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PERFORMANCE INDICATORS

Generic PIs:

- Steady state value $x_{ss} = \lim_{t\to\infty} x(t)$

- steady state error

$$e(t) = f(t) - x(t)$$

excitation (forcing function) — response

$$e_{ss} = \lim_{t\to\infty} e(t).$$

Specific PIs

Specific PIs depend on system order and on excitation type.

Examples of specific PIs:

- rise time
- delay time
- settling time
- maximum overshoot.
- decay time / half-time

etc.

1st order system PIs

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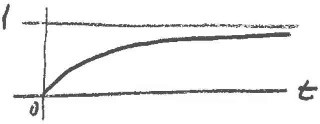
1^{st} Order System x_{ss} and e_{ss}

$$T\dot{x} + x = f(t) \; ; \quad G(s) = \frac{1}{Ts+1}$$

Step $f(t) = 1$



$f(t) = 1 \; , t > 0$

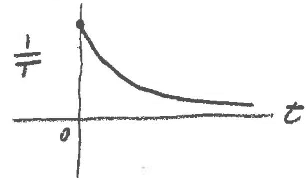
$$x(t) = 1 - e^{-t/T} ; \quad \boxed{x_{ss} = \lim_{t\to\infty} x(t) = 1}$$

$$e(t) = 1 - (1 - e^{-t/T}) = e^{-t/T}$$

steady state error

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{t\to\infty} e^{-t/T} = \boxed{0}$$

Impulse $f(t) = \delta(t)$



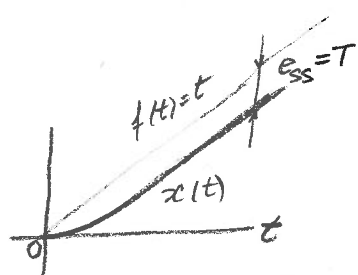
$f(t) = \delta(t)$

$$x(t) = \frac{1}{T} e^{-t/T} ; \quad \boxed{x_{ss} = 0}$$

$$e(t) = \delta(t) - \frac{1}{T} e^{-t/T} .$$

$$e_{ss} = \lim_{t\to\infty} e(t) = \boxed{0}$$

Ramp $f(t) = t$



$f(t) = t$

$$x(t) = t - T(1 - e^{-t/T}) \qquad \boxed{x_{ss} = \infty} \quad \text{NO steady state value}$$

$$e(t) = T(1 - e^{-t/T}) .$$

$$e_{ss} = \lim_{t\to\infty} e(t) = T - T \lim_{t\to\infty} \cancel{e^{-t/T}}^{\,0} = \boxed{T}$$

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1st order system

x_{ss} and e_{ss} by FVT

Recall: Final Value Theorem $\quad x_{ss} = \lim_{s\to 0} s\,X(s)$

$$G(s) = \frac{1}{Ts+1} \qquad e_{ss} = \lim_{s\to 0} s\,E(s)$$

$$X(s) = G(s)\,F(s) = \frac{1}{Ts+1}F(s)$$

$$E(s) = F(s) - X(s) = F(s) - G(s)F(s)$$

$$E(s) = \left(1 - G(s)\right)F(s) = \frac{Ts}{Ts+1}F(s)$$

Step $\quad F(s) = 1/s$

$$X(s) = \frac{1}{Ts+1}\cdot\frac{1}{s}, \qquad x_{ss} = \lim_{s\to 0} \cancel{s}\,\frac{1}{Ts+1}\cdot\frac{1}{\cancel{s}} = 1$$

$$E(s) = \frac{T\cancel{s}}{Ts+1}\cdot\frac{1}{\cancel{s}} \qquad e_{ss} = \lim_{s\to 0} s\,\frac{T}{Ts+1} = 0$$

