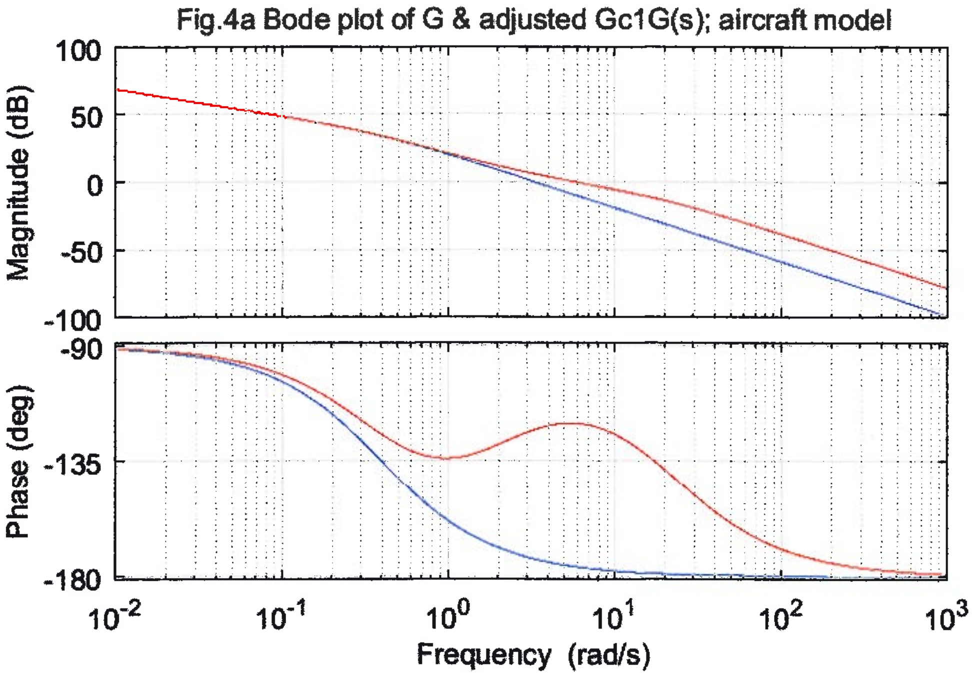
Fig.4a Bode plot of G & adjusted Gc1G(s); aircraft model
Magnitude (dB)
100
50
0
-50
-100
Phase (deg)
-90
-135
-180
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/s)

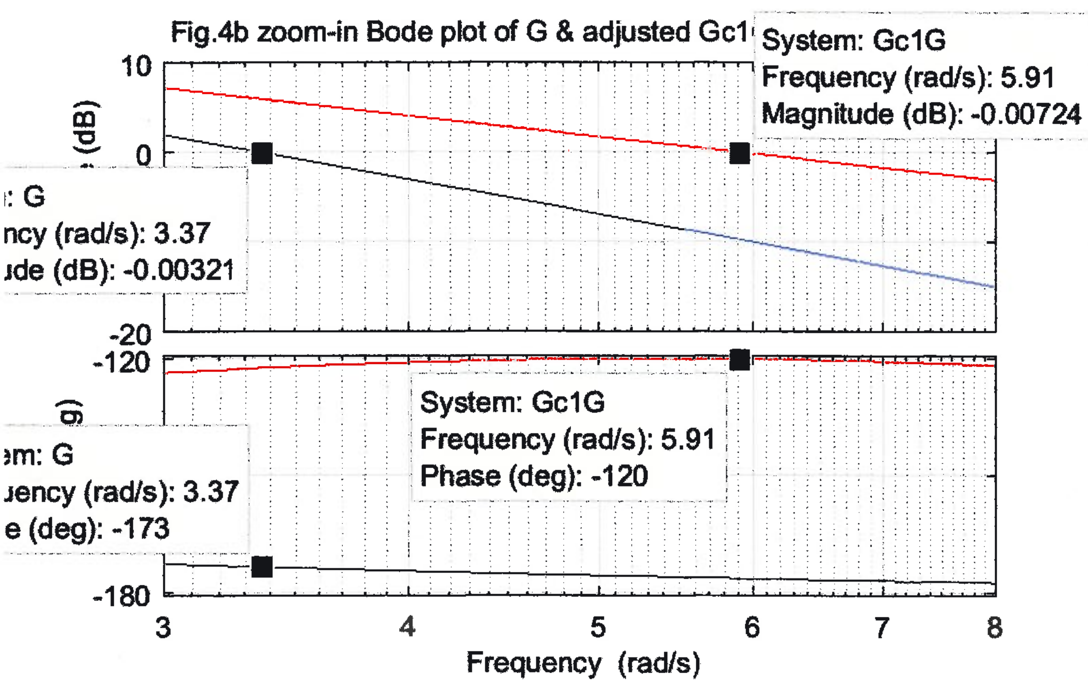
Fig.4b zoom-in Bode plot of G & adjusted Gc1
System: Gc1G
Frequency (rad/s): 5.91
Magnitude (dB): -0.00724
10
0
-20
: G
ncy (rad/s): 3.37
ude (dB): -0.00321
-120
-180
System: Gc1G
Frequency (rad/s): 5.91
Phase (deg): -120
em: G
uency (rad/s): 3.37
e (deg): -173
3
4
5
6
7
8
Frequency (rad/s)

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12
E

(c) Time response behavior of the compensated system.

Fig 5 displays the time response of the compensated system $(G_{c1}G)_{CL}$ compared with the response of the original system G_{CL}. It can be observed that the compensated system has a fast rise time and a small overshoot

$t_r = 0.296$ sec

$M_p = 17\%$ (12)

The original system had a longer rise time, a much higher overshoot, and took a long time to settle.

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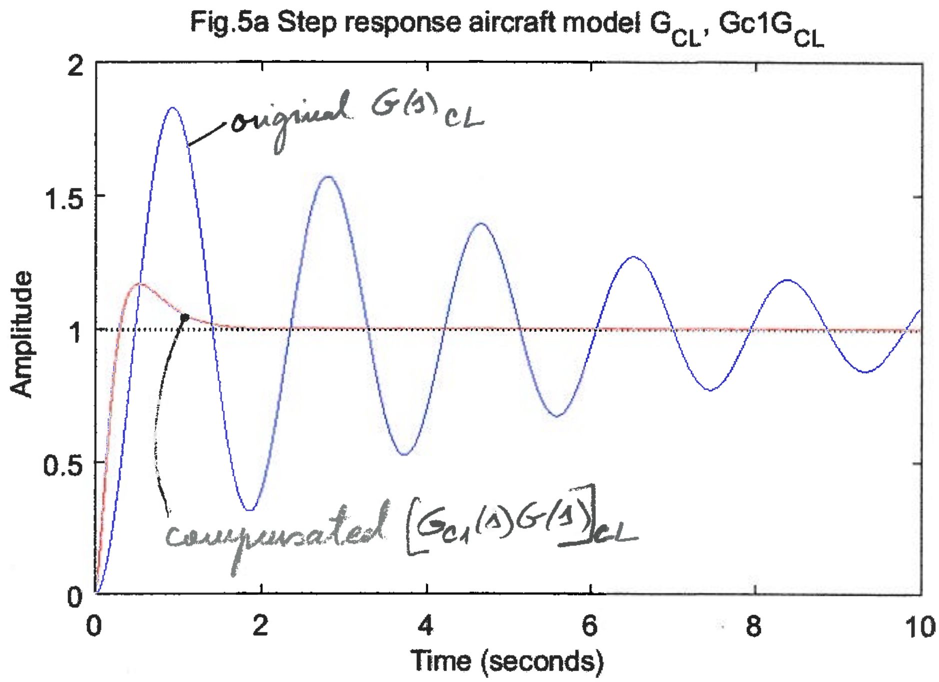
Fig.5a Step response aircraft model G_{CL}, $Gc1G_{CL}$
original $G(s)_{CL}$
compensated $[G_{c1}(s)G(s)]_{CL}$
Amplitude
Time (seconds)

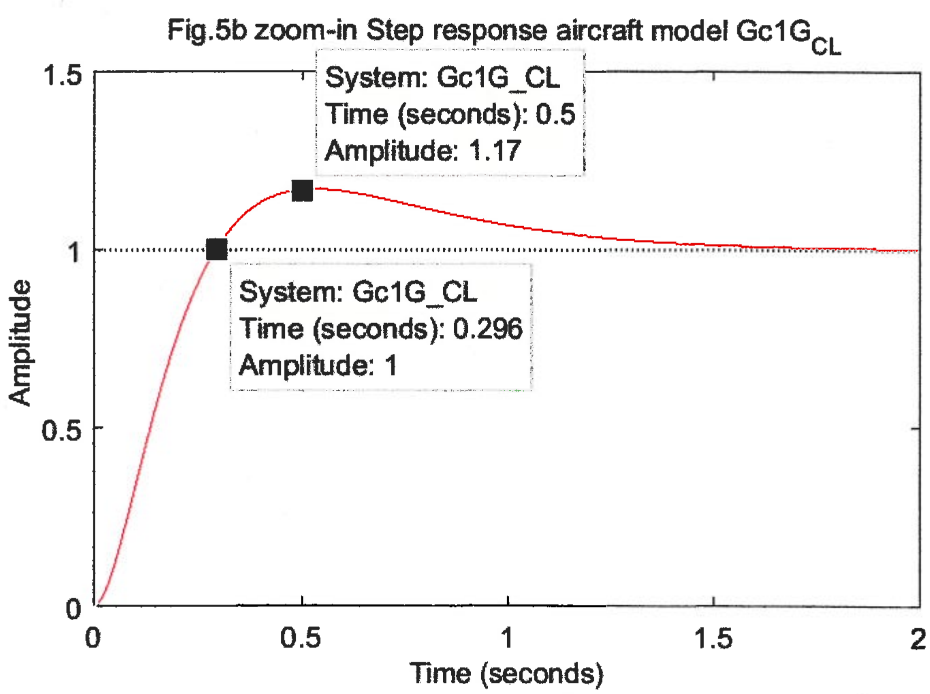
Fig.5b zoom-in Step response aircraft model $Gc1G_{CL}$
System: Gc1G_CL
Time (seconds): 0.5
Amplitude: 1.17
System: Gc1G_CL
Time (seconds): 0.296
Amplitude: 1
Amplitude
Time (seconds)

