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0161018

Standard form of inhomogeneous eq$^{\underline{n}}$

Divide Eq. (1) by m, i.e.,

$$\frac{1}{m}(1): \quad \ddot{x} + \frac{k}{m}x = \frac{1}{m}f^* \qquad (12)$$

Recall $\frac{k}{m} = \omega_n^2$ (13)

Define $f(t) = \frac{1}{k}f^*(t)$ normalized forcing function (14)

$$(14): \quad \frac{1}{m}f^*(t) \underset{(14)}{=} \frac{1}{m}kf(t) \underset{(13)}{=} \omega_n^2 f(t) \qquad (15)$$

$(13),(15) \longrightarrow (12)$:

$$\ddot{x} + \omega_n^2 x = \omega_n^2 f(t) \qquad (16)$$

2nd order inhomogeneous ODE in standard form

Forced vibration response

$$x(t) = x_c(t) + x_p(t) = C\sin(\omega_n t + \varphi) + x_p(t) \qquad (17)$$

$(17) \longrightarrow (16)$:

$$\ddot{x}_c + \ddot{x}_p + \omega_n^2(x_c + x_p) = \omega_n^2 f(t)$$

$$\underbrace{\left(\ddot{x}_c + \omega_n^2 x_c\right)}_{=0} + \ddot{x}_p + \omega_n^2 x_p = \omega_n^2 f(t)$$

$$\ddot{x}_p + \omega_n^2 x_p = \omega_n^2 f(t) \qquad (18)$$

Need to find $C, \varphi, x_p(t)$; not easy!

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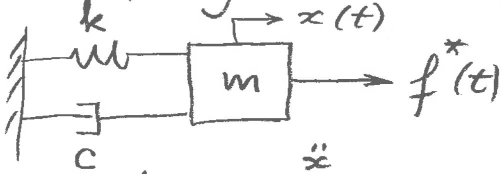
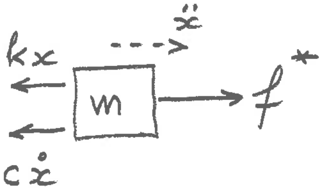
20161019

Spring-mass-damper oscillator



FBD



NLM $\quad m\ddot{x} = -kx - c\dot{x} + f^*$

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f^*(t) \qquad \text{EOM} \qquad (1)$$

- Damped forced vibration eqⁿ.
- 2ⁿᵈ order inhomogeneous ODE

Solution $x(t)$ of Eq.(1) consists of the sum of complementary solⁿ $x_c(t)$ and of particular solⁿ $x_p(t)$ where the complementary solⁿ satisfies the homogeneous eqⁿ while the particular solution satisfies the inhomogeneous eqⁿ.

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2016/10/17 Damped free vibration (homogeneous eq^n)

Set to zero the RHS of Eq. (1) to get

$$m\ddot{x} + c\dot{x} + kx = 0 \quad \text{(damped free vibration eq}^n\text{)} \qquad (2)$$

Assume $\quad x(t) = C e^{pt} \qquad (3)$

Hence
$$\left.\begin{aligned} \dot{x} &= pCe^{pt} = px \\ \ddot{x} &= p^2 C e^{pt} = p^2 x \end{aligned}\right\} \qquad (4)$$

(3), (4) → (2) :

$$mp^2 x + cpx + kx = 0$$

or
$$mp^2 + cp + k = 0 \quad \text{characteristic eq}^n \qquad (5)$$

$$p_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

$$p_{1,2} = -\frac{c}{2m} \pm i\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} \qquad (6)$$

Critical damping, c_{cr}

Define critical damping c_{cr} as the value of c to make zero the term under the square root sign, i.e.,

$$\frac{k}{m} - \left(\frac{c_{cr}}{2m}\right)^2 = 0$$

$$c_{cr}^2 = \frac{4m^2 k}{m} = 4mk \qquad c_{cr} = 2\sqrt{mk} \qquad (7)$$

10/16/2017

Damping ratio, ζ

Define damping ratio ζ as the ratio between damping c and critical damping c_{cr}, i.e.,

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{mk}} \quad \text{damping ratio} \qquad (8)$$

Natural frequency, ω_n

Recall the definition

$$\omega_n^2 = \frac{k}{m}, \quad \omega_n = \sqrt{\frac{k}{m}} \qquad (9)$$

Damped frequency, ω_d

Calculate

$$\frac{c}{2m} \underset{(8)}{=} \frac{\zeta c_{cr}}{2m} \underset{(7)}{=} \zeta \frac{2\sqrt{mk}}{2m} = \zeta\sqrt{\frac{k}{m}} \underset{(9)}{=} \zeta\omega_n \qquad (10)$$