Impulse $\quad F(s) = 1$

$$X(s) = \frac{1}{Ts+1}, \qquad x_{ss} = \lim_{s\to 0} s\,\frac{1}{Ts+1} = 0$$

$$E(s) = \frac{Ts}{Ts+1}, \qquad e_{ss} = \lim_{s\to 0} s\,\frac{Ts}{Ts+1} = 0$$

Ramp $\quad F(s) = 1/s^2$

$$X(s) = \frac{1}{Ts+1}\cdot\frac{1}{s^2}, \qquad x_{ss} = \lim_{s\to 0} \cancel{s}\,\frac{1}{Ts+1}\cdot\frac{1}{s^{\cancel{2}}} \to \infty$$

No ss value!

$$E(s) = \frac{T\cancel{s}}{Ts+1}\,\frac{1}{s^{\cancel{2}}}, \qquad e_{ss} = \lim_{s\to 0} \cancel{s}\,\frac{T}{Ts+1}\,\frac{1}{\cancel{s}} = T$$

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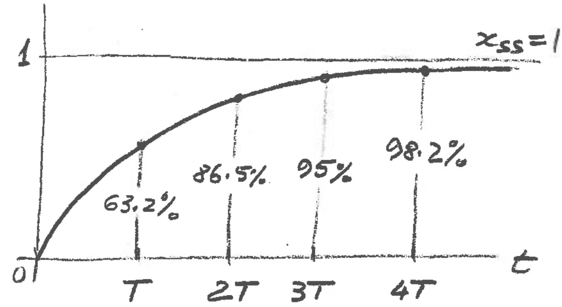
1^{st} order system specific PIs

Step response

Specific PIs

$$x(t) = 1 - e^{-t/T}$$



Rise time, t_r

- rise to 95% of x_{ss} $\quad t_{r\_95\%} = 3T$
- —— 98% ———— $\quad t_{r\_98\%} \simeq 4T$

In general, $1 - e^{-t/T} = x \longrightarrow t = -T \ln(1-x)$

Delay time, t_d

- Rise to 50% of x_{ss}

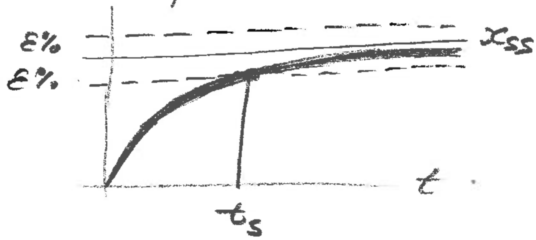
$$x(t) = 1 - e^{-t/T} = 0.5$$

$$e^{-t/T} = 0.5 \longrightarrow t_d = -T \ln 0.5 = 0.693T$$

$$\approx 0.7T$$

Settling time t_s

Time to get within ε% of x_{ss}



- $\varepsilon\% = 2\%$

$$t_s^{2\%} \approx 4T$$

- $\varepsilon\% = 5\%$

$$t_s^{5\%} = 3T$$

$$t_s^{\varepsilon} = -T \ln \varepsilon$$

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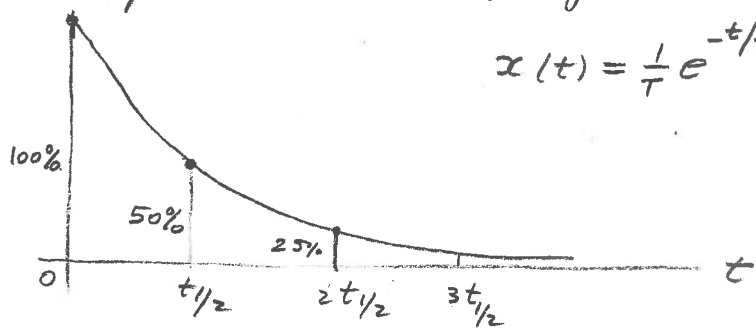
Impulse response specific PI

$$x(t) = \frac{1}{T} e^{-t/T}$$



Half-life decay time

$$e^{-t/T} = \frac{1}{2} \longrightarrow t_{1/2} = -T \ln \frac{1}{2} \cong 0.693T$$

Note : signal continues to decay by half of its value after each additional $t_{1/2}$.

