

4 Time Series Response

4.1 Transfer Functions Block Diagrams

Transfer functions can be created using:

- Polynomial model (numerator / denominator)

$$G(s) = \frac{B(s)}{A(s)} = \frac{b_ms^m + b_{m-1}s^{m-1} + \dots + b_0}{a_ns^n + a_{n-1}s^{n-1} + \dots + a_0} \quad (4.1)$$

- n = order of transfer function models where $m < n$

- Zero-pole-gain model (numerator / denominator)

$$G(s) = k \frac{(s - z_1)(s - z_2) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)} \quad (4.2)$$

- zeros: z_1, z_2, \dots, z_m roots of $B(s) = 0$
- poles: p_1, p_2, \dots, p_n roots of $A(s) = 0$
- gain: k

- Time constant model

$$G(s) = \frac{k}{s^N} \cdot \frac{(T_a + 1)(T_b + 1) \cdots}{(T_1 + 1)(T_2 + 1) \cdots} \quad (4.3)$$

- N = type of transfer function model

4.1.1 Create transfer function using numerator and denominator coefficients

MATLAB can be used to create continuous-time single-input, single-output (SISO) transfer functions from their numerator and denominator coefficients using `tf`. To use the `tf` function, you must have the Control System Toolbox licensed and installed. To find out if you do, type: `ver control` in your Command Window or a script.

Method I

Create the transfer function

$$G(s) = \frac{s}{s^2 + 3s + 2} \quad (4.4)$$

Listing 1: MATLAB code for Method I.

```
1 num = [1 0];
2 dem = [1 3 2];
3 G = tf(num, dem)
4 % To use the tf function, you must have the Control System Toolbox
5 % licensed and installed. To find out if you do, type:
6 % ver control in your Command Window or a script.
```

where `num` and `dem` are the numerator and denominator polynomial coefficients in descending powers of s . For example, `dem = [1 3 2]` represents the denominator polynomial $\frac{s}{s^2 + 3s + 2}$.

`G` is a `tf` model object, which is a data container for representing the transfer function in polynomial form.

Method II

Alternatively, you can specify the transfer function $G(s)$ as an expression in s -domain.

1. Create a transfer Function model for the variable s
2. Specify $G(s)$ as a ratio of polynomials in s

Listing 2: MATLAB code for Method II.

```
1 s = tf('s');
2 G = s/(s^2+3*s+2)
```

Therefore, the full expression of $G(s)$ can be written as

$$G(s) = \frac{B(s)}{A(s)} = \frac{s}{s^2 + 3s + 2} = \frac{b_1s + b_0}{a_1s^2 + a_2s + a_0} \quad (4.5)$$

where

$$B(s) = b_1s + b_0 \quad (4.6)$$

resulting in $b_1 = 1$, and $b_0 = 0$; or, $B = [1 \ 0]$. Moreover,

$$A(s) = a_1s^2 + a_2s + a_0 \quad (4.7)$$

where $a_1 = 1$, $a_2 = 3$, and $a_0 = 1$; or, $A = [1 \ 3 \ 1]$.

4.1.2 Create transfer function using Zeros, Poles, and Gain

MATLAB can be used to create continuous-time single-input, single-output (SISO) transfer functions in factored form using `zpk`. Create the factored transfer function

$$G(s) = 5 \frac{s}{(s-1-i)(s-1-i)(s-2)} \quad (4.8)$$

Listing 3: MATLAB code to create a transfer function using Zeros, Poles, and Gain.

```
1 Z = 0;
2 P = [-1-1i -1+1i -2];
3 K = 5;
4 G = zpk(Z,P,K)
```

where Z and P are zeros and poles (the roots of the numerator and denominator respectively). K is the gain of the factored form. Solving for the poles p_1 , p_2 , and p_3 of $G(s)$;

$$G(s) = 5 \frac{s}{(s-1-i)(s-1-i)(s-2)} \quad (4.9)$$

$$= 5 \frac{s-0}{[s-(-1-i)][s-(-1-i)][s-(-2)]} \quad (4.10)$$

where $K = 5$, $s-0 = s-z_1 \rightarrow z_1 = 0$, and $Z = [0]$. Therefore,

$$[s-(-1-i)][s-(-1-i)][s-(-2)] = (s-p_1)(s-p_2)(s-p_3) \quad (4.11)$$

this leads to

$$G(s) = k \frac{s-z_1}{(s-p_1)(s-p_2)(s-p_3)} \quad (4.12)$$

therefore, $G(s)$ has a real pole at $s = -2$ and a pair of complex poles as $s = -1 \pm i$. The vector $P = [-1-1i \ -1+1i \ -2]$ specifies these pole locations.

4.2 Order versus Type

A system has a “Type” and an “Order”, which have different meanings.

- Order = n , highest exponent of s in the denominator. n is the number of poles.
- Type = N , exponent of the factored out s in the denominator. N is the number of poles in origin ($p = 0$).

Consider the 1st-order mass-damper system (no stiffness) as shown in figure X with the transfer function

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s} \quad (4.13)$$

The transfer function can easily be written in the basic form

$$G(s) = \frac{b_0}{a_2 s^2 + a_1 s} \quad (4.14)$$

where $a_2 = 1$, $a_1 = 2\zeta\omega_n$, $b_0 = \omega_n^2$. There the presence of a_2 means it's a 2nd order system. To solve for the type of the system, s must be factored out of the denominator, leading to:

$$\begin{aligned} G(s) &= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s} \\ &= \frac{\omega_n^2}{s(s + 2\zeta\omega_n)} \\ &= \frac{\frac{\omega_n}{2\zeta}}{s} \cdot \frac{1}{\frac{1}{2\zeta\omega_n}s + 1} \\ &= \frac{K}{s^N} \cdot \frac{1}{T_1 s + 1} \end{aligned} \quad (4.15)$$

where $K = \frac{\omega_n}{2\zeta}$, $N = 1$, and $T_1 = \frac{1}{2\zeta\omega_n}$. $N = 1$ means that is a Type 1 system. Therefore, this is a 2nd order system of Type 1, “Type” and “Order” have different meanings. Table 4.2 reports the types and orders for different transfer functions.

need to find some real-world examples of orders and types.

make a figure

Table 1: Examples of types and orders for different transfer functions.

transfer function	Type	Order
$G(s) = \frac{1}{Ts+1}$	0	1
$G(s) = \frac{1}{cs} = \frac{1/c}{s}$	1	1
$G(s) = \frac{1}{Js^2} = \frac{1/J}{s^2}$	2	2
$G(s) = \frac{1}{Js^2+cs} = \frac{K/c}{s} \cdot \frac{1}{J/c s+1}$	1	2
$G(s) = \frac{K}{J^4} \frac{T_a s + 1}{(T_1 s + 1)(T_2 s + 1)(T_3 s + 1)}$		
$= \frac{b_1 s + b_0}{s^4(a_3 s^3 + a_2 s^2 + a_1 s + a_0)}$	4	7
$= \frac{b_1 s + b_0}{(a_3 s^7 + a_2 s^6 + a_1 s^5 + a_0 s^4)}$		

