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2016/01/07

Differentiation Property

$$\mathcal{L} f'(t) = s F(s) \quad \text{if } f(0) = 0 \quad (1)$$

$$\mathcal{L} f''(t) = s^2 F(s) \quad \text{if } f'(0) = f(0) = 0 \quad (2)$$

$$\vdots$$

$$\mathcal{L} f^{(n)}(t) = s^n F(s) \quad \text{if } f^{(n-1)}(0) = \dots = f'(0) = f(0) = 0 \quad (n)$$

Proof

$$(1): \mathcal{L} f'(t) = \int_0^{\infty} f'(t) e^{-st} dt = f e^{-st} \Big|_0^{\infty} - \int_0^{\infty} f(-s) e^{-st} dt$$

Integration by parts

$$u dv = d(uv) - v du$$

$$\underbrace{e^{-st}}_{u} \underbrace{f' dt}_{dv} = \underbrace{f}_{v} \underbrace{-s e^{-st}}_{du}$$

$$= f(\infty) e^{-\infty} - f(0) + s \underbrace{\int_0^{\infty} f e^{-st} dt}_{F(s)} = \underbrace{s F(s)}_{QED} - f(0) \quad \text{QED}$$

(2): denote $g(t) = f'(t)$

$$G(s) = \mathcal{L} g(t) = \mathcal{L} f'(t) = s F(s)$$

$$\mathcal{L} f''(t) = \mathcal{L} g'(t) = s G(s) = s^2 F(s)$$

QED

(n): by induction ...

Bottom line: "to differentiate, multiply by s ."

provided $f(0) = 0, f'(0) = 0, \dots$

Else: need to subtract them $\mathcal{L} f'(t) = s F(s) - f(0)$ etc.

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Integration Property

$$\mathcal{L}\left(\int_0^t f(t^*) dt^*\right) = \frac{1}{s} F(s)$$

Proof Denote $g(t) = \int_0^t f(t^*) dt^*$

Then $g'(t) = f(t)$, $g(0) = 0$

$$\mathcal{L} \text{ LHS: } \mathcal{L} g'(t) = s \mathcal{L} g(t) = s \mathcal{L} \left(\int_0^t f(t^*) dt^* \right)$$

$$\mathcal{L} \text{ RHS: } \mathcal{L} f(t) = F(s)$$

$$\mathcal{L} \text{ LHS} = \mathcal{L} \text{ RHS: } s \mathcal{L} \left(\int_0^t f(t^*) dt^* \right) = F(s)$$

divide by s to get

$$\mathcal{L} \left(\int_0^t f(t^*) dt^* \right) = \frac{1}{s} F(s) \quad \text{Q.E.D.}$$

Bottom line: "To integrate, divide by s "

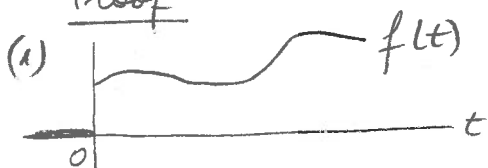
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Shift Properties

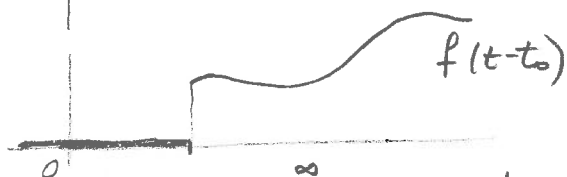
Shift in t : $\mathcal{L} f(t-t_0) = e^{-t_0 s} F(s)$ (1)

Shift in s : $\mathcal{L} e^{s_0 t} f(t) = F(s-s_0)$ (2)

Proof



Function $f(t)$ is zero for -ve argument



Function $f(t-t_0)$ is zero for $t < t_0$.

$$\mathcal{L} f(t-t_0) = \int_0^{\infty} f(t-t_0) e^{-st} dt = \int_0^{t_0} 0 \cdot e^{-st} dt + \int_{t_0}^{\infty} f(t-t_0) e^{-st} dt$$

$$= \int_{t_0}^{\infty} f(t-t_0) e^{-st} dt \quad \begin{array}{l} \text{change} \\ \text{of variable} \end{array} \quad \left| \begin{array}{l} t^* = t - t_0 \rightarrow t = t^* + t_0 \\ dt^* = dt \end{array} \right.$$

$$= \int_0^{\infty} f(t^*) e^{-s(t^*+t_0)} dt^* = e^{-st_0} \underbrace{\int_0^{\infty} f(t^*) e^{-st^*} dt^*}_{F(s)}$$

$F(s) \cdot QED$

(2) $\mathcal{L} e^{s_0 t} f(t) = \int_0^{\infty} e^{s_0 t} f(t) e^{-st} dt = \int_0^{\infty} f(t) e^{-(s-s_0)t} dt$

$F(s-s_0)$

QED

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Final Value Theorem Steady-state Response

$$x_{ss} = x(\infty) = \lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s).$$