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Lead_compensator_aircraft_20180510.m

```
% Phase compensator design: aircraft model
%% initialization
clc %clear command window
clear %removes all variables from workspace; release memory
format compact
close all %closes all figures
s=tf('s');
%% original aircraft model
display('Phase margin: aircraft model')
K=114;              % gain
J=10;               % inertia
c=4;                % damping
display('input data')
display([K J c],'    K    | J  |    c')
G=K/(J*s^2+c*s)     % G(s)
display('=========================')
%% specs: MIL-DTL-9490E margin requirements
GM=10;    % gain margin spec, dB
PM=60;    % phase margin spec, deg
display([GM PM],'GM, dB | PM, deg')
figure(1)
subplot(2,1,1)
bode(G)
grid
title('Bode plot of original G(s); aircraft model')
subplot(2,1,2)
d1=0;d2=1;N=1e3; w=logspace(d1,d2,N);
bode(G,w)
grid
title('zoom-in Bode plot of original G(s); aircraft model')
display('=========================')
% READ ON PLOT: phase in deg for |G|=0 dB
% phi=-173;
phi=input('Input phase in deg when |G|=0 dB, phi=');
gamma=phi-(-180);    % gamma = phase margin, deg
% display([phi],'phi = phase at |G|=0 dB point, deg')
display([gamma],'gamma = phase margin, deg')
display('=========================')
%% add compensator Gc(s)
display('Phase compensator design: aircraft model')
phi_m=PM-gamma % maximum compensator phase that need to be obtained
alpha=(1-sind(phi_m))/(1+sind(phi_m)) % compensator attenuation factor alpha
display(1/sqrt(alpha),'1/sqrt(alpha)') % expected gain rise from compensator
Gc_wm=mag2db(1/sqrt(alpha)); % expected gain rise from compensator, dB
G_wm=-Gc_wm;
display(G_wm,'G_wm, dB')
figure(2) % Bode plot to find the freq wm
d1=log10(3);d2=log10(12);N=1e3; w=logspace(d1,d2,N);
bode(G,w)
grid
title('zoom-in Bode plot of |G| to find freq wm; aircraft model')
% READ ON PLOT: frequecy for which |G|=G_wm
% wm=5.83
wm=input('Input frequency in rad/s when |G|=G_wm, wm=');
T=1/wm/sqrt(alpha) % time constant of the compensator
```

```
%% Test compensated system
figure(3)
subplot(2,1,1)
Gc=(s*T+1)/(s*alpha*T+1)
GcG=Gc*G;
bode(G,GcG)
grid
title('Bode plot of G & GcG(s); aircraft model')
subplot(2,1,2)
d1=log10(3);d2=log10(8);N=1e3; w=logspace(d1,d2,N);
bode(G,GcG,w)
grid
title('zoom-in Bode plot of G & GcG(s); aircraft model')
% READ ON PLOT: phase in deg for |GcG|=0 dB
% phi_c=-123;
phi_c=input('Input phase in deg when |GcG|=0 dB, phi_c=');
gamma_c=phi_c-(-180);    % gamma_c = phase margin of GcG, deg
display([phi_c],'phi_c = phase at |GcG|=0 dB point, deg')
display([gamma_c],'gamma_c = phase margin of GcG, deg')
display('=========================')
%% Adjust compensator to reach PM requirements
display('Adjustment of phase compensator: aircraft model')
dphi=PM-gamma_c  % additional phase shift needed
phi_m1=phi_m+dphi   % adjusted phase shift
alpha1=(1-sind(phi_m1))/(1+sind(phi_m1)) % adjusted alpha
Gc1_wm=mag2db(1/sqrt(alpha1)); % expected gain rise from compensator, dB
G_wm1=-Gc1_wm;
display(G_wm1,'G_wm1, dB')
figure(4) % Bode plot to find the freq wm for |G_wm|dB=-|Gc_wm|dB
d1=0;d2=1;N=1e3; w=logspace(d1,d2,N);
bode(G,w)
grid
title('zoom-in Bode plot of G(s) to find new freq wm1; aircraft model')
% READ ON PLOT: frequecy for which |G|=-Gc_wm1, dB
% wm1=6.1;
wm1=input('Input frequency rad/s when |G|=G_wm1, wm1=');
T1=1/wm1/sqrt(alpha1) % updated time constant of the compensator
%% Final compensator design
Gc1=(s*T1+1)/(s*alpha1*T1+1)
Gc1G=Gc1*G;
figure(5)
subplot(2,1,1)
bode(G,Gc1G)
grid
title('Bode plot of G & adjusted Gc1G(s); aircraft model')
subplot(2,1,2)
d1=log10(3);d2=log10(8);N=1e3; w=logspace(d1,d2,N);
bode(G,Gc1G,w)
grid
title('zoom-in Bode plot of G & adjusted Gc1G(s); aircraft model')
phi_c1=input('Input phase in deg for |Gc1G|=0 dB, phi_c1=');
gamma_c1=phi_c1-(-180);    % gamma_c = phase margin of GcG, deg
display([phi_c1],'phi_c1 = phase at |Gc1G|=0 dB point, deg')
display([gamma_c1],'gamma_c = phase margin of GcG, deg')
%% Plot step response response of the original and compensated systems
figure(6)
Tfinal=10; dt=0.01; t=0:dt:Tfinal;
G_CL=feedback(G,1);
Gc1G_CL=feedback(Gc1G,1);
subplot(2,1,1)
step(G_CL,Gc1G_CL,t)
title('step response aircraft model G_C_L, Gc1G_C_L')
subplot(2,1,2)
Tzoom=2; Nt=1e3; dt=Tzoom/Nt; tz=0:dt:Tzoom;
step(Gc1G_CL,tz,'r')
title('zoom-in step response aircraft model Gc1G_C_L')
ylim([0 1.5])
```

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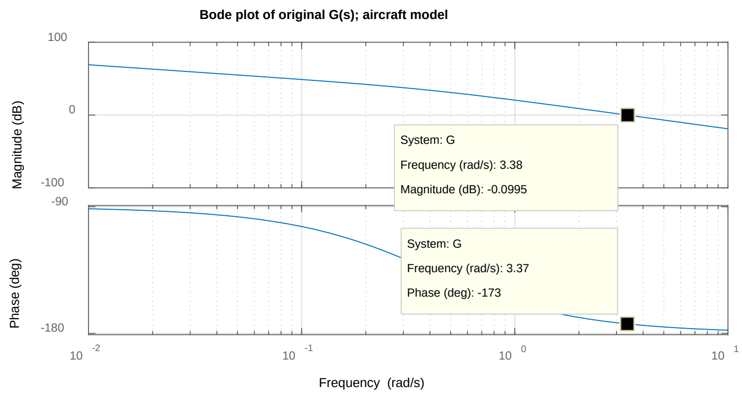
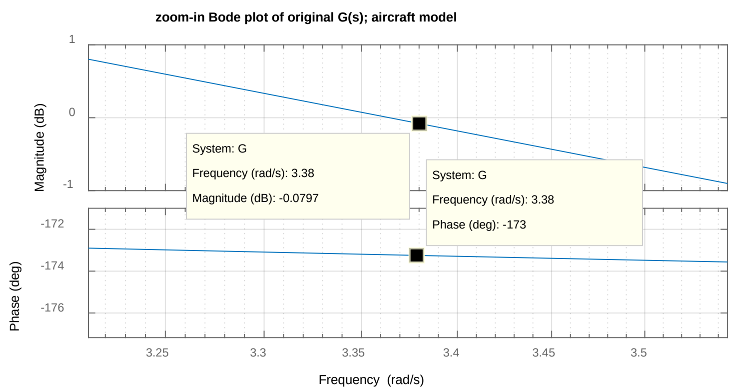
```
Phase margin: aircraft model
input data
    K   | J  |    c =
   114    10      4

G =

      114
  ------------
  10 s^2 + 4 s

Continuous-time transfer function.

=========================
GM, dB | PM, deg =
    10    60
=========================
```






```
Input phase in deg when |G|=0 dB, phi=-173
gamma = phase margin, deg =
     7
```