4.3 Time Response

Time response calculations are obtained using the Laplace transforms where the Laplace transform is

$$X(s) = G(s)F(s) \quad (4.16)$$

and the time response is

$$x(t) = \mathcal{L}^{-1}[X(s)] \quad (4.17)$$

4.3.1 Time Series Response for a Step Function

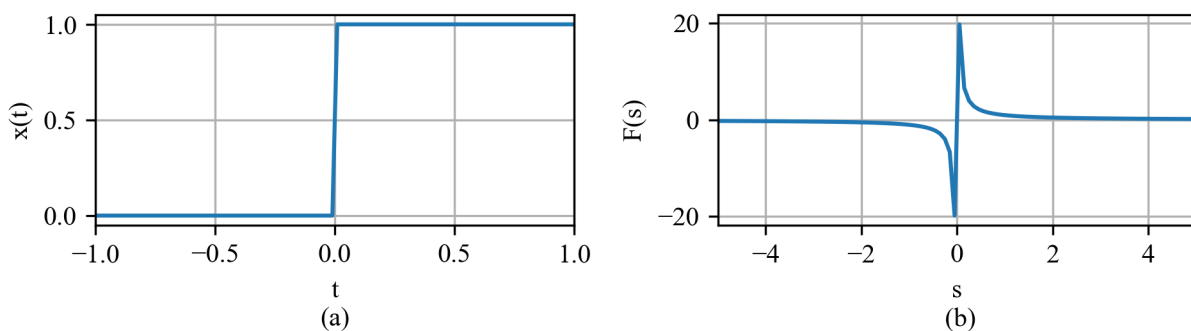


Figure 4.1: Step function; showing the (a) time domain and; (c) the s-space.

A step function is expressed as the following Laplace pair:

$$\text{LT pair} = \begin{cases} f(t) & 1(t), \quad t > 0 \\ F(s) & \frac{1}{s} \end{cases} \quad (4.18)$$

therefore, the time response of the systems is expressed as

$$x(t) = \mathcal{L} \left[G(s) \frac{1}{s} \right]^{-1} \quad (4.19)$$

In MATLAB, this is expressed as:

Listing 4: MATLAB code for the time-series response of a step function.

```
1 x_t = step(G)
```

4.3.2 Time Series Response for a Impulse Function

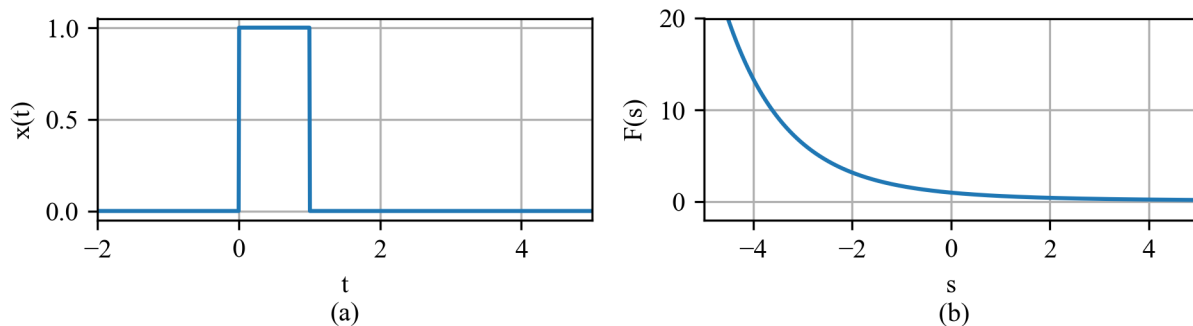


Figure 4.2: Pulse function; showing the (a) time domain and; (c) the s-space.

An impulse function is expressed as the following Laplace pair:

$$\text{LT pair} = \begin{cases} f(t) & p(t; \tau) \\ F(s) & \frac{1 - e^{-st\tau}}{s\tau} \end{cases} \quad (4.20)$$

therefore, the time response of the systems is expressed as

$$x(t) = \mathcal{L} \left[G(s) \frac{1 - e^{-st\tau}}{s\tau} \right]^{-1} \quad (4.21)$$

In MATLAB, this is expressed as:

Listing 5: MATLAB code for the time-series response of a step function.

```
1 x_t = impulse(G)
```

4.3.3 Time Series Response for a Ramp Function

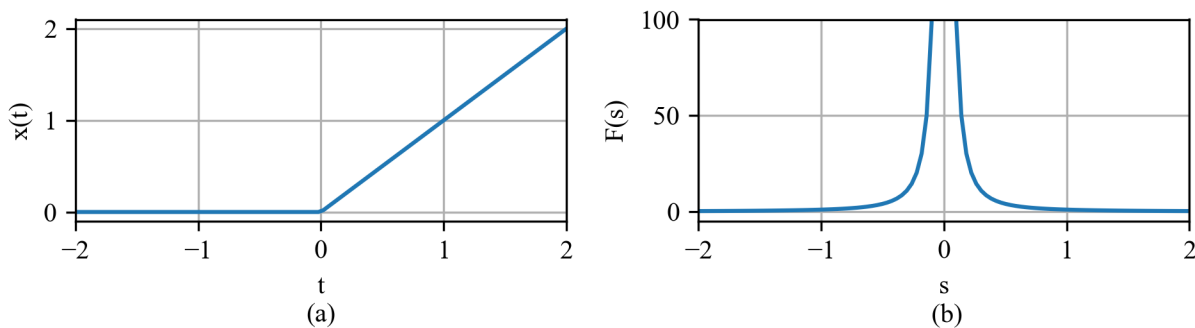


Figure 4.3: Ramp function; showing the (a) time domain and; (c) the s-space.

An ramp function is expressed as the following Laplace pair:

$$\text{LT pair} = \begin{cases} f(t) & t, \quad t > 0 \\ F(s) & \frac{1}{s^2} \end{cases} \quad (4.22)$$

therefore, the time response of the systems is expressed as

$$x(t) = \mathcal{L} \left[G(s) \frac{1}{s^2} \right]^{-1} \quad (4.23)$$

In MATLAB, this is expressed as:

Listing 6: MATLAB code for the time-series response of a step function.

```
1 x_t = impulse(G/(s^2))
```

Note that for the MATLAB code, we used the property:

$$X(s) = G(s) \frac{1}{s^2} = \left(\frac{G(s)}{s^2} \right) \cdot 1 \quad (4.24)$$

where 1 is the Laplace transform of an impulse. Note that ramp is not an option in MATLAB as this command is already used to generate a time-series ramp signal.

4.4 1st Order System Time Response

The first order equation of motion is

$$T\dot{x}(t) + x(t) = f(t) \quad (4.25)$$

where $x(0) = 0$ is the initial condition and T is a time constant for the first order system. The Laplace transform gives us

$$\begin{aligned} x &\rightarrow X(s) \\ \dot{x} &\rightarrow sX(s) \\ f(t) &\rightarrow F(s) \end{aligned} \quad (4.26)$$

therefore, the s-domain equation is:

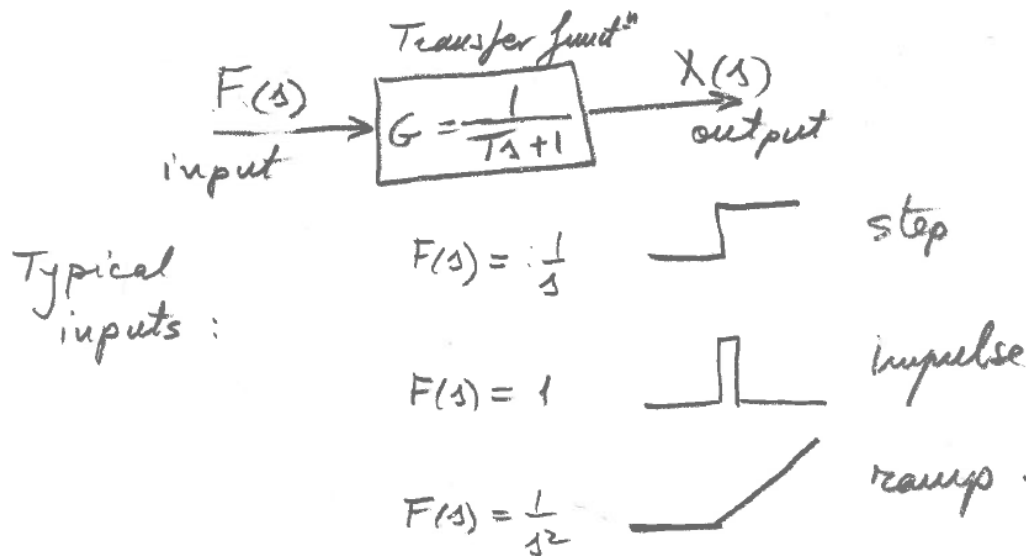
$$TsX(s) + X(s) = F(s) \quad (4.27)$$

where:

$$X(s) = \frac{F(s)}{Ts + 1} = \frac{1}{Ts + 1} F(s) = G(s)F(s) \quad (4.28)$$

therefore, the transfer function is:

$$G(s) = \frac{1}{Ts + 1} \quad (4.29)$$



4.4.1 Step response of a 1st Order System

$$X(s) = G(s)F(s) \quad (4.30)$$

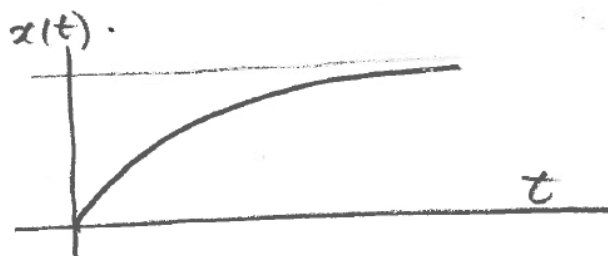
$$= \frac{1}{Ts + 1} \cdot \frac{1}{s}$$

$$= \frac{1}{s(Ts + 1)} \quad (4.31)$$

Therefore, solving for $\mathcal{L}[X(s)]^{-1}$ yields

$$x(t) = 1 - e^{-t/T} \quad (4.32)$$

or



4.4.2 Impulse response of a 1st Order System

$$X(s) = G(s)F(s) \quad (4.33)$$

$$= \frac{1}{Ts+1} \cdot 1$$

$$= \frac{1}{Ts+1} \quad (4.34)$$

Therefore, solving for $\mathcal{L}[X(s)]^{-1}$ yields

$$x(t) = \frac{1}{T}e^{-t/T} \quad (4.35)$$

or

4.4.3 Ramp response of a 1st Order System

$$X(s) = G(s)F(s) \quad (4.36)$$

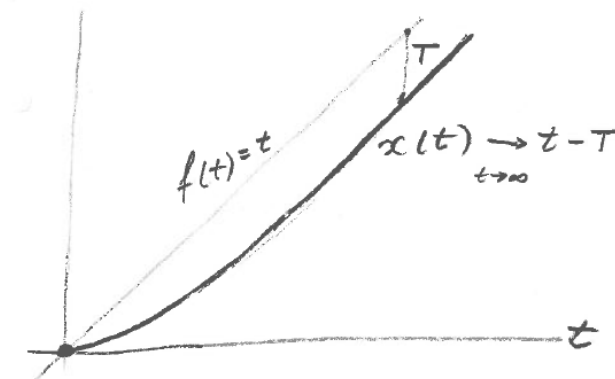
$$= \frac{1}{Ts+1} \cdot \frac{1}{s^2}$$

$$= \frac{1}{s^2(Ts+1)} \quad (4.37)$$

Therefore, solving for $\mathcal{L}[X(s)]^{-1}$ yields

$$\begin{aligned} x(t) &= t - T + Te^{-t/T} \\ &= t - T(1 - e^{-t/T}) \end{aligned} \quad (4.38)$$

or



Moreover,

$$x(t) = t - T + Te^{-t/T} \xrightarrow[t \rightarrow \infty]{} t - T \quad (4.39)$$

4.4.4 Summary of the First Order System Responses

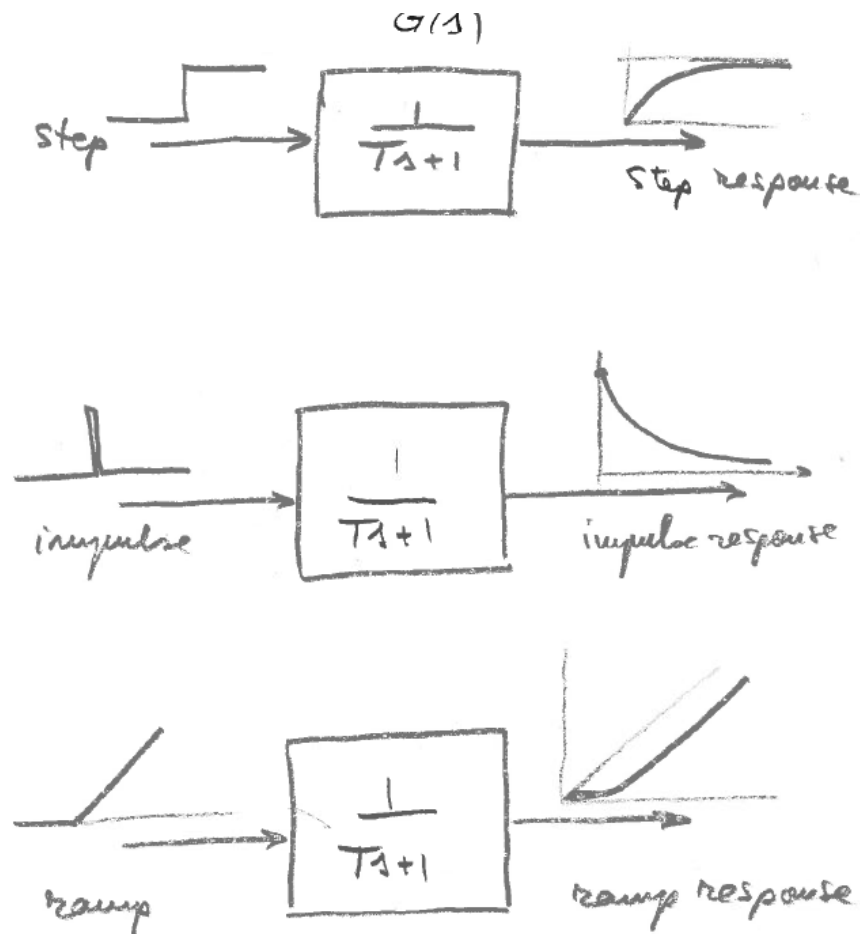


Figure 4.4: A summary of the first order system responses.

Listing 7: MATLAB code for time series responses of 1st order system.

```

1  %{
2  This program studies time response of 1st order systems
3  %}
4  clc
5  clear
6  close all
7
8  format compact
9
10 %% Given data
11 T=2.5; % time response for the 1D system
12
13 %% time range setup
14 T_max = 10;           % run the test to 10 seconds
15 dt = T_max*1e-4;      % find the delta-t value
16 t = 0:dt:T_max;       % build the time vector
17
18 %% Define system
19 B = [1];
20 A = [T 1];
21 G = tf(B,A);
22
23 %% create the figure environment
24 figure(1)
25
26 %% step response
27 subplot(3,1,1)
28 hold on
29 step(G,t)
30 ylim([0 1.2])
31
32 %% Impulse response
33 subplot(3,1,2)
34 impulse(G,t)
35
36 %% Ramp response
37 F_ramp = tf([1],[1 0 0])
38 subplot(3,1,3)
39 impulse(G*F_ramp,t)
40 title('Ramp Response') % need to set manually.