\uparrow steady state response \uparrow final value

$$x_{ss} = \lim_{s \rightarrow 0} sX(s)$$

Proof Start with LT of the derivative $\dot{x}(t)$, i.e.,

$$\mathcal{L} \dot{x}(t) = sX(s) \quad , \quad x(0) = 0 \quad (1)$$

But, by definition

$$\mathcal{L} \dot{x}(t) = \int_0^{\infty} \dot{x}(t) e^{-st} dt \quad (2)$$

$$(1) \equiv (2): \quad \int_0^{\infty} \dot{x}(t) e^{-st} dt = sX(s) \quad (3)$$

Take limit $s \rightarrow 0$ of Eq. (3), i.e.,

$$\lim_{s \rightarrow 0} \left[\int_0^{\infty} \dot{x}(t) e^{-st} dt \right] = \lim_{s \rightarrow 0} sX(s) \quad (4)$$

$$\text{but } \lim_{s \rightarrow 0} e^{-st} = e^0 = 1 \quad (5)$$

$$(5) \rightarrow (4) \text{ LHS: } \int_0^{\infty} \dot{x}(t) dt = x(t) \Big|_0^{\infty} = x(\infty) = x_{ss}$$

$$\text{LHS} = \text{RHS:} \quad x_{ss} = \lim_{s \rightarrow 0} sF(s)$$

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INVERSE LAPLACE TRANSFORM

Assume $X(s) = \frac{B(s)}{A(s)}$ (1)

$A(s), B(s)$ polynomials in s :

$$A(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_0 : \text{degree } n$$

$$B(s) = b_m s^m + b_{m-1} s^{m-1} + \dots + b_0 : \text{degree } m < n$$

Then $X(s)$ can be expanded in partial fractions, i.e.,

$$X(s) = \frac{z_1}{s-p_1} + \frac{z_2}{s-p_2} + \dots + \frac{z_n}{s-p_n} = \sum_{i=1}^n \frac{z_i}{s-p_i} \quad (2)$$

Recall $\mathcal{L} e^{p_0 t} = \frac{1}{s-p_0}$ (pole p_0)

Hence, the iLT of (2) is a sum, i.e.,

$$x(t) = z_1 e^{p_1 t} + z_2 e^{p_2 t} + \dots + z_n e^{p_n t} = \sum_{i=1}^n z_i e^{p_i t} \quad (3)$$

The values p_1, p_2, \dots, p_n are the roots of the denominator $A(s)$. They are called "poles". They may be complex numbers.

The values z_1, z_2, \dots, z_n are called "residues"

$$z_i = \lim_{s \rightarrow p_i} [(s-p_i) X(s)] \quad (4)$$

Use MATLAB function "residue" to find p_i & z_i 45/523

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ILT with complex poles.

When complex poles appear, they are in conjugate pairs, i.e.,

$$p_{1,2} = \sigma \pm i\omega_d \quad (1)$$

$$\text{PFE: } X(s) = X_1(s) + X_2(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} \quad (2)$$

$$\text{It can be shown that } r_1 = -ir_0, r_2 = ir_0 \quad (3)$$

$$X_1(s) = \frac{-ir_0}{s-p_1}, \quad p_1 = \sigma + i\omega_d \quad (4)$$

$$x_1(t) = \mathcal{L}^{-1} X_1(s) = \mathcal{L}^{-1} \frac{-ir_0}{s-p_1} = -ir_0 e^{p_1 t} \quad (5)$$

$$= -ir_0 e^{(\sigma + i\omega_d)t} = -ir_0 e^{\sigma t} e^{i\omega_d t}$$

$$= -ir_0 e^{\sigma t} (\cos \omega_d t + i \sin \omega_d t) \quad (6)$$

$$= r_0 e^{\sigma t} (-i \cos \omega_d t + \sin \omega_d t)$$

Similarly

$$X_2(s) = \frac{ir_0}{s-p_2}, \quad p_2 = \sigma - i\omega_d \quad (7)$$

$$x_2(t) = ir_0 e^{\sigma t} e^{-i\omega_d t} = ir_0 e^{\sigma t} (\cos \omega_d t - i \sin \omega_d t) \quad (8)$$

$$= r_0 e^{\sigma t} (i \cos \omega_d t + \sin \omega_d t)$$

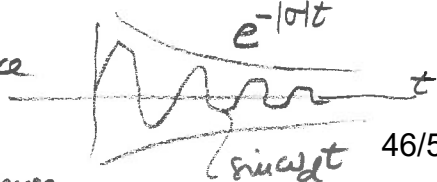
$$x_1(t) + x_2(t) =$$

$$r_0 e^{\sigma t} (-i \cos \omega_d t + \sin \omega_d t) + r_0 e^{\sigma t} (i \cos \omega_d t + \sin \omega_d t) = 2r_0 e^{\sigma t} \sin \omega_d t$$

We expect $\sigma < 0$, hence

$$\sigma = -|\sigma|$$

Poles in LHS for stable response.



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MATLAB instruction residue

$$X(s) = \frac{B(s)}{A(s)}$$

$$B(s) = b_m s^m + b_{m-1} s^{m-1} + \dots + b_0$$

$$B = [b_m \ b_{m-1} \ \dots \ b_0]$$

$$A(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_0$$

$$A = [a_n \ a_{n-1} \ \dots \ a_0]$$

$$[r, p, k] = \text{residue}(B, A)$$

where: p = vector of poles (roots of $A(s)$)

r = vector of residues at the poles

k = vector of direct term coeff.