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More precise estimates

Rise time to x:

$$x_{ss}(1 - e^{-t/T}) = x$$

$$e^{-t/T} = 1 - x/x_{ss}$$

$$-t/T = \ln(1 - x/x_{ss})$$

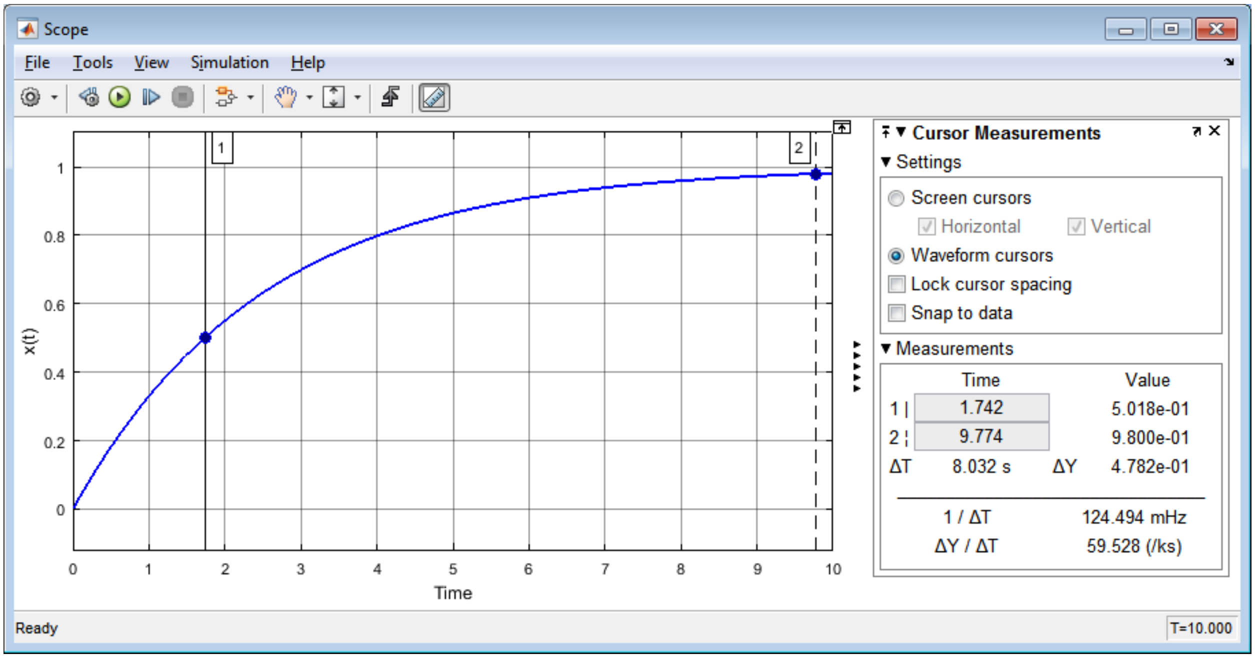
$$t = -T \ln\left(1 - \frac{x}{x_{ss}}\right)$$

4.3 Use Trace to Measure Performance Indicators

Press the ‘Cursor Measurements’ button to activate the cursors.

4.3.1 Delay time t_d measurement

Place the first cursor around the point where ‘Value’ measurement is closest to 0.5. $x(t)=0.5$. Read the ‘Time’ value. This is estimate for t_d. It gives the value $t_d = 1.742$ sec.



4.3.2 Settling time t_s measurement

The second cursor can be used to get the settling time t_s. We are going to use the 2% definition of t_d. This means that the response should be around 0.98, or 9.8e-1. Reading the corresponding time value, we get $t_s = 9.774$ sec.