```

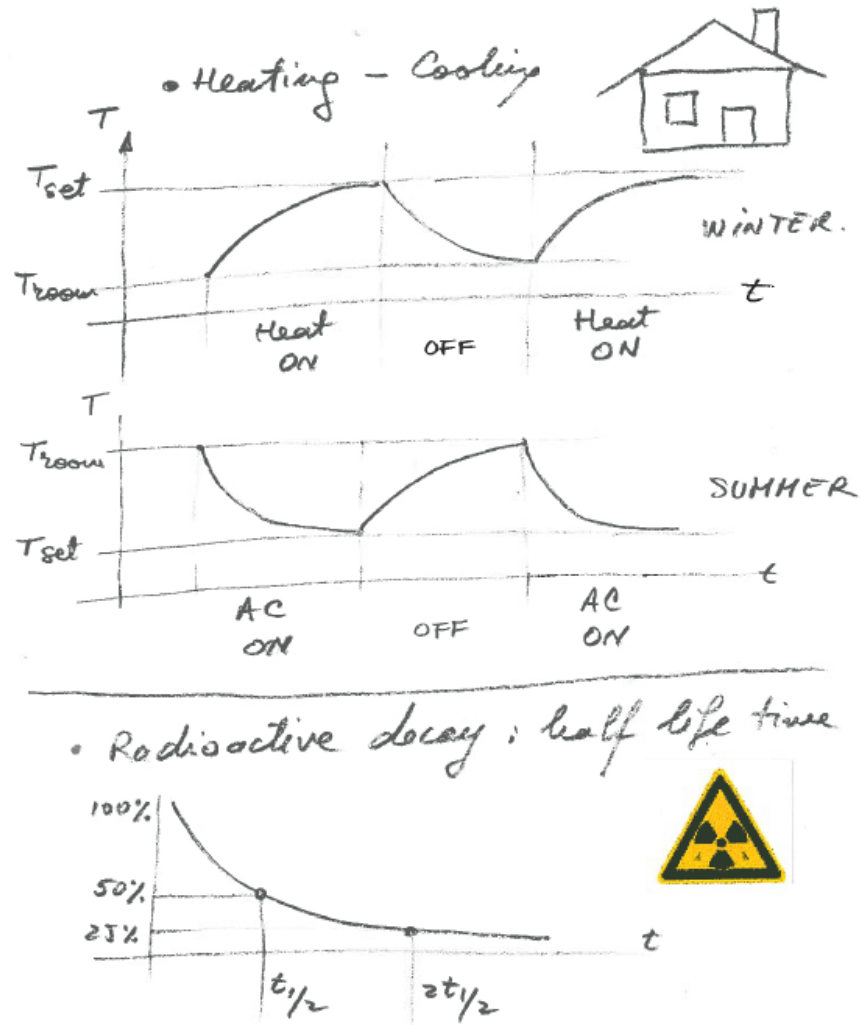


Figure 4.5: Examples of first order systems in the real world.

4.5 2nd Order System Time Response

The ordinary differential equation for the equation of motion of a 2nd order system can be expressed as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = \omega_n^2f(t) \quad (4.40)$$

with the initial conditions $\ddot{x}(0) = 0$, and $\dot{x}(0) = 0$. The Laplace transform gives us

$$\begin{aligned} x &\rightarrow X(s) \\ \dot{x} &\rightarrow sX(s) \\ \ddot{x} &\rightarrow s^2X(s) \\ f(t) &\rightarrow F(s) \end{aligned} \quad (4.41)$$

Taking the Laplace transform of the equation of motion yields

$$s^2X(s) + 2\zeta\omega_nsX(s) + \omega_n^2X(s) = \omega_n^2F(s) \quad (4.42)$$

Pulling $X(s)$ out of the first equation results in

$$(s^2 + 2\zeta\omega_n s + \omega_n^2)X(s) = \omega_n^2 F(s) \quad (4.43)$$

next, we can solve for $X(s)$

$$X(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} F(s) \quad (4.44)$$

As $X(s) = G(s)F(s)$, it we can show that

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (4.45)$$

4.5.1 Step Response for a 2nd Order System

$$X(s) = G(s)F(s) \quad (4.46)$$

$$\begin{aligned} &= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s} \\ &= \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \end{aligned} \quad (4.47)$$

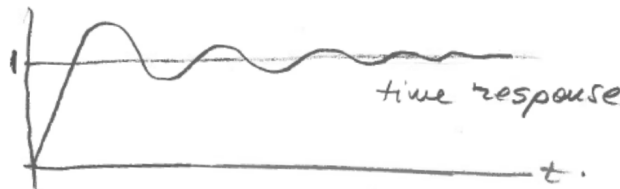
Therefore, solving for $\mathcal{L}[X(s)]^{-1}$ yields

$$x(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) \quad (4.48)$$

where

$$\begin{aligned} \phi &= \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta} \\ &= \sin^{-1} \sqrt{1-\zeta^2} \end{aligned} \quad (4.49)$$

or



Proof

$$7] x_c(t) = e^{-\zeta \omega_n t} (A \cos \omega_d t + B \sin \omega_d t) ; A, B \quad \text{ODE derivation}$$

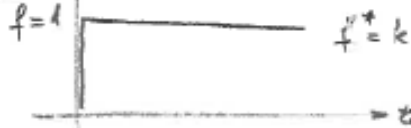
$$= e^{-\zeta \omega_n t} C \sin(\omega_d t + \varphi) ; C, \varphi$$

$$\ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = \omega_n^2 f(t) ; \omega_n^2 f = \frac{f^*}{m} \rightarrow f = \frac{f^*}{k}$$

Step input, $f(t) = 1(t)$

$$x_p(t) = D, \quad \dot{x}_p - \ddot{x}_p = 0$$

$$\omega_n^2 D = \omega_n^2 \rightarrow D = 1$$



$$x_p(t) = 1$$

$$x(t) = e^{-\zeta \omega_n t} C \sin(\omega_d t + \varphi) + 1$$

$$\dot{x}(t) = (-\alpha \sin \varphi + \omega_d \cos \varphi) C e^{-\alpha t}$$

$$x(0) = C \sin \varphi + 1 = 0$$

$$\dot{x}(0) = (-\alpha \sin \varphi + \omega_d \cos \varphi) = 0 \rightarrow \tan \varphi = \frac{\omega_d}{\alpha}$$

$$\tan \varphi = \frac{\omega_n \sqrt{1-\zeta^2}}{\zeta \omega_n} = \frac{\sqrt{1-\zeta^2}}{\zeta} \rightarrow \varphi = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\sin^2 \varphi = \frac{\tan^2 \varphi}{1 + \tan^2 \varphi} = \frac{1 - \zeta^2}{\zeta^2 + 1 - \zeta^2} = 1 - \zeta^2$$

$$\sin \varphi = \sqrt{1 - \zeta^2} \quad \varphi = \sin^{-1} \sqrt{1 - \zeta^2}$$

$$C = -\frac{1}{\sin \varphi} = -\frac{1}{\sqrt{1 - \zeta^2}}$$

$$x(t) = 1 - \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \sin(\omega_d t + \varphi) \quad \text{unit response}$$

for $f^* = k$

For $f^* = F_0$

$$x(t) = \frac{F_0}{k} \left[1 - \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \sin(\omega_d t + \varphi) \right] = x_{st} \cdot x(t)$$

$$x_{st} = F_0/k \quad \text{static displacement}$$

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14) Step response $f(t) = 1(t)$, $F(s) = \frac{1}{s}$

$$X(s) = \frac{\omega_n^2}{(s+\alpha)^2 + \omega_d^2} \cdot \frac{1}{s} = \frac{A}{s} + \frac{Ds+E}{(s+\alpha)^2 + \omega_d^2}$$