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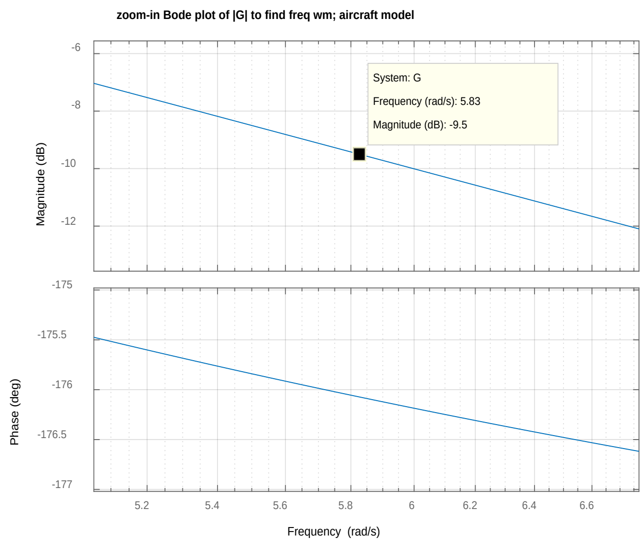
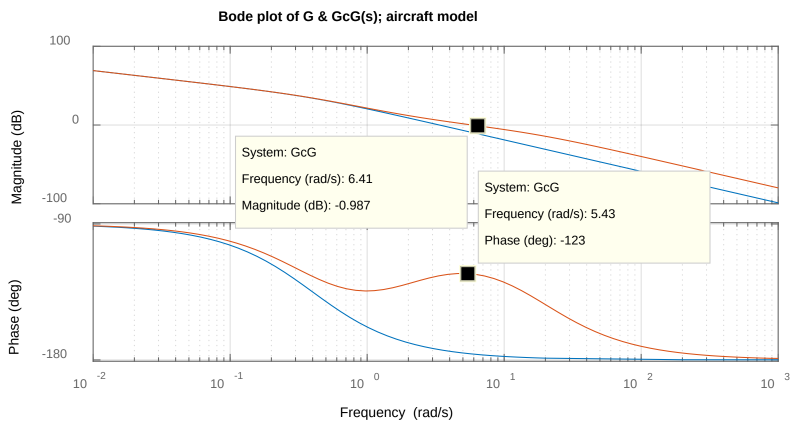
```
Phase compensator design: aircraft model
phi_m =
    53
alpha =
    0.1120
1/sqrt(alpha) =
    2.9887
G_wm, dB =
   -9.5096
```

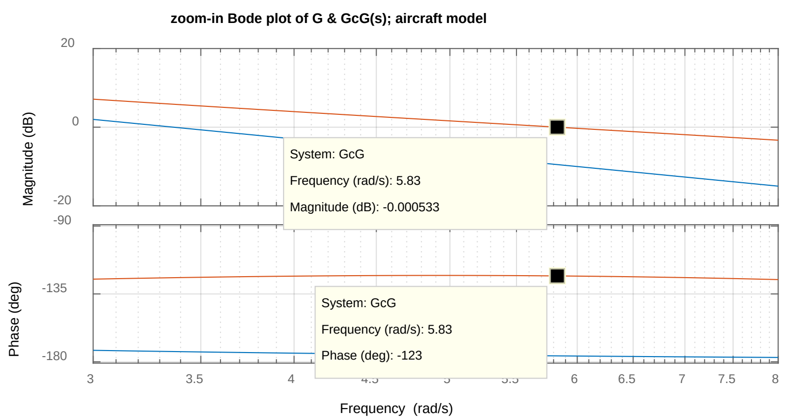



```
Input frequency in rad/s when |G|=G_wm, wm=5.83
T =
    0.5126
```

430/523

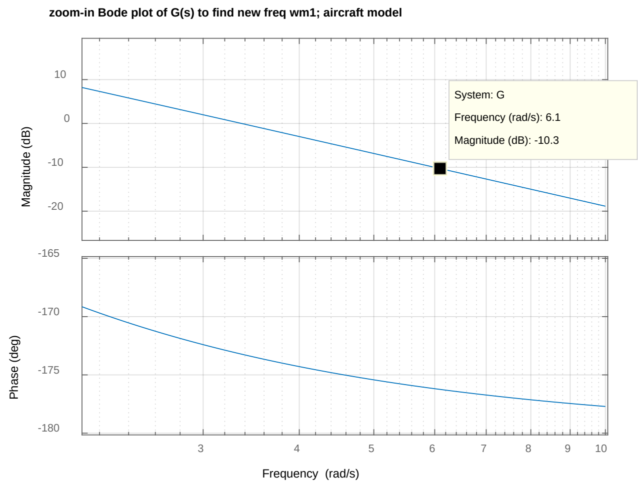
```
Gc =

  0.5126 s + 1
  -------------
  0.05739 s + 1
```





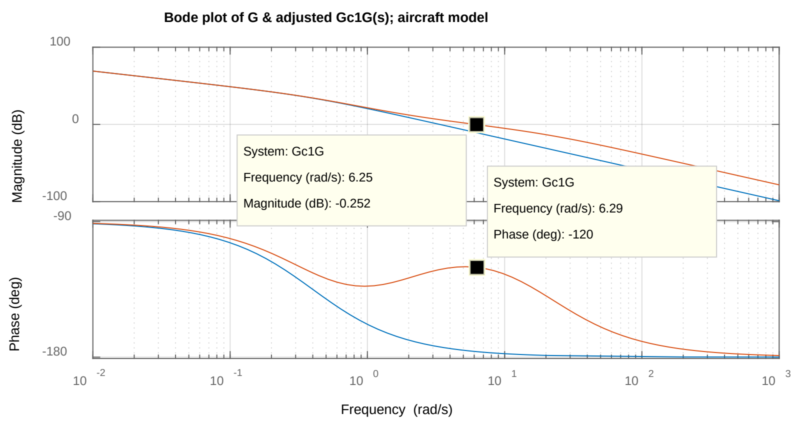

```
Input phase in deg when |GcG|=0 dB, phi_c=-123
phi_c = phase at |GcG|=0 dB point, deg =
  -123
gamma_c = phase margin of GcG, deg =
    57
-------------------------
```

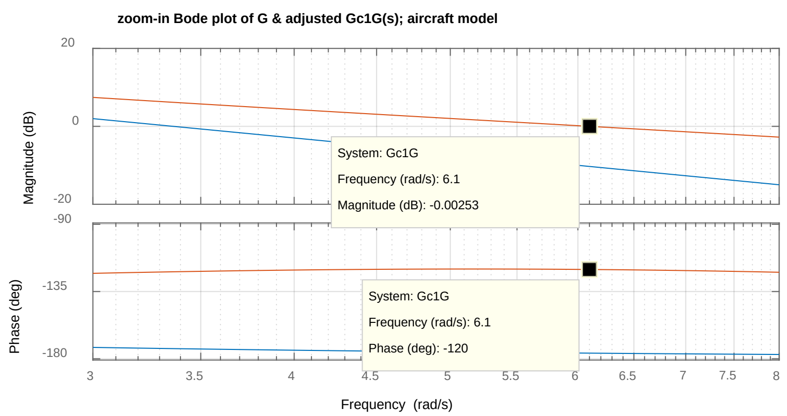
```
Adjustment of phase compensator: aircraft model
dphi =
     3
phi_m1 =
    56
alpha1 =
    0.0935
G_wm1, dB =
  -10.2932
```




```
Input frequency rad/s when |G|=G_wm1, wm1=6.1
T1 =
    0.5362
```

```
Gc1 =

  0.5362 s + 1
  -------------
  0.05012 s + 1
```






```
Input phase in deg for |Gc1G|=0 dB, phi_c1=-120
phi_c1 = phase at |Gc1G|=0 dB point, deg =
  -120
gamma_c = phase margin of GcG, deg =
    60
```

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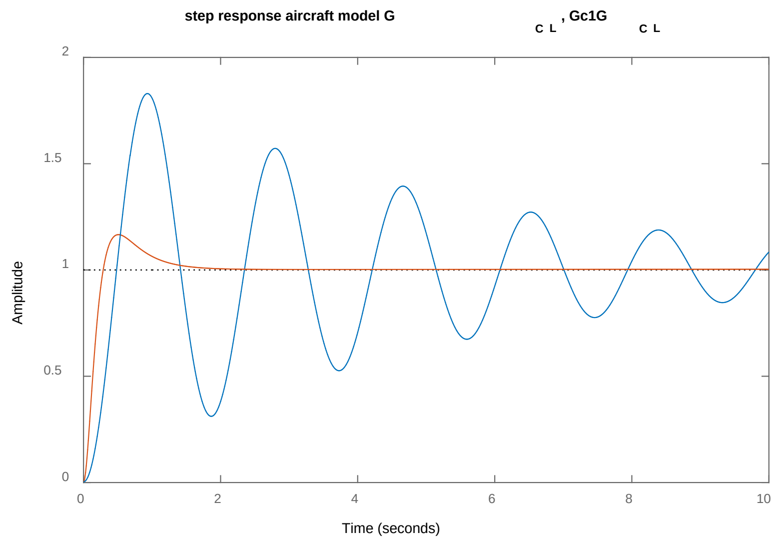
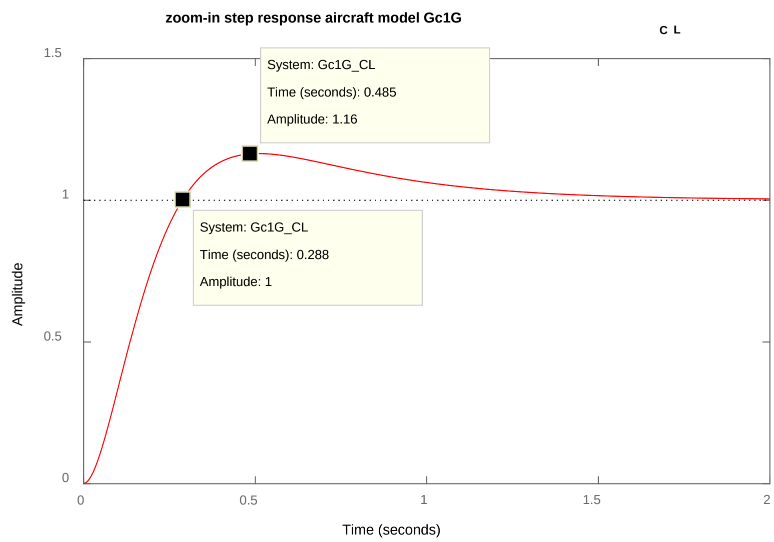
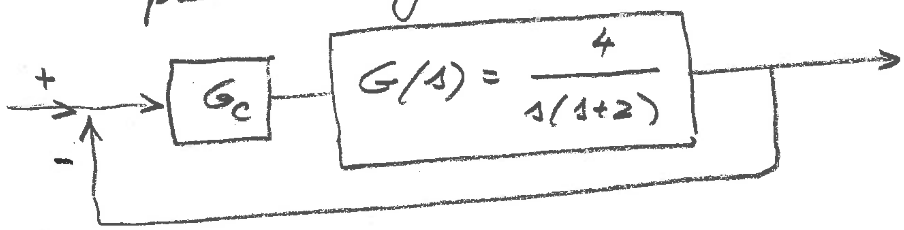

step response aircraft model G
C L
, Gc1G
C L
Amplitude
Time (seconds)
0
0.5
1
1.5
2
0
2
4
6
8
10


zoom-in step response aircraft model Gc1G
C L
System: Gc1G_CL
Time (seconds): 0.485
Amplitude: 1.16
System: Gc1G_CL
Time (seconds): 0.288
Amplitude: 1
Amplitude
Time (seconds)
0
0.5
1
1.5
0
0.5
1
1.5
2

C7

Ex. 11.6

Design a lead compensator to reduce servomotor ramp error and meet phase and gain margin requirements.