2nd order system PIs

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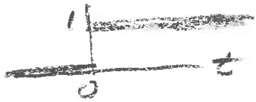
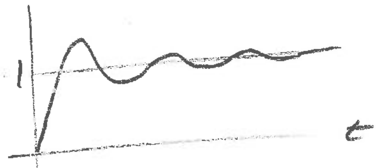
2^{nd} order system x_{ss} and e_{ss}

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 f(t) \;;\quad G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

step $f(t) = 1$, $t > 0$





$$x(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \varphi)$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \qquad \varphi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \sin^{-1}\sqrt{1-\zeta^2}$$

$$x_{ss} = \lim_{t\to\infty} x(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} \lim_{t\to\infty} \cancelto{0}{e^{-\zeta\omega_n t}} \sin(\omega_d t + \varphi)$$

$$x_{ss}^{step} = 1$$

$$e(t) = f(t) - x(t) = \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \varphi)$$

$$e_{ss} = \lim_{t\to\infty} e(t) = \frac{1}{\sqrt{1-\zeta^2}} \lim_{t\to\infty} \cancelto{0}{e^{-\zeta\omega_n t}} \sin(\omega_d t + \varphi) = 0$$

$$e_{ss}^{step} = 0$$

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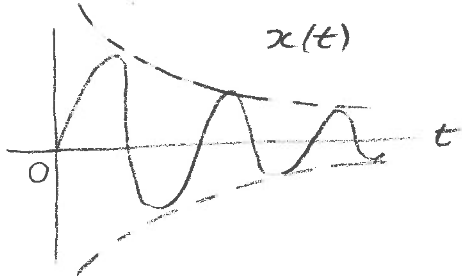
Impulse $f(t)=\delta(t)$



$$x(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_d t$$

$$x_{ss} = \lim_{t\to\infty} \frac{\omega_n}{\sqrt{1-\zeta^2}} \cancelto{0}{e^{-\zeta\omega_n t}} \sin\omega_d t$$

$$\boxed{x_{ss}^{impulse} = 0}$$

$$e(t) = f(t) - x(t) = \delta(t) - \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\omega_d t$$

$$e_{ss} = \lim_{t\to\infty}\left(\cancelto{0}{\delta(t)} - \frac{\omega_n}{\sqrt{1-\zeta^2}} \cancelto{0}{e^{-\zeta\omega_n t}} \sin\omega_d t \right)$$

$$\boxed{e_{ss}^{impulse} = 0}$$

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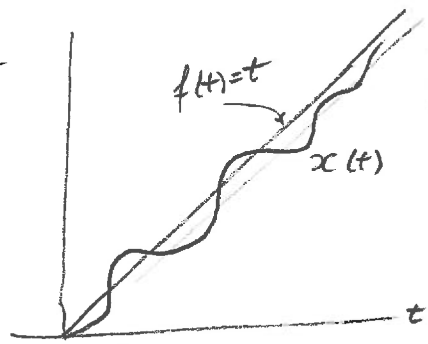
Ramp $f(t) = t$



$$x(t) = t - \frac{2\zeta}{\omega_n}\left[1 + \frac{1}{\sin\varphi_1} e^{-\zeta\omega_n t} \sin(\omega_d t - \varphi_1)\right]$$

$$\varphi_1 = \tan^{-1} \frac{2\zeta\sqrt{1-\zeta^2}}{1-2\zeta^2}$$

$$x_{ss} = \lim_{t\to\infty} x(t) = t - \frac{2\zeta}{\omega_n} \longrightarrow \infty$$

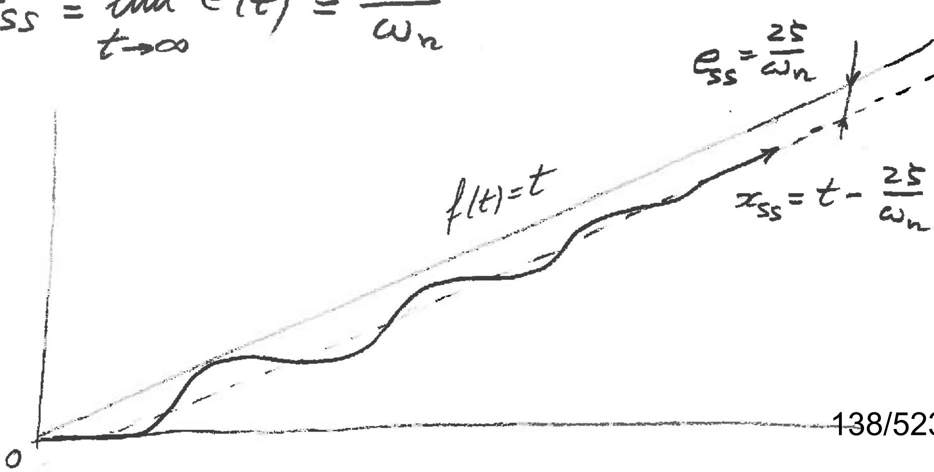
No fixed x_{ss} value
Grows continously!