(9), (10) $\rightarrow$ (6):

$$p_{1,2} = -\zeta\omega_n \pm i\sqrt{\omega_n^2 - \zeta^2\omega_n^2}$$

$$= -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$$

$$p_{1,2} = -\zeta\omega_n \pm i\omega_d \qquad (11)$$

where

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \quad \text{damped frequency} \qquad (12)$$

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2016/10/19

Damped free vibration response

(11) → (3) :

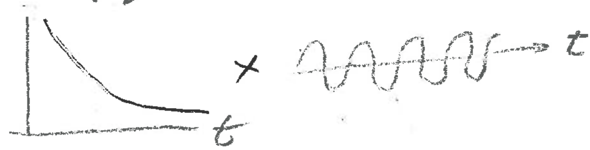
$$x(t) = C_1 e^{(-\zeta\omega_n + i\omega_d)t} + C_2 e^{(-\zeta\omega_n - i\omega_d)t}$$

$$= e^{-\zeta\omega_n t}\left(C_1 e^{i\omega_d t} + C_2 e^{-i\omega_d t}\right)$$

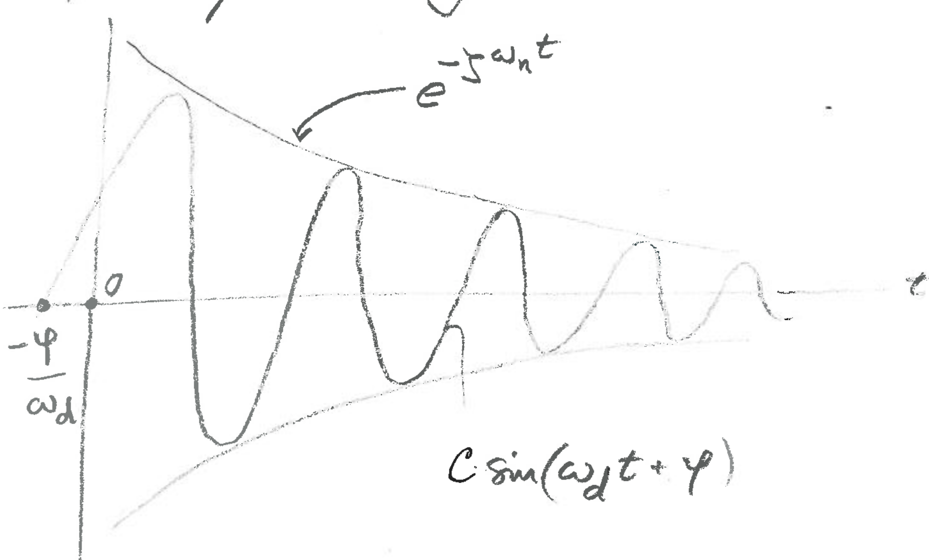
or

$$x_c(t) = C e^{-\zeta\omega_n t} \sin(\omega_d t + \varphi) \qquad (13)$$

natural freq. ω_n damped freq. ω_d



- Free damped vibration response
- Complementary sol^{n} $x_c(t)$



Damped oscillatory motion

Standard form of damped forced vibration equation

Divide Eq. (1) by m, i.e.

$$\frac{1}{m}(1): \quad \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{1}{m}f^{*}(t) \qquad (14)$$

Calulate

$$\frac{c}{m} \underset{(8)}{=} \frac{\zeta c_{cr}}{m} \underset{(7)}{=} \frac{\zeta\, 2\sqrt{mk}}{m} \underset{(9)}{=} 2\zeta\omega_n \qquad (15)$$

Define $\quad f(t) = \frac{1}{k} f^{*}(t) \quad$ normalized forcing function $\qquad (16)$

Calculate

$$\frac{1}{m} f^{*}(t) \underset{(16)}{=} \frac{1}{m} k f(t) \underset{(9)}{=} \omega_n^2 f(t) \qquad (17)$$

(15), (17) $\rightarrow$ (14) :

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \omega_n^2 f(t) \qquad (18)$$

- Damped force vibration eq^{n} in standard form
- 2nd order ODE in standard form.

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2016 10 19

Damped forced vibration response.

$$x(t) = x_c(t) + x_p(t)$$
$$= e^{-\zeta\omega_n t} C \sin(\omega_d t + \varphi) + x_p(t) \qquad (19)$$

(19) → (18) :

$$\underbrace{\left(\ddot{x}_c + 2\zeta\omega_n \dot{x}_c + \omega_n^2 x_c\right)}_{=0}$$
$$+ \ddot{x}_p + 2\zeta\omega_n \dot{x}_p + \omega_n^2 x_p = \omega_n^2 f(t) \qquad (20)$$

Need to find $C, \varphi, x_p(t)$

NOT easy!