<u>Given</u> : $G(s) = \frac{4}{s(s+2)}$

<u>Find</u> :

(a) static velocity error const. K_V

phase margin, γ

gain margine, $(K_g)_{dB}$

(b) design compensator to achieve:

$K_V \geq 20\,/\text{sec}$

$\gamma \geq 50^\circ$ $(PM = 50^\circ)$

$(K_g)_{dB} \geq 10\,dB$ $(GM = 10\,dB)$

<u>Solution</u>

(a) characterize current system

- $K_V = \lim_{s \to 0} s\,G(s) = \lim_{s \to 0} s \frac{4}{s(s+2)} = \frac{4}{2} = 2/\text{sec}.$
- $\varphi(G = 0\,dB) = -128^\circ$; $\gamma = -128^\circ + 180^\circ = 52^\circ$
- $(K_g)_{dB} = \infty$ because $\angle G$ never crosses -180° line

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C7a

Fig.1a Bode plot of original G(s); Example 11.6

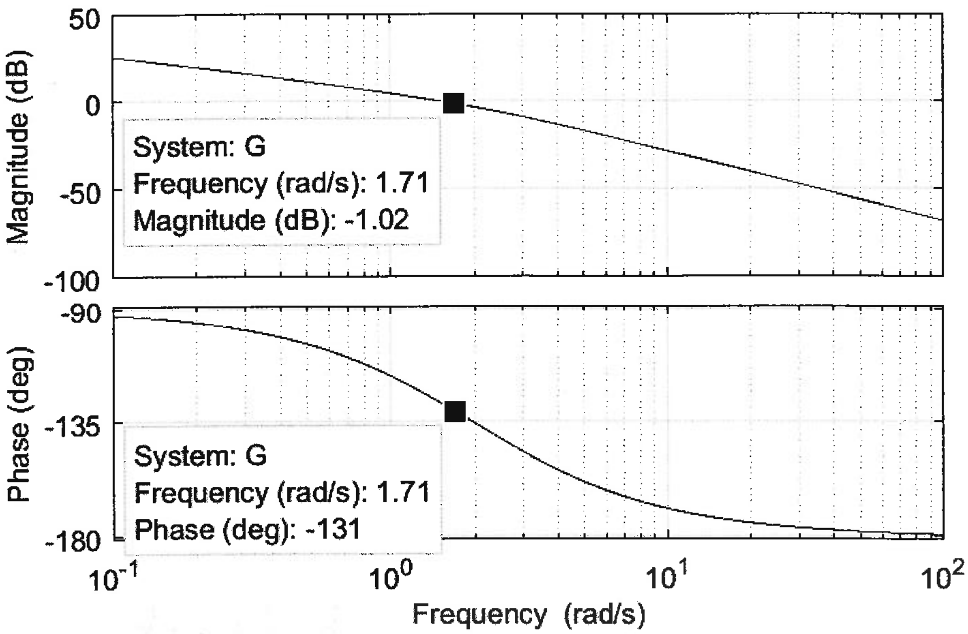
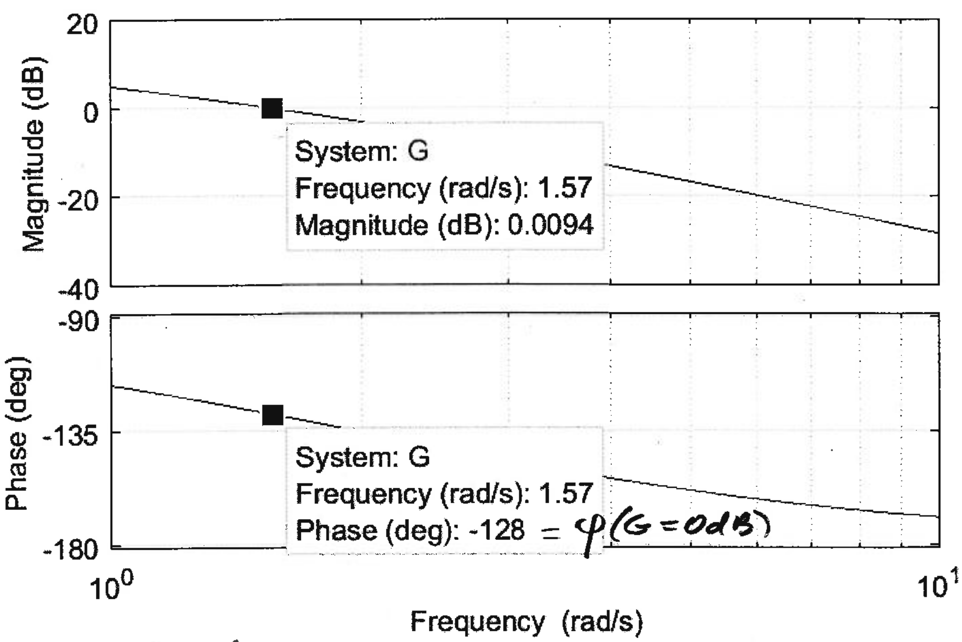


Fig.1b zoom-in Bode plot of original G(s); Example 11.6



$\gamma = -128 + 180 = 52°$

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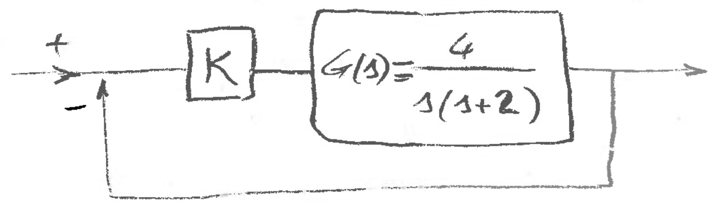
C8/ (b) Design P-controller to improve K_v



Roadmap

(1) add K to improve K_V to $K_V^* = 20/sec$

(2) Notice that adding K shifts magnitude plot upward and spoils the phase margin.

(3) add compensator G_c to improve phase and bring phase back to $\gamma^* \cong 50°$

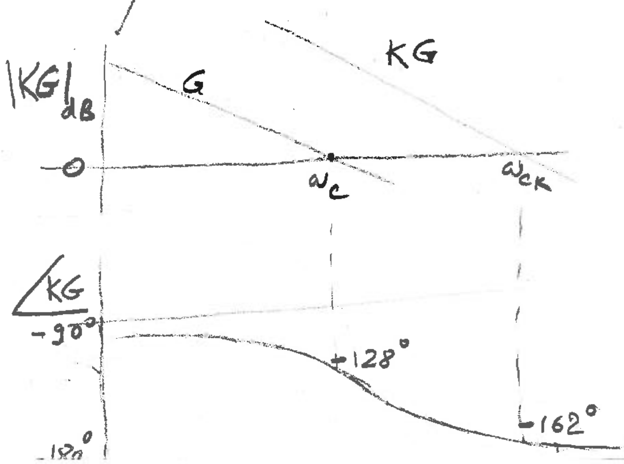
Design

D1. Calculate K needed to bring K_v to value $K_v^* = 20/s$

$$K_v^* = \lim_{s \to 0} s\, G^*(s) = \lim_{s \to 0} K \cdot \frac{4}{s(s+2)} = 2K = 20$$

$$\rightarrow \boxed{K = 10} \qquad K_{dB} = 20\,dB$$

D2. Draw Bode diagram for new system. We find that gain crossover freq ω_c has moved to the right and the phase margin has decreased



$\omega_{ck} = 6.16$ rad/sec.

$\varphi(\omega_{ck}) = -162°$

$\gamma_K = 18° < 50°$

$(-162° + 180° = 18°)$

Need to add 32° !