$$e(t) = f(t) - x(t)$$

$$= t - \left\{t - \frac{2\zeta}{\omega_n}\left[1 + \frac{1}{\sin\varphi_1} e^{-\zeta\omega_n t} \sin(\omega_d t - \varphi_1)\right]\right\}$$

$$e(t) = \frac{2\zeta}{\omega_n}\left[1 + \frac{1}{\sin\varphi_1} e^{-\zeta\omega_n t} \sin(\omega_d t - \varphi_1)\right]$$

$$e_{ss} = \lim_{t\to\infty} e(t) = \frac{2\zeta}{\omega_n}$$



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2nd order system x_{ss} and e_{ss} by FVT

$$x_{ss} = \lim_{s\to 0} s X(s)$$

$$X(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} F(s)$$

$$e_{ss} = \lim_{s\to 0} s E(s)$$

$$E(s) = F(s) - X(s) = \left(1 - \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right) F(s)$$

$$E(s) = \frac{s^2+2\zeta\omega_n s}{s^2+2\zeta\omega_n s+\omega_n^2} F(s)$$

Step $F(s) = \frac{1}{s}$

$$x_{ss} = \lim_{s\to 0} \cancel{s} \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \frac{1}{\cancel{s}} = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$e_{ss} = \lim_{s\to 0} \cancel{s} \frac{s^2+2\zeta\omega_n s}{s^2+2\zeta\omega_n s+\omega_n^2} \frac{1}{\cancel{s}} = \frac{0}{\omega_n^2} = 0$$

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Impulse $F(s)=1$

$$x_{ss}=\lim_{s\to 0} s\,\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\cdot 1 = 0$$

$$e_{ss}=\lim_{s\to 0} s\,\frac{s^2+2\zeta\omega_n s}{s^2+2\zeta\omega_n s+\omega_n^2}\cdot 1 = 0$$

Ramp $F(s)=\frac{1}{s^2}$

$$x_{ss}=\lim_{s\to 0} \cancel{s}\,\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\cdot\frac{1}{s^{\cancel{2}}}=\lim_{s\to 0}\frac{1}{s}=\infty$$

No ss value!

$$e_{ss}=\lim_{s\to 0} \cancel{s}\,\frac{s^{\cancel{2}}+2\zeta\omega_n \cancel{s}}{s^2+2\zeta\omega_n s+\omega_n^2}\cdot\frac{1}{\cancel{s^2}}$$

$$=\lim_{s\to 0}\frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{2\zeta\omega_n}{\omega_n^2}=\frac{2\zeta}{\omega_n}$$

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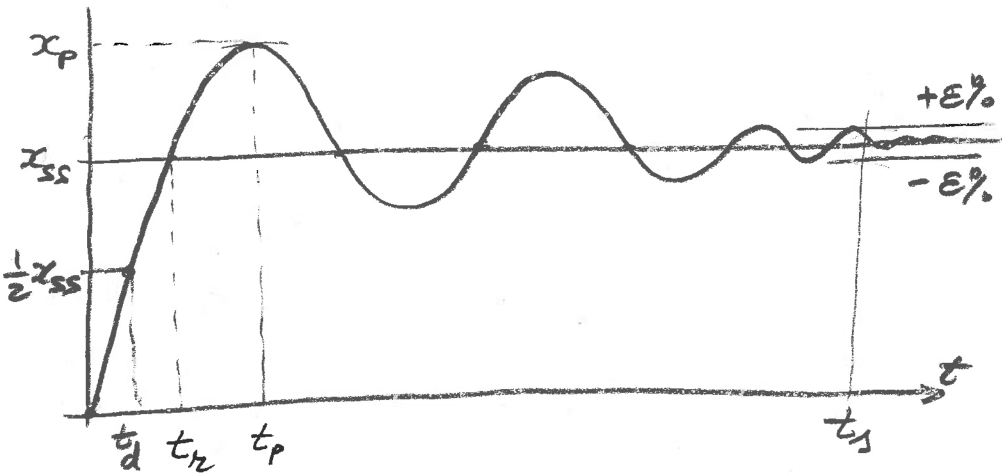
2^{nd} order syst. Specific PI_s