ILT by
partial
fraction
expansion

$$A(s+\alpha)^2 + A\omega_d^2 + Ds^2 + Es = \omega_n^2$$

$$A(s^2 + 2s\alpha + \alpha^2 + \omega_d^2) + Ds^2 + Es = 1$$

$$s^2: A + D = 0$$

$$s^1: 2\alpha A + E = 0$$

$$s^0: A(\alpha^2 + \omega_d^2) = \omega_n^2 \quad A(\cancel{s^2}\omega_n^2 + (1-\cancel{s^2})\omega_n^2) = \omega_n^2$$

$$A = 1$$

$$D = -A = -1$$

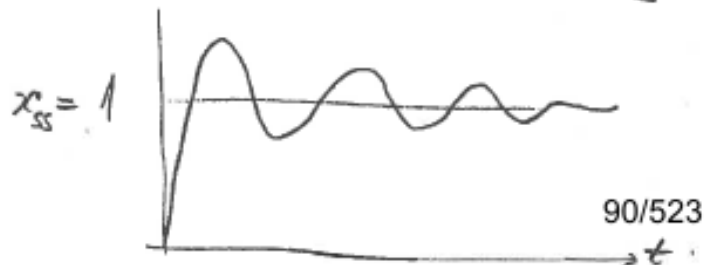
$$E = -2\alpha A = -2\alpha$$

$$X(s) = \left[\frac{1}{s} - \frac{s+2\alpha}{(s+\alpha)^2 + \omega_d^2} \right] = \left[\frac{1}{s} - \frac{\alpha + (s+\alpha)}{(s+\alpha)^2 + \omega_d^2} \right]$$

$$x(t) = 1 - e^{-\alpha t} \left[\frac{\alpha}{\omega_d} \sin \omega_d t + \cos \omega_d t \right]$$

$$x(t) = 1 - e^{-\alpha t} \left(\frac{s}{\sqrt{1-\zeta^2}} \sin \omega_d t + \cos \omega_d t \right)$$

$$x(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta \omega_n t} \sin(\omega_d t + \varphi) \quad \varphi = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$$



4.5.2 Impulse response of a 2nd Order System

$$X(s) = G(s)F(s) \quad (4.50)$$

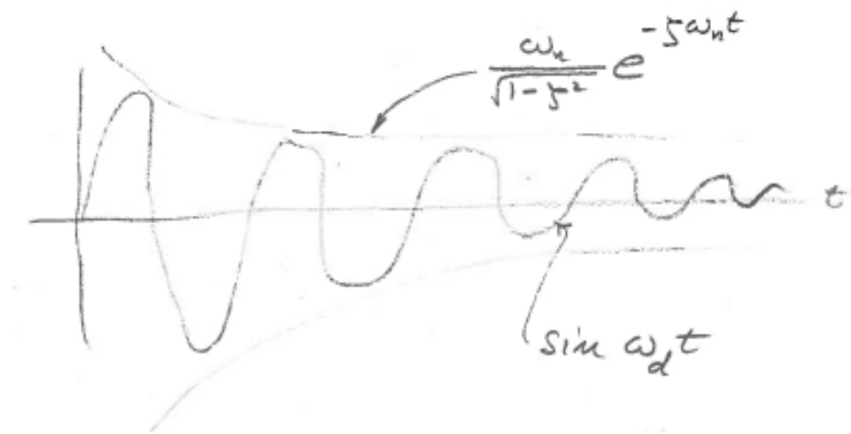
$$= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot 1$$

$$= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (4.51)$$

Therefore, solving for $\mathcal{L}[X(s)]^{-1}$ yields

$$x(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t) \quad (4.52)$$

or



Proof

13a) Long hand solution

ILT by partial fraction expansion

$$\frac{X}{\omega_n^2} = \frac{1}{s^2(s+p_1)(s+p_2)} = \frac{A}{s+p_1} + \frac{B}{s+p_2}$$

$$As + Ap_2 + Bs + Bp_1 = 1$$

$$s^1: A + B = 0 \rightarrow B = -A$$

$$s^0: Ap_2 + Bp_1 = 1 \rightarrow A = \frac{1}{p_2 - p_1}, B = \frac{1}{p_1 - p_2}$$

$$\frac{1}{(s+p_1)(s+p_2)} = \frac{1}{p_2 - p_1} \left(\frac{1}{s+p_1} - \frac{1}{s+p_2} \right)$$

$$x(t) = \frac{1}{p_2 - p_1} \left(e^{-p_1 t} - e^{-p_2 t} \right)$$

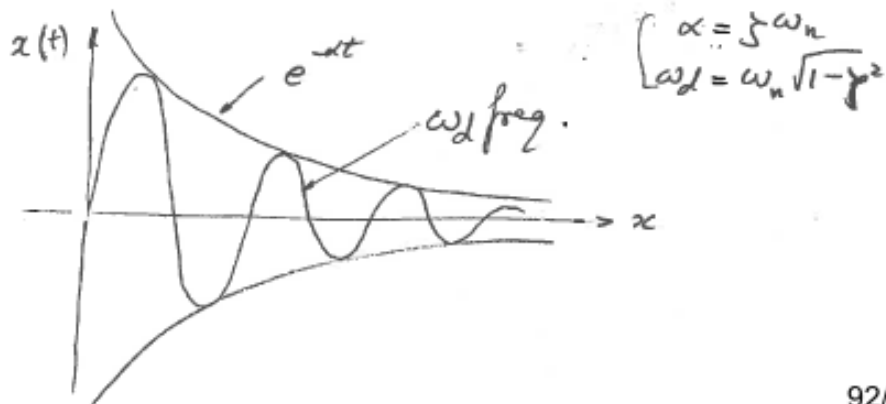
$$p_{1,2} = \zeta \omega_n \mp i \omega_n \sqrt{1 - \zeta^2} = \alpha \mp i \omega_d$$

$$p_2 - p_1 = i 2 \omega_d$$

$$\frac{x(t)}{\omega_n^2} = \frac{-1}{2i \omega_d} \left[e^{-(\alpha - i \omega_d)t} - e^{-(\alpha + i \omega_d)t} \right]$$

$$= \frac{1}{2i \omega_d} e^{-\alpha t} \left[e^{i \omega_d t} - e^{-i \omega_d t} \right]$$

$$= \frac{1}{\omega_d} e^{-\alpha t} \frac{e^{i \omega_d t} - e^{-i \omega_d t}}{2i} = \frac{e^{-\alpha t}}{\omega_d} \sin \omega_d t$$



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Alternative way

$$\frac{1}{(s+p_1)(s+p_2)} \xrightarrow{\mathcal{L}^{-1}} \frac{1}{p_2-p_1} (e^{-p_1 t} - e^{-p_2 t})$$