437/523

c8a

Fig.2a Bode plot of G & KG(s);

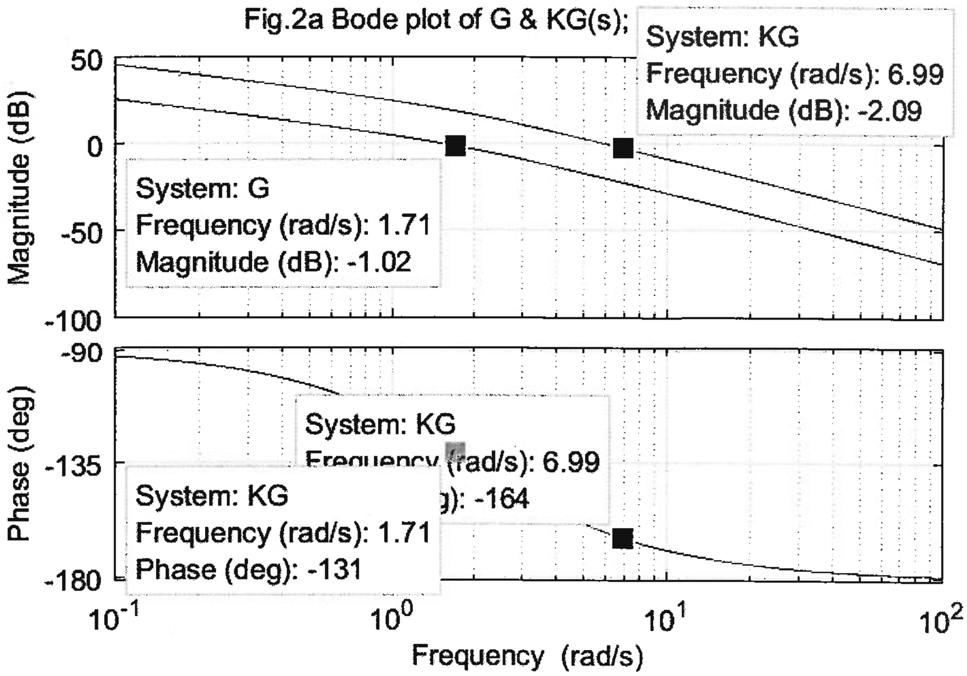
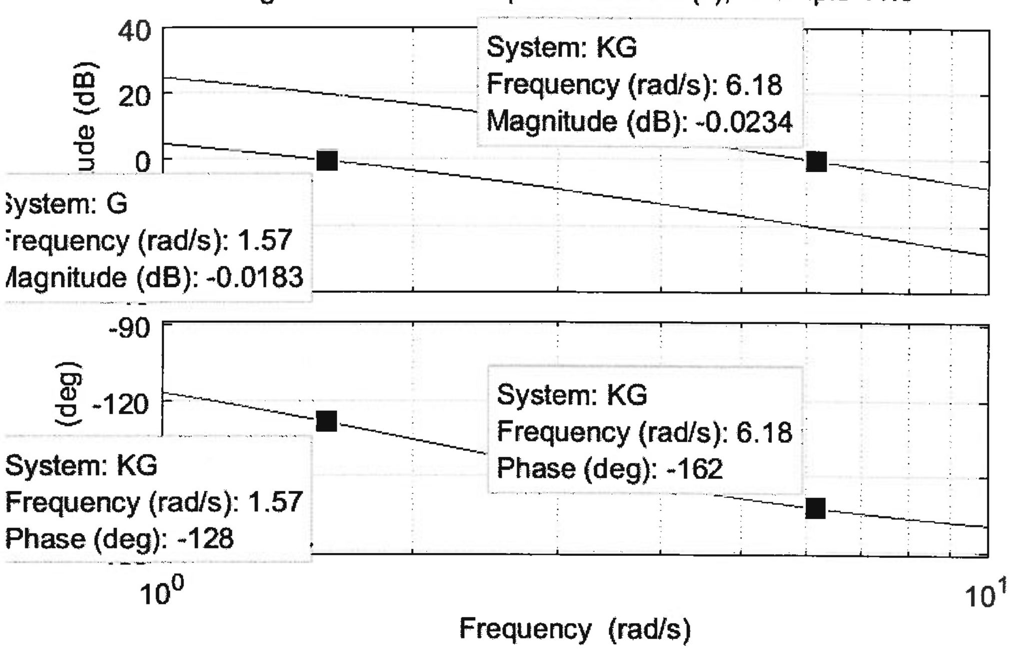


Fig.2b zoom-in Bode plot of G & KG(s); Example 11.6



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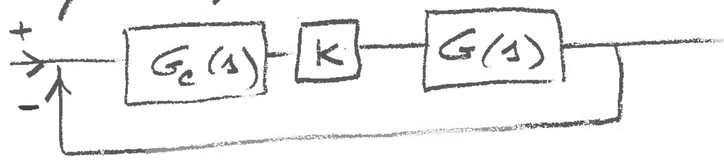
C9 Design compensator to improve γ.



We need lead compensator

$$G_c(s) = \frac{Ts + 1}{\alpha T s + 1}$$

We need to improve the phase margin from $\gamma = 18^\circ$ to $\gamma_M = 50^\circ$, i.e., we need the phase compensator to add $\varphi_m = 32^\circ$

Recall $\alpha = \left.\frac{1 - \sin\varphi_m}{1 + \sin\varphi_m}\right|_{\varphi_m = 32^\circ} = 0.3073$

To calculate ω_m, recall

$$|G_c(\omega_m)| = \frac{1}{\sqrt{\alpha}} = 1.8040 = 5.125\,dB$$

We identify ω_u as the point on the Bode plot of $KG(\omega)$ where

$$|KG(\omega_m)|_{dB} = -|G_c(\omega_n)|_{dB} = -5.125\,dB.$$

From Bode plot, we read

$$\omega_m = 8.38 \text{ rad/sec for } |KG|_{dB} = -5.13dB$$

Hence $T = \frac{1}{\omega_m\sqrt{\alpha}} = 0.2153$ sec.

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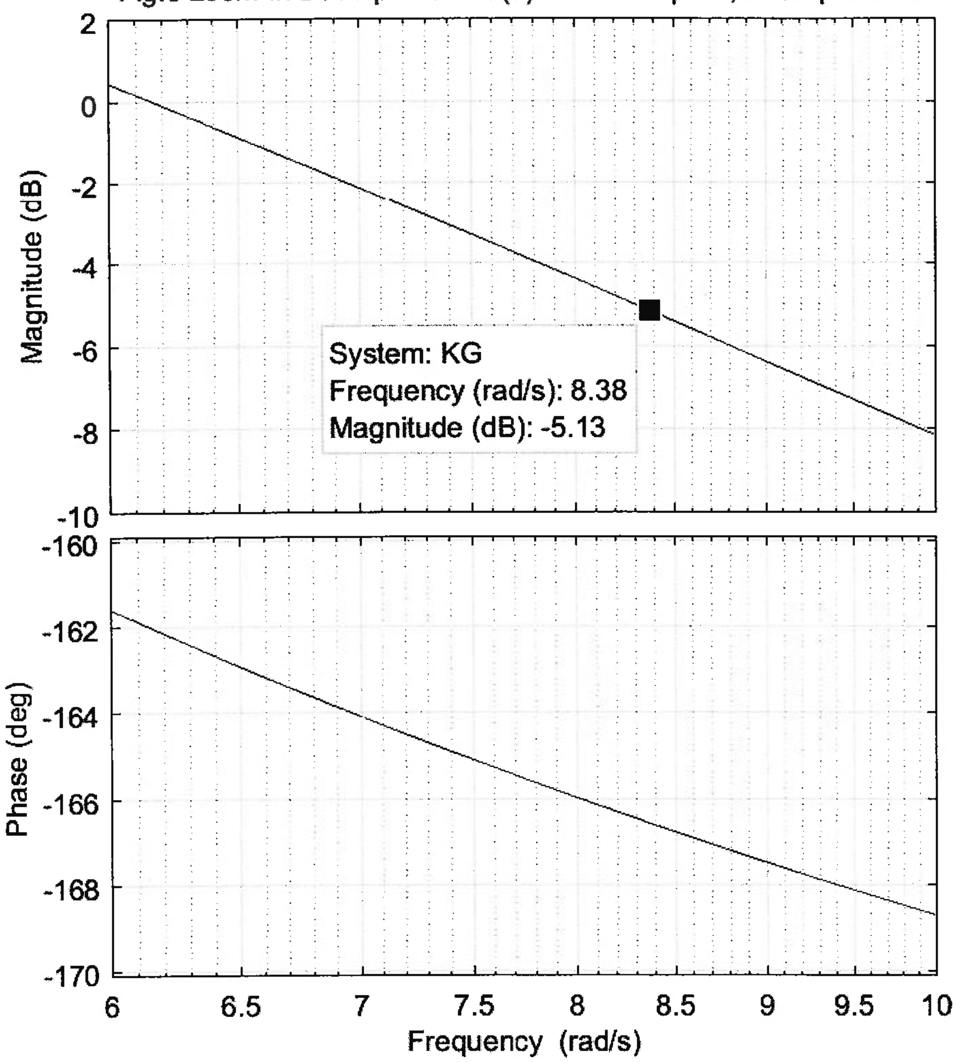
C9a

Fig.3 zoom-in Bode plot of KG(s) to find freq wm; Example 11.6



$\omega_m = 8.38$ rad/sec for $|KG|_{dB} = -5.13$dB

440/523

C96

```
Phase compensator design: Exampe 11.6

phi =

  -128


gamma =

    52


alpha =

    0.3073


1/sqrt(alpha =

    1.8040


Gc_wm, dB =

    5.1250


T =

    0.2153


Gc =

  0.2153 s + 1
  -------------
  0.06615 s + 1
```