t_r = rise time < 0 — 100% Theory; 5% — 95% MATLAB

t_p = peak time

x_p = peak value

M_p = max. percentage overshoot

$$M_p = \left(\frac{x_p}{x_{ss}} - 1\right) 100\%$$

t_s = settling time for a given $\varepsilon\%$

t_d = delay time ; time to rise to $\frac{1}{2}x_{ss}$ for the first time

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Procedure

Given: 2nd order sys. $G(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

ω_n = freq

ζ = damping.

1. Calculate:

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

$$\varphi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \sin^{-1}\sqrt{1-\zeta^2}$$

2. calculate performance indicators:

$$t_r = \frac{\pi - \varphi}{\omega_d}$$

$$t_p = \frac{\pi}{\omega_d}$$

$$x_p = 1 + e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}$$

$$M_p = e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}$$

$$t_s \simeq \frac{4}{\zeta\omega_n} \qquad \zeta << 1 \ ; \ \pm 2\%$$

Proof of 2nd order sys. P.I.

Recall the time response:

$$x(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \varphi) \qquad (1)$$

- rise time t_r $\quad 0 \longrightarrow 100\%$

$x(0) = 0 \qquad x(t_r) = 1$

$x(t_r) = 0$

Need to have $\sin(\omega_d t_r + \varphi) = 0$ in Eq. (1).

$$\omega_d t_r + \varphi = \pi \rightarrow t_r = \frac{\pi - \varphi}{\omega_d}.$$

- peak time t_p

$\frac{dx}{dt} = 0$

$$\left.\frac{dx}{dt}\right|_{t=t_p} = 0$$

$$\frac{dx}{dt} = 0 \rightarrow \frac{d}{dt}\left[e^{-\zeta\omega_n t_p} \sin(\omega_d t_p + \varphi)\right] = 0$$

$$-\zeta\omega_n \cancel{e^{-\zeta\omega_n t_p}} \sin(\quad) + \cancel{e^{-\zeta\omega_n t_p}} \omega_d \cos(\quad) = 0$$

$$\zeta\omega_n \sin(\omega_d t + \varphi) = \omega_d \cos(\omega_d t + \varphi)$$

$$\tan(\omega_d t_p + \varphi) = \frac{\omega_d}{\zeta\omega_n} = \frac{\sqrt{1-\zeta^2}}{\zeta} = \tan\varphi$$

Must find t_p such that

$$\tan(\omega_d t_p + \varphi) = \tan\varphi$$

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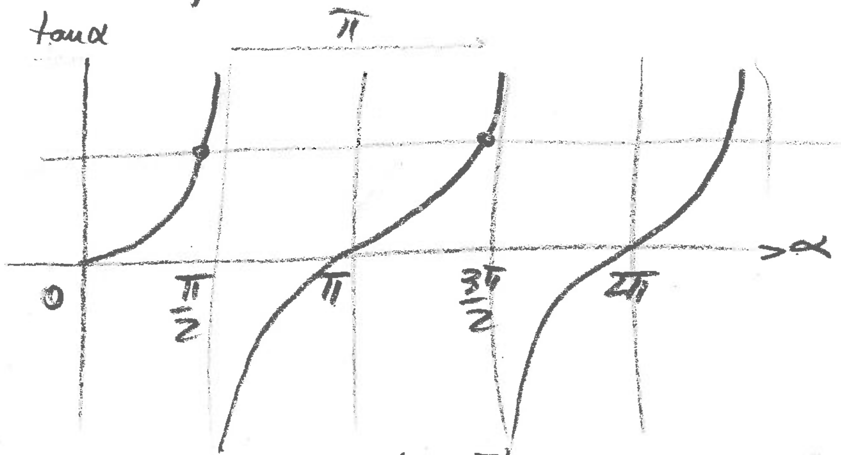
The function $\tan\alpha$ repeats itself after $\pi, 2\pi, 3\pi, \cdots$