Another way Residue Theorem

$$\frac{1}{(s+p_1)(s+p_2)} = \frac{a_1}{s+p_1} + \frac{a_2}{s+p_2}$$

$$a_1 = \left[(s+p_1) \frac{1}{(s+p_1)(s+p_2)} \right]_{s=-p_1} = \frac{1}{-p_1+p_2}$$

$$a_2 = \left[(s+p_2) \frac{1}{(s+p_1)(s+p_2)} \right]_{s=-p_2} = \frac{1}{p_1-p_2} = -a_1$$

4.5.3 Ramp response of a 2nd Order System

$$X(s) = G(s)F(s) \quad (4.53)$$

$$= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s^2}$$

$$= \frac{\omega_n^2}{s^2(s^2 + 2\zeta\omega_n s + \omega_n^2)} \quad (4.54)$$

Therefore, solving for $\mathcal{L}[X(s)]^{-1}$ yields

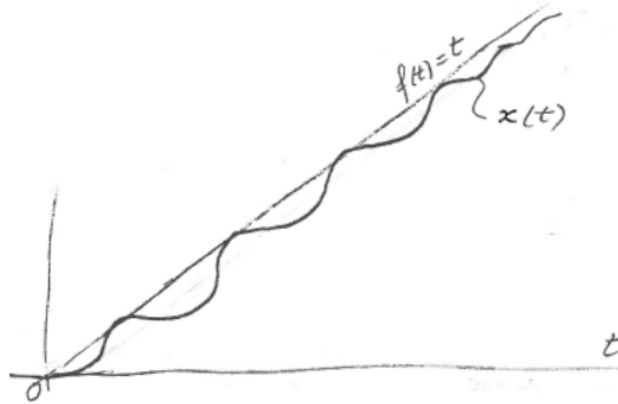
$$x(t) = t - \frac{2\zeta}{\omega_n} \left(1 + \frac{1}{\sin \gamma_1} e^{-\zeta\omega_n t} \sin(\omega_d t - \gamma_1) \right) \quad (4.55)$$

where

$$\gamma_1 = \tan^{-1} \frac{2\zeta \sqrt{1-\zeta^2}}{1-2\zeta^2} \quad (4.56)$$

$$= \sin^{-1} 2\zeta \sqrt{1-\zeta^2} \quad (4.57)$$

or



Moreover,

$$\begin{aligned} \sin^2 \gamma_1 &= \frac{\tan^2 \gamma_1}{1 + \tan^2 \gamma_1} \\ &= \frac{4s^2(1-\zeta^2)}{(1-2\zeta^2) + 4\zeta^2(1-\zeta^2)} \\ &= \frac{4s^2(1-\zeta^2)}{1-4s^2+4s^4+4s^2-4s^4} \\ &= 4s^2(1-\zeta^2) \end{aligned} \quad (4.58)$$

and

$$\sin \gamma_1 = 2\zeta \sqrt{1-\zeta^2} \quad (4.59)$$

Proof

PROOF of 2nd order system Ramp response

ODE solution

$$x_p(t) = Dt + E$$

$$\dot{x}_p = D; \ddot{x}_p = 0$$

$$\ddot{x}_p + 2\zeta\omega_n \dot{x}_p + \omega_n^2 x_p = \omega_n^2 f(t)$$

$$2\zeta\omega_n D + \omega_n^2 (Dt + E) = \omega_n^2 t$$

$$t^0: 2\zeta\omega_n D + \omega_n^2 E = 0 \rightarrow E = -\frac{2\zeta}{\omega_n} D$$

$$t^1: \omega_n^2 D = \omega_n^2 \rightarrow D = 1 \rightarrow E = -\frac{2\zeta}{\omega_n}$$

$$x_p(t) = t - \frac{2\zeta}{\omega_n}$$

$$x(t) = e^{-\zeta\omega_n t} C \sin(\omega_d t + \varphi_1) + t - \frac{2\zeta}{\omega_n}$$

$$\dot{x}(t) = (-\zeta\omega_n \sin\varphi_1 + \omega_d \cos\varphi_1) C e^{-\zeta\omega_n t} + 1$$

$$x(0) = 0: C \sin\varphi_1 - \frac{2\zeta}{\omega_n} = 0 \rightarrow C \omega_n \sin\varphi_1 = 2\zeta \quad (a)$$

$$\dot{x}(0) = 0: (-\zeta\omega_n \sin\varphi_1 + \omega_d \cos\varphi_1) C + 1 = 0$$

$$C(\omega_d \cos\varphi_1 - \zeta\omega_n \sin\varphi_1) = 1$$

$$\omega_d \cos\varphi_1 - \zeta\omega_n \sin\varphi_1 = \frac{1}{C}$$

$$(a): \quad 2\zeta^2 - C \omega_d \cos\varphi_1 = 1$$

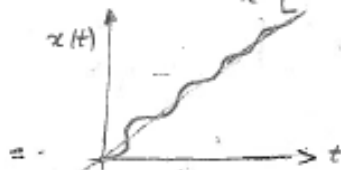
$$C \omega_n \sqrt{1-\zeta^2} \cos\varphi_1 = 2\zeta^2 - 1$$

$$C \omega_n \cos\varphi_1 = -\frac{1-2\zeta^2}{\sqrt{1-\zeta^2}} \quad (b)$$

$$\frac{(a)}{(b)} = \tan\varphi_1 = -\frac{2\zeta\sqrt{1-\zeta^2}}{1-2\zeta^2}$$

$$\varphi_1 = -\tan^{-1} \frac{2\zeta\sqrt{1-\zeta^2}}{1-2\zeta^2}, \quad C = \frac{2\zeta}{\omega_n} \cdot \frac{1}{\sin\varphi_1}$$

$$x(t) = t - \frac{2\zeta}{\omega_n} \left[1 - \frac{1}{\sin\varphi_1} e^{-\zeta\omega_n t} \sin(\omega_d t + \varphi_1) \right]$$



Note: φ_1 is different from $\varphi = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$ of step response

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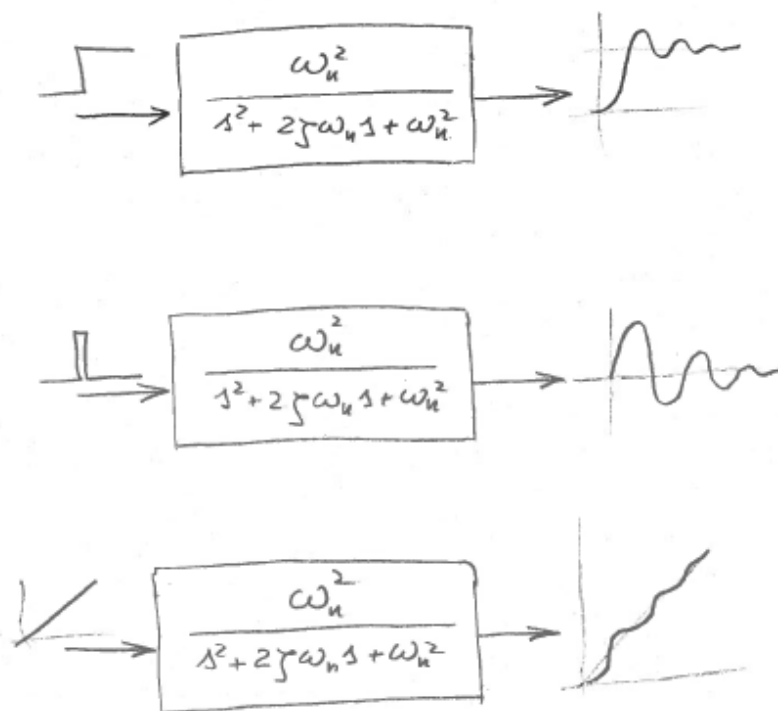
4.5.4 Summary of the Second Order System Responses

Figure 4.6: A summary of the second order system responses.