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cgc

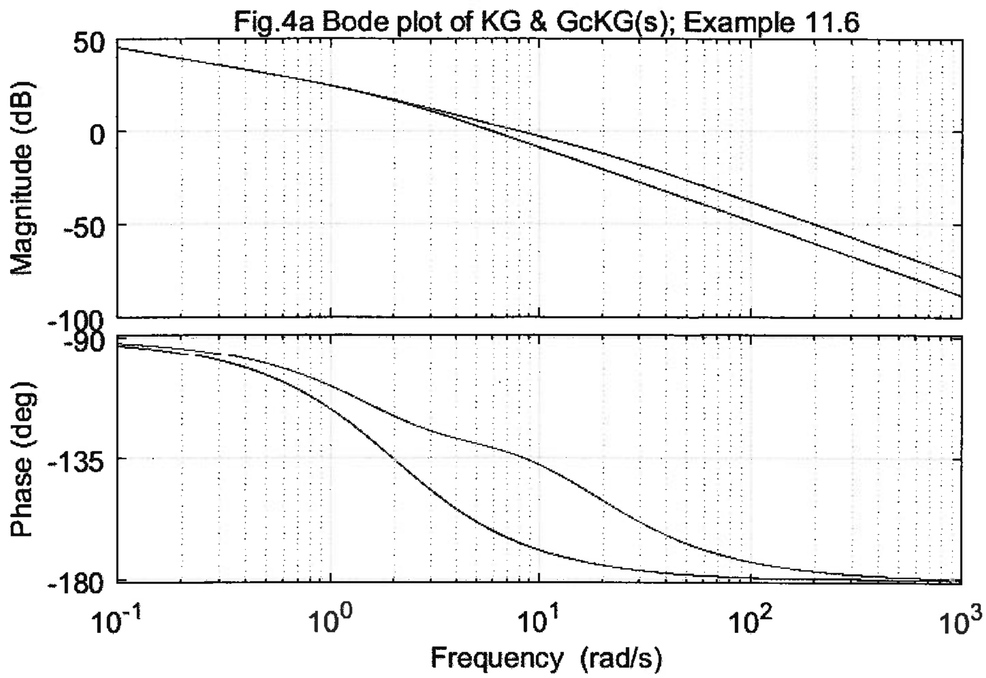
Fig.4a Bode plot of KG & GcKG(s); Example 11.6
Magnitude (dB)
Phase (deg)
Frequency (rad/s)

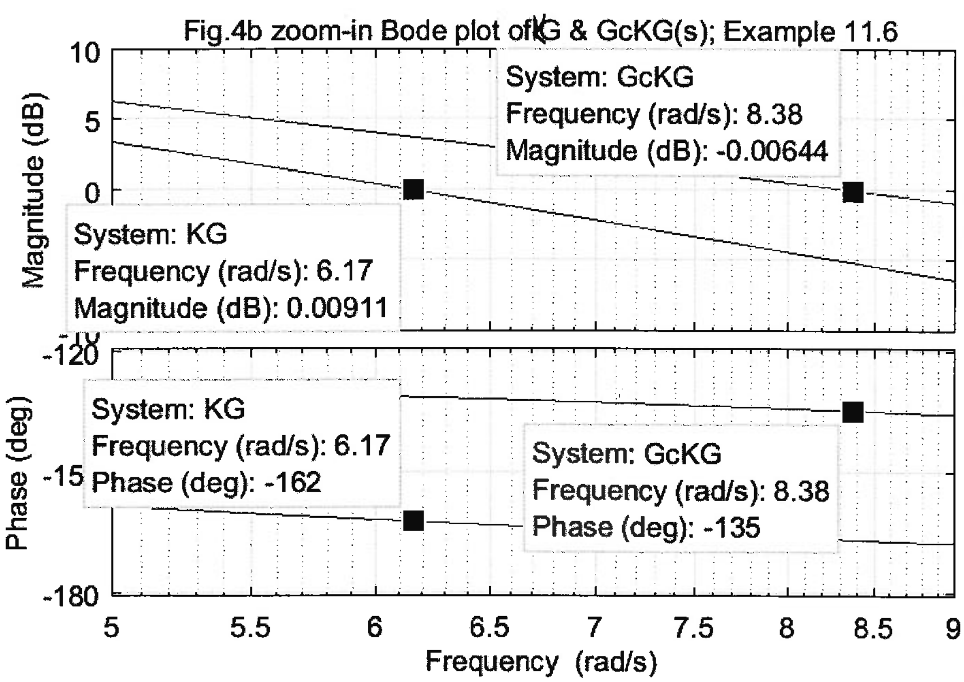
Fig.4b zoom-in Bode plot of KG & GcKG(s); Example 11.6
System: GcKG
Frequency (rad/s): 8.38
Magnitude (dB): -0.00644
System: KG
Frequency (rad/s): 6.17
Magnitude (dB): 0.00911
System: KG
Frequency (rad/s): 6.17
Phase (deg): -162
System: GcKG
Frequency (rad/s): 8.38
Phase (deg): -135
Magnitude (dB)
Phase (deg)
Frequency (rad/s)

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C10 Test the compensator

Fig 4 shows that the compensator is not sufficient because the compensated phase margin is

$$\gamma_c = -135 + 180 = 45^\circ$$

We need 50°; thus, we need to adjust the compensator to get to 50°. We need an additional 5° of compensation, i.e., $\Delta\varphi = 5^\circ$. The new φ_m is

$$\varphi_{m_1} = \varphi_m + \Delta\varphi = 32 + 5 = 37^\circ$$

The new α is $\quad \alpha_1 = \dfrac{1 - \sin\varphi_{m_1}}{1 + \sin\varphi_{m_1}} = 0.2486$

We keep $T_1 = T$ and calculate new compensator

$$G_{c_1} = \frac{T_1 s + 1}{\alpha_1 T_1 s + 1}$$

With this new compensator, the Bode plot is as shown in Fig. 5. The phase at $G_{c_1} K G = 0\,dB$ is -130°

The margin is $\gamma_{c_1} = -130^\circ + 180^\circ = 50^\circ = \gamma_M$

We have met the specification!

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C10a

```
dphi =

     5


phi_m1 =

    37


alpha1 =

    0.2486


T1 =

    0.2153


Gc1 =

  0.2153 s + 1
  -------------
  0.05352 s + 1
```

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C10b

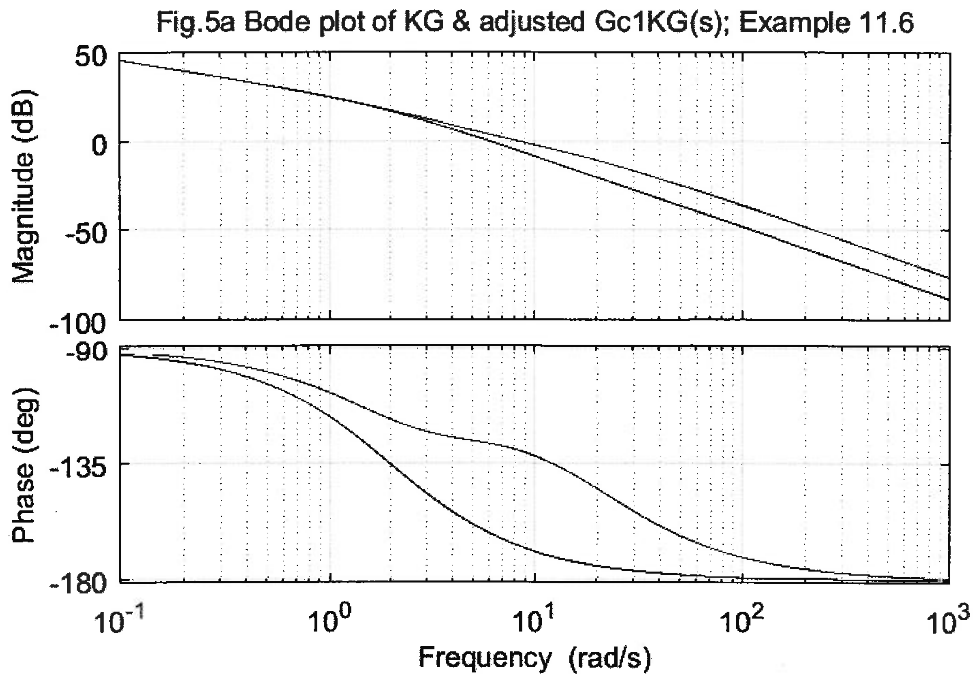
Fig.5a Bode plot of KG & adjusted Gc1KG(s); Example 11.6
50
0
-50
-100
Magnitude (dB)
-90
-135
-180
Phase (deg)
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/s)

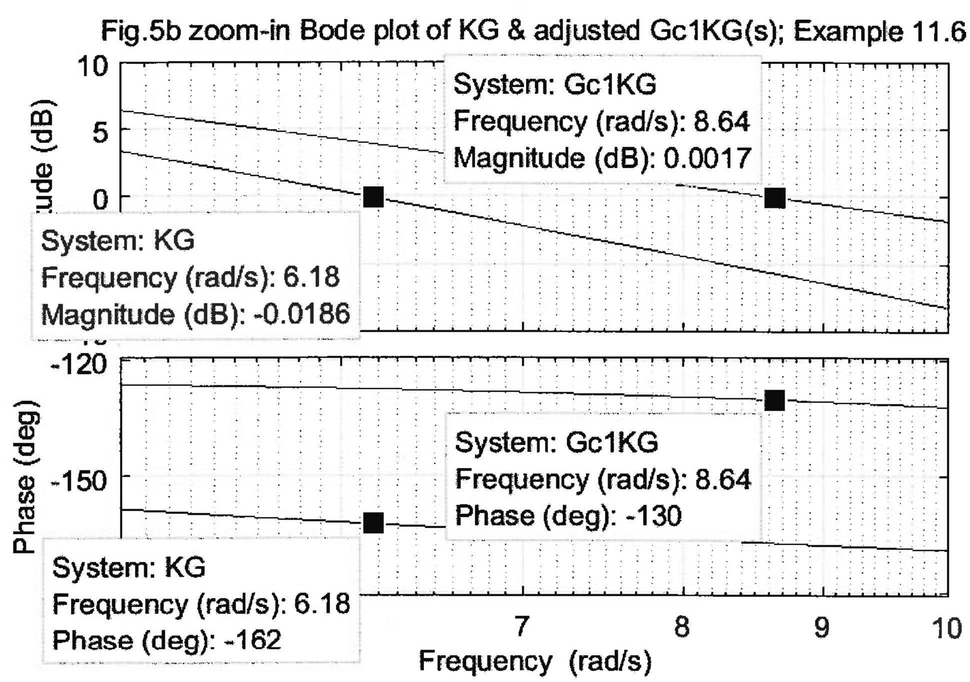
Fig.5b zoom-in Bode plot of KG & adjusted Gc1KG(s); Example 11.6
10
5
0
System: Gc1KG
Frequency (rad/s): 8.64
Magnitude (dB): 0.0017
System: KG
Frequency (rad/s): 6.18
Magnitude (dB): -0.0186
-120
-150
Phase (deg)
System: Gc1KG
Frequency (rad/s): 8.64
Phase (deg): -130
System: KG
Frequency (rad/s): 6.18
Phase (deg): -162
7
8
9
10
Frequency (rad/s)