$\tan\alpha = \tan(\alpha+\pi) = \tan(\alpha+2\pi) \cdots$

Hence: $\omega_d t_p + \varphi = \varphi + \pi \Rightarrow t_p \omega_d = \pi$

$$t_p = \frac{\pi}{\omega_d}$$

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Peak value, x_p

$$x_p = x(t_p) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n \frac{\pi}{\omega_d}} \underbrace{\sin(\varphi+\pi)}_{-\sin\varphi}$$

$$x_p = 1 + \frac{1}{\cancel{\sqrt{1-\zeta^2}}}\left(e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}\right)\cancel{\sqrt{1-\zeta^2}}$$

$(\sin\varphi = \sqrt{1-\zeta^2})$

$$\boxed{x_p = 1 + e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}}$$

Max. overshoot, M_p

$$M_p = \frac{x_p - x_{ss}}{x_{ss}} = \frac{1 + e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi} - 1}{1}$$

$$\boxed{M_p = e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}}$$

$\zeta = 0.4 \longrightarrow M_p \simeq 25\%$

$ 0.8 \longrightarrow 1.5\%$

$0.4 < \zeta < 0.8$ usual range

$25\% < M_p < 1.5\%$

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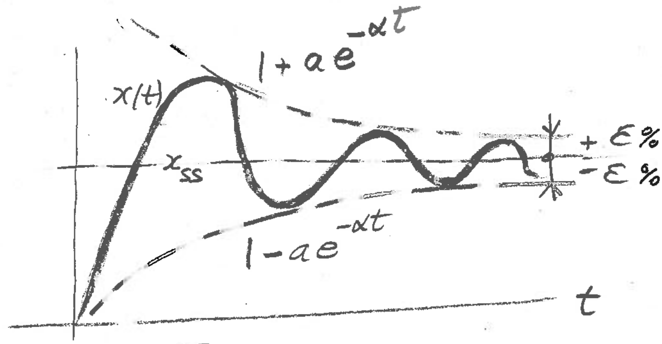
Settling time, t_s

$|x_{ss} - x(t_s)| < \Delta$

and

$|x_{ss} - x(t > t_s)| < \Delta$

$\Delta = \varepsilon \cdot x_{ss}$



$$x(t) = 1 - \underbrace{\frac{1}{\sqrt{1-\zeta^2}}}_{a} e^{\overbrace{-\zeta\omega_n}^{\alpha} t} \sin(\omega_d t + \varphi)$$

$$x(t) = 1 - a e^{-\alpha t} \sin(\omega_d t + \varphi) \quad ; \; x_{ss} = 1$$

Envelopes: $1 \pm a e^{-\alpha t}$

Settling condition: $a e^{-\alpha t} = \Delta$

Approx. calculation $a = 1/\sqrt{1-\zeta^2} \Big|_{\zeta \ll 1} \simeq 1$

$$e^{-\alpha t} = \frac{\Delta}{a} = \varepsilon$$

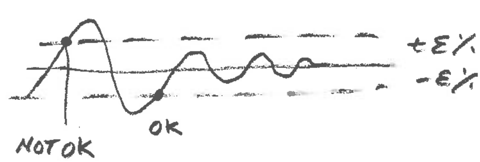
For $\varepsilon = 2\%$

$$-\alpha t = \ln 0.02 = -3.9 \simeq -4$$

$$t_s = \frac{4}{\alpha} = \frac{4}{\zeta\omega_n} \quad \left\{ \begin{array}{l} \zeta \ll 1 \\ \varepsilon = 2\% \end{array} \right.$$