Listing 8: MATLAB code for time series responses of 2nd order system.

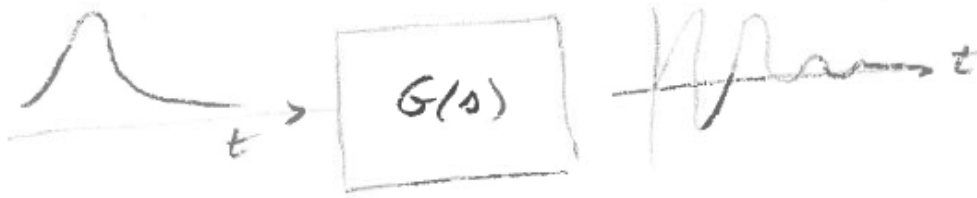
```

1  %%
2  This program studies time response of 2nd order systems
3  %%
4
5  clc
6  clear
7  close all
8
9  format compact
10
11 %% Given data
12 fn = 5; % time response for the 1D system
13 wn = 2*pi()*fn
14 z = 0.035
15
16 %% time range setup
17 T_max = 10; % run the test to 10 seconds
18 dt = T_max*1e-4; % find the delta-t value
19 t = 0:dt:T_max; % build the time vector
20
21 %% Define system
22 B = [wn^2];
23 A = [1 2*z*wn wn^2];
24 G = tf(B,A);
25
26 %% create the figure enviornment
27 figure(1)
28 xlim([0 1])
29
30 %% step response
31 subplot(3,1,1)
32 hold on
33 step(G,t)
34 ylim([0 2])
35
36 %% Impulse response
37 subplot(3,1,2)
38 impulse(G,t)
39 ylim([-35 35])
40
41 %% Ramp response
42 F_ramp = tf([1],[1 0 0])
43 subplot(3,1,3)
44 impulse(G*F_ramp,t)
45 xlim([0 1])
46 ylim([0 1])
47 title('Ramp Response') % need to set manually.

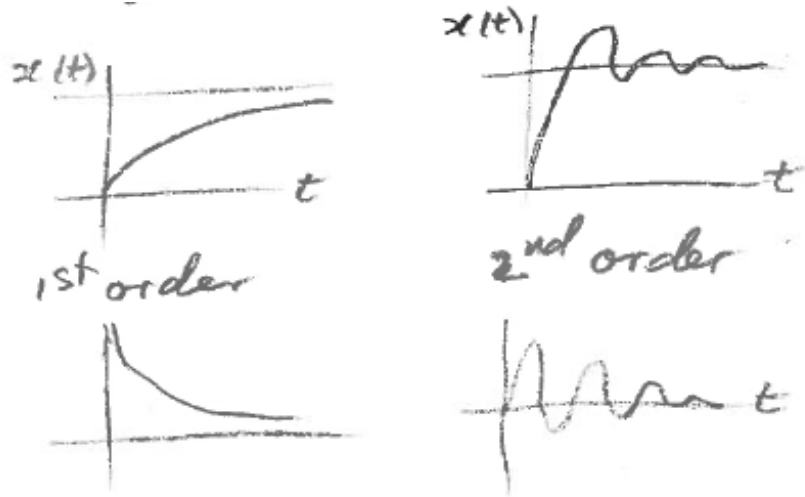
```

4.6 Stability of response

A system is stable if any stable input excitation produces a stable output response.



A response is stable if it remains bounded at $t \rightarrow \infty$



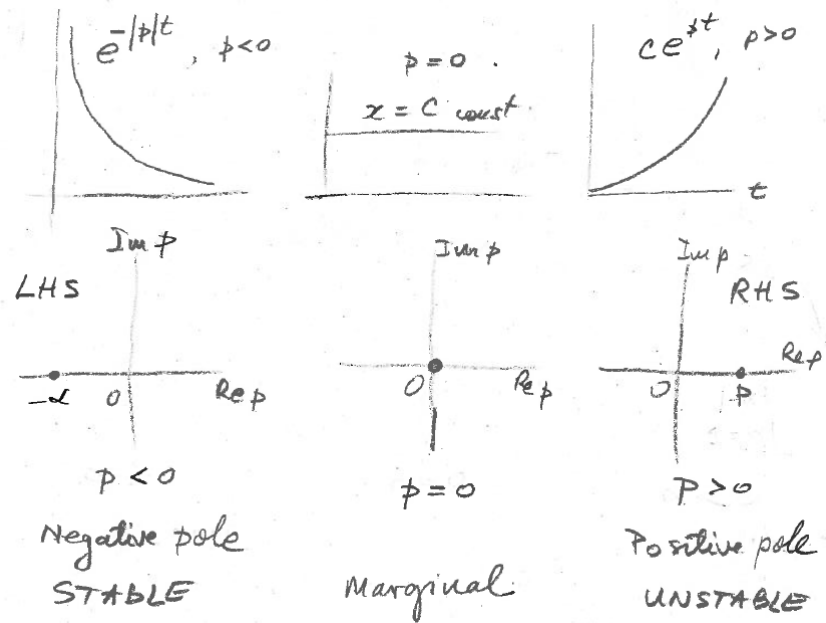
4.6.1 Stability of 1st-Order Responses

$$X(s) = \frac{K}{s - p} \quad (4.60)$$

$$x(t) = Ke^{pt} \quad (4.61)$$

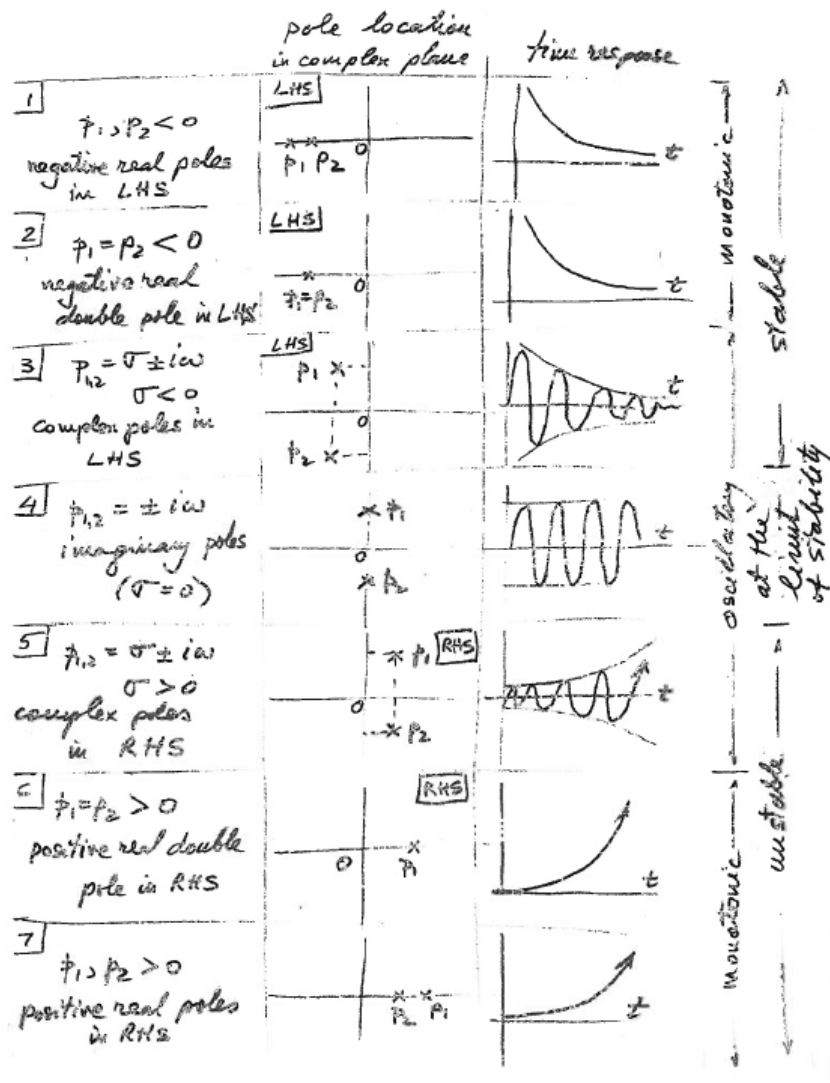
where p is the pole of $X(s)$.