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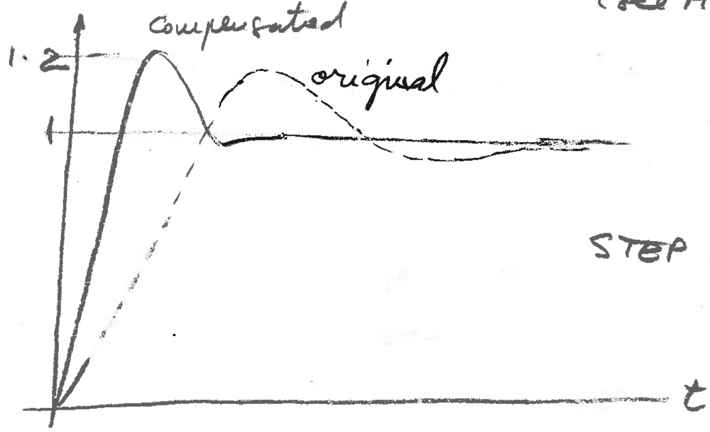
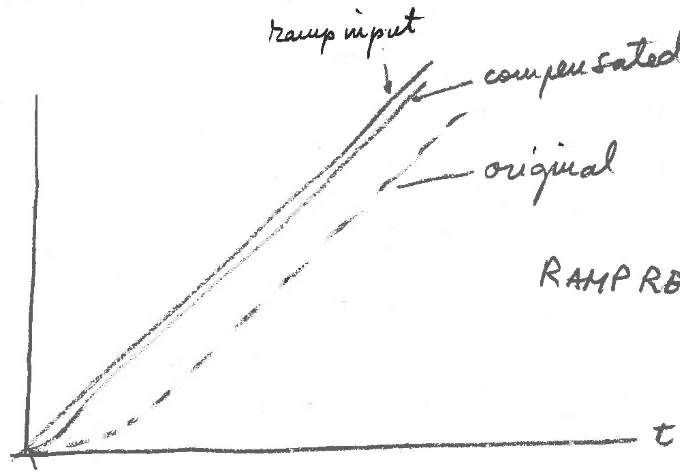
C11) Plot step response and ramp response of the original and compensated systems

(see MATLAB)



STEP RESPONSE



RAMP RESPONSE

Discussion

Ramp response has improved dramatically

Step response has a faster rise time and a faster settling time, but it has a slightly higher overshoot

C 11a

Fig.6a Step response Example 11.6

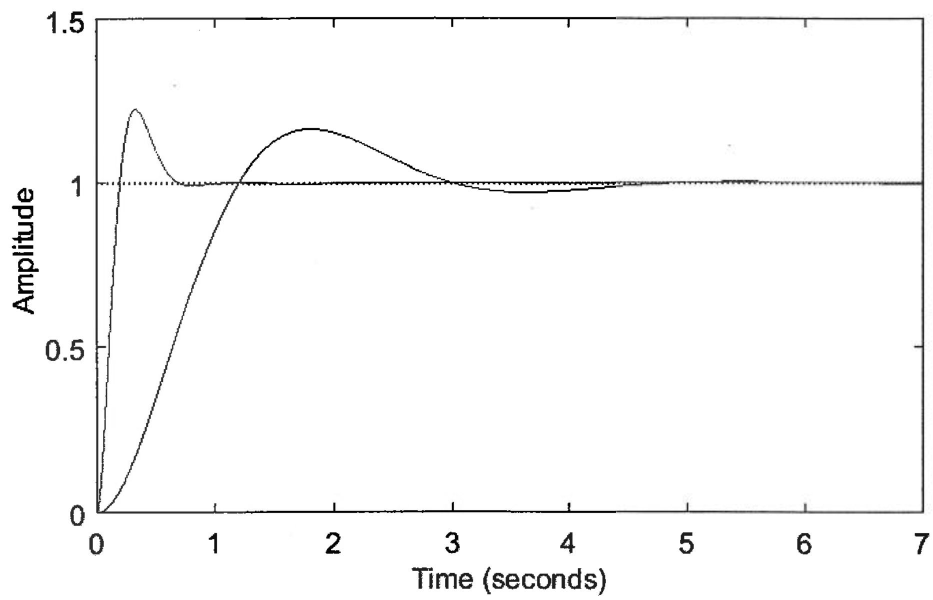
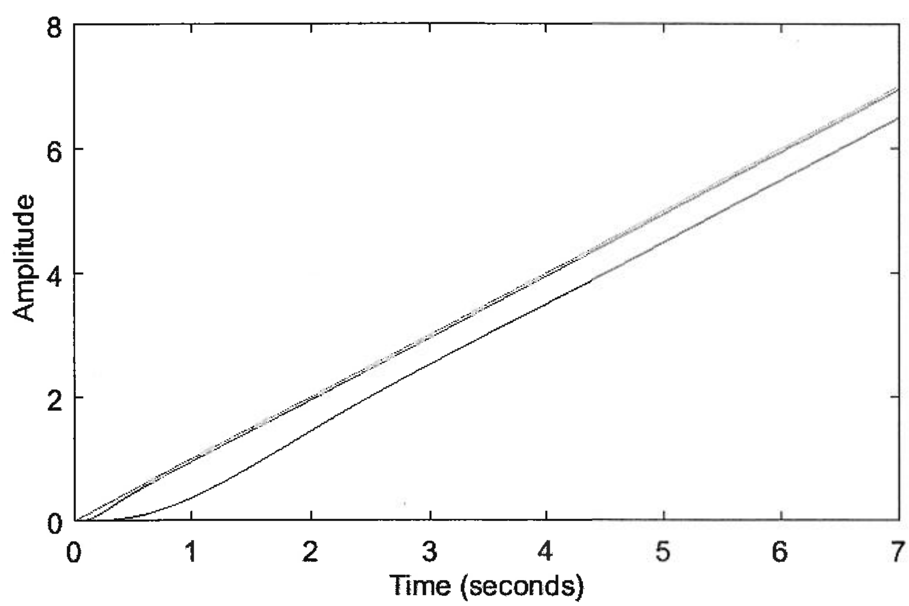


Fig.6b Ramp response Example 11.6



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8.8 Lag Compensators

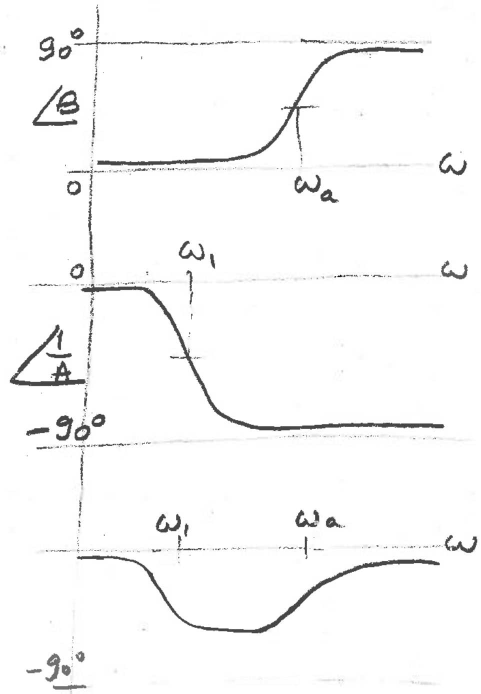
C4

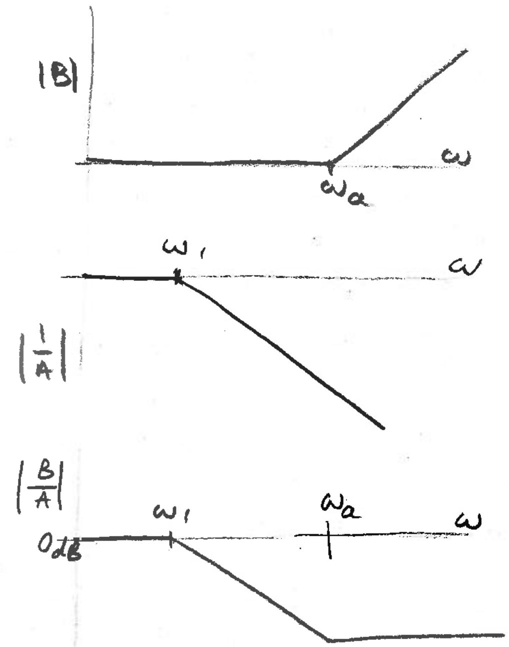
LAG COMPENSATOR

$$G_c = \frac{sT_a+1}{sT_1+1}$$

$\omega_1 < \omega_a$, $T_a < T_1$

— Phase — — Gain —



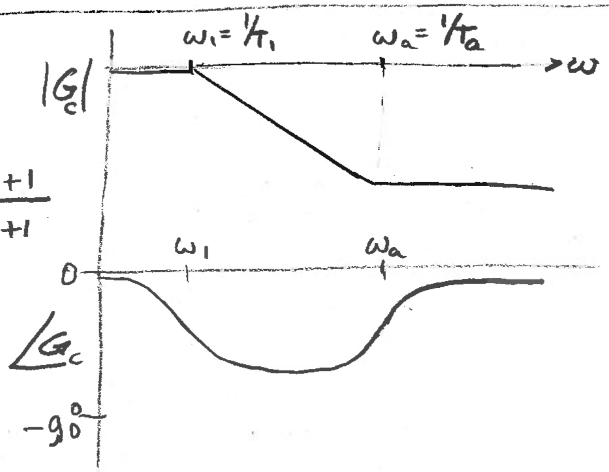


Adds "lag" between ω_1 & ω_a

Low pass filter
(higher freq. are reduced)