In our work, $m < n$, then $k = []$ void matrix

Partial fraction expansion (PFE):

$$X(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \dots \quad (1)$$

Note: If $m \geq n$, then define direct term as

$$k_1 s^2 + k_2 s^{2-1} + \dots + k_{q+1}$$

and add it to the end of (1) to get $X(s)$

Example

$$X(s) = \frac{1}{s+2} + \frac{2}{s+1} = \frac{(s+1) + 2(s+2)}{(s+1)(s+2)}$$

$$= \frac{3s+5}{s^2+3s+2} = \frac{B(s)}{A(s)}$$

(a) Define:

$$B = [3 \ 5] ; A = [1 \ 3 \ 2]$$

$$[k, p, k] = \text{residue}(B, A)$$

$$k_1 = 1 \quad p_1 = -2 \quad k = \text{void.}$$

$$k_2 = 2 \quad p_2 = -1$$

$$\frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} = \frac{1}{s-(-2)} + \frac{2}{s-(-1)} = \frac{1}{s+2} + \frac{2}{s+1}$$

checks!

(b) Define:

$$k = [1 \ 2] ; p = [-2 \ -1] ; k = 0$$

$$[b, a] = \text{residue}(k, p, k)$$

$$b = [3 \ 5]$$

$$a = [1 \ 3 \ 2]$$

checks!

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Convolution property of Laplace Transform

$$\mathcal{L}^{-1} G(s) F(s) = \mathcal{L}^{-1} F(s) G(s)$$

$$= \int_0^t \underbrace{f(\tau)}_{\text{excitation function}} \underbrace{g(t-\tau)}_{\text{impulse response of the system}} d\tau \quad (1)$$

Convolution expresses the system response $x(t)$ to a complicated excitation $f(\tau)$ as an integral using the impulse response $g(t)$ shifted by τ to $(t-\tau)$, i.e.,

$$x(t) = (f * g)(t) = \int_0^t f(\tau) g(t-\tau) d\tau \quad (2)$$

The integral in Eq. (2) is not easily computed in time domain; however, Laplace transform makes it easy because:

1. calculate $F(s) = \mathcal{L} f(t)$, $G(s) = \mathcal{L} g(t)$
2. multiply $F(s) G(s)$
3. take inverse Laplace transform to get $x(t)$

$$\mathcal{L}^{-1} F(s) G(s) = (f * g)(t) = \underline{x(t)}$$

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DOMINANT POLES

Given:

$$X(s) = \frac{B(s)}{A(s)} = \frac{z_1}{s-p_1} + \frac{z_2}{s-p_2} + \dots + \frac{z_k}{s-p_k} + \dots \quad (1)$$

$p_k \in \mathbb{C}$ complex number

$p_k = \sigma_k + i\omega_k$, conjugate pairs. (2a)

or

$$p_k \in \mathbb{R} \quad p_k = \sigma_k \quad (2b)$$

Perform $\mathcal{L}^{-1} E_g(1)$:

$$x(t) = z_1 e^{p_1 t} + z_2 e^{p_2 t} + \dots + z_k e^{p_k t} + \dots \quad (3)$$

We are interested in the long term behavior of $x(t)$, i.e., to find

$$x_{ss}(t) = \lim_{t \rightarrow \infty} x(t) \quad \text{steady state response}$$

Note that every term in the expansion will have

$$\text{the form } z_k e^{p_k t} = \underbrace{z_k e^{\sigma_k t}}_{\text{exponential function}} \cdot \underbrace{e^{i\omega_k t}}_{\text{harmonic oscillation}}.$$

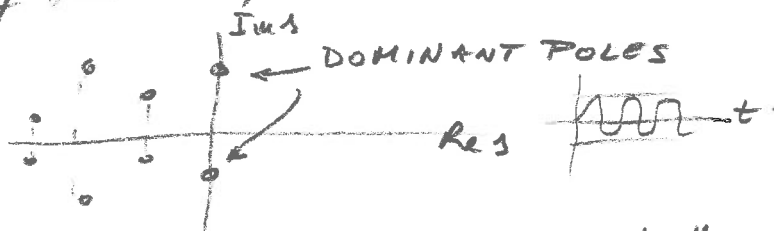
We distinguish the following possible cases:

A. If at least one σ_k is +ve ($\sigma_k > 0$) then $e^{\sigma_k t} \rightarrow \infty$, UNSTABLE system.

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B. If system is STABLE, i.e., all $\sigma_k \leq 0$,
(all poles in LHS) then:

B1. If terms with $\sigma_k = 0$ exist, then these
will survive as sustained oscillations
while the rest have died out. This means
that poles situated on the imaginary
axis, if they exist, are dominant poles



B2. If terms with $\sigma_k = 0$ do NOT exist, then
the dominant poles are the poles closest
to the imaginary axis because they die
hardest having small σ_k values (small
damping)

