TRANSFER FUNCTION SOL$^{\underline{N}}$ OF ODEs

Example: 2^{nd} order ODE in standard form:

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \omega_n^2 f(t) \qquad (1)$$

$\mathcal{L}$ Eq(1):

$$s^2 X + 2\zeta\omega_n s X + \omega_n^2 X = \omega_n^2 F(s)$$

or

$$(s^2 + 2\zeta\omega_n s + \omega_n^2) X(s) = \omega_n^2 F(s) \qquad (2)$$

Solution of Eq.(2) yields

$$X(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} F(s) \qquad (3)$$

Define

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad (4)$$

Transfer function

(4) → (3):

$$X(s) = G(s) F(s) \qquad (5)$$

$$F(s) \longrightarrow \boxed{G(s)} \longrightarrow X(s)$$

Block diagram

This concept is generalised to higher order ODEs.

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Transfer Function Models

- Polynomial model : (numerator / denominator)

$$G(s) = \frac{B(s)}{A(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

$n =$ order of TF model $\qquad m < n$

- Zero-pole-gain model (z-p-k)

$$G(s) = K \frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$$

- zeros : $z_1, z_2, \ldots, z_m$ roots of $B(s)=0$
- poles : $p_1, p_2, \ldots, p_n$ roots of $A(s)=0$
- gain : K

- Time constant model

$$G(s) = \frac{K}{s^N} \cdot \frac{(T_a s+1)(T_b s+1)\ldots}{(T_1 s+1)(T_2 s+1)\ldots}$$

$N =$ Type of TF model.

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Create Transfer Function Using Numerator and Denominator Coefficients

This example shows how to create continuous-time single-input, single-output (SISO) transfer functions from their numerator and denominator coefficients using `tf`.

Create the transfer function $G(s) = \dfrac{s}{s^2 + 3s + 2}$:

Method I

```
num = [1 0];
den = [1 3 2];
G = tf(num,den);
```

`num` and `den` are the numerator and denominator polynomial coefficients in descending powers of s. For example, `den = [1 3 2]` represents the denominator polynomial $s^2 + 3s + 2$.

`G` is a `tf` model object, which is a data container for representing transfer functions in polynomial form.

Method II

> **Tip** Alternatively, you can specify the transfer function $G(s)$ as an expression in s:
>
> 1. Create a transfer function model for the variable s.
>
> ```
> s = tf('s');
> ```
>
> 2. Specify $G(s)$ as a ratio of polynomials in s.
>
> ```
> G = s/(s^2 + 3*s + 2);
> ```

$$G(s) = \frac{B(s)}{A(s)} = \frac{s}{s^2+3s+2} = \frac{b_1 s + b_0}{a_2 s^2 + a_1 s + 2}$$

$$B(s) = s = b_1 s + b_0$$

$$b_1 = 1 \quad b_0 = 0 \qquad B = [1 \quad 0]$$

$$A(s) = s^2 + 3s + 2$$

$$= a_2 s^2 + a_1 s + a_0$$

$$a_2 = 1 \quad a_1 = 3 \quad a_0 = 1 \qquad A = [1 \quad 3 \quad 1]$$

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Create Transfer Function Model Using Zeros, Poles, and Gain

This example shows how to create single-input, single-output (SISO) transfer functions in factored form using `zpk`.

Create the factored transfer function $G(s) = 5\frac{s}{(s+1+i)(s+1-i)(s+2)}$:

```
Z = [0];
P = [-1-1i -1+1i -2];
K = 5;
G = zpk(Z,P,K);
```

`Z` and `P` are the zeros and poles (the roots of the numerator and denominator, respectively). `K` is the gain of the factored form. For example, $G(s)$ has a real pole at $s = -2$ and a pair of complex poles at $s = -1 \pm i$. The vector `P = [-1-1i -1+1i -2]` specifies these pole locations.

$$G(s) = 5\frac{s}{(s+1+i)(s+1-i)(s+2)}$$

$$= 5\frac{(s-0)}{[s-(-1-i)][s-(-1+i)][s-(-2)]}$$

$$= K\frac{s-z_1}{(s-p_1)(s-p_2)(s-p_3)}$$

$K = 5$

$s-0 = s-z_1 \longrightarrow z_1 = 0 \qquad Z = [0]$

$[s-(-1-i)][s-(-1+i)][s-(-2)] = (s-p_1)(s-p_2)(s-p_3)$

$$\left.\begin{aligned} p_1 &= -1-i \\ p_2 &= -1+i \\ p_3 &= -2 \end{aligned}\right\} \quad P = [-1-i \quad -1+i \quad -2]$$

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Order vs. Type

Example: mass-damper system (no stiffness):

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s} = \frac{\omega_n^2}{s(s+2\zeta\omega_n)} = \frac{\omega_n/2\zeta}{s} \cdot \frac{1}{\left(\frac{1}{2\zeta\omega_n}\right)s + 1}$$

or

$$G(s) = \frac{b_0}{a_2 s^2 + a_1 s} = \frac{K}{s^N} \frac{1}{T_1 s + 1}$$

where $\boxed{a_2 = 1}$

$a_1 = 2\zeta\omega_n$

$b_0 = \omega_n^2$

$\rightarrow$ $\boxed{\text{2nd order system}}$

$K = \frac{\omega_n}{2\zeta}$

$\boxed{N = 1}$

$T_1 = \frac{1}{2\zeta\omega_n}$

$\rightarrow$ $\boxed{\text{Type 1 system}}$

This is a 2nd order system of Type 1

"Type" and "Order" have different meaning!

More Type-Order examples

	Type	order
$G(s) = \dfrac{1}{Ts+1}$	0	0
$G(s) = \dfrac{1}{cs} = \dfrac{1/c}{s}$	1	1
$G(s) = \dfrac{1}{Js^2} = \dfrac{1/J}{s^2}$	2	2
$G(s) = \dfrac{K}{Js^2+cs} = \dfrac{K/c}{s} \cdot \dfrac{1}{\frac{J}{c}s+1}$	1	2
$G(s) = \dfrac{K}{s^4} \dfrac{T_a s+1}{(T_1 s+1)(T_2 s+1)(T_3 s+1)}$	4	7

$$= \frac{b_1 s + b_0}{s^4 (a_3 s^3 + a_2 s^2 + a_1 s + a_0)}$$

$$= \frac{b_1 s + b_0}{a_3 s^7 + a_2 s^6 + a_1 s^5 + a_0 s^4}$$

"Order" $= n$, highest exponent of s in the denominator

n = number of poles

Type $= N$, exponent of the factored out s in the denominator.

N = number of poles in Origin ($p=0$)