The stability is dictated by the sign of p , i.e. its location in the complex p -plane. If $P < 0$ (or p is in the left-hand-side), the system is stable. Therefore, if a disturbing force is applied, the system will return to its initial state.



4.6.2 Stability of 2nd-Order Responses

$$X(s) = \frac{k(s - z_1)}{(s - p_1)(s - p_2)} \quad (4.62)$$



4.6.3 Stability of Higher-order Responses

Starting with a general expression for the output of a system

$$X(s) = \frac{B(s)}{A(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_0} \quad (4.63)$$

partial fraction expansion results in

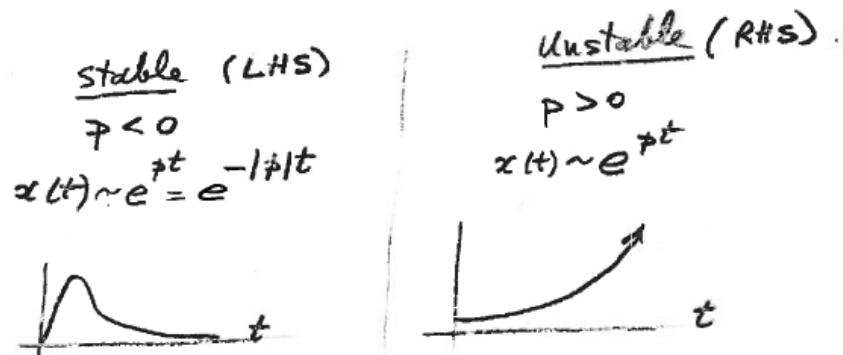
$$X(s) = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + \dots + \frac{r_n}{s - p_n} \quad (4.64)$$

where p_1, p_2, \dots, p_n are the poles of the system (i.e. roots of $A(s) = 0$) and r_1, r_2, \dots, r_n are the residues of the system. Note that the poles can either be real or complex. Again, MATLAB can be used to solve for the roots, poles, and gains of the system using $[r, p, k] = \text{residue}(B, A)$. The real poles can be

- single pole: $\frac{r}{s-p} \xrightarrow{\mathcal{L}^{-1}} re^{pt}$

- double poles: $\frac{r}{(s-p)^2} \xrightarrow{\mathcal{L}^{-1}} r t e^{pt}$
- multiple poles: $\frac{r}{(s-p)^j} \xrightarrow{\mathcal{L}^{-1}} \frac{1}{(j-1)!} t^{j-1} e^{pt}$

where stable and unstable responses are



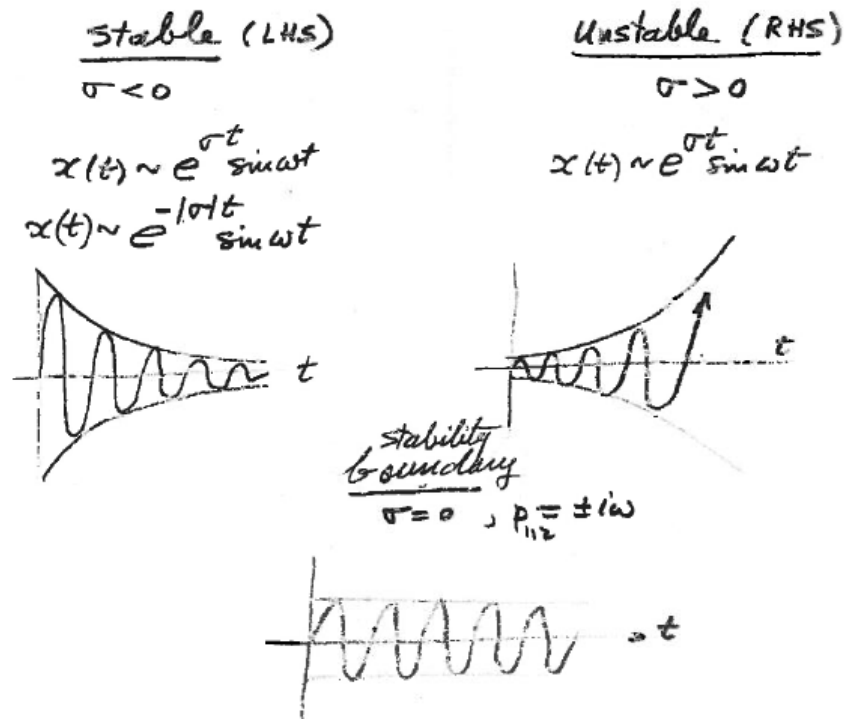
The complex poles are always in conjugate pairs, and are governed by $p_{1,2} = \sigma \pm i\omega$ where

- first pole: $\frac{1}{s-p_1} = \frac{1}{s-(\sigma+i\omega)} \xrightarrow{\mathcal{L}^{-1}} e^{(\sigma+i\omega)t} = e^{\sigma t} e^{i\omega t}$
- second pole: $\frac{1}{s-p_2} = \frac{1}{s-(\sigma-i\omega)} \xrightarrow{\mathcal{L}^{-1}} e^{(\sigma-i\omega)t} = e^{\sigma t} e^{-i\omega t}$

Using Euler's formula, this results in

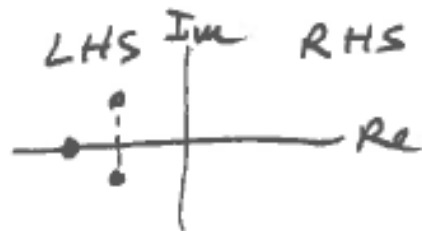
$$x(t) = e^{\sigma t} (e^{i\omega t} - e^{-i\omega t}) = e^{\sigma t} \sin(\omega t) \quad (4.65)$$

where stable and unstable responses are



4.6.4 Absolute Stability

A necessary and sufficient condition for a system to be stable is that its poles are placed in the Left hand side of the complex plane.



4.6.5 Marginal Stability

If the poles are purely imaginary (i.e. placed on the imaginary axis) then the system has marginal stability.

- bounded impulse response, the system is stable
- unbounded impulse response, or other inputs, the system is not stable.

4.6.6 Relative Stability

Would a stable system still be stable if its parameters are slightly changed? What margin of safety is there?