$$G_c(s) = \frac{sT_a+1}{sT_1+1}$$

$T_a < T_1$



LP filter

Phase lag

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c5

LAG-LEAD COMPENSATOR

(NOTCH FILTER)

$$G_c = \frac{sT_a+1}{sT_1+1}\cdot\frac{sT_b+1}{sT_2+1}$$

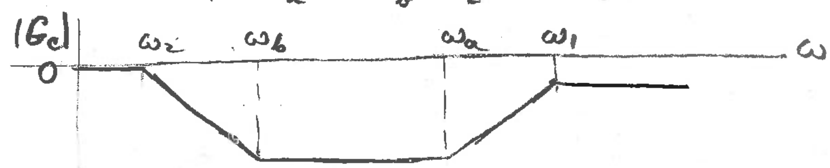
lead lag

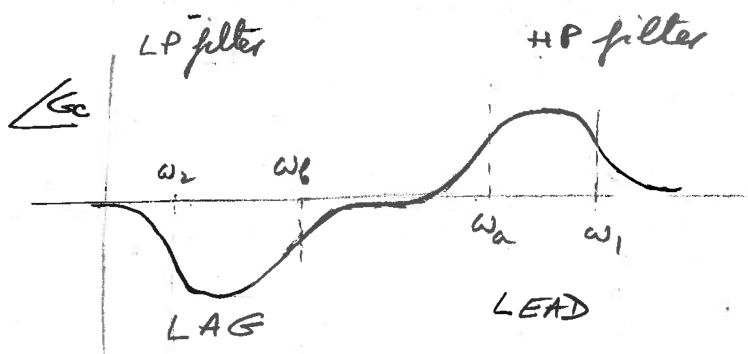
$G_c(i0) = 1$

$G_c(i\infty) = \frac{T_a}{T_1}\cdot\frac{T_b}{T_2}$

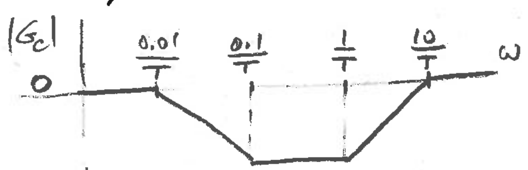
$T_1 < T_a < T_b < T_2$

$\omega_1 > \omega_a > \omega_b > \omega_2$





Example (Fig 11.35, p 648) $G_c = \frac{s+1}{0.1Ts+1}\cdot\frac{10Ts+1}{100Ts+1}$

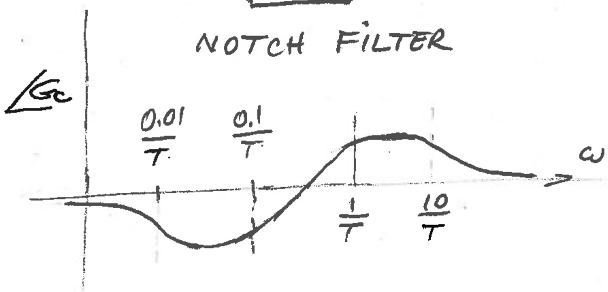


T: $0.1T < T < 10T < 100T$

ω: $\frac{10}{T} > \frac{1}{T} > \frac{0.1}{T} > \frac{0.01}{T}$

$G(i0) = 1$ (0 dB)

$G(i\infty) = 1$ (0 dB)



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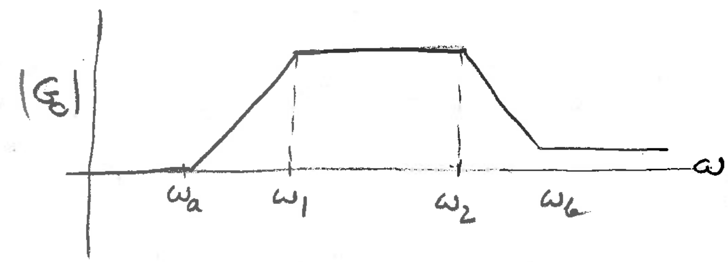
C6

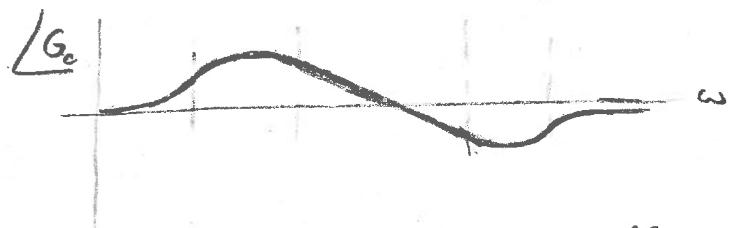
LEAD–LAG COMPENSATOR
(BAND PASS FILTER)

$$G_c = \frac{sT_a + 1}{sT_1 + 1} \cdot \frac{sT_b + 1}{sT_2 + 1}$$

$$\omega_a < \omega_1 < \omega_2 < \omega_b$$

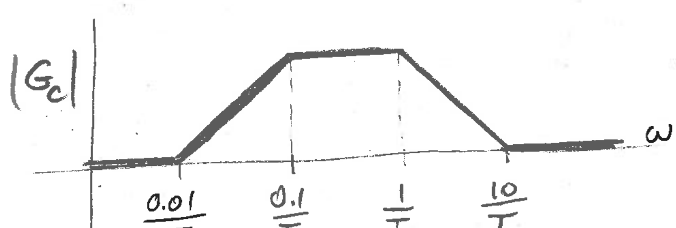
$$T_a > T_1 > T_2 > T_b$$



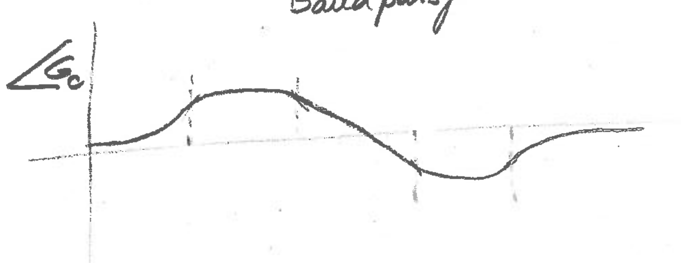


Example : Band pass filter

$$G_c = \frac{100Ts+1}{10Ts+1} \cdot \frac{0.1Ts+1}{Ts+1}$$



Band pass filter



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C6a

Notch filter

Example

$$G_c = \frac{Ts+1}{0.1Ts+1}\cdot\frac{10Ts+1}{100Ts+1}$$

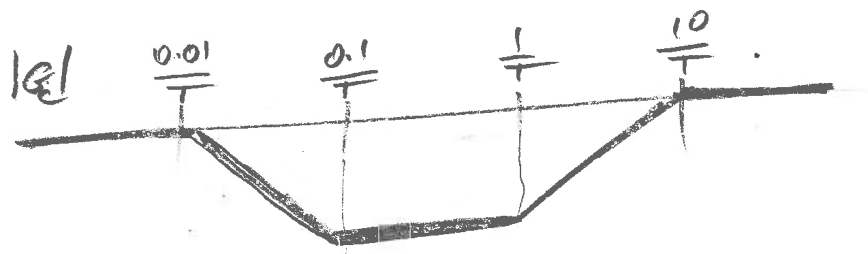
$$\begin{matrix} T_1 & T_a & T_b & T_2 \\ 0.1T < & T < & 10T < & 100T \end{matrix}$$

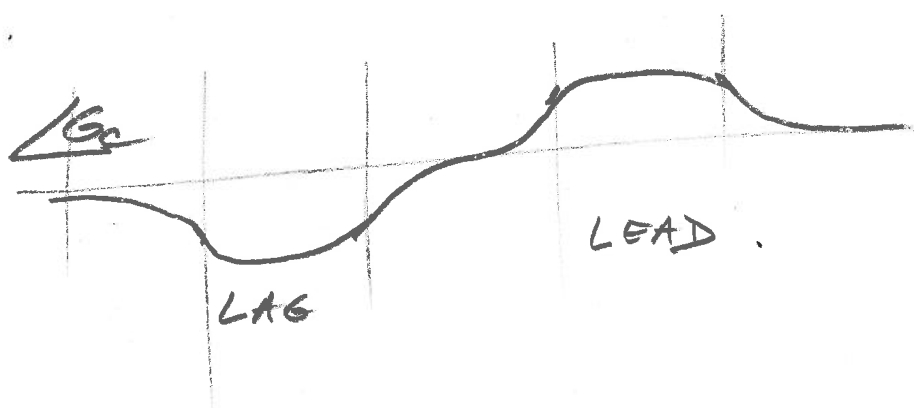
$$\omega_1 > \omega_a > \omega_b > \omega_2$$

$$\frac{10}{T} > \frac{1}{T} > \frac{0.1}{T} > \frac{0.01}{T}$$

$$\omega_2 < \omega_b < \omega_a < \omega_1$$

$$\frac{0.01}{T} \quad \frac{0.1}{T} \quad \frac{1}{T} \quad \frac{10}{T}$$





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