Definition of settling time:

"Get within $\pm\varepsilon\%$ of x_{ss} and stay so"



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Effect of ζ & ω_n on performance

ω_n : shortens: rise time, t_r
peak time, t_p
settling time, t_s

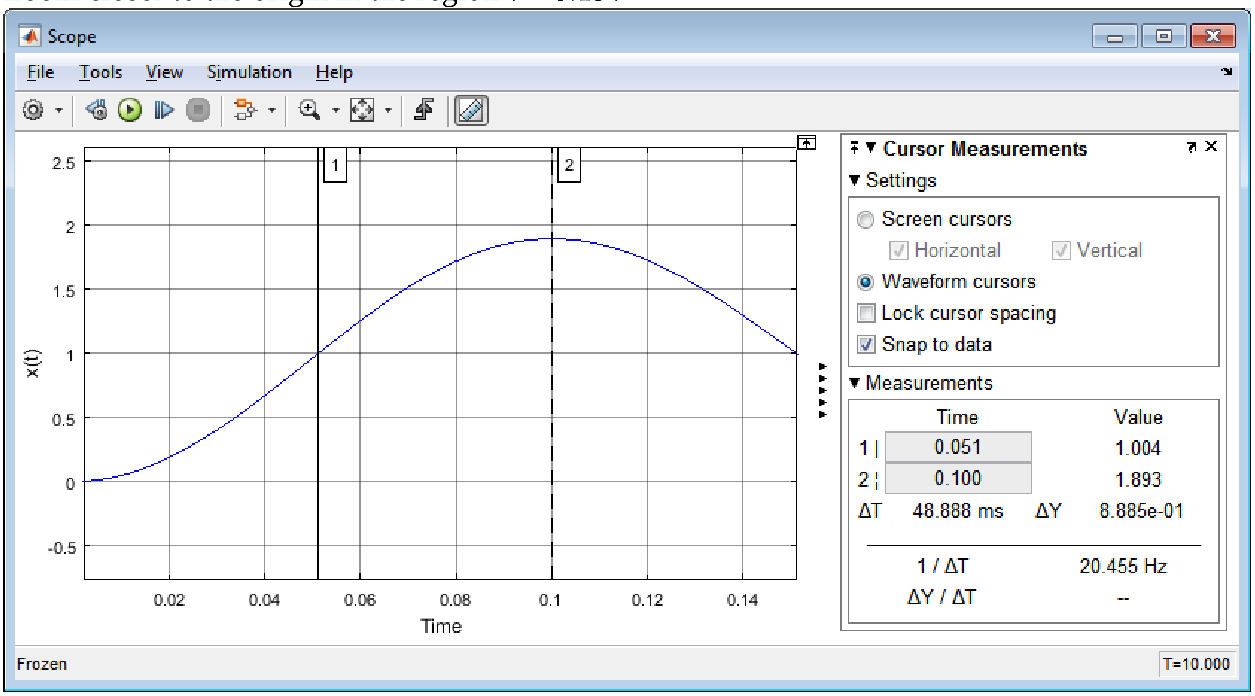
ζ : reduces max overshoot, M_p
shortens settling time

4.4 USE TRACE TO MEASURE PERFORMANCE INDICATORS

Release the 'Zoom' button and press the 'Cursor Measurements' button to activate the cursors. In the 'Cursor Measurements → Settings' check 'Snap to data' box.

4.4.1 Measurement of t_r, t_p, x_p, M_p

Zoom closer to the origin in the region $t < 0.15$.



Use the first cursor to find the first crossing of $x_{ss} = 1$. Read the time as $t_r = 0.051$ sec

Place the second cursor at peak value. Read the peak time $t_p = 0.100$ sec and peak amplitude $x_p = 1.893$. Calculate $M_p = 89.3